國立臺灣大學管理學院資訊管理學系

碩士論文

Department of Information Management
College of Management
National Taiwan University

Master’s Thesis

考量內生需求、資源分配與需求競食之設施選址

A Competitive Facility Location Problem with Endogenous Demand and Resource Allocation

陳柄瑞
Ping-Jui Chen

指導教授：孔令傑 博士
Advisor：Ling-Chieh Kung, Ph.D.

中華民國 113 年 7 月
July 2024

doi:10.6342/NTU202402305

謝辭

首先，我要向我的指導教授致以最誠摯的謝意。在這段學習與研究的過程中，孔令傑老師的悉心指導與不懈支持讓我受益匪淺。無論是在研究方向的確定還是論文的撰寫上，老師總是耐心地為我解答疑惑，提供寶貴的建議，並且在我遇到困難時給予鼓勵。老師的專業知識與敬業精神深深影響了我，讓我在學術道路上更加堅定和自信。

此外，我也要感謝我的兩位學弟陳瑾叡和吳驊祐。他們在實驗數據的收集與分析、演算法開發等方面給予了我極大的幫助。兩位所提供的豐富資料為我的研究奠定了堅實的基礎。

感謝大家的支持與幫助，這份論文的完成離不開你們的努力與奉獻。我將銘記這段難忘的合作經歷。

陳柄瑞 謹識
于臺大資訊管理學研究所
民國一百一十三年七月

doi:10.6342/NTU202402305

摘要

競爭性設施選址問題長年以來受到了廣泛的討論。在一般的設施選址問題中，決策者需要決定設施的位址、以及分配哪些使用者該前往哪些設施。然而，當我們討論到競爭性設施時，我們會發現各設施對於消費者的效用取決於決策者投放於設施的資源以及設施與消費者的距離。在這種情況下，使用者的行為便不能被決策者強制決定，而是消費者會依據設施的效用來選擇前往的設施。此外，決策者除了透過建設設施來增加其吸引力之外，也能將不同的有限資源投放於設施中。不同的資源被分配至不同設施時將產生不同的吸引力，這成為另一個設施及資源差異性的來源。將不同的資源種類分配方式納入考慮能讓我們的問題更貼近現實情況。

在我們的研究中，我們考慮一個具有不同資源類型、各資源類型有限、資源吸引力於各設施不同的競爭性設施選址問題，決策者需要決定設施的位址、數量、以及分配於各設施的資源種類，目標是吸引儘可能多的總服務人數以最大化利潤。為解決此問題，我們建立了一個混合整數規劃模型並開發一個啟發式演算法，透過數值實驗，可以看到我們的演算法能在可接受的時間範圍內得到接近最佳解的結果。

關鍵字：設施選址、競爭性設施、資源分配、內生性需求、混合整數規劃模型

doi:10.6342/NTU202402305

Abstract

The competitive facility location (CFL) problem has been widely discussed for many years. In a typical facility location problem, decision makers need to determine the locations of facilities and allocate which users should go to which facilities. However, when we discuss CFL problems, we find that the utility of each facility for consumers depends on the resources allocated by decision makers to the facilities and the distance between the facilities and consumers. In this case, user behavior cannot be forcibly determined by decision makers; instead, consumers choose facilities based on their utility. In addition to increasing attractiveness level of facilities by constructing additional enhancements, decision makers can allocate different limited resources to the facilities. Different allocations of resources to different facilities result in varying attractiveness, creating another source of facility and resource heterogeneity. Considering different resource types in the allocation process makes our problem more realistic.

In our study, we consider a competitive facility location problem with different resource types, limited quantities for each resource type, and varying resource attractiveness for different facilities. Decision makers need to determine the locations and quantities of facilities and allocate resource types to each facility, aiming to attract as many served users as possible to maximize profit. To solve this problem, we developed a mixed-integer programming model and a heuristic algorithm. Through numerical experiments, we observed that our algorithm produces results close to the optimal solution within an acceptable time frame.

doi:10.6342/NTU202402305

Keywords: Facility location, Competitive facility, Resources allocation, Endogenous demand, Mixed-integer programming model

doi:10.6342/NTU202402305

Contents

謝辭 **i**

摘要 **ii**

Abstract **iii**

1 Introduction **1**

1.1 Background and motivation 1

1.2 Research objectives 4

1.3 Research plan 5

2 Literature Review **6**

2.1 CFL problem without diminishing marginal benefit 7

2.2 CFL problem with diminishing marginal benefit 8

2.3 Diminishing marginal benefit and attractiveness decision 9

3 Problem Description and Formulation **11**

doi:10.6342/NTU202402305

4 The Algorithm **20**

4.1 Overview 20

4.2 The Lagrangian Relaxation-based Algorithm with Dimension Reduction . 22

4.2.1 Relaxation of The Original Problem 22

4.2.2 Simplification of the model 23

4.2.3 The Iteration Process for Optimizing the Lagrangian Multipliers . 27

4.2.4 Adjusting Infeasible Solutions 29

5 Numerical Study **35**

5.1 Experiment Setting 35

5.2 Benchmarks 36

5.2.1 The Continuous Relaxation of The Original Problem 36

5.2.2 The Greedy Heuristic Algorithm 37

5.3 Solution Performance Evaluation 41

5.4 Time Performance Evaluation 43

6 Conclusion **46**

doi:10.6342/NTU202402305

List of Figures

4.1 The flowchart of the Lagrangian relaxation-based algorithm with dimension reduction 21

4.2 The flowchart of the infeasible solutions adjusting process 31

5.1 The flowchart of the first stage of GHA 38

5.2 The flowchart of the second stage of GHA 39

doi:10.6342/NTU202402305

List of Tables

3.1 List of decision variables 15

3.2 List of sets and parameters 16

5.1 Instance size definition 35

5.2 Solution performance of different problem sizes 42

5.3 Time performance of different problem sizes (in seconds) 44

5.4 Computational time of the iteration process in LRADR (in seconds) . . . 45

doi:10.6342/NTU202402305

Chapter 1

Introduction

1.1 Background and motivation

The problems associated with competitive facility location (CFL) have been extensively examined over the span of several decades. In a standard competitive facility location scenario, a decision maker must determine optimal locations for establishing facilities from a predefined set. Also, the decision maker strategically positions them to rival existing or future facilities of competing firm(s) in the market. The decision maker aims to gain a competitive market share while the consumers in the market would choose a facility to be served. The primary goals of this problem typically encompass profit maximization, cost minimization, and other related objectives from the perspective of the decision maker.

In many cases, the quantity and placement of facilities frequently influence consumers' inclination to purchase a product or engage in a service. Indeed, in numerous instances, consumer demands may surge only when an adequate number of facilities are established.

doi:10.6342/NTU202402305

The public bicycle sharing systems around the world serve as an example. These systems can only achieve success once a sufficient number of rental sites have been established and an ample supply of bicycles has been provided.

In the context of the CFL problem, the cannibalization effect occurs when a company introduces new facilities to compete with existing facilities for the same market share. The new facilities attract customers who would have otherwise chosen one of the existing facilities. Such an effect is also widely discussed in the previous studies (Aboolian et al., 2007; Küçükaydin et al., 2011; Yu, 2019). In simpler terms, the cannibalization effect suggests that the presence of a new facility may draw customers away from the existing facilities from all firms rather than attracting entirely new customers. This may result in a shift of market share within the company's own network of facilities, potentially diminishing the overall sales or performance growth.

For a decision maker addressing the CFL problem, managing the cannibalization effect becomes crucial. The decision maker needs to plan on how to introduce new facilities in a way that minimizes the negative impact on existing facilities, ensuring that the overall market share is increased rather than just redistributed among the company's own facilities. Balancing the placement and characteristics of new facilities is essential to optimize the competitive position of the entire network.

As mentioned above, the cannibalization effect implies that a new facility may divert customers from existing facilities of all firms, rather than bringing in entirely new customers. However, we still aim to model the market expansion effect—capturing additional demand from customers who were not adequately served by the existing facilities. Several works also mention the market expansion effect (Aboolian et al., 2007; Lin and

doi:10.6342/NTU202402305

Tian, 2021). We reckon that market demand should be endogenous (dependent on decisions) rather than exogenous, so that the market demand is elastic and increases with the total attractiveness of all facilities.

Beside all the aforementioned factors, we believe that there is another factor that should be considered in CFL problems. Take vehicle-sharing systems as another noteworthy illustration. Unlike public bicycle sharing systems with only one type of bicycle, there are usually many different types of vehicles for customers to choose from in vehicle-sharing systems. Customers would only be willing to use the vehicle-sharing service when there are enough preferred cars nearby and plenty of spaces to park around destinations. Also, different vehicle types are with varying attractiveness for customers. As a result, the types of resources significantly influence the decisions made by the decision maker.

For the same vehicle-sharing systems example, the numbers of different vehicle types are limited for decision maker. Therefore, a decision maker has to not only determine the number and locations of rental sites but also consider how to distribute all cars on hand to appropriate locations based on types of cars. This is a problem involving resources allocation. Here are some other examples of similar scenarios that may involve resources allocation. When building gyms, we must not only decide on the number and locations but also allocate limited resources like fitness trainers strategically to attract the maximum number of consumers. When constructing hospitals, we also have to allocate specific resources to capture as much demand as possible, given that the numbers of certain extremely expensive equipment are limited.

Although works of the CFL problem with attractiveness decision are proposed (Küçükaydin et al., 2011; Lin and Tian, 2021), resources allocation is not involved. Therefore, we

doi:10.6342/NTU202402305

are going to discuss a CFL problem that a decision maker determines the number and locations of facilities as well as allocates limited resources to them in order to gain a competitive market share.

1.2 Research objectives

In this study, we explore a competitive facility location problem involving resource allocation. A decision maker first selects locations to build facilities from a set of candidate locations. At the same time, the decision maker determines how to allocate the available resources to the established facilities. The total amount of each resource on hand are limited. There are also upper bounds for the total amount of all resources and the total amount of each resource for each facility. The resources have varying attractiveness values when assigned to different facilities. In addition to allocating resources to facilities, the decision maker can also enhance the attractiveness level of each facility by adding additional attractiveness. All resources and additional attractiveness enhancements come with a cost. Groups of customers with varying population sizes are located in different places. Customers would patronize the built facilities according to Huff's gravity rule. The demand from each group of customers would be split to each facility according to the ratio of facility attractiveness and distance from facility. The goal is to maximize profit by attracting a maximum number of customers while simultaneously competing with existing facilities.

We consider a decision maker entering a market with existing facilities built by a single firm or multiple firms. Customers in the market would choose a facility to be served. The

doi:10.6342/NTU202402305

gravity rule proposed by Huff (1964) will be employed to model the patronizing behavior of customers in this research. A customer is more likely to choose a facility based on its attractiveness, with the likelihood decreasing as distance from the facility increases. Facilities of all firms are competing with each other to capture as much demand as possible. Meanwhile, we assume that the competitors already exist and would not react to the decision maker. We also assume that the decisions are permanent, meaning that the locations and the distribution of resources would not be changed.

1.3 Research plan

In the next chapter, we will review some relevant literature about competitive facility location problem. We will discuss the classical CFL problem and the CFL problem with and without endogenous demands as well as attractiveness decision. In Chapter 3, we will formulate our competitive facility location problem with resource allocation as an integer program and prove its NP-hardness. The proposed algorithm will be presented in Chapter 4. The numerical study and conclusion will be discussed in Chapters 5 and 6, respectively.

doi:10.6342/NTU202402305

Chapter 2

Literature Review

The Competitive Facility Location (CFL) problem diverges from classical facility location problems in the operations research field due to their inclusion of competition among facilities associated with different firms. The existence of competition is the primary factor that distinguishes the CFL problem from traditional location problems (Küçükaydin et al., 2011). The first competitive facility location problem is proposed by Hotelling (1929). Hotelling considers the problem of locating two competing facilities along a segment, such as two ice-cream vendors along a beach strip. The distribution of purchasing power along the segment is assumed to be uniform, and customers choose the closest facility. In this case, competition exists among different firms. In a more general setting, competition may also exist among facilities built by a single firm or multiple firms. The subsequent discussion will cover various examples. Problem categories differ depending on the objective function and constraints. In the following discussion, we will put emphasis on whether the CFL problem considers diminishing marginal benefits and attractiveness decision.

doi:10.6342/NTU202402305

2.1 CFL problem without diminishing marginal benefit

In classical CFL problems, decision makers typically assume that customers always choose the nearest facility when seeking optimal locations for opening new facilities. However, Ljubić and Moreno (2018) claim that customers have a preference for being served by facilities based on their personal preferences, which may not always be known to decision makers. Hence, random utility models are commonly employed to predict customer behavior and the multinomial logit model is used to forecast the captured demand in their study. Drezner et al. (2018) propose a CFL model that extends the gravity model by assuming randomly distributed attractiveness of facilities as well.

Küçükaydin et al. (2011) investigate a scenario in which a company or franchise penetrates a market by establishing new facilities in the presence of existing competitor facilities. The objective is to determine not only the optimal location but also attractiveness level of each facility to maximize the overall profit. Also, the competitor has the ability to respond by modifying the attractiveness level of its current facilities with the goal of maximizing its profit. A bi-level mixed-integer non-linear programming model is introduced to depict this situation. Yu (2019) also proposes a scenario that two competing firms, a leader and a follower are involved, and the leader aims to establish p chain facilities from a set of potential locations while expecting the follower to respond by initiating q facilities. Moreover, this study adopts the partially proportional rule with a threshold instead of Huff ’ s gravity-based rule to characterize consumer behavior

doi:10.6342/NTU202402305

patterns. A bi-level programming model is formulated to describe this problem.

The works mentioned above do not consider the diminishing marginal benefit in customer demand; in other words, they assume market demand is exogenous rather than endogenous (decision dependent). As a result, we will delve deeper into research regarding endogenous market demands.

2.2 CFL problem with diminishing marginal benefit

Berman and Krass (2002) propose the spatial interaction model with variable expenditures, and Aboolian et al. (2007) follow the approach to consider the problem of locating a set of new facilities that compete for customer demand, both among themselves and with existing facilities, aiming to capture the effects of “market expansion” and “market cannibalization.” Also, Kung and Liao (2018) investigate a profit-maximizing service provider’s CFL problem. The service provider decides where to build facilities by considering the stand-alone benefit of a single facility and the network benefit between a pair of facilities. To capture the diminishing marginal benefit property, a consumer’s willingness to use the service is modeled as a non-decreasing concave function of the sum of all benefits.

Although the works mentioned in this section examine the market expansion effect and the network effect, none of them consider that the attractiveness level of the facilities is a decidable factor with an impact on customers' decisions to patronize. Even the attractiveness of the facilities is taken into consideration, it is not decidable. They assume that the attractiveness of the facilities is the same or follows a certain distribution. Yet, as

doi:10.6342/NTU202402305

the decision maker, we should have the ability to decide how much to invest in enhancing the attractiveness level of the facilities we build, aiming to attract customers and compete with competitors. Although the attractiveness decision is considered by the study of Küçükaydin et al. (2011) mentioned in section 2.1, the market demand is assume to be exogenous in their work.

2.3 Diminishing marginal benefit and attractiveness decision

Lin and Tian (2021) try to solve a competitive facility location problem where a company enters a market with established competitor facilities. The explicit consideration includes the market expansion effect resulting from the introduction of new facilities and the supplementary decision on zone-specific attractiveness level aiming at enhancing the overall attraction of a facility to a specific customer zone. In short, this study allows decision maker to determine or adjust the attractiveness level of the facilities.

In our study, we also consider diminishing marginal benefit and attractiveness decision. However, there is one key factor that differentiates our study from Lin and Tian (2021). In the work of Lin and Tian (2021), the trade-off is solely between investment cost and attractiveness enhancement. Nevertheless, it may not be sufficiently complex to model scenarios that involve resource allocation. Therefore, the decision maker determines the attractiveness level of the facilities by allocating different resources to them, rather than arbitrarily assigning attractiveness level in our model. There are also relevant constraints with respect to different resources. In certain instances, the same level of at-

doi:10.6342/NTU202402305

tractiveness enhancement may be achieved by investing different resources, regardless of whether the costs of resources are equal. By considering attractiveness of facilities with resources allocation, we can model a more complex scenario that could not be addressed in previous research.

doi:10.6342/NTU202402305

Chapter 3

Problem Description and Formulation

In this section, the problem statement and formation of our competitive facility location problem with resources allocation are provided.

We consider a decision maker deciding the optimal facility locations to allocate specific resources, aiming to establish facilities with varying attractiveness level. The goal is to maximize profit by attracting as many customers as possible while competing with existing facilities. Let $J = \{1, 2, 3, ..., |J|\}$ denote the set of locations where a facility may be built, and $K = \{1, 2, 3, ..., |K|\}$ represent the set of resource types that the decision maker could allocate to the facilities. For ease of exposition, we may call the facility built at location j as facility j from time to time. The decision maker have to choose locations to build facilities with specific resources. To model this, let $y_j \in \{0, 1\}$ be 1 if a facility is built at location j, otherwise y_j is 0, and $x_{jk} \geq 0$ represents the amount of type k resource allocated to facility j. Assume that U_{jk}^{LT} is the maximum amount of

doi:10.6342/NTU202402305

resource k could be allocated to facility j. According to above definitions, we have the constraints

$$x_{jk} \leq U_{jk}^{LT} y_i \quad \forall j \in J, k \in K, \tag{3.1}$$

which ensures resources can only be allocated to the facility that is built and the maximum amount of the allocated resources are restricted as well.

There is also a maximum allocation limit for each resource and a maximum capacity for each facility, assumed to be U_k^T and U_j^L, respectively. Then we have

$$\sum_{j \in J} x_{jk} \leq U_k^T, \quad \forall k \in K \tag{3.2}$$

$$\text{and} \sum_{k \in K} x_{jk} \leq U_j^L, \quad \forall j \in J, \tag{3.3}$$

which state that total allocation for each resource must not exceed the limit and that resources for each facility cannot surpass the capacity.

We also assume that each unit of resource k would yield attractiveness V_{jk} when allocated to facility j. Furthermore, we have the allowance to enhance the attractiveness level of the built facilities. Let a_j represent the additional attractiveness added to facility j. Then the attractiveness level of facility j would be $\sum_{k \in K} V_{jk} x_{jk} + a_j$ and the following constraint

$$a_j \leq A y_j, \quad \forall j \in J \tag{3.4}$$

doi:10.6342/NTU202402305

ensures additional attractiveness can only be added to built facilities, where A is the maximum amount of additional attractiveness.

To formulate the decision maker ' s objective function, we still need to define the customer set and the competitor set, as well as make some necessary assumptions. We define $I = \{1, 2, 3, ..., |I|\}$ as the set of customer locations. Similarly, we call customer at location i as customer i. The maximum demand for customer i is H_i. We also define $L = \{1, 2, 3, ..., |L|\}$ as the set of existing competitor facility locations and calling the competitor facility at location l as competitor l. According to Huff ' s gravity-based rule, the utility of facility j for customer i is defined as

$$u_{ij} = \frac{E_j(\sum_{k \in K} V_{jk} x_{jk} + a_j)}{D_{ij}^2}, \tag{3.5}$$

where D_{ij} is the distance between customer i and facility j and function $E_j(\cdot)$ is non-decreasing, concave, and passes through the origin. The function $E_j(\cdot)$ is introduced here to model the diminishing marginal utility of yield attractiveness. Also, the utility of competitor l for customer i is defined as

$$\bar{u}_{il} = \frac{\bar{A}_l}{\bar{D}_{il}^2}, \tag{3.6}$$

where $\bar{A}_l$ is the attractiveness level of competitor l and $\bar{D}_{il}$ is the distance between customer i and competitor l. The total attractiveness of all facilities to customer i is

$$TA_i = \sum_{j \in J} \frac{E_j(\sum_{k \in K} V_{jk} x_{jk} + a_j)}{D_{ij}^2} + \sum_{l \in L} \frac{\bar{A}_l}{\bar{D}_{il}^2}. \tag{3.7}$$

doi:10.6342/NTU202402305

We assume that the probability of a customer i choosing facility j is

$$P_{ij} = \frac{E_j(\sum_{k\in K} V_{jk}x_{jk} + a_j)/D_{ij}^2}{TA_i}, \tag{3.8}$$

where the denominator signifies the combined attractiveness contributed by both our facilities and the competitor's facilities to customer i, whereas the numerator represents the attractiveness level of our facility j to customer i.

Furthermore, the effective demand of customer i, $\tilde{H}_i$, is assumed to be elastic and to increase with the total attractiveness of all facilities to customer i. Then the effective demand of customer i would be

$$\tilde{H}_i = H_i \cdot G_i(TA_i), \tag{3.9}$$

where function $G(\cdot)$ is a non-decreasing, concave, and non-linear function that passes through the origin. Next, the total demand from customer i captured by our facilities would be represented by

$$W_i(x, a) = H_i \cdot G_i(TA_i) \times \frac{\sum_{j\in J} E_j(\sum_{k\in K} V_{jk}x_{jk} + a_j)/D_{ij}^2}{TA_i}. \tag{3.10}$$

We have everything we need to formulate the decision maker ' s objective function. The total profit from attracted customer is

$$\sum_{i\in I} W_i(x, a) - \sum_{j\in J}(F_j y_j + C_j a_j + \sum_{k\in K} B_{jk}x_{jk}), \tag{3.11}$$

doi:10.6342/NTU202402305

where F_j is the fixed cost of building facility j, C_j is the cost of adding a unit of extra attractiveness at facility j, and B_{jk} represents the cost of allocating a unit of type k resource to facility j.

Collectively, we may formulate the decision maker ' s problem as

$$
\begin{aligned}
\max \quad & \sum_{i \in I} W_i(x, a) - \sum_{j \in J} (F_j y_j + C_j a_j + \sum_{k \in K} B_{jk} x_{jk}) \\
\text{s.t.} \quad & x_{jk} \leq U_{jk}^{LT} y_j \quad \forall j \in J,\ \forall k \in K \\
& \sum_{j \in J} x_{jk} \leq U_k^T \quad \forall k \in K \\
& \sum_{k \in K} x_{jk} \leq U_j^L \quad \forall j \in J \\
& a_j \leq A y_j \quad \forall j \in J \\
& x_{jk} \geq 0 \quad \forall j \in J,\ \forall k \in K && (3.12) \\
& a_j \geq 0 \quad \forall j \in J && (3.13) \\
& y_j \in \{0, 1\} \quad \forall j \in J. && (3.14)
\end{aligned}
$$

Table 3.1 introduces all the decision variables mentioned above, and Table 3.2 introduces all the sets and parameters mentioned above.

Decision variables	
y_j	A binary variable that verifies whether a facility is built at location j
x_{jk}	A non-negative variable that verifies the units of type k resource allocated to facility j
a_j	A non-negative variable that verifies the units of extra attractiveness added to facility j

Table 3.1: List of decision variables

doi:10.6342/NTU202402305

Sets	
I	The set of customer locations, $I = \{1, 2, 3, ..., \lvert I\rvert\}$
J	The set of candidate facility locations, $J = \{1, 2, 3, ..., \lvert J\rvert\}$
K	The set of resource type, $K = \{1, 2, 3, ..., \lvert K\rvert\}$
L	The set of existing competitor facility locations, $L = \{1, 2, 3, ..., \lvert L\rvert\}$
Parameters	
U_{jk}^{LT}	The maximum amount of resources k could be allocated to facility j
U_k^T	The maximum allocation limit for resources k
U_j^L	The maximum capacity of facility j
V_{jk}	The yielded attractiveness of a unit of type k resource allocated to facility j
H_i	The maximum demand for customer i
D_{ij}	The distance between customer i and facility j
$\bar{D}_{ij}$	The distance between customer i and competitor l
$\bar{A}_l$	The attractiveness of competitor facility l
F_j	The fixed cost of building facility j
C_j	The cost of adding a unit of extra attractiveness at facility j
B_{jk}	The cost of allocating a unit of type k resource to facility j

Table 3.2: List of sets and parameters

Furthermore, we can show that our optimization problem exhibits some kind of convexity in the following theorem.

Theorem 1. *The program defined in (3.1) to (3.4) and (3.11) to (3.14) is a convex program if variables y_j are relaxed to be continuous within* $[0, 1]$.

Proof. First, all the constraints are linear constraints if variables y_j are relaxed to be continuous within $[0, 1]$, so the feasible region is convex. Secondly, we are going to prove that (3.11) is a concave function. Let

$$r_i = \sum_{l \in L} \left[\bar{A}_l / \bar{D}_{il}^2\right],$$

doi:10.6342/NTU202402305

which implies

$$TA_i(x,a) = \sum_{j\in J}\left[E_j(\sum_{k\in K} V_{jk}x_{jk} + a_j)/D_{ij}^2\right] + r_i.$$

Then $TA_i(x,a)$ is the sum of concave functions since $E_j(\cdot)$ is non-decreasing, concave, and passes through the origin. As a result, $TA_i(x,a)$ is concave. We then evaluate $W_i(x,a)$. We define

$$\widetilde{W}_i(Y) = \tilde{H}_i \times p_{ij} = H_i \cdot G_i(Y) \times \left(1 - \frac{r_i}{Y}\right),$$

then we have

$$W_i(x,a) = \widetilde{W}_i(TA_i(x,a)) = \tilde{H}_i \times p_{ij} = H_i \cdot G_i(TA_i) \times \left(1 - \frac{r_i}{TA_i}\right).$$

We may derive the first-order derivative of $\widetilde{W}_i(Y)$ with respect to Y as

$$\widetilde{W}_i'(Y) = H_i\left[G_i'(Y)\left(1 - \frac{r_i}{Y}\right) + G_i(Y)\frac{r_i}{Y_i^2}\right].$$

Given that $G_i(\cdot)$ is non-decreasing, concave, and passes through the origin, and that $W_i'(Y) \geq 0$ for $Y \geq r_i$, it follows that $W_i(Y)$ is increasing if $Y \geq r_i$. Moreover, we have

$$\begin{aligned}\widetilde{W}_i''(Y) &= H_i\left[G_i''(Y)\left(1 - \frac{r_i}{Y}\right) + G_i'(Y)\frac{2r_i}{Y^2} - G_i(Y)\frac{2r_i}{Y^3}\right] \\ &= H_i\left[G_i''(Y)\left(1 - \frac{r_i}{Y}\right) + \frac{2r_i}{Y^3}\left[G_i'(Y)\cdot Y - G_i(Y)\right]\right].\end{aligned}$$

doi:10.6342/NTU202402305

Let

$$\eta(Y) = G_i'(Y) \cdot Y - G_i(Y),$$

then we know that

$$\eta(0) = 0$$

and

$$\eta'(Y) = G_i''(Y) \cdot Y \leq 0 \text{ for } Y \geq 0.$$

In other words, since $\eta(\cdot)$ is non-increasing and passes through the origin, $\eta(Y)$ will be less than or equal to 0 for any Y greater than or equal to 0. As a result, $\widetilde{W}''(Y) \leq 0$ for $Y \geq 0$, which means $\widetilde{W}_i(Y)$ is concave when Y is non-negative. Then we can know that $W_i(x, a)$ is also concave since it is the composition function of $\widetilde{W}_i(Y)$ and $TA_i(x, a)$ where $W_i(TA_i)$ is concave and non-decreasing while $TA_i(x, a)$ is concave and greater than or equal to r_i. Knowing that $W_i(x, a)$ is concave, we can further infer that (3.11) is also concave since it is the sum of multiple concave functions if variables y_j are relaxed to be continuous within $[0, 1]$. Finally, the feasible region is proved to be convex, and (3.11) is also demonstrated to be a concave function. Consequently, we are maximizing a concave function, which makes it a convex program. □

Before we move to the next section, the solvability of the problem also needs to be considered. Kung and Liao (2018) establish in Proposition 1 that their facility location

doi:10.6342/NTU202402305

problem is NP-hard under conditions of diminishing marginal benefits, even in scenarios without network effect. That is,

$$\max_{x_i \in \{0,1\}} g\left(\sum_{i \in I} s_i x_i\right) - \sum_{i \in I} h_i x_i \tag{3.15}$$

is NP-hard as long as $g(\cdot)$ is concave and non-linear. Nevertheless, our competitive facility location problem with resource allocation can be viewed as equivalent after appropriate simplification. If we fix the cost of adding extra attractiveness C_j, the cost of allocating resources B_{jk}, the attractiveness of competitor facility $\bar{A}_l$, and the units of extra attractiveness added to facility a_j to be 0, and also fix the distance D_{ij} between customer and facility, the number of resource types $|K|$, the number of customer location $|I|$ to be 1, and set the function $E_j(\cdot)$ as linear and passing through the origin as well as relax all the constraints without the binary constraint (3.1) while setting U_{jk}^{LT} as 1, then our problem will be

$$\max_{y_j \in \{0,1\}} H \cdot G\left(\sum_{j \in J} V_j y_j\right) - \sum_{j \in J} F_j y_i \tag{3.16}$$

The two problems (3.15) and (3.16) can be viewed as equivalent. If a problem is NP-hard after being simplified, it indicates that the original, more complex problem is also NP-hard. Therefore, since our problem is NP-hard after the above simplification, our original competitive facility location problem with resource allocation is also NP-hard.

doi:10.6342/NTU202402305

Chapter 4

The Algorithm

4.1 Overview

Developing heuristic algorithms is a common approach to solving NP-hard problems. In this chapter, we propose a Lagrangian relaxation-based algorithm with dimension reduction (LRADR) for our specific problem. In each iteration of our proposed algorithm, we obtain the optimal solution under the given Lagrangian multipliers by solving the relaxed convex programming problem. We then update the Lagrangian multipliers based on the optimal solution obtained from the relaxed program, aiming to achieve a tighter upper bound in subsequent iterations. The iterations continue until the termination conditions are met. However, the solution obtained from the iterations may not be feasible for the original problem. Consequently, some appropriate adjustments are incorporated into our proposed algorithm. The details of the Lagrangian relaxation-based algorithm with dimension reduction will be discussed in the following sections. The flowchart of the algorithm is in Figure 4.1.

doi:10.6342/NTU202402305

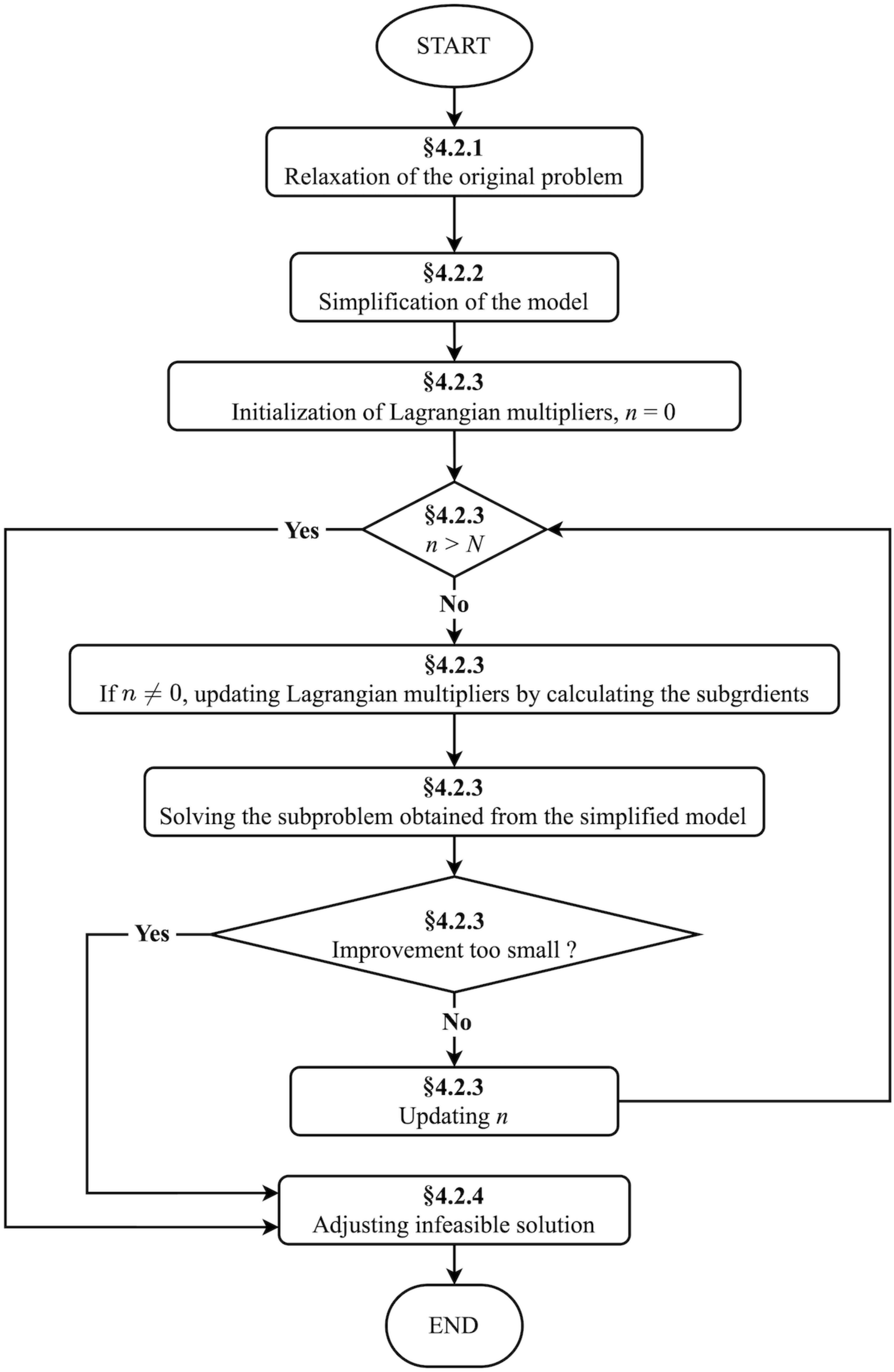

Figure 4.1: The flowchart of the Lagrangian relaxation-based algorithm with dimension reduction

doi:10.6342/NTU202402305

4.2 The Lagrangian Relaxation-based Algorithm with Dimension Reduction

4.2.1 Relaxation of The Original Problem

To address the original non-lienar integer problem more efficiently, our approach involves developing a heuristic algorithm to avoid the time-consuming process of solving problems with integer variables. As a result, we initially relax constraints (3.1) and (3.4) that contain integer variables. Furthermore, we observe that constraint (3.2) connects different facilities and constraint (3.3) connects different resources, which can pose challenges when devising a heuristic algorithm. Therefore, constraints (3.2) and (3.3) will also be relaxed.

We relax the original problem by introducing Lagrangian multipliers $\lambda_{jk} \geq 0,\ \mu_j \geq 0,\ \rho_k \geq 0$ and $\sigma_j \geq 0$ for dualizing constraints (3.1), (3.2), (3.1) and (3.4), respectively. The method for determining the values of the Lagrangian multipliers will be discussed in the following sections. Then the relaxed problem can be formulated as

$$\begin{aligned} L(\lambda, \mu, \rho, \sigma) = \max\ & \sum_{i \in I} W_i(x, a) - \sum_{j \in J}(F_j y_j + C_j a_j + \sum_{k \in K} B_{jk} x_{jk}) \\ & + \sum_{j \in J}\sum_{k \in K} \lambda_{jk}(U^{LT}_{jk} y_j - x_{jk}) + \sum_{j \in J} \mu_j (U^L_j - \sum_{k \in K} x_{jk}) \\ & + \sum_{k \in K} \rho_k (U^T_k - \sum_{j \in J} x_{jk}) + \sum_{j \in J} \sigma_j (A y_j - a_j) \qquad (4.1) \\ \text{s.t. } & x_{jk} \geq 0 \quad \forall j \in J,\ \forall k \in K \\ & a_j \geq 0 \quad \forall j \in J \end{aligned}$$

doi:10.6342/NTU202402305

$$y_j \in \{0,1\} \quad \forall j \in J.$$

4.2.2 Simplification of the model

To facilitate the observation of the relationships between various variables and coefficients, we further refined the terms of the model after relaxation as

$$\begin{aligned} L(\lambda,\mu,\rho,\sigma) = \max \ & \sum_{i\in I} W_i(x,a) \\ & + \sum_{j\in J}\left[-F_j + \sum_{k\in K}\lambda_{jk}U_{jk}^{LT} + \sigma_j A\right] y_j \\ & + \sum_{j\in J}\left[-C_j - \sigma_j\right] a_j \\ & + \sum_{j\in J}\sum_{k\in K}\left[-B_{jk} - \lambda_{jk} - \mu_j - \rho_k\right] x_{jk} \\ & + \sum_{j\in J}\mu_j U_j^L + \sum_{k\in K}\rho_k U_k^T \qquad (4.2) \\ \text{s.t. } & x_{jk} \geq 0 \quad \forall j \in J,\ \forall k \in K \\ & a_j \geq 0 \quad \forall j \in J \\ & y_j \in \{0,1\} \quad \forall j \in J. \end{aligned}$$

Observing the model above, we find that the value of the decision variables y_j is actually determined solely by its coefficient $\left[-F_j + \sum_{k\in K}\lambda_{jk}U_{jk}^{LT} + \sigma_j A\right]$ since y_j is not part of $W_i(x,a)$. To maximize the objective value, y_j should be 1 if its coefficient is positive, otherwise y_j should be 0. As a result, the decision variable y_j is no longer necessary to be included in the optimization model since it can be determined directly

doi:10.6342/NTU202402305

by observing its coefficients. Let's define a set J_B to simplify our notation:

$$J_B = \left\{ j : -F_j + \sum_{k \in K} \lambda_{jk} U_{jk}^{LT} + \sigma_j A \geq 0 \right\}. \tag{4.3}$$

That is, in the optimal solution, $y_j = 1$ for $j \in J_B$ and $y_j = 0$ otherwise. Therefore, the value of y_j is known without needing to be included in the model.

Furthermore, we find that it is actually not effective for the relaxed model to add extra attractiveness and allocate resources to a facility simultaneously, since the constraints regarding the maximum allocation limit for each resource and the maximum capacity for each facility are relaxed. Recall the objective function (4.2) of the relaxed model where

$$W_i(x, a) = \tilde{H}_i \times p_i = H_i \cdot G_i(TA_i) \times \left[1 - \frac{\sum_{l \in L} \left[\bar{A}_l / \bar{D}_{il}^2 \right]}{TA_i} \right]$$

and

$$TA_i = \sum_{j \in J} \frac{E_j(\sum_{k \in K} V_{jk} x_{jk} + a_j)}{D_{ij}^2} + \sum_{l \in L} \frac{\bar{A}_l}{\bar{D}_{il}^2}.$$

We can take $-C_j - \sigma_j$ as the unit cost of adding extra attractiveness to facility j and $-B_{jk} - \lambda_{jk} - \mu_j - \rho_k$ as the unit cost of allocating type k resource to facility j. At the same time, the attractiveness yielded by adding a unit of extra attractiveness and allocating a unit of resource k to facility j are 1 and V_{jk} respectively. Because there are no limits on the amount of extra attractiveness and resource allocation, the model will choose to achieve the same total attractiveness in the most cost-effective manner according to the ratio of yielded attractiveness to the cost spent. This can be done either by only adding extra attractiveness without allocating any resources or by only allocating a single type

doi:10.6342/NTU202402305

of resource without adding extra attractiveness, depending on which option is more cost-effective at each facility. Then we can define several sets accordingly to simplify our notation. More precisely, we define

$$k_j^* = \arg\max_{k \in K} \frac{V_{jk}}{B_{jk} + \lambda_{jk} + \mu_j + \rho_k} \ \forall j \in J \tag{4.4}$$

and

$$J = J_1 \cup J_2, \tag{4.5}$$

where

$$J_1 = \left\{ j \in J : \frac{1}{C_j + \sigma_j} \geq \frac{V_{jk}}{B_{jk_j^*} + \lambda_{jk_j^*} + \mu_j + \rho_{k_j^*}} \right\}, \tag{4.6}$$

$$J_2 = \left\{ j \in J : \frac{1}{C_j + \sigma_j} < \frac{V_{jk_j^*}}{B_{jk_j^*} + \lambda_{jk_j^*} + \mu_j + \rho_{k_j^*}} \right\}. \tag{4.7}$$

That is, it is more cost-effective to add extra attractiveness instead of allocating resources to the facility for $j \in J_1$, and to allocate type k_j^* resource instead of allocating any other resources or adding extra attractiveness to the facility for $j \in J_2$.

In conclusion, in the optimal solution, the model would choose to add extra attractiveness without allocating any resources to the facility for all $j \in J_1$, and it would choose to allocate resource k_j^* without allocating any other resources or adding any extra attractiveness to the facility for all $j \in J_2$. Consequently, we can further simplify the variables and refine the model to facilitate the solving process. The aforementioned (3.7)

doi:10.6342/NTU202402305

and (3.10) will be refined as

$$W_i(x,a) = \tilde{H}_i \times p_i = H_i \cdot G_i(TA_i) \times \left[1 - \frac{\sum_{l\in L}\left[\bar{A}_l/\bar{D}_{il}^2\right]}{TA_i}\right] \tag{4.8}$$

where

$$TA_i = \sum_{j\in J_1}\left[E_j(a_j)/D_{ij}^2\right] + \sum_{j\in J_2}\left[E_j(V_{jk_j^*}x_{jk_j^*})/D_{ij}^2\right] + \sum_{l\in L}\left[\bar{A}_l/\bar{D}_{il}^2\right], \tag{4.9}$$

and the relaxed model (4.2) and the constraints (3.12), (3.13) will be refined as

$$\begin{aligned} L(\lambda,\mu,\rho,\sigma) = \max\ & \sum_{i\in I} W_i(x,a) \\ & + \sum_{j\in J_1}\left[-C_j - \sigma_j\right] a_j \\ & + \sum_{j\in J_2}\left[-B_{jk_j^*} - \lambda_{jk_j^*} - \mu_j - \rho_{k_j^*}\right] x_{jk_j^*} \\ & + \sum_{j\in J}\mu_j U_j^L + \sum_{k\in K}\rho_k U_k^T \end{aligned} \tag{4.10}$$

$$\text{s.t. } a_j \geq 0 \quad \forall j \in J_1 \tag{4.11}$$

$$x_{jk_j^*} \geq 0 \quad \forall j \in J_2. \tag{4.12}$$

According to Theorem 1, this is also a continuous convex program so that can be solved by a typical solver. We summarize the above relation in the following theorem.

Theorem 2. *The optimal solution to the program defined in (4.2) and (3.12) to (3.14) is equivalent to the optimal solution to the program defined in (4.10) and (4.11) to (4.12) while* $y_j = 1$ *for all* $j \in J_B$ *and* $y_j = 0$ *otherwise.*

doi:10.6342/NTU202402305

4.2.3 The Iteration Process for Optimizing the Lagrangian Multipliers

In the iteration process of Lagrangian Relaxation (LR), we are trying to solve the problem

$$w^* = \min_{\lambda \geq 0, \mu \geq 0, \rho \geq 0, \sigma \geq 0} L(\lambda, \mu, \rho, \sigma). \tag{4.13}$$

Since $L(\lambda, \mu, \rho, \sigma)$ is the upper bound of the original problem given the positive Lagrangian multipliers according to weak duality of Lagrangian relaxation, w^* represents the tightest upper bound that is closest to the original optimal value among all $L(\lambda, \mu, \rho, \sigma)$. The algorithm we use to find w^* is called the subgradient algorithm. That is, in every iteration, we solve the optimal $L(\lambda, \mu, \rho, \sigma)$ value by the solver given Lagrangian multipliers $\lambda, \mu, \rho, \sigma$ and find a subgradient of $L(\lambda, \mu, \rho, \sigma)$ to update the Lagrangian multipliers accordingly. This process aims to make the $L(\lambda, \mu, \rho, \sigma)$ value smaller and smaller in the following iterations. The whole process continues until the termination conditions are met. The steps of the subgradient algorithm are as follows:

1. Let $n = 0$. Get the initial Lagrangian multipliers $\lambda^0,\ \rho^0,\ \mu^0,$ and σ^0, where

$$\lambda^0 = \left[\left[\lambda^0_{jk}...\lambda^0_{j|K|}\right] ... \left[\lambda^0_{|J|k}...\lambda^0_{|J||K|}\right]\right],$$

$$\rho^0 = \left[\rho^0_k...\rho^0_{|K|}\right],$$

$$\mu^0 = \left[\mu^0_j...\mu^0_{|J|}\right],$$

$$\sigma^0 = \left[\sigma^0_j...\sigma^0_{|J|}\right].$$

doi:10.6342/NTU202402305

The λ^0_{jk}, ρ^0_k, μ^0_j, and σ^0_j above can be any non-negative value.

2. Check if n reaches the predetermined maximum iteration number N. If it does, stop.

3. If $n \neq 0$, let

$$\begin{aligned}
\lambda^n_{jk} &= \lambda^{n-1}_{jk} - \frac{\delta}{n+4} \cdot h^\lambda_{jk}(y_{(n-1)}, x_{(n-1)}),\\
\rho^n_k &= \rho^{n-1}_k - \frac{\delta}{n+4} \cdot h^\rho_k(x_{(n-1)}),\\
\mu^n_j &= \mu^{n-1}_j - \frac{\delta}{n+4} \cdot h^\mu_j(x_{(n-1)}),\\
\sigma^n_j &= \sigma^{n-1}_j - \frac{\delta}{n+4} \cdot h^\sigma_j(y_{(n-1)}, a_{(n-1)}).
\end{aligned}$$

The step size, $\frac{\delta}{n+4}$, will decrease as the iteration number increases where δ should be non-negative. $h^\lambda(\cdot)$, $h^\rho(\cdot)$, $h^\mu(\cdot)$, and $h^\sigma(\cdot)$ are derived form the relaxed constraints (3.1), (3.2), (3.3), and (3.4) where

$$\begin{aligned}
h^\lambda_{jk}(y_{(n-1)}, x_{(n-1)}) &= x_{jk} - U^{LT}_{jk} y_j,\\
h^\rho_k(x_{(n-1)}) &= \sum_{j\in J} x_{jk} - U^T_k,\\
h^\mu_j(x_{(n-1)}) &= \sum_{k\in K} x_{jk} - U^L_j,\\
h^\sigma_j(y_{(n-1)}, a_{(n-1)}) &= a_j - Ay_j.
\end{aligned}$$

4. Solve $L(\lambda^n, \mu^n, \rho^n, \sigma^n)$ and get $y_{(n)}$, $a_{(n)}$, and $x_{(n)}$.

5. If $n \neq 0$, check if the improvement is too small, that is, $\frac{L(\lambda^n,\mu^n,\rho^n,\sigma^n)-L(\lambda^{n-1},\mu^{n-1},\rho^{n-1},\sigma^{n-1})}{L(\lambda^{n-1},\mu^{n-1},\rho^{n-1},\sigma^{n-1})} < \epsilon$, where ϵ is the predetermined threshold. If this condition is met, stop.

doi:10.6342/NTU202402305

6. Let $n = n + 1$.

7. Repeat steps 2 through 6 until the termination conditions are met.

After completing the steps above, the minimum value $L(\lambda^{n^*}, \mu^{n^*}, \rho^{n^*}, \sigma^{n^*})$ is the closest value we can obtain to w^*, and the corresponding $y_{(n^*)}$, $a_{(n^*)}$, and $x_{(n^*)}$ is the solution.

4.2.4 Adjusting Infeasible Solutions

The solution obtained from the iteration process, which we call the initial infeasible solution, may not be feasible for the original non-lienar integer problem. As a result, some necessary modifications are needed. The initial infeasible solution may be infeasible due to violations of constraints (3.1), (3.2), (3.3), and (3.4). The details of the violations are as follows:

- Violation of constraint (3.1): The type-k resource is allocated to facility j while facility j is not built, or the amount of allocated resource exceeds the limit.
- Violation of constraint (3.2): The total amount of type-k resource allocated to the facilities exceeds the limit.
- Violation of constraint (3.3): The total amount of resources allocated to facilities j exceeds the capacity.
- Violation of constraint (3.4): The extra attractiveness is added to facility j while facility j is not built, or the amount of added extra attractiveness exceeds the limit.

The process of adjusting the initial infeasible solution to make it feasible will be divided into two stages. In the first stage, we will obtain several fundamental feasible solutions,

doi:10.6342/NTU202402305

which will be further combined and improved in the second stage to get the final solution. The flowchart of the adjustment process is in Figure 4.2.

In the first stage, we will obtain two fundamental feasible solutions. These solutions are named *Solution Reduced* and *Solution Expanded*. The only difference between these two fundamental feasible solutions lies in the first step of the obtaining process. Steps 2 to 5 are identical for both solutions. The detailed steps are as follows:

1. (a) For *Solution Reduced*: For each facility j, if resources are allocated ($x_{jk} > 0$ for any $k \in K$) or extra attractiveness is added ($a_j > 0$) to an unbuilt one ($y_j = 0$), the amount of allocated resources or added extra attractiveness to the facility will be set to 0 (i.e., x_{jk} and a_j will be set to 0).

   (b) For *Solution Expanded*: For each facility j, if resources are allocated ($x_{jk} > 0$ for any $k \in K$) or extra attractiveness is added ($a_j > 0$) to an unbuilt one ($y_j = 0$), the facility will be set as built (i.e., y_j will be set to 1).

2. For the built facilities, if the extra attractiveness added to it exceeds the limit A, the amount of the added extra attractiveness will be set to the value A.

doi:10.6342/NTU202402305

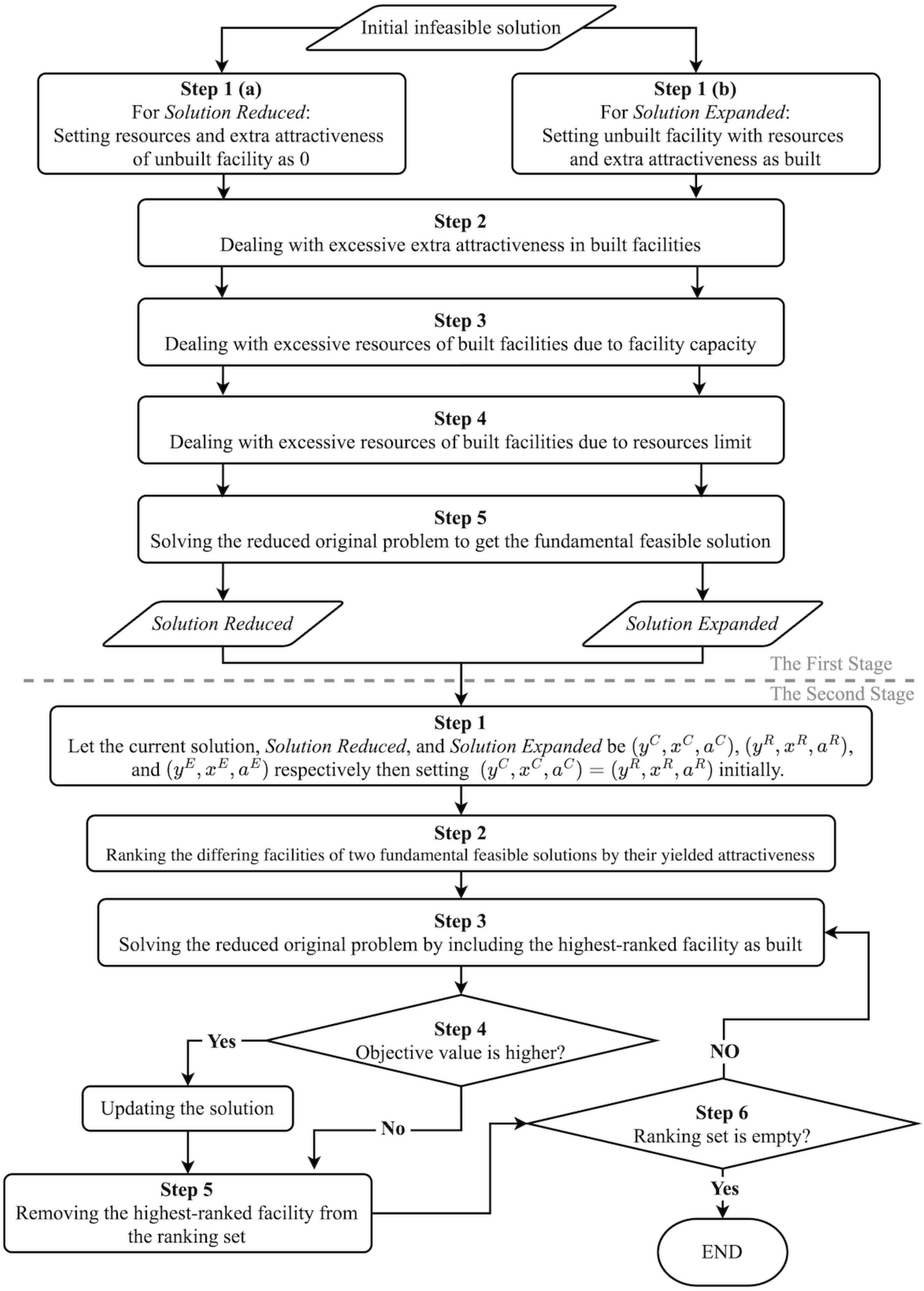

Figure 4.2: The flowchart of the infeasible solutions adjusting process

3. For the built facilities and all types of resources, if the amount of type k resource allocated to facility j exceeds the limit U_{jk}^{LT} or the facility capacity U_j^L, the amount of type k resource allocated to facility j will be set to $\min(U_{jk}^{LT}, U_j^L)$.

doi:10.6342/NTU202402305

4. For all types of resources, if the total amount of allocated type k resource, denoted as $Total_k$, exceeds the maximum allocation limit for type k resource U_k^T, the amount of type k resource allocated to each facility will be reduced proportionally. This reduction is achieved by multiplying the allocation amount at each facility by $\frac{U_k^T}{Total_k}$.

5. Finally, we solve the reduced original problem by the solver given the built facility. That is, we solve the original problem but fix the built facilities as the ones obtained from step 1, instead of allowing the model to determine which facilities to build using integer variables. The reduced original problem will then be an convex program with significantly fewer variables, making it easier to solve. Ultimately, the fundamental feasible solutions are obtained.

After the steps above, the obtained solutions are already feasible. However, the solutions can be further improved to get a higher objective value in the second stage.

Next comes the second stage. Between the two methods mentioned in stage 1, *Solution Expanded* is expected to build no fewer facilities than *Solution Reduced*. The facilities built in *Solution Reduced* will also be built in *Solution Expanded*. In stage 1, the stations that differ between the two methods will either be fully built (*Solution Expanded*) or not built at all (*Solution Reduced*). However, the decision of whether to build the differing facilities should be more flexible. As a result, we propose a selection algorithm with the following steps.

1. Let the current solution, *Solution Reduced*, and *Solution Expanded* be (y^C, x^C, a^C), (y^R, x^R, a^R), and (y^E, x^E, a^E) respectively. Set the starting current solution as *Solution Reduced*. That is, set (y^C, x^C, a^C) = (y^R, x^R, a^R) initially.

doi:10.6342/NTU202402305

2. We define the set of differing facilities as

$$J_{\text{diff}} = \left\{ j \in J : y_j^R = 0, y_j^E = 1 \right\}. \tag{4.14}$$

For all $j \in J_{\text{diff}}$, we rank the facilities by their ratio of yielded attractiveness and building cost, which is $\frac{\sum_{k \in K} V_{jk} x_{jk} + a_j}{F_j}$, in descending order. The rank of the differing facilities is further defined as a set

$$R = \{R_j \mid j \in J_{\text{diff}}\}, \tag{4.15}$$

where $R_j = 1$ for the facility with the highest yielded attractiveness and $R_j = |J_{\text{diff}}|$ for the facility with the lowest yielded attractiveness.

3. Solve the reduced original problem mentioned in the first stage by including the highest-ranked facility as built. That is, we solve the original problem but fix the built facilities as the ones in the set

$$N = \left\{ j \in J : y_j^C = 1 \right\} \cup \left\{ \underset{j \in J_{\text{diff}}}{\arg\min}\, R_j \right\}. \tag{4.16}$$

4. Check if the objective value increases. If it does, update the current solution (y^C, x^C, a^C) as the one obtained from step 3.

5. Remove $\min(R)$ from R.

6. Verify if $R = \phi$. If so, the final solution is achieved, which is (y^C, x^C, a^C); otherwise,

doi:10.6342/NTU202402305

return to step 3.

After completing the two-stage adjustment process, we obtain the final feasible solution.

doi:10.6342/NTU202402305

Chapter 5

Numerical Study

5.1 Experiment Setting

To demonstrate the experimental results of the algorithm under different problem scales, we evaluate its performance by considering the size of several factors. The factors include the number of customer locations $|I|$, the number of candidate facility locations $|J|$, the number of resource types $|K|$, and the number of existing competitor facility locations $|L|$. We define the instance sizes as Table 5.1.

Instance size	$\|I\|$	$\|J\|$	$\|K\|$	$\|L\|$
Small	3	25	1	4
Medium	6	225	3	9
Large	9	625	5	16
Extra Large	12	1225	7	25

Table 5.1: Instance size definition

At the same time, the aforementioned functions $G_i(\cdot)$ and $E_j(\cdot)$ are set as $G_i(x) = 1 - 2^{(-0.02x)}\ \forall i \in I$ and $E_j(x) = 40 - 40 \times 2^{(-0.05x)}\ \forall j \in J$.

doi:10.6342/NTU202402305

The problem sizes above represent the four scenarios of our experiment, and we generate 100 instances for each scenario. The experiments were conducted on a MacBook Pro equipped with an Apple M1 Pro chip, a 10-core CPU, and 16 GB of unified memory. The Lagrangian relaxation-based algorithm with dimension reduction for all instances and the original non-lienar integer problem for small instances were implemented and solved using LINDO API 15.0 via Python 3.10.

5.2 Benchmarks

Besides the Lagrangian relaxation-based algorithm with dimension reduction, we also develop two other methods to get the benchmarks of the solution. We solve the continuous relaxation of the original problem to obtain the upper bound value. Additionally, we designed a greedy heuristic to attempt solving the problem, aiming to compare its performance with our Lagrangian relaxation-based algorithm with dimension reduction.

5.2.1 The Continuous Relaxation of The Original Problem

To better evaluate the performance of our Lagrangian relaxation-based algorithm with dimension reduction, we aim to obtain the upper bound value of the problem by solving the continuous relaxation of the original problem (CROP). That is, by relaxing the only integer variable y_j from the original problem to be linear within the interval $[0, 1]$, the problem becomes a convex programming without integer variables that is easy to solve by a typical solver. However, the feasible region of the continuously relaxed problem expands after the relaxation. Consequently, the objective value of the problem increases, making

doi:10.6342/NTU202402305

it a suitable candidate for an upper bound. We also solve CROP using LINDO API 15.0 via Python 3.10. The detailed results of this benchmark will be further discussed in the next section.

5.2.2 The Greedy Heuristic Algorithm

The greedy heuristic algorithm (GHA) we propose is a relatively intuitive yet not optimized solution. It employs the concepts of greedy search and iterates until convergence. According to the experimental results, GHA can serve as a benchmark for comparison with LRADR we proposed. The main concept of GHA is to build the currently most effective facility with a certain level of resources and extra attractiveness in each iteration. Despite its relative intuitiveness, the process of GHA is not simple. The flowchart of the GHA are divided into two stages as Figure 5.1 and Figure 5.2.

Before elaborating on the detailed steps of GHA, we first define the "maximum effective attractiveness" as θ, where $E(\theta)$ reaches 99% of the maximum value of the $E(\cdot)$ function since the growth of the function $E(x)$ becomes negligible as x increases. In our experimental setting, the maximum value of the function $E(x) = 40 - 40 \cdot 2^{-0.05x}$ is 40, so θ should be the value that makes $E(\theta) = 40 \times 0.99 = 39.6$. Numerically, we can calculate the value of θ is around 132.877. As a result, θ will be considered the maximum level of allocated resources and additional attractiveness for each facility.

Furthermore, we introduce the concept of "completely built". Given an starting solution (y^S, x^S, a^S), to build an additional facility $\widehat{j}$ completely with attractiveness π, we follow this procedure:

doi:10.6342/NTU202402305

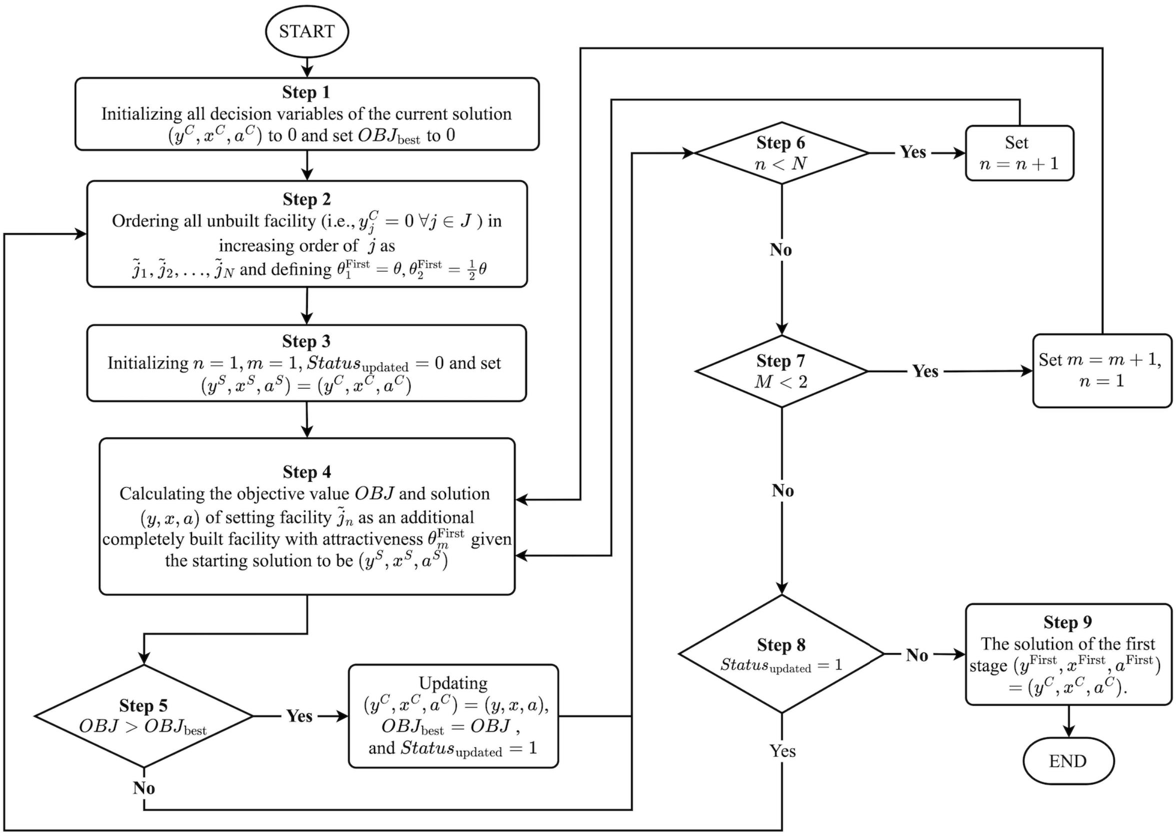

Figure 5.1: The flowchart of the first stage of GHA

(a) Set $y = y^S$ and update $y_{\widehat{j}} = 1$.

(b) Allocate as much resource k with the highest $V_{\widehat{j}k}$ to facility $\widehat{j}$ as possible , stopping when one of the constraints (3.1), (3.2), or (3.3) is not satisfied, or until the attractiveness of facility $\widehat{j}$ reaches the value π. That is, let $k^*_{\widehat{j}} = \arg\max_{k \in K} V_{\widehat{j}k}$, $\widehat{U}^{LT}_{\widehat{j}k^*_{\widehat{j}}}$ be the remaining resources $k^*_{\widehat{j}}$ could be allocated to facility $\widehat{j}$, $\widehat{U}^T_{k^*_{\widehat{j}}}$ be the remaining allocation limit for resources $k^*_{\widehat{j}}$, and $\widehat{U}^L_{\widehat{j}}$ be the remaining capacity of facility $\widehat{j}$, then set $x = x^S$ and $x_{\widehat{j}k^*_{\widehat{j}}}$ will be updated as $\min\left(\widehat{U}^{LT}_{\widehat{j}k^*_{\widehat{j}}}, \widehat{U}^T_{k^*_{\widehat{j}}}, \widehat{U}^L_{\widehat{j}}, \frac{\pi}{V_{\widehat{j}k^*_{\widehat{j}}}}\right)$.

(c) Add as much extra attractiveness to facility $\widehat{j}$ as possible until the the constraints (3.4) is not satisfied, or until the attractiveness of facility j reaches the value π. That is, set $a = a^S$ and update a_j to be $\min\left(A, \pi - x_{\widehat{j}k^*_{\widehat{j}}} V_{\widehat{j}k^*_{\widehat{j}}}\right)$.

doi:10.6342/NTU202402305

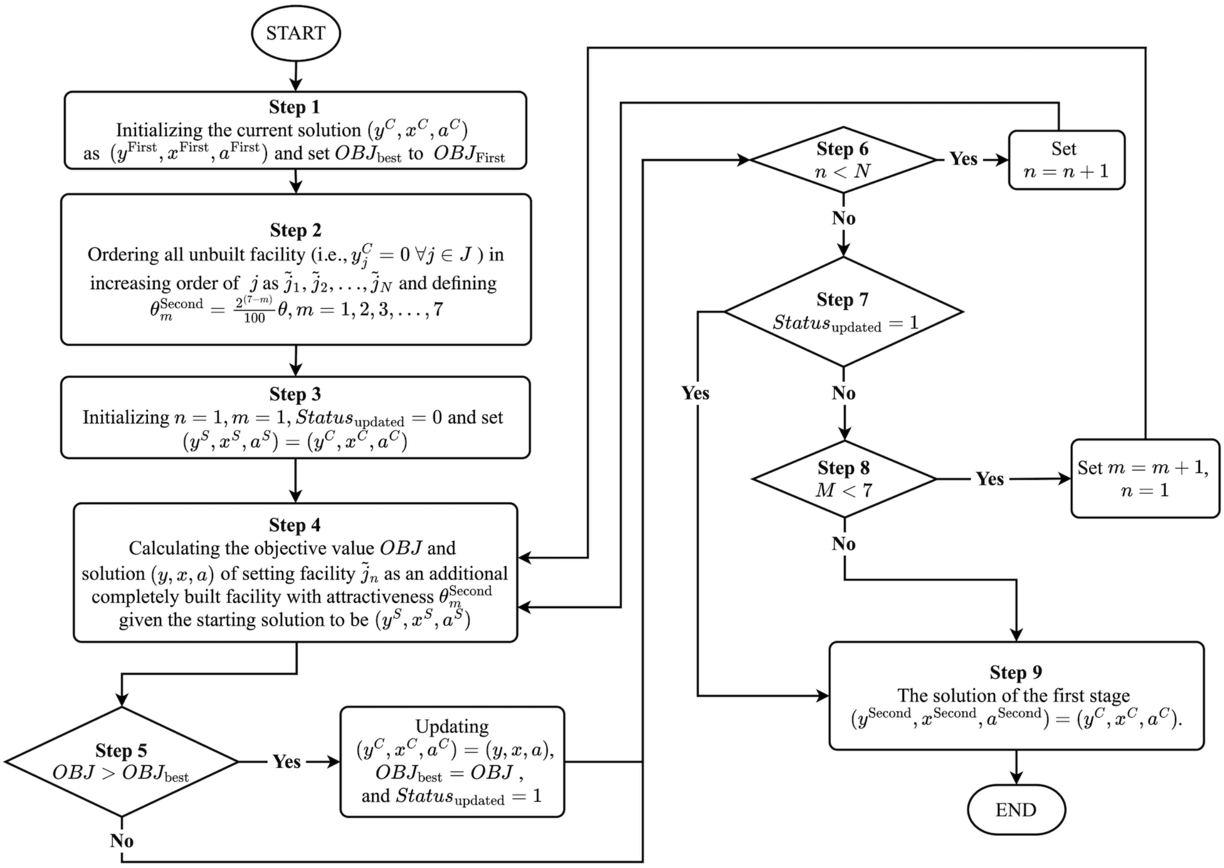

Figure 5.2: The flowchart of the second stage of GHA

(d) (y, x, a) will be the solution of building an additional facility $\widehat{j}$ completely with attractiveness π given an starting solution (y^S, x^S, a^S).

Then the detailed steps of the first stage of GHA are as follows.

1. Initialize all decision variables of the current solution (y^C, x^C, a^C) to 0, and set the best objective value currently found OBJ_{best} as 0.

2. For each unbuilt facility (i.e., $y_j^C = 0 \; \forall j \in J$), we order them in increasing order of j and rename them as $\widetilde{j}_1, \widetilde{j}_2, \ldots, \widetilde{j}_N$. Also, we define $\theta_1^{\text{First}} = \theta$ and $\theta_2^{\text{First}} = \frac{1}{2}\theta$.

3. Initialize $n = 1$, $m = 1$, $Status_{\text{updated}} = 0$, and set $(y^S, x^S, a^S) = (y^C, x^C, a^C)$.

4. We calculate the objective value of setting facility $\widetilde{j}_n$ as an additional completely

doi:10.6342/NTU202402305

built facility with attractiveness θ_m^{First} given the starting solution to be (y^S, x^S, a^S). The corresponding objective value and solution are OBJ and (y, x, a) .

5. Check if $OBJ > OBJ_{\text{best}}$. If it does, update $(y^C, x^C, a^C) = (y, x, a)$, $OBJ_{\text{best}} = OBJ$, and $Status_{\text{updated}} = 1$.

6. Check if $n < N$. If it does, set $n = n + 1$ and go back to step 4; otherwise, proceed to step 7.

7. Check if $m < 2$. If they do, set $m = m + 1, n = 1$ and go back to step 4; otherwise, proceed to step 8.

8. Check if $Status_{\text{updated}} = 1$. If it does, go back to step 2; otherwise, move to step 9.

9. The first stage of GHA ends. The solution of the first stage is $(y^{\text{First}}, x^{\text{First}}, a^{\text{First}}) = (y^C, x^C, a^C)$ and the objective value is $OBJ_{\text{First}} = OBJ_{\text{Best}}$.

However, the solution obtained from the first stage can be further improved by considering to build additional facility completely with more detailed maximum effective attractiveness level. Nevertheless, it would be very time-consuming unless we simplify the stopping condition to make the process stop earlier. As a result, we just update the solution for one more detailed maximum effective attractiveness level in the second stage. Then the steps of the second stage of GHA are as follows.

1. Initialize the current solution (y^C, x^C, a^C) as $(y^{\text{First}}, x^{\text{First}}, a^{\text{First}})$ and the best objective value currently found OBJ_{best} as OBJ_{First}.

doi:10.6342/NTU202402305

2. For each unbuilt facility (i.e., $y_j^C = 0 \ \forall j \in J$), we order them in increasing order of j and rename them as $\widetilde{j}_1, \widetilde{j}_2, \ldots, \widetilde{j}_N$. Also, we define $\theta_m^{\text{Second}} = \frac{2^{(7-m)}}{100}\theta, m = 1, 2, 3, \ldots, 7$.

3. Initialize $n = 1$, $m = 1$, $Status_{\text{updated}} = 0$, and set $(y^S, x^S, a^S) = (y^C, x^C, a^C)$.

4. We calculate the objective value of setting facility $\widetilde{j}_n$ as an additional completely built facility with attractiveness θ_m^{Second} given the starting solution to be (y^S, x^S, a^S). The corresponding objective value and solution are OBJ and (y, x, a) .

5. Check if $OBJ > OBJ_{\text{best}}$. If it does, update $(y^C, x^C, a^C) = (y, x, a)$, $OBJ_{\text{best}} = OBJ$, and $Status_{\text{updated}} = 1$.

6. Check if $n < N$. If it does, set $n = n + 1$ and go back to step 4; otherwise, proceed to step 7.

7. Check if $Status_{\text{updated}} = 1$. If it does, move to step 9; otherwise, proceed to step 8.

8. Check if $m < 7$. If it does, set $m = m + 1, n = 1$ and go back to step 4; otherwise, move to step 9.

9. The final solution is obtained, which is (y^C, x^C, a^C) with objective value OBJ_{best}.

5.3 Solution Performance Evaluation

As mentioned in the previous section, we solve the continuous relaxation of the original problem to obtain the upper bound value since it's difficult to find the solution of the

doi:10.6342/NTU202402305

original problem, especially for the larger instances. However, if we can assess the difference between the objective values of the original problems and the continuously relaxed ones, we will have a better understanding of the solution's performance. Therefore, we solve both the original problems and the continuously relaxed ones for the small-sized instances to calculate the integrality gap. For the 100 small-sized instances, the average value of the integrality gap is approximately 10.6%. That is, the objective value of the continuously relaxed problem is about 10.6% higher than the objective value of the original problem on average. Having this information, we will gain a better understanding of the comparisons for the performance among the algorithms and the continuous relaxation of the original problem discussed below.

In Table 5.2, we evaluate the solution performance of LRADR by comparing it with benchmarks GHA and CROP. Define γ_{LR} as the objective value obtained by LRADR, γ_G as the objective value obtained by GHA, and γ_{CR} as the objective value obtained by CROP. Then we define the average value of $\frac{\gamma_{LR}}{\gamma_{CR}}$ as AVG_{LR} and the minimum value as MIN_{LR}. Similarly, we define the average value of $\frac{\gamma_G}{\gamma_{CR}}$ as AVG_G and the minimum value as MIN_G.

Instance size	LRADR		GHA	
	AVG_{LR}	MIN_{LR}	AVG_G	MIN_G
Small	0.8790	0.4360	0.7466	0.2455
Medium	0.8404	0.6778	0.7796	0.5486
Large	0.8350	0.6493	0.7728	0.6254
Extra Large	0.8149	0.6885	0.7774	0.6704

Table 5.2: Solution performance of different problem sizes

Considering the existence of the integrality gap, we can infer that the solutions ob-

doi:10.6342/NTU202402305

tained by LRADR are very close to the optimal solution. The integrality gap provides a measure of how much the solution to the relaxed problem deviates from the solution to the integer problem. In our experiments, we observe that the integrality gap is around 10.6%. Additionally, AVG_{LR} values are greater than 0.81 for all instance size, which suggests that the solutions found by LRADR are not only feasible but also highly efficient, aligning closely with the optimal solutions. Consequently, LRADR proves to be a robust method for tackling our competitive facility location problem, offering solutions that are both practical and near-optimal within a reasonable computation time.

Moreover, we observe that both the average and worst-case performance of LRADR are superior to GHA across all instance sizes. This consistent performance highlights the effectiveness of LRADR, demonstrating its robustness in various scenarios. Thus, LRADR proves its utility despite our extensive efforts in developing GHA. The relatively intuitive greedy heuristic algorithm falls short in effectively addressing such a complex problem, failing to provide solutions with the same level of quality and efficiency as LRADR. This underscores the importance of LRADR in solving our competitive facility location problems.

5.4 Time Performance Evaluation

In this section, we assess the computational time of our algorithm across various problem sizes. For the small-size instances, the original problem remains solvable, and thus, the computational time for solving the original problem is included as a result. For the rest of the instance sizes, only the computational times for solving LRADR and GHA are

doi:10.6342/NTU202402305

presented, as the original problem cannot be solved within a reasonable time frame. Table 5.3 displays the mean and median of the computational time across different problem sizes.

Instance size	OPT		LRADR		GHA	
	Mean	Median	Mean	Median	Mean	Median
Small	8.75	7.43	53.92	71.36	0.15	0.14
Medium	–	–	101.35	72.01	23.08	22.77
Large	–	–	344.31	289.56	333.16	338.88
Extra Large	–	–	1023.39	869.25	2394.76	2107.28

Table 5.3: Time performance of different problem sizes (in seconds)

We find that the computational time for solving the original problem is acceptable for small instances, with GHA performing even faster. However, this is primarily due to the small size of the instances. Indeed, LRADR may not be as useful for small instances. Nevertheless, its advantages become more apparent for larger instances, which typically represent more realistic scenarios. For medium instances, while the computational time of LRADR is longer than that of GHA, it remains acceptable, taking no more than a few minutes. Concurrently, GHA's computational time grows much faster as the instance size increases. For large instances, the computational times for LRADR and GHA are nearly identical. For extra large instances, the computational times for GHA are more than double those of LRADR. Consequently, in terms of time performance, LRADR appears more suitable for larger, more realistic scenarios.

Moreover, we also conduct an experiment to compare the computational time of the iteration process in our LRADR under the condition of whether the model simplification

doi:10.6342/NTU202402305

is applied. We calculate the average time of the iteration process under three conditions: simplification is fully applied, simplification is applied to y_j only, and simplification is totally not applied. We name the three conditions as full simplification, partial simplification, and no simplification, and we let AVG_{full}, AVG_{partial}, and AVG_{none} represent the average time of the iteration process under these conditions, respectively, in Table 5.4.

Instance size	AVG_{full}	AVG_{partial}	AVG_{none}
Small	40.99	43.54	39.06
Medium	107.44	150.90	182.84
Large	221.44	353.93	679.57
Extra Large	549.35	1211.80	–

Table 5.4: Computational time of the iteration process in LRADR (in seconds)

In small instances, the average computational time is similar since the instance size is too small to show the difference. However, the average computational time for full simplification (AVG_{full}) is nearly half of that for no simplification (AVG_{none}) in medium-sized instances and one-third in large-sized instances. Additionally, the average computational time for full simplification (AVG_{full}) is significantly shorter than that for partial simplification ($\text{AVG}_{\text{partial}}$). Although the average computational time for no simplification (AVG_{none}) is too long to be calculated for extra large instances, the average computational time for partial simplification ($\text{AVG}_{\text{partial}}$) is still more than double that of full simplification (AVG_{full}). Therefore, we can conclude that simplification is indeed effective in reducing the computational time of the iteration process in our LRADR.

doi:10.6342/NTU202402305

Chapter 6

Conclusion

This research tackles the challenge of determining optimal facility locations in a competitive facility location problem while considering resource distribution. We formulate the problem as a convex non-linear integer program with the objective of maximizing profits by attracting the highest possible number of customers in competition with existing facilities. Recognizing the problem's NP-hard complexity, we devise a heuristic algorithm that utilizes Lagrangian relaxation, iteratively fine-tuning Lagrangian multipliers to find solutions. Furthermore, we propose a straightforward greedy heuristic algorithm trying to address the issue. Our numerical study show that our approach can achieve near-optimal solutions in a reasonable time frame.

However, there are some factors that are not considered in our research, which can be a starting point for future studies. For instance, we do not consider the reactions of competitors. Including competitors' behaviors in the model would make it more complex but also more realistic. Drawing inspiration from existing research, bi-level models are likely to be useful in addressing such types of problems. Another research avenue involves

doi:10.6342/NTU202402305

exploring customer preferences. In our model, customers are drawn based on the attractiveness of the facility and their closeness to it. If customer preferences are included, the model can be applied to a more complex and realistic scenario.

doi:10.6342/NTU202402305

Bibliography

Aboolian, R., O. Berman, D. Krass. 2007. Competitive facility location model with concave demand. *European Journal of Operational Research* **181**(2) 598–619.

Berman, O., D. Krass. 2002. Locating multiple competitive facilities: Spatial interaction models with variable expenditures. *Annals of Operations Research* **111**(1) 197–225.

Drezner, T., Z. Drezner, D. Zerom. 2018. Competitive facility location with random attractiveness. *Operations Research Letters* **46**(3) 312–317.

Hotelling, H. 1929. Stability in competition. *The Economic Journal* **39**(153) 41–57.

Huff, D. 1964. Defining and estimating a trading area. *Journal of Marketing* **28**(3) 34–38.

Kung, L.-C., W.-H. Liao. 2018. An approximation algorithm for a competitive facility location problem with network effects. *European Journal of Operational Research* **267**(1) 176–186.

Küçükaydin, H., N. Aras, I. Kuban Altınel. 2011. Competitive facility location problem with attractiveness adjustment of the follower: A bilevel programming model and its solution. *European Journal of Operational Research* **208**(3) 206–220.

doi:10.6342/NTU202402305

Lin, Y.-H., Q. Tian. 2021. Generalized Benders decomposition for competitive facility location with concave demand and zone-specialized variable attractiveness. *Computers Operations Research* **130** 105236.

Ljubić, I., E. Moreno. 2018. Outer approximation and submodular cuts for maximum capture facility location problems with random utilities. *European Journal of Operational Research* **266**(1) 46–56.

Yu, W. 2019. A leader-follower model for discrete competitive facility location problem under the partially proportional rule with a threshold. *PLoS One* **14**(12) e0225693.

doi:10.6342/NTU202402305