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邊界凍結滲流模型及其變體之探討

Boundary Frozen Percolation and a Variant

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邊界凍結滲流模型及其變體之探討

Boundary Frozen Percolation and a Variant

本論文係陳鈞麒君（R12221015）在國立臺灣大學數學系完成之碩士學位論文，於民國 114 年 5 月 26 日承下列考試委員審查通過及口試及格，特此證明

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摘要

我們研究了 Makowiec 在其碩士論文 [7] 中提出的邊界凍結滲流模型，該模型定義在三角格子的有限子圖 $B(N)$ 上，其中 $B(N)$ 表示以原點為中心、邊長為 $2N$ 的平行四邊形區域。在此模型中，所有格點初始為空，並隨著時間從 0 漸進到 1 逐漸被佔據。當某個佔據簇首次觸及 $B(N)$ 的邊界時，即視為凍結。利用 [17] 中發展的技術，我們證明了一個比 Makowiec 在其猜想 3.1 中提出的結果更強的結論：當模型考慮在區域 $B(N)$ 中時，原點在時間 1 屬於凍結簇的機率隨著 $N \to \infty$ 趨近於零。

我們進一步引入並分析一個變體模型，稱為穿越凍結滲流模型。在此模型中，若某個佔據簇包含一條橫向（左到右）或縱向（上到下）貫穿 $B(N)$ 的路徑，則該簇立即凍結。透過與邊界凍結滲流相同的方法，我們同樣證明：在此模型中，原點在時間 1 屬於凍結簇的機率也會隨著 $N \to \infty$ 而趨近於零。

關鍵字：凍結滲流模型、近臨界滲流，自組織臨界性

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Abstract

We study the boundary frozen percolation introduced by Makowiec in [7], which is defined on the finite subgraph $B(N)$ of the triangular lattice. Here, $B(N)$ denotes the parallelogram centered at the origin with side length $2N$. In this model, all the sites are initially vacant and become occupied as time evolves from 0 to 1. An occupied cluster becomes frozen as soon as it touches the boundary of $B(N)$. Using techniques developed in [17], we prove a result stronger than Conjecture 3.1 of [7]: the probability of the origin being frozen at time 1 goes to zero as $N \to \infty$.

We further introduce and analyze a variant model, called crossing frozen percolation, where occupied clusters freeze as soon as containing either a horizontal or vertical crossing of $B(N)$. Applying the same methods, we show that in this model as well, the probability of the origin being frozen at time 1 tends to zero as $N \to \infty$.

Keywords: frozen percolation, near-critical percolation, self-organized criticality

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Chapter 1 Introduction

The concept of frozen bond percolation was originally introduced by Aldous in [1], where clusters on the infinite binary tree are prevented from growing once they become infinite. Whether such a process can be consistently defined on other infinite graphs is not obvious. In particular, Benjamini and Schramm showed that it cannot be constructed on $\mathbb{Z}^2$ (see also Remark (i) following Theorem 1 in [18]). As a way to overcome this limitation, one can instead study a finite-parameter version of the model, where the process is defined for a fixed value N, and then analyze its limiting behavior as $N \to \infty$.

On the triangular lattice, several variants of frozen percolation have been studied, such as diameter-frozen, volume-frozen, and boundary-frozen percolation. In all cases, clusters freeze when they become sufficiently large, though the precise criterion varies. This thesis focuses on boundary frozen site percolation on the triangular lattice, first introduced by Makowiec [7]. We confirm Conjecture 3.1 from Makowiec' s thesis and further investigate a related model with a different freezing condition.

1.1 Introduction to Frozen Percolation

We begin by describing the general framework of frozen site percolation on a graph $G = (V, E)$. Each vertex $v \in V$ is assigned a random activation time τ_v, where the

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collection $(\tau_v)_{v\in V}$ consists of i.i.d. uniformly distributed random variables on the interval $[0,1]$. The system evolves continuously in time $t \in [0,1]$. Initially, all vertices are vacant. A vertex v becomes occupied at time τ_v, unless it has a neighbor whose occupied cluster already satisfies a given freezing condition. Once an occupied cluster meets this condition, it becomes frozen. A vertex is frozen if it belongs to a frozen cluster or is adjacent to one; otherwise, it is non-frozen. The frozen status of a vertex is time-dependent; unless otherwise stated, we evaluate it at time $t = 1$. In particular, if v is frozen and u is adjacent to v, then u must either be frozen as well or remain vacant, as it cannot become occupied without violating the freezing condition.

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^2$. The triangular lattice $\mathbb{T} = (V, E)$ is defined by the vertex set

$$V = \left\{x + ye^{\frac{\pi}{3}i} \in \mathbb{C} : x, y \in \mathbb{Z}\right\},$$

where we identify $\mathbb{C} \cong \mathbb{R}^2$, and the edge set

$$E = \{\{u, v\} : u, v \in V \text{ with } \|u - v\| = 1\}.$$

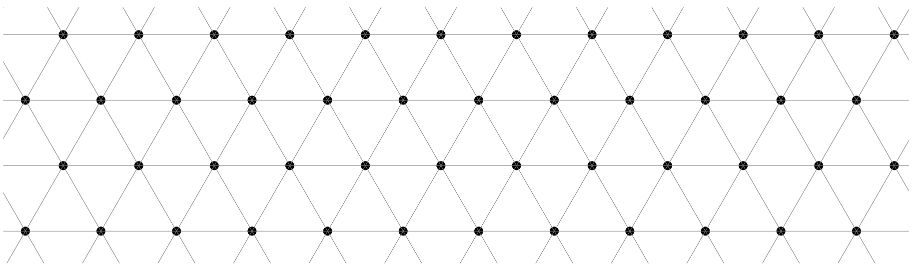

Figure 1.1: Triangular lattice.

Let $\|\cdot\|_\infty$ denote the infinity norm with respect to the basis $\left\{1, e^{\frac{\pi}{3}i}\right\}$. Under this norm, the set of vertices at distance at most N from the origin forms a parallelogram,

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which we denote by

$$B(N) = [-N, N] \times [-N, N].$$

We call $B(N)$ the box of size N centered at the origin. For any vertex $v \in V$, we denote by $B(v; N) := v + B(N)$ the box of size N centered at v.

For integers $1 \leq n < m$, the annulus centered at $u \in V$ with inner radius n and outer radius m is defined by

$$A(u; n, m) := B(u; m) \setminus B(u; n-1).$$

In particular, we write $A(n, m)$ when the annulus is centered at the origin.

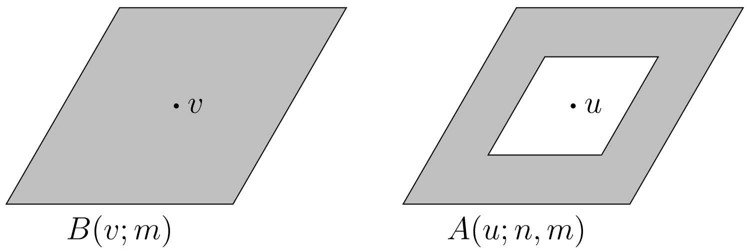

Figure 1.2: Shaded regions illustrate $B(v; m)$ and $A(u; n, m)$ as sets of vertices.

We next define several variants of frozen site percolation on the triangular lattice. As previously noted, our interest lies in the asymptotic behavior of these processes as the parameter $N \to \infty$.

Diameter Frozen Percolation

For $W \subset V$ on the triangular lattice, define the diameter by

$$\mathrm{diam}(W) := \sup_{u,v \in W} \|u - v\|_\infty.$$

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In diameter frozen site percolation with parameter N on the triangular lattice, an occupied cluster becomes frozen as soon as its diameter is at least N.

Van den Berg et al. studied the diameter frozen bond percolation with parameter N on the binary tree in [14], showing that, as $N \to \infty$, this process converges (in an appropriate sense) to Aldous' process in [1]. Moreover, in [13], van den Berg et al. showed that on the square lattice, there exists a uniform lower bound on the probability that the diameter of the occupied cluster containing the origin lies within the interval $(1-\epsilon)N$ and $(1-\epsilon')N$ for some $0 < \epsilon' < \epsilon < 1$. An analogous result holds for the N-parameter diameter frozen site percolation process on the triangular lattice.

Theorem (Theorem 1.1 in [13]). *Let $\mathscr{C}_N$ denote the occupied cluster containing the origin at time 1 in the diameter frozen bond percolation process with parameter N on the square lattice. Then for any $0 < \epsilon' < \epsilon < 1$,*

$$\liminf_{N \to \infty} \mathbb{P}_N^D \left(\mathrm{diam}(\mathscr{C}_N) \in ((1-\epsilon)N, (1-\epsilon')N)\right) > 0,$$

where $\mathbb{P}_N^D$ denotes the associated probability measure.

When this event occurs, the origin remains non-frozen at time 1. Therefore, the probability that the origin is frozen at time 1 is uniformly bounded away from 1. Moreover, Kiss [5] proved a stronger result, showing that this probability actually tends to zero as $N \to \infty$.

Theorem (Theorem 1.1 in [5]). *As $N \to \infty$, the probability that the origin belongs to a frozen cluster in the N-parameter diameter frozen site percolation process tends to zero.*

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Volume Frozen Percolation

In volume frozen site percolation on the triangular lattice, an occupied cluster freezes as soon as it contains at least N vertices. Van den Berg et al. [16] conjectured that the probability of the origin being frozen tends to zero as $N \to \infty$, analogous to the behavior in diameter frozen percolation. This conjecture was later confirmed in [15].

Theorem (Theorem 1.1 in [15]). *In volume frozen percolation on the triangular lattice with parameter $N \geq 1$,*

$$\mathbb{P}_N^V\,(\textit{the origin is frozen at time } 1) \overset{N\to\infty}{\longrightarrow} 0,$$

where $\mathbb{P}_N^V$ denotes the associated probability measure.

Both papers examine the non-monotonic behavior of volume frozen percolation, which arises at certain ”exceptional scales” $m_k(N)$. The behavior of the process in $B(m)$, for $m = m(N)$, differs significantly depending on the choice of m. If $m(N)$ is close to $m_k(N)$, the probability that the origin is frozen remains bounded away from 0. In contrast, if $m(N)$ is far from $m_k(N)$, the probability tends to zero as $N \to \infty$. This non-monotonicity makes the analysis of volume frozen percolation and the proof of the above theorem particularly challenging. Notably, such exceptional scales do not arise in other frozen percolation models.

Boundary Frozen Percolation

We now define boundary frozen site percolation on the triangular lattice. Fix $N \geq 1$, and consider the subgraph $\mathbb{T}_N = (V_N, E_N)$ induced by the vertex set $V_N := B(N) \cap V$.

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For simplicity, we remove from E_N all edges $\{u, v\}$ such that both endpoints lie on the inner boundary $\partial B(N)$ (see Section 2.1 for the definition of $\partial B(N)$). With slight abuse of notation, we continue to denote the modified edge set by E_N.

The freezing condition is defined as follows: an occupied cluster freezes as soon as it intersects the boundary $\partial B(N)$. That is, at time τ_v, a vertex v becomes occupied unless one of its neighbors already belongs to a frozen cluster. We denote the corresponding probability measure by $\mathbb{P}_N^B$. Since the model can be viewed as a finite-range interacting particle system, the process is well-defined for every $N \in \mathbb{N}$, and there exists an associated probability space $\left(\Omega_N, \mathcal{F}_N, \mathbb{P}_N^B\right)$.

Analogous to the diameter and volume frozen percolation processes, Makowiec [7] studied the asymptotic behavior of the probability that the origin is frozen at time $t = 1$ as $N \to \infty$. He conjectured the existence of a constant $C > 0$ such that

$$\inf_{N \geq 1} \mathbb{P}_N^B \left(\text{the origin is not frozen at time } t = 1\right) > C.$$

Moreover, he aimed to show the stronger statement:

$$\mathbb{P}_N^B \left(\text{the origin is frozen at time } t = 1\right) \overset{N\to\infty}{\longrightarrow} 0.$$

In [17], the authors studied a forest fire process on finite subdomains of the triangular lattice, where ignitions originate at the boundary. Although their setting differs from ours in that it does not require surrounding vertices of frozen clusters to remain vacant, their method—based on the existence of deep paths confined within cones of opening angle less than π—can be adapted. With suitable modifications, we apply this method to establish the following result.

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Theorem 1.1 (Modified from Theorem 1.1 in [17]). *Consider boundary frozen percolation in $B(N)$ for $N \geq 1$. Then*

$$\mathbb{P}_N^B\,(\textit{the origin is frozen at time } t = 1) \overset{N\to\infty}{\longrightarrow} 0.$$

1.2 A Variant Model

We now consider a different freezing condition: an occupied cluster freezes as soon as it contains an occupied horizontal or vertical crossing of $B(N)$ (see Section 2.1 for the definition of a crossing). We refer to this model as crossing frozen percolation, and denote the associated probability measure by $\mathbb{P}_N^C$. As in the boundary frozen percolation, one can show that the probability that the origin is frozen at time $t = 1$ tends to zero as $N \to \infty$.

Theorem 1.2. *Consider crossing frozen percolation in $B(N)$ for $N \geq 1$. Then*

$$\mathbb{P}_N^C\,(\textit{the origin is frozen at time } t = 1) \overset{N\to\infty}{\longrightarrow} 0. \tag{1.1}$$

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Chapter 2 Tools from Near-Critical Percolation

In this chapter, we introduce some basic concepts and tools in percolation theory. We also provide an overview of both classical and contemporary results that are essential for analyzing frozen percolation models. Detailed proofs are mostly omitted, with references to the relevant literature provided instead.

2.1 Notations

Consider the triangular lattice $\mathbb{T} = (V, E)$. Let $\Omega = \{0, 1\}^V$ be the sample space, and consider the σ-algebra generated by cylinder sets. For a given $p \in [0, 1]$, each vertex is independently assigned to be occupied with probability p and vacant with probability $1 - p$. If a vertex v is occupied, we write $\omega_v = 1$; if it is vacant, we write $\omega_v = 0$. Thus, $\omega = (\omega_v)_{v \in V} \in \Omega$ represents a (site) percolation configuration.

We write $u \sim v$ if $\{u, v\} \in E$. A sequence of vertices $(v_1, v_2, \ldots, v_s)$ is called a (self-avoiding) path if $v_i \sim v_{i+1}$ for all $i = 1, \ldots, s - 1$, and $v_i \neq v_j$ for all $i, j \in 1, \ldots, s$ with $i \neq j$. If $v_s \sim v_1$, the sequence $(v_1, v_2, \ldots, v_s, v_1)$ is called a circuit. A path (respectively, circuit) π is said to be occupied if every vertex in the path (respectively,

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circuit) is occupied. Conversely, if every vertex in the path (respectively, circuit) is vacant, then π is called a vacant path (respectively, circuit). We say that a path (or circuit) π is of length n if it contains n vertices.

We say that two sets of vertices S_1 and S_2 are connected if there exist $u \in S_1$ and $v \in S_2$ such that there is an occupied path π from u to v. We denote this connection by $u \leftrightarrow v$. If $S_1 = \{u\}$, we write $u \leftrightarrow S_2$ instead of $\{u\} \leftrightarrow S_2$, and similarly when $S_2 = \{v\}$. If the path π is required to use only vertices from a subset $R \subseteq V$, we denote this event by $\{S_1 \leftrightarrow S_2 \text{ in } R\}$. For any vertex $v \in V$, the occupied cluster containing v is defined as

$$\mathscr{C}(v) = \{u \in V : u \leftrightarrow v\}.$$

For a set of vertices $S \subseteq V$, we denote its complement by $S^c := V \setminus S$. The outer boundary $\partial^{\text{out}} S$ of S is defined as

$$\partial^{\text{out}} S = \{u \in S^c : \text{there exists } v \in S \text{ such that } v \sim u\},$$

and the inner boundary ∂S of S is defined as

$$\partial S = \{v \in S : \text{there exists } u \in S^c \text{ such that } v \sim u\}.$$

Consider the basis given by 1 and $e^{i\pi/3}$. Let $x_1 < x_2$ and $y_1 < y_2$ be integers. The parallelogram

$$R = [x_1, x_2] \times [y_1, y_2]$$

is defined as the set of vertices in the convex hull of the corners $x_j + y_k e^{i\pi/3}$ for $j, k = 1, 2$. For such a parallelogram R, we denote its left, right, top, and bottom inner boundaries by

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$\partial_L R$, $\partial_R R$, $\partial_T R$, and $\partial_B R$, respectively, each being a part of the inner boundary ∂R (see Figure 2.1).

We say that there exists an occupied horizontal crossing in R if the event $\{\partial_L R \leftrightarrow \partial_R R \text{ in } R\}$ occurs, and an occupied vertical crossing if $\{\partial_T R \leftrightarrow \partial_B R \text{ in } R\}$ occurs. These events are denoted by $\mathcal{C}_H(R)$ and $\mathcal{C}_V(R)$, respectively. We similarly define vacant crossings, denoted by $\mathcal{C}_H^*(R)$ and $\mathcal{C}_V^*(R)$.

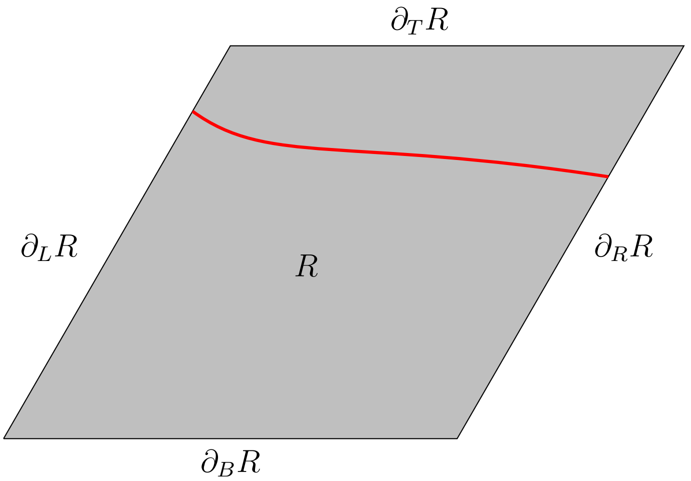

Figure 2.1: The parallelogram R and its boundary. The red path illustrates a horizontal crossing of R.

In the half-plane percolation setting, all the preceding definitions extend naturally by considering their restriction to the intersection with the chosen half-plane. For instance, the upper half-plane of the triangular lattice is denoted by $\mathbb{T}_{\mathbb{H}} = (V_{\mathbb{H}}, E_{\mathbb{H}})$, where

$$V_{\mathbb{H}} = \{v = v_1 + v_2 e^{\frac{\pi}{3} i} \in V : v_2 \geq 0\},$$

and

$$E_{\mathbb{H}} = \{\{u, v\} : u, v \in V_{\mathbb{H}},\ \|u - v\| = 1\}.$$

Fix $1 \leq n < m$. We define the event that there exists an occupied circuit surrounding

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$B(v;n)$ and contained in $B(v;m)$ as

$$\mathcal{O}(v;n,m) := \{\exists \text{ an occupied circuit in } A(v;n,m)\}.$$

For the vacant case, we also define

$$\mathcal{O}^*(v;n,m) := \{\exists \text{ a vacant circuit in } A(v;n,m)\}.$$

2.2 Useful Tools in Percolation Theory

In the following, we summarize a selection of fundamental results in percolation theory that are instrumental for our analysis.

Definition 2.1. *Let $\omega, \omega' \in \Omega$. We write $\omega \leq \omega'$ if, for every vertex $v \in V$, we have $\omega_v \leq \omega'_v$. An event $\mathcal{E} \subseteq \Omega$ is said to be increasing if, for any $\omega \in \mathcal{E}$, it holds that $\omega' \in \mathcal{E}$ whenever $\omega' \geq \omega$.*

Theorem 2.2 (FKG inequality)**.** *Let A and B be two increasing events. Then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\, \mathbb{P}_p(B).$$

Proof. See Theorem 2.4 in [2]. □

On the triangular lattice, we have by duality that, for all $n \in \mathbb{N}$,

$$\mathbb{P}_{\frac{1}{2}}\left(\mathcal{C}_H([0,n] \times [0,n])\right) = \frac{1}{2}.$$

In particular, these crossing probabilities are uniformly bounded from below. The Russo–Seymour–Welsh (RSW) theorem provides lower bounds for such crossing probabilities

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in parallelograms. Combined with the FKG inequality, this allows us to establish uniform bounds for many important structures.

Theorem 2.3 (Russo–Seymour–Welsh)**.** *For any $k \geq 2$, there exists a constant $\delta(k) > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}_{\frac{1}{2}}\left(\mathcal{C}_H\left([0, kn] \times [0, n]\right)\right) \geq \delta(k).$$

Proof. See Russo [9] and Seymour and Welsh [10]. A modern treatment appears in Theorem 2 and Corollary 3 of [8]. □

We next describe a standard coupling for percolation. Assign to each vertex $v \in V$ a random variable τ_v, where the collection $(\tau_v)_{v \in V}$ consists of i.i.d. variables uniformly distributed on $[0, 1]$. For any fixed $p \in [0, 1]$, we say that v is p-occupied if $\tau_v \leq p$, and p-vacant otherwise.

Under this coupling, percolation at level p corresponds to declaring all p-occupied vertices as occupied. The associated probability measure is denoted by $\mathbb{P}_p$.

We are particularly interested in the probability that the origin is connected to points arbitrarily far away. This motivates the definition of the percolation probability:

$$\theta(p) := \lim_{n \to \infty} \mathbb{P}_p\left(0 \leftrightarrow \partial B(n)\right).$$

By construction, $\theta(p)$ is a non-decreasing function of p, with $\theta(0) = 0$ and $\theta(1) = 1$. This function exhibits a phase transition, and the value at which the transition occurs is called the critical parameter, defined by

$$p_c := \sup\{p : \theta(p) = 0\}.$$

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Theorem 2.4. *For Bernoulli site percolation on the triangular lattice, the critical parameter is given by $p_c = \frac{1}{2}$. Moreover, $\theta(p_c) = 0$.*

Proof. See [3] for the case of bond percolation on $\mathbb{Z}^2$. The proof can be adapted with minor modifications to site percolation on the triangular lattice. □

Next, we introduce the characteristic length, which describes the scale at which percolation behavior begins to differ from criticality. For $\epsilon \in \left(0, \frac{1}{2}\right)$, define

$$L_\epsilon(p) := \begin{cases} \min\left\{n \geq 1 : \mathbb{P}_p\left(\mathcal{C}_V([0,n] \times [0,n])\right) \leq \epsilon\right\}, & \text{if } p < \frac{1}{2}, \\ \min\left\{n \geq 1 : \mathbb{P}_p\left(\mathcal{C}_V^*([0,n] \times [0,n])\right) \leq \epsilon\right\}, & \text{if } p > \frac{1}{2}, \end{cases}$$

and set $L_\epsilon\left(\frac{1}{2}\right) := \infty$. By duality, $L_\epsilon(p) = L_\epsilon(1-p)$. Moreover, using the coupling, one can show that $L_\epsilon(p)$ is increasing for $p < \frac{1}{2}$, and decreasing for $p > \frac{1}{2}$.

Theorem 2.5. *For any $\epsilon, \epsilon' \in \left(0, \frac{1}{2}\right)$, we have*

$$L_\epsilon(p) \asymp L_{\epsilon'}(p) \quad \textit{as } p \to \frac{1}{2}.$$

Proof. See Corollary 37 in [8]. □

From Theorem 2.5, we fix $\epsilon = 10^{-4}$ for convenience, which is sufficiently small, and omit the subscript ϵ in the notation $L(p)$ in what follows.

Theorem 2.6 (Exponential decay)**.** *There exist universal constants $c_1, c_2 > 0$ such that for all $p < p_c$ and $n \geq 1$,*

$$\mathbb{P}_p\left(\mathcal{C}_H([0,n] \times [0,n])\right) \leq c_1 e^{-c_2 n/L(p)}. \tag{2.1}$$

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By duality, the corresponding lower bound holds for $p > p_c$: for all $n \geq 1$,

$$\mathbb{P}_p\left(\mathcal{C}_H([0,n] \times [0,n])\right) \geq 1 - c_1 e^{-c_2 n / L(p)}. \tag{2.2}$$

Proof. See Lemma 39 in [8]. □

2.3 Arm Events

Arm events play a crucial role in the study of critical and near-critical percolation. These events concern the existence of multiple disjoint occupied or vacant paths (called arms) crossing an annulus $A(n, N)$, where $n < N$. Arm events can be combined to describe more complex structures.

Let $j \geq 1$, and consider a color sequence $\sigma = (\sigma_1, \ldots, \sigma_j)$, where each $\sigma_i \in \{o, v\}$ denotes an occupied or vacant state, respectively. Two sequences are identified if they are equal up to a cyclic permutation.

Fix $j \geq 1$ and a color sequence $\sigma = (\sigma_1, \ldots, \sigma_j)$. For any two integers $n < m$, the j-arm event with color sequence σ is defined as

$$A_{j,\sigma}(n,m) := \left\{ \begin{array}{l} \text{There exist } j \text{ disjoint paths } \pi_1, \ldots, \pi_j \text{ from } \partial B(n) \text{ to } \partial B(m), \\ \text{ordered counterclockwise and colored according to } \sigma \end{array} \right\}.$$

In the half-plane setting $\mathbb{T}_{\mathbb{H}}$, we define $A^H_{j,\sigma}(n,m)$ as the event that j disjoint arms, with colors specified by σ, connect $\partial^{\text{in}} B(n)$ to $\partial^{\text{in}} B(m)$ using only sites in $\mathbb{T}_{\mathbb{H}}$.

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We define the following probabilities at criticality:

$$\pi_1(n,m) := \mathbb{P}_{\frac{1}{2}}\left(A_{1,o}(n,m)\right), \quad \pi_4(n,m) := \mathbb{P}_{\frac{1}{2}}\left(A_{4,ovov}(n,m)\right),$$

which correspond to the 1-arm and 4-arm events, respectively. When $n = 0$, we write $\pi_1(m) := \pi_1(0,m)$ and $\pi_4(m) := \pi_4(0,m)$.

Similarly, for the upper half-plane $\mathbb{T}_{\mathbb{H}}$, we define

$$\pi_1^+(n,m) := \mathbb{P}_{\frac{1}{2}}\left(A_{1,o}^H(n,m)\right), \quad \pi_4^+(n,m) := \mathbb{P}_{\frac{1}{2}}\left(A_{4,ovov}^H(n,m)\right).$$

When $n = 0$, we write $\pi_1^+(m) := \pi_1^+(0,m)$ and $\pi_4^+(m) := \pi_4^+(0,m)$.

Theorem 2.7. *Fix $j \geq 1$ and $\sigma \in \{o,v\}^j$. Then the following results hold:*

1. ***Extendability**: For any $p \in [0,1]$ and $1 \leq n \leq m \leq L(p)$,*

$$\mathbb{P}_p\left(A_{j,\sigma}(n,2m)\right),\ \mathbb{P}_p\left(A_{j,\sigma}\left(\tfrac{n}{2},m\right)\right) \asymp \mathbb{P}_p\left(A_{j,\sigma}(n,m)\right) \quad \text{as } p \to \tfrac{1}{2}. \tag{2.3}$$

2. ***Quasi-multiplicativity**: For any $p \in [0,1]$, if $n_0(j) \leq n_1 < n_2 < n_3 \leq L(p)$ and $n_2 \geq 4n_1$, then*

$$\mathbb{P}_p\left(A_{j,\sigma}(n_1,n_2)\right)\mathbb{P}_p\left(A_{j,\sigma}(n_2,n_3)\right) \asymp \mathbb{P}_p\left(A_{j,\sigma}(n_1,n_3)\right) \quad \text{as } p \to \tfrac{1}{2}, \tag{2.4}$$

where $n_0(j)$ is the smallest nonnegative integer such that $A_{j,\sigma}(n_0(j),N) \neq \emptyset$ for all $N \geq n_0(j)$.

3. ***Arm events near criticality**: For any $p \in (0,1)$ and $n_0(j) \leq n \leq m \leq L(p)$,*

$$\mathbb{P}_p\left(A_{j,\sigma}(n,m)\right) \asymp \mathbb{P}_{\frac{1}{2}}\left(A_{j,\sigma}(n,m)\right) \quad \text{as } p \to \tfrac{1}{2}. \tag{2.5}$$

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Moreover, the same results hold in the half-plane percolation setting.

Proof. See Proposition 12, Proposition 16, Proposition 17, and Theorem 27 in [8]. □

Theorem 2.8. *On the triangular lattice, the following estimates hold:*

1. $$|p_c - p| L(p)^2 \pi_4(L(p)) \asymp 1 \quad \text{as } p \to \tfrac{1}{2}. \tag{2.6}$$

2. $$\theta(p) \asymp \pi_1(L(p)) \quad \text{as } p \downarrow \tfrac{1}{2}. \tag{2.7}$$

Proof. See Theorem 2 and (4.5) in [4], or (7.25) and Proposition 34 in [8]. □

Theorem 2.9. *Fix $j \geq 1$ and a color sequence σ (non-constant if $j \geq 2$). For any $n, N \geq n_0(j)$,*

$$\mathbb{P}_{\frac{1}{2}}\left(A_{j,\sigma}(n, N)\right) = \left(\frac{n}{N}\right)^{\alpha_j + o(1)} \quad \text{as } N \to \infty, \tag{2.8}$$

where $\alpha_1 = \frac{5}{48}$ and $\alpha_j = \frac{j^2-1}{12}$ for $j \geq 2$. We call α_j the j-arm exponent. In particular, combining this with Theorem 2.8, we obtain the critical exponents for θ and L:

$$L(p) = \left|p - \tfrac{1}{2}\right|^{-\frac{4}{3}+o(1)} \quad \text{as } p \to \tfrac{1}{2}, \quad \text{and} \quad \theta(p) = \left(p - \tfrac{1}{2}\right)^{\frac{5}{36}+o(1)} \quad \text{as } p \downarrow \tfrac{1}{2}. \tag{2.9}$$

Proof. These estimates are derived from the relationship between arm events and Schramm–Loewner Evolution (SLE) with parameter $\kappa = 6$. Together with Kesten's scaling relation [4], this connection allows one to establish both the existence and exact values of the multichromatic arm exponents. See Theorem 1.1 in [6] and Theorem 4 in [12] for further details. □

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Theorem 2.10. *Fix $j \geq 1$ and a color sequence σ (non-constant if $j \geq 2$). For any $n, N \geq n_0(j)$,*

$$\mathbb{P}_{\frac{1}{2}}\left(A^H_{j,\sigma}(n,N)\right) = \left(\frac{n}{N}\right)^{\beta_j + o(1)} \quad \text{as } N \to \infty, \tag{2.10}$$

where $\beta_j = \frac{j(j+1)}{6}$ for $j \geq 1$. We call β_j the half-plane j-arm exponent.

Proof. See Theorem 3 in [12]. □

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Chapter 3 Proofs of Main Results

In this chapter, we prove Theorems 1.1 and 1.2.

For Theorem 1.1, although the model in [17] also stops clusters from growing once they touch the boundary, it does not require the outer boundary of a frozen cluster to remain vacant. This difference is minor, and the methods from [17] can still be applied with only small modifications. The proof of Theorem 1.1 can likewise be adjusted, with minor modifications, to establish Theorem 1.2.

3.1 Some Preliminaries

Fix $\alpha \in (0, \frac{\pi}{2}]$ and $N \geq 1$. For any $v \in \partial B(N)$, let $\mathcal{C}^{(\alpha)}(v)$ denote the intersection of $B(N)$ with the closed cone of opening angle 2α, apex at v, and is directed inward within $B(N)$. For $0 \leq n < m$, the truncated cone is given by

$$\mathcal{C}^{(\alpha)}(v; n, m) := \mathcal{C}^{(\alpha)}(v) \cap A(v; n, m).$$

We similarly define the 1-arm event, denoted by $A_1\left(\mathcal{C}^{(\alpha)}(v; n, m)\right)$, as the event that there exists an occupied path in $\mathcal{C}^{(\alpha)}(v; n, m)$ connecting $\mathcal{C}^{(\alpha)}(v; n, m) \cap \partial B(v; n)$ to $\mathcal{C}^{(\alpha)}(v; n, m) \cap \partial B(v; m)$. In the case $n = 0$, we take $\partial B(v; 0)$ to be $\{v\}$.

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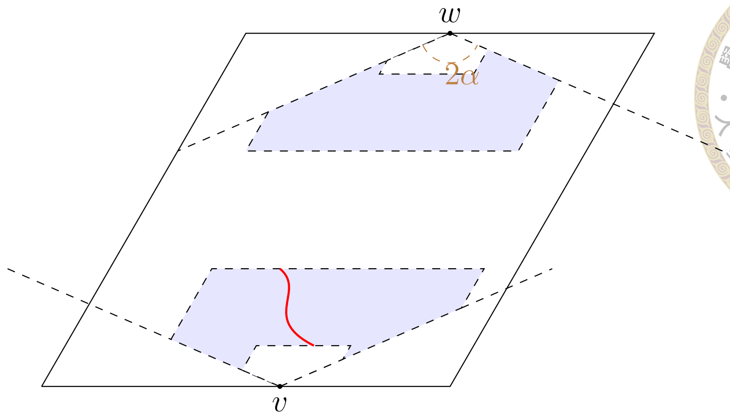

Figure 3.1: The region $\mathcal{C}^{(\alpha)}(w; n, m)$ and the event $A_1\left(\mathcal{C}^{(\alpha)}(v; n, m)\right)$

Let $0 \leq n < m \ll N$, and suppose that $v \in \partial B(N)$ lies at distance at least $2m$ from any corner of $B(N)$. Under this assumption, the truncated cone region $\mathcal{C}^{(\alpha)}(v; n, m)$ does not intersect the boundary of $B(N)$. We define the 1-arm probabilities at criticality in the cone region by

$$\pi_1^{(\alpha)}(m) := \mathbb{P}_{\frac{1}{2}}\left(A_1(\mathcal{C}^{(\alpha)}(v; 0, m))\right), \quad \text{and} \quad \pi_1^{(\alpha)}(n, m) := \mathbb{P}_{\frac{1}{2}}\left(A_1(\mathcal{C}^{(\alpha)}(v; n, m))\right).$$

By translation invariance, these definitions are well-defined. Furthermore, the 1-arm probabilities in cones satisfy analogues of the "extendability," "quasi-multiplicativity," and "1-arm exponent" properties known from the full-plane setting.

Theorem 3.1. *Let $\alpha \in \left(0, \frac{\pi}{2}\right]$. Then the following estimates hold:*

1. *There exist constants $c_1, c_2 > 0$ (depending on α) such that for all $0 \leq r < R$,*

$$c_1\, \pi_1^{(\alpha)}(r, R) \leq \pi_1^{(\alpha)}(r, 2R), \quad \pi_1^{(\alpha)}\left(\frac{r}{2}, R\right) \leq c_2\, \pi_1^{(\alpha)}(r, R). \tag{3.1}$$

2. *There exist constants $c_3, c_4 > 0$ (depending on α) such that for all $0 \leq r < s < R$*

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with $s \geq 4r$,

$$c_3\, \pi_1^{(\alpha)}(r, R) \leq \pi_1^{(\alpha)}(r, s)\, \pi_1^{(\alpha)}(s, R) \leq c_4\, \pi_1^{(\alpha)}(r, R). \tag{3.2}$$

3. *For any $\alpha \in \left(0, \frac{\pi}{2}\right)$, define*

$$\kappa_\alpha := \frac{\pi}{6\alpha}. \tag{3.3}$$

Then, for any $\eta > 0$, there exist constants $c_5, c_6 > 0$ (depending on α and η) such that for all $1 \leq r < R$,

$$c_5 \left(\frac{r}{R}\right)^{\kappa_\alpha + \eta} \leq \pi_1^{(\alpha)}(r, R) \leq c_6 \left(\frac{r}{R}\right)^{\kappa_\alpha - \eta}. \tag{3.4}$$

Proof. The first two assertions follow from the Russo–Seymour–Welsh theorem and the FKG inequality. The third is a consequence of the conformal invariance of critical percolation in the scaling limit [11]. □

Given $m \geq 1$ and $v \in \partial B(N)$, we define

$$F_m(v) := \left\{\tau_v \in [0, p_c] \text{ and } A_1(\mathcal{C}^{(\pi/2)}(v; 1, m)) \text{ occurs at time } \tau_v\right\}.$$

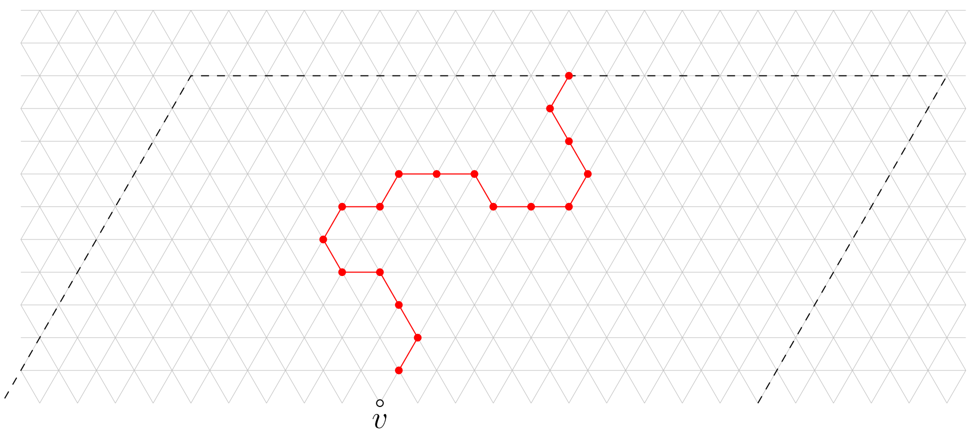

Figure 3.2: A configuration illustrating the event $F_m(v)$ occurring at time τ_v

Given $p \in [0, 1]$, we define $\mathcal{F}_p$ to be the set of all vertices $w \in B(N)$ such that there

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exists a p-occupied path from w to $\partial B(N)$. Moreover, we define $\mathcal{L}_N$ as the collection of all vertices $v \in \partial B(N)$ satisfying

$$\|v - z\| \geq \frac{N}{3} \quad \text{for all } z \in \{(\pm N, \pm N)\}.$$

That is, $\mathcal{L}_N$ consists of the boundary vertices that are not too close to the corners of $B(N)$.

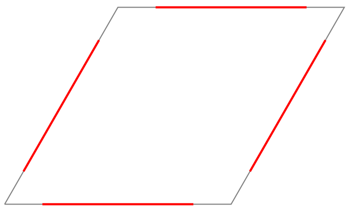

Figure 3.3: Illustration of the set $\mathcal{L}_N$.

Lemma 3.2. *Let $\eta > 0$ and $N \in \mathbb{N}$. There exists a constant $c(\eta) > 0$ such that for any $1 \leq n \leq \frac{N}{3}$ and any $v \in B(N)$,*

$$\mathbb{P}_N^B(F_n(v)) \leq c(\eta)\, n^{-\frac{13}{12}+\eta}. \tag{3.5}$$

Proof. Without loss of generality, we may embed $B(N)$ into the upper half-plane $\mathbb{T}_{\mathbb{H}}$ by applying translation, rotation, and reflection as needed, so that the vertex v is mapped to the origin and $B(N)$ lies within $\mathbb{T}_{\mathbb{H}}$ (see Figure 3.4).

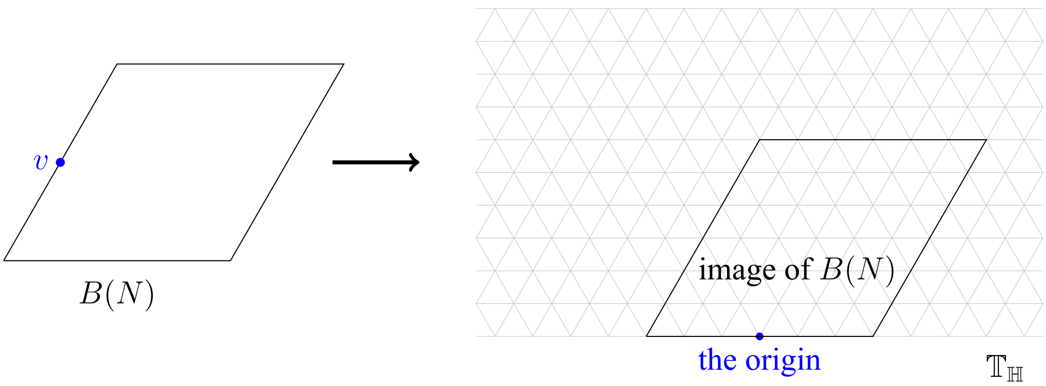

Figure 3.4: The embedding of $B(N)$ in $\mathbb{T}_{\mathbb{H}}$

Let τ_O denote the τ-value of the origin in the upper half-plane $\mathbb{T}_{\mathbb{H}}$. We define the

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event

$$\tilde{F}_n := \left\{\tau_O \in [0, p_c],\ \text{and } A^H_{1,o}(0,n) \text{ occurs at time } \tau_O\right\}.$$

Then, by the embedding, for any $v \in B(N)$ and $1 \le n \le \frac{N}{3}$,

$$\mathbb{P}^B_N(F_n(v)) \le \mathbb{P}(\tilde{F}_n),$$

where $\mathbb{P}$ denotes the percolation probability on $\mathbb{T}_{\mathbb{H}}$. It thus suffices to show that

$$\mathbb{P}(\tilde{F}_n) \le c(\eta)\, n^{-\frac{13}{12}+\eta}.$$

Assume, without loss of generality, that n is large enough so that $p_c - \delta > \frac{3}{4}p_c$, and select $\delta > 0$ such that $L(p_c - \delta) = n$. Define $J \ge 0$ to be the integer for which

$$p_c - 2^{J+1}\delta \le \tfrac{3}{4}p_c < p_c - 2^J\delta.$$

We partition the event $\{\tau_O \in [0, p_c)\}$ into the subevents

$$\{\tau_O \in [p_c - \delta, p_c)\}, \quad \{\tau_O \in [0, p_c - 2^{J+1}\delta)\}, \quad \text{and } \{\tau_O \in [p_c - 2^{j+1}\delta, p_c - 2^j\delta)\}$$

for $0 \le j \le J$. Then we consider the sum of the upper bounds of the probabilities of $\tilde{F}_n$ over these events.

- For $\tau_O \in [p_c - \delta, p_c)$, since $L(\tau_O) \ge n$, we can use the following upper bound:

$$\mathbb{P}\left(\{\tau_O \in [p_c - \delta, p_c)\} \cap \{\text{at time } p_c,\ A^H_{1,o}(1,n) \text{ occurs}\}\right) \le \delta\pi^+_1(1,n).$$

- If $\tau_O \in (0, p_c - 2^{J+1}\delta)$, then using Theorem 2.6, the FKG inequality, and the RSW

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theorem, we obtain:

$$\mathbb{P}\left(\{\tau_O \in (0, p_c - 2^{J+1}\delta)\} \cap \left\{\text{at time } \tfrac{3}{4}p_c,\ A^H_{1,o}(1,n) \text{ occurs}\right\}\right) \leq p_c\, c_1 e^{-c_2 n / L(\frac{3}{4}p_c)}.$$

- For an arbitrary $0 \leq j \leq J$, if $\tau_O \in (p_c - 2^{j+1}\delta, p_c - 2^j\delta)$, then we use the following bound:

$$\begin{aligned}
&\mathbb{P}\left(\left\{\tau_O \in (p_c - 2^{j+1}\delta, p_c - 2^j\delta)\right\} \cap \left\{\text{at time } p_c - 2^j\delta,\ A^H_{1,o}(1,n) \text{ occurs}\right\}\right) \\
&\leq \mathbb{P}\left(\tau_O \in (p_c - 2^{j+1}\delta, p_c - 2^j\delta)\right) \\
&\qquad \times \mathbb{P}\left(\text{at time } p_c,\ A^H_{1,o}(1, L(p_c - 2^{j+1}\delta)) \text{ occurs}\right) \\
&\qquad \times \mathbb{P}\left(\text{at time } p_c - 2^j\delta,\ A^H_{1,o}(L(p_c - 2^j\delta), n) \text{ occurs}\right) \\
&\leq 2^j\delta\, \pi_1^+(1, L(p_c - 2^{j+1}\delta))\, c_1 e^{-c_2 n / L(p_c - 2^j\delta)}.
\end{aligned}$$

Here, the constants $c_1, c_2 > 0$ are independent of n and j.

Next, for any $1 \leq j \leq J$, we analyze the behavior of $L(p)$ using equation (2.6):

$$\begin{aligned}
\frac{L(p_c - 2^j\delta)}{L(p_c - \delta)} &\asymp \left(\frac{\delta\pi_4(L(p_c - \delta))}{2^j\delta\pi_4(L(p_c - 2^j\delta))}\right)^{\frac{1}{2}} \\
&= 2^{-\frac{j}{2}}\left(\frac{\pi_4(L(p_c - \delta))}{\pi_4(L(p_c - 2^j\delta))}\right)^{\frac{1}{2}} \\
&\asymp 2^{-\frac{j}{2}}\left(\pi_4(L(p_c - 2^j\delta), L(p_c - \delta))\right)^{\frac{1}{2}} \quad \text{(quasi-multiplicativity)} \\
&= 2^{-\frac{j}{2}}\left(\frac{L(p_c - 2^j\delta)}{L(p_c - \delta)}\right)^{\frac{1}{2}(\frac{5}{4} + o(1))} \quad \text{(since } \alpha_4 = \tfrac{5}{4}\text{)},
\end{aligned} \tag{3.6}$$

which implies:

$$\frac{L(p_c - 2^j\delta)}{L(p_c - \delta)} \asymp (2^{-\frac{j}{2}})^{-\frac{8}{3} + o(1)}.$$

Therefore, for any $\eta > 0$, there exist constants $c_1' = c_1'(\eta), c_2' = c_2'(\eta) > 0$ such that

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for any $1 \le j \le J$,

$$c_1'(2^j)^{-\frac{4}{3}-\eta} n \le L(p_c - 2^j\delta) \le c_2'(2^j)^{-\frac{4}{3}+\eta} n. \tag{3.7}$$

We now estimate the total probability by summing over all subevents:

$$\begin{aligned}\mathbb{P}(\tilde{F}_n) &\le \delta\pi_1^+(1,n) + p_c\, c_1 e^{-c_2 n/L(\frac{3}{4}p_c)} \\ &\quad + \sum_{j=0}^{J} 2^j \delta\, \pi_1^+(1, L(p_c - 2^{j+1}\delta))\, c_1 e^{-c_2 n/L(p_c-2^j\delta)}.\end{aligned} \tag{3.8}$$

To bound $\pi_1^+(1, L(p_c - 2^{j+1}\delta))$, we apply quasi-multiplicativity and the exponent $\beta_1 = \frac{1}{3}$. For $j \in \{0, \dots, J\}$,

$$\pi_1^+(1, L(p_c - 2^{j+1}\delta)) \le c_3 \frac{\pi_1^+(1, L(p_c - \delta))}{\pi_1^+(L(p_c - 2^{j+1}\delta), L(p_c-\delta))} \le c_3'(2^{j+1})^{\frac{4}{9}+\eta}\pi_1^+(n),$$

x for some constants $c_3, c_3' = c_3'(\eta)$.

By (3.7), we also have:

$$e^{-c_2 n/L(p_c - 2^j\delta)} \le e^{-c_2''(2^j)^{\frac{4}{3}-\eta}},$$

for some $c_2'' = c_2''(\eta) > 0$.

Since $n = L(p_c - \delta) = \delta^{-\frac{4}{3}+o(1)}$, we obtain $\delta = n^{-\frac{3}{4}+o(1)}$. Using (2.10), there exist constants $c_4, c_5 > 0$ depending only on η such that:

$$\delta \le c_4 n^{-\frac{3}{4}+\frac{\eta}{2}}, \quad \text{and} \quad \pi_1^+(n) \le c_5 n^{-\frac{1}{3}+\frac{\eta}{2}}.$$

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Substituting these into (3.8) yields:

$$\mathbb{P}_N^B(F_n(v)) \le \mathbb{P}(\tilde{F}_n) \le c\,n^{-\frac{3}{4}+\frac{\eta}{2}} \cdot n^{-\frac{1}{3}+\frac{\eta}{2}} = c\,n^{-\frac{13}{12}+\eta}. \tag{3.9}$$

□

Definition 3.3. *Fix $N \ge 1$. For $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and $\delta < \frac{1}{10}$, we say that $v \in \partial B(N)$ is an $(\alpha, \delta N)$-cone site if the following conditions hold:*

1. $v \in \mathcal{L}_N$.
2. $\mathcal{F}_{p_c} \cap \partial^{\text{out}}\mathcal{C}^{(\alpha)}(v; 1, \delta N) = \emptyset$.
3. $A_1\left(\mathcal{C}^{(\alpha)}(v; 1, \delta N)\right)$ *occurs at time* p_c *and* $\tau_v \in (p_c, \frac{3}{4})$.
4. *Writing* $w_1, w_2 \in \partial B(N)$ *for the two neighbors of* v, *we have* $\tau_{w_1}, \tau_{w_2} > \tau_v$.

We write $\text{Cone}_N(\alpha, \delta N)$ *for the set of all* $(\alpha, \delta N)$*-cone sites in* $\partial B(N)$.

The restriction $\delta < \frac{1}{10}$ ensures that for any two distinct vertices $v, v' \in \mathcal{L}_N$ on different sides of $\partial B(N)$, we have $B(v; \delta N) \cap B(v'; \delta N) = \emptyset$. In fact, $\frac{1}{10}$ can be replaced by any sufficiently small positive constant.

Lemma 3.4. *Let $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and $\delta \in \left(0, \frac{1}{10}\right)$. Then there exist constants $c_1, c_2 > 0$, depending only on α, such that for all $N \ge 1$ and $v, v' \in \mathcal{L}_N$,*

(i)

$$c_1 \pi_1^{(\alpha)}(\delta N) \le \mathbb{P}_N^B\left(v \in \text{Cone}_N(\alpha, \delta N)\right) \le \frac{1}{2}\pi_1^{(\alpha)}(\delta N).$$

(ii)

$$\mathbb{P}_N^B\left(v, v' \in \text{Cone}_N(\alpha, \delta N)\right) \le c_2 \pi_1^{(\alpha)}(\delta N) \cdot \pi_1^{(\alpha)}\left(\|v - v'\| \wedge \delta N\right).$$

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Proof. Without loss of generality, we assume that N is sufficiently large. For part (i), the upper bound follows by considering only condition (3) in Definition 3.3:

$$\mathbb{P}_N^B\left(v \in \mathrm{Cone}_N(\alpha, \delta N)\right) \leq \frac{1}{4} \cdot \pi_1^{(\alpha)}(1, \delta N) = \frac{1}{2}\pi_1^{(\alpha)}(\delta N).$$

To obtain the lower bound, consider the event that all vertices $v' \in \partial B(N) \setminus \{v\}$ with $\|v - v'\| \leq d_0$ are $\frac{3}{4}$-vacant, where $d_0 \geq 1$ is to be chosen later. This event has probability $(\frac{1}{4})^{2d_0}$. Under this condition and $\tau_v \in (p_c, \frac{3}{4})$, the cluster $\mathcal{F}_{p_c}$ cannot reach within distance $d_0 + 1$ of v.

Let $v' \in \partial B(N)$ with $\|v - v'\| = k \geq d_0 + 1$. We distinguish two cases:

- If v and v' lie on the same side of $\partial B(N)$, then any p_c-occupied path from v' to $\partial^{\text{out}}\mathcal{C}^{(\alpha)}(v; 1, \delta N)$ must have length at least $k \cos \alpha$.
- If v and v' lie on different sides, then any such path must be of length at least δN since the shortest connection between sides is on the order of the box size.

Define

$$E_{d_0} := \left\{ \begin{array}{c} \text{No } p_c\text{-occupied path connects } \partial^{\text{out}}\mathcal{C}^{(\alpha)}(v; 1, \delta N) \\ \text{to any } v' \in \partial B(N) \text{ with } \|v - v'\| \geq d_0 + 1 \end{array} \right\}.$$

Using Lemma 3.2 and a union bound, we obtain

$$\mathbb{P}_N^B(E_{d_0}^c) \leq 2 \sum_{k=d_0+1}^{2N} c(k \cos \alpha)^{-\frac{13}{12}+\eta} + 6N \cdot c(\delta N)^{-\frac{13}{12}+\eta} \leq \frac{C}{(d_0)^{\frac{1}{12}-\eta}},$$

for some constant $C = C(\alpha, \eta) > 0$ and all sufficiently large N.

We now choose $\eta = \frac{1}{24}$ and then fix d_0 large enough so that the right-hand side of the

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previous inequality is at most $\frac{1}{2}$. Then $\mathbb{P}_N^B(E_{d_0}) \geq \frac{1}{2}$. Since the events

$$E_{d_0}, \quad \{\tau_v \in (p_c, \tfrac{3}{4})\}, \quad \text{and} \quad A_1(\mathcal{C}^{(\alpha)}(v; 1, \delta N))$$

are mutually independent, we have

$$\begin{aligned}
\mathbb{P}_N^B\left(v \in \mathrm{Cone}_N(\alpha, \delta N)\right) &\geq \mathbb{P}_N^B\left(E_{d_0} \cap \{\tau_v \in (p_c, \tfrac{3}{4})\} \cap A_1(\mathcal{C}^{(\alpha)}(v; 1, \delta N))\right) \\
&= \mathbb{P}_N^B(E_{d_0}) \cdot \mathbb{P}_N^B\left(\tau_v \in (p_c, \tfrac{3}{4})\right) \cdot \pi_1^{(\alpha)}(1, \delta N) \\
&\geq \tfrac{1}{2} \cdot \left(\tfrac{1}{4}\right) \cdot \pi_1^{(\alpha)}(1, \delta N) = \left(\tfrac{1}{4}\right)^{2d_0+1} \pi_1^{(\alpha)}(\delta N),
\end{aligned}$$

as claimed.

For part (ii), let $d = \|v - v'\|$. If $d \geq 2\delta N \sin\alpha$, then the cones $\mathcal{C}^{(\alpha)}(v; 1, \delta N)$ and $\mathcal{C}^{(\alpha)}(v'; 1, \delta N)$ are disjoint, and hence the corresponding 1-arm events are independent. Therefore,

$$\begin{aligned}
\mathbb{P}_N^B\left(v, v' \in \mathrm{Cone}_N(\alpha, \delta N)\right) &\leq \mathbb{P}_N^B\left(v \in \mathrm{Cone}_N(\alpha, \delta N)\right) \times \mathbb{P}_N^B\left(v' \in \mathrm{Cone}_N(\alpha, \delta N)\right) \\
&\leq \left(\pi_1^{(\alpha)}(1, \delta N)\right)^2 \\
&\leq 4\left(\pi_1^{(\alpha)}(\delta N)\right)^2.
\end{aligned}$$

If instead $d < 2\delta N \sin\alpha$, then the annuli

$$\mathcal{C}^{(\alpha)}\left(v; 1, \tfrac{d}{4}\right), \quad \mathcal{C}^{(\alpha)}\left(v'; 1, \tfrac{d}{4}\right), \quad \mathcal{C}^{(\alpha)}\left(v; \tfrac{d}{2}, \delta N\right)$$

are disjoint, and the corresponding 1-arm events occur simultaneously at time p_c. By the

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quasi-multiplicativity property,

$$\begin{aligned}\mathbb{P}_N^B\left(v, v' \in \mathrm{Cone}_N(\alpha, \delta N)\right) &\leq \mathbb{P}_N^B\left(A_1^{(\alpha)}\left(v; 1, \tfrac{d}{4}\right)\right) \cdot \mathbb{P}_N^B\left(A_1^{(\alpha)}\left(v'; 1, \tfrac{d}{4}\right)\right) \\ &\quad \cdot \mathbb{P}_N^B\left(A_1^{(\alpha)}\left(v; \tfrac{d}{2}, \delta N\right)\right) \\ &\leq \left(\pi_1^{(\alpha)}\left(1, \tfrac{d}{4}\right)\right)^2 \cdot \pi_1^{(\alpha)}\left(\tfrac{d}{2}, \delta N\right) \\ &\leq c_2' \pi_1^{(\alpha)}(\delta N) \cdot \pi_1^{(\alpha)}(d),\end{aligned}$$

for some constant $c_2' = c_2'(\alpha) > 0$. Taking $c_2 = \max\{4, c_2'(\alpha)\}$ completes the proof.

$\square$

The following estimate will be useful for the proof of Lemma 3.6:

Lemma 3.5. *For any $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and any $n \geq 1$, there exists a constant $C = C(\alpha) > 0$ such that*

$$\sum_{k=1}^{n} \left(\pi_1^{(\alpha)}(k, n)\right)^{-1} \leq Cn. \tag{3.10}$$

Proof. Fix $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. By equation (3.3), we have

$$\kappa_\alpha = \frac{\pi}{6\alpha} < 1.$$

Choose $\eta = \eta(\alpha) > 0$ such that $\kappa_\alpha + 2\eta = 1$. Then, by equation (3.4), there exists a constant $c_1 = c_1(\alpha) > 0$ such that for all $1 \leq k \leq n$,

$$\pi_1^{(\alpha)}(k, n) \geq c_1 \left(\frac{k}{n}\right)^{\kappa_\alpha + \eta} = c_1 \left(\frac{k}{n}\right)^{1-\eta}.$$

Taking reciprocals and summing, we obtain

$$\sum_{k=1}^{n} \left(\pi_1^{(\alpha)}(k, n)\right)^{-1} \leq \frac{1}{c_1} n^{1-\eta} \sum_{k=1}^{n} k^{-(1-\eta)} \leq \frac{1}{c_1} n^{1-\eta} \cdot c_2 n^{\eta} = \frac{c_2}{c_1} n,$$

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for some constant $c_2 = c_2(\alpha) > 0$, where the second inequality follows from a Riemann sum approximation. This completes the proof. $\square$

Lemma 3.6. *Fix $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and $\delta \in \left(0, \frac{1}{10}\right)$. For each $N \geq 1$, let $V_N = V_N^{\alpha,\delta}$ denote the number of $(\alpha, \delta N)$-cone sites. Then for any $\eta > 0$, there exist positive constants $c_1, c_2, c_3 > 0$ depending on α and η such that for all sufficiently large N,*

$$\mathbb{P}_N^B \left(V_N \geq c_1 N \pi_1^{(\alpha)}(\delta N) \right) \geq 1 - c_2 N \cdot (\delta N)^{-\frac{13}{12}+\eta} - c_3 \delta.$$

Proof. By Lemma 3.4, there exists a constant $c = c(\alpha) > 0$ such that

$$\mathbb{E}_N^B V_N \geq c N \pi_1^{(\alpha)}(\delta N). \tag{3.11}$$

We define a localized variant $\widetilde{I}_N(v)$ of the event $I_N(v) := \{v \in \mathrm{Cone}_N(\alpha, \delta N)\}$, which depends on the parameters α and δ, by restricting attention to paths contained within the region

$$R_{N,\delta}(v) := B(v; \delta N) \cap B(N).$$

In particular, we replace the condition $\mathcal{F}_{p_c} \cap \partial^{\mathrm{out}} \mathcal{C}^{(\alpha)}(v; 1, \delta N) = \emptyset$ in Definition 3.3 with the localized condition $\mathcal{F}'_{p_c}(v) \cap \partial^{\mathrm{out}} \mathcal{C}^{(\alpha)}(v; 1, \delta N) = \emptyset$, where $\mathcal{F}'_{p_c}(v)$ denotes the set of vertices $u \in B(N)$ for which there exists a p_c-occupied path from u to $\partial B(N)$ entirely contained within $R_{N,\delta}(v)$. (See Figure 3.5.)

Let $\widetilde{V}_N$ denote the number of vertices v for which the localized event $\widetilde{I}_N(v)$ occurs. Since $I_N(v) \subseteq \widetilde{I}_N(v)$, it follows that

$$\widetilde{V}_N \geq V_N.$$

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Consequently, if $\widetilde{I}_N(v)$ occurs but $I_N(v)$ does not, then there exists no p_c-occupied path γ within $R_{N,\delta}(v)$ connecting some boundary vertex $v' \in \partial B(N)$ to $\partial^{\text{out}}\mathcal{C}^{(\alpha)}(v;1,\delta N)$.

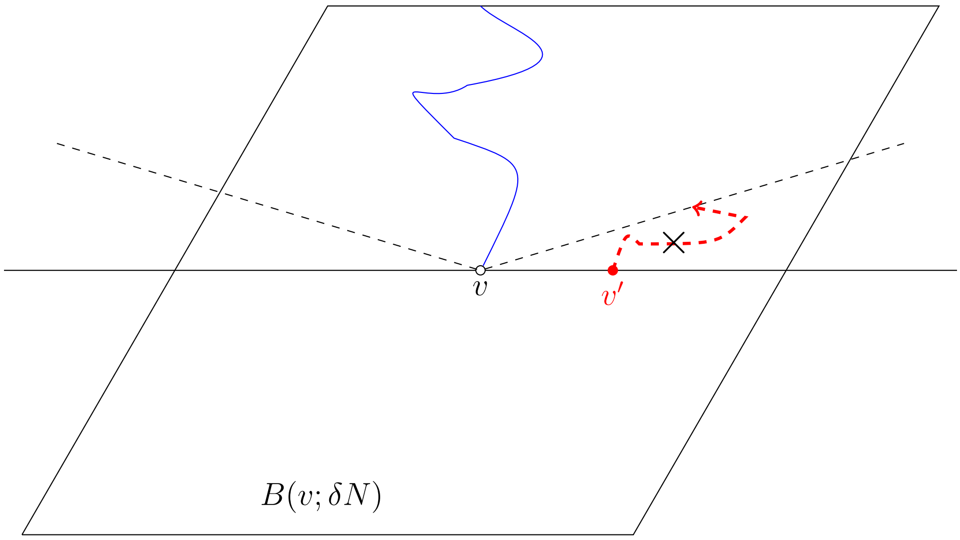

Figure 3.5: The event $\widetilde{I}_N(v)$ occurs if no vertex $v' \in \partial B(N) \cap B(v;\delta N)$ is connected to $\mathcal{C}^{(\alpha)}(v;1,\delta N)$ by a p_c-occupied path that is entirely contained within the region $R_{N,\delta}(v)$.

We distinguish between two cases. If $\|v - v'\| \geq \frac{1}{2}\delta N$, then γ must have length at least $\frac{\cos\alpha}{2}\delta N$ to intersect both v' and $\partial_{\text{out}}\mathcal{C}^{(\alpha)}(v;1,\delta N)$. If instead $\|v - v'\| \leq \frac{1}{2}\delta N$, then γ must exit $R_{N,\delta}(v)$, implying that it connects v' to a vertex at distance at least $\frac{\sqrt{3}}{4}\delta N$. Hence, by applying Lemma 3.2, there exist constants $c_2', c_2 > 0$, depending only on α and η, such that

$$\mathbb{P}_N^B(V_N \neq \tilde{V}_N) \leq c_2' N \cdot \left(\frac{\delta N \cos\alpha}{2}\right)^{-\frac{13}{12}+\eta} \leq c_2 N \cdot (\delta N)^{-\frac{13}{12}+\eta}.$$

We now estimate the variance of $\tilde{V}_N$. For all $v, v' \in \mathcal{L}_N$ satisfying $\|v - v'\| \geq 2\delta N$, the associated events $\tilde{I}_N(v)$ and $\tilde{I}_N(v')$ are independent. This yields the following upper

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bound:

$$
\begin{aligned}
\operatorname{Var}(\tilde{V}_N) &= \sum_{v\in\mathcal{L}_N}\sum_{v'\in\mathcal{L}_N} \operatorname{Cov}\big(\tilde{I}_N(v), \tilde{I}_N(v')\big) \\
&\le \sum_{v\in\mathcal{L}_N}\sum_{\|v'-v\|\le 2\delta N} \mathbb{P}_N^B\left(\tilde{I}_N(v)\cap\tilde{I}_N(v')\right) \\
&\le \sum_{v\in\mathcal{L}_N} 2\sum_{d=0}^{2\delta N} c^{(1)}\pi_1^{(\alpha)}(\delta N)\cdot\pi_1^{(\alpha)}(d\wedge\delta N) \quad \text{(by Lemma 3.4)} \\
&= \sum_{v\in\mathcal{L}_N} 2\left[\sum_{d=0}^{\delta N} c^{(1)}\pi_1^{(\alpha)}(\delta N)\cdot\pi_1^{(\alpha)}(d) + \sum_{d=\delta N+1}^{2\delta N} c^{(1)}\pi_1^{(\alpha)}(\delta N)^2\right] \\
&\le \sum_{v\in\mathcal{L}_N} 2\left[\sum_{d=0}^{\delta N} c^{(2)}\pi_1^{(\alpha)}(\delta N)^2\cdot\left(\pi_1^{(\alpha)}(d,\delta N)\right)^{-1} + c^{(1)}\delta N\pi_1^{(\alpha)}(\delta N)^2\right] \\
&\le 8N\cdot 2\left[c^{(2)}\pi_1^{(\alpha)}(\delta N)^2\cdot\sum_{d=0}^{\delta N}\left(\pi_1^{(\alpha)}(d,\delta N)\right)^{-1} + c^{(1)}\delta N\pi_1^{(\alpha)}(\delta N)^2\right] \\
&\le c^{(4)}\delta\left(N\pi_1^{(\alpha)}(\delta N)\right)^2,
\end{aligned}
$$

where we applied the quasi-multiplicativity property (2.4) in the fifth line. All constants $c^{(i)} = c^{(i)}(\alpha) > 0$, for $i = 1,\dots,4$, depend only on α. Moreover, since the proof of Lemma 3.4 does not rely on condition 2 of Definition 3.3, the same argument applies to the events $\tilde{I}_N(\cdot)$ without modification.

Applying Chebyshev's inequality, together with the first-moment estimate (3.11) and the fact that $\tilde{V}_N \ge V_N$, we conclude that

$$
\mathbb{P}_N^B\left(\tilde{V}_N \ge \frac{c}{2}N\pi_1^{(\alpha)}(\delta N)\right) \ge 1 - \frac{4\operatorname{Var}(\tilde{V}_N)}{\left(cN\pi_1^{(\alpha)}(\delta N)\right)^2} \ge 1 - c_3\delta,
$$

for some constant $c_3 = c_3(\alpha) > 0$. Setting $c_1 := \frac{c}{2}$, we deduce

$$
\mathbb{P}_N^B\left(V_N \ge c_1 N\pi_1^{(\alpha)}(\delta N)\right) \ge 1 - c_2 N(\delta N)^{-\frac{13}{12}+\eta} - c_3\delta,
$$

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which completes the proof. $\square$

Lemma 3.7. *For all* $\delta \in \left(0, \frac{1}{13}\right)$, *we have:*

$$\text{for all } \beta > \frac{3}{4}(1-\delta), \quad \mathbb{P}_N^B\left(\mathcal{F}_{p_c+N^{-\beta}} \cap B(N - N^{1-\delta}) \neq \emptyset\right) \xrightarrow{N\to\infty} 0. \tag{3.12}$$

Proof. Fix $\delta \in \left(0, \frac{1}{13}\right)$. Following the same reasoning as in the proof of Lemma 3.2, we obtain a bound analogous to (3.5). For each $\eta > 0$, the probability that there exists a $(p_c + N^{-\beta})$-occupied path beginning at some vertex $v \in \partial B(N)$ and extending to a distance of at least $N^{1-\delta}$ is at most

$$\left(N^{1-\delta}\right)^{-\frac{3}{4}+\frac{\eta}{2}} \cdot \left(N^{1-\delta}\right)^{-\frac{1}{3}+\frac{\eta}{2}},$$

for all sufficiently large N. Since $\beta > \frac{3}{4}(1-\delta)$, we have

$$L(p_c + N^{-\beta}) = N^{\frac{4}{3}\beta + o(1)} \gg N^{1-\delta}.$$

Applying a union bound yields

$$\mathbb{P}_N^B\left(\mathcal{F}_{p_c+N^{-\beta}} \cap B(N - N^{1-\delta}) \neq \emptyset\right) \leq cN \cdot \left(N^{1-\delta}\right)^{-\frac{13}{12}+\eta}. \tag{3.13}$$

Since $\delta < \frac{1}{13}$, we may choose $\eta > 0$ sufficiently small so that the bound in (3.13) tends to zero as $N \to \infty$, completing the proof. $\square$

3.2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

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We aim to show that for any $\varepsilon > 0$,

$$\mathbb{P}_N^B\left(\text{the origin is frozen at time } t = 1\right) \leq \varepsilon$$

for all sufficiently large N. Fix $\delta \in \left(0, \frac{1}{13}\right)$ and $\beta \in \left(\frac{3}{4}(1-\delta), \frac{3}{4}\right)$. Set $\underline{p} := p_c + N^{-\beta}$, and define

$$E_1 := \left\{\mathcal{F}_{\underline{p}} \cap B(N - N^{1-\delta}) = \emptyset\right\}.$$

By Lemma 3.7, there exists $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$,

$$\mathbb{P}_N^B(E_1) \geq 1 - \frac{\varepsilon}{5}. \tag{3.14}$$

Let $\beta' \in \left(\beta, \frac{3}{4}\right)$. Consider the event

$$E_2 := \left\{\text{at time } \underline{p},\ \mathcal{O}\left(O; N - N^{4\beta'/3},\ N - N^{1-\delta}\right) \text{ occurs}\right\}.$$

Using (2.9), we have $L(\underline{p}) = N^{\frac{4}{3}\beta + o(1)} \ll N^{\frac{4}{3}\beta'}$ as $N \to \infty$, so by (2.2), there exists $N_2 \in \mathbb{N}$ such that for all $N \geq N_2$,

$$\mathbb{P}_N^B(E_2) \geq 1 - \frac{\varepsilon}{5}. \tag{3.15}$$

We now assume that both events E_1 and E_2 occur, and we denote by $\mathcal{O}$ an occupied circuit as specified in the definition of E_2.

We set $\alpha = \frac{\pi}{2}/(1+\delta)$, so by (3.3), we have $\alpha_1^{(\alpha)} = \frac{1}{3}(1+\delta)$. Define

$$E_3 := \left\{\text{the number of } (\alpha, \delta N)\text{-cone sites is at least } N^{\frac{2}{3}-\delta}\right\}.$$

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Applying Lemma 3.6 with the specific choice $\eta = \frac{1}{24}$, there exists $N_3 \in \mathbb{N}$ such that for all $N \geq N_3$,

$$\mathbb{P}_N^B(E_3) \geq 1 - \frac{\varepsilon}{5}. \tag{3.16}$$

Assuming that the event $E_1 \cap E_2 \cap E_3$ occurs, we make the following observations:

(a) Since $\delta N \cos\alpha \geq N^{\frac{4\beta'}{3}}$ for all sufficiently large N, each $(\alpha, \delta N)$-cone site v has a neighbor $w \in B(N) \setminus \partial B(N)$ such that there exists a p_c-occupied path from w to the circuit $\mathcal{O}$.

(b) All such $(\alpha, \delta N)$-cone sites are $\underline{p}$-vacant. Indeed, suppose this were not the case: then there would exist a cone site v with $\tau_v \in (p_c, \underline{p})$. This would imply the existence of a $\underline{p}$-occupied path from v to $\partial B(N - N^{1-\delta})$, contradicting the event E_1.

(c) Let $\overline{p} := p_c + N^{-\frac{2}{3}+2\delta} > \underline{p}$, and define the event

$$E_4 := \{\text{there exists at least one } (\alpha, \delta N)\text{-cone site with } \tau\text{-value in } (p_c, \overline{p})\}.$$

Since there are at least $N^{\frac{2}{3}-\delta}$ cone sites, there exists $N_4 \in \mathbb{N}$ such that for all $N \geq N_4$,

$$\mathbb{P}_N^B\left(E_4 \mid E_1 \cap E_2 \cap E_3\right) \geq 1 - \frac{\varepsilon}{5}. \tag{3.17}$$

If E_4 occurs and v is a cone site satisfying the condition, then by observation (a), either:

- v becomes occupied at time τ_v and causes the circuit $\mathcal{O}$ to freeze, or
- before time τ_v, the occupied cluster containing w and $\mathcal{O}$ has already frozen, so that v remains vacant at time 1.

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In either case, the circuit $\mathcal{O}$ is frozen.

Finally, define

$$E_5 := \left\{ \text{at time } \overline{p},\ \mathcal{O}^* \left(O; 0, \tfrac{N}{2}\right) \text{ occurs} \right\}.$$

Since

$$L(\overline{p}) = N^{\frac{8}{9} - \frac{8}{3}\delta + o(1)} \ll N \quad \text{as } N \to \infty,$$

there exists $N_5 \in \mathbb{N}$ such that for all $N \geq N_5$,

$$\begin{aligned} \mathbb{P}_N^B(E_5) &= 1 - \mathbb{P}_N^B \left(\left\{ \text{at time } \overline{p},\ A_{1,o}\left(0, \tfrac{N}{2}\right) \text{ occurs} \right\} \right) \\ &\geq 1 - \mathbb{P}_N^B \left(\text{at time } \overline{p},\ A_{1,o}\left(0, L(\overline{p})\right) \text{ occurs} \right) \\ &\geq 1 - 4c_1 e^{-c_2 N/(2L(\overline{p}))} \quad \text{(by (2.1))} \\ &\geq 1 - \frac{\varepsilon}{5}. \end{aligned} \tag{3.18}$$

If $\bigcap_{i=1}^{5} E_i$ occurs, then the origin is not frozen at time $t = 1$. Hence, for all

$$N \geq N_0 := \max_{1 \leq i \leq 5} N_i,$$

we have

$$\mathbb{P}_N^B \left(\text{the origin is not frozen at time } t = 1 \right) \geq 1 - \varepsilon,$$

which completes the proof.

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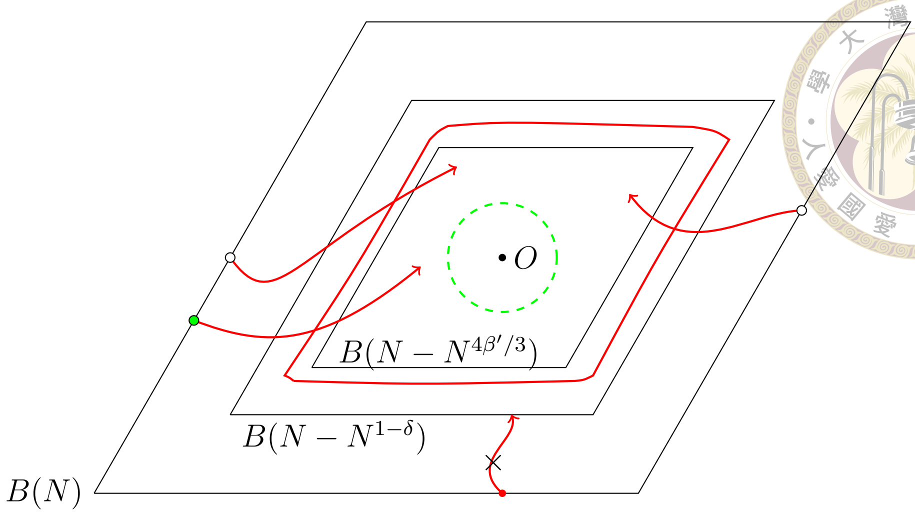

Figure 3.6: Illustration of the event $\bigcap_{i=1}^{5} E_i$ occurring. The color red indicates time $\underline{p}$, and green indicates $\overline{p}$. Solid lines represent occupied paths, while dashed lines represent vacant paths.

3.3 Proof of Theorem 1.2

In this section, we highlight the modifications to Theorem 1.1 needed to prove Theorem 1.2. The remaining arguments follow directly from the proof of Theorem 1.1.

It suffices to show that for any $\varepsilon > 0$,

$$\mathbb{P}_N^C(\text{the origin is frozen at time } t = 1) \leq \varepsilon$$

for all sufficiently large N. We consider the same events $E_i' = E_i$, $i = 1, 2, 5$, as before, and note that for $i = 1, 2, 5$,

$$\mathbb{P}_N^C(E_i') \geq 1 - \frac{\varepsilon}{5} \tag{3.19}$$

for all sufficiently large N. We also denote by $\mathcal{O}'$ an occupied circuit in E_2'.

Next, let $V_N^1, V_N^2, V_N^3, V_N^4$ denote the number of $(\alpha, \delta N)$-cone sites along the four sides of $\partial B(N)$. By symmetry, the variables V_N^i, $i = 1, 2, 3, 4$, are identically distributed.

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Using the same argument as in the proof of Lemma 3.6, we can apply it to the modified quantity V_N^1. Then for each $\eta > 0$, there exist constants $c_1, c_2, c_3 > 0$, depending only on α and η, such that for all sufficiently large N,

$$\mathbb{P}_N^C\left(V_N^1 \geq c_1 N \pi_1^{(\alpha)}(\delta N)\right) \geq 1 - c_2 N (\delta N)^{-\frac{13}{12}+\eta} - c_3 \delta.$$

We take the same values $\alpha = \frac{\pi}{2}/(1+\delta)$ and $\eta = \frac{1}{24}$, and define

$$E_3' := \left\{\text{on each side of } \partial B(N),\ \text{the number of } (\alpha, \delta N)\text{-cone sites} \geq N^{\frac{2}{3}-\delta}\right\}.$$

We obtain that

$$\mathbb{P}_N^C(E_3') \geq 1 - \frac{\varepsilon}{5} \tag{3.20}$$

for all sufficiently large N.

Similar to observations (a) and (b) in Section 3.2, the same conclusions hold for crossing frozen percolation. Therefore, we define

$$E_4' := \{\text{on each side of } \partial B(N),\ \text{at least one of these cone sites is } \overline{p}\text{-occupied}\}.$$

Assuming that $E_1' \cap E_2' \cap E_3'$ occurs, we obtain the analogue of (3.17):

$$\mathbb{P}_N^C(E_4' \mid E_1' \cap E_2' \cap E_3') \geq 1 - \frac{\varepsilon}{5} \tag{3.21}$$

for all sufficiently large N. As in observation (c) of Section 3.2, this implies that $\mathcal{O}'$ is frozen.

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Finally, if $\bigcap_{i=1}^{5} E_i'$ occurs, then the origin is not frozen at time $t = 1$. Hence,

$$\mathbb{P}_N^C \left(\text{the origin is not frozen at time } t = 1\right) \geq 1 - \varepsilon$$

for all sufficiently large N, which completes the proof.

Remark 3.8. Crossing boundary frozen percolation can be regarded as a special case of a more general freezing rule, in which an occupied cluster freezes upon containing two boundary vertices. Using the same arguments, similar results hold for any $k \in \mathbb{N}$, under the condition that a cluster freezes once it contains k boundary vertices.

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