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非緊緻黎曼流形上的幾何流

Geometric flows on complete noncompact Riemannian manifolds

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摘要

本論文探討非緊流形上的幾何流的存在性與唯一性，主要關注 Ricci 流、曲線縮短流（curve shortening flow）和 Yamabe 流。

在第 II 章中，我們研究具有對稱性的 *Ricci* 流。我們證明在曲率衰減條件 $|\mathrm{Rm}(g(t))| \leqslant c/t$ 並附加某些條件下，Killing 向量場在 Ricci 流中得以保持（定理 1, 2），其中唯一性在這個假設下仍然是個未解問題（近期已被解決 [Lee25]）。此外，我們給出了在初使條件為旋轉對稱的完備度量且其 warped 函數單調遞增，Ricci 流的短時間存在性（定理 3）。值得注意的是，此存在性定理不須施加任何曲率條件。

在第 III 章中，我們證明平面 $\mathbb{R}^2$ 上圖形曲線縮短流的唯一性，其中初始條件條件落在 $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}\backslash K)$，其中 $K \subset \mathbb{R}$ 為一有界子集，並且在無窮遠處快速衰減（定理 7）。雖然針對一般初始條件的存在性已經有相關研究，但目前唯一性僅在 $\mathcal{L}^{p>1}_{\mathrm{loc}}(\mathbb{R})$ 初始條件下獲得證明。本研究在 $p = 1$ 的某些情況下證實了唯一性。

在第 IV 章中，我們證明具有非負 *Ricci* 曲率且局部共形平坦的完備流形的間隙定理（定理 8）。我們通過研究在**不假設初始度量為的曲率有界**的情況下，Yamabe 流的長時間存在性（定理 13），以獲得該間隙結果。

關鍵字：非緊緻流形、Ricci 流、曲線縮短流、Yamabe 流、間隙定理

Abstract

This thesis explores the *existence and uniqueness of geometric flows on noncompact manifolds*, focusing on Ricci flow, curve shortening flow, and Yamabe flow.

In Chapter II, we study the property of *Ricci flow with symmetry*. We confirm Killing field preservation under curvature decay $|\mathrm{Rm}(g(t))| \leqslant c/t$ with additional conditions (Theorems 1, 2), where the uniqueness problem of Ricci flow remains open in this scenario (Recently it has been solved by Man-Chun Lee [Lee25]). We also establish the short-time existence for Ricci flow from a complete, rotationally symmetric metric with a non-decreasing warped function, without imposing curvature conditions (Theorem 3).

In Chapter III, we prove the *uniqueness of graphical curve shortening flow on* $\mathbb{R}^2$ for $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}\backslash K)$ initial data with fast decay at infinity (Theorem 7). While the existence results for the rough initial data are known, uniqueness has been established only for $\mathcal{L}^{p>1}_{\mathrm{loc}}(\mathbb{R})$ data. Our work provides some affirmative evidence when $p = 1$.

In Chapter IV, we prove a *gap theorem for complete locally conformally flat mani-*

folds with nonnegative Ricci curvature (Theorem 8) by investigating the long-time solution of Yamabe flow under nonnegative Ricci curvature *without assuming bounded curvature on the initial metric* (Theorem 13).

Keywords: open manifold, Ricci flow, curve shortening flow, Yamabe flow, gap theorem

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Chapter I Introduction

Geometric flows are a fundamental theme in geometric analysis. Intuitively, they are heat-type equations that deform geometric structures, often increasing their symmetry. This concept has led to significant breakthroughs in geometry.

For instance, in the intrinsic geometry, *Ricci flow* has been a powerful tool in revisiting the uniformization theorem [Ham88], establishing the Poincaré conjecture [Per02, Per03a, Per03b], and proving the differentiable sphere theorem [BS09], etc. Another important intrinsic geometric flow is the *Yamabe flow*, which evolves a metric in the direction of the gradient of the normalized $\mathcal{L}^2$-norm of scalar curvature. The Yamabe flow is expected to deform a given metric into one that is conformal to the original and has constant scalar curvature, thereby resolving the Yamabe problem [Bre05, Bre07]. In extrinsic geometry, the *mean curvature flow* and *curve shortening flow* (the one-dimensional mean curvature flow) play a significant role in understanding the structure of submanifolds, particularly in studying the geometry of convex hypersurfaces [HS09, Has20]. One notable geometric application is the proof of the famous *three geodesics theorem*, which was established using the curve shortening flow [Gra89].

A natural question that arises in the study of geometric flows is: **Why does the flow exist?** Namely, given an initial metric (or submanifold), why should we expect the existence of a solution to these geometric flows, given an initial metric or submanifold? Furthermore, **why is this solution unique?** In the case of a closed manifold (or closed submanifold), the well-posedness follows from the well-posedness of uniform parabolic equations. It is important to note that even though the Ricci flow is not initially a *strongly parabolic equation*, by applying the *De-Turck trick* [DeT83], the equation becomes equivalent to a strongly parabolic equation. However, things become more challenging and complex in the non-compact setting. To date, both existence and uniqueness are unclear in the complete non-compact setting. A fundamental and remarkable short-time existence of Ricci flow [Shi89] was proven by W.-X. Shi assuming complete and **bounded curvature assumption**, and a similar result holds for Yamabe flow [CZ02]. So far, the exploration of existence without assuming bounded curvature has been incomplete, some partial progress appears under the condition on curvature lower bound [Lai19, CRW15, CHL24]. Same for the uniqueness problem, there are few results on the uniqueness of Geometric flow in the complete non-compact setting [CZ06, Lee19, Kot14, LM21, Lee25].

This thesis mainly discusses the **existence and uniqueness of geometric flow on noncompact manifold**, including the Ricci flow, curve shortening flow, and Yamabe flow. We divide it into three chapters according to three different geometric flows.

In Chapter II, we focus on two matters in Ricci flow: *Preservation of Killing vector field along Ricci flow* and *Existence of Ricci flow from the rotationally symmetric metric on* $\mathbb{R}^{n+1}$. The first is a direct consequence of the uniqueness of Ricci flow, and related to the uniqueness under some symmetric assumption [LT05]. We confirm this property under the curvature decayed $|\mathrm{Rm}(g(t))| \leqslant c/t$ on $\mathcal{M}^n \times (0,T]$ with some additional conditions (Theorem 1, 2), where uniqueness in this setting remains open until very recently [Lee25]. The second is a short-time existence of Ricci flow starting from a **complete and rotation-**

ally symmetric metric on $\mathbb{R}^{n+1}$ with non-decreasing warped function (Theorem 3). More precisely, a rotationally symmetric metric g on $\mathbb{R}^{n+1}$ means outside the origin, g can be viewed as a warped of $\mathbb{R}^+ \times_f S^n$ for some smooth function $f : \mathbb{R}^+ \to \mathbb{R}^+$. Our existing result only requires f to be non-decreasing and smooth (in the sense of metric) at the origin, which doesn't impose any curvature condition on g.

In Chapter III, we prove the uniqueness of *graphical curve shortening flow on* $\mathbb{R}^2$ for $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{C}^0_{\text{loc}}(\mathbb{R} \backslash K)$-initial data assuming fast decayed near infinity (Theorem 7). Notably, the existence of the graphical curve shortening flow had been developed for rough initial data [ST24, Sob25]. Unlike the existence, the full picture of uniqueness is only known for $\mathcal{L}^{p>1}_{\text{loc}}(\mathbb{R})$-initial data [Sob24, CK20], and our result provides an affirmative evidence on the uniqueness for $\mathcal{L}^1_{\text{loc}}(\mathbb{R})$-initial data.

In Chapter IV, we prove a gap theorem for complete and *locally conformally flat manifold* with nonnegative Ricci curvature (Theorem 8). Inspired by some previous works [CZ02, Ma16], we intend to investigate a long-time Yamabe flow starting from the given metric that preserves nonnegative Ricci condition and has bounded curvature on the time interval $[a, \infty) \subset (0, \infty)$ for all $a > 0$. Here, the difficulty is to establish the long-time solution of Yamabe flow *without assuming bounded curvature* (Theorem 13). To overcome this problem, we adopt a similar idea in Ricci flow [CL23], using heat kernel estimates and a *small average scalar curvature condition* to derive a priori curvature estimates (Lemma 20).

Chapter II Rotationally symmetry and Ricci flow

II.1 Introduction

Given a smooth manifold without boundary $\mathcal{M}^n$, a *Ricci flow* is a family of smooth Riemannian metrics $(g(t))_{t\in I}$ satisfying

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(g(t)), \tag{II.1}$$

on $\mathcal{M}^n \times I$, where $I \subset \mathbb{R}$ is some interval. This concept was first introduced by R. Hamilton [Ham82] in 1983 and he proved that a closed 3-manifold with positive Ricci curvature is diffeomorphic to a spherical space form by evolving the Ricci flow from the given metric. Following Hamilton's work, geometers developed many deep connections between geometry and topology using the Ricci flow. The most famous and celebrated work is the Poincaré conjecture, which G. Perelman proved in 2002 through his groundbreaking works on the Ricci flow [Per02, Per03b, Per03a].

On the closed manifold, the existence and uniqueness of a Ricci flow solution with a given initial metric were first established by R. Hamilton [Ham82] using the Nash-Moser iteration method. Later, D. Deturck [DeT83] provided another approach using the "fixed

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gauge method," which simplified the original Hamilton's proof. As a result, these fundamental properties are well understood in the compact setting. However, those properties only progress partially in the non-compact setup.

The first existence result in the complete non-compact setting was established by W.-X. Shi [Shi89] in 1989, under an additional assumption that the initial metric has bounded curvature assumption. In recent years, some recent results have removed the bounded curvature assumption; instead, they assumed some curvature lower bounds, for instance, [Lai19, LT24], etc. A remarkable achievement in 2D is the construction of *instantaneously complete Ricci flow* starting from rough initial data. This was first established by G. Giesen and P. Topping [GT11] and later extended by P. Topping and H. Yin [TY24].

For the uniqueness property of Ricci flow, the first result in the non-compact setting is due to B.-L. Chen and X.-P. Zhu [CZ06, Theorem 1.1.] in 2006, with the extra assumption that the curvature is bounded on the whole spacetime. Their main idea is to build up the Harmonic heat flow coupling with Ricci flow and use this to conclude the uniqueness, which is not that straightforward as well. In the two-dimensional case, P. Topping [Top15, Theorem 1.1.] confirmed the uniqueness problem without any further conditions. The proof heavily relied on the fact that the Ricci flow in 2D can be viewed as the logarithmic heat equation in isothermal coordinates, a property specific to 2D. In the past decade, several uniqueness results have been established under conditions on curvature decay, $|\mathrm{Rm}(g(t))|_{g(t)} \leqslant c/t$, along with either uniform metric condition $c^{-1}g_0 \leqslant g(t) \leqslant cg_0$ on $\mathcal{M}^n \times (0,T]$ for some $c > 1$, or a polynomial growth bound of $g(0)$ on $\mathcal{M}^n$, see [LM21, Theorem 1.3.], [Lee19, Theorem 1.2.] and [Kot14]. These results are inspired by [Kot14], which introduced a new energy approach analogous to the classical proof of the uniqueness of the heat equation of exponential to quadratic growth. In March 2025, M.-C.

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Lee resolved the uniqueness problem of Ricci flow under the scaling invariant estimate using the local harmonic map heat flow method [Lee25].

Our first result concerns *whether the property of being a Killing vector field is preserved under Ricci flow*. This follows directly from the uniqueness of Ricci flow. Indeed, given an isometry $F : (\mathcal{M}^n, g_0) \to (\mathcal{M}^n, g_0)$ and a solution of Ricci flow $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ with $g(0) = g_0$, $(\mathcal{M}^n, F^*g(t))_{t\in[0,T]}$ is also a solution of Ricci flow with $F^*g(0) = g_0$. Therefore, F remains an isometry on $(\mathcal{M}^n, g(t))$ for all $t \in [0,T]$. Since the flow of a Killing vector field forms a one-parameter subgroup of isometries, it follows from the above discussion that the property of being a Killing vector field is preserved under Ricci flow, provided the solution of Ricci flow is unique.

Inspired by the work in [LT05, Section 1.2.], they used an alternative approach, which didn't require the uniqueness result of Ricci flow, to prove that, if the complete Ricci flow $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ has bounded curvature on the entire space-time $\mathcal{M}^n \times [0,T]$, then any bounded Killing vector field w.r.t. $g(0)$ remains a bounded Killing vector field w.r.t. $g(t)$ for all $t \in [0,T]$. The first result of this paper continues their idea and replaces the bounded curvature condition with a weaker condition

$$|\operatorname{Rm}(g(t))|_{g(t)} \leqslant \frac{c}{t}, \tag{II.2}$$

on $\mathcal{M}^n \times (0,T]$ for some $c > 0$. As we discussed before, the uniqueness of Ricci flow had been established by Lee in 2025, but our results were completed during my first year of master's degree, which is earlier than that paper, and we adopted a different method. Thereby, we still present the results in this thesis.

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Theorem 1. Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a complete Ricci flow with the interior estimate

$$|\mathrm{Rm}(g(t))| \leqslant \frac{c}{t}, \tag{II.3}$$

on $\mathcal{M}^n \times (0,T]$. Suppose that X_0 is a bounded Killing vector field on $(\mathcal{M}^n, g(0))$, then X_0 is also a bounded Killing vector field on $(\mathcal{M}^n, g(t))$ for all $t \in [0,T]$.

Suppose we allow the exponential growth of the Killing vector field X_0. In contrast to the bounded case, we prove the same result as in Theorem 1 but with additional assumptions, which ensure the upper bound of the backward heat kernel, referring to Proposition 2 for the complete statement.

Theorem 2. Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a complete Ricci flow with the following conditions:

$$\begin{cases} |\mathrm{Rm}(g(t))| \leqslant \frac{c}{t}, \text{ on } \mathcal{M}^n \times (0,T]; \\ \mathrm{inj}_{g(t)}(x) \geqslant \sqrt{\frac{t}{c}}, \text{ for all } (x,t) \in \mathcal{M}^n \times (0,T]; \\ \mathrm{scal}(x,0) \geqslant K, \text{ on } \mathcal{M}^n; \\ \mathrm{Vol}_{g(0)}(B_{g(0)}(x,r)) \leqslant v_0 r^n e^{Lr^2}, \text{ for all } x \in \mathcal{M}^n \text{ and } r \geqslant 0, \end{cases} \tag{II.4}$$

for some constants $c, L, v_0 > 0$ and $K \in \mathbb{R}$. Suppose that X_0 is a Killing vector field w.r.t. $g(0)$ and is with exponential growth, i.e., there is a point $p \in \mathcal{M}^n$ and constants $A, B \in \mathbb{R}, \varepsilon \in (0,1]$ such that

$$|X_0|^2_{g(0)}(x) \leqslant \exp(Ad_0(x,p)^{2(1-\varepsilon)} + B), \tag{II.5}$$

for all $x \in \mathcal{M}^n$. Then X_0 is also a Killing vector field w.r.t. $g(t)$ with exponential growth.

Remark 1. In fact, by the volume comparison theorem, assuming $\mathrm{Ric}(x,0) \geqslant K$ implies the conditions $\mathrm{scal}(x,0) \geqslant K$ and $\mathrm{Vol}_{g(0)}(B_{g(0)}(x,r)) \geqslant v_0 r^n e^{Lr}$ for all $x \in \mathcal{M}^n$ and $r > 0$.

At first glance, the assumption (II.4) does not seem natural to require. However, most of the results of existence guarantee these conditions. In practice, if $(\mathcal{M}^n, g)$ is a complete manifold such that

$$\begin{cases} \mathrm{Rm}(g)) + \alpha_0 \mathcal{I} \in \mathcal{C}_{\mathrm{PIC1}} \\ \mathrm{Vol}_g(B_g(p,1)) \geqslant v_0 > 0 \quad \text{for all } p \in \mathcal{M}^n, \end{cases} \tag{II.6}$$

for some constants $\alpha_0 \in (0,1]$ and $v_0 > 0$, then by [Lai19, Corollary 1.2. and Lemma 3.4.], there exists a Ricci flow $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ with $g(0) = g$ satisfying (II.4).

Our second result investigates the existence of rotationally symmetric Ricci flows starting from a rotationally symmetric metric with a non-decreasing warping function. Intuitively, the increasing warped function ensures no minimal hypersphere, preventing the neck singularity from happening under Ricci flow. According to this observation, we prove the following.

Theorem 3. Let $(\mathbb{R}^{n+1}, g := ds^2 + f(s)^2 g_{\mathrm{std}})$ be a complete and rotationally symmetric manifold. Suppose that $f_s(s) \geqslant 0$ on $\mathbb{R}^{n+1}$. Then there exist two constants $\hat{\Lambda}(n, g|_{\{|s|\leqslant 4\}}) > 0$ and $\hat{T}(n, g|_{\{|s|\leqslant 4\}}) > 0$ and a complete and rotationally symmetric Ricci flow $(\mathbb{R}^{n+1}, g(t))_{t\in[0,\hat{T}]}$ starting from g such that

$$|\mathrm{Rm}(g(t))| \leqslant \frac{\hat{\Lambda}}{t} \tag{II.7}$$

on $\mathbb{R}^{n+1} \times (0, \hat{T}]$.

It is worth noting that our result **does not require any curvature conditions**. Consequently, this theorem allows us to construct examples of Ricci flows in higher dimensions

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that originate from spaces with unbounded curvature on both sides.

Let's outline the structure of this chapter. In sections II.2 and II.3, we focus on the proof of Theorem 1 and 2, respectively. In section II.4, we discuss some elementary properties of the complete and rotationally symmetric metric on $\mathbb{R}^{n+1}$. In section II.5, we complete the proof of Theorem 3. Finally, in section II.6, we establish a uniform κ-noncollapsed property for the complete and rotationally symmetric Ricci flow on $\mathbb{R}^{n+1}$.

II.2 Proof of Theorem 1

Let's first explain our strategy. The first step is to construct a solution $X \in \mathcal{C}^\infty(\mathcal{M}^n \times [0, T])$ such that

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)} - \mathrm{Ric}_{g(t)}\right) X = 0 \quad \text{on } \mathcal{M}^n \times [0, T], \\ X(x, 0) = X_0(x) \quad \text{on } \mathcal{M}^n. \end{cases} \tag{II.8}$$

Note that if we consider a natural connection D on spacetime such that $D_{\partial_t} g(t) = 0$ on $\mathcal{M}^n \times [0, T]$, then the equation (II.8) is equivalent to a heat-type equation $D_{\partial_t} X = \Delta_{g(t)} X$. Therefore, equation (II.8) provides a natural way to flow a vector field along the Ricci flow. In section II.2.1, we will show that if $|X_0|$ is bounded on $(\mathcal{M}^n, g_0)$, then the solution of (II.8) exists.

After constructing the time-dependent vector field X, the second step is to consider a time-dependent Killing tensor $h_{ij} := \nabla_i X_j + \nabla_j X_i$. As in the computation in [LT05, Section 1.2.], the norm of h satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) |h|^2 = -2|\nabla h|^2 + 4 \sum_{i,j,k,l} \mathrm{Rm}_{ijkl}\, h_{il} h_{jk} \leqslant 4|\mathrm{Rm}||h|^2. \tag{II.9}$$

Note that $h(\cdot, 0) = 0$ follows from the fact that X_0 is a Killing vector field w.r.t. $g(0)$. Thus, if the maximal principle holds, we obtain $h \equiv 0$, that is, $\nabla_i X_j + \nabla_j X_i = 0$ for all i, j. After taking ∇_j and summing over j, we get $\Delta X^i + \mathrm{Ric}^i_k X^k = 0$. This shows that $X(\cdot, t) = X_0$ for all $t \in [0, T]$, confirming our assertion.

Finally, we need to explain how the maximal principle holds. To do so, we apply a local maximal principle [LT22, Corollary 3.2.]. It requires $|\nabla X|^2 \leqslant c/t$ on $\mathcal{M}^n \times (0, T]$ for some constant $c > 0$, we will prove it in Section II.2.2.

II.2.1 Construct a solution to (II.8) when X_0 is bounded

In this subsection, we prove the existence of the solution of (II.8) when the initial vector $|X_0|$ is bounded. The idea is to consider the local Dirichlet problem of (II.8) with respect to a given compact exhaustion $\Omega_1 \Subset \Omega_2 \Subset \cdots$ with $\bigcup_k \Omega_k = \mathcal{M}^n$, and adopt the Bernstien's trick to show that for any nonnegative integer $m \geqslant 0$ and $R \geqslant 1$, there is a constant $C(m, R)$ such that $|\nabla^m X_{\Omega_k}|(x, t) \leqslant C(m, R)$ for all $k \geqslant 1$ and $(x, t) \in \Omega_R \times [0, T]$. Finally, the Arzela-Ascoli theorem is applied to obtain a smooth subsequential limit of X.

Proposition 1. Let $(\mathcal{M}^n, g(t))_{t \in [0,T]}$ be a complete Ricci flow with a scaling invariant bound $|\mathrm{Rm}|(t) \leqslant c/t$ on $\mathcal{M}^n \times (0, T]$ for some $c > 0$. Then the solution of (II.8) exists if $|X_0|$ is bounded. Moreover, we have $|X| \leqslant \sup_{\mathcal{M}^n} |X_0|_{g(0)}$ on $\mathcal{M}^n \times [0, T]$.

Proof of Proposition 1. By assumption, write $|X_0| \leqslant K_0$ on $\mathcal{M}^n$. Fix $R \gg 1$ and $p \in \mathcal{M}^n$. Choose a smooth cut-off function $\phi_R : \mathcal{M}^n \to [0, 1]$ such that $\phi_R(x) = 1$ on $B_0(p, R)$,

$\phi(x) = 0$ on $B_0(p, 2R)^c$, and $|\nabla \phi_R| \leqslant 2/R$. Consider a Dirichlet problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)} - \mathrm{Ric}_t\right) X_R = 0 \quad \text{on } B_0(p, 2R) \times [0, T], \\ X_R(x, 0) = \phi_R(x) X_0(x) \quad \text{on } B_0(p, 2R), \\ X_R(x, t) = 0 \quad \text{on } \partial B_0(p, 2R) \times [0, T]. \end{cases} \tag{II.10}$$

The standard linear parabolic theory shows that X_R exists as long as $g(t)$ exists. Since all derivatives of curvature are bounded on $B_0(p, 2R) \times [0, T]$ (depends on R), the solution X_R exists and is smooth. Note that

$$(\partial_t - \Delta_{g(t)})|X_R|^2 = -2|\nabla X_R|^2 \leqslant 0.$$

By the standard maximal principle, we conclude that

$$|X_R|^2_{g(t)}(x, t) \leqslant \max_{\mathcal{M}^n} |X_0|^2_0 := k_0^2, \tag{II.11}$$

on $\mathcal{M}^n \times [0, T]$, and is worth noting that R.H.S. is independent of R.

For simplicity, for any two tensors A, B, the notation $A * B$ denotes a linear combination of some contraction of $A \otimes B$. Then we have the following well-known fact.

Lemma 1. [She06, Lemma 6.2.] If A is a tensor quantity that satisfies a heat-type evolution equation

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) A = F,$$

under the Ricci flow (where F is a tensor of the same type as A), then the following

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equations hold:

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|A|^2=-2|\nabla A|^2+F*A+\operatorname{Ric}*A*A, \tag{II.12}$$

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)\nabla A=\nabla F+\operatorname{Rm}*\nabla A+\nabla\operatorname{Ric}*A. \tag{II.13}$$

Applying Lemma 1, we can conclude that

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|\nabla^m X_R|^2=-2|\nabla^{m+1}X_R|^2+\sum_{k=0}^{m}\nabla^m X_R*\nabla^k X_R*\nabla^{m-k}\operatorname{Rm}.$$

We adopt the Bernstein's trick.

Claim 1. Fix $r>1$, for any $m\geqslant 0$, there is a constant $C_{r,m}$ such that $|\nabla^m X_R|^2\leqslant C_{r,m}$ on $B_0(p,2r)\times[0,T]$ for all $R\geqslant 1$.

Proof of Claim 1. We use the induction argument on m. The case $m=0$ follows from (II.11). For $m>1$, we have

$$\begin{cases}\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|\nabla^{m-1}X_R|^2\leqslant -2|\nabla^m X_R|^2+C,\\[2ex] \left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|\nabla^m X_R|^2\leqslant -2|\nabla^{m+1}X_R|^2+C|\nabla^m X_R|^2+C,\end{cases}$$

where C is a constant depending on r and m, and C may vary from line to line. Define $f(x,t):=|\nabla^m X_R|^2(|\nabla^{m-1}X_R|^2+\Lambda)$, where Λ is a large positive number to be determined later, then

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f &\leqslant (-2|\nabla^{m+1}X_R|^2 + C|\nabla^m X_R|^2 + C)(|\nabla^{m-1}X_R|^2 + \Lambda) + (-2|\nabla^m X_R|^2 + C)|\nabla^m X_R|^2 \\
&\quad + 2|\nabla|\nabla^m X_R|^2||\nabla|\nabla^{m-1}X_R|^2| \\
&\leqslant (-2|\nabla^{m+1}X_R|^2 + C|\nabla^m X_R|^2 + C)(|\nabla^{m-1}X_R|^2 + \Lambda) + (-2|\nabla^m X_R|^2 + C)|\nabla^m X_R|^2 \\
&\quad + C|\nabla^{m+1}X_R|^2 + |\nabla^m X_R|^4 \\
&\leqslant [-\Lambda + C]\, |\nabla^{m+1}X_R|^4 - |\nabla^m X_R|^4 + C(1+\Lambda)|\nabla^m X_R|^2 + C(1+\Lambda) \\
&\leqslant -\left(\frac{1}{\Lambda + C}\right)^2 f^2 + C(\Lambda) \\
&:= -\alpha f^2 + \beta.
\end{aligned}$$

for some $\Lambda \gg C$, the second inequality follows from Cauchy's inequality. Define $g := \phi_r f$. Since we are looking at the compact domain $\overline{B_0(p,2r)} \times [0,T]$, all derivatives of curvature are bounded above by constants independent of R. Let $M := \sup_{\overline{B_0(p,2r)} \times [0,T]} g(x,t)$ be the supremum of g and $(x_0,t_0) \in \overline{B_0(p,2r)} \times [0,T]$ satisfy $g(x_0,t_0) = M$. If $M = 0$ or $t_0 = 0$, then the assertion holds. Otherwise, we have $M > 0$ and $t_0 \in (0,T]$, and this implies $\phi_r(x_0) > 0$. The maximal principle shows that

$$\begin{aligned}
0 &\leqslant \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) g(x_0,t_0) \\
&= f(x_0,t_0)\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\phi_r(x_0) + \phi_r(x_0)\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f(x_0,t_0) - 2\langle \nabla g(x_0,t_0), \nabla \log \phi_r(x_0)\rangle \\
&\quad + 2\frac{|\nabla\phi_r|^2(x_0,t_0)}{\phi_r(x_0)} g(x_0,t_0) \\
&\leqslant C\frac{1}{\phi_r(x_0)} g(x_0,t_0) + \frac{1}{\phi_r(x_0)}\left[-\alpha g^2(x_0,t_0) + \beta\right] - 0 + C\frac{C}{\phi_r(x_0)} g(x_o,t_0),
\end{aligned}$$

and this implies that $-\alpha M^2 + CM + C \leqslant 0 \Rightarrow M \leqslant C$, for some C that depends on $|\phi|_{C^2(\overline{B_0(p,2r)} \times [0,T])}$ but is independent of R. Hence, we have completed the proof. □

According to this claim, we know that there is a subsequential limit X such that $X_{R_k} \to X$ locally smooth on $\mathcal{M}^n \times [0, T]$. Hence, X satisfies (II.8). Also, (II.11) implies the global $\mathcal{C}^0$ bound

$$|X|_{g(t)}(x, t) \leqslant \max_{\mathcal{M}^n} |X_0|_{g_0} = k_0. \tag{II.14}$$

□

Remark 2. In the case of $|X_0|_{g_0}$ bounded, we can remove the assumption $|\mathrm{Rm}| \leqslant \frac{c}{t}$. It suffices to assume the completeness of each time slice.

II.2.2 $\mathcal{C}^1$-estimate of X

In this subsection, we establish a $\mathcal{C}^1$-estimate of X. Although this estimate $|\nabla X|^2 \leqslant K/t$ explodes as t tends to 0, it is sufficient to achieve our goals.

Lemma 2. Let X be defined as above and $k_0 := \sup_{\mathcal{M}^n} |X_0|_{g(0)} < +\infty$. Then there exists a constant $K = K(n, c, k_0, T) > 0$ such that

$$|\nabla X|^2_{g(t)}(x, t) \leqslant \frac{K}{t}, \tag{II.15}$$

on $\mathcal{M}^n \times (0, T]$.

Proof of Lemma 2. By Lemma 1, we have

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) |X|^2 = -2|\nabla X|^2 \\ \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) |\nabla X|^2 = -2|\nabla^2 X|^2 + \nabla X * \nabla X * \mathrm{Rm} + \nabla X * X * \nabla \mathrm{Rm}, \end{cases} \tag{II.16}$$

Combining (II.16) with Shi's first-order estimate [CCG+07, Theorem 14.13.] and Cauchy's

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inequality, the above equations become

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|\nabla X|^2 \leqslant -2|\nabla^2 X|^2+\frac{C}{t}|\nabla X|^2+\frac{C}{t^2}, \tag{II.17}$$

for some constant $C=C(n,k_0,c)>0$. As before, we perform Bernstein's trick again, considering $f=|\nabla X|^2(|X|^2+\Lambda)$. Then, we have the inequality

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)f \leqslant -Cf^2+\frac{C}{t^2} \tag{II.18}$$

for some $C=C(n,k_0,c)>0$. Now we recall the distance distortion estimates of Hamilton and Perelman:

Lemma 3. [Per02, Lemma 8.3.] There is a constant $\beta=\beta(n)$ satisfying the following. For any Ricci flow $(\mathcal{M}^n,g(t))_{t\in[0,T]}$, with the estimate $\mathrm{Ric}(x,t)\leqslant\frac{(n-1)c}{t}$ on $\mathcal{M}^n\times(0,T]$, the distance function $d(x,t):=d_t(x,x_0)$ satisfies

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)d(x,t)\geqslant -\beta\sqrt{c}t^{-\frac{1}{2}}, \tag{II.19}$$

whenever $d(x,t)\geqslant\sqrt{t}$, and the inequality holds in the viscosity sense. Also, we have

$$B_t(x_0,L-\beta\sqrt{ct})\subset B_s(x_0,L-\beta\sqrt{cs}), \tag{II.20}$$

for any $L>0$ and $0\leqslant s\leqslant t\leqslant T$.

Thence, if we choose δ sufficiently large, then the modified distance $\tilde{d}(x,t)=d(x,t)+\delta\sqrt{ct}$ satisfies

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)\tilde{d}(x,t)\geqslant 0, \tag{II.21}$$

in the viscosity sense, whenever $d(x,t)\geqslant\sqrt{t}$. Accordingly, we choose a smooth cut-off

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function $\phi : \mathbb{R}_{\geqslant 0} \to [0,1]$ such that $\phi = 1$ on $[0,1]$, vanishes outside $[0,2]$, and satisfies $|\phi'|^2 \leqslant 1000\phi$. Define $\Phi(x,t) := \phi\left(\tilde{d}(x,t)/R\right)$ for $R := 10(1+\delta\sqrt{c})\sqrt{T}$. Then we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\Phi(x,t) \leqslant -R^{-2}\phi''(\tilde{d}(x,t)), \tag{II.22}$$

on $\mathcal{M}^n \times [0,T]$ in the viscosity sense. Define $G(x,t) := \phi(x,t)f(x,t)t$. Since G is compactly supported on $\mathcal{M}^n \times [0,T]$, there exists a point $(x',t') \in \mathcal{M}^n \times [0,T]$ so that $G(x',t') = \sup_{(x,t)\in\mathcal{M}^n\times[0,T]} G(x,t) =: M \geqslant 0$. If $M = 0$, then the assertion holds for any $K > 0$ near x_0. Otherwise, the maximal principle (we only need (II.22) holds in the viscosity sense) shows that

$$\begin{aligned} 0 \leqslant \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) G(x',t') &\leqslant -R^{-2}\frac{\phi''}{\phi}M - C\frac{M^2}{t'\phi} + \frac{C}{t'\phi} + 2MR^{-2}\frac{\phi'^2}{\phi^2} \\ \Rightarrow & -CM^2 + Ct'M + C \geqslant 0 \\ \Rightarrow & -CM^2 + CM + C \geqslant 0 \\ \Rightarrow & M \leqslant K, \end{aligned}$$

for some constant $K = K(n,k_0,c,R,T) = K(n,c,k_0,T) > 0$. This implies that $|\nabla X|^2 \leqslant K/t$ on $B_T(x_0,T) \times [0,T]$. We confirm the assertion since x_0 is arbitrary. $\square$

II.2.3 Complete the Proof of Theorem 1

Finally, we define a time-dependent $(0,2)$-tensor $h_{ij} = \nabla_i X_j + \nabla_j X_i$. Similar to the calculation as [LT05, Section 1.2.], we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|h|^2 \leqslant \mathrm{Rm} * h * h \leqslant C|\mathrm{Rm}||h|^2, \tag{II.23}$$

for some constant $C = C(n) > 0$. Lemma 2 implies that $|h|^2 \leqslant K/t$ on $\mathcal{M}^n \times (0, T]$ for some constant $K = K(n, c, k_0, T) > 0$. Also, $|\mathrm{Rm}| \leqslant c/t$ follows from our assumption. Hence, a local maximal principle results in [LT22, Corollary 3.2.], confirms that $h \equiv 0$ on $\mathcal{M}^n \times [0, T]$. Similar to the argument as [LT05, Lemma 2], we conclude that $X = X_0$ for all $t \in [0, T]$ and we accomplish the proof. □

II.3 Proof of Theorem 2

The main difficulty in dealing with an unbounded Killing vector X_0 arises when attempting to apply the standard maximal principle directly to (II.11). We cannot obtain a global uniform $\mathcal{C}^0$ upper bound for X_R that is independent of R. Instead of that, we employ the **Dirichlet heat kernel G_Ω with respect to the heat equation** to establish a local uniform $\mathcal{C}^0$ bound. We recall two common heat kernels on the field of Ricci flow.

Definition 1 (Dirichlet heat kernel $G_\Omega(x, t; y, s)$ with respect to the heat equation)**.** Fixed $\Omega \Subset \mathcal{M}^n$. For any $x, y \in \Omega$ and $0 \leqslant s < t \leqslant T$, we denote **the Dirichlet heat kernel $G_\Omega(x, t; y, s)$ with respect to the heat equation**, which is a positive smooth function that satisfies

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t),x}\right) G_\Omega(x, t; y, s) = 0 \quad \text{for all } x, y \in \Omega \text{ and } 0 \leqslant s < t \leqslant T, \\ \lim_{t \searrow s^+} G_\Omega(x, t; y, s) = \delta_y(x) \quad \text{for all } x, y \in \Omega \text{ and } s \in [0, T), \\ \left(\frac{\partial}{\partial s} + \Delta_{g(s),y} - \mathrm{scal}_g(y, s)\right) G_\Omega(x, t; y, s) = 0 \quad \text{for all } x, y \in \Omega \text{ and } 0 \leqslant s < t \leqslant T, \\ \lim_{s \nearrow t^-} G_\Omega(x, t; y, s) = \delta_x(y) \quad \text{for all } x, y \in \Omega \text{ and } t \in (0, T], \\ G_\Omega(x, t; y, s) = 0 \;\; \text{whenever either } x \in \partial\Omega \text{ or } y \in \partial\Omega. \end{cases} \tag{II.24}$$

Definition 2 (Dirichlet heat kernel $\tilde{G}_\Omega(x,t;y,s)$ with respect to the backward heat equation)**.** Fixed $\Omega \Subset \mathcal{M}^n$. For any $x, y \in \Omega$ and $0 \leqslant s < t \leqslant T$, we denote **the Dirichlet heat kernel** $\tilde{G}_\Omega(x,t;y,s)$ **with respect to the backward heat equation**, which is a positive smooth function that satisfies

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t),x} - \mathrm{scal}_g(x,t)\right) \tilde{G}_\Omega(x,t;y,s) = 0 \quad \text{for all } x, y \in \Omega \text{ and } 0 \leqslant s < t \leqslant T, \\ \lim_{t \searrow s^+} \tilde{G}_\Omega(x,t;y,s) = \delta_y(x) \quad \text{for all } x, y \in \Omega \text{ and } s \in [0,T), \\ \left(\frac{\partial}{\partial s} + \Delta_{g(s),y}\right) \tilde{G}_\Omega(x,t;y,s) = 0 \quad \text{for all } x, y \in \Omega \text{ and } 0 \leqslant s < t \leqslant T, \\ \lim_{s \nearrow t^-} \tilde{G}_\Omega(x,t;y,s) = \delta_x(y) \quad \text{for all } x, y \in \Omega \text{ and } t \in (0,T], \\ \tilde{G}_\Omega(x,t;y,s) = 0 \text{ whenever either } x \in \partial\Omega \text{ or } y \in \partial\Omega. \end{cases} \tag{II.25}$$

In the following proof, we will mainly rely on upper bounds of the Dirichlet heat kernel $G_\Omega(x,t;y,s)$ with respect to the heat equation. To this end, in this section, we assume that there are constants $c, L > 0$ and $K \in \mathbb{R}$ such that the Ricci flow $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ satisfies the following conditions:

$$\begin{cases} |\mathrm{Rm}(g(t))| \leqslant \frac{c}{t}, \text{ on } \mathcal{M}^n \times (0,T]; \\ \mathrm{inj}_{g(t)}(x) \geqslant \sqrt{\frac{t}{c}}, \text{ for all } (x,t) \in \mathcal{M}^n \times (0,T] \text{ with } B_t\left(x, \sqrt{\frac{t}{c}}\right) \Subset \mathcal{M}^n; \\ \mathrm{scal}(g(0)) \geqslant K, \text{ on } \mathcal{M}^n; \\ \mathrm{Vol}_{g(0)}(B_{g(0)}(x,r)) \leqslant v_0 r^n e^{Cr^2}, \text{ for all } x \in \mathcal{M}^n \text{ and } r \geqslant 0. \end{cases} \tag{II.26}$$

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II.3.1 Construct a solution to (II.8) when X_0 is exponentially growth

In this section, we prove the following existence result for equation (II.8) when the initial vector $|X_0|$ is exponentially growth.

Theorem 4. Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a complete Ricci flow satisfying (II.4). Given a vector field X_0, if there are constants $A > 0$, $B \in \mathbb{R}$, $\varepsilon \in [0, 1]$, and a point $p \in \mathcal{M}^n$ such that for any $x \in \mathcal{M}^n$, then we have

$$|X_0|_{g(0)}(x)^2 \leqslant \exp(Ad_0(x,p)^{2(1-\varepsilon)} + B), \tag{II.27}$$

then there exists a smooth solution X of (II.8) with the bound

$$|X|^2_{g(t)}(x,t) \leqslant \exp(A'd_0(x,p)^{2(1-\varepsilon)} + B'), \tag{II.28}$$

for all $x \in \mathcal{M}^n$, for some constants $A', B' \in \mathbb{R}$ which depend on $n, c, K, T, A, B, \varepsilon, L$.

Proof of Theorem 4. We only give the proof for $\varepsilon = 0$, since the proofs are the same for $\varepsilon \in (0, 1]$. By (II.20) and the volume comparison, we may assume that $T \leqslant T(n, c, K, A, L, \epsilon)$ for some small number since we can iterate our solution to the entire lifespan. We consider the following PDE:

Fix $R \gg 1$ and $p \in \mathcal{M}^n$. Choose a smooth cut-off function $\phi_R : \mathcal{M}^n \to [0, 1]$ such that $\phi_R(x) = 1$ on $B_0(p, R)$, $\phi(x) = 0$ on $B_0(p, 2R)^c$, and $|\nabla\phi_R| \leqslant 2/R$. Consider a Dirichlet problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)} - \text{Ric}_t\right) X_R = 0 \ \text{, on } B_0(p, 5R) \times [0, T], \\ X_R(x, 0) = \phi_R(x)X_0(x) \ \text{, on } B_0(p, 8R), \\ X_R(x, t) = 0 \ \text{, on } \partial B_0(p, 8R) \times [0, T]. \end{cases} \tag{II.29}$$

Then we also have $\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|X_R|^2 \leqslant 0$. Now, we introduce the following heat kernel estimate with respect to the heat equation coupled with Ricci flow.

Proposition 2. For any $n, c > 0$ and K, there exists $C(n, c, K, T) > 0$ such that the following is true: Suppose $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ is a Ricci flow satisfying the conditions (II.4). Let $p \in \mathcal{M}^n$ be a fixed point so that $B_t(p, 4r) \Subset \mathcal{M}^n$ for some $r \geqslant 1$ for all $t \in [0, T]$. Let Ω be a domain with smooth boundary such that $\Omega \Subset B_t(p, r)$ for all $t \in [0, T]$. Then the Dirichlet heat kernel $G_\Omega(x, t; y, s)$ with respect to the heat equation on $\Omega \times \Omega \times [0, T]$ satisfies

$$G_\Omega(x, t; y, 0) \leqslant \frac{C}{t^{\frac{n}{2}}} \exp\left(-\frac{d_0^2(x, y)}{Ct}\right), \tag{II.30}$$

for all $0 < t \leqslant T$ and $x, y \in \Omega$.

Proof of Proposition 2. By [ST22, Lemma 8.1.], there is a constant $K' = K'(n, c, T, K)$ such that $\mathrm{scal}_g(t) \geqslant K'$ on $\mathcal{M}^n \times [0, T]$. Let $\tilde{G}_\Omega(x, t; y, s)$ be the Dirichlet heat kernel with respect to the backward heat equation on $\Omega \times [0, T]$. By [LT22, Proposition 4.1.], we have

$$\tilde{G}_\Omega(x, t; y, s) \leqslant \frac{C}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{d_s^2(x, y)}{C(t-s)}\right).$$

Since we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t),x}\right) e^{-K't}\tilde{G}_\Omega(x, t; y, 0) \geqslant 0,$$

and $\lim_{t\nearrow 0^+} e^{-K't}\tilde{G}_\Omega(x, t; y, 0) - G_\Omega(x, t; y, 0) = 0$, by standard maximal principle and (II.24), we conclude that

$$G_\Omega(x, t; y, 0) \leqslant e^{-K't}\tilde{G}_\Omega(x, t; y, 0) \leqslant \frac{Ce^{|K'T|}}{t^{\frac{n}{2}}} \exp\left(-\frac{d_0^2(x, y)}{Ct}\right).$$

□

Now we fix $r > 0$ and $x \in B_0(p, r)$, assuming $|X_0|^2(y) \leqslant f(d_0(x, y))$ for all $y \in \mathcal{M}^n$. Consider the function

$$h(x,t) := \int_{B_0(p,2R)} \phi_R(y)|X_0|^2(y)G_\Omega(x,t;y,0)d\,\mathrm{Vol}_{g(0)}(y),$$

then the function h satisfies $(\partial_t - \Delta_t)h = 0$ on $\Omega \times [0, T]$ and $|X_R|^2 = h$ on the parabolic boundary of $\Omega \times [0, T]$. Hence, the standard maximal principle shows that $|X_R|^2 \leqslant h$ on $\Omega \times [0, T]$. By Proposition 2, we have

$$\begin{aligned}
|X_R|^2(x,t) &\leqslant \int_{B_0(p,2R)} \phi_R(y)|X_0|^2(y)G_\Omega(x,t;y,0)d\,\mathrm{Vol}_{g(0)}(y)\\
&\leqslant \int_{B_0(p,2R)} \phi_R(y)|X_0|^2(y)\frac{C}{t^{\frac{n}{2}}}\exp\left(-\frac{d_0^2(x,y)}{Ct}\right)d\,\mathrm{Vol}_{g(0)}(y)\\
&\leqslant \int_{B_0(x,2R+r)} f(d_0(x,y))\frac{C}{t^{\frac{n}{2}}}\exp\left(-\frac{d_0(x,y)^2}{Ct}\right)d\,\mathrm{Vol}_{g(0)}(y)\\
&= \int_0^{2R+r} f(z)\frac{C}{t^{\frac{n}{2}}}\exp\left(-\frac{z^2}{Ct}\right)\mathrm{Vol}\left(\partial B_0(x,z)\right)dz\\
&= f(2R+r)\frac{C}{t^{\frac{n}{2}}}\exp\left(-\frac{(2R+r)^2}{Ct}\right)\mathrm{Vol}\,B_0(x,2R+r)\\
&\quad - \int_0^{2R+r}\left[f'(z)\frac{C}{t^{\frac{n}{2}}} - 2zf(z)\frac{C}{t^{\frac{n}{2}+1}}\right]\exp\left(-\frac{z^2}{Ct}\right)\mathrm{Vol}\left(B_0(x,z)\right)dz,
\end{aligned}$$

where the third line follows from the co-area formula. Now we choose $f(z) = e^{\epsilon z^2+B}e^{\epsilon z^2}$, for some $T < (2C(1 + L + \epsilon))^{-1}$. Note that the volume assumption is $\mathrm{Vol}(B_0(x, r)) \leqslant$

$Cr^n e^{Lr^2}$, combining together we get

$$
\begin{aligned}
|X_R|^2(x,t) \leqslant{}& C\frac{(2R+r)^n}{t^{\frac{n}{2}}}\exp\left(-\frac{(2R+r)^2}{2Ct}+L(2R+r)^2\right)e^{\epsilon r^2+B}\\
&+ e^{\epsilon r^2+B}\int_0^{2R+r} z e^{\epsilon z^2}\frac{C}{t^{\frac{n}{2}+1}}z^n\exp\left(-\frac{z^2}{Ct}+Lz^2\right)dz\\
\leqslant{}& e^{\epsilon r^2+B}\left[C+C\int_0^{2R+r}\left(\frac{z}{\sqrt{t}}\right)^{n+2}\exp\left(-\frac{z^2}{2Ct}+Lz^2\right)\frac{dz}{z}\right]\\
={}& e^{\epsilon r^2+B}\left[C+C\int_0^{\frac{2R+r}{\sqrt{t}}} z^{n+1}e^{-\frac{z^2}{2C}+Lz^2t}dz\right]\\
\leqslant{}& e^{\epsilon r^2+B}\left[C+C\int_0^{\infty} z^{n+1}e^{-\frac{z^2}{2C}+Lz^2T}dz\right]\\
\leqslant{}& Ce^{\epsilon r^2+B}
\end{aligned}
$$

for some C independent of R and r. Hence, we have shown that for any bounded region $\Omega \Subset \mathcal{M}^n$, there is a constant $C_\Omega > 0$ such that $|X_R|(x,t) \leqslant C_\Omega$ for all R and $(x,t) \in \Omega \times [0,T]$. In fact, we demonstrate that there is a constant $C = C(n,c,K,T) > 0$ such that

$$
|X_R|^2(x,t) \leqslant Ce^{\epsilon d_0(x,p)^2+B}, \tag{II.31}
$$

for all R and $x \in \mathcal{M}^n$. Repeat the argument as in Claim 1, we conclude that there is a smooth vector field X such that there is a subsequence X_{R_n} that converges to X locally smoothly as $R_n \to \infty$ and the smooth convergence ensures that X satisfies (II.8). □

II.3.2 Local maximal principle

This section improves the local maximal principle in [LT22, Theorem 1.1.]. We follow the same approach as [LT19], which is choosing the cut-off function carefully to perform the estimation.

Theorem 5. Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a Ricci flow which is possibly incomplete. Suppose that $\mathrm{Ric}(g(t)) \leqslant \alpha t^{-1}$ on $\mathcal{M}^n \times (0, T]$ for some $\alpha > 0$. Let $\varphi(x,t)$ be a continuous function on $\mathcal{M}^n \times [0,T]$ which satisfies $\varphi(x,t) \leqslant \alpha t^{-1}$ on $\mathcal{M}^n \times (0,T]$ and

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \varphi\Big|_{(x_0,t_0)} \leqslant L(x_0,t_0)\varphi(x_0,t_0), \tag{II.32}$$

whenever $\varphi(x_0,t_0) > 0$ in the sense of barrier, for some continuous function $L(x,t)$ on $\mathcal{M}^n \times [0,T]$ with $L(x,t) \leqslant \alpha t^{-1}$. Suppose $p \in \mathcal{M}^n$ and $R > 0$ such that $B_0(p,R) \Subset \mathcal{M}^n$ and $\varphi(x,0) \leqslant 0$ on $B_0(p,R)$. Then for any $\varepsilon \in (0,1)$, there exists $\hat{T}(n,\alpha,\varepsilon) > 0$ such that for $t \in (0, \min\{T, R^2\hat{T}\}]$,

$$\varphi(p,t) \leqslant R^{-2} \exp\left(-\frac{R^{2(1-\varepsilon)}}{t^{1-\varepsilon}}\right). \tag{II.33}$$

Proof of Theorem 5. The proof is the same as [LT22, Theorem 1.1.] but with a different cut-off function. First of all, we may assume $R = 2$ by scaling. For $a > 0$, we define $\psi : [0,\infty) \to [0,1]$ a smooth non-increasing function such that

$$\psi(s) := \begin{cases} 1 \quad \text{for } 0 \leqslant s \leqslant \frac{1}{2}; \\ \exp(-\frac{1}{(1-s)^a}) \quad \text{for } \frac{3}{4} \leqslant s \leqslant 1; \\ 0 \quad \text{for } s \geqslant 1. \end{cases} \tag{II.34}$$

and such that $\psi'' \geqslant -c\psi$ for some constant $c > 0$. For $s \in (3/4, 1)$, we have

$$|\psi'|^2 = a^2 \left(-\log\psi\right)^{\frac{2a+2}{a}} \psi^2.$$

Then there is a constant $C > 0$ such that

$$|\psi'|^2 \leqslant C\left[1 + \left(-\log\psi\right)^{\frac{2a+2}{a}}\right]\psi^2,$$

for all $s \in [0, \infty)$. Now, we may assume $L(x,t)$ is nonnegative by replacing L with $|L|$, and $B_t(p, \frac{3}{2}) \Subset \mathcal{M}^n$ for all $t \in [0, \min\{T, T_1(n,\alpha)\}]$ by Lemma 3. Similar to [LT22, Theorem 1.1.], there exists constant $c_1(n) > 0$ such that, if we consider the cut-off function

$$\Psi_r(x,t) := \exp(-Cr^{-2}t)\psi\left(\frac{d_t(x,p) + c_1\alpha\sqrt{t}}{r}\right), \tag{II.35}$$

where $r \in (0,1]$ is a constant, then following the discussion in [LT22, Theorem 1.1.], we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\Psi_r \leqslant 0, \tag{II.36}$$

in the sense of barrier on $B_0(p,2) \times [0, \min\{T, T_1\}]$, where $T_1 := \min\{\frac{r}{2}(c_1\alpha+1)^{-2}, C^{-1}r^2\log 2, 1\}$ is a constant depending on n, r, α. Let $\eta(t) \geqslant 0$ be a smooth function with $\eta(t) > 0$ for all $t > 0$. Consider the function

$$F = -\Psi_r\varphi + \eta. \tag{II.37}$$

Suppose that $F(\cdot, 0) > 0$ on $B_0(p,2)$. If $F \leqslant 0$ on some $(x,t) \in B_0(p,2) \times [0, \min\{T, T_1\}]$, then there is a point $(x_0, t_0) \in B_0(p,2) \times [0, \min\{T, T_1\}]$ such that $F(x_0,t_0) = 0$ and $F(x,t) \geqslant 0$ on $B_0(p,2) \times [0, t_0]$. In particular, we get $\varphi(x_0,t_0) > 0$. Again, following the same strategy as [LT22, Theorem 1.1.], we get

$$\eta' \leqslant L\eta + 2\eta\frac{|\nabla\Psi_r|^2}{\Psi_r^2}, \tag{II.38}$$

at (x_0,t_0). By our construction of φ, we get

$$\frac{|\nabla\Psi_r|^2}{\Psi_r^2} = r^{-2}\frac{|\psi'|^2}{\psi^2} \leqslant Cr^{-2}\left[1 + (-\log\psi)^{\frac{2a+2}{a}}\right] \leqslant Cr^{-2} + Cr^{-2}\left(\log\frac{\varphi}{\eta}\right)^{\frac{2a+2}{a}}, \tag{II.39}$$

where the last inequality follows from $\psi \in [0,1]$. Hence, we get

$$\eta'(t_0) \leqslant \begin{cases} \eta(t_0)\left[L_0 + 2Cr^{-2}\left(\log \frac{a_0}{\eta(t_0)}\right)^{\frac{2a+2}{a}}\right] \\ \eta(t_0)\left[\frac{\alpha}{t_0} + 2Cr^{-2} + 2Cr^{-2}\left(\log \frac{\alpha}{t_0\eta(t_0)}\right)^{\frac{2a+2}{a}}\right], \end{cases} \tag{II.40}$$

where $L_0 = \max_{B_0(p,2)\times[0,T]} L + 2Cr^{-2}$ and $a_0 = \max_{B_0(p,2)\times[0,T]} |\varphi|$. Now we use an inductive argument to prove the following statement.

Claim 2. For any $k \in \mathbb{N}$, there exists a constant $\tau_k > 0$ such that if $\varepsilon \in (2^{-k}, 1)$, then

$$\varphi(x,t) \leqslant 2\exp(-t^{-(1-\varepsilon)}), \tag{II.41}$$

on $B_t(p, 2^{-k} - c_1\alpha\sqrt{t})$ and $t \in (0, \min\{T, \tau_k\}]$.

Proof of Claim 2. For $k = 1$, we consider $\eta(t) = \delta t + \exp(-\frac{1}{t^{1-\varepsilon}})$ with small constant $\delta << 1$ and $r = 1$, then $F > 0$ near $t = 0$ by [LT22, Theorem 1.1.]. Therefore. the first line of (II.40) implies

$$\begin{aligned} \delta + (1-\varepsilon)t_0^{\varepsilon-2}\exp(-t_0^{\varepsilon-1}) &\leqslant (\delta t_0 + \exp(-t_0^{\varepsilon-1}))\left(L_0 + 2C\left(\log \frac{a_0}{\delta t_0 + \exp(-t_0^{\varepsilon-1})}\right)^{\frac{2a+2}{a}}\right) \\ &\leqslant (\delta t_0 + \exp(-t_0^{\varepsilon-1}))\left(L_0 + 2C\left(\log a_0 + t_0^{\varepsilon-1}\right)^{\frac{2a+2}{a}}\right) \\ &\leqslant (\delta t_0 + \exp(-t_0^{\varepsilon-1}))\left(L_0 + 2^{\frac{2a+2}{a}}C(\log a_0)^{\frac{2a+2}{a}} + 2^{\frac{2a+2}{a}}Ct_0^{(\varepsilon-1)\frac{2a+2}{a}}\right). \end{aligned} \tag{II.42}$$

Note that $t_0 \leqslant 1$, so we get

$$\frac{1-\varepsilon}{t_0} \leqslant \frac{\delta + (1-\varepsilon)t_0^{\varepsilon-2}\exp(-t_0^{\varepsilon-1})}{\delta t_0 + \exp(-t_0^{\varepsilon-1})} \leqslant L' + 2^{\frac{2a+2}{a}}Ct_0^{(\varepsilon-1)\frac{2a+2}{a}}, \tag{II.43}$$

for some constant L' depending on L_0, a, C, a_0. Since $1-\varepsilon \in (0, \frac{1}{2})$, there is a constant $a = a(\varepsilon) > 0$ such that $(1-\varepsilon)\frac{2a+2}{a} \leqslant 1$. This implies that $t_0 \geqslant \tau$ for some τ which is independent of δ. Therefore, letting $\tau \to 0^+$, we obtain that $\varphi(x,t) \leqslant 2\exp(-t^{-(1-\varepsilon)})$ on $B_t(p, \frac{1}{2} - c_1\alpha\sqrt{t})$ near $t = 0$.

Now, we assume that the assertion holds for k. For $\varepsilon \in (2^{-(k+1)}, 1)$, we consider $\eta(t) = \delta\exp(-t^{-(1-2^{-k})}) + \exp(-t^{-(1-\varepsilon)})$ with small $\delta << 1$ and $r = 2^{-k}$. By the induction hypothesis, $F > 0$ near $t = 0$. Then the first line of (II.40) implies

$$\begin{aligned} &\frac{\delta 2^{-k} t_0^{-(1-2^{-k})-1}\exp(-t_0^{-(1-2^{-k})}) + (1-\varepsilon) t_0^{\varepsilon-2}\exp(-t_0^{\varepsilon-1})}{\delta\exp(-t_0^{-(1-2^{-k})}) + \exp(-t_0^{\varepsilon-1})} \\ &\leqslant L_0 + 2^{2k+1}C\left(\log\frac{a_0}{\delta\exp(-t_0^{-(1-2^{-k})}) + \exp(-t_0^{\varepsilon-1})}\right)^{\frac{2a+2}{a}} \\ &\leqslant L' + \tilde{C} t_0^{(\varepsilon-1)\frac{2a+2}{a}}, \end{aligned} \tag{II.44}$$

where L' and $\tilde{C}$ are constants independent of δ. For the L.H.S. of (II.44), we have

$$\frac{\delta 2^{-k} t_0^{-(1-2^{-k})-1}\exp(-t_0^{-(1-2^{-k})}) + (1-\varepsilon) t_0^{\varepsilon-2}\exp(-t_0^{\varepsilon-1})}{\delta\exp(-t_0^{-(1-2^{-k})}) + \exp(-t_0^{\varepsilon-1})} \geqslant (1-\varepsilon)2^{-k} t_0^{-(1-2^{-k-1})-1}. \tag{II.45}$$

Combining (II.44) and (II.45), we obtain

$$(1-\varepsilon)2^{-k} \leqslant L' t_0^{2-2^{-k-1}} + \tilde{C} t_0^{2-2^{-k-1}+(\varepsilon-1)\left(2+\frac{2}{a}\right)}. \tag{II.46}$$

Choose $a = 2^{k+2}(1-\varepsilon) > 0$ so that

$$2 - 2^{-k-1} + (\varepsilon - 1)\left(2 + \frac{2}{a}\right) > 0. \tag{II.47}$$

Hence, there is a $\tau_k > 0$ which is independent of δ, such that $t_0 \geqslant \tau_k$. In particular, we obtain that $\varphi(x,t) \leqslant 2\exp(-t^{-(1-\varepsilon)})$ on $B_t(p, 2^{-(k+1)} - c_1\alpha\sqrt{t})$ and $t \in (0, \min\{T, \tau_k\}]$. Hence, we complete the proof of Claim 2. $\square$

Now, we improve the constant τ_k to $\hat{T}$, where $\hat{T}$ is independent of L_0 and a_0. Choose $\eta(t) = \frac{1}{2}\exp(-t^{-(1-\varepsilon)})$ and $r = 2^{-k}$ if $\varepsilon \in [2^{-k}, 2^{-k+1})$. By Claim 2, $F > 0$ near $t = 0$. The second line of (II.40) implies

$$\begin{aligned}(1-\varepsilon)t_0^{-2+\varepsilon} &\leqslant \frac{\alpha}{t_0} + 2^{1-2k}C + 2^{1-2k}C\left(\log\frac{\alpha}{t_0\exp(-t_0^{-(1-\varepsilon)})}\right)^{\frac{2a+2}{a}} \\ &\leqslant \alpha t_0^{-1} + C' + C' t_0^{-(1-\varepsilon)\frac{2a+2}{a}},\end{aligned} \tag{II.48}$$

where C' is a constant only depending on ε, α, n. Taking $= \frac{4(1-\varepsilon)}{\varepsilon} > 0$, then there is a constant $\hat{T} > 0$ such that $t_0 \geqslant \hat{T}$. In other words, we complete the proof. $\square$

Unfortunately, we are not able to prove Theorem 5 for the case $\varepsilon = 0$. It seems to be correct by the classical maximal principle result, but we don't have the proof now.

II.3.3 $\mathcal{C}^1$-estimate of exponential growth X

Similar to the proof of Theorem 1, we need to establish a $\mathcal{C}^1$-estimate for X. However, the difference is that the upper bound becomes $f(d_0(x,p))/t$ if $|X|^2(x,t) \leqslant f(d_0(x,p))$ for all $x \in \mathcal{M}^n$. Thanks to the Theorem 5, this is sufficient to apply the maximal principle if $f(r) = \mathrm{O}(e^{r^{2(1-\varepsilon)}})$ as $r \to \infty$ for some $\varepsilon \in (0,1]$. From now on, we will assume that the Killing vector field X_0 has the exponential of almost quadratic growth.

Proposition 3. Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a Ricci flow satisfying (II.4) and X be constructed as above. Suppose there is a point $p \in \mathcal{M}^n$, $\varepsilon \in [0,1]$, and $A, B \geqslant 0$ so that $|X_0|^2(x) \leqslant \exp(Ad_0(x,p)^{2(1-\varepsilon)} + B)$ for all $x \in \mathcal{M}^n$, then there are constants A'', B'' which depend on $A, B, n, c, K, \varepsilon, L$ such that

$$|\nabla X|^2(x,t) \leqslant \frac{\exp(A''d_0(x,p)^{2(1-\varepsilon)} + B'')}{t},$$

on $\mathcal{M}^n \times (0, T]$.

Proof of Proposition 3. Fix $R > 1$. Then it is sufficient to prove that

$$|\nabla X|^2(x,t) \leqslant \frac{\exp(A'' R^{2(1-\varepsilon)} + B'')}{t},$$

on $B_0(p, R) \times (0, T]$. To prove this, we follow the same strategy as in Section II.2.2, but we perform more refined estimates. The Shi's first local estimate [CCG+07, Theorem 14.13.] and (II.28) show that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) |\nabla X|^2 \leqslant -2|\nabla^2 X|^2 + \frac{C}{t}|\nabla X|^2 + \frac{\exp(A' R^{2(1-\varepsilon)} + B')}{t^2},$$

on $B_0(p, 2R) \times (0, T]$. Consider the function $f(x,t) := t|\nabla X|^2(|X|^2 + \Lambda \exp(A' R^{2(1-\varepsilon)} + B'))$ for some $\Lambda > 0$, to be determined later. Therefore, on $B_0(p, 2R) \times (0, T]$, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f \leqslant{} & \frac{f}{t} + t\left[-2|\nabla^2 X|^2 + \frac{C}{t}|\nabla X|^2 + \frac{\exp(A' R^{2(1-\varepsilon)} + B')}{t^2}\right] \big(|X|^2 + \Lambda \exp(A' R^{2(1-\varepsilon)} \\
& + B')\big) - 2t|\nabla X|^4 + 8t|\nabla^2 X||\nabla X|^2|X| \\
\leqslant{} & \frac{f}{t} + t\left[-2|\nabla^2 X|^2 + \frac{C}{t}|\nabla X|^2 + \frac{\exp(A' R^{2(1-\varepsilon)} + B')}{t^2}\right] (|X|^2 + \Lambda \exp(A' R^{2(1-\varepsilon)} \\
& + B')) - 2t|\nabla X|^4 + t|\nabla X|^4 + C\exp(A' R^{2(1-\varepsilon)} + B')|\nabla^2 X|^2 \\
\leqslant{} & t\left[-2\Lambda \exp(A' R^{2(1-\varepsilon)} + B') + C\exp(A' R^{2(1-\varepsilon)} + B')\right] |\nabla^2 X|^2 - t|\nabla X|^4 \\
& + \frac{C(1+\Lambda)f}{t} + \frac{C(1+\Lambda)\exp(2A' R^{2(1-\varepsilon)} + 2B')}{t},
\end{aligned}$$

for some C independent of R. Hence, if we choose $\Lambda \gg C$, independent of R, then we have

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right) f \leqslant \frac{-\alpha \exp(-2A'R^{2(1-\varepsilon)}-2B')f^2+\beta\exp(2A'R^{2(1-\varepsilon)}+2B')}{t}, \tag{II.49}$$

on $B_0(p,2R)\times(0,T]$, for some $\alpha,\beta>0$ independent of R. Finally, we consider $g(x,t):=\phi\left(\frac{\tilde{d}_t(x,p)}{DR}\right)f(x,t)$, where $D>>1$ would be determined later, $\tilde{d}$ is the modified distance, and $\phi:\mathbb{R}_{\geqslant 0}\to[0,1]$ is a smooth cut-off function defined with $|\phi'|^2\leqslant 1000\phi$, see (II.21) to the precise definition. Since g has compactly support in $\mathcal{M}^n\times[0,T]$, there exists $(x',t')\in\mathcal{M}^n\times[0,T]$ that realized the maximal value $M=g(x',t')=\sup g(x,t)$. If $M=0$, then the assertion holds. If $M>0$, then we have

$$\begin{aligned}0&\leqslant\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)g\\&\leqslant-\frac{\phi''M}{D^2R^2\phi}+\frac{-\alpha\exp(-2A'R^{2(1-\varepsilon)}-2B')M^2+\beta\exp(2A'R^{2(1-\varepsilon)}+2B')}{\phi t}+2\frac{M\phi'^2}{D^2R^2\phi^2},\end{aligned}$$

and we multiply ϕt on both sides, this shows that

$$\begin{aligned}&0\leqslant-\alpha\exp(-2A'R^{2(1-\varepsilon)}-2B')M^2+C(n,D,T)R^{-2}M+\beta\exp(2A'R^{2(1-\varepsilon)}+2B')\\ \Rightarrow M\leqslant&\ \frac{CR^{-2}+\sqrt{C^2R^{-4}+4\alpha\beta}}{\alpha\exp(-2A'R^{2(1-\varepsilon)}-2B')}\leqslant C'\exp(2A'R^{2(1-\varepsilon)}+2B'),\end{aligned}$$

for some $C'=C'(n,D,T,\alpha,\beta)>0$. Now we recall an expanding ball lemma, which was proved in [He17, Lemma 3.5.] and had a slightly weaker version in [LT22, Lemma 2.2.].

Lemma 4. [LT22, Lemma 2.2.] For any $n\in\mathbb{N}$ and $c,v_0,\sigma>0$, there exists $\Lambda(n,c,v_0,\sigma)>1$ and $R_0(n,v_0,c,\sigma)>0$ with the following property. Let $(\mathcal{M}^n,g(t))_{t\in[0,T]}$ be a Ricci flow solution with $T\leqslant 1$. Suppose that $p\in\mathcal{M}^n$ with $B_0(p,R)\Subset\mathcal{M}^n$ for some $R\geqslant R_0$ such that

1. $\mathrm{scal}(g(0)) \geqslant -\sigma$ on $B_0(p,R)$.
2. $\mathrm{Vol}_0\, B_0(x,r) \leqslant v_0 r^n$ for all $B_0(x,r) \subset B_0(p,R)$ and $r \in (0,1]$.
3. $|\mathrm{Rm}(g(t))| \leqslant \frac{c}{t}$ for all $x \in B_t(p,R)$ and $t \in [0,T]$.
4. $\frac{\mathrm{Vol}_t\, B_t(x,\sqrt{t})}{t^{\frac{n}{2}}} \geqslant c^{-1}$ for all $B_t(x,\sqrt{t}) \subset B_0(p,R)$ and $t \in (0,T]$.

Then for all $t \in [0,T]$, we have

$$B_0(p,\mu^{-1}R) \subset B_t(p,\frac{1}{2}R), \tag{II.50}$$

Note that the third condition follows from the assumption of the lower bound of the injectivity radius. Choose $D = \mu + \delta\sqrt{cT}$, then there are constants A'', B'' depending on $n, c, K, T, A, R_0, \mu, \varepsilon$ such that

$$|\nabla X|^2(x,t) \leqslant \frac{f(x,t)}{t\exp(A'R^{2(1-\varepsilon)}+B')} \leqslant \frac{M}{t\exp(A'R^{2(1-\varepsilon)}+B')} \leqslant \frac{\exp(A''R^{2(1-\varepsilon)}+B'')}{t},$$

for all $(x,t) \in B_0(p,R) \times (0,T]$. This confirms the assertion. □

II.3.4 Complete the proof of Theorem 2

Finally, we define a $(0,2)$-tensor $h_{ij} = \nabla_i X_j + \nabla_j X_i$, and the evolution equation becomes

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|h|^2 \leqslant C|\mathrm{Rm}||h|^2, \tag{II.51}$$

on $\mathcal{M}^n \times [0,T]$, Note that $h(\cdot,0) = 0$ follows from the assumption. Our tactic is to apply the maximal principle to deduce $h \equiv 0$. According to [LT22, Corollary 3.2.], for $r >> 1$, consider $\varphi(x,t) := e^{-r^{(2-\varepsilon)}}|h|^2(x,t)$. Then on $B_0(p,r) \times (0,T)$, we have $|\mathrm{Rm}| \leqslant c/t$ and

$$\varphi(x,t) = \frac{|h|^2(x,t)}{e^{r^{2-\varepsilon}}} \leqslant \frac{C}{t}, \tag{II.52}$$

for some C independent of r. Therefore, Theorem 5 indicates that

$$|h|^2(p,t) \leqslant r^2 \exp\left(r^{2-\varepsilon} - \frac{r^{2(1-\varepsilon)}}{t^{1-\varepsilon}}\right) \to 0,$$

as $r \to \infty$ for all $t \in [0,T]$. This implies that $h(p,t) = 0$ on $[0,T]$. Since $p \in \mathcal{M}^n$ is arbitrary, we obtain that $h \equiv 0$ on $\mathcal{M}^n \times [0,T]$. In the end, Tian's and Lu's argument confirms that X_0 is a Killing vector field with respect to $g(t)$ (See [LT05, Section 1.2.]), combining with the expanding ball lemma 4, we complete the proof of Theorem 2. ☐

II.4 The existence results of rotationally symmetric Ricci flow on $\mathbb{R}^{n+1}$

A rotationally symmetric metric on $\mathbb{R}^{n+1}$ is a Riemannian metric $g = ds^2 + f(s)^2 g_{\text{std}}$ with some smooth warped function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\lim_{s\to 0} f^{(1)}(s) = 1 \text{ and } \lim_{s\to 0} f^{(2k)}(s) = 0, \tag{II.53}$$

for all $k \geqslant 0$, where g_{std} is the standard spherical metric on S^n. As shown in the introduction, one of the existence of Ricci flows starting with g is given by the conditions $\mathrm{Rm}(g) + \alpha_0 \mathcal{I} \in \mathcal{C}_{\text{PIC1}}$ and volume is weakly non-collapsed. The following proposition shows whether g satisfies the weakly volume non-collapsed.

Proposition 4 (Volume Ratio bounds)**.** Let $g = ds^2 + f(s)^2 g_{\text{std}}$ be a complete and rotationally symmetric metric on $\mathbb{R}^{n+1}$. Suppose there exists a constant $\delta > 0$ such that

$$f(s) \geqslant \delta s \chi_{[0,1]}(s) + \delta \chi_{(1,\infty)}(s), \tag{II.54}$$

on $\mathbb{R}^{n+1}$. Then there exists a constant $v = v(\delta, n) > 0$ such that $\frac{\mathrm{Vol}_g(B_g(x,r))}{r^{n+1}} \geqslant v$ for all $x \in \mathbb{R}^{n+1}$ and $r \in (0,1]$. In particular, if $f_s > 0$ on $\mathbb{R}^{n+1}$ and $f_s \geqslant \delta$ on $|s| \leqslant 1$, then the assumption holds.

Proof. Due to the rescaling property, it suffices to deal with the case $r = 1$. For $x = o$, the co-area formula implies

$$\mathrm{Vol}_g(B_g(o, \frac{1}{4})) = \int_0^{\frac{1}{4}} f(s)^n \omega_n ds \geqslant \delta^n \int_0^{\frac{1}{4}} s^n \omega_n ds = \frac{\left(\frac{1}{4}\right)^{n+1} \delta^n \omega_n}{n+1} =: v_1, \tag{II.55}$$

where ω_n is the volume of unit sphere S^n. Note that $B_g(o, \frac{1}{4}) \subset B_g(x, 1)$ for all $x \in B_g(o, \frac{1}{2})$, this shows $\mathrm{Vol}_g(B_g(x,1)) \geqslant v_1$ for all $x \in B_g(o, \frac{1}{2})$. For $x \notin B_g(o, \frac{1}{4})$, $f(x) \geqslant \frac{\delta}{4}$. It remains to the case $x \notin B_g(x, \frac{1}{2})$. Note that

$$A_x^{\frac{1}{4}} \subset B_g(x, \frac{1}{2}), \tag{II.56}$$

where

$$A_x^r := \left\{ (s', \theta') \in \mathbb{R}^+ \times S^n : |s' - s| < r \text{ and } d_{g_{\mathrm{std}}}(\theta, \theta') < \frac{r}{f(s')} \right\}, \tag{II.57}$$

for all $r \geqslant 0$. By the co-area formula again,

$$\begin{aligned} \mathrm{Vol}_g(A_x^{\frac{1}{4}}) &= \int_{s-\frac{1}{4}}^{s+\frac{1}{4}} f(s')^n \, \mathrm{Vol}_{g_{\mathrm{std}}} \left(B_{g_{\mathrm{std}}} \left(\theta, \frac{1}{4f(s')} \right) \right) ds' \\ &= \int_{s-\frac{1}{4}}^{s+\frac{1}{4}} f(s')^n \int_0^{\min\left\{\frac{1}{4f(s')}, \pi\right\}} \omega_{n-1} \sin^{n-1}(t) dt ds'. \end{aligned}$$

If $f(s') < 1/2\pi$, then

$$\int_0^{\min\left\{\frac{1}{4f(s')},\pi\right\}} \omega_{n-1}\sin^{n-1}(t)dt \geqslant \frac{\omega_n}{2}. \tag{II.58}$$

Otherwise, we have

$$\begin{aligned}\int_0^{\min\left\{\frac{1}{4f(s')},\pi\right\}} \omega_{n-1}\sin^{n-1}(t)dt &\geqslant \omega_{n-1}\int_0^{\frac{1}{4f(s')}}\left(t\cos\frac{1}{4f(s')}\right)^{n-1}dt \\ &= \frac{\omega_{n-1}}{n}\left[\cos\left(\frac{1}{4f(s')}\right)\right]^{n-1}\left(\frac{1}{4f(s')}\right)^{n},\end{aligned} \tag{II.59}$$

combining with $f(s') \geqslant \frac{\delta}{4}$, we get

$$\int_0^{\min\left\{\frac{1}{4f(s')},\pi\right\}} \omega_{n-1}\sin^{n-1}(t)dt \geqslant \frac{\omega_{n-1}\cos^{n-1}(1/\delta)}{n4^n}f(s')^{-n}. \tag{II.60}$$

Finally, (II.58) and (II.60) imply

$$\begin{aligned}\mathrm{Vol}_g(A_x^{\frac{1}{4}}) &= \int_{s-\frac{1}{4}}^{s+\frac{1}{4}} f(s')^n \int_0^{\min\left\{\frac{1}{4f(s')},\pi\right\}} \omega_{n-1}\sin^{n-1}(t)dtds' \\ &\geqslant \int_{s-\frac{1}{4}}^{s+\frac{1}{4}} f(s')^n \min\left\{\frac{\omega_n}{2}, \frac{\omega_{n-1}\cos^{n-1}(1/\delta)}{n4^n}f(s')^{-n}\right\}ds' \\ &\geqslant \int_{s-\frac{1}{4}}^{s+\frac{1}{4}} \min\left\{\frac{\omega_n\delta^n}{2\cdot 4^n}, \frac{\omega_{n-1}\cos^{n-1}(1/\delta)}{n4^n}\right\}ds' \\ &= \frac{1}{2}\min\left\{\frac{\omega_n\delta^n}{2\cdot 4^n}, \frac{\omega_{n-1}\cos^{n-1}(1/\delta)}{n4^n}\right\} =: v_2.\end{aligned} \tag{II.61}$$

Taking $v := \min\{v_1, v_2\}$, only depends on δ, n, we have shown that $\mathrm{Vol}_g(B_g(x,1)) \geqslant v_0$ for all $x \in \mathbb{R}^n$, which confirms the assertion. $\square$

We now have a sufficient condition for weakly volume non-collapsed, so we lack whether we have $\mathrm{Rm}(g) + \alpha_0\mathcal{I} \in \mathcal{C}_{\mathrm{PIC1}}$. We adopt the same notations as in [AK04, Section 2] to derive the curvature of metric g. Let K and L be the sectional curvatures of the 2-

plane perpendicular to and tangent to the sphere $\{x\} \times S^n$ respectively. Then

$$K = -\frac{f''(s)}{f(s)} \quad \text{and} \quad L = \frac{1 - f'(s)^2}{f(s)^2}. \tag{II.62}$$

Also, the Ricci curvature and scalar curvature are

$$\mathrm{Ric}(g) = \frac{-nf''(s)}{f(s)} ds^2 + \left[-f(s)f''(s) + (n-1)(1 - f'(s)^2)\right] g_{\mathrm{std}}, \tag{II.63}$$

and

$$\mathrm{scal}(g) = n \left(-2\frac{f''(s)}{f(s)} + (n-1)\frac{1 - f'(s)^2}{f(s)^2} \right). \tag{II.64}$$

Note that $(\mathbb{R}^{n+1}, g)$ is locally conformally flat so that the curvature operator can be written by

$$\mathrm{Rm}(g) = W(g) + \frac{1}{n-1}\left(\mathrm{Ric}(g) - \frac{\mathrm{scal}(g)}{2n} g\right) \owedge g = \frac{1}{n-1}\left(\mathrm{Ric}(g) - \frac{\mathrm{scal}(g)}{2n} g\right) \owedge g, \tag{II.65}$$

where $\owedge$ is the Kulkarni–Nomizu product. Since $\mathcal{I} = g \owedge g$, we get

$$\mathrm{Rm}(g) + \mathcal{I} = \frac{1}{n-1}\left\{\left[\mathrm{Ric}(g) - \left(\frac{\mathrm{scal}(g)}{2n} - (n-1)\right) g\right] \owedge g\right\} =: A_g \owedge g. \tag{II.66}$$

By the definition of the Kulkarni–Nomizu product, $\mathrm{Rm}(g) + \mathcal{I} \in \mathcal{C}_{\mathrm{PIC1}}$ is equivalent to

$$A_g(e_1, e_1) + A_g(e_2, e_2) + 2A_g(e_3, e_3) \geqslant 0, \tag{II.67}$$

for all orthonormal three g-frames $\{e_1, e_2, e_3\}$. By a direct computation, we get

$$A_g = \left(1 - \frac{f''(s)}{f(s)} - \frac{1 - f'(s)^2}{2f(s)^2}\right) ds^2 + \left(\frac{1 - f'(s)^2}{2} + f(s)^2\right) g_{\mathrm{std}}. \tag{II.68}$$

So one can easily check that the following condition is equivalent to $\mathrm{Rm}(g) + \mathcal{I} \in \mathcal{C}_{\mathrm{PIC1}}$.

$$\begin{cases} \frac{1-f'(s)^2}{2f(s)^2} + 1 \geqslant 0 \quad \text{for all } s > 0, \\ \\ 2 - \frac{f''(s)}{f(s)} \geqslant 0 \quad \text{for all } s > 0. \end{cases} \tag{II.69}$$

We focus on the first condition. It implies $f_s \leqslant \sqrt{1+2f^2}$ for all $s \geqslant 0$. This implies $f(s) \leqslant \sinh(\sqrt{2}s)/\sqrt{2}$ and $f_s(s) \leqslant \cosh(\sqrt{2}s)$ for all $s \geqslant 0$. Therefore, the lower curvature bounds restrict the warped function's growth. In summary, as a special case of [Lai19, Corollary 1.2.], the following result holds:

Example 1. Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a smooth function satisfying (II.53) and (II.69). Suppose that $f(s) = \mathrm{O}\left(e^{\sqrt{2}s}\right)$ as $s \to \infty$ for some $\varepsilon > 0$ hold. Then for any $n \geqslant 4$, there is a complete and rotationally symmetric Ricci flow $(\mathbb{R}^{n+1}, g(t))_{t \in [0,T]}$ with initial metric $g(0) = ds^2 + f(s)^2 g_{\mathrm{std}}$.

The above discussion expounds we shouldn't require the Ricci curvature bounded from below at the initial time if we allow the growth of warped function $f(s, 0)$ faster than the exponential growth. However, in the assumption of Theorem 3, one could make the growth of f as large as we want.

II.5 Proof of Theorem 3

II.5.1 Pseudolocality theorem

The first key observation is the pseudolocality theorem for rotationally symmetric Ricci flow, which follows from the non-collapsing property. The validity of this theorem is supported by the generalized Hamilton-Ivey estimate in this context.

Lemma 5 (Pseudolocality theorem)**.** Suppose that $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ is a (connected) Ricci flow so that for some $x_0 \in \mathcal{M}^n$,

1. $B_t(x_0, 2) \Subset \mathcal{M}^n$ for $t \in [0, T]$;
2. $|\mathrm{W}(g(x,t))| \leqslant \frac{c}{t}$ for all $x \in B_t(x_0, 2)$ and $t \in [0, T]$, for some $c \geqslant 0$;
3. $\frac{\mathrm{Vol}_t(x,r)}{r^n} \geqslant v_1 > 0$ for all $t \in [0, T]$ and $x \in B_t(x_0, 1)$ and $r \in (0, 1)$.

Then there exists two constants $a = a(v_1, c, n) > 0$ and $\hat{T} = \hat{T}(v_1, c, n)$ so that for all $t \in (0, \min\{T, \hat{T}\}]$ we have

$$|\mathrm{Rm}(g(x_0, t))| \leqslant \frac{a}{t}. \tag{II.70}$$

Proof of Lemma 5. We argue by contradiction. Suppose the conclusion is false. Then, there exists $v_0 > 0$ and $c \geqslant 0$ such that for any $a_i \to \infty$ and $T_i \to 0$, there exists a sequence of Ricci flow $(\mathcal{M}_i^n, g_i(t))_{t\in[0,T_i]}$ and $x_i \in \mathcal{M}_i^n$ satisfying the assumptions, but the assertion fails in arbitrary small time. By the smoothness of Ricci flow, we may find $t_i \in (0, T_i]$ so that

1. $B_{g_i(t)}(x_i, 2) \Subset \mathcal{M}_i^n$ for $t \in [0, t_i]$;
2. $|\mathrm{W}(g_i(x,t))| \leqslant \frac{c}{t}$ for all $t \in [0, t_i]$ and $x \in B_{g_i(t)}(x_i, 2)$;
3. $\frac{\mathrm{Vol}_t(x,r)}{r^n} \geqslant v_1 > 0$ for all $t \in [0, t_i]$, $x \in B_t(x_i, 1)$ and $r \in (0, 1]$;
4. $|\mathrm{Rm}(g_i(x_i, t))| < a_i/t$ for all $t \in (0, t_i)$;
5. $|\mathrm{Rm}(g_i(x_i, t_i))| = a_i/t_i$.

We may assume $a_i t_i \to 0$. By 4 and $a_i t_i \to 0$, [ST22, Lemma 5.1.] implies that for i large enough, there is a constant $\beta = \beta(n) > 0$, $\tilde{t}_i \in (0, t_i]$, and $\tilde{x}_i \in B_{g_i(\tilde{t}_i)}(x_i, \frac{3}{4} - \frac{1}{2}\beta\sqrt{a_i \tilde{t}_i})$

so that

$$|\mathrm{Rm}(g_i(x,t))| \leqslant 4|\mathrm{Rm}(g_i(\tilde{x}_i,\tilde{t}_i))| =: 4Q_i, \tag{II.71}$$

whenever $d_{g_i(\tilde{t}_i)}(x,\tilde{x}_i) < \frac{1}{8}\beta a_i Q_i^{-\frac{1}{2}}$ and $\tilde{t}_i - \frac{1}{8}a_i Q_i^{-1} \leqslant t \leqslant \tilde{t}_i$ where $\tilde{t}_i Q_i \geqslant a_i \to \infty$. Now, consider the parabolic scaling centered at $(\tilde{x}_i,\tilde{t}_i)$, that is, define $\tilde{g}_i(t) := Q_i g_i(Q_i^{-1}t + \tilde{t}_i)$ for all $t \in [-\frac{1}{8}a_i, 0]$. In the proof of [ST22, Lemma 5.1.], the parabolic domain $B_{\tilde{g}_i(0)}(\tilde{x}_i, \frac{1}{8}\beta a_i) \times [-\frac{1}{8}a_i, 0]$ is contained in the region that assumption holds. Then we get

1. $|\mathrm{Rm}(\tilde{g}_i(\tilde{x}_i, 0))| = 1$;
2. $|\mathrm{Rm}(\tilde{g}_i(x,t))| \leqslant 4$ for all $(x,t) \in B_{\tilde{g}_i(0)}(\tilde{x}_i, \frac{1}{8}\beta a_i) \times [-\frac{1}{8}a_i, 0]$;
3. $\frac{\mathrm{Vol}_{\tilde{g}_i(t)}(x,r)}{r^n} \geqslant v_1 > 0$ for all $(x,t) \in B_{\tilde{g}_i(0)}(\tilde{x}_i, \frac{1}{8}\beta a_i) \times [-\frac{1}{8}a_i, 0]$ and $r \in (0, \sqrt{Q_i}]$;
4. $|\mathrm{W}(\tilde{g}_i(x,t))| \leqslant c(t + Q_i\tilde{t}_i)^{-1}$ for all $(x,t) \in B_{\tilde{g}_i(0)}(\tilde{x}_i, \frac{1}{8}\beta a_i) \times [-\frac{1}{8}a_i, 0]$.

Now, by Hamilton's compactness theorem and Cheeger-Gromov-Taylor classical result, after passing to a subsequence, there is a geometric limit

$$(\mathcal{M}_i^n, \tilde{g}_i(t), (\tilde{x}_i, 0))_{t\in[-\frac{1}{8}a_i,0]} \to (\mathcal{N}^n, g_\infty(t), (x_\infty, 0))_{t\leqslant 0}, \tag{II.72}$$

converge in the sense of Hamilton-Cheeger-Gromov. Also, $(\mathcal{N}^n, g_\infty(t))_{t\leqslant 0}$ is a non-flat, with bounded curvature (less than 4), LCF on each time-slice, $\mathrm{AVR}(g_\infty(t)) \geqslant v_0$ for all $t \leqslant 0$, ancient solution of complete Ricci flow. By the generalized Hamilton-Ivey estimate [Zha15, Theorem 1.1.], $(\mathcal{N}^n, g_\infty(t))$ has nonnegative curvature operator for all $t \leqslant 0$. But this contradicts to [BCRW19, Lemma 4.2.]. $\square$

II.5.2 A priori estimate for volume ratio

Now, we aim to prove that the condition 3 is preserved under the curvature decayed $|\mathrm{Rm}(g(t))| \leqslant c/t$. We refer to the following lemma about the lower bound of scalar curvature being preserved under the curvature decayed assumption.

Lemma 6. [ST22, Lemma 8.1.] For any constants $c, K > 0$, $\gamma \in (0,1)$ and $n \in \mathbb{N}$, there exists a constant $\hat{T}(c, K, \gamma, n) > 0$ satisfying the following. Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a Ricci flow. Suppose that $B_{g(t)}(x_0, 1) \Subset \mathcal{M}^n$ for some $x_0 \in \mathcal{M}^n$ for all $t \in [0,T]$ and

1. $\mathrm{scal}(g(0)) \geqslant -K$ on $B_{g(0)}(x_0, 1)$.
2. $\mathrm{Ric}(g(t)) \leqslant \frac{c}{t}$ on $B_{g(t)}(x_0, \sqrt{t})$ for all $t \in (0,T]$.

Then

$$\mathrm{scal}(g(t)) \geqslant -2K, \tag{II.73}$$

on $B_{g(0)}(x_0, 1-\gamma)$ for all $t \in [0, \min\{T, \hat{T}\}]$.

We can prove a priori estimate for volume ratio by combining Lemma 6 and Proposition 4.

Lemma 7 (Noncollapsing Property)**.** Let $(\mathbb{R}^{n+1}, g(t) := ds(t)^2 + f(s(t),t)^2 g_{\text{std}})_{t\in[0,T]}$ be a complete and rotationally symmetric Ricci flow with bounded curvature. Suppose that $f_s(s,0) \geqslant 0$ holds for all $s > 0$. Then $f_s \geqslant 0$ holds on $\mathbb{R}^{n+1} \times [0,T]$. Furthermore, assume further that there is a constant $c > 0$ such that $|\mathrm{Rm}(g(t))| \leqslant c/t$ on $\mathbb{R}^{n+1} \times [0,T]$. Then there exists two constants $v = v(n, g(0)|_{B_{g(0)}(o,4)}) > 0$ and $\hat{T} = \hat{T}(n, c, g(0)|_{B_{g(0)}(o,4)})$ such that

$$\frac{\mathrm{Vol}_{g(t)} B_{g(t)}(x,r)}{r^{n+1}} \geqslant v, \tag{II.74}$$

holds on $\mathbb{R}^{n+1} \times [0, \min\{T, \hat{T}\}]$ and for all $r \in (0, 1]$.

Remark 3. It's crucial that v is independent of c in the proof of Theorem 3.

Proof of Lemma 7. The first assertion follows from [Di 21, Lemma 3.1.]. It suffices to confirm the last assertion. Let $K > 0$ be a constant such that $\operatorname{scal}(g(0)) \geqslant -K$ on $B_{g(0)}(o, 4)$. By Lemma 6, $\operatorname{scal}(g(t)) \geqslant -2K$ on $B_0(o, 3) \times [0, \min\{T, T_1\}]$ for some $T_1 = T_1(n, c, K) > 0$. Note that the evolution equation of f_s is

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f_s = \frac{\operatorname{scal}(g(t))}{n} f_s, \tag{II.75}$$

we refer it to [Di 21, Lemma 3.1.]. Let $\delta := \min\{f_s(s, 0) : s \in [0, 3]\} \geqslant 0$ be a constant. If $\delta = 0$, then we may rescale the metric so that $\delta > 0$ becomes positive since $f_s(0) = 1$ and our dependency is about $g(0)|_{|s| \leqslant 4}$. Thus $\delta > 0$. Define $\phi(s(t), t) := \delta(1 - 2Kt) - f_{s(t)}(s(t), t)$, then (II.75) implies

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \phi &= \frac{\operatorname{scal}(g(t))}{n} \phi - \frac{\delta(1 - 2Kt) \operatorname{scal}(g(t))}{n} - 2\delta K \\ &\leqslant \frac{\operatorname{scal}(g(t))}{n} \phi + 2\delta K \left(\frac{1 - 2Kt}{n} - 1\right) \\ &\leqslant \frac{\operatorname{scal}(g(t))}{n} \phi, \end{aligned} \tag{II.76}$$

on $B_0(o, 3) \times [0, \min\{T, T_1, \frac{1}{4K}\}]$. Note that $\phi < \frac{\delta}{2}$ and $\phi(0) \leqslant 0$ holds on $B_0(o, 3) \times [0, \min\{T, T_1, \frac{1}{4K}\}]$, by Theorem 5, we obtain $\phi \leqslant \frac{\delta}{4}$ on $B_0(o, 2) \times [0, \min\{T, T_1, \frac{1}{4K}, T_2\}]$ for some constant $T_2(n, c, \delta) > 0$. Therefore, we get $f_s \geqslant \frac{\delta}{4}$ on $B_0(o, 2) \times [0, \min\{T, \hat{T}\}]$ for some $\hat{T}(n, c, K, \delta) > 0$. Combining this with $f_s > 0$, we get $f(s(t), t) \geqslant \frac{\delta}{4} s(t) \chi_{[0,1]}(s(t)) + \frac{\delta}{4} \chi_{(1,\infty)}(s(t))$ on $\mathbb{R}^{n+1} \times [0, \min\{T, \hat{T}\}]$. Thus, we complete the proof by applying Proposition 4 on $\mathbb{R}^{n+1} \times [0, \min\{T, \hat{T}\}]$.

□

Recall Shi's existence theorem of Ricci flows under the bounded curvature assumption.

Lemma 8. [Shi89, Theorem 1.1.] There exists two dimension constants $\Lambda(n), a(n) > 0$ such that, for any complete non-compact manifold $(\mathcal{M}^n, g)$ with curvature bound $|\mathrm{Rm}| \leqslant k_0$, there exists a complete Ricci flow $(\mathcal{M}^n, g(t))_{t\in[0,\frac{\Lambda}{k_0}]}$ starting from g with

$$|\mathrm{Rm}(g(t))|_{g(t)} \leqslant ak_0 \leqslant \frac{a\Lambda}{t}, \tag{II.77}$$

on $\mathcal{M}^n \times [0, \frac{\Lambda}{k_0}]$.

II.5.3 Complete the proof of Theorem 3

Proof of Theorem 3. For any $k > 10$, consider the rotationally symmetric metric $g_k = ds^2 + f_k(s)^2 g_{\text{std}}$ such that

$$\begin{cases} f_k(s) = f(s), \text{ for all } s \in [0, k]; \\ f_k \text{ is linear outside } [0, k+1]; \\ \frac{\partial}{\partial s} f_k(s) > 0, \text{ on } \mathbb{R}^{n+1}. \end{cases} \tag{II.78}$$

Then $(\mathbb{R}^{n+1}, g_k)$ is a complete and rotationally symmetric with bounded curvature. By Lemma 8 and the uniqueness theorem of Ricci flow with bounded curvature [CZ06, Theorem 1.1.], there is a complete and rotationally symmetric Ricci flow $(\mathbb{R}^{n+1}, g_k(t))_{t\in[0,T_k]}$ starting from g_k with bounded curvature and satisfying $|\mathrm{Rm}(g_k(t))| \leqslant c/t$, where $c = a\Lambda$ is the dimensional constant in Lemma 8. Note that T_k could tend to zero a priori, with no uniform curvature bound in spacetime. However, we claim they can be extended to a uniform time $\hat{T}$ and with instantaneously uniform curvature bound.

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Claim 3. There exist two constants $\hat{T}(n, g|_{|s|\leqslant 4}), \hat{\Lambda}(n, g|_{|s|\leqslant 4}) > 0$ such that $(\mathbb{R}^{n+1}, g_k(t))$ could be extended until $\hat{T}$ and satisfy

$$|\mathrm{Rm}(g_k(t))| \leqslant \frac{\hat{\Lambda}}{t}, \tag{II.79}$$

on $\mathbb{R}^{n+1} \times (0, \hat{T}]$.

As long as the claim holds, combined with Hamilton's compactness theorem and $g_k(0)$ converges to g locally smoothly, there is a Ricci flow $(\mathbb{R}^{n+1}, g(t))_{t\in[0,\hat{T}]}$ starting from g such that there is a sequence $k_i \to \infty$ so that $(\mathbb{R}^{n+1}, g_{k_j}(t))_{t\in[0,\hat{T}]}$ converges to $(\mathbb{R}^{n+1}, g(t))_{t\in[0,\hat{T}]}$ in the Hamilton-Cheeger-Gromov sense. By virtue of the Claim, $|\mathrm{Rm}(g(t))| \leqslant \hat{\Lambda}/t$ holds on $\mathbb{R}^{n+1} \times (0, \hat{T}]$. Then the distance distortion lemma (Lemma 3) implies $(\mathbb{R}^{n+1}, g(t))_{t\in[0,\hat{T}]}$ is a complete Ricci flow. Note that $g(t)$ is rotationally symmetric if

$$\frac{\partial}{\partial\theta^i} \text{ is a Killing vector field w.r.t. } g(t), \tag{II.80}$$

for all $i = 1, \cdots, n$, where $\frac{\partial}{\partial\theta^i}$ is the rotation vector on $\mathbb{R}^{n+1}$. By virtue of Hamilton-Cheeger-Gromov convergence, $g(t)$ is rotationally symmetric for all $t \in [0, \hat{T}]$. It remains to confirm the claim. From now on, we omit the index k for convenience and set $t_1 := T_k$. Since $(\mathbb{R}^{n+1}, g(t))_{t\in[0,t_1]}$ has bounded curvature and $g(0) = g$ if $|s| \leqslant k$, by Lemma 7, there are two constants $T_1(n, g|_{|s|\leqslant 4}), v(n, g|_{|s|\leqslant 4}) > 0$ such that

$$f_s > 0 \text{ and } \frac{\mathrm{Vol}_{g(t)} B_{g(t)}(x, r)}{r^{n+1}} \geqslant v, \tag{II.81}$$

on $\mathbb{R}^{n+1} \times [0, \min\{t_1, \hat{T}\}]$ and for all $r \in (0, 1]$. Note that $\mathrm{W}(g(t)) = 0$ for all $t \in [0, t_1]$, by Lemma 5 (Setting $c = 0$), there are two constants $T_2(n,0,v), \hat{\Lambda}(n,0,v) > 0$ such that

$$|\mathrm{Rm}(g(t))| \leqslant \frac{\hat{\Lambda}}{t}, \tag{II.82}$$

on $\mathbb{R}^{n+1} \times (0, \min\{t_1, T_1, T_2\}]$. Choosing $\hat{T} := \min\{T_1, T_2, T_3\} > 0$ for some T_3 to be determined later. If $t_1 \geqslant \hat{T}$, then we done. Otherwise, we inductively extended the flow in the following way. Set $t_2 := t_1$ and $i \geqslant 2$. Suppose that $(\mathbb{R}^{n+1}, \tilde{g}_i(t))_{t\in[0,t_i]}$ is a complete and rotationally symmetric Ricci flow with the following

1. $\tilde{g}_i(t)$ has bounded curvature and satisfies (II.82) on $\mathbb{R}^{n+1} \times (0, t_i]$.
2. $t_{i-1} \leqslant t_i < \hat{T}$ and $\tilde{g}_i(t)$ is an extension of $\tilde{g}_{i-1}(t)$.

Now, at $t = t_i$, $(\mathbb{R}^{n+1}, \tilde{g}_i(t_i))$ is a complete and rotationally symmetric manifold with curvature bound $\hat{\Lambda}/t_i$, by Lemma 8 and uniqueness theorem again, there is a complete and rotationally symmetric Ricci flow $(\mathbb{R}^{n+1}, h(t))_{t\in[0,\frac{\Lambda}{\hat{\Lambda}}t_i]}$ starting from $\tilde{g}_i(t_i)$ with curvature bound $a\frac{\hat{\Lambda}}{t_i}$, where a is the constant in Lemma 8. Extend $\tilde{g}_i$ by h, we construct an extension $\tilde{g}_{i+1}(t)$ on $[0, t_{i+1} := (1 + \frac{\Lambda}{\hat{\Lambda}})t_i]$ with curvature bound

$$|\mathrm{Rm}(\tilde{g}_{i+1}(t))| \leqslant \frac{\max\{\hat{\Lambda}, a(\hat{\Lambda} + \Lambda)\}}{t}, \tag{II.83}$$

on $\mathbb{R}^{n+1} \times (0, t_{i+1}]$. By Lemma 7, there is a constant $T_3(n, \max\{\hat{\Lambda}, a(\hat{\Lambda} + \Lambda)\}, g|_{|s|\leqslant 4}) = T_3(n, g|_{|s|\leqslant 4}) > 0$ such that (II.81) holds for $\tilde{g}_{i+1}(t)$ on $\mathbb{R}^{n+1} \times [0, \min\{t_{i+1}, T_3\}]$. By Lemma 5 (Again, setting $c = 0$.), (II.82) holds for $\tilde{g}_{i+1}(t)$ on $\mathbb{R}^{n+1} \times (0, \min\{t_{i+1}, \hat{T}\}]$. If $t_{i+1} \geqslant \hat{T}$, then $(\mathbb{R}^{n+1}, \tilde{g}_{i+1}(t))_{t\in[0,\hat{T}]}$ satisfies the assertion in the Claim. Otherwise, $(\mathbb{R}^{n+1}, \tilde{g}_{i+1}(t))_{t\in[0,t_{i+1}]}$ satisfies the induction hypothesis 1 and 2. If this process never stop, then $t_i = (1 + \frac{\Lambda}{\hat{\Lambda}})^{i-2}t_2 < \hat{T}$ for all $i \geqslant 2$, which encounters a contradiction. Therefore, there exists an extension satisfying the claim and we complete the proof. □

Remark 4. We should be careful that, in our construction, f_s could be zero at some point. A priori, we only have $f_s \geqslant 0$ on $\mathbb{R}^{n+1} \times [0, \hat{T}]$. Nevertheless, if $f_s = 0$ at some point (p, t_0) with $t_0 > 0$, then the strong maximal principle would imply $f_s \equiv 0$ on $\mathbb{R}^{n+1} \times [0, t_0]$, which is impossible. Therefore, the warped function f would instantaneously satisfy $f_s > 0$.

II.6 Entropy on complete and Rotationally symmetric Ricci flow

In this section, we study the Perelman-$\mathcal{W}$ entropy on the open manifold $(\mathcal{M}^n, g)$ with the following condition:

1. $n \geqslant 4$.
2. $\mathcal{M}^n$ is simply connected.
3. $(\mathcal{M}^n, g)$ is locally conformally flat.
4. The scalar curvature is bounded below and above, i.e. $\mathrm{scal}_{\mathrm{min}} \leqslant \mathrm{scal}$ on $\mathcal{M}^n$.

One basic example is the complete and rotationally symmetric metric on $\mathbb{R}^n$ or $\mathbb{R} \times S^{n-1}$ ($n \geqslant 3$) with scalar curvature bounded from below. Note that both situations occur on the complete and rotationally symmetric Ricci flow by [Che09, Corollary 2.3.]. Under these assumptions, we can show that the following inequality holds on $(\mathcal{M}^n, g)$.

Lemma 9. Suppose that $(\mathcal{M}^n, g)$ is a simply connected and locally conformally flat manifold. Then we have

$$Q(S^n)\left(\int_{\mathcal{M}^n} \phi^{\frac{2n}{n-2}} d\mathrm{vol}_g\right)^{\frac{n-2}{2n}} \leqslant \left(\int_{\mathcal{M}^n} |\nabla\phi|^2 + \frac{n-2}{4(n-1)}\,\mathrm{scal}(g)\phi^2 d\mathrm{vol}_g\right)^{\frac{1}{2}}, \quad \text{(II.84)}$$

for all $\phi \in \mathcal{C}_c^\infty(\mathcal{M}^n)$, where $Q(S^n)$ is the Yamabe constant of S^n with standard round metric.

Proof of Lemma 9. It suffices to show that the Yamabe constant $Q(\mathcal{M}^n)$ of $(\mathcal{M}^n, g)$ equals to $Q(S^n)$. Since $\mathcal{M}^n$ is simply connected, there is a conformal immersion $\pi : \mathcal{M}^n \to S^n$. Hence the assertion holds by [YS88, Proposition 2.2.]. ☐

This type of Sobolev inequality leads to a lower bound on Perelman's $\mathcal{W}$-functional. Recall that on an oriented Riemannian manifold $(\mathcal{M}^n, g)$, the Perelman's $\mathcal{W}$-functional is defined by

$$\mathcal{W}(g, f, \tau) := (4\pi\tau)^{-\frac{n}{2}} \int_{\mathcal{M}^n} [\tau(4|\nabla f|^2 + \mathrm{scal}(g) f^2) - 2f^2 \log f - nf^2] d\mathrm{vol}_g, \tag{II.85}$$

where $\tau > 0$ and $f \in \mathcal{C}_c^\infty(\mathcal{M}^n)$ satisfies

$$(4\pi\tau)^{-\frac{n}{2}} \int_{\mathcal{M}^n} f^2 d\mathrm{vol}_g = 1. \tag{II.86}$$

Note that an easy computation shows that $\mathcal{W}(g, f, \tau) = \mathcal{W}(\tau^{-1} g, f, 1)$. The entropy $\mu(\mathcal{M}^n, g, \tau)$ is defined by the infimum of (II.85) among all $f \in \mathcal{C}_c^\infty(\mathcal{M}^n)$ satisfies (II.86). Then the following proposition ensures that $\mu(\mathcal{M}^n, g, \tau)$ is finite under our assumption.

Proposition 5. Let $(\mathcal{M}^n, g)$ be defined as above. Then there exists a constant $C_2 = C_2(n), C_3 = C_3(n) \in (0, +\infty)$ so that

$$\mathcal{W}(g, f, \tau) \geqslant \frac{2(4\pi\tau)^{1-\frac{n}{2}}}{3\pi} \int_{\mathcal{M}^n} |\nabla f|^2 d\mathrm{vol}_g - C_2 + C_3\, \mathrm{scal}_{\min} \tau, \tag{II.87}$$

for all $\tau > 0$ and $0 < f \in \mathcal{C}_c^\infty(\mathcal{M}^n)$ satisfies (II.86). Moreover, $\mu(\mathcal{M}^n, g, \tau)$ is bounded on any compact set $\tau \in I \subset \mathbb{R}^+$, and the ν-entropy

$$\nu(\mathcal{M}^n, g, \tau_0) := \inf_{\tau \in (0,\tau_0]} \mu(g, \tau) > -\infty, \tag{II.88}$$

is finite for all $\tau_0 > 0$.

Proof of Proposition 5. We follow the proof of [Top06, Lemma 8.1.8.]. After the scaling $g \mapsto \tau^{-1} g$ (Hence $\mathrm{scal}(\tau^{-1}g) \geqslant \mathrm{scal}_{\min} \tau$), it suffices to prove that

$$\begin{aligned}\mathcal{W}\left(g, f, \frac{1}{4\pi}\right) - \frac{2}{3\pi}\int_{\mathcal{M}^n} |\nabla f|^2 d\mathrm{vol}_g &= \int_{\mathcal{M}^n} \frac{1}{3\pi}|\nabla f|^2 + \frac{1}{4\pi}\mathrm{scal}(g) f^2 - 2f^2 \log f - nf^2 d\mathrm{vol}_g \\ &\geqslant -C_2 + C_3 \, \mathrm{scal}_{\min} \tau,\end{aligned} \tag{II.89}$$

for all $0 < f \in \mathcal{C}_c^\infty(\mathcal{M}^n)$ with $\|f\|_{\mathcal{L}^2} = 1$. Since $\log x$ is concave, by Jensen's inequality, we have

$$\frac{4}{n-2}\int_{\mathcal{M}^n} \log f f^2 d\,\mathrm{Vol}_g = \int_{\mathcal{M}^n} \log f^{\frac{4}{n-2}} f^2 d\,\mathrm{Vol}_g \leqslant \log\left(\int_{\mathcal{M}^n} f^{\frac{2n}{n-2}} d\,\mathrm{Vol}_g\right). \tag{II.90}$$

Combining Lemma 9 and (II.90), we get

$$\begin{aligned}\frac{4}{n-2}\int_{\mathcal{M}^n} \log f f^2 d\,\mathrm{Vol}_g &\leqslant \log\left[Q(S^n)^{-1}\left(\int_{\mathcal{M}^n} |\nabla f|^2 + \frac{n-2}{4(n-1)}\mathrm{scal}(g) f^2 d\mathrm{vol}_g\right)^{\frac{1}{2}}\right] \\ &= -\log Q(S^n) + \frac{1}{2}\log\left[\int_{\mathcal{M}^n} |\nabla f|^2 + \frac{n-2}{4(n-1)}\mathrm{scal}(g) f^2 d\mathrm{vol}_g\right]\end{aligned} \tag{II.91}$$

Now we choose $\epsilon := 2/3$,

$$\begin{aligned}\int_{\mathcal{M}^n} \frac{1-\epsilon}{\pi}|\nabla f|^2 + \frac{1}{4\pi}\mathrm{scal}(g) f^2 d\mathrm{vol}_g &= \frac{1}{3\pi}\int_{\mathcal{M}^n} |\nabla f|^2 + \frac{3}{4}\mathrm{scal}(g) f^2 d\mathrm{vol}_g \\ &\geqslant \frac{1}{3\pi}\int_{\mathcal{M}^n} |\nabla f|^2 + \frac{n-2}{4(n-1)}\mathrm{scal}(g) f^2 d\mathrm{vol}_g \\ &\quad + \frac{2n-1}{12(n-1)\pi}\mathrm{scal}_{\min} \tau,\end{aligned} \tag{II.92}$$

where the last inequality follows from $\mathrm{scal}(\tau^{-1}g) \geqslant \tau\,\mathrm{scal}_{\min}$ on $\mathcal{M}^n$. Combining all of the estimates, we obtain

$$\begin{aligned}
&\int_{\mathcal{M}^n} \frac{1}{3\pi}|\nabla f|^2 + \frac{1}{4\pi}\,\mathrm{scal}(g) f^2 - 2f^2 \log f - nf^2 d\mathrm{vol}_g \\
\geqslant& -n + \frac{n-2}{2}\log Q(S^n) + \frac{2n-1}{12(n-1)\pi}\,\mathrm{scal}_{\min}\tau + \frac{1}{3\pi}\int_{\mathcal{M}^n}|\nabla f|^2 + \frac{n-2}{4(n-1)}\,\mathrm{scal}(g) f^2 d\mathrm{vol}_g \\
&- \frac{n-2}{2}\log\int_{\mathcal{M}^n}|\nabla f|^2 + \frac{n-2}{4(n-1)}\,\mathrm{scal}(g) f^2 d\mathrm{vol}_g \\
\geqslant& -C_2 + C_3\,\mathrm{scal}_{\min}\tau,
\end{aligned} \tag{II.93}$$

where the last inequality follows from the function $\frac{t}{3\pi} - \frac{n-2}{2}\log t$ is bounded below for $t > 0$, and this completes the proof. □

Consequently, we found the following property for complete and rotationally symmetric Ricci flow.

Corollary 1. Let $(\mathbb{R}^{n+1}, g(t))_{t\in[0,T]}$ be a complete and rotationally symmetric Ricci flow with $n \geqslant 3$. Then there is a constant $\kappa(n) > 0$ such that $\nu(\mathbb{R}^{n+1}, g(t), t) \geqslant -\kappa$ for all $t \in (0, T]$.

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Chapter III Uniqueness of 1D Ecker-Huisken flow with $\mathcal{L}^1$-decayed

III.1 Introduction

The curve-shortening flow, a gradient flow of the length functional, and one of the simplest geometric flows have been studied for a long time. In contrast to flows in higher dimensions, the existence and behavior of the non-compact curve-shortening flow exhibit favorable properties.

This article will study the property around the 1D Ecker-Huisken flow. the 1D Ecker-Huisken flow is the **one-dimensional graphical curve shortening flow**. Since the flow is graphical, it is equivalent to the following definition if we express the flow as the map $(x, t) \mapsto (x, y(x, t))$.

Definition 3 (Ecker-Huisken Flow)**.** A function $y : \mathbb{R} \times [0, T) \to \mathbb{R}$ is a solution of the 1D Ecker-Huisken flow (or 1D-graphical mean curvature flow) with $\mathcal{L}^p_{\text{loc}}$(or $\mathcal{W}^{1,1}_{\text{loc}}$)-initial data if y is smooth on $\mathbb{R} \times (0, T)$ and satisfying

$$y_t = \frac{y_{xx}}{1 + y_x^2} = (\arctan y_x)_x, \tag{III.1}$$

on $\mathbb{R} \times (0, T)$, and $y(\cdot, t)$ converges to $y(\cdot, 0)$ in the sense of $\mathcal{L}^p_{\text{loc}}$(or $\mathcal{W}^{1,1}_{\text{loc}}$) as $t \to 0$.

Thanks to this viewpoint, it's reasonable to ask the existence of the 1D Ecker-Huisken flow in the rough initial data setting. There are several results about the existence of the 1D Ecker-Huisken flow, we refer the readers to [ST24]. In [Sob24, Theorem 4.3.1], it developed a long-time existence theory for the non-atomic real-valued Radon measure $\nu \in \mathcal{P}(\mathbb{R})$, that is, if we write $\nu = y_0\mathcal{L}^1 + \nu_{\text{sing}}$ by the Lebesgue-Radon-Nikodym decomposition, and suppose that $y_0 \in \mathcal{L}^p_{\text{loc}}(\mathbb{R} \backslash \operatorname{supp} \nu_{\text{sing}})$ for some $p \in [1, \infty)$, then there is a smooth function $y : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ such that y satisfies (III.1) and

$$\begin{aligned} y(\cdot, t)\mathcal{L}^1 &\rightharpoonup \nu, \text{ weakly on } \mathcal{L}^1(\mathbb{R}); \\ y(\cdot, t) &\to y_0, \text{ strongly in } \mathcal{L}^p_{\text{loc}}(\mathbb{R} \backslash \text{supp}(\nu_{\text{sing}})), \end{aligned} \tag{III.2}$$

as $t \to 0$. Since the existence theory has been well-studied, it raises the following nature question.

Question 1. When is the solution unique?

There are some situations which had been explored, for instance, when the initial data is $\mathcal{L}^p_{\text{loc}}(\mathbb{R})$ with $\mathbf{p > 1}$, A. Sobnack proved the uniqueness result in his unpublished Ph. D. thesis [Sob24, Theorem 4.3.1]. Arjun also mentioned that his method is the same as [CK20, Proposition 3.1] but chose the cut-off function more carefully. Their idea is to derive the following $\mathcal{L}^p$ estimate.

Theorem 6. ([Sob24, Proposition 4.3.38], [CK20, Proposition 3.1]) Let $y^1, y^2 \in \mathcal{C}^\infty((-R, R) \times [0, T))$

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be two smooth solutions of the 1D Ecker-Huisken flow. Then for any $0 < r < R$ and $\delta > 0$, there is a cut-off function φ supported in $[-R, R]$ with $\varphi \equiv 1$ on $[-r, r]$ such that for all $p \in [1, \infty)$,

$$\|(y^1(t)-y^2(t))\varphi\|_{\mathcal{L}^p((-R,R))} \leqslant \|(y^1(s)-y^2(s))\varphi\|_{\mathcal{L}^p((-R,R))} + \frac{2\pi(1+\delta)p}{(R-r)^{1-\frac{1}{p}}}(t-s), \qquad \text{(III.3)}$$

for all $0 \leqslant s \leqslant t \leqslant T$.

Roughly speaking, the relation of (III.3) and the uniqueness is plugging $s = 0$ and $R \to \infty$, then the same initial data leads the R.H.S. tending to zero. To generalize this result to the rough initial data, we need the solution with $\mathcal{L}^p_{\text{loc}}$-converging to the initial data. And obviously for $p = 1$, (III.3) lack of the information about R and r. In the case of $\mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R})$-initial data, P. Daskalopoulos and M. Saez proved the uniqueness of the 1D Ecker-Huisken flow in [DS23, Theorem 1.1]. Similar to the above, they investigated the following quantitative

$$\frac{d}{dt}\int_{-2R}^{2R}(y^1_x(t)-y^2_x(t))_+\varphi\, d\mathcal{H}^1 \leqslant \frac{C_\delta}{R^{1+\delta}}\left(\int_{-2R}^{2R}(y^1_x(t)-y^2_x(t))_+\varphi\, d\mathcal{H}^1\right)^{\delta}, \qquad \text{(III.4)}$$

for all $t \in [0, T]$ and $\delta \in (0, 1)$ and an appropriate choice of $\varphi \in \mathcal{C}^\infty_c((-2R, 2R); [0, 1])$ with $\varphi = 1$ on $[-R, R]$. Although they only considered $\mathcal{C}^{0,1}_{\text{loc}}(\mathbb{R})$-initial data, their argument holds for the initial data with weaker regularity.

Let's outline the structure of this chapter. In section III.2, we explore a uniqueness result for a rapidly decayed $\mathcal{L}^1$-initial data **with the assumption that is $\mathcal{C}^0_{\text{loc}}$-convergence near infinity**. In section III.3, we prove some quantitative estimate similar to (III.3) only in terms of $\mathcal{L}^1$ information, but with some additional condition on y^i.

III.2 Uniqueness of rapidly decayed $\mathcal{L}^1$-initial data

The goal in this section is to prove the following:

Theorem 7. Suppose that $y^i : [0,T) \to \mathbb{R}$ are two solutions of the 1D Ecker-Huisken flow with same initial $\mathcal{L}^1$-data y_0. If $y^i(\cdot,t) \xrightarrow{\mathcal{C}^0_{\text{loc}}(K^c)} y_0$ as $t \to 0$ for some compact set $K \subset \mathbb{R}$ and there is a non-increasing function $f \in \mathcal{L}^1([1,\infty))$ such that $y_0(x) = \mathrm{O}(f(|x|))$ holds as $|x| \to \infty$, then $y^1 \equiv y^2$ on $\mathbb{R} \times [0,T)$.

Remark 5. 1. For example, we can choose $f(t) := t^{-(1+\epsilon)}$ for any $\epsilon > 0$ or $f(t) = 1/(t \log t)$.

2. A priori, T could be $+\infty$. But it suffices to prove this result in the case $T < \infty$ for all T. So from now on, we may assume $T < \infty$.

3. We may replace the condition of $\mathcal{C}^0_{\text{loc}}(K^c)$ with a requirement that the solution satisfies a certain maximum principle or exhibits specific growth behavior in spacetime. Moreover, we note that the solution constructed in [ST24] satisfies the decay assumption, as shown in Lemma 10.

Before we go to the proof, we first study the benefits of this kind of initial data. The first lemma gives some partial answers.

Lemma 10 (Asymptotic behavior on space-time)**.** Let $y : \mathbb{R} \times [0,T)$ be a solution of the 1D Ecker-Huisken flow with an $\mathcal{L}^1$-initial data y_0. Assume that $y^i(\cdot,t) \xrightarrow{\mathcal{C}^0_{\text{loc}}(K^c)} y_0$ as $t \to 0$ for some compact set $K \subset \mathbb{R}$ and there is a non-increasing function $f \in \mathcal{L}^1([1,\infty))$ such that $y_0(x) = \mathrm{O}(f(|x|))$ holds as $|x| \to \infty$. Then there are some constants $C, C', C'' > 0$

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(depends on T) such that

$$|y(x,t)| \leqslant Cf\left(\frac{|x|}{2}\right) + C' \exp(-C''x^2) \tag{III.5}$$

on $(\mathbb{R}\backslash K') \times [0,T)$ for some compact subset $K' \subset \mathbb{R}$

We separate the proof of this lemma in two steps. The first is to use the avoidance principle to get an upper bound $|x|^{-1}$, the assumption of locally uniform convergence near infinity ensures the validity of the avoidance principle. It's worth noting that the global avoidance principle only holds for **uniformly proper curve shortening flow**, which is not in our ansatz since we only require the $\mathcal{L}^1$ convergence in the middle part. Instead, we apply the local avoidance principle (in other words, the maximal principle for **graphical curve shortening flows**) on a closed interval, which means that we need to deal with the behavior at the endpoints. The idea is to replace the Great Circle with the Grim Reaper for comparison. The second is to improve the height estimate via the function $W : \mathbb{R}^+ \to \mathbb{R}^+$ (constructed in [ST24, Section 4]).

Proof of Lemma 10. By symmetry, we only prove the case that $x > 0$.

Step 1, $|x|^{-1}$-estimate for y.

Say $|y_0(x)| \leqslant Cf(x)$ and $y(\cdot,t) \overset{t\searrow 0}{\to} y_0$ locally uniformly when $|x| > R_1$. For $R > R_1 > 1$, consider an upward grim reaper S centered at $(2R, Cf(R))$ with width $2R$. More precisely, S can be parametrized as

$$(x,t) \mapsto \left(x-2R, Cf(R) + \frac{t}{R} - R\log\left(\cos\frac{x-2R}{R}\right) =: \gamma(x,t)\right), \tag{III.6}$$

on $(R,3R) \times [0,\infty)$. The assumption causes $\gamma(x,0) > y_0(x)$ on $[R,3R]$. By the uniform convergence of y on $[R,3R] \times [0,T)$, there is a region $\Omega := (R+\delta, 3R-\delta) \times (0,T)$ such

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that $\gamma > y$ on the parabolic boundary of Ω. Applying the avoidance principle on Ω, this forces that $\gamma > y$ on Ω. By the symmetric again, these imply that

$$|y(2R,t)| < \gamma(2R,t) = \frac{t}{R} + Cf(R). \tag{III.7}$$

From this estimate, we deduce that $|y(x,t)| \leqslant (2t)/x + Cf(\frac{|x|}{2})$ for all $|x| > R_1$ and $t \in [0,T]$.

Step 2. Improve the estimate.

Fix $R > R_1$ and $\delta > 0$. Let $W : (0,\infty) \to (0,\infty)$ be the smooth function with the property that $t \mapsto (x, t^{\frac{1}{2}} W(t^{-\frac{1}{2}} x))$ is a self-similar solution of the 1D Ecker-Huisken flow on $\mathbb{R}^+$. We quote some results in [ST24, Section 4]. First of all, W satisfies

$$-\arctan W'(x) = \frac{1}{2} x W(x) + \int_x^\infty W(s)ds. \tag{III.8}$$

Furthermore, thanks for the [ST24, Lemma 4.1.],

$$W(x) \leqslant Ce^{-x^2}, \tag{III.9}$$

for $x > 1$. Consider the map

$$(x,t) \mapsto \left(x + R, (t+\delta)^{\frac{1}{2}} W((t+\delta)^{-\frac{1}{2}} x) + Cf(R) =: y_\delta(x,t) \right), \tag{III.10}$$

for all $(x,t) \in (0,\infty) \times (0,\infty)$. Then $y_\delta(x,0) > y_0(x+R)$ on $\mathbb{R}^+$. Consider the parabolic neighborhood $\Omega_T := (0,K) \times [0,T)$ for some large $K > 0$ and $T < \infty$. As K is large enough, by Step 1., we know

$$y_\delta(K,t) > Cf(R) \geqslant Cf\left(\frac{K+R}{2}\right) + \frac{2T}{K+R} \geqslant y(K+R,t), \tag{III.11}$$

for all $t \in [0, T)$. By the avoidance principle, we conclude $y_\delta(x,t) > y(x+R,t)$ on Ω_T. In particular,

$$y(2R,t) \leqslant y_\delta(R,t) \leqslant Cf(R) + \frac{C(t+\delta)}{e^{\frac{R^2}{t+\delta}}} \leqslant Cf(R) + \frac{C(T+\delta)}{e^{\frac{R^2}{T+\delta}}} \leqslant Cf\left(\frac{2R}{2}\right) + C'e^{-C''R^2}, \tag{III.12}$$

for some universal constants $C, C', C'' > 0$ (may depends on T). This completes the proof. ☐

After we received the $\mathcal{L}^\infty$ bounds in the whole spacetime near infinity, the following lemma is to comprehend the instantaneous regularity in the spacetime in terms of the local $\mathcal{L}^\infty$ bound.

Lemma 11 (Interior regularity of E-H Flow)**.** Let $y : (-2,2) \times [0,T] \to \mathbb{R}$ be a smooth solution of the 1D Ecker-Huisken flow. Suppose that there exists constants $M > 0$ and $\Lambda > 0$ such that $|y(x,t)| \leqslant M$ and $|y_x(x,t)| \leqslant \Lambda$ on $(-1,1) \times [0,T]$. Then there exists a constant $C_1 = C_1(\Lambda, T) > 0$ such that

$$|y_x(0,t)| \leqslant \frac{C_1 M}{\sqrt{t}}, \tag{III.13}$$

on $t \in (0,T)$. Furthermore, for all $\delta \in (0,T)$, if $M << 1$ small enough depending on Λ, C_1 and δ, then there is a constant $C_1' = C_1'(\Lambda, \delta, T) > 0$ such that

$$|y_{xx}(0,t)| \leqslant \frac{C_1' M}{\sqrt{t-\delta}}, \tag{III.14}$$

holds for all $t \in (\delta, T)$.

Proof of Lemma 11. For simplicity, we define an operator $\mathcal{L} := \partial_t - \frac{\partial_{xx}}{1+w^2}$ and $w := y_x$. By assumption, $\mathcal{L}$ is a uniformly elliptic operator and $\mathcal{L}y = 0$. A direct computation shows that for any $\alpha \in \mathbb{R}$,

$$\mathcal{L}(y+2M)^{\alpha} = -\alpha(\alpha-1)\frac{(y+2M)^{\alpha-2}w^2}{1+w^2}, \tag{III.15}$$

$$\mathcal{L}w^2 = -\frac{2+6w^2}{(1+w^2)^2}w_x^2, \tag{III.16}$$

and

$$\mathcal{L}w_x^2 = -\frac{2}{1+w^2}w_{xx}^2 - \frac{12w}{(1+w^2)^2}w_x^2 w_{xx} + \frac{12w^2-4}{(1+w^2)^3}w_x^4. \tag{III.17}$$

Again, we adopt the Bernstein's trick here. Choose an even cut-off function $\varphi : (-1,1) \to [0,1]$ such that

$$\varphi(x) := \begin{cases} 1, \text{ if } x \in (-\frac{1}{2}, \frac{1}{2}); \\ \text{non-increasing, if } x \in (\frac{1}{2}, \frac{3}{4}); \\ \exp\left(-\frac{1}{1-x^2}\right), \text{ if } |x| \in (\frac{3}{4}, 1). \end{cases} \tag{III.18}$$

Then there exists a constant $C > 0$ such that $\varphi'' \geqslant -C\varphi$. Define $\Psi(x,t) := \exp(-Ct)\varphi(x)$, we obtain

$$\mathcal{L}\Psi \leqslant 0. \tag{III.19}$$

Now, consider $F_1(x,t) := w^2(y+2M)^{\alpha}$. Then

$$\begin{aligned} \mathcal{L}F_1 &= w^2\mathcal{L}(y+2M)^{\alpha} + (y+2M)^{\alpha}\mathcal{L}w^2 - 2\frac{2ww_x \cdot \alpha(y+2M)^{\alpha-1}w}{1+w^2}, \\ &= -\alpha(\alpha-1)\frac{w^2(y+2M)^{\alpha}}{1+w^2}\left(\frac{w}{y+2M}\right)^2 - 2\frac{1+3w^2}{(1+w^2)^2}(y+2M)^{\alpha}w^2\left(\frac{w_x}{w}\right)^2 \\ &\quad - 4\alpha\frac{w^2(y+2M)^{\alpha}}{1+w^2}\frac{w_x}{y+2M} \\ &= \frac{w^2(y+2M)^{\alpha}}{1+w^2}\left[-\alpha(\alpha-1)\left(\frac{w}{y+2M}\right)^2 - 4\alpha\frac{w_x}{w}\cdot\frac{w}{y+2M} - 2\frac{1+3w^2}{1+w^2}\left(\frac{w_x}{w}\right)^2\right]. \end{aligned} \tag{III.20}$$

If we choose $\alpha = -1 + \epsilon$ with $\epsilon \in (0, 1)$, then we get

$$\mathcal{L}F_1 \leqslant \frac{-\epsilon(1-\epsilon)F_1^2}{(1+\Lambda^2)(3M)^{1+\epsilon}}. \tag{III.21}$$

Therefore, we consider the function $G_1(x, t) := t\Psi(x, t)F_1(x, t)$ and we obtain

$$\mathcal{L}G_1 = \frac{G_1}{t} + t\left[F_1\mathcal{L}\Psi + \Psi\mathcal{L}F_1 - 2\frac{\Psi_x \cdot F_{1,x}}{1+w^2}\right] \leqslant \frac{G_1}{t} + t\left[-\frac{\epsilon(1-\epsilon)\Psi F_1^2}{(1+\Lambda^2)(3M)^{1+\epsilon}} - \frac{2\Psi_x \cdot F_{1,x}}{1+w^2}\right]. \tag{III.22}$$

Since G_1 vanishes on the parabolic boundary of $(-1, 1) \times [0, T)$, if the maximal value is positive, then the maximal point $(x_0, t_0) \in (-1, 1) \times (0, T)$ is in the interior region. (If the maximal value is zero then $y_x(0, t) = 0$). At the maximal point (x_0, t_0), we have

$$\begin{aligned} 0 &\leqslant \mathcal{L}G_1(x_0, t_0) \\ &\leqslant \frac{G_1(x_0, t_0)}{t_0} + t_0\left[-\frac{\epsilon(1-\epsilon)G_1(x_0, t_0)^2}{(1+\Lambda^2)(3M)^{1+\epsilon}t_0^2\Psi(x_0, t_0)} + 2\frac{|\Psi_x|^2 G_1(x_0, t_0)}{t_0\Psi(x_0, t_0)}\right]. \end{aligned} \tag{III.23}$$

Multiply $t_0\Psi(x_0, t_0)$ and choose $C > 0$ such that $|\Psi_x|^2 \leqslant C$, then

$$\begin{aligned} & 0 \leqslant G_1(x_0, t_0) - \frac{\epsilon(1-\epsilon)G_1(x_0, t_0)^2}{(1+\Lambda^2)(3M)^{1+\epsilon}} + 2CG_1(x_0, t_0) \\ \implies \quad & G_1(x_0, t_0) \leqslant \frac{(1+2C)(1+\Lambda^2)3^{1+\epsilon}}{\epsilon(1-\epsilon)}M^{1+\epsilon} \\ \implies \quad & y_x(0, t)^2 = \frac{e^{Ct}G_1(0, t)}{t(y(0, t) + 2M)^{-1+\epsilon}} \leqslant \frac{e^{CT}G_1(x_0, t_0)}{t \cdot (3M)^{-1+\epsilon}} \leqslant \frac{9e^{CT}(1+2C)(1+\Lambda^2)}{\epsilon(1-\epsilon)}\frac{M^2}{t}. \end{aligned} \tag{III.24}$$

Similar to the above, for $\alpha \in (-1, 0)$, we have

$$\mathcal{L}(w + 2\Lambda)^\alpha = -\frac{\alpha(w + 2\Lambda)^{\alpha-1}w_x^2}{1+w^2}\left(\frac{2w}{1+w^2} + \frac{\alpha-1}{w+2\Lambda}\right). \tag{III.25}$$

We make a new ansatz that $w^2 \leqslant \frac{4+\beta}{48-\beta}$ holds for some $\beta \in (-4, 0)$ and translate the time interval from (δ, T) to $(0, T-\delta)$. The previous result implies this ansatz holds if

$M^2 \leqslant C_2(\delta, T, \Lambda)\frac{4+\beta}{48-\beta}$ and we may replace Λ with $C_3(\delta, T, \Lambda)M$. This leads (III.17) to satisfy

$$\mathcal{L}w_x^2 \leqslant -\frac{1}{1+w^2}w_{xx}^2 + \frac{\beta}{(1+w^2)^2}w_x^4, \tag{III.26}$$

which follows from the Arithmetic-Geometric inequality. Consider $F_2(x,t) := w_x^2(w + 2\Lambda)^\alpha$ for some $\alpha \in (\frac{\beta}{2}, 0)$ would be determined later. Then

$$\begin{aligned}
\mathcal{L}F_2 &= (w+2\Lambda)^\alpha \mathcal{L}w_x^2 + w_x^2\mathcal{L}(w+2\Lambda)^\alpha - 4\alpha\frac{(w+2\Lambda)^{\alpha-1}w_x^2 w_{xx}}{1+w^2} \\
&\leqslant (w+2\Lambda)^\alpha\Bigg[-\frac{1}{1+w^2}w_{xx}^2 + \frac{\beta}{(1+w^2)^2}w_x^4 - w_x^4\frac{\alpha}{(1+w^2)(w+2\Lambda)}\left(\frac{2w}{1+w^2} + \frac{\alpha-1}{w+2\Lambda}\right) \\
&\quad -4\alpha\frac{w_{xx}w_x^2}{(1+w^2)(w+2\Lambda)}\Bigg] \\
&= \frac{F_2}{1+w^2}\Bigg[-\left(\frac{w_{xx}}{w_x}\right)^2 + \frac{\beta(w+2\Lambda)^2}{1+w^2}\left(\frac{w_x}{w+2\Lambda}\right)^2 - \left(\frac{2\alpha w(w+2\Lambda)}{1+w^2} + \alpha(\alpha-1)\right)\left(\frac{w_x}{w+2\Lambda}\right)^2 \\
&\quad -4\alpha\frac{w_{xx}}{w+2\Lambda}\Bigg] \\
&\leqslant \frac{F_2}{1+w^2}\left[-\left(\frac{w_{xx}}{w_x}\right)^2 - 4\alpha\frac{w_{xx}}{w+2\Lambda} + \left(\frac{(\beta-2\alpha)w + 2\beta\Lambda}{1+w^2}(w+2\Lambda) - \alpha(\alpha-1)\right)\left(\frac{w_x}{w+2\Lambda}\right)^2\right] \\
&\leqslant \frac{F_2}{1+w^2}\left[-\left(\frac{w_{xx}}{w_x}\right)^2 - 4\alpha\frac{w_{xx}}{w+2\Lambda} - \alpha(\alpha-1)\left(\frac{w_x}{w+2\Lambda}\right)^2\right],
\end{aligned} \tag{III.27}$$

where we use $(\beta - 2\alpha)(w+\Lambda) + (\beta+2\alpha)\Lambda \leqslant 0$ in the last inequality. Choose $\alpha = -\frac{1}{3} + \epsilon \in (\max\{-\frac{1}{3}, \frac{\beta}{2}\}, 0)$, the equation becomes

$$\mathcal{L}F_2 \leqslant -(1-3\epsilon)\epsilon\frac{F_2^2}{(1+\Lambda^2)(w+2\Lambda)^{\frac{4}{3}+\epsilon}} \leqslant -(1-3\epsilon)\epsilon\frac{F_2^2}{(1+\Lambda^2)(3C_3M)^{\frac{5}{3}+\epsilon}}. \tag{III.28}$$

Similar to the above, define $G_2 := t\Psi F_2$ and apply the maximal principle, the result shows that

$$y_{xx}(0,t+\delta)^2 = w_x(0,t)^2 = \frac{e^{C(T-\delta)}G_2(0,t)(w(0,t)+2C_3M)^{\frac{1}{3}-\epsilon}}{t} \leqslant \frac{C_3M^{\frac{1}{3}-\epsilon+\frac{5}{3}+\epsilon}}{t} = \frac{C_3M^2}{t}. \tag{III.29}$$

Note that we translate the time interval so the first equality needs to shift δ. This confirms our assertions. □

Now we are able to prove our main result.

Proof of Theorem 7. First, we investigate the following asymptotic behavior for y^i:

Claim 4. For any constant $\delta > 0$, there exists some constants $C_\delta, C'_\delta, C'' > 0$ such that

$$|y^i_t|(x,t) + |y^i_x|(x,t) \leqslant C_\delta f\left(\frac{|x|-1}{2}\right) + C'_\delta \exp(-C''(|x|-1)^2), \tag{III.30}$$

for all $t \in [\delta, T-\delta]$ as $|x| >> 1$ large enough.

Proof of Claim 4. Thanks for the symmetry, we only discuss the case $x > 0$. By Lemma 10, we obtain $\|y^i\|_{\mathcal{L}^\infty(\mathbb{R}\times[\delta/4,T-\delta/4])} < +\infty$. According to [Sob24, Corollary 2.6.10](or originally, [ES92, Corollary 5.3]), there is a constant $\Lambda > 0$ such that $|y^i_x(x,t)| \leqslant \Lambda$ on $\mathbb{R}\times[\delta/2, T-\delta/2]$. By Lemma 10 and Lemma 11, on the neighborhood $(R-2, R+2)\times[\delta/2, T-\delta/2]$, there is a constant $C_1(\Lambda, T, \delta) > 0$ such that

$$|y^i_x(R,t)| \leqslant C_1 \|y^i\|_{\mathcal{L}^\infty\left((R-1,R+1)\times[\frac{\delta}{2},T-\frac{\delta}{2}]\right)} \leqslant C_1\left[Cf\left(\frac{R-1}{2}\right) + C'e^{-C''(R-1)^2}\right], \tag{III.31}$$

on $(\mathbb{R}\backslash K')\times[3\delta/4, T-3\delta/4]$ for some universal constant $C > 0$. Therefore, there is a time-independent constant R_2 such that for $R > R_2$, $|y^i_x| << 1$ is small enough on $(\mathbb{R}\backslash[-R_2, R_2])\times[3\delta/4, T-3\delta/4]$. By Lemma 11 again, there is a constant $C_2 = C_2(R_2, C_1, \delta, \Lambda) > 0$ such that

$$|y^i_{xx}(R,t)| \leqslant C_2\left[Cf\left(\frac{R-1}{2}\right) + C'e^{-C''(R-1)^2}\right], \tag{III.32}$$

on $(\mathbb{R}\backslash(K' \cup [-R_2, R_2])) \times [\delta, T-\delta]$. Since $|y^i_t| \leqslant |y^i_{xx}|$, these achieve our claim. ☐

The second claim is these asymptotic behaviors imply the preservation of the integral.

Claim 5. For any $t \in [0, T)$,

$$\int_{\mathbb{R}} y^1(x,t)dx = \int_{\mathbb{R}} y^2(x,t)dx = \int_{\mathbb{R}} y_0(x)dx. \tag{III.33}$$

Proof of Claim 5. Fix $t_0 \in (0, T)$. Claim 4 implies there is a constant $C = C(t_0) > 0$ and a compact set $K \subset \mathbb{R}$ such that

$$|y^i_t|(x,t) + |y^i_x|(x,t) \leqslant Cf\left(\frac{|x|-1}{2}\right) + C\exp(-Cx^2), \tag{III.34}$$

on $(\mathbb{R}\backslash K) \times [\frac{t_0}{2}, \frac{3t_0}{2}]$. Note that the R.H.S. is integrable w.r.t. x. Hence, we conclude

$$\left.\frac{d}{dt}\right|_{t=t_0} \int_{\mathbb{R}} y^i(x,t)dx = \int_{\mathbb{R}} y^i_t(x,t_0)dx = \int_{\mathbb{R}} (\arctan y^i_x(x,t_0))_x dx = 0, \tag{III.35}$$

where the first and the last equalities follow from (III.34). Therefore, the quantitative $\int_{\mathbb{R}} y^i(x,t)dx$ is a constant on $(0,T)$. By the $\mathcal{L}^1(\mathbb{R})$-convergence, this shows that this constant is $\int_{\mathbb{R}} y_0$, which confirms the assertion. ☐

We consider the partial accumulative function $\mathcal{A}^i(x,t) := \int_{-\infty}^{x} y_i(z,t)dz$ for all $(x,t) \in \mathbb{R} \times [0,T)$ and $i = 1,2$. It's well-defined since $y_i(\cdot,t) \in \mathcal{L}^1(\mathbb{R})$ for all $i = 1,2$ and $t \in [0,T)$. The assumption $y_i(\cdot,t) \to y_0$ as $t \to 0^+$ in the $\mathcal{L}^1$-sense implies $\lim_{t\to 0^+} \mathcal{A}^i(x,t) = \mathcal{A}^i(x,0)$ uniformly for all $x \in \mathbb{R}$. Lemma 10 shows that

$$\lim_{x\to-\infty} \mathcal{A}^1(x,t) - \mathcal{A}^2(x,t) = 0, \tag{III.36}$$

for all $t \in [0,T)$. According to Claim 5,

$$\lim_{x\to\infty} \mathcal{A}^1(x,t) - \mathcal{A}^2(x,t) = \lim_{x\to\infty} \int_x^{\infty} y_2(z,t) - y_1(z,t)dz = 0, \tag{III.37}$$

for all $t \in (0,T)$. Now, as how we proved the Claim 5, we calculate the evolution equation of $\mathcal{A}^i$, that is

$$\mathcal{A}^i_t(x,t) = \int_{-\infty}^{x} y^i_t(z,t)dz = \int_{-\infty}^{x} (\arctan y^i_z)_z dz = \arctan y^i_x(x,t) = \arctan \mathcal{A}^i_{xx}(x,t), \tag{III.38}$$

for all $(x,t) \in \mathbb{R}\times(0,T)$. Consider the function $\mathcal{B}^i(x,t) = e^{-\varepsilon t}\mathcal{A}^i(x,t)$ for some $\varepsilon > 0$, then we get

$$(\mathcal{B}^1 - \mathcal{B}^2)_t = -\varepsilon(\mathcal{B}^1 - \mathcal{B}^2) + e^{-\varepsilon t}(\arctan \mathcal{A}^1_{xx} - \arctan \mathcal{A}^2_{xx}), \tag{III.39}$$

for all $(x,t) \in \mathbb{R}\times(0,T)$. Also, we have $(\mathcal{B}^1 - \mathcal{B}^2)(x,0) = 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to-\infty}(\mathcal{B}^1 - \mathcal{B}^2)(x,t) = \lim_{x\to\infty}(\mathcal{B}^1 - \mathcal{B}^2)(x,t) = 0$ for all $t \in (0,T)$. Note that $\lim_{t\to 0^+}(\mathcal{B}^1 - \mathcal{B}^2)(x,t) = 0$ uniformly for all $x \in \mathbb{R}$ and

$$-\pi \leqslant (\mathcal{A}^1-\mathcal{A}^2)_t \leqslant \pi \Rightarrow -\pi t \leqslant (\mathcal{A}^1-\mathcal{A}^2)(x,t) \leqslant \pi \Rightarrow -\pi e^{-\varepsilon t} t \leqslant (\mathcal{B}^1-\mathcal{B}^2)(x,t) \leqslant \pi e^{-\varepsilon t} t, \tag{III.40}$$

for all $(x,t) \in \mathbb{R}\times(0,T)$. Therefore, for any $\hat{T} \in (0,T)$, the supremum $M = \sup_{(x,t)\in\mathbb{R}\times[0,\hat{T}]}(\mathcal{B}^1 - \mathcal{B}^2)(x,t) \geqslant 0$ is finite. Suppose that $M > 0$, we claim the maximum value is achieved. It suffices to show that $(\mathcal{B}^1 - \mathcal{B}^2)^{-1}\left([\frac{M}{2}, M]\right) \cap \mathbb{R}\times[0,\hat{T}]$ is bounded. We use the following inductive argument to accomplish it.

Note that $(\mathcal{B}^1 - \mathcal{B}^2)(x,t) \leqslant \frac{M}{2}$ for all $(x,t) \in \mathbb{R}\times[0,\frac{M}{2\pi}]$. Since $(\mathcal{B}^1 - \mathcal{B}^2)(x,\frac{M}{2\pi}) \to 0$ as $|x| \to \infty$, there exists a number $R_1 > 0$ such that $(\mathcal{B}^1 - \mathcal{B}^2)(x,\frac{M}{2\pi}) \leqslant \frac{M}{4}$ for all

$|x| \geqslant R_1$. Thanks to $|(\mathcal{A}^1 - \mathcal{A}^2)_t| \leqslant \pi$, we get $(\mathcal{B}^1 - \mathcal{B}^2)(x,t) \leqslant \frac{M}{2}$ for all $|x| \geqslant R_1$ and $t \in [\frac{M}{2\pi}, \frac{2M}{2\pi}]$. We play the same game on $[\frac{2M}{2\pi}, \frac{3M}{2\pi}]$, that is, choose a number $R_2 > 0$ such that $(\mathcal{B}^1 - \mathcal{B}^2)(x, \frac{2M}{2\pi}) \leqslant \frac{M}{4}$ for all $|x| \geqslant R_2$, and we get $(\mathcal{B}^1 - \mathcal{B}^2)(x,t) \leqslant \frac{M}{2}$ for all $|x| \geqslant R_2$ and $t \in [\frac{2M}{2\pi}, \frac{3M}{2\pi}]$. Continue this process, since $\hat{T}$ is finite, it stops at some k-th steps, which reflects that

$$(\mathcal{B}^1 - \mathcal{B}^2)^{-1}\left([\frac{M}{2}, M]\right) \cap \mathbb{R} \times [0, \hat{T}] \subset \bigcup_{\ell=1}^{k} \left[-R_\ell, R_\ell] \times [\frac{\ell M}{2\pi}, \frac{(\ell+1)M}{2\pi}]\right], \tag{III.41}$$

which confirms the assertion.

Therefore, we may choose $(x_0, t_0) \in \mathbb{R} \times [0, \hat{T}]$ such that $M = (\mathcal{B}^1 - \mathcal{B}^2)(x_0, t_0)$. Then we have

$$\begin{cases} (\mathcal{B}^1 - \mathcal{B}^2)_t(x_0, t_0) \geqslant 0; \\ (\mathcal{B}^1 - \mathcal{B}^2)_{xx}(x_0, t_0) \leqslant 0; \\ (\mathcal{A}^1 - \mathcal{A}^2)_{xx}(x_0, t_0) \leqslant 0, \end{cases} \tag{III.42}$$

where the last inequality follows from the fact x_0 also achieve maximal for function $(\mathcal{A}^1 - \mathcal{A}^2)(\cdot, t)$. Substitute (x_0, t_0) into (III.39), we get

$$0 \leqslant -\varepsilon M + e^{-\varepsilon t_0}(\arctan \mathcal{A}^1_{xx} - \arctan \mathcal{A}^2_{xx})(x_0, t_0) \leqslant -\varepsilon M. \tag{III.43}$$

The last inequality follows because the function arctan is increasing. This causes a contradiction. Hence, we get $\mathcal{A}^1 \leqslant \mathcal{A}^2$ on $\mathbb{R} \times [0, T)$ since $\hat{T} \in (0, T)$ is an arbitrary constant. By symmetry, this implies $\mathcal{A}^1 \equiv \mathcal{A}^2$ on $\mathbb{R} \times [0, T)$. Then the Fundamental Theorem of Calculus ensures that $y_1(x,t) = \mathcal{A}^1_x(x,t) = \mathcal{A}^2_x(x,t) = y_2(x,t)$ for all $(x,t) \in \mathbb{R} \times [0, T)$, which completes the proof. □

In the end, we list some properties that we proved in the Theorem 7.

Proposition 6. Let $y : [0, T) \to \mathbb{R}$ be a solution of the 1D Ecker-Huisken flow with the initial $\mathcal{L}^1$-data y_0. If $y(\cdot, t) \xrightarrow{\mathcal{C}^0_{\text{loc}}(K^c)} y_0$ as $t \to 0$ for some compact set $K \subset \mathbb{R}$ and there is a non-increasing function $f \in \mathcal{L}^1([1, \infty))$ such that $y_0(x) = \mathrm{O}(f(|x|))$ holds as $|x| \to \infty$, then

$$\int_{\mathbb{R}} y(x,t)dx = \int_{\mathbb{R}} y_0(x)dx, \tag{III.44}$$

for all $t \in [0, T)$. Furthermore, if we consider the accumulative function $\mathcal{A}(x,t) := \int_{-\infty}^{x} y(z,t)dz$, then it satisfies

$$\mathcal{A}_t(x,t) = \arctan \mathcal{A}_{xx}(x,t) \tag{III.45}$$

on $\mathbb{R} \times (0, T)$, and

$$\lim_{t \to 0} \mathcal{A}(x,t) = \mathcal{A}(x,0), \tag{III.46}$$

uniformly on $\mathbb{R}$.

Proof of Proposition 6. The proof lies in the proof of Theorem 7. So we omit the proof here. □

Remark 6. As noted in Remark 3, the solution constructed by P. Topping and A. Sobnack also satisfies Proposition 6, since it arises from a sequence of smooth solutions with the same decay behavior on the initial function.

III.3 Some quantitative estimates

In this section, we provide two a priori estimates for $\mathcal{L}^1$-initial data, but with some additional assumption.

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Lemma 12. Given the constants $0 < r < R$ and $\lambda \in (\frac{1}{3}, 1)$. Let $y^i : [-R, R] \times [0, T) \to \mathbb{R}$ be two solutions of Ecker-Huisken flow with $\mathcal{L}^1$-initial data. Suppose that $y^1(x,t) \geqslant y^2(x,t)$ and $y^1_x(x,t) \geqslant y^2_x(x,t)$ hold for all $(x,t) \in (-R, R) \times [0, T]$. Then there exists a constant $C = C(\lambda) > 0$ such that

$$\left(\int_{-R}^{R} (y^1(x,t) - y^2(x,t))\varphi(x)dx\right)^{1-\lambda} \leqslant \left(\int_{-R}^{R} (y^1(x,0) - y^2(x,0))\varphi(x)dx\right)^{1-\lambda} + C\frac{t}{(R-r)^{2\lambda}}, \tag{III.47}$$

where $\varphi : [-R, R] \to [0, 1]$ is a cut-off function with the form

$$\varphi(x) := \begin{cases} 1, \text{ if } x \in [-r, r]; \\ \frac{R-x}{R-r}, \text{ if } x \in [r, R]; \\ \frac{R+x}{R-r}, \text{ if } x \in [-R, -r]. \end{cases} \tag{III.48}$$

Proof of Lemma 12. Define another cut-off function $\psi : [r, R] \to \mathbb{R}_{\geqslant 0}$ by

$$\psi(x) := \begin{cases} \frac{(x-r)^2}{2(R-r)}, \text{ if } x \in [r, r + \frac{R-r}{4}]; \\ -\frac{(x-r)^2}{2(R-r)} + (x - r) - \frac{3}{16}(R - r), \text{ if } x \in [r + \frac{R-r}{4}, r + \frac{R-r}{2}]; \\ \psi(R + r - x), \text{ if } x \in [\frac{R+r}{2}, R]. \end{cases} \tag{III.49}$$

Then $|\psi_x| \leqslant \varphi$ on $[r, R]$. Now, we establish an elementary inequality of arctan.

Lemma 13. For any $\lambda \in (0, 1)$, there exists a constant $C(\lambda) > 0$ such that

$$|\arctan a - \arctan b| \leqslant C(\lambda)|a - b|^{\lambda}, \tag{III.50}$$

for all $a, b \in \mathbb{R}$.

Proof of Lemma 13. Since arctan is monotone-increasing, it's enough to prove that $\arctan a -$

$\arctan b \leqslant C(\lambda)(a-b)^{\lambda}$ for $a \geqslant b$. Fix $b \in \mathbb{R}$. This inequality holds for $a = b$ and $a - b > (C(\lambda)/\pi)^{-1/\lambda}$. For $0 < a - b < (C(\lambda)/\pi)^{-1/\lambda}$, notice that

$$\frac{d}{da}\arctan a = \frac{1}{1+a^2} \leqslant 1, \tag{III.51}$$

and

$$\frac{d}{da}C(\lambda)(a-b)^{\lambda} = \lambda C(\lambda)(a-b)^{-1+\lambda} \geqslant \lambda C(\lambda)\left(\frac{C(\lambda)}{\pi}\right)^{\frac{1}{\lambda}-1} = C(\lambda)^{\frac{1}{\lambda}}\frac{\lambda}{\pi^{\frac{1}{\lambda}-1}}. \tag{III.52}$$

Choose $C(\lambda) := \frac{\lambda^{-\lambda}}{\pi^{\lambda-1}}$, then we get $\frac{d}{da}($ R.H.S. $-$ L.H.S.$) \geqslant 0$ for $a-b \in (0, (C(\lambda)/\pi)^{-1/\lambda})$. Hence, we complete the proof. $\square$

Now, for $t \in [0,T)$, we define $I(t) := \int_{-R}^{R} \varphi(x)(y^1(x,t) - y^2(x,t))dx$. Then

$$\begin{aligned} \frac{d}{dt}I(t) &= -\int_{-R}^{R}\left[\arctan y_x^1 - \arctan y_x^2\right]\varphi_x dx \\ &\leqslant \int_{r}^{R}\left[\arctan y_x^1 - \arctan y_x^2\right]|\varphi_x| + \int_{-R}^{-r}\left[\arctan y_x^1 - \arctan y_x^2\right]|\varphi_x|. \end{aligned} \tag{III.53}$$

Now we fixed $\lambda \in (1/3, 1)$. For the first term of the R.H.S., by Lemma 13, we have

$$\begin{aligned} \int_{r}^{R}\left[\arctan y_x^1 - \arctan y_x^2\right]|\varphi_x| &\leqslant C(\lambda)\int_{r}^{R}(y_x^1 - y_x^2)^{\lambda}\psi^{\lambda}\cdot\frac{(-\varphi_x)}{\psi^{\lambda}} \\ &\overset{\text{Hölder}}{\leqslant} C(\lambda)\left(\int_{r}^{R}(y_x^1 - y_x^2)\psi\right)^{\lambda}\left(\int_{r}^{R}\left(\frac{-\varphi_x}{\psi^{\lambda}}\right)^{\frac{1}{1-\lambda}}\right)^{1-\lambda} \\ &\overset{\text{I.B.P.}}{\leqslant} C(\lambda)(R-r)^{-1}\left(\int_{r}^{R}(y^1 - y^2)(-\psi_x)\right)^{\lambda}\left(\int_{r}^{R}\psi^{\frac{-\lambda}{1-\lambda}}\right)^{1-\lambda} \\ &\leqslant \frac{2C(\lambda)}{R-r}I(t)^{\lambda}\left(\int_{r}^{\frac{R+r}{2}}\psi^{\frac{-\lambda}{1-\lambda}}\right)^{1-\lambda}. \end{aligned} \tag{III.54}$$

Hence, by symmetry, we have

$$\frac{d}{dt}I(t) \leqslant \frac{4C(\lambda)}{R-r}I(t)^{\lambda}\left(\int_r^{\frac{R+r}{2}} \psi^{\frac{-\lambda}{1-\lambda}}\right)^{1-\lambda} \Rightarrow \frac{d}{dt}I(t)^{1-\lambda} \leqslant \frac{C'(\lambda)}{R-r}\left(\int_r^{\frac{R+r}{2}} \psi^{\frac{-\lambda}{1-\lambda}}\right)^{1-\lambda}. \tag{III.55}$$

By the direct computation, since $\lambda \in (1/3, 1)$, we have

$$\begin{aligned}
\int_r^{\frac{R+r}{2}} \psi^{\frac{-\lambda}{1-\lambda}} &= \int_r^{r+\frac{R-r}{4}} \left[\frac{(x-r)^2}{2(R-r)}\right]^{\frac{-\lambda}{1-\lambda}} + \int_{r+\frac{R-r}{4}}^{\frac{R+r}{2}} \left[-\frac{(x-r)^2}{2(R-r)} + (x-r) - \frac{3}{16}(R-r)\right]^{\frac{-\lambda}{1-\lambda}} \\
&\leqslant (2(R-r))^{\frac{\lambda}{1-\lambda}} \int_0^{\frac{R-r}{4}} t^{\frac{-2\lambda}{1-\lambda}} dt + \int_{r+\frac{R-r}{4}}^{\frac{R+r}{2}} \left[\frac{R-r}{32}\right]^{\frac{-\lambda}{1-\lambda}} \\
&\leqslant C(\lambda)(R-r)^{\frac{\lambda}{1-\lambda}}\, t^{\frac{1-3\lambda}{1-\lambda}}\Big|_{t=0}^{\frac{R-r}{4}} + C(\lambda)(R-r)^{\frac{1-2\lambda}{1-\lambda}} \\
&= C(\lambda)(R-r)^{\frac{1-2\lambda}{1-\lambda}}.
\end{aligned} \tag{III.56}$$

Combining (III.55) and (III.56), we get

$$\frac{d}{dt}I(t)^{1-\lambda} \leqslant \frac{C''(\lambda)}{(R-r)^{2\lambda}} \Rightarrow I(t)^{1-\lambda} \leqslant I(0)^{1-\lambda} + \frac{C''(\lambda)t}{(R-r)^{2\lambda}}. \tag{III.57}$$

This completes the proof. □

Using the same idea, we can deal with the case that $y^1(x,t) \geqslant y^2(x,t)$ for all $(x,t) \in [-R,R] \times [0,T)$, and

$$\begin{cases}
\limsup_{x\to -R}(y_x^1(x,t) - y_x^2(x,t)) \leqslant 0, \text{ for all } t \in (0,T); \\
\liminf_{x\to R}(y_x^1(x,t) - y_x^2(x,t)) \geqslant 0, \text{ for all } t \in (0,T); \\
|\{x \in [-R,R] : y_x^1(x,t) = y_x^2(x,t)\}| \leqslant 1, \text{ for all } t \in (0,T).
\end{cases} \tag{III.58}$$

In that case, if $p \in [-R,R]$ satisfies $y_x^1(p,t) = y_x^2(p,t)$ for some $t \in (0,T)$, then $y^1(p,t) - y^2(p,t) = \min_{x\in[-R,R]} y^1(x,t) - y^2(x,t)$. This leads our argument to work.

Lemma 14. Given the constants $0 < r < R$ and $\lambda \in (\frac{1}{3}, 1)$. Let $y^i : [-R, R] \times [0, T) \to \mathbb{R}$ be two solutions of Ecker-Huisken flow with $\mathcal{L}^1$-initial data. Suppose that $y^1(x,t) \geqslant y^2(x,t)$ and (III.58) hold for all $(x,t) \in (-R, R) \times [0, T]$. Then there exists a constant $C = C(\lambda) > 0$ such that

$$\left(\int_{-R}^{R} (y^1(x,t) - y^2(x,t))\varphi(x)dx\right)^{1-\lambda} \leqslant \left(\int_{-R}^{R} (y^1(x,0) - y^2(x,0))\varphi(x)dx\right)^{1-\lambda} + C\frac{t}{(R-r)^{2\lambda}}, \tag{III.59}$$

where $\varphi : [-R, R] \to [0, 1]$ is the function defined as (III.48).

Proof of Lemma 14. As we did in the proof of Lemma 12, we have

$$\frac{d}{dt}I(t) = -\int_{r}^{R} \left[\arctan y_x^1 - \arctan y_x^2\right] \varphi_x - \int_{-R}^{-r} \left[\arctan y_x^1 - \arctan y_x^2\right] \varphi_x. \tag{III.60}$$

If there is no point $p \in [-R, -r] \cup [r, R]$ such that $y_x^1(p,t) = y_x^2(p,t)$, then we can do the same argument as in the proof of Lemma 12 since we only need to make sure the sign of $y_x^1 - y_x^2$ doesn't change in those two intervals. This means that we have

$$\frac{d}{dt}I(t)^{1-\lambda} \leqslant \frac{C(\lambda)t}{(R-r)2\lambda}. \tag{III.61}$$

Otherwise, we may assume $p \in [r, R]$ such that $y_x^1(p,t) = y_x^2(p,t)$. The above discussion shows that p attains the minimum of $y^1(\cdot,t) - y^2(\cdot,t)$ over $[-R, R]$. So we have

$$0 \leqslant (y^1(p,t) - y^2(p,t))\psi(p) \leqslant \psi(p)\frac{\int_{-R}^{R}(y^1(x,t) - y^2(x,t))\varphi(x)dx}{\int_{-R}^{R}\varphi(x)dx} \leqslant \frac{3(R-r)}{16}\frac{I(t)}{R}. \tag{III.62}$$

This implies that

$$\begin{aligned}\int_r^R |y_x^1 - y_x^2|\psi &= \int_r^p (y_x^2 - y_x^1)\psi + \int_p^R (y_x^1 - y_x^2)\psi \\ &= 2(y^2(p,t) - y^1(p,t))\psi(p) + \int_r^p (y^1 - y^2)\psi_x + \int_p^R (y^2 - y^1)\psi_x \\ &\leqslant \frac{3(R-r)}{8R} I(t) + \int_r^R (y^1 - y^2)|\psi_x| \\ &\leqslant \frac{3}{8} I(t) + I(t) \\ &= \frac{11}{8} I(t).\end{aligned} \tag{III.63}$$

Therefore, making the same argument as before, we obtain

$$\frac{d}{dt} I(t) \leqslant \frac{4C(\lambda)}{R-r} \left(\frac{11 I(t)}{8}\right)^{\lambda} \left(\int_r^{\frac{R+r}{2}} \psi^{\frac{-\lambda}{1-\lambda}}\right)^{1-\lambda} \leqslant \frac{C'''(\lambda)}{(R-r)^{2\lambda}} I(t)^{\lambda}. \tag{III.64}$$

This confirms our assertion. □

Unfortunately, we lack the estimate which is only concerned with the $\mathcal{L}^1$-data.

Chapter IV Gap Theorem on locally conformally flat manifold using Yamabe flow

This is a joint work with Prof. Man-Chun Lee at CUHK and we upload the preprint on arXiv. [HL25]

IV.1 Introduction

A Riemannian manifold $(\mathcal{M}^n, g)$ is *locally conformally flat* if, for any point $p \in \mathcal{M}^n$, there exists a coordinate chart conformal to the open set in Euclidean space. This is a high-dimensional analog of isothermal coordinate and allows for elegant structural results with curvature lower bounds, see also [YS88, CZ02, CH04, Ma16, Che25]. In [CH04], under the assumption of nonnegative Ricci curvature, these manifolds must be either *globally conformally equivalent to* $\mathbb{R}^n$ *or* S^n/Γ or *locally isometric to* $\mathbb{R} \times S^{n-1}$ *or flat*. This successful topological classification lays the foundation for further classification at the geometric level.

In [CZ02], Chen and Zhu demonstrated that for a complete locally conformally flat manifold $(\mathcal{M}^n, g)$ with bounded and nonnegative Ricci curvature, it must be flat if there is a function $\varepsilon : \mathbb{R} \to \mathbb{R}$ with $\lim_{r\to 0} \varepsilon(r) = 0$ such that

$$r^2 \fint_{B_g(x_0,r)} \mathrm{scal}(g(y))\, d\,\mathrm{Vol}_g\, y \leqslant \varepsilon(r), \tag{IV.1}$$

for some $x_0 \in \mathcal{M}^n$ and for all $r > 0$, where $\fint_S := \mathrm{Vol}(S)^{-1} \int_S$ is the average of the integral. It is remarkable that the right-hand side of (IV.1) is scaling-invariant and we refer to this type of condition as a *average scalar curvature condition*. Their approach is to evolve Yamabe flow and apply the Harnack inequality in [Cho92] to conclude the flatness. In [Ma16], the author relaxed condition (IV.1) to the global version

$$\int_0^\infty r \fint_{B_g(x_0,r)} \mathrm{scal}(g(y))\, d\,\mathrm{Vol}_g\, y dr < +\infty, \tag{IV.2}$$

holds for some $x_0 \in \mathcal{M}^n$, but required the manifold to be *non-parabolic*. Notice that following an example in [CH04, Section 1], there is a nontrivial, complete, and locally conformally flat manifold $(\mathcal{M}^n, g)$ with nonnegative Ricci curvature, whose Ricci curvature is at most quadratic growth and has maximal volume growth. Therefore, it is reasonable to require some curvature behavior around infinity. Both results [CZ02, Ma16] assume bounded curvature to obtain the short-time existence of Yamabe flow with initial metric g, see also [CZ02, Section 2].

The main result in this paper removes the bounded curvature assumption but requires a **small average scalar curvature at any point**.

Theorem 8. For $n \geqslant 5$, there is a constant $\varepsilon = \varepsilon(n) > 0$ such that for any complete,

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non-compact, and locally conformally flat manifold $(\mathcal{M}^n, g)$, if $(\mathcal{M}^n, g)$ satisfies

$$\operatorname{Ric} \geqslant 0 \quad \text{and} \quad \int_0^\infty r \fint_{B_g(x,r)} \operatorname{scal}(y) d\operatorname{Vol}_g(y) dr \leqslant \varepsilon, \tag{IV.3}$$

for all $x \in \mathcal{M}^n$, then $(\mathcal{M}^n, g)$ is flat.

Motivated by Chan-Lee's work in Ricci flow [CL23], the uniform small curvature assumption allows for the construction of a long-time solution of Yamabe flow with $|\operatorname{Rm}(g(t))| \leqslant c/t$ and nonnegative Ricci tensor for all $t \geqslant 0$. However, several technical challenges must be addressed. The major difficulty is that **Yamabe flow is NOT a super Ricci flow.** Consequently, we can not apply Perelman-Hamilton's distance lemma [Per02, Lemma 8.3] in general. In [Che25], under the almost Ricci pinching assumption, Chen established an almost distance distortion lemma [Che25, Lemma 3.2]. However, this result does not apply to the setting we consider.

In our approach, although we lack a comparable estimate for the time-dependent distance $d_{g(t)}(x, y)$, the uniform average scalar curvature provides a **uniform bound on the conformal factor** of the solution to the Yamabe flow. This observation plays a crucial role in establishing the heat kernel upper bound and the local maximal principle, see Theorem 11 and Theorem 12 for more precisely statements. The small average scalar curvature then ensures that we can construct a solution of Yamabe flow with curvature bound $|\operatorname{Rm}(g(t))| \leqslant 1/t$ (Theorem IV.3), inspired by the results in Ricci flow, see [CL23, LT24, Che25]. However, a key subtlety arises when applying Hochard's lemma [Hoc19, Corollaire IV.1.2] or the argument in [CL23, Section 4], as these approaches result in a loss of control over the conformal factor. Instead, we employ the **Dirichlet-type Yamabe flow** to iteratively construct the solution, ensuring that the conformal factor remains controlled throughout our inductive process.

We note that while there exists a long-time solution to the Yamabe flow starting from a complete, locally conformally flat manifold with certain additional conditions, this solution does not necessarily satisfy the instantaneously bounded curvature property [Ma16, Theorem 5 and Proposition 4].

We outline how we achieve our goal by separating the proof into several steps.

Step 1 (Section IV.3)**.** Suppose g is non-flat. Take an open exhaustion $\Omega_1 \Subset \Omega_2 \Subset \cdots$ with smooth compact boundary and $\bigcup \Omega_k = \mathcal{M}^n$. Consider the Dirichlet problem for the Yamabe flow

$$\begin{cases} \frac{\partial}{\partial t} u(x,t)^{\frac{n+2}{n-2}} = \frac{(n+2)(n-1)}{n-2} \left(\Delta_g u(x,t) - \frac{n-2}{4(n-1)} \mathrm{scal}(g(x)) u(x,t) \right), \text{ if } (x,t) \in \Omega \times (0,\infty); \\ u(x,t) > 0, \text{ if } (x,t) \in \Omega \times (0,\infty); \\ u(x,0) = 1, \text{ if } x \in \Omega; \\ u(x,t) = 1, \text{ if } (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{IV.4}$$

for $\Omega = \Omega_k$. Note that the metric $g(t) = u^{\frac{4}{n-2}}(t) g$ is equivalent to the Yamabe flow equation with initial metric g. We can prove the long-time existence under the assumption (IV.3). Moreover, we can show that there is a constant $C = C(n) > 0$ and a smooth nonnegative function $w : \mathcal{M}^n \to [0,\infty)$ so that

$$e^{-C\varepsilon} \leqslant e^{-w(x)} \leqslant u \leqslant 1, \tag{IV.5}$$

on $\Omega \times (0,\infty)$. In other words, the conformal factor is uniformly bounded throughout spacetime.

Step 2 (Section IV.4, IV.5)**.** In Section IV.4, we establish an upper bound for the heat kernel assuming that the curvature decays at a rate of c/t and the metric remains uniformly equivalent. Our approach follows the heat kernel estimate in [BCRW19, Proposition 3.1]. In Section IV.5, using the heat kernel estimate derived in Section IV.4, we investigate a local maximal principle in a form similar to [LT22, Theorem 1.2].

Step 3 (Section IV.6, IV.7)**.** In Section IV.6, we construct a long-time solution of Yamabe flow with curvature decayed $|\mathrm{Rm}|(x,t) \leqslant 1/t$ (IV.4) . This approach ensures that the solution satisfies $|\mathrm{Rm}(g(t))| \leqslant 1/t$ and $\mathrm{Ric}(g(t)) \geqslant 0$ on $\mathcal{M}^n \times (0,\infty)$. Finally, in Section IV.7, since the constructed solution has an instantaneous curvature bound and nonnegative Ricci curvature, we can apply the Harnack inequality due to Bennett Chow [Cho92] and conclude flatness using the same argument as in [Ma16, Theorem 1].

IV.2 Preliminary of locally conformally flat Yamabe Flow

The advantage of locally conformally flat Yamabe flow is that the property of locally conformally flat is preserved under Yamabe flow, so the evolution equations of curvature tensor become simpler.

Lemma 15. Under a locally conformally flat Yamabe flow, the evolution equations of scalar curvature and Ricci curvature are

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta_{g(t)}\right) \mathrm{scal} = \mathrm{scal}^2, \tag{IV.6}$$

and

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta_{g(t)}\right) \mathrm{Ric}_{ij} = \frac{1}{n-2} B_{ij}, \tag{IV.7}$$

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where

$$B_{ij} = (n-1)|\operatorname{Ric}|^2 g_{ij} + n\mathrm{scal}\operatorname{Ric}_{ij} - n(n-1)\operatorname{Ric}^2_{ij} - \mathrm{scal}^2 g_{ij}. \tag{IV.8}$$

If we denote $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ to be the eigenvalue of Ric, then (IV.8) becomes

$$B_{ii} = \frac{1}{2}\sum_{j,k\neq i}(\lambda_k - \lambda_j)^2 + (n-2)\sum_{j\neq i}(\lambda_j - \lambda_i)\lambda_i \tag{IV.9}$$

These are all basic computations, we refer to the detailed proof to [Cho92, Lemma 2.2. and Lemma 2.4.]. By Lemma 15, we derive the evolution equation of the smallest eigenvalue for the Ricci tensor.

Lemma 16. Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a locally conformally flat Yamabe flow. Let ℓ be the nonnegative continuous function such that

$$\ell(x,t) := \min\{L \geqslant 0 : \operatorname{Ric}(x,t) + Lg(g,t) \geqslant 0\}, \tag{IV.10}$$

for all $(x,t) \in \mathcal{M}^n \times [0,T]$. Then

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta_{g(t)}\right)\ell \leqslant 2\mathrm{scal}\ell + n\ell^2, \tag{IV.11}$$

in the barrier sense.

Proof of Lemma 16. Consider the tensor $A = \operatorname{Ric} + \ell g$, then clearly $A \geqslant 0$ for all $(x,t) \in \mathcal{M}^n \times [0,T]$. Fix a point $(q,\tau) \in \mathcal{M}^n \times [0,T]$. If $A(q,\tau) > 0$ strictly for some $(q,\tau) \in \mathcal{M}^n \times [0,T]$, then $\ell = 0$ in a neighborhood of (q,τ) and the assertion holds trivially. We may assume that there is a unit vector $v \in T_q\mathcal{M}^n$ so that $A(v,v) = 0$. We constantly extend v along t and parallel extend v to the space direction with connection ∇^t, then we get $\partial_t v = \nabla^t v = \Delta v = 0$ at (q,τ). Note that by the definition of ℓ, v is the smallest eigenvector for $\mathrm{Ric}(q,\tau)$ and $\ell(q,\tau) = -\lambda_1$. By (IV.9), we can easily find that

$$B(v,v)|_{(q,\tau)} \geqslant -(n-2)\mathrm{scal}(q,\tau)\ell(q,\tau) - n(n-2)\ell(q,\tau)^2. \tag{IV.12}$$

Choose the barrier to be $\phi := -\mathrm{Ric}(v,v)$ in the neighborhood of (q,τ). Then

$$\begin{aligned}\left(\frac{\partial}{\partial t} - (n-1)\Delta_t\right)\phi(q,\tau) &= -\left[\left(D_{\frac{\partial}{\partial t}} - (n-1)\Delta_t\right)\mathrm{Ric}\right](v,v)\Big|_{(q,\tau)} \\ &= -\frac{B(v,v)}{n-2}(q,\tau) - \mathrm{scal}(q,\tau)\,\mathrm{Ric}(v,v) \\ &\leqslant 2\,\mathrm{scal}(q,\tau)\ell(q,\tau) + n\ell(q,\tau)^2.\end{aligned} \tag{IV.13}$$

This completes the proof. □

The following is the existence of Yamabe flow with bounded curvature.

Theorem 9. [Che25, Theorem 2.4.] Let $(\mathcal{M}^n, g)$ be an open manifold with bounded curvature $|\mathrm{Rm}(g)| \leqslant K$. Then there exist constants $\beta = \beta(n), \Lambda = \Lambda(n) > 0$ and a short-time solution of Yamabe flow starting with g such that $|\mathrm{Rm}|(g(t)) \leqslant \Lambda K$ for all $t \in [0, \frac{\beta}{K}]$. Furthermore, the conformal factor $u(x,t)^{\frac{4}{n-2}}$ is bounded from above by 2 and below by $\frac{1}{2}$ on $\mathcal{M}^n \times [0, \frac{\beta}{K}]$.

In this article, instead of describing the global existence of the Yamabe flow, we mainly use the local version of the existence of the Dirichlet-type Yamabe flow.

Theorem 10. [Che25, local version of Theorem 2.4.] Let $(\mathcal{M}^n, g)$ be a Riemannian manifold, not necessarily complete. Let $\Omega \Subset \mathcal{M}^n$ be a smooth compact domain. Then there exist two constants $\beta = \beta(n) > 0$ and $\Lambda = \Lambda(n) > 0$ such that if $\sup_\Omega |\mathrm{Rm}(g)|_g \leqslant r^{-2}$

holds for some $r > 0$, then there exists a smooth function $u \in \mathcal{C}^\infty(\Omega \times [0, \beta r^2])$ such that

$$\begin{cases} \frac{\partial}{\partial t} u(x,t)^{\frac{n+2}{n-2}} = \frac{(n+2)(n-1)}{n-2} \left(\Delta_g u(x,t) - \frac{n-2}{4(n-1)} \mathrm{scal}(g(x)) u(x,t) \right), \text{ if } (x,t) \in \Omega \times (0, \beta r^2]; \\ u(x,t) > 0, \text{ if } (x,t) \in \Omega \times (0, \beta r^2]; \\ u(x,0) = 1, \text{ if } x \in \Omega; \\ u(x,t) = 1, \text{ if } (x,t) \in \partial\Omega \times (0, \beta r^2]. \end{cases} \tag{IV.14}$$

Furthermore, we have

$$\sup_{\Omega_r \times [0, \beta r^2]} |\mathrm{Rm}(g(t))|_{g(t)} \leqslant \Lambda r^{-2} \text{ and } \frac{1}{2} \leqslant \min_{\Omega \times [0, \beta r^2]} u(x,t) \leqslant \max_{\Omega \times [0, \beta r^2]} u(x,t) \leqslant \frac{3}{2}, \tag{IV.15}$$

where $\Omega_r := \{x \in \Omega : \mathrm{dist}_0(x, \partial\Omega) > r\}$.

In Cheng's argument [Che25, Section 2], one can mollify the argument in the local setting, that is, to yield the interior curvature bound away from the boundary. We leave the detailed proof to the interested reader. In fact, according to the curvature equation $(\partial_t - (n-1)\Delta)\,\mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}$ for Yamabe flow, we can show a curvature bound in terms of the initial curvature bound assuming a uniform bound of conformal factor, which is the following statement.

Proposition 7. Let $(\mathcal{M}^n, g(t) = u(t)^{\frac{4}{n-2}} g(0))$ be a locally conformally flat Yamabe flow and $\Omega \Subset \mathcal{M}^n$ satisfy $|\mathrm{Rm}(g(0))| \leqslant r^{-2}$ on Ω for some $r > 0$. Suppose that $a^{-1} \leqslant u(t) \leqslant a$ holds on Ω for some $a > 1$. Then there are some constants $c(n,a), \hat{T}(n,a) > 0$ such that

$$|\mathrm{Rm}(g(x,t))| \leqslant c r^{-2} \leqslant \frac{c\hat{T}}{t}, \tag{IV.16}$$

for all $t \in (0, \min\{T, r^2 \hat{T}\}]$ and $x \in \Omega_r := \{x \in \Omega : \mathrm{dist}_0(x, \partial\Omega) \geqslant r\}$.

Proof of Proposition 7. The argument is the same as the argument in [Che25, Section 2], and the proof is omitted here. ☐

IV.3 Long-Time Solution of (IV.4)

We use the same argument as [Ma16] did but give a more precise estimation to accomplish **Step 1.**. First, under the assumption (IV.3), if the manifold is non-flat, then there is a ball $B_g(p,R)$ such that scal $\neq 0$ on $B_g(p,R)$. Therefore, for any $x \in \mathcal{M}^n$, we have

$$\int_{d_g(x,p)+R}^{\infty}\frac{r}{\mathrm{Vol}_g(x,r)}dr \leqslant \frac{1}{\int_{B_g(p,R)}\mathrm{scal}(y)dy}\int_0^{\infty} r\fint_{B_g(x,r)}\mathrm{scal}(y)d\,\mathrm{Vol}_g(y)dr \leqslant \frac{\varepsilon}{\int_{B_g(p,R)}\mathrm{scal}(y)dy}. \tag{IV.17}$$

This implies that under the assumption (IV.3) and non-flat, $(\mathcal{M}^n, g)$ is non-parabolic. This gives a useful estimate for the green function, that is, by a result in [LY86], there is a constant $C = C(n) > 0$ such that

$$C^{-1}\int_{d_g(x,y)}^{\infty}\frac{r}{\mathrm{Vol}_g(x,r)}dr \leqslant G(x,y) \leqslant C\int_{d_g(x,y)}^{\infty}\frac{r}{\mathrm{Vol}_g(x,r)}dr, \tag{IV.18}$$

for all $x \neq y$. Back to (IV.4), the short-time existence follows from the smooth initial condition. To obtain a long-time existence, it suffices to prove that u is uniformly bounded in $\omega \times (0,T)$. Since $\mathrm{scal}(g) \geqslant 0$ holds, by (IV.4) and the maximal principle, we get

$$u(x,t) \leqslant 1, \tag{IV.19}$$

on $\Omega \times (0,T)$. Now, consider the positive function $w \in \mathcal{C}^{\infty}(\mathcal{M}^n)$ satisfies

$$\Delta w = -\frac{n-2}{4(n-1)}\mathrm{scal}(g_0). \tag{IV.20}$$

Then

$$\frac{d}{dt}(\log u + w) \geqslant (n-1)u^{-\frac{4}{n-2}}\Delta_g(\log u + w). \tag{IV.21}$$

Since $\log u + w \geqslant 0$ on the parabolic neighborhood of $\Omega \times (0, T)$. Applying the minimal principle again, we get

$$e^{-w(x)} \leqslant u(x,t) \leqslant 1, \tag{IV.22}$$

for all $(x,t) \in \Omega \times [0,T)$. This implies we can extend the life span of the solution to a whole nonnegative time. By the representation formula of the Green function and (IV.18), we get

$$\begin{aligned} w(x) &= \frac{n-2}{4(n-1)} \int_{\mathcal{M}^n} G(x,y)\mathrm{scal}_g(y) d\,\mathrm{Vol}(y) \\ &\leqslant C \int_{\mathcal{M}^n} \mathrm{scal}_g(y) \int_{d_g(x,y)}^{\infty} \frac{r}{\mathrm{Vol}_g(x,r)} dr d\,\mathrm{Vol}(y) \\ &= C \int_0^{\infty} \frac{r}{\mathrm{Vol}_g(x,r)} \int_{B_g(x,r)} \mathrm{scal}_g(y) d\,\mathrm{Vol}(y) dr \\ &\leqslant C\varepsilon, \end{aligned} \tag{IV.23}$$

and this implies $e^{-C\varepsilon} \leqslant u(x,t) \leqslant 1$ on $\Omega \times [0,\infty)$ for some constant C only depends on n, which means that the conformal factor u is uniformly bounded on $\Omega \times [0,\infty)$. Taking an exhaustion $\Omega_1 \Subset \Omega_2 \Subset \cdots$, since the conformal factor is uniformly equivalent, u_k converges to a smooth and long-time solution of Yamabe flow of u satisfying (IV.4) on $\mathcal{M}^n$ by standard Schauder estimate. In summary, we prove the following.

Proposition 8. Let $(\mathcal{M}^n, g)$ be defined as above. Then there exists a unique smooth positive solution $u \in \mathcal{C}^\infty(\mathcal{M}^n \times (0,+\infty))$ of (IV.4) with $\Omega = \mathcal{M}^n$. Moreover, there exists a dimensional constant $C = C(n) > 0$ so that

$$e^{-C\varepsilon} \leqslant u(x,t) \leqslant 1, \tag{IV.24}$$

for all $(x,t) \in \mathcal{M}^n \times [0,+\infty)$.

IV.4 An Upper Bound of Heat Kernel

In this section, we aim to derive the heat kernel's upper bound. First, we investigate the definition of the heat kernel.

Definition 4. On a Yamabe flow $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ and an open set $\Omega \Subset \mathcal{M}^n$ with smooth boundary, we define **the adjoint Dirichlet heat kernel** $K_\Omega(x,t;y,s)$ **with respect to the backward heat equation** to be a smooth function so that K_Ω is positive in the interior, and for $0 \leqslant s < t \leqslant T$,

$$\begin{cases} \left(\frac{\partial}{\partial t} - (n-1)\Delta_{g(t),x} - \frac{n}{2}\mathrm{scal}_{g(t),x}\right) K_\Omega(x,t;y,s) = 0, \text{ on } \Omega\times\Omega\times(s,T]; \\ \lim_{t\to s^+} K_\Omega(x,t;y,s) = \delta_y(x), \text{ for } y\in\Omega; \\ K_\Omega(x,t;y,s) = 0, \text{ if either } x\in\partial\Omega \text{ or } y\in\partial\Omega, \end{cases} \tag{IV.25}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial s} + (n-1)\Delta_{g(s),y}\right) K_\Omega(x,t;y,s) = 0, \text{ on } \Omega\times\Omega\times[0,s); \\ \lim_{s\to t^-} K_\Omega(x,t;y,s) = \delta_x(y), \text{ for } x\in\Omega; \\ K_\Omega(x,t;y,s) = 0, \text{ if either } x\in\partial\Omega \text{ or } y\in\partial\Omega, \end{cases} \tag{IV.26}$$

where δ_y is the Dirac measure at y.

Like Ricci flow, we aim to achieve an upper bound of the heat kernel where only some scaling invariant assumptions, for instance, the curvature decay $|\mathrm{Rm}| \leqslant \frac{c}{t}$. But the biggest difference is that **Yamabe flow is not a super Ricci flow**, we cannot apply the classical distance distortion lemma to detect the changing of distance function between two different time slices. Therefore, we make an additional assumption, which is usually

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unachievable but holds in our case: that uniform boundness of conformal factor $c^{-1} \leqslant u(x,t) \leqslant c$.

Theorem 11. Let $A \geqslant 1$ be a constant. Then there exists a constant $C = C(n, A) > 0$ such that the following holds: For any Yamabe flow $(\mathcal{M}^n, g(t))_{t \in [0,1]}$ with the following properties:

$$|\mathrm{Rm}(g(t))| \leqslant \frac{A}{t} \text{ and } A^{-1} g(0) \leqslant g(t) \leqslant A g(0), \tag{IV.27}$$

for all $(x,t) \in \mathcal{M}^n \times (0,1]$. Let $p \in \mathcal{M}^n$ be some point with $B_t(p, 4r) \Subset \mathcal{M}^n$ for some $r \geqslant 1$ for all $t \in [0,1]$. Suppose that $\Omega \Subset \mathcal{M}^n$ is a relatively compact domain with a smooth boundary and $\Omega \Subset B_t(p, r)$ for all $t \in [0,1]$ and $K_\Omega(x,t;y,s)$ is the adjoint Dirichlet heat kernel with respect to the backward heat equation. Then the heat kernel satisfies

$$K_\Omega(x,t;y,s) \leqslant \frac{C}{\mathrm{Vol}_t(B_t(x, \sqrt{t-s}))} \exp\left(-\frac{d_0(x,y)^2}{C(t-s)}\right), \tag{IV.28}$$

for all $0 \leqslant s < t \leqslant 1$ and $x, y \in \Omega$.

Remark 7. The size of the geodesic ball ensures the existence of minimizing geodesic joining $x, y \in \Omega$.

The proof is inspired by the groundbreaking work about heat kernel estimate in [BCRW19] and many other improvements in Ricci flow, e.g. [CL23, LT22]. Before proving Theorem 11, we first consider the case that curvature is bounded in the whole spacetime. This lemma was implicitly proved in [CTY11].

Lemma 17. [Heat kernel upper bounds in bounded curvature case] Let $(\mathcal{M}^n, g(t))_{t \in [0,1]}$ be a Yamabe flow such that $\sup_{\mathcal{M}^n \times [0,1]} |\nabla \mathrm{Rm}| + |\mathrm{scal}| + |\nabla \mathrm{scal}| + |\Delta \mathrm{scal}| < \infty$. Let $p \in \mathcal{M}^n$ be a point with $B_0(p, 2(r+1)) \Subset \mathcal{M}^n$ for some $r \geqslant 1$. Suppose that $\Omega \Subset B_0(p, r)$ is an open set with a smooth boundary and $K_\Omega(x,t;y,s)$ is the adjoint Dirichlet heat kernel with respect to the backward heat equation. If $\sup_{\mathcal{M}^n\times[0,1]}|\mathrm{Rm}|\leqslant M$ for some constant $M>0$, then we have

$$K_\Omega(x,t;y,s)\leqslant\frac{C}{\mathrm{Vol}_\tau(B_\tau(x,\sqrt{t-s}))}\exp\left(-\frac{d_\tau(x,y)^2}{C(t-s)}\right),\tag{IV.29}$$

and

$$K_\Omega(x,t;y,s)\leqslant\frac{C}{\mathrm{Vol}_\tau(B_\tau(y,\sqrt{t-s}))}\exp\left(-\frac{d_\tau(x,y)^2}{C(t-s)}\right),\tag{IV.30}$$

for all $0\leqslant s<t\leqslant 1$ and $\tau\in[0,1]$, where C depends on n,M.

Proof of Lemma 17. The proof is the same as [LT22, Lemma 4.1.] and [CL23, Lemma 2.1.] since it doesn't use any Ricci flow property. So we omit the proof. ☐

Using Lemma 17, we can now proceed as in [BCRW19][Proposition 3.1.] but instead **distance distortion lemma** with **uniformly equivalence of distance**, to obtain our assertion.

Proof of Theorem 11. From now on, the constant C would only depend on n and A and may vary from line to line. After a parabolic rescaling and time-shifting, we may assume $t=1$ and $s=0$. Therefore, it's equivalent to show that

$$K_\Omega(x,1;y,0)\leqslant\frac{C}{\mathrm{Vol}_1(B_1(x,1))}\exp\left(-\frac{d_0(x,y)^2}{C}\right),\tag{IV.31}$$

since the metrics are all equivalent. Take a subdivision $\{[t_{k+1},t_k]\}$ of $(0,1]$ with $t_k=16^{-k}$. Then by Lemma 17, after we rescale the metric on $[t_{k+1},t_k]$, we get

$$K_\Omega(x,t_k;y,t_{k+1})\leqslant\frac{C}{\mathrm{Vol}_\tau(B_\tau(x,\sqrt{t_k-t_{k+1}}))}\exp\left(-\frac{d_\tau(x,y)^2}{C(t_k-t_{k+1})}\right),\tag{IV.32}$$

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and

$$K_\Omega(x,t_k;y,t_{k+1}) \leqslant \frac{C}{\mathrm{Vol}_\tau(B_\tau(y,\sqrt{t_k-t_{k+1}}))}\exp\left(-\frac{d_\tau(x,y)^2}{C(t_k-t_{k+1})}\right), \tag{IV.33}$$

for all $k \geqslant 0$, $\tau \in [t_{k+1},t_k]$, and $x,y \in \Omega$. In particular, we put $k=0$ and $\tau = t_1$,

$$K_\Omega(x,1;y,t_1) \leqslant \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\exp\left(-\frac{d_{t_1}(x,y)^2}{C(1-t_1)}\right), \tag{IV.34}$$

for all $y \in \Omega$. By maximal principle on $\Omega \times [0,t_1]$, we get

$$K_\Omega(x,1;\cdot,\cdot) \leqslant \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))} \text{ on } \Omega\times[0,t_1]. \tag{IV.35}$$

Let $d \geqslant 1$ be a large constant depending on n, A to be determined later. In fact, by volume comparison, (IV.35) proves the (IV.31) for $d_0(x,y) \leqslant 4d$, so the remaining problem is for $d_0(x,y) > 4d$. Let $r_k := 4d(1-2^{-k})$ for all $k \geqslant 0$ and we make the following claim.

Claim 6. Define a_k by

$$a_k := \begin{cases} \sup_{\Omega\backslash B_0(x,r_k)} K_\Omega(x,1;\cdot,t_k), \text{ if } \Omega\backslash B_0(x,r_k) \neq \varnothing; \\ 0, \text{ otherwise.} \end{cases} \tag{IV.36}$$

Then there exist two constants $C = C(n,A) > 0$ and $\underline{d} = \underline{d}(n,A) > 1$ so that

$$a_{k+1} \leqslant \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\exp\left(-\frac{d^2}{C}\right), \tag{IV.37}$$

for all $k \geqslant 1$ and $d \geqslant \underline{d}$.

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Note that by the continuity of K_Ω, we have

$$\lim_{k\to\infty} a_{k+1} \geqslant \sup_{\Omega\backslash B_0(x,4d)} K_\Omega(x,1;\cdot,0). \tag{IV.38}$$

If $d_0(x,y) \leqslant 4\underline{d}$, then previous discussion implies the assertion holds. For $d_0(x,y) > 4\underline{d}$, take $d = \frac{d_0(x,y)}{4}$. According to this claim, we get

$$\begin{aligned} K_\Omega(x,1;y,0) &\leqslant \lim_{k\to\infty} a_{k+1} \\ &\leqslant \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))} \exp\left(-\frac{d_0(x,y)^2}{16C}\right) \\ &\leqslant \frac{C}{\mathrm{Vol}_1 B_1(x,1)} \exp\left(-\frac{d_0(x,y)^2}{16C}\right), \end{aligned} \tag{IV.39}$$

where the last inequality follows from the volume comparison and the equivalent of metrics. It remains to prove the claim.

Proof of Claim 6. We give an inductive argument on k. $k = 0$ holds by (IV.34). For $k > 0$, the semi-group property shows that

$$K_\Omega(x,1;z,t_{k+1}) = \int_\Omega K_\Omega(x,1;y,t_k) K_\Omega(y,t_k;z,t_{k+1}) d\,\mathrm{Vol}_{t_k}(y), \tag{IV.40}$$

for $z \in \Omega\backslash B_0(x,r_{k+1})$. We separate the integral into the integral over $B_{t_k}(z,\sqrt{A}(r_{k+1} - r_k)) \cap \Omega$ and $\Omega\backslash B_{t_k}(z,\sqrt{A}(r_{k+1}-r_k))$. For $y \in B_{t_k}(z,\sqrt{A}(r_{k+1}-r_k)) \cap \Omega$, we have

$$d_0(x,y) \geqslant d_0(x,z) - d_0(y,z) \geqslant r_{k+1} - \frac{\sqrt{A}(r_{k+1}-r_k)}{\sqrt{A}} \geqslant r_k, \tag{IV.41}$$

this implies $y \in \Omega\backslash B_0(x,r_k)$. Therefore, by the definition of a_k, we have

$$\int_{B_{t_k}(z,\sqrt{A}(r_{k+1}-r_k))\cap\Omega} K_\Omega(x,1;y,t_k) K_\Omega(y,t_k;z,t_{k+1}) d\,\mathrm{Vol}_{t_k}(y) \leqslant a_k. \tag{IV.42}$$

On the other hands, by (IV.33), (IV.35), and volume comparison, we get

$$
\begin{aligned}
&\int_{\Omega\setminus B_{t_k}(z,\sqrt{A}(r_{k+1}-r_k))} K_\Omega(x,1;y,t_k)K_\Omega(y,t_k;z,t_{k+1})\,d\,\mathrm{Vol}_{t_k}(y)\\
\leqslant{}& \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\int_{\Omega\setminus B_{t_k}(z,\sqrt{A}(r_{k+1}-r_k))}\frac{C}{\mathrm{Vol}_{t_k}(B_{t_k}(z,\sqrt{t_k-t_{k+1}}))}\exp\left(-\frac{d_{t_k}(z,y)^2}{C(t_k-t_{k+1})}\right)d\,\mathrm{Vol}_{t_k}(y)\\
\leqslant{}& \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\frac{1}{\mathrm{Vol}_{t_k}(B_{t_k}(z,\sqrt{t_k-t_{k+1}}))}\int_{\sqrt{A}(r_{k+1}-r_k)}^{\mathrm{diam}_{t_k}(\Omega)}\exp\left(-\frac{r^2}{C(t_k-t_{k+1})}\right)\mathrm{Vol}_{t_k}(\partial B_{t_k}(z,r))dr\\
\leqslant{}& \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\frac{\mathrm{Vol}_{t_k}(B_{t_k}(z,\sqrt{A}(r_{k+1}-r_k)))}{\mathrm{Vol}_{t_k}(B_{t_k}(z,\sqrt{t_k-t_{k+1}}))}\exp\left(-\frac{A(r_{k+1}-r_k)^2}{C(t_k-t_{k+1})}\right)\\
&+\frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\int_{\sqrt{A}(r_{k+1}-r_k)}^{\infty}\frac{2r}{C(t_k-t_{k+1})}\exp\left(-\frac{r^2}{C(t_k-t_{k+1})}\right)\frac{\mathrm{Vol}_{t_k}(B_{t_k}(z,r))}{\mathrm{Vol}_{t_k}(B_{t_k}(z,\sqrt{t_k-t_{k+1}}))}dr\\
\leqslant{}& \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\exp\left(-\frac{4^{-k}d^2}{C(t_k-t_{k+1})}\right).
\end{aligned}
\tag{IV.43}
$$

Combining all of these estimates and (IV.34), we obtain

$$
\begin{aligned}
a_{k+1}&\leqslant a_k+\frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\exp\left(-\frac{4^{-k}d^2}{C(t_k-t_{k+1})}\right)\\
&\leqslant \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\exp\left(-\frac{A^{-1}d^2}{C(1-t_1)}\right)+\frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\sum_{i=1}^{k}\exp\left(-\frac{4^i d^2}{C}\right)\\
&\leqslant \frac{C}{\mathrm{Vol}_{t_1}(B_{t_1}(x,\sqrt{1-t_1}))}\exp\left(-\frac{d^2}{C}\right)\left[1+\sum_{i=1}^{\infty}\exp\left(-\frac{(4^i-1)d^2}{C}\right)\right].
\end{aligned}
\tag{IV.44}
$$

Hence, we must only choose $\underline{d}\geqslant\sqrt{C}$ to complete the proof. □

□

Remark 8. We can replace the Yamabe flow with a more general geometric flow satisfying the control $|\frac{\partial}{\partial t}g(t)|+|\mathrm{Rm}(g(t))|\leqslant\frac{A}{t}$ and the uniform equivalence of metric $A^{-1}d_0\leqslant d_t\leqslant Ad_0$. The proof is the same, and we leave it to interested readers to verify.

IV.5 Local Maximal Principle

Using the function we constructed in the previous section, we can study the Ricci curvature along the locally conformally flat Yamabe flow under the scaling-invariant estimate $|\mathrm{Rm}| \leqslant \frac{c}{t}$ for $t > 0$ and the uniformly equivalent condition. The following statement is a local maximal principle for the Yamabe flow and will be applied several times in this article.

Theorem 12. Let $(\mathcal{M}^n, g(0))$ be a complete Riemannian manifold with $n \geqslant 5$. Let $(B_{g(0)}(p, 2a^{\frac{2}{n-2}}), g(t) = u(t)^{\frac{4}{n-2}} g(0))_{t\in[0,T]}$ be an locally conformally flat Yamabe flow with initial metric $g(0)$ for some $p \in \mathcal{M}^n$ and a constant $a > 1$. Suppose that

$$|\mathrm{Rm}(g(x,t))| \leqslant \frac{a}{t}; \tag{IV.45}$$

$$\mathrm{Ric}(g(x,0)) \geqslant 0; \tag{IV.46}$$

$$\mathrm{scal}(g(x,t)) \geqslant -1; \tag{IV.47}$$

$$a^{-1} \leqslant u(x,t) \leqslant a, \tag{IV.48}$$

holds on $B_{g(0)}(p, 2a^{\frac{2}{n-2}}) \times [0,T]$. Let ℓ be a non-negative continuous function on $B_{g(0)}(p, 2a^{\frac{2}{n-2}}) \times [0,T]$ with $\ell \leqslant \frac{a}{t}$ on $B_{g(0)}(p, 2a^{\frac{2}{n-2}}) \times (0,T]$. Assume that ℓ satisfies

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta_{g(t)}\right)\ell\Big|_{(x_0,t_0)} \leqslant 2\mathrm{scal}(g(x_0,t_0))\ell(x_0,t_0) + K\ell(x_0,t_0)^2, \tag{IV.49}$$

in the barrier sense for some constant $K \geqslant 0$ and $\ell(x_0,t_0) > 0$. Suppose that $\ell(x,0) \leqslant 0$ holds on $B_0(p, 2a^{\frac{2}{n-2}})$. Then there exists $\Lambda(n,a,K) > 0$ such that for $t \in [0,T]$,

$$\ell(p,t) \leqslant \Lambda. \tag{IV.50}$$

Proof of Theorem 12. First of all, we consider a stronger but scaling-invariant statement. We fix $\Lambda > a$, which would be determined later. We claim the following:

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Claim 7. There is a constant $\Lambda > 0$ depending on n, a such that

$$\ell(x,t) \leqslant \frac{\Lambda}{\mathrm{dist}_0(x, \partial B_0(p,1))^2}, \tag{IV.51}$$

on $B_0(p,1) \times [0,T]$.

It suffices to prove the claim. By the smoothness of the solution, there is the largest time $t_0 \in [0,T]$(may depend on the flow) such that (IV.51) holds on $B_0(p,1) \times [0,t_0]$. We aim to show that if $t_0 < T$, then Λ has an upper bound that only depends on a, n. Assume there is a time $t_0 < T$ and a point $x_0 \in B_0(p,1)$ such that $\ell(x_0,t_0) = \Lambda \mathrm{dist}_0(x_0, \partial B_0(x_0,1))^{-2}$ and $\ell(x,t) \leqslant \Lambda \mathrm{dist}_0(x, \partial B_0(x,1))^{-2}$ on $B_0(p,1) \times [0,t_0]$. Choose $\rho_0 := \mathrm{dist}_0(x_0, \partial B_0(x_0,1))$. Hence, on $B_0(x_0, \frac{1}{2}\rho_0) \times [0,t_0]$, we have

$$\ell(x,t) \leqslant \frac{\Lambda}{\mathrm{dist}_0(x, \partial B_0(p,1))^2} \leqslant 4\Lambda \rho_0^{-2}. \tag{IV.52}$$

Now we perform the following parabolic scaling:

$$\begin{aligned} \tilde{g}(x,t) &:= \rho_0^{-2} g\left(x, \rho_0^2 t\right); \\ \tilde{\ell}(x,t) &:= \rho_0^2 \ell\left(x, \rho_0^2 t\right); \\ \tilde{t_0} &:= \rho_0^{-2} t_0. \end{aligned} \tag{IV.53}$$

Since all other quantities are scaling invariant and $\tilde{t_0} \leqslant \frac{a}{\Lambda}$, we may assume $\rho_0 = 1$ and $x_0 = p$. So $\ell \leqslant 4\Lambda$ on $B_0(p, \frac{1}{2}) \times [0,t_0]$, $\ell(p,t_0) = \Lambda$ and $t_0 \leqslant \frac{a}{\Lambda} \leqslant 1$. By Lemma 16, if we set $F(x,t) := \exp(-nK\Lambda t)\ell(x,t)^{\frac{n}{4}}$, then

$$\Box F(x,t) \leqslant \frac{n}{2}\mathrm{scal}(g(t))F(x,t) - \frac{(n-1)(n-4)}{n}F(x,t)|\nabla_t \log F|^2, \tag{IV.54}$$

in the sense of a barrier on $B_0(p, \frac{1}{2}) \times [0,t_0]$, where $\Box := \frac{\partial}{\partial t} - (n-1)\Delta_{g(t)}$ is the abbreviation of heat operator. Also, $F(x,t) \leqslant (4\Lambda)^{\frac{n}{4}}$ on $B_0(x_0, \frac{1}{2}) \times [0, t_0]$. Now for $\epsilon \in (0,1)$, take a non-increasing cut-off function $\varphi : [0,\infty) \to [0,1]$ with $\varphi \equiv 1$ on $[0, \frac{1}{2}]$, $\operatorname{supp} \varphi \subset [0, \frac{3}{4}]$ and $|\varphi'|^2 \leqslant \varphi^{2-\epsilon}$. Define $\Phi(x,t) := \varphi\left(4d_0(x,x_0)^2\right)$. Then

$$\begin{aligned}\Box\Phi(x,t) = &-4(n-1)u^{-\frac{4}{n-2}}\varphi'\left(\Delta_0 d_0(x,x_0)^2 + 2\nabla_0 \log u \cdot \nabla_0 d_0(x,x_0)^2\right)\\ &-16(n-1)u^{-\frac{4}{n-2}}\varphi''|\nabla_0 d_0(x,x_0)^2|^2.\end{aligned} \tag{IV.55}$$

Also, for any $\alpha > 1 + \frac{3}{n-1}$, the evolution equation of u^α is

$$\begin{aligned}\Box u^\alpha &= \alpha u^{\alpha - \frac{4}{n-2}}\left[(2-(n-1)(\alpha-1))|\nabla_0 \log u|^2 - \frac{n-2}{4}\mathrm{scal}(g(0))\right]\\ &\leqslant \alpha(2-(n-1)(\alpha-1))|\nabla_t \log u|^2 u^\alpha + \frac{(n-1)\alpha a^{\alpha-\frac{4}{n-2}}}{4}.\end{aligned} \tag{IV.56}$$

By the Laplacian comparison on $g(0)$, we have $\Delta_0 d_0(x,x_0)^2 \leqslant 2n$ in the sense of barrier. Combining with (IV.48) and $\varphi'' \geqslant -c\varphi$ for some universal constant $c > 0$, we get

$$\begin{aligned}\Box(\Phi u^\alpha) \leqslant &-C_1\Phi u^\alpha|\nabla_t \log u|^2 + \Phi u^\alpha\left[C_2\left(1+\frac{|\varphi'|}{\varphi}\right) + 8(n-1)\left|\nabla_t \log u \cdot \nabla_t d_0(x,x_0)^2\right|\frac{|\varphi'|}{\varphi}\right]\\ &+ 8\Phi u^\alpha(n-1)\frac{|\varphi'|}{\varphi}\alpha\left|\nabla_t d_0(x,x_0)^2 \cdot \nabla_t \log u\right|\\ \leqslant &-\frac{C_1}{2}\Phi u^\alpha|\nabla_t \log u|^2 + C_2\Phi u^\alpha\left(1+\varphi^{-\epsilon}\right).\end{aligned} \tag{IV.57}$$

for some constant $C_1, C_2 > 0$ depending on n, c, α, a, where we apply the Cauchy-Schwartz inequality in the last inequality. This induces us to consider the quantity $G(x,t) := \Phi(x,t)(F(x,t) + \Lambda^{\frac{n-1}{4}}u(x,t)^\alpha)$. By (IV.54), (IV.55) and (IV.57),

$$\begin{aligned}\Box G \leqslant\, &\frac{n}{2}\mathrm{scal}(g(t))G + FC\left(|\varphi'| + \varphi + |\varphi'||\nabla_t \log u|\right) - \frac{(n-1)(n-4)}{n}\Phi F|\nabla_t \log F|^2\\ &-2(n-1)\Phi F\nabla_t \log F \cdot \nabla_t \log \Phi + \Lambda^{\frac{n-1}{4}}\left[-\frac{C_1}{2}\Phi u^\alpha|\nabla_t \log u|^2 + C_2\Phi u^\alpha\left(1+\varphi^{-\epsilon}\right)\right]\\ \leqslant\, &\frac{n}{2}\mathrm{scal}(g(t))G + CF(\Phi^{1-\epsilon} + \Phi) + C\frac{F^2}{\Lambda^{\frac{n-1}{4}}}\Phi^{1-\epsilon} + \Lambda^{\frac{n-1}{4}}C\Phi u^\alpha + \Lambda^{\frac{n-1}{4}}C\Phi^{1-\epsilon}u^\alpha,\end{aligned} \tag{IV.58}$$

where C is a constant only depending on n, ϵ, a, α, we apply the Cauchy-Schwartz inequality in the last line. Now we define C as the constant that only depends on n, ϵ, a, α and may change from line to line. Since $F \leqslant (4\Lambda)^{\frac{n}{4}}$ and $\Phi \leqslant 1$,

$$\begin{aligned}\Box G &\leqslant \frac{n}{2}\mathrm{scal}(g(t))G + CF(\Phi^{1-\epsilon}+\Phi) + C\frac{F^2}{\Lambda^{\frac{n-1}{4}}}\Phi^{1-\epsilon} + \Lambda^{\frac{n-1}{4}}C\Phi u^{\alpha} + \Lambda^{\frac{n-1}{4}}C\Phi^{1-\epsilon}u^{\alpha} \\ &\leqslant \frac{n}{2}\mathrm{scal}(g(t))G + C\Lambda^{\frac{n+1}{4}} + C\Lambda^{\frac{n-1}{4}}.\end{aligned} \tag{IV.59}$$

Now, let $K := K_{\Omega}$ be the heat kernel which be defined in Definition 4 with some smooth domain $B_0(p, \frac{\sqrt{3}}{4}) \subset \Omega \subset B_0(p, \frac{1}{2})$. By Theorem 11,

$$K(x,t;y,s) \leqslant \frac{C}{\mathrm{Vol}_t(x, \sqrt{t-s})}\exp\left(-\frac{d_0(x,y)^2}{C(t-s)}\right) \leqslant \frac{C}{\mathrm{Vol}_0(x, C\sqrt{t-s})}\exp\left(-\frac{d_0(x,y)^2}{C(t-s)}\right), \tag{IV.60}$$

for all $x, y \in \Omega$ and $0 \leqslant s < t \leqslant 1$, the last inequality follows from the Bishop-Gromov volume comparison. Therefore, for $(x,t) \in \Omega \times [0, t_0]$, (IV.59) and (IV.60) imply

$$\begin{aligned}G(x,t) &= \lim_{s\to t^-}\int_{\Omega}K(x,t;y,s)G(y,s)d\,\mathrm{Vol}_s(y) \\ &= \int_0^t \frac{\partial}{\partial s}\int_{\Omega}K(x,t;y,s)G(y,s)d\,\mathrm{Vol}_s(y)ds + \int_{\Omega}K(x,t;y,0)G(y,0)d\,\mathrm{Vol}_0(y) \\ &= \int_0^t\int_{\Omega}K(x,t;y,s)\left(\Box_{s,y} - \frac{n}{2}\mathrm{scal}(g(s))\right)G(y,s)d\,\mathrm{Vol}_s(y)ds \\ &\quad + \int_{\Omega}K(x,t;y,0)\varphi(y,0)d\,\mathrm{Vol}_0(y) \\ &\leqslant C(\Lambda^{\frac{n+1}{4}} + \Lambda^{\frac{n-1}{4}})\int_0^t\int_{B_0(x,1)}\frac{1}{\mathrm{Vol}_0(x, C\sqrt{t-s})}\exp\left(-\frac{d_0(x,y)^2}{C(t-s)}\right)d\,\mathrm{Vol}_0(y)ds \\ &\quad + \Lambda^{\frac{n-1}{4}}\int_{B_0(x,1)}\frac{1}{\mathrm{Vol}_0(x, C\sqrt{t})}\exp\left(-\frac{d_0(x,y)^2}{Ct}\right)d\,\mathrm{Vol}_0(y).\end{aligned} \tag{IV.61}$$

Note that K and G vanish on $\partial\Omega$ so there is no boundary term above. For the last term, if we denote $A(x,w)$ the g_0-area of $\partial B_0(x,w)$, by the co-area formula,

$$\begin{aligned}
&\int_0^t \int_{B_0(x,1)} \frac{1}{\mathrm{Vol}_0(x, C\sqrt{t-s})} \exp\left(-\frac{d_0(x,y)^2}{C(t-s)}\right) d\,\mathrm{Vol}_0(y) ds \\
=& \int_0^t \int_0^1 \frac{A(x,w)}{\mathrm{Vol}_0(x, C\sqrt{s})} \exp\left(-\frac{w^2}{Cs}\right) dwds \\
=& \int_0^t \frac{\mathrm{Vol}_0(x,1)}{\mathrm{Vol}_0(x, C\sqrt{s})} \exp\left(-\frac{1}{Cs}\right) + \int_0^1 \frac{2w}{Cs} \frac{\mathrm{Vol}_0(x,w)}{\mathrm{Vol}_0(x, C\sqrt{s})} \exp\left(-\frac{w^2}{Cs}\right) dwds \\
\leqslant& \int_0^t \max\{1, (C\sqrt{s})^{-\frac{n}{2}}\} \exp\left(-\frac{1}{Cs}\right) ds + \frac{2}{C} \int_0^t \int_0^\infty z \max\{1, z^n\} \exp\left(-\frac{z^2}{C}\right) dzdw \\
\leqslant& Ct.
\end{aligned} \tag{IV.62}$$

and

$$\begin{aligned}
&\int_{B_0(x,1)} \frac{1}{\mathrm{Vol}_0(x, C\sqrt{t})} \exp\left(-\frac{d_0(x,y)^2}{Ct}\right) d\,\mathrm{Vol}_0(y) \\
=& \int_0^1 \frac{A(x,w)}{\mathrm{Vol}_0(x, C\sqrt{t})} \exp\left(-\frac{w^2}{Ct}\right) dw \\
=& \frac{\mathrm{Vol}_0(x,1)}{\mathrm{Vol}_0(x, C\sqrt{t})} \exp\left(-\frac{1}{Ct}\right) + \frac{2}{C} \int_0^1 \frac{w}{t} \frac{\mathrm{Vol}_0(x,w)}{\mathrm{Vol}_0(x, C\sqrt{t})} \exp\left(-\frac{w^2}{Ct}\right) dw \\
\leqslant& \sup_{z>0} z^n \exp(-Cz^2) + C \int_0^\infty z \max\{1, z^n\} \exp\left(-\frac{z^2}{C}\right) dz \\
=:& \hat{C}.
\end{aligned} \tag{IV.63}$$

Therefore, we get $G(x,t) \leqslant C(\Lambda^{\frac{n+1}{4}} + \Lambda^{\frac{n-1}{4}})t + \hat{C}\Lambda^{\frac{n-1}{4}}$ on $\Omega \times [0, t_0]$. Since $\Lambda t_0 \leqslant a$, at (p, t_0), we have

$$\exp(-nKa)(4\Lambda)^{\frac{n}{4}} \leqslant G(p,t_0) \leqslant C(\Lambda^{\frac{n+1}{4}} + \Lambda^{\frac{n-1}{4}})t_0 + \hat{C}\Lambda^{\frac{n-1}{4}} \leqslant Ca(\Lambda^{\frac{n-3}{4}} + \Lambda^{\frac{n-5}{4}}) + \hat{C}\Lambda^{\frac{n-1}{4}}. \tag{IV.64}$$

Choose $\Lambda := \exp(4nKa) \max\{(2Ca + \hat{C} + 1)^4 4^n, a\}$, then we get a contradiction. □

Now we provide a quantitative estimate for the lower bound of scalar curvature along Yamabe flow.

Lemma 18. Suppose g_0 is a metric on $\mathcal{M}^n$ and $x_0 \in \mathcal{M}^n$ is a point such that

1. $B_{g_0}(x_0, r) \Subset \mathcal{M}^n$;

2. $\mathrm{Ric}(g_0) \geqslant -(n-1)r^{-2}$.

Let $g(t)$ be a smooth solution to the Yamabe flow on $\mathcal{M}^n \times [0, Tr^2]$ (not necessarily complete) with $g(0) = g_0$ such that

$$\alpha^{-1} g_0 \leqslant g(t) \leqslant \alpha g_0$$

on $\mathcal{M}^n \times (0, Tr^2]$ for some $\alpha > 1$ and $r > 0$. Then there is $\Lambda_1(n, \alpha) > 0$ such that for all $t \in (0, Tr^2]$, $\mathrm{scal}(g(x_0, t)) \geqslant -\Lambda_1 r^{-2}$.

Proof. By rescaling, we might assume $r = 1$. This follows from a simpler modification of the proof of Lemma 12. We consider the test function

$$F = \phi\left(d_{g_0}(x, x_0)\right) \cdot \varphi + L_0 u^{\gamma} \tag{IV.65}$$

where $\varphi = \mathrm{scal}_-$, L_0 is a large constant, $g(t) = u^{\frac{4}{n-2}} g_0$ and ϕ is smooth non-increasing function $\phi : [0, +\infty) \to [0, 1]$ such that $\phi \equiv 1$ on $[0, \frac{1}{4}]$, vanishes outside $[0, 1]$ and satisfies $|\phi'|^2 \leqslant 10^3 \phi, \phi'' \geqslant -10^3 \phi$. Then we have

$$\begin{aligned} \Box F \leqslant & -(n-1)\Delta_{g(t)}\phi \cdot \varphi - 2\langle \nabla\phi, \nabla\varphi \rangle_{g(t)} - \phi\varphi^2 \\ & + \frac{\gamma(n-2)}{4} u^{\gamma - \frac{4}{n-2}} L_0 \varphi - L_0 u^{\gamma} |\nabla \log u|^2_{g(t)} \end{aligned} \tag{IV.66}$$

in the sense of barrier. Here we have used the evolution equation of u, see (IV.56). We might assume the function to be smooth when applying the maximum principle.

We now simplify the evolution inequality. We will use C_i to denote any constant

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depending only on n, α. Using the same derivation of (IV.57) and choice of ϕ,

$$-(n-1)\Delta_{g(t)}\phi \leqslant C_1\left(1+\phi^{1/2}|\nabla \log u|_{g(t)}\right) \tag{IV.67}$$

On the other hand at its maximum point, $\nabla F = 0$ so that

$$\phi\nabla\varphi + \varphi\nabla\phi + L_0\gamma u^{\gamma-1}\nabla u = 0$$

and thus,

$$\begin{aligned} -2\langle\nabla\phi, \nabla\varphi\rangle &= 2\phi^{-1}\langle\nabla\phi, \varphi\nabla\phi + L_0\gamma u^{\gamma-1}\nabla u\rangle \\ &\leqslant C_2\varphi + C_2L_0\phi^{-1/2}|\nabla \log u|_{g(t)}. \end{aligned} \tag{IV.68}$$

Substituting it back to the evolution equation of F yields

$$\begin{aligned} \Box F \leqslant\ & C_1\varphi\left(1+\phi^{1/2}|\nabla \log u|\right) + C_2\varphi + C_2L_0\phi^{-1/2}|\nabla \log u| \\ & - \phi\varphi^2 + C_3L_0\varphi - L_0u^{\gamma}|\nabla \log u|^2 \\ \leqslant\ & C_4L_0\varphi + C_4\phi^{-1}L_0 + C_4L_0^{-1}\varphi^2\phi - \phi\varphi^2 \end{aligned} \tag{IV.69}$$

at its maximum. Hence if we choose $L_0 = 2C_4$, then at its interior maximum point inside the support of ϕ, we have

$$(\phi\varphi)^2 \leqslant C_5\varphi\phi + C_5 \leqslant \frac{1}{2}(\phi\varphi)^2 + C_6. \tag{IV.70}$$

That said, for all $(x,t) \in B_{g_0}(x_0, 1) \times [0, T]$,

$$F(x,t) \leqslant \sqrt{2C_6} + L_0 := C_7. \tag{IV.71}$$

Evaluating at x_0 gives us the result. $\square$

By Lemma 16, we can derive a Ricci lower bound along the locally conformally flat Yamabe flow as a consequence of the above theorem.

Lemma 19. Let $(\mathcal{M}^n, g(0))$ be a complete manifold with non-negative Ricci curvature. Suppose that $(B_0(p,1), g(t))_{t\in[0,T]}$ is a locally conformally flat Yamabe flow with conformal factor $u(x,t)^{\frac{4}{n-2}} := g(x,t)/g(x,0)$, and there is a constant $a > 1$ such that on $B_0(p,1) \times [0,T]$,

$$|\mathrm{Rm}(g(x,t))| \leqslant \frac{a}{t}; \tag{IV.72}$$

$$\mathrm{Ric}(g(x,0)) \geqslant 0; \tag{IV.73}$$

$$\mathrm{scal}(g(x,t)) \geqslant -1; \tag{IV.74}$$

$$a^{-1} \leqslant u(x,t) \leqslant a. \tag{IV.75}$$

Then there is a constant $\Lambda > 0$ depending on n, a such that

$$\mathrm{Ric}(g(p,t)) \geqslant -\Lambda, \tag{IV.76}$$

on $[0,T]$.

Proof of Lemma 19. It is nothing but a combination of Lemma 16 and Theorem 12 with $K = n$. $\square$

Lemma 20. Let $(\mathcal{M}^n, g(0))$ be a complete manifold with non-negative Ricci curvature. Suppose that $(B_0(p,r), g(t))_{t\in[0,T]}$ is a locally conformally flat Yamabe flow with conformal factor $u(x,t)^{\frac{4}{n-2}} := g(x,t)/g(x,0)$, and there is a constant $a > 1$ such that on

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$B_0(p,r) \times [0,T]$,

$$|\mathrm{Rm}(g(x,t))| \leqslant \frac{a}{t}; \tag{IV.77}$$

$$\mathrm{Ric}(g(x,0)) \geqslant 0; \tag{IV.78}$$

$$\mathrm{scal}(g(x,t)) \geqslant -r^{-2}; \tag{IV.79}$$

$$a^{-1} \leqslant u(x,t) \leqslant a. \tag{IV.80}$$

Assume that there is a small constant $\varepsilon << 1$ depending on n and a with

$$\int_0^\infty r \fint_{B_{g(0)}(x,r)} \mathrm{scal}(g(0)) d\,\mathrm{Vol}_0(y) dr \leqslant \varepsilon, \tag{IV.81}$$

holds for all $x \in \mathcal{M}^n$. Then there is a constant $\hat{T} = \hat{T}(n,a,\varepsilon) > 0$ such that

$$|\mathrm{Rm}|(p,t) \leqslant \frac{1}{t}, \tag{IV.82}$$

for all $t \in (0, \min\{r^2\hat{T}, T\}]$.

Before we prove this lemma, we record an important observation in [CL23, Lemma 3.1.]. Given a complete Riemannian manifold $(\mathcal{N}^m, h)$, for $x_0 \in \mathcal{N}^m$ and $r > 0$, define

$$\begin{cases} k_h(x_0,r) := r^{-2} \fint_{B_h(x_0,r)} \mathrm{scal}(h(x,s)) d\,\mathrm{Vol}_h(x); \\ f_h(x_0,r) := \int_0^r s \cdot k_h(x_0,s) ds. \end{cases} \tag{IV.83}$$

For instance, the assumption of the main theorem is $f_g(x,r) \leqslant \varepsilon$ for all $x \in \mathcal{M}^n$ and $r > 0$.

Lemma 21. [CL23, Lemma 3.1.] Suppose that $(\mathcal{M}^n, g)$ is a complete Riemannian manifold with nonnegative Ricci curvature. Then

$$k_g(x,r) \leqslant 2^n(f_g(x,2r) - f_g(x,r)) \tag{IV.84}$$

for all $x \in \mathcal{M}^n$ and $r > 0$.

Therefore, combine the Lemma 21 and (IV.3),

$$k_{g(0)}(x,r) \leqslant 2^n\varepsilon, \tag{IV.85}$$

for all $x \in \mathcal{M}^n$ and $r > 0$.

Proof of Lemma 20. May assume $r = 1$ by scaling. Choose a smooth domain $B_{g(0)}(p,\frac{1}{2}) \subset \Omega \subset B_{g(0)}(p,\frac{3}{4})$, $\phi : B_{g(0)}(p,1/2) \to [0,1]$ with $\operatorname{supp}\phi \subset B_{g(0)}(p,1/4)$ and consider a smooth nonnegative function v with

$$v(x,t) := \int_\Omega \phi(d_{g(0)}(y,p))K_\Omega(x,t;y,0)\mathrm{scal}(g(0,y))d\,\mathrm{Vol}_0(y), \tag{IV.86}$$

for all $x \in B_0(p,\frac{1}{2})$ and $t \in [0,T]$, where K_Ω is the Dirichlet conjugate heat kernel on Ω. Then $v(x,0) = \mathrm{scal}(g(x,0))$ on Ω with equation

$$\begin{aligned}\square\left(\mathrm{scal} - e^{\frac{n-4}{2}t}v\right)_+ &= \mathrm{scal}^2 - \frac{n}{2}\mathrm{scal}\cdot e^{\frac{n-4}{2}t}v - \frac{n-4}{2}e^{\frac{n-4}{2}t}v \\ &\leqslant 2\mathrm{scal}\left(\mathrm{scal} - e^{\frac{n-4}{2}t}v\right)_+ - \left[\left(\frac{n}{2} - 2\right)\mathrm{scal} + \frac{n-4}{2}\right]e^{\frac{n-4}{2}t}v \\ &\leqslant 2\mathrm{scal}\left(\mathrm{scal} - e^{\frac{n-4}{2}t}v\right)_+,\end{aligned} \tag{IV.87}$$

in the sense of barrier. By (IV.60) and the co-area formula, for $x \in B_0(p, \frac{1}{2})$,

$$\begin{aligned}
v(x,t) &\leqslant \int_{B_0(x,1)} \frac{C}{\mathrm{Vol}_0(x, C\sqrt{t})} \exp\left(-\frac{d_0(x,y)^2}{Ct}\right) \mathrm{scal}(g(y,0)) d\,\mathrm{Vol}_0\, y \\
&\leqslant C \frac{\mathrm{Vol}_0(x,1)}{\mathrm{Vol}_0(x, C\sqrt{t})} \exp\left(-\frac{1}{Ct}\right) k_{g(0)}(x,1) + \frac{C}{t}\int_0^1 \frac{k_{g(0)}(x,r)}{r} \frac{\mathrm{Vol}_0(x,r)}{\mathrm{Vol}_0(x,C\sqrt{t})} \exp\left(-\frac{r^2}{Ct}\right) dr \\
&\leqslant C\varepsilon + \frac{C}{t}\int_0^1 \frac{k_{g(0)}(x,r)}{r} dr \\
&\leqslant C(t^{-1}+1)\varepsilon,
\end{aligned} \tag{IV.88}$$

where the second to the last inequality follows from the Bishop-Gromov and $C > 0$ is a constant depending on n and a. We choose $\varepsilon \leqslant \frac{1}{4n(n-1)C}$ and $T_1 := T_1(a, n, \varepsilon) > 0$ such that $v(x,t) \leqslant C(t^{-1}+1)\varepsilon \leqslant \frac{1}{4n(n-1)t}$ on $B_0(p, \frac{1}{2}) \times (0, \min\{T_1, T\}]$. Therefore, by virtue of Theorem 12, there is a constant $\Lambda = \Lambda(n,a) > 0$ such that

$$\mathrm{scal}(p,t) = \mathrm{scal}(p,t) - e^{\frac{n-4}{2}t} v(p,t) + e^{\frac{n-4}{2}t} v(p,t) \leqslant \Lambda + \frac{1}{2n(n-1)t}, \tag{IV.89}$$

for all $t \in (0, \min\{T_1, T\}]$. According to the nonnegative scalar curvature and $\mathrm{Ric} + \Lambda \geqslant 0$ (Lemma 19),

$$|\mathrm{Rm}|(p,t) \leqslant (n-1)|\mathrm{Ric}|(p,t) \leqslant (n-1)(|\mathrm{Ric} + \Lambda g| + \sqrt{n}\Lambda)(p,t) \leqslant (n-1)(n\,\mathrm{scal}(p,t) + C_n\Lambda) \leqslant \frac{1}{2t} + C_n\Lambda, \tag{IV.90}$$

on $(0, \min\{T_1, T\}]$. Choose $\hat{T} = \min\{(2C_n\Lambda)^{-1}, T_1\} > 0$ depending on a, n, ε. Then we confirm the assertion. ☐

IV.6 Long-time solution of Yamabe flow with $1/t$ decayed

Theorem 13. Let $(\mathcal{M}^n, g)$ be a non-flat complete Riemannian manifold with the assumption (IV.3). Then for $\varepsilon < \varepsilon(n)$ small enough, there exists a complete locally conformally flat Yamabe flow $(\mathcal{M}^n, g(t))_{t\geqslant 0}$ with $g(0) = g$ such that

$$|\mathrm{Rm}(g(t))|_{g(t)} \leqslant \frac{1}{t} \text{ and } \mathrm{Ric}(g(t)) \geqslant 0, \tag{IV.91}$$

on $\mathcal{M}^n \times (0, \infty)$. Moreover, the bound $\exp(-w(x)) \leqslant u(x,t) \leqslant 1$ holds on $\mathcal{M}^n \times [0, \infty)$, where $w : \mathcal{M}^n \times \mathbb{R}$ is defined as in Section IV.3.

Proof of Theorem 13. Fix $R > 0$. Consider a smooth domain Ω such that $B_g(p, R + 3) \subset \Omega$ for a fixed point $p \in \mathcal{M}^n$ and choose $r \in (0,1)$ such that $|\mathrm{Rm}(g)| \leqslant r^{-2}$ on Ω. Then Proposition 7 and Proposition 8 show that the equation (IV.14) has a long-time solution, and there exist some constants $a(n,\varepsilon), c_1(n,a), T_0(n,a) > 0$ such that $a^{-1} \leqslant \exp(-w(x)) \leqslant u(x,t) \leqslant 1$ on $\Omega \times [0,\infty)$ and $|\mathrm{Rm}(g(t))| \leqslant c_1 r^{-2} \leqslant c_1 T_0/t$ on $\Omega_r \times (0, r^2 T_0]$, where $w : \mathcal{M}^n \to \mathbb{R}$ is the function which defined in Section IV.3. By Lemma 18, $\mathrm{scal}(g(x,t)) \geqslant 0$ on $\Omega \times [0,\infty)$ by letting $r \to \infty$. By Lemma 20, there is a constant $T_1(n, a, c_1 T_0) = T_1(n,\varepsilon) > 0$ so that $|\mathrm{Rm}(g(t))| \leqslant 1/t$ on $\Omega_{2r} \times [0, r^2 \min\{T_1, T_0\}]$.

Now we set $t_1 := r^2 \min\{T_0, T_1\}$, $R_1 := R + 3 - 2r$ and $\hat{T} = \hat{T}(n, \max\{a, c_2(1 + T_2)\}, \varepsilon)$ is the constant in Lemma 20 and $c_2(n, a^2), T_2(n, a^2)$ are the constants in Proposition 7. The purpose is to show a region in spacetime containing $B_0(p, R) \times (0, T)$ with $|\mathrm{Rm}(g(t))| \leqslant 1/t$ for some T independent of r, p and R. We perform an inductive construction of the Yamabe flow. First, set $t_{k+1} := (1 + T_2)t_k$ and $R_{k+1} := R_k - 2\left(a^{\frac{2}{n-2}} + \sqrt{\frac{T_2+1}{\hat{T}}}\right) t_k^{\frac{1}{2}}$. The induction hypothesis is the following: There is a sequence of locally conformally flat Yamabe flows $(\Omega(k), g(t))_{t\in[0,t_k]}$ such that the following holds:

1. $\Omega(k) \subset \Omega(k-1)$ is a smooth domain.

2. $|\mathrm{Rm}(g(t))| \leqslant 1/t$, $\exp(-w(x)) \leqslant u(t) \leqslant 1$ and $\mathrm{scal}(g(t)) \geqslant 0$ on $\Omega(k) \times (0, t_k]$.

3. $B_0(p, R_k) \subset \Omega(k)$.

Choosing $\Omega(1) := \Omega_{2r}$ and $[0, t_1]$, then the assertion holds for $k = 1$. For $k > 1$, $|\text{Rm}(g(t_{k-1}))| \leqslant 1/t_{k-1}$ follows from the induction hypothesis. Now, Proposition 7 implies $|\text{Rm}(g(x,t))| \leqslant c_2/t_{k-1}$ for all $x \in \Omega' := \{x \in \Omega(k-1) : \text{dist}_{g(t_{k-1})}(x, \partial\Omega(k-1)) \geqslant t_{k-1}^{1/2}\}$ and $t \in [t_{k-1}, t_{k-1} + t_{k-1}T_2]$. Therefore,

$$|\text{Rm}(g(x,t))| \leqslant \frac{\max\{1, c_2(1+T_2)\}}{t}, \tag{IV.92}$$

on $\Omega' \times [0, (1+T_2)t_{k-1}]$. For $L > 0$ and $x \in \Omega'$ with $B_0(x, Lt_{k-1}^{1/2}) \subset \Omega'$, by Lemma 20,

$$|\text{Rm}(g(x,t))| \leqslant \frac{1}{t}, \tag{IV.93}$$

for all $t \in (0, \min\{L^2 t_{k-1}\hat{T}, (1+T_2)t_{k-1}\}]$. Taking $L^2 = \frac{1+T_2}{\hat{T}}$, by the fact $u(x,t) \geqslant a^{-1}$, we get that for $\text{dist}_g(x, \partial\Omega(k-1)) \geqslant \left(a^{\frac{2}{n-2}} + \sqrt{\frac{T_2+1}{\hat{T}}}\right) t_k^{\frac{1}{2}}$ and $x \in \Omega(k-1)$, $|\text{Rm}(g(x,t))| \leqslant 1/t$ for $t \in (0, t_k]$. Choose $\Omega(k) \subset \Omega(k-1)$ a smooth domain such that $B_0(p, R_k) \subset \Omega(k)$ and $\text{dist}_g(x, \partial\Omega(k-1)) \geqslant \left(a^{\frac{2}{n-2}} + \sqrt{\frac{T_2+1}{\hat{T}}}\right) t_k^{\frac{1}{2}}$ for all $x \in \Omega(k)$. This completes the induction argument.

Note that

$$R_{k+1} = R + 3 - 2r - 2\left(a^{\frac{2}{n-2}} + \sqrt{\frac{T_2+1}{\hat{T}}}\right) \sum_{j=1}^{k} t_j^{\frac{1}{2}} \to -\infty, \tag{IV.94}$$

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as $k \to \infty$. Hence, there exists a $i > 0$ such that $R_{i+1} < R \leqslant R_i$. Then

$$\begin{aligned} R &> R + 3 - 2r - 2\left(a^{\frac{2}{n-2}} + \sqrt{\frac{T_2+1}{\hat{T}}}\right)\sum_{j=1}^{i} t_j^{\frac{1}{2}} \\ &\geqslant R + 1 - 2\left(a^{\frac{2}{n-2}} + \sqrt{\frac{T_2+1}{\hat{T}}}\right)\sqrt{t_i}\sum_{j=0}^{\infty}(1+T_2)^{-\frac{j}{2}} \\ &=: R + 1 - \Lambda\sqrt{t_i}, \end{aligned} \tag{IV.95}$$

for some constant $\Lambda(n, \varepsilon) > 0$ independent of i, R, r and p. Therefore, $t_i \geqslant \Lambda^{-2}$ and $|\mathrm{Rm}(g(t))| \leqslant 1/t$ on $B_0(p, R) \times (0, \Lambda^{-2})$. Since $R > 0$ and $p \in \mathcal{M}^n$ are arbitrary and Λ is uniform, these imply $|\mathrm{Rm}(g(t))| \leqslant 1/t$ on $\mathcal{M}^n \times (0, \infty)$ by the parabolic rescaling argument. By Lemma 19 and the parabolic scaling again, we yield an immortal solution of locally conformally flat Yamabe flow $(\mathcal{M}^n, g(t))_{t\in[0,\infty)}$ with $g(0)$ and satisfy $|\mathrm{Rm}(g(t))| \leqslant 1/t$ and $\mathrm{Ric}(g(t)) \geqslant 0$ on $\mathcal{M}^n \times (0, \infty)$. This completes the proof. ☐

IV.7 Proof of Theorem 8

Proof of Theorem 8. If not, then Theorem 13 provides a complete long-time solution of locally conformally flat Yamabe flow $(\mathcal{M}^n, g(t) := u(t)^{\frac{4}{n-2}} g)_{t\geqslant 0}$ satisfying (IV.91) and $\exp(-w(x)) \leqslant u(x,t) \leqslant 1$. Since $\mathrm{scal}(g(t)) \geqslant 0$ holds on $\mathcal{M}^n \times [0, \infty)$ and $\mathrm{scal}(g) \not\equiv 0$ on $\mathcal{M}^n$, by the equation (IV.6) and standard maximal principle, $\mathrm{scal}(g(t)) > 0$ on $\mathcal{M}^n \times (0, \infty)$. Now we recall the following Harnack inequality due to Bennet Chow.

Theorem 14. ([Cho92], [Ma16, Theorem 7]) Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a complete and locally conformally flat Yamabe flow with bounded curvature. Suppose that $\mathrm{Ric}(g(t)) \geqslant 0$

holds on $\mathcal{M}^n \times [0,T]$. Then for any 1-form X, $Z(g(t),X,t) \geqslant 0$ on $\mathcal{M}^n \times [0,T]$, where

$$Z(g,X,t) := \frac{\partial}{\partial t}\operatorname{scal}(g(t)) + g(\nabla^g \operatorname{scal}(g), X) + \frac{1}{2(n-1)}\operatorname{Ric}(g)(X,X) + \frac{\operatorname{scal}(g)}{t}, \tag{IV.96}$$

for any metric g, 1-form X and $t > 0$.

The original version in [Cho92] is only available for closed locally conformally flat Yamabe flow. But as the discussion in [Cho92], this statement remains true as long as Hamilton's tensor maximal principle holds, which includes the case of complete Yamabe flow with bounded curvature. By this Harnack inequality, we get $Z(g(t),X,t) \geqslant 0$ for all $X \in \Omega(\mathcal{M}^n)$ and $t > 0$ by the curvature bound $|\operatorname{Rm}(g(t))| \leqslant 1/t$. Taking $X = -\nabla^{g(t)} \log \operatorname{scal}(g(t))$, then

$$\frac{\partial}{\partial t}\operatorname{scal}(g(t)) + \frac{\operatorname{scal}(g(t))}{t} \geqslant \frac{|\nabla \operatorname{scal}|^2}{2\operatorname{scal}} \geqslant 0 \Rightarrow \frac{\partial}{\partial t}(t\operatorname{scal}(g(t))) \geqslant 0, \tag{IV.97}$$

on $\mathcal{M}^n \times (0,\infty)$. Hence, for $t \geqslant 1$ and $\tau \in [\sqrt{t}, t]$, we have $\tau \operatorname{scal}(g(\tau)) \geqslant \sqrt{t}\operatorname{scal}(g(\sqrt{t}))$. The equation of Yamabe flow implies

$$\int_0^t \operatorname{scal}(g(x,\tau))d\tau = -\frac{4}{n-2}\log u(x,t) \leqslant \frac{4}{n-2}w(x), \tag{IV.98}$$

for all $x \in \mathcal{M}^n$. Then

$$\frac{1}{2}\sqrt{t}\operatorname{scal}(g(\sqrt{t}))\log t = \int_{\sqrt{t}}^t \frac{\sqrt{t}\operatorname{scal}(g(\sqrt{t}))}{\tau}d\tau \leqslant \int_{\sqrt{t}}^t \operatorname{scal}(g(\tau))d\tau \leqslant \frac{4}{n-2}w, \tag{IV.99}$$

on $\mathcal{M}^n$. Hence,

$$0 \leqslant \sqrt{t}\operatorname{scal}(g(x,\sqrt{t})) \leqslant \frac{8}{n-2}\frac{w(x)}{\log t} \to 0, \tag{IV.100}$$

as $t \to \infty$. But the middle term is monotone non-decreasing in t, we conclude $\operatorname{scal}(g(x,t)) =$

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0 on $\mathcal{M}^n \times [1, \infty)$. By the monotonicity again, scal $\equiv 0$ on $\mathcal{M}^n \times [0, \infty)$. This confirms g to be a flat metric and we complete the proof. $\square$

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