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一個關於傑勒西單調性公式的評注

A Note on the Geroch Monotonicity Formula

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A Note on the Geroch Monotonicity Formula

本論文係__簡培育__（姓名）__R07221010__（學號）在國立臺灣大學__數學__（系）完成之碩士學位論文，於民國__113__年__1__月__26__日承下列考試委員審查通過及口試及格，特此證明。

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As the academic contribution of my thesis is relatively limited, this acknowledgement page is perhaps the most important part of my thesis. At least, in no way would I want to forget to do two things if I ever stood a chance of completing my studies at NTU: thank people who helped me along the way and convey a message to those in need of help.

First of all, I am substantially indebted to Professor Ai-Nung Wang, who accepted the role of being my advisor during my graduate studies. I remember the day I wrote him an email to seek mentorship. I told him about my old age, non-math background, and the fact that I had been declined by several potential advisors. And for some reasons, I might not be able to quit my job after receiving mentorship. But he simply replied to my mail with an OK. That OK really made a strong impression on me. In addition to his warm acceptance, I am especially thankful that he allowed me to present my thesis in the form of a study note. Without his crucial decision, I could not have started to write my thesis and meet the requirement of graduation.

Besides Professor Wang, I feel indebtedness to a great many people, with whom I corresponded to address research issues and article errata. These are senior researchers and authors who have established a fine reputation in mathematics or physics; therefore, I do not think my humble thesis is worthy of having their names listed. Despite that, my heartfelt gratitude extends to these unnamed persons. But for their generosity and prompt assistance, I would have been stuck in the endless circle of false reasoning on and on. I hope one day I can become a man who is qualified to pass their good deeds on to the next generations.

Lastly, I must say something to people who are still suffering or ready to suffer. During my studies at NTU, I heard many times about students dropping out of school or committing suicide. The cause of these tragic events could be multi-faceted, and I believe these people made their decisions out of overwhelming desperation. So, if you ever find yourself in need of a second opinion or advice, please do not hesitate to reach out to me. You have got my Student ID, and I am sure you know how to contact me by email.

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Abstract in Chinese

廣義相對論中的霍金準局部能量爲 S. W. Hawking 在1968年提出的概念，其在逆平均曲率流下的單調性以隱晦的方式初見於一篇1973年的文章，文章作者爲 Robert Geroch ，因此該性質一般稱作傑勒西單調性公式。標誌著非負的時變率，這個公式在近代許多幾何流、數學相對論的文獻中被明白揭示並證明，其中一篇文獻是 Gerhard Huisken 與 Alexander Polden 在1996年完成的工作，此二人證明公式的手法爲取得幾個演化方程後再求能量的時變率。在這份評注中，我們將詳述 Huisken 與 Polden 如何在那篇1996年的文章中證明傑勒西單調性公式。

關鍵詞：準局部能量、霍金能量、幾何演化方程、逆平均曲率流、傑勒西單調性

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Abstract in English

The Hawking quasi-local energy in general relativity is a notion proposed by S. W. Hawking in 1968. Its monotonicity under inverse mean curvature flow was first suggested in a 1973 article authored by Robert Geroch, commonly known as the Geroch monotonicity formula. As a non-negative time derivative, this formula is explicitly stated and proved in many of the modern references on mathematical relativity and geometric flows, including an article composed by Gerhard Huisken and Alexander Polden in 1996. Huisken and Polden proved the formula by taking the time derivative of the energy function after some evolution equations were developed. In this note, we shall present a detailed exposition of how Huisken and Polden prove the Geroch monotonicity formula in the 1996 article.

Keywords: quasi-local energy, Hawking energy, geometric evolution equation, inverse mean curvature flow, Geroch monotonicity

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1 Introduction

Since the invention of the general theory of relativity, Einstein's novel ideas about gravity have been continuously reshaping our knowledge of the universe. While this theory gained its credibility and applicability through a number of experimental observations, it brings some bizarre consequences that deeply confuse theorists. One of them is non-locality of gravitational energy. According to the equivalence principle, the energy density of a gravitational field can always be reduced to zero using a suitable frame of reference (see, e.g., [12, chapter 20] and [15, chapter 4]). Despite the fact that gravitational energy cannot be localized in general relativity, evidence such as gravitational radiation poses a need to measure gravitational energy. Rather than seek the meaningless notion of the energy density of a gravitational field, relativists endeavor to measure the gravitational energy contained in a finite region, thereby introducing the notion of quasi-local energy. Depending on various theoretical needs, a multitude of definitions of quasi-local energy have been theorized in the course of many decades, such as the Bartnik energy, the Penrose energy, and the Brown-York energy. Each of them has its own strengths and weaknesses, and an ultimate, completely satisfactory definition is still in pursuit. For a review article on this topic, we recommend László B. Szabados for his extensive and in-depth discussion [13].

It is worth noting that "quasi-local mass" and "quasi-local energy" are used interchangeably in some of the literature on general relativity while others make a clear distinction between these two terms. Throughout this note, we shall adhere to the use of "quasi-local energy" and confine our attention to its mathematical essence.

In 1968, English physicist Stephen Hawking proposed a definition of quasi-local energy using a tetrad formalism [5]. This definition provides an intuitive way of measuring gravitational energy at a quasi-local level, by considering the bending of light rays passing through a region [13, section 6]. In the language of mathematical relativity, it is typically presented as follows.

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Definition 1.1. Given a closed hypersurface Σ in a Riemannian 3-manifold embedded in the 4-dimensional spacetime, we define its Hawking energy to be

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \mathrm{d}\mu_{\Sigma} \right), \tag{1.1}$$

where $|\Sigma|$ is the area of Σ, H is the mean curvature (scalar) of Σ, and $\mathrm{d}\mu_{\Sigma}$ is the Riemannian volume form on Σ.

Though it was mentioned earlier that "quasi-local mass" and "quasi-local energy" can be used interchangeably in literature, we have to remind readers that the term "Hawking mass" is preferred by some people (see, for example, [8] and [9]) when they refer to (1.1). We shall not follow in their footsteps with, however, the mass symbol m kept as a matter of course.

Apart from its connection to light bending, Hawking energy has some other properties that make it a candidate for quasi-local energy. In [3, section 4], Douglas Eardley listed a number of properties for quasi-local energy to obey if it is to address the problem of non-locality of gravitational energy. And Hawking energy satisfies some of them as follows:

1. A point in a spacetime, in the sense of a shrinking region, contains no energy.
2. A metric 2-sphere in the Minkowski spacetime contains no energy.
3. In an asymptotically flat spacetime, the energy contained in the coordinate sphere of radius r tends to the ADM energy as $r \to \infty$.

The third property of Hawking energy is worthy of note since it suggests that Hawking energy provides a reasonable estimate of the total energy.

Incidentally, despite having an intimate relationship with the ADM energy, Hawking energy, in general, does not share the enchanting property of positivity. When embedded in the Euclidean space, Σ can be shown to possess negative Hawking energy [11, pp. 122-123]. See also [2] to examine an unusual circumstance under which Hawking energy is sure to attain positive values.

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Hawking energy bears another property that will engage our attention in the rest of this note: Hawking energy is monotonically increasing under inverse mean curvature flow. This property stemmed from [4, APPENDIX], in which American physicist Robert Geroch attempts to prove the positive-energy theorem, thus making the property known as the Geroch monotonicity formula. To have a real understanding of this formula, we start with the definition of inverse mean curvature flow.

Definition 1.2. Let $\Phi_0 : \Sigma \to M$ be a smooth embedding of a hypersurface Σ in a Riemannian 3-manifold M and consider a time interval $[0, T)$ for some $T > 0$. Σ is said to evolve by *inverse mean curvature flow* (IMCF) if there exists a one-parameter family of smooth embeddings $\Phi : \Sigma \times [0, T) \to M$ that solves the initial-value problem

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = \dfrac{1}{H}\nu, & (1.2) \\ \Phi(\cdot, 0) = \Phi_0, & (1.3) \end{cases}$$

where H is the mean curvature and ν is the unit outer normal.

Via the time-dependent embedding $\Phi_t := \Phi(\cdot, t)$, Σ can be visually identified as the hypersurface $\Sigma_t := \Phi_t(\Sigma)$ moving at a speed of $1/H$. With this concept in mind, we are ready to usher in the monotonicity formula for the Hawking energy $m(\Sigma_t)$.

Theorem 1.3 (Geroch monotonicity formula)**.** *Let $(M, \overline{g})$ be a Riemannian 3-manifold with non-negative scalar curvature $\mathsf{R}_{\overline{g}}$. If Σ_t is a connected closed hypersurface in M evolving from Σ by IMCF, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} m(\Sigma_t) \geqslant 0. \tag{1.4}$$

This formula can be traced back to 1973, when Geroch outlined the proof of non-negativity of ADM energy in [4]. Though he did not explicitly write (1.4) in the proof, his argument about an integral W did contribute to the discovery of the formula. In fact, W can be cast into a form that resembles (1.1) if we apply the Gauss-Bonnet theorem. And the inequality $\frac{\mathrm{d}}{\mathrm{d}t} W \geqslant -\frac{1}{2} W$ he obtained as a by-product can be turned into (1.4) once we know area grows exponentially under IMCF. Thus, the monotonicity formula was credited to Geroch and was named the Geroch monotonicity formula.

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While the monotonicity formula was only implied in Geroch's work, it is explicitly derived in many of the modern references on mathematical relativity and geometric flows. Some quick references include [6, section 6] and [11, Theorem 4.27] among others. And what we would like to do with this note is expand on the derivation in [6], where Gerhard Huisken and Alexander Polden drew the conclusion in Theorem 1.3 by carrying out the differentiation. It is noteworthy that Dan Lee later conceived a similar derivation in [11] with more details. However, Lee based his derivation on X_t, "the first-order deformation vector field," which is somewhat obscure to the author of this note (see [11, section 2.2]). We shall not adopt this approach and mentioned it only for readers' interest.

The rest of this note is organized as follows. In section 2, we will examine some geometric evolution equations to lay the foundation of the proof. We will also include a minimum knowledge of Riemannian geometry for later references. After gathering all the ingredients, we will proceed to section 3 to demonstrate how the Geroch monotonicity formula is proved in [6]. Finally, we will arrange section 4 to briefly discuss the level-set formulation in [7, 8], with which Huisken and Tom Ilmanen investigated the Geroch monotonicity formula in a weak sense.

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2 Preliminaries

In this section, we will prepare ourselves for the proof of the Geroch monotonicity formula, which involves a basic understanding of Riemannian geometry and geometric flows. Meanwhile, we manage to build a system of notation and terminology that is viable throughout this note. Many great volumes serve our purposes, and we will be writing with reference to [10, 1, 14].

Now let us begin with the evolution equations of some geometric quantities. We do not focus on any particular type of geometric flow, so these equations apply not only to IMCF but also to other geometric flows such as mean curvature flow and Gauss curvature flow. Specifically, we consider the scenario of Definition 1.2 with the role of $1/H$ replaced by a symmetric homogeneous function φ of the principal curvatures λ_1, λ_2:

$$\frac{\partial \Phi}{\partial t} = \varphi \nu \tag{2.1}$$

In this scenario and hereafter, all quantities, including the manifolds on which these quantities are defined, are understood to be smooth, that is, to be of class C^∞. And we always assume both the hypersurface Σ and the ambient manifold M are oriented. Now, if $\overline{g}$ is the Riemannian metric on the ambient manifold M, the embedding $\Phi_t : \Sigma \to M$ will induce a time-dependent Riemannian metric $\Phi_t^* \overline{g}$ on Σ, so it makes sense to discuss the evolution of $g := \Phi_t^* \overline{g}$. As it turns out in [6, Theorem 3.2(i)], if h_{ij} represent the components of the scalar-valued second fundamental form A in local coordinates $\{x^i\}$, that is,

$$h_{ij} := A(\partial_i, \partial_j) = \overline{g}\left(\overline{\nabla}_{\partial_i} \nu, \mathrm{d}\Phi_t(\partial_j)\right), \tag{2.2}$$

then the components of g in $\{x^i\}$ will evolve as follows.

Proposition 2.1.

$$\frac{\partial}{\partial t} g_{ij} = 2\varphi h_{ij} \tag{2.3}$$

A brief derivation of (2.3) has been included in [6, Lemma 7.4]; however, some of the

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steps in the derivation seem to carry no immediate justification, which we will point out in a moment. From now on, all barred quantities will be used to denote geometric features of $\overline{g}$, and if any index repeats as a subscript and as a superscript in a monomial, this monomial will be summed over the index according to the Einstein summation convention. In the following proof, we will adopt the convention to view Φ_t as a parametrization of the hypersurface Σ_t and denote by $\partial\Phi \big/ \partial x^i$ the differential of Φ_t acting on the i-th coordinate vector field $\partial \big/ \partial x^i$.

Proof of Proposition 2.1. By definition, we know

$$\begin{aligned} g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= (\Phi_t^*\overline{g})\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &= \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \frac{\partial\Phi}{\partial x^j}\right), \end{aligned} \tag{2.4}$$

but differentiating (2.4) directly can be cumbersome and tedious. Fortunately, the differentiation can be carried out with ease if we use normal coordinates, whose definition and properties can be found in [10, pp. 131-133]. To do so, we let $p \in \Sigma$ be given and choose normal coordinates $\left\{x^i\right\}$ centered at p and normal coordinates $\{y^\alpha\}$ centered at $\Phi_t(p)$. Now, the Christoffel symbols in these coordinates vanish at the center, so the Gauss-Weingarten equations[†]

$$\frac{\partial^2\Phi^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial\Phi^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\gamma}^\alpha \frac{\partial\Phi^\beta}{\partial x^i}\frac{\partial\Phi^\gamma}{\partial x^j} = -h_{ij}\nu^\alpha, \tag{2.5}$$

$$\frac{\partial\nu^\alpha}{\partial x^i} + \overline{\Gamma}_{\beta\gamma}^\alpha \frac{\partial\Phi^\beta}{\partial x^i}\nu^\gamma = h_{ij}g^{jk}\frac{\partial\Phi^\alpha}{\partial x^k} \tag{2.6}$$

reduce to

$$\frac{\partial^2\Phi^\alpha}{\partial x^i \partial x^j} = -h_{ij}\nu^\alpha, \tag{2.7}$$

$$\frac{\partial\nu^\alpha}{\partial x^i} = h_{ij}g^{jk}\frac{\partial\Phi^\alpha}{\partial x^k} \tag{2.8}$$

at the center of the normal coordinates. The reduced Weingarten equations (2.8) are in

[†]These are equations obtained from unwinding (2.2) with the Greek alphabet signifying coordinates in M. See Appendix A for a short derivation.

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our interest and will be used later on. Returning to (2.4), we see

$$\frac{\partial}{\partial t}g_{ij} = \overline{g}\left(\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x^i}, \frac{\partial\Phi}{\partial x^j}\right) + \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x^j}\right). \tag{2.9}$$

Notice that (2.9) does not make its appearance in the proof of [6, Lemma 7.4]: the two terms on the right-hand side of (2.9) seem to be combined to give

$$\frac{\partial}{\partial t}g_{ij} = 2\overline{g}\left(\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x^i}, \frac{\partial\Phi}{\partial x^j}\right), \tag{2.10}$$

which can be proved to be legitimate, as our computation will suggest. For the present, let us not approve of (2.10), because we cannot simply switch the roles of i, j on the right-hand side of (2.9) even though $\overline{g}$ is symmetric. Instead, we compute

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= \overline{g}\left(\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x^i}, \frac{\partial\Phi}{\partial x^j}\right) + \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x^j}\right) \\
&= \overline{g}\left(\frac{\partial}{\partial x^i}\frac{\partial\Phi}{\partial t}, \frac{\partial\Phi}{\partial x^j}\right) + \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \frac{\partial}{\partial x^j}\frac{\partial\Phi}{\partial t}\right) \\
&= \overline{g}\left(\frac{\partial}{\partial x^i}(\varphi\nu), \frac{\partial\Phi}{\partial x^j}\right) + \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \frac{\partial}{\partial x^j}(\varphi\nu)\right) \\
&= \overline{g}\left(\frac{\partial\varphi}{\partial x^i}\nu + \varphi\frac{\partial\nu}{\partial x^i}, \frac{\partial\Phi}{\partial x^j}\right) + \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \frac{\partial\varphi}{\partial x^j}\nu + \varphi\frac{\partial\nu}{\partial x^j}\right) \\
&= \overline{g}\left(\varphi\frac{\partial\nu}{\partial x^i}, \frac{\partial\Phi}{\partial x^j}\right) + \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \varphi\frac{\partial\nu}{\partial x^j}\right).
\end{aligned}$$

The last equality holds because each $\frac{\partial\Phi}{\partial x^i}$ is perpendicular to ν. Now plug (2.8) into this equation to find

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= \overline{g}\left(\varphi h_{ij}g^{jk}\frac{\partial\Phi}{\partial x^k}, \frac{\partial\Phi}{\partial x^j}\right) + \overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \varphi h_{ji}g^{ik}\frac{\partial\Phi}{\partial x^k}\right) \\
&= \varphi h_{ij}g^{jk}\overline{g}\left(\frac{\partial\Phi}{\partial x^k}, \frac{\partial\Phi}{\partial x^j}\right) + \varphi h_{ji}g^{ik}\overline{g}\left(\frac{\partial\Phi}{\partial x^i}, \frac{\partial\Phi}{\partial x^k}\right) \\
&= \varphi h_{ij}g^{jk}g_{kj} + \varphi h_{ji}g^{ik}g_{ik} \\
&= 2\varphi h_{ij}
\end{aligned}$$

with the last equality guaranteed by the symmetry of A. Next, choose an arbitrary system

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of coordinates $\{\widehat{x}^i\}$ around p and recall changing coordinates by, say,

$$\widehat{h}_{\alpha\beta} = h_{ij}\frac{\partial x^i}{\partial \widehat{x}^\alpha}\frac{\partial x^j}{\partial \widehat{x}^\beta}.$$

Then, to prove (2.3) in a general setting, we simply observe that

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{g}_{\alpha\beta} &= \frac{\partial x^i}{\partial \widehat{x}^\alpha}\frac{\partial x^j}{\partial \widehat{x}^\beta}\frac{\partial}{\partial t}g_{ij} = \frac{\partial x^i}{\partial \widehat{x}^\alpha}\frac{\partial x^j}{\partial \widehat{x}^\beta}(2\varphi h_{ij}) \\ &= 2\varphi\left(\frac{\partial x^i}{\partial \widehat{x}^\alpha}\frac{\partial x^j}{\partial \widehat{x}^\beta}h_{ij}\right) = 2\varphi\widehat{h}_{\alpha\beta}.\end{aligned}$$

Note that all hatted quantities signify local expressions in $\{\widehat{x}^i\}$. □

In what follows, we designate $\mathrm{d}\mu_g$ as the Riemannian volume form of (Σ, g) and $\mathsf{Ric}_{\overline{g}}$ as the Ricci curvature of $(M, \overline{g})$. Depending on different contexts, we sometimes write $\mathrm{d}\mu_g, \mathsf{Ric}_{\overline{g}}$ respectively as $\mathrm{d}\mu_\Sigma, \mathsf{Ric}_M$. To be consistent with Definition 1.2, we reserve the capital H for the mean curvature so that

$$H = \mathrm{tr}_g A = \lambda_1 + \lambda_2.$$

Letting $C^\infty(\Sigma)$ be the set of all smooth functions on Σ, we define Δ as the Laplace-Beltrami operator on $C^\infty(\Sigma)$ given by $\Delta u = \mathrm{div}(\mathrm{grad}\, u)$. And, with A denoting the scalar-valued second fundamental form, we declare the norm $\|A\|$ to be the square root of the inner product $\langle A|A\rangle$. In fact, if $\{e_1, e_2\}$ is an orthonormal frame, we can write

$$\|A\| = \sqrt{\sum_{i,j=1}^{2}\left(A(e_i, e_j)\right)^2}.$$

If, in addition, $\{e_1, e_2\}$ consists of eigenvectors of the shape operator $S := \{g^{ik}h_{kj}\}$, then

$$\|A\| = \sqrt{\sum_{i,j=1}^{2}\left\langle S(e_i)\middle|e_j\right\rangle^2} = \sqrt{\lambda_1^2 + \lambda_2^2} = \sqrt{\frac{1}{2}(\lambda_1 - \lambda_2)^2 + \frac{1}{2}H^2}. \tag{2.11}$$

The proposition below are consequences of Proposition 2.1, whose derivations are already explicit and complete in [6, section 7.1]. We simply cite them for later use.

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Proposition 2.2.

$$\frac{\partial}{\partial t}\mathrm{d}\mu_g = H\varphi \mathrm{d}\mu_g \tag{2.12}$$

$$\frac{\partial H}{\partial t} = -\Delta\varphi - \left(\|A\|^2 + \mathsf{Ric}_{\overline{g}}(\nu,\nu)\right)\varphi \tag{2.13}$$

Let $\mathfrak{X}(\Sigma)$ be the set of all smooth vector fields on (Σ, g). Then the Riemann curvature tensor of Σ is defined to be the covariant 4-tensor field Rm_Σ that acts on $X, Y, Z, W \in \mathfrak{X}(\Sigma)$ by

$$\mathsf{Rm}_\Sigma(X, Y, Z, W) = \left\langle -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z \middle| W\right\rangle_g .$$

In local coordinates, the components of Riemann curvature tensor are related to Ricci curvature by

$$\mathsf{Ric}_\Sigma(\partial_i, \partial_j) = g^{k\ell}\mathsf{Rm}_\Sigma(\partial_i, \partial_k, \partial_j, \partial_\ell).$$

Meanwhile, the definition of Riemann curvature tensor immediately leads to its symmetry and skew-symmetry (or anti-symmetry) in some of the arguments. We include these symmetries for the sake of proof. For $X, Y, Z, W \in \mathfrak{X}(\Sigma)$, [10, Proposition 7.12] states that:

Proposition 2.3.

(a) $\mathsf{Rm}_\Sigma(X, Y, Z, W) = \mathsf{Rm}_\Sigma(Z, W, X, Y)$

(b) $\mathsf{Rm}_\Sigma(X, Y, Z, W) = -\mathsf{Rm}_\Sigma(Y, X, Z, W)$

(c) $\mathsf{Rm}_\Sigma(X, Y, Z, W) = -\mathsf{Rm}_\Sigma(X, Y, W, Z)$

On the other hand, the famous Gauss equation states that:

Proposition 2.4. *If Σ is a Riemannian submanifold of M, and $\mathbf{A}$ is the vector-valued second fundamental form, then the Riemann curvature tensors of Σ and M will respect*

$$\mathsf{Rm}_M(W, X, Y, Z) = \mathsf{Rm}_\Sigma(W, X, Y, Z) + \langle \mathbf{A}(W, Z)|\mathbf{A}(X, Y)\rangle - \langle \mathbf{A}(W, Y)|\mathbf{A}(X, Z)\rangle$$

for all $X, Y, Z, W \in \mathfrak{X}(\Sigma)$.

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Be advised that we choose $\mathbf{A} = -A\nu$ whenever A is the scalar second fundamental form and ν is the unit outer normal. A detailed proof of this proposition can be found in [10, Theorem 8.5]; what really concerns us is its implication when Σ happens to be a hypersurface in M. Before we start, let us recall from Theorem 1.3 that $\mathsf{R}_{\overline{g}}$ represents the scalar curvature of $(M, \overline{g})$, which is defined by

$$\mathsf{R}_{\overline{g}} = \mathsf{tr}_{\overline{g}}\mathsf{Ric}_M$$

and is also denoted by R_M. Then:

Corollary 2.5. *Let Σ be a hypersurface in a Riemannian n-manifold M. Then*

$$\mathsf{R}_M = \mathsf{R}_\Sigma + 2\mathsf{Ric}_M(\nu, \nu) + \|A\|^2 - H^2. \tag{2.14}$$

Proof. The proof is almost by definition. Choose an orthonormal frame $\{e_1, \ldots, e_{n-1}\}$ for Σ. Then, because $\mathbf{A} = -A\nu$, the Gauss equation applied to $\{e_i\}$ reads

$$\begin{aligned} \mathsf{Rm}_M(e_i, e_j, e_i, e_j) &= \mathsf{Rm}_\Sigma(e_i, e_j, e_i, e_j) + A(e_i, e_j)A(e_j, e_i) - A(e_i, e_i)A(e_j, e_j) \\ &= \mathsf{Rm}_\Sigma(e_i, e_j, e_i, e_j) + \big(A(e_i, e_j)\big)^2 - A(e_i, e_i)A(e_j, e_j). \end{aligned} \tag{2.15}$$

By definition, we see

$$\begin{aligned} \mathsf{R}_\Sigma = g^{ij}g^{k\ell}\mathsf{Rm}_\Sigma(e_i, e_k, e_j, e_\ell) &= \delta^{ij}\delta^{k\ell}\mathsf{Rm}_\Sigma(e_i, e_k, e_j, e_\ell) \\ &= \sum_{i,\ell=1}^{n-1} \mathsf{Rm}_\Sigma(e_i, e_\ell, e_i, e_\ell). \end{aligned}$$

On the other hand, recall that H is the trace of A with respect to g:

$$H = \mathsf{tr}_g A = g^{ij}h_{ij} = \sum_{i=1}^{n-1} A(e_i, e_i)$$

This helps to establish

$$H^2 = \sum_{i,j=1}^{n-1} A(e_i, e_i)A(e_j, e_j).$$

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Thus, summing (2.15) over i, j from 1 to $n-1$ yields

$$\sum_{i,j=1}^{n-1} \mathsf{Rm}_M(e_i, e_j, e_i, e_j) = \mathsf{R}_\Sigma + \|A\|^2 - H^2. \tag{2.16}$$

We are then motivated to express the summation in (2.16) as $\mathsf{R}_M - 2\mathsf{Ric}_M(\nu, \nu)$, which we begin by letting $e_n = \nu$ to make $e_1, e_2, \ldots, e_{n-1}, e_n$ form an orthonormal frame for M. Next, change the ending index of the summation from $n-1$ into n, being sure to offset extra terms thus produced:

$$\begin{aligned}\sum_{i,j=1}^{n-1} \mathsf{Rm}_M(e_i, e_j, e_i, e_j) &= \sum_{i,j=1}^{n} \mathsf{Rm}_M(e_i, e_j, e_i, e_j) - \sum_{i=1}^{n} \mathsf{Rm}_M(e_i, e_n, e_i, e_n) \\ &\quad - \sum_{j=1}^{n} \mathsf{Rm}_M(e_n, e_j, e_n, e_j) + \mathsf{Rm}_M(e_n, e_n, e_n, e_n) \\ &= \mathsf{R}_M - \sum_{i=1}^{n} \mathsf{Rm}_M(e_i, e_n, e_i, e_n) - \sum_{j=1}^{n} \mathsf{Rm}_M(e_n, e_j, e_n, e_j)\end{aligned}$$

Note that $\mathsf{Rm}_M(e_n, e_n, e_n, e_n)$ contributes nothing because of the skew-symmetry of Riemann curvature tensor. This skew-symmetry is again used to establish

$$\begin{aligned}&- \sum_{i=1}^{n} \mathsf{Rm}_M(e_i, e_n, e_i, e_n) - \sum_{j=1}^{n} \mathsf{Rm}_M(e_n, e_j, e_n, e_j) \\ &\qquad = -2 \sum_{i=1}^{n} \mathsf{Rm}_M(e_n, e_i, e_n, e_i) \\ &\qquad = -2\delta^{ij} \mathsf{Rm}_M(e_n, e_i, e_n, e_j) \\ &\qquad = -2\overline{g}^{ij} \mathsf{Rm}_M(e_n, e_i, e_n, e_j) = -2\mathsf{Ric}_M(e_n, e_n).\end{aligned}$$

(2.16) then concludes with

$$\mathsf{R}_M - 2\mathsf{Ric}_M(\nu, \nu) = \mathsf{R}_\Sigma + \|A\|^2 - H^2.$$

□

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Remark 2.6. In the context of Hawking energy, the ambient manifold M in Corollary 2.5 is three-dimensional, so the scalar curvature R_Σ is two times the Gauss curvature K, which renders (2.14) as

$$\mathsf{R}_M = 2K + 2\mathsf{Ric}_M(\nu, \nu) + \|A\|^2 - H^2.$$

In light of (2.13), we see

$$\begin{aligned} \frac{\partial H}{\partial t} &= -\Delta\varphi - \left[\|A\|^2 + \frac{1}{2}\left(\mathsf{R}_M - 2K - \|A\|^2 + H^2\right)\right]\varphi \\ &= -\Delta\varphi + \frac{1}{2}\left(2K - \mathsf{R}_M - \|A\|^2 - H^2\right)\varphi. \end{aligned} \tag{2.17}$$

It might be abrupt to introduce the Gauss curvature K into (2.14), but we aimed to forge links between (2.13) and the Gauss-Bonnet theorem, which plays an integral part in our proof of the Geroch monotonicity formula. In the following theorem and hereafter, a closed manifold will always mean a compact manifold without boundary.

Theorem 2.7 (Gauss-Bonnet theorem)**.** *Let (Σ, g) be a closed Riemannian 2-manifold. If K is the Gauss curvature of Σ, and $\chi(\Sigma)$ is the Euler characteristic of Σ, then*

$$\int_\Sigma K \mathrm{d}\mu_g = 2\pi\chi(\Sigma). \tag{2.18}$$

We have no intention to prove this famous theorem, which has been treated properly in many textbooks. For a standard proof by triangulation, one can consult [10, Theorem 9.7]. While a written proof is not provided here for (2.18), we are interested in one particular component of this equation: $\chi(\Sigma)$. It is a topological property of Σ that can be evaluated according to the genus k of Σ. If Σ is connected and orientable, then $\chi(\Sigma) = 2 - 2k$; if Σ is connected and non-orientable, then $\chi(\Sigma) = 2 - k$. Either way, since k is non-negative, we can see $\chi(\Sigma) \leqslant 2$, an inequality that will be useful in the future.

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3 Proof of the formula

Now we are in a position to prove Theorem 1.3, which states that the Hawking energy of a connected closed hypersurface Σ will increase monotonically under IMCF as long as Σ evolves in a Riemannian 3-manifold of non-negative scalar curvature. We have to emphasize again that our proof does not contain any original idea; its purpose is merely to serve as an explanatory supplement to the proof in [6, section 6].

Proof of (1.4). Based on the definition of IMCF, our geometric quantities evolve by (2.1) with $\varphi = H^{-1}$. Then (2.17) becomes

$$\frac{\partial H}{\partial t} = -\Delta(H^{-1}) + \frac{1}{2}\left(2K - \mathsf{R}_M - \|A\|^2 - H^2\right) H^{-1}. \tag{3.1}$$

On the other hand, after substituting $\varphi = H^{-1}$ into (2.12), we obtain

$$\frac{\partial}{\partial t}\mathrm{d}\mu_g = \mathrm{d}\mu_g, \tag{3.2}$$

which in turn gives the rate at which the area of (Σ, g) changes with t:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Sigma| = \int_\Sigma \frac{\partial}{\partial t}\mathrm{d}\mu_g = \int_\Sigma \mathrm{d}\mu_g = |\Sigma| \tag{3.3}$$

Next, we invoke (3.1) and (3.2) to see

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma H^2 \mathrm{d}\mu_g \\
&\quad = \int_\Sigma \frac{\partial}{\partial t}\left(H^2 \mathrm{d}\mu_g\right) \\
&\quad = \int_\Sigma \left(2H\frac{\partial H}{\partial t}\mathrm{d}\mu_g + H^2\frac{\partial}{\partial t}\mathrm{d}\mu_g\right) \\
&\quad = \int_\Sigma \left\{2H\left[-\Delta(H^{-1}) + \frac{1}{2}\left(2K - \mathsf{R}_M - \|A\|^2 - H^2\right)H^{-1}\right]\mathrm{d}\mu_g + H^2\mathrm{d}\mu_g\right\} \\
&\quad = \int_\Sigma \left[-2H\Delta(H^{-1}) + 2K - \mathsf{R}_M - \|A\|^2\right]\mathrm{d}\mu_g.
\end{aligned} \tag{3.4}$$

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Let us isolate the integral of $-2H\Delta(H^{-1})$ to do integration by parts. Since we are taking care of a closed hypersurface, the boundary term will not be coming:

$$\begin{aligned}\int_\Sigma -2H\Delta(H^{-1})\mathrm{d}\mu_g &= -2\int_\Sigma H\,\mathrm{div}(\mathrm{grad}(H^{-1}))\mathrm{d}\mu_g \\ &= -2\left[0-\int_\Sigma \left\langle \mathrm{grad}(H^{-1})\middle|\mathrm{grad}\,H\right\rangle \mathrm{d}\mu_g\right] \\ &= 2\int_\Sigma \left\langle -H^{-2}\,\mathrm{grad}\,H\middle|\mathrm{grad}\,H\right\rangle \mathrm{d}\mu_g \\ &= \int_\Sigma -2H^{-2}\|\mathrm{grad}\,H\|^2\mathrm{d}\mu_g\end{aligned}$$

Thus, upon integration by parts, (3.4) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma H^2\mathrm{d}\mu_g \\ &\quad= \int_\Sigma \left(-2H^{-2}\|\mathrm{grad}\,H\|^2 + 2K - \mathsf{R}_M - \|A\|^2\right)\mathrm{d}\mu_g \\ &\quad= \int_\Sigma 2K\mathrm{d}\mu_g + \int_\Sigma \left(-2H^{-2}\|\mathrm{grad}\,H\|^2 - \mathsf{R}_M - \|A\|^2\right)\mathrm{d}\mu_g \\ &\quad= \underbrace{\int_\Sigma 2K\mathrm{d}\mu_g}_{①} + \underbrace{\int_\Sigma \left[-2H^{-2}\|\mathrm{grad}\,H\|^2 - \mathsf{R}_M - \frac{1}{2}(\lambda_1-\lambda_2)^2 - \frac{1}{2}H^2\right]\mathrm{d}\mu_g}_{②}\,.\end{aligned}$$

Note that we have included (2.11) in the last equality. To evaluate the integral ①, we can apply the Gauss-Bonnet theorem (2.18) to see

$$① = 4\pi\chi(\Sigma) \leqslant 8\pi,$$

where the inequality holds because Σ is connected. On the other hand, because the scalar curvature of M is non-negative, our estimate of the integral ② is that

$$② \leqslant \int_\Sigma -\frac{1}{2}H^2\mathrm{d}\mu_g.$$

Adding up these two estimates, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma H^2\mathrm{d}\mu_g \leqslant 8\pi - \frac{1}{2}\int_\Sigma H^2\mathrm{d}\mu_g. \tag{3.5}$$

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So far, we have been occupied with the ingredients needed to estimate the time derivative of Hawking energy. These ingredients are precisely (3.3) and (3.5). With both of them, we can derive the Geroch monotonicity formula simply by differentiating the energy function:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}m(\Sigma) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{\frac{|\Sigma|}{16\pi}}\left(1-\frac{1}{16\pi}\int_\Sigma H^2\mathrm{d}\mu_g\right)\right] \\
&= \left(1-\frac{1}{16\pi}\int_\Sigma H^2\mathrm{d}\mu_g\right)\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\frac{|\Sigma|}{16\pi}} \\
&\qquad + \sqrt{\frac{|\Sigma|}{16\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\left(1-\frac{1}{16\pi}\int_\Sigma H^2\mathrm{d}\mu_g\right) \\
&= \left(1-\frac{1}{16\pi}\int_\Sigma H^2\mathrm{d}\mu_g\right)\cdot\frac{1}{2\sqrt{\frac{|\Sigma|}{16\pi}}}\cdot\frac{1}{16\pi}\cdot|\Sigma| \\
&\qquad + \sqrt{\frac{|\Sigma|}{16\pi}}\left(0-\frac{1}{16\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma H^2\mathrm{d}\mu_g\right) \\
&= \frac{1}{2}\sqrt{\frac{|\Sigma|}{16\pi}}\left(1-\frac{1}{16\pi}\int_\Sigma H^2\mathrm{d}\mu_g\right) \\
&\qquad - \frac{1}{16\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma H^2\mathrm{d}\mu_g \\
&\geqslant \frac{1}{2}\sqrt{\frac{|\Sigma|}{16\pi}}\left(1-\frac{1}{16\pi}\int_\Sigma H^2\mathrm{d}\mu_g\right) \\
&\qquad - \frac{1}{16\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\left(8\pi-\frac{1}{2}\int_\Sigma H^2\mathrm{d}\mu_g\right) \\
&= 0
\end{aligned}
$$

The inequality completes our proof.

□

Remark 3.1. The application of the Gauss-Bonnet theorem suggests the inability to generalize the argument to other dimensions.

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4 Weak formulation

In this miniature section, we would like to say a few words to introduce a weak version of the Geroch monotonicity formula (1.4). It was developed by Huisken and Ilmanen as the cornerstone of [7, 8], where they managed to prove the Riemannian Penrose inequality. Since we just finished the proof of the strong version, it might be best to immediately bring in the statement of the weak version and see why it is a monotonicity formula. To be sure, this will come with a fair share of terminology that is unknown to us, but it should mean us no harm because we will not venture into the proof.

Theorem 4.1 (Geroch Monotonicity Formula 5.8 of [8])**.** *Let M be a complete 3-manifold and let E_0 be a precompact open subset of M that has C^1 boundary and realizes*

$$\int_{\partial E_0} \|A\|^2 \mathrm{d}\mu_{\partial E_0} < \infty.$$

Suppose $\{E_t\}_{t>0}$ weakly solves (1.2) with E_0 as the initial condition. If E_0 is a minimizing hull, then for $0 \leqslant r < s$,

$$\begin{aligned} m(\Sigma_s) - m(\Sigma_r) \geqslant \frac{1}{(16\pi)^{3/2}} \int_r^s |\Sigma_t|^{1/2} \Bigg\{ & 16\pi - 8\pi\chi(\Sigma_t) \\ & + \int_{\Sigma_t} \left[2H^{-2}\|\operatorname{grad} H\|^2 + \mathsf{R}_M + (\lambda_1 - \lambda_2)^2 \right] \mathrm{d}\mu_{\Sigma_t} \Bigg\} \mathrm{d}t, \end{aligned} \tag{4.1}$$

provided that E_s is precompact.

As a key element of the theorem, E_t represents a collection of strict sub-level sets that relates to IMCF in a way we will explain later. To see why (4.1) is a monotonicity formula for Hawking energy, observe that the right-hand side of the inequality will be non-negative if $\mathsf{R}_M \geqslant 0$ and $\chi(\Sigma_t) \leqslant 2$, which can be accomplished by requiring that the scalar curvature of M be non-negative and that Σ_t be connected.

The groundwork for Theorem 4.1 lies in a level-set formulation of IMCF. To be precise, Huisken and Ilmanen make a level-set ansatz in [7, 8] by assuming the IMCF in

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Definition 1.2 is given by level sets of a function u on M with

$$\Sigma_t = \partial E_t,$$

where E_t denotes the strict sub-level set $\{u < t\}$. With this ansatz, weak IMCF can be formulated by first observing that if $\operatorname{grad} u$ is nowhere zero, then (1.2) can be translated into

$$\operatorname{div}_M \left(\frac{\operatorname{grad} u}{\|\operatorname{grad} u\|} \right) = \|\operatorname{grad} u\|. \tag{4.2}$$

As we will see, this equation effectively compels the mean curvature to remain non-negative, a strategy Huisken and Ilmanen used to address singularities of classical IMCF. We shall make no further progress in the topic of weak IMCF and let us close the topic by showing how (4.2) implicates classical IMCF. To wit, assume Φ solves IMCF as described in Definition 1.2. Then, since Σ_t is fulfilled by a level set of u, we have $t = u(\Phi(p,t))$. Applying the chain rule to this equation, we see

$$\begin{aligned} 1 = \frac{\mathrm{d}}{\mathrm{d}t} t = \frac{\mathrm{d}}{\mathrm{d}t} u(\Phi(p,t)) &= \left\langle \operatorname{grad} u \middle| \frac{\partial \Phi}{\partial t} \right\rangle \\ &= \left\langle \operatorname{grad} u \middle| \frac{1}{H} \frac{\operatorname{grad} u}{\|\operatorname{grad} u\|} \right\rangle \\ &= \frac{1}{H} \|\operatorname{grad} u\|. \end{aligned} \tag{4.3}$$

Then remember that the mean curvature H can be alternatively expressed as[†]

$$\operatorname{div}_M \left(\frac{\operatorname{grad} u}{\|\operatorname{grad} u\|} \right). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain (4.2).

[†]This is a global expression that is valid on all of Σ_t because we have assumed $\operatorname{grad} u$ is nowhere zero; however, if H is to be evaluated locally, we need only assume u is a local defining function for Σ_t [10, Problem 8-2]. Be that as it may, we shall offer a derivation in Appendix B using the original assumption.

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Appendix A The Gauss-Weingarten equations

In this appendix, we provide a derivation for the Gauss-Weingarten equations (2.5) and (2.6). As mentioned before, they are obtained from unwinding the coordinate representation (2.2). Let us begin by deriving the Weingarten equations (2.6).

$$
\begin{aligned}
h_{ij} &= A(\partial_i, \partial_j) = A(\partial_j, \partial_i) = \overline{g}\left(\overline{\nabla}_{\partial_j}\nu, \mathrm{d}\Phi_t(\partial_i)\right) \\
&= \left\langle \overline{\nabla}_{\partial_j}(\nu^\alpha \partial_\alpha) \middle| \partial_i \Phi^\delta \partial_\delta \right\rangle = \partial_i \Phi^\delta \left\langle \partial_j \nu^\alpha \partial_\alpha + \nu^\alpha \overline{\nabla}_{\partial_j} \partial_\alpha \middle| \partial_\delta \right\rangle \\
&= \partial_i \Phi^\delta \left\langle \partial_j \nu^\alpha \partial_\alpha + \nu^\alpha \partial_j \Phi^\beta \overline{\nabla}_{\partial_\beta} \partial_\alpha \middle| \partial_\delta \right\rangle \\
&= \partial_i \Phi^\delta \left\langle \partial_j \nu^\alpha \partial_\alpha + \nu^\gamma \partial_j \Phi^\beta \overline{\Gamma}^\alpha_{\beta\gamma} \partial_\alpha \middle| \partial_\delta \right\rangle \\
&= \partial_i \Phi^\delta \left(\partial_j \nu^\alpha + \nu^\gamma \partial_j \Phi^\beta \overline{\Gamma}^\alpha_{\beta\gamma} \right) \overline{g}_{\alpha\delta} \qquad \text{(A.1)}
\end{aligned}
$$

Now we pause for a second to see (2.4) change into

$$
g_{ki} = \overline{g}(\partial_k \Phi, \partial_i \Phi) = \overline{g}(\partial_k \Phi^\alpha \partial_\alpha, \partial_i \Phi^\delta \partial_\delta) = \partial_k \Phi^\alpha \partial_i \Phi^\delta \overline{g}_{\alpha\delta},
$$

which helps to express (A.1) as

$$
h_{ij} \partial_k \Phi^\alpha = \left(\partial_j \nu^\alpha + \nu^\gamma \partial_j \Phi^\beta \overline{\Gamma}^\alpha_{\beta\gamma} \right) g_{ki}.
$$

Therefore,

$$
h_{ij} g^{jk} \partial_k \Phi^\alpha = \left(\partial_j \nu^\alpha + \nu^\gamma \partial_j \Phi^\beta \overline{\Gamma}^\alpha_{\beta\gamma} \right) \delta^j_i = \partial_i \nu^\alpha + \nu^\gamma \partial_i \Phi^\beta \overline{\Gamma}^\alpha_{\beta\gamma}.
$$

Let us proceed with the derivation of the Gauss equations (2.5). Taking the Gauss formula [10, Theorem 8.2] for granted, we know

$$
\overline{\nabla}_{\partial_i} \partial_j = \nabla_{\partial_i} \partial_j + \mathbf{A}(\partial_i, \partial_j),
$$

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where **A** is the vector-valued second fundamental form. Since $\mathbf{A} = -A\nu$, the right-hand side reads

$$\begin{aligned}\nabla_{\partial_i}\partial_j + \mathbf{A}(\partial_i, \partial_j) &= \Gamma_{ij}^k \partial_k - h_{ij}\nu^\alpha \partial_\alpha \\ &= \left(\Gamma_{ij}^k \partial_k \Phi^\alpha - h_{ij}\nu^\alpha\right) \partial_\alpha. \end{aligned} \tag{A.2}$$

As to the left-hand side, we introduce the mixed-derivative notation $\partial_{ij}^2 \coloneqq \partial_i \partial_j$ to compute

$$\begin{aligned}\overline{\nabla}_{\partial_i}\partial_j = \overline{\nabla}_{\partial_i}(\partial_j \Phi^\alpha \partial_\alpha) &= \partial_{ij}^2 \Phi^\alpha \partial_\alpha + \partial_j \Phi^\gamma \overline{\nabla}_{\partial_i} \partial_\gamma \\ &= \partial_{ij}^2 \Phi^\alpha \partial_\alpha + \partial_j \Phi^\gamma \partial_i \Phi^\beta \overline{\nabla}_{\partial_\beta} \partial_\gamma \\ &= \left(\partial_{ij}^2 \Phi^\alpha + \partial_j \Phi^\gamma \partial_i \Phi^\beta \overline{\Gamma}_{\beta\gamma}^\alpha\right) \partial_\alpha. \end{aligned} \tag{A.3}$$

Then equating (A.2) with (A.3) yields

$$\partial_{ij}^2 \Phi^\alpha + \partial_j \Phi^\gamma \partial_i \Phi^\beta \overline{\Gamma}_{\beta\gamma}^\alpha = \Gamma_{ij}^k \partial_k \Phi^\alpha - h_{ij}\nu^\alpha,$$

which is nothing but a rearrangement of (2.5).

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Appendix B Mean curvature as divergence

This appendix is intended for a derivation of the mean-curvature formula (4.4). In our derivation, the following linear-algebra lemma will be fundamental.

Lemma B.1. *Let T be a linear operator on a finite-dimensional inner product space $(V, \langle\cdot|\cdot\rangle)$, let ν be a unit vector in V, and define a linear operator U on V by*

$$U(x) = T(x) - \langle x|\nu\rangle\, T(\nu).$$

If T maps $\nu^\perp := \{\nu\}^\perp$ into $\nu^\perp$, then $\mathsf{tr}(T|_{\nu^\perp}) = \mathsf{tr}(U)$.

This lemma is actually a hint given by John Lee in his book, and we will demonstrate how it can possibly lead to (4.4). To validate the lemma, we simply make some elementary observations:

(i) $T = U$ on $\nu^\perp$.

(ii) $V = \mathsf{span}(\nu) \oplus \nu^\perp$

(iii) $\mathsf{tr}(U) = \mathsf{tr}(U|_{\mathsf{span}(\nu)}) + \mathsf{tr}(U|_{\nu^\perp})$

(iv) $U = \mathbf{0}$ on $\mathsf{span}(\nu)$.

After the lemma is proved, the derivation of (4.4) boils down to one thing: what should act the role of the operator T? Basic Riemannian geometry suggests that T should be the total covariant derivative

$$\overline{\nabla}\nu : X \mapsto \overline{\nabla}_X \nu$$

of the unit normal vector field $\nu := {}^{\mathrm{grad}\,u}/_{\|\mathrm{grad}\,u\|}$. That way, $\mathsf{tr}(T)$ will be none other than $\mathrm{div}_M \nu$, and $T|_{\nu^\perp}$ will become the shape operator $S := \left\{g^{ik} h_{kj}\right\}$, whose trace is equal to the mean curvature H. Now it remains to show $\mathsf{tr}(U) = \mathrm{div}_M \nu$ with U defined as in Lemma B.1. Let us begin by choosing an orthonormal frame $\{e_1, \ldots, e_{n-1}\}$ for Σ_t and

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set $e_n = \nu$. Then

$$\begin{aligned}
\mathsf{tr}(U) &= \sum_{i=1}^{n} \langle U(e_i)|e_i\rangle = \sum_{i=1}^{n} \langle T(e_i) - \langle e_i|\nu\rangle\, T(\nu)|e_i\rangle \\
&= \sum_{i=1}^{n} \langle T(e_i)|e_i\rangle - \sum_{i=1}^{n} \langle e_i|\nu\rangle \langle T(\nu)|e_i\rangle \\
&= \mathsf{tr}(T) - \langle e_n|\nu\rangle \langle T(\nu)|e_n\rangle \\
&= \mathrm{div}_M \nu - 1 \cdot \langle T(\nu)|\nu\rangle .
\end{aligned}$$

To see $\langle T(\nu)|\nu\rangle = 0$, a straightforward computation will suffice:

$$\begin{aligned}
\langle T(\nu)|\nu\rangle &= \left\langle \overline{\nabla}_\nu \nu \middle| \nu \right\rangle \\
&= \frac{1}{2}\left(\left\langle \overline{\nabla}_\nu \nu \middle| \nu \right\rangle + \left\langle \nu \middle| \overline{\nabla}_\nu \nu \right\rangle\right) \\
&= \frac{1}{2}\overline{\nabla}_\nu \langle \nu|\nu\rangle = \frac{1}{2}\overline{\nabla}_\nu 1 = \frac{1}{2}\nu(1) = \frac{1}{2}\cdot 0
\end{aligned}$$

Finally, we apply Lemma B.1 to conclude $H = \mathrm{div}_M \nu$.

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