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# 代數纖維空間中正則除子與反正則除子的正性表現

# Positivity of Canonical and Anticanonical Divisors in Algebraic Fibre Spaces

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# 摘要


本論文的主要目的是探討在代數纖維空間中正則除子與反正則除子的正性表現，首先我們會證明給定一個代數纖維空間 $f : X \to Y$, 若對 $X$ 上的奇異點有一些限制，$X$ 的反正則除子 $-K_X$ 為一擬有效/$\mathbb{Q}$-有效/巨大除子，且 $-K_X$ 之相對應的的漸進基底點集合並不支配 $Y$ 的話，則 $-K_Y$ 也會是擬有效/$\mathbb{Q}$-有效/巨大除子。作為本結果的推論，我們可以證明若 $X$ 僅具備川又對數端末 (klt) 奇異點/對數正規 (lc) 奇異點，則 $f$ 的相對反正則除子 $-K_{X/Y}$ 必然不是豐富除子/巨大 nef 除子。

第二部分我們會先證明對於滿足特定條件的代數纖維空間，其介於 $X$ 以及 $F$ 的反正則切斷環之間的限制映射會是一個賦次環之間的環單射。該定理為江尻-權業單射性定理的一個變形。做為此一單射性定理的應用, 我們會證明一個反正則除子版本的飯高猜想，其具體敘述如下：給定一代數纖維空間 $f : X \to Y$, 若 $X$ 僅具有 klt $\mathbb{Q}$-Gorenstein 奇異點, $-K_X$ 為一 $\mathbb{Q}$-有效除子, 且其穩定基底點集合 $\mathbf{B}(-K_X)$ 並不支配 $Y$ 的話，則對於 $f$ 的一般纖維 $F$, 我們有以下的不等式

$$\kappa(-K_X) \leq \kappa(-K_F) + \kappa(-K_Y)。$$

於第三部分我們則會證明一般化非消滅猜想的部分特例並討論其與飯高猜想的關聯。一般化非消滅猜想敘述為給定一 klt 對 $(X, \Delta)$, $X$ 上的一個 nef $\mathbb{Q}$-Cartier 除子 $L$, 以及一非負的有理數 $t$, 若 $K_X + \Delta + tL$ 為一擬有效 $\mathbb{Q}$-因子, 則其會與某一有效 $\mathbb{Q}$-因子數值同值。於本論文中我們會證明對於有非負小平次元的三維多樣




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體, 若其小平次元與 $q(X)$ 兩不變量當中有至少一個大於零, 則該多樣體滿足一般化非消滅猜想。做為此結論的應用, 我們可以證明對於代數纖維空間 $f: X \to Y$, 若 $X$ 的維度不超過七維且已知 $X$ 有非負的小平維度, 則該代數纖維空間滿足飯高猜想, 也即是滿足以下不等式

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

最後, 我們會再額外補充一些有關反正則除子具有良好正性的代數多樣體的一些相關討論。

**關鍵字：**代數纖維空間、正則除子、反正則除子、弱正性定理、漸進基底點、反正則飯高維度、飯高猜想、一般化非消滅猜想、nef 縮小映射



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Abstract

In this thesis, we will discuss the positivity behavior of the canonical divisor and the anticanonical divisor in algebraic fibre spaces. At first, we will prove for an algebraic fibre space $f : X \to Y$, if $X$ has mild singularities, $-K_X$ is pseudoeffective (resp. effective, big), and the corresponding asymptotic base locus of $-K_X$ does not dominate $Y$, then $-K_Y$ is also pseudoeffective (resp. effective, big). As a corollary, we can prove if $X$ has klt (resp. lc) singularities, then the relative anticanonical divisor $-K_{X/Y}$ could not be nef and big (resp. could not be ample).

Secondly, we will prove for certainly algebraic fibre spaces, the restriction map between the anticanonical section ring of $X$ and $F$ is an injective graded ring homomorphism. This theorem is a variant of Ejiri-Gongyo's Injectivity Theorem. As an application, we will prove an "anticanonical version" of the Iitaka Conjecture. The statement is: if $X$ has at worst klt $\mathbb{Q}$-Gorenstein singularities, and $-K_X$ is $\mathbb{Q}$-effective with the stable base locus $\mathbf{B}(-K_X)$ does not dominate $Y$, the for a general fibre $F$ of $f$, we have the



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inequality

$$\kappa(-K_X) \leq \kappa(-K_F) + \kappa(-K_Y).$$

Thirdly, we will prove some special cases of the Generalized Nonvanishing Conjecture and discuss its relation with the Iitaka Conjecture. The Generalized Nonvanishing Conjecture states that for a klt pair $(X, \Delta)$, a nef $\mathbb{Q}$-Cartier divisor $L$ on $X$, and a non-negative rational number $t$, if $K_X + \Delta + tL$ is pseudoeffective, then it should numerically equivalent to an effective $\mathbb{Q}$-divisor. In this thesis, we will prove that for threefolds with non-negative Kodaira dimension, if either the Kodaira dimension or the irregularity is positive, then the Generalized Nonvanishing Conjecture holds. As an application, we can show the Iitaka conjecture is true if $X$ has non-negative Kodaira dimension and the dimension of $X$ is at most seven, that is, for an algebraic fibre space $f : X \to Y$ with general fibre $F$, if dim $X \leq 7$ and $\kappa(X) \geq 0$, then we have

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

Finally, we will give some other miscellanies results about varieties whose anticanonical divisor has good positivities.

**Keywords:** Algebraic fibre space, canonical divisor, anticanonical divisor, weakly positivity theorem, asymptotic base locus, anti-canonical Iitaka dimension, Iitaka Conjecture, Generalized Nonvanishing Conjecture, nef reduction map.


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# Contents





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# Chapter 1 Introduction

Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties, that is, a projective morphism between normal projective varieties with connected fibres. A natural and important question is to discuss the relation between $K_X$ and $K_Y$, the canonical divisors of $X$ and $Y$. For example, one classical example of such a question is the well-known Iitaka Conjecture, which states that

$$\kappa(X) \geq \kappa(F) + \kappa(Y),$$

where $F$ denotes a general fibre, and $\kappa$ denotes the Kodaira dimension of varieties, which is the Iitaka dimension of the canonical divisor. Note that the main purpose of the Iitaka Conjecture is to relate the ”positivities” of $K_X, K_Y$ and $K_F$. In particular, since the Kodaira dimension is one of the most important birational invariants of normal projective varieties, this conjecture is very important for the study of the ”birational geometry of algebraic fibre spaces”.

Conversely, in the world of varieties with ”negative” canonical divisors, especially in the case of Fano varieties and related varieties, it is also very natural and important to compare the positivities of $-K_X$ and $-K_Y$, the anticanonical divisor of $X$ and $Y$. Although despite the Kodaira dimension, the Iitaka dimension of anticanonical divisor is not a birational invariant in general, this invariant still gives us much information about the geometry of varieties with effective anticanonical divisor. For example, Ejiri and Gongyo proved that for a variety with nef anticanonical divisor, the rational dimension is always not less than the Iitaka dimension of the anticanonical divisor (cf. [EG19]). On the other


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hand, in the Minimal Model Mrogram, it is expected that every projective variety with non-negative Kodaira dimension is birationally equivalent to a good minimal model, and varieties with negative dimension will be birational to a variety which has a Mori fibre spaces structure. Here, a Mori fibre space is an algebraic fibre space whose general fibres are Fano varieties with Picard number 1. Therefore, the study of algebraic fibre spaces with positive $-K_X$, $-K_Y$, and $-K_F$ is also closely related to the study of the minimal model program.

In this thesis, we will discuss that in an algebraic fibre space $f : X \to Y$ with general fibre $F$, how does the positivity of $\pm K_X$, $\pm K_F$, and $\pm K_Y$ affect each other. To study this question, one of the most important classical tool is the Weakly Positivity Theorem, which is developed and generalized by many mathematicians (cf. [Cam04, Fuj17, Fuj78, Kaw81, Vie83]). The main result of the Weakly Positivity Theorem is for sufficiently divisible positive integer $m$, the sheaf $f_*\mathcal{O}_X(m(K_{X/Y}))$ is weakly positive in the sense of [EG19] (or [Vie83]), where $K_{X/Y} = K_X - f^*K_Y$. Thus, by the "positivity" of $K_{X/Y}$, we should expect that under some assumptions, if $K_Y$ has good positivity, then so does $K_X$. Conversely, we should also expect the positivity of $-K_X$ will affect the positivity of $-K_Y$. In this thesis, we will discuss both the canonical divisor case and the anticanonical divisor case of the above problem.

For the anticanonical divisor, we will mainly discuss how the positivity of $-K_X$ affects $-K_Y$. In the special case that $f$ is a smooth morphism, by works of Kollár-Miyaoka-Mori, Fujino-Gongyo, and Birkar-Chen, it is proved that if $-K_X$ is ample (resp. nef and big, nef, semiample), then so does $-K_Y$ (cf. [KMM92a], [FG12], [BC16]). However, a morphism being smooth is a very strong condition. For general algebraic fibre spaces, one interesting result is a theorem of Meng Chen and Qi Zhang (cf. [CZ13]). They proved that if $X$ has mild singularities and $-K_X$ is nef, then $-K_Y$ is pseudo-effective. This statement is originally conjectured by Demailly-Peternell-Schneider in [DPS01], where the original conjecture states more generally that if the "non-nef locus" of $-K_X$ does not dominate $Y$, then $-K_Y$ is pseudo-effective. After the theorem of Chen-Zhang, in [EG19] and [EIM20], Ejiri-Gongyo and Ejiri-Iwai-Matsumura generalized this result in many situ-


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ations. More precisely, Ejiri and Gongyo generalize this result for sub-log canonical pairs $(X, \Delta)$ such that $-(K_X + \Delta)$ is nef. Also, when $Y$ has at worst canonical singularities, Ejiri-Iwai-Matsumura generalize this result for pseudoeffective anticanonical divisor with $\mathbf{B}_{-}(-K_X)$ does not dominate $Y$, where $\mathbf{B}_{-}(-K_X)$ is the restricted base locus. Moreover, Ejiri, Iwai, and Matsumura explicitly describe the "upper bound" of the restricted base locus of $-K_Y$. In Theorem 4.1.1 of this thesis, by furthermore generalizing the ideas of Ejiri-Gongyo and Ejiri-Iwai-Matsumura, we will prove the pseudoeffectiveness part of $-K_Y$ in the theorem of Ejiri-Iwai-Matsumura still holds even if we do not assume $Y$ has at worst canonical singularities.

Note that for a pseudoeffecitve divisor $D$, $\mathbf{B}_{-}(D) = \emptyset$ is equivalent to $D$ is nef. Also, intuitively, a pseudoeffective divisor can be thought of as a divisor that is "nef at almost everywhere". Similarly, a big divisor can also be thought of as a divisor that is "ample at almost everywhere", and a big divisor $D$ is ample is also equivalent to $\mathbf{B}_{+}(D) = \emptyset$. Therefore, as an analog statement, it is very natural to ask that if $-K_X$ is big with $\mathbf{B}_{+}(-K_X)$ does not dominate $Y$, then whether $-K_Y$ is big. In [EIM20], Ejiri-Iwai-Matsumura gives an affirmative answer for the above question under the assumption that $Y$ has at worst canonical singularities. Moreover, as the pseudoeffective case, they give an explicit description of the upper bound of $\mathbf{B}_{+}(-K_Y)$. In Theorem 4.1.7 of this thesis, we will also prove the bigness of $-K_Y$ without assuming $Y$ has at worst canonical singularities. As a corollary, we can prove that if $X$ has klt (resp. lc) singularities, then $-K_{X/Y}$ can not be nef and big (resp. ample) (cf. Theorem 4.1.8). Note that when $Y$ is a curve, these two results had been proved by Araujo-Druel (cf. [CD13]), and when $Y$ has at worst canonical singularities, the non-ampleness of $-K_{X/Y}$ is also proved by Ejiri-Iwai-Matsumura in any dimension. Our works generalized these results in more general situations.

Moreover, as one more analog statement, it is natural to ask that if $-K_X$ is effective with $\mathbf{B}(-K_X)$ does not dominate $Y$, then whether $-K_Y$ is effective. For this question, by using Ambro's canonical formula ([Amb05]), we also give a positive answer to the above question (cf. Theorem 4.2.2). Moreover, by observing the above results, we may wonder that, under some assumptions on the asymptotic base locus of $-K_X$, is there exists


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an "anticanonical version" of the Iitaka Conjecture? Intuitively, by the Weakly Positive Theorem, we may expect there could be an "inverse" inequality of the Iitaka Conjecture, that is, we may expect the inequality

$$\kappa(-K_X) \leq \kappa(-K_F) + \kappa(-K_Y)$$

holds under some good assumptions. Note that if there is no assumption for the asymptotic base locus, then by looking for some special ruled surfaces, this inequality fails even if in dimension 2 (see Example 4.2.8). However, we can prove that if $X$ has at worst klt $\mathbb{Q}$-Gorenstein singularities, and $-K_X$ is effective with stable base locus $\mathbf{B}(-K_X)$ does not dominate $Y$, then the above inequality holds (cf. Theorem 4.2.1). The idea of our proof is the following: By modifying the proof of Ejiri-Gongyo's injectivity theorem (cf. [EG19, Theorem 1.2]), we can show that if $\kappa(-K_Y) = 0$, then under our assumption, the map $R(X, -K_X) \to R(F, -K_F)$ is an injective graded ring homomorphism (cf. Theorem 4.2.4), which implies $\kappa(-K_X) \leq \kappa(-K_F)$. For general cases, we take the Iitaka fibration of $-K_Y$, and restrict the original morphism $f : X \to Y$ on the preimage of the fibre of the Iitaka fibration to reduce it to the case $\kappa(-K_Y) = 0$.

For the canonical divisor, in this thesis we will mainly discuss about the Iitaka Conjecture. At this moment, Iitaka Conjecture is proved in the case if $\dim X \leq 6$, $\dim Y \leq 2$, or $Y$ has maximal Albanese dimension (cf. [Bir09], [Cao18], [CP17], [HPS18], [Kaw82]). Moreover, it is also well-known that if the fibre has good minimal models, then the Iitaka Conjecture holds (cf. [Kaw85]). However, the general case of the Iitaka Conjecture is still an open problem. In this thesis, we will prove some special cases of the Iitaka Conjecture by modifying Birkar's proof.

At first, for an algebraic fibre space $f : X \to Y$ with general fibre $F$ has Kodaira dimension 0, there is a canonical bundle formula proved by Fujino-Mori ([FM00]), which states that by replacing $X$ and $Y$ with sufficiently higher birational models, there exists a klt pair $(Y, B)$ and a nef divisor $L$ on $Y$ such that $\kappa(X) = \kappa(Y, K_Y + B + L)$. In general, such a combination of a klt pair with a nef $\mathbb{Q}$-divisor is called a "generalized pair". Recently, for such generalized pairs, it is conjectured by Lazić-Matsumura-


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Peternell-Tsakanikas-Xie (cf.[LMP+22]) that if $K_Y + B + tL$ is pseudoeffective for some non-negative rational number $t$, then $K_Y + B + tL$ will numerically equivalent to an effective $\mathbb{Q}$-divisor. In particular, if $\kappa(Y) \geq 0$, then for any $t \geq 0$, $K_Y + B + tL$ will numerically equivalent to an effective $\mathbb{Q}$-divisor. This conjecture is called the Generalized Nonvanishing Conjecture. Originally, this conjecture is introduced by Lazić-Peternell in [LP20], where the original version states that for a klt pair $(X, \Delta)$ and a nef Cartier divisor $L$, if $K_X + \Delta$ is pseudoeffective, then for any non-negative rational number $t$, $K_X + \Delta + tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. In this thesis, by studying the nef reduction map and the Albanese morphism, we can prove the original version of Generalized Nonvanishing Conjecture (that is, $(X, \Delta)$ is a klt pair such that $K_X + \Delta$ is pseudoeffective) is true for almost all cases in dimension 3. More precisely, we can prove this version of Generalized Nonvanishing Conjecture in dimension 3 if either the Kodaira dimension of the threefold is positive, or the irregularity is positive, or the nef divisor $L$ does not have maximal nef dimension (cf. Theorem 5.1.1).

On the other hand, by modifying Birkar's proof, we can prove that the Generalized Nonvanishing Conjecture implies the Iitaka Conjecture for algebraic fibre spaces whose general fibres have Kodaira dimension 0 (cf. Theorem 5.2.1). Combining with our (partial) result of the Generalized Nonvanishing Conjecture in dimension 3, we can prove for $\dim X \leq 7$, almost all cases of the Iitaka conjecture hold. More precisely, the Iitaka Conjecture for $\dim X = 7$ holds if either $\kappa(X) \geq 0$, or $q(X) > 0$, or the base $Y$ is not a threefold with $\kappa(Y) = q(Y) = 0$ (cf. Theorem 5.2.3).

This thesis is organized as follows: chapter 2 is the preliminaries. We will give some well-known basic properties about general algebraic geometry, birational geometry, and some recent developments of the Minimal Model Program. In Chapter 3, we will introduce known properties of algebraic fibre spaces. Some parts of this section are well-known classical results, and others have already proved in these several decades. In Chapters 4 and chapter 5, we will prove some new results about the positivity of (anti)canonical divisors in algebraic fibre spaces. Chapter 4 focuses on the anticanonical divisors, especially the Kodaira dimension of anticanonical divisors. On the other hand, chapter 5 is mainly


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discussed with the canonical divisors, especially for the Iitaka Conjecture and the Generalized Nonvanishing Conjecture. Chapter 6 includes some other miscellanies results about varieties with positive anticanonical divisor. Finally, in Chapter 7, we will discuss some related topics and open questions that are possible to study in the future.


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# Chapter 2 Preliminaries

In this section, we will give a brief review of fundamental tools in algebraic geometry, birational geometry, and the Minimal Model Program.

## 2.1 Algebraic geometry and birational geometry background

In this section, we will introduce some well-known basic knowledge of algebraic geometry and birational geometry. The main reference of this section is [KM98].

### 2.1.1 Conventions and notations

In this thesis, we work over $\mathbb{C}$. A $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) is a rational combination (resp. real combination) of prime divisors. A $\mathbb{Q}$-divisor $D$ is called $\mathbb{Q}$-Cartier (resp. $\mathbb{Q}$-effective) if there is a positive integer $m$ s,t, $mD$ is linearly equivalent to a Cartier divisor (resp. effective divisor.) A morphism $f : X \to Y$ is called an *algebraic fibre space* if it is a surjective projective morphism with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Note that for fields of characteristic 0, this condition is equivalent to $f$ having connecdted fibres.

For a $\mathbb{Q}$-divisor $D = \sum d_i D_i$ on $X$, we will denote

$$D^+ := \sum_{d_i>0} d_i D_i, \qquad D^- := \sum_{d_i<0} d_i D_i,$$


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and that

$$D^h := \sum_{f(\mathrm{Supp} D_i)=Y} d_i D_i, \qquad D^v := \sum_{f(\mathrm{Supp} D_i)\neq Y} d_i D_i.$$

### 2.1.2 Basic notions in birational geometry

In this subsection, we will introduce basic notions of birational geometry, including the Kodaira dimension, Iitaka dimension, semiample fibration, Iitaka fibration, and augmented irregularity.

**Definition 2.1.1.** *(Iitaka dimension) Let $X$ be a projective variety, and $D$ be a $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. The Iitaka dimension of $D$, denoted by $\kappa(X, D)$ (or by $\kappa(D)$ if there is no ambiguity), is defined by:*

$$\kappa(X, D) := \sup\{\dim \mathrm{Im}(\phi_{|mD|}) | m \in \mathbb{Z}_{>0}\}.$$

*Here, $\phi_{|mD|}$ is the rational map $X \dashrightarrow \mathbb{P}^N$ defined by $x \to [s_0(x), ..., s_N(x)]$, where $\{s_i\}$ is a basis of $H^0(X, \lfloor mD \rfloor)$. If $D$ is not $\mathbb{Q}$-effective, then $H^0(X, \lfloor mD \rfloor)$ is always empty, so we define $\kappa(D) = -\infty$.*

If $\kappa(X, D) \geq 0$, then there is another equivalent definition of the Iitaka dimension:

**Proposition 2.1.2.** *(cf.[Uen75]):*

$$\kappa(X, D) := \sup\{k \in \mathbb{N} | \limsup_{m\to\infty} \frac{\dim H^0(X, \lfloor mD \rfloor)}{m^k} > 0\}.$$

**Definition 2.1.3.** *(Kodaira dimension) Let $X$ be a smooth projective variety. The Kodaira dimension of $X$, denoted $\kappa(X)$, is the Iitaka dimension of the canonical divisor $K_X$. If $X$ is not smooth, then we define $\kappa(X)$ to be the Kodaira dimension of a resolution of singularities of $X$.*

Now, we can introduce the semiample fibration and the Iitaka fibration.


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**Theorem 2.1.4** (Semiample fibration). *Let $X$ be a normal projective variety, and $D$ be a $\mathbb{Q}$-Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$. Assume $D$ is semiample, then for sufficiently divisible positive integers $m$, $\phi_{|mD|}$ is isomorphic to a fixed algebraic fibre space $\phi : X \to Y$ between normal projective varieties such that* $\dim Y = \kappa(X, D)$. *Moreover, let $F$ be a general fibre of $\phi$, then $D|_F \sim_{\mathbb{Q}} 0$.*

**Theorem 2.1.5** (Iitaka fibration). *Let $X$ be a normal projective variety, and $D$ be a $\mathbb{Q}$-Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$. Then for sufficiently divisible positive integers $m$, $\phi_{|mD|}$ are birational equivalent to a fixed algebraic fibre space $\phi : X' \to Y'$ between normal projective varieties such that* $\dim Y = \kappa(X, D)$. *That is, there is a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "\phi"] & X \arrow[d, dashed, "\phi_{|mD|}"] \\ Y' \arrow[r, dashed, "\mu"] & Y. \end{tikzcd}$$

*such that both $\pi$ and $\mu$ are birational. Moreover, let $F$ be a very general fibre of $\phi$, then $\kappa(F, D|_F) = 0$.*

At the end of this subsection, we introduce the definition of augmented irregularity:

**Definition 2.1.6.** *Let $X$ be a normal projective variety, the augmented irregularity of $X$, denoted by $\tilde{q}(X)$, is defined by*

$$\tilde{q}(X) := \max\{q(X') |\ X' \to X \text{ is quasi étale }\}.$$

### 2.1.3 The cone of divisors, the cone of 1-cycles, and the Mori cone

In this subsection, we will briefly recall some basic properties of the cone of divisor and Mori cone. At first, we state the following well-known ampleness criterion of ($\mathbb{Q}$-)Cartier divisors.

**Theorem 2.1.7.** *(Nakai-Moishezon Criterion, cf. [KM98, Theorem 1.37]) A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A$ on a proper scheme is ample if and only if for every closed subscheme $Z \subset X$, we have $A^{\dim Z}.Z > 0$.*


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Next, we will introduce some special classes of $\mathbb{Q}$-divisors.

**Definition 2.1.8.** *A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $X$ is called* nef *if $D.C \geq 0$ for every integral curve $C \subset X$.*

**Definition 2.1.9.** *A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is called big if $\kappa(X, D) = \dim X$.*

The following are some basic properties of nef and big $\mathbb{Q}$-divisors.

**Proposition 2.1.10.** *(cf. [Laz04, Corollary 2.2.6]) A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is big if and only if there is an ample $\mathbb{Q}$-divisor $A$ such that $D - A$ is $\mathbb{Q}$-effective.*

**Proposition 2.1.11.** *(cf. [KM98, Proposition 2.61]) A nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is big if and only $D^n > 0$, where $n = \dim X$.*

Next, we will introduce the numerical equivalent classes of 1-cycles.

**Definition 2.1.12.** *We say two $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ and $D'$ are numerically equivalent, denoted $D \equiv D'$, if for every integral curve $C \subset X$, we have $D.C = D'.C$.*

**Definition 2.1.13.** *We say two 1-cycle $C$ and $C'$ are numerically equivalent, denoted $D \equiv D'$, if for every Cartier divisor $D$ on $X$, we have $D.C = D.C'$.*

**Lemma 2.1.14.** *(cf. [Laz04, Lemma 1.1.18]) Let $D_1, ..., D_k$ and $D'_1, ..., D'_k$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on $X$. Assume for every $i$, $D_i \equiv D'_i$, then for every $k$-dimensional integral closed subscheme $Z$ of $X$, we have $D_1D_2...D_kZ = D'_1D'_2...D'_kZ$, where $D_1D_2...D_kZ$ denotes the intersection product of $D_1, ..., D_k$ with $Z$.*

Combining the above lemma and the Nakai-Moishezon criterion, we can see that $A \equiv A'$ and $A$ is ample, then $A'$ is also ample. As a corollary, bigness is also a numerical property.

Now we can introduce the cone of divisors.

**Definition 2.1.15.** *(Cone of divisor) Let $N^1(X)$ be the real vector space generated by the set of numerically equivalent classes of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$, which is a finite-dimensional $\mathbb{R}$-vector space.*



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(1) *The ample cone of $X$, denoted $\mathrm{Amp}(X)$, is the cone in $N^1(X)$ which is generated by all the numerically equivalent classes of ample divisors;*

(2) *The nef cone of $X$, denoted $\mathrm{Nef}(X)$, is the cone in $N^1(X)$ which is generated by all the numerically equivalent classes of nef divisors;*

(3) *The big cone of $X$, denoted $\mathrm{Big}(X)$, is the cone in $N^1(X)$ which is generated by all the numerically equivalent classes of big divisors;*

(4) *The effective cone of $X$, denoted $\mathrm{Eff}(X)$, is the cone in $N^1(X)$ which is generated by all the numerically equivalent classes of effective divisors;*

(5) *The pseudo-effective cone of $X$, denoted $\mathrm{Pse}(X)$, is the closure of $\mathrm{Eff}(X)$.*

**Definition 2.1.16.** *A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is called pseudo-effective if $[D] \in \mathrm{Pse}(X)$, where $[D]$ is the numerically equivalent class of $D$ in $N^1(X)$.*

**Theorem 2.1.17.** *(cf. [Laz04]) $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$, $\mathrm{Pse}(X) = \overline{\mathrm{Big}(X)}$. $\mathrm{Int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$, $\mathrm{Int}(\mathrm{Pse}(X)) = \mathrm{Big}(X)$. In particular, $\mathrm{Nef}(X)$ and $\mathrm{Pse}(X)$ are closed, and $\mathrm{Amp}(X)$ and $\mathrm{Big}(X)$ are open.*

**Definition 2.1.18.** *(The cone of curve) Let $N_1(X)$ be the set of numerically equivalent classes of 1-cycles $X$. By the definition of numerical equivalence, the pair*

$$N^1(X) \times N_1(X) \to \mathbb{Q}$$

*defined by the intersection product $(D, C) \to D.C$, is a perfect pairing. In particular, $\dim N^1(X) = \dim N_1(X)$. Moreover, we define the following:*

(1) *$NE(X)$ is the cone in $N_1(X)$ which is generated by all the numerically equivalent classes of effective curves;*

(2) *The Mori cone of $X$, denoted by $\overline{NE}(X)$, is the closure of $NE(X)$ in $N_1(X)$.*

**Theorem 2.1.19.** *(Kleinmen's Criterion) A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is ample if and only if for every $[C] \in \overline{NE}(X) - \{0\}$, we have $D.C > 0$.*


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### 2.1.4 Singularities of pairs

In this subsection, we will introduce the notion of singularities of pairs.

**Definition 2.1.20.** *A* pair $(X, \Delta)$ *contains a normal projective variety $X$ and a $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. A pair is called a* log pair *if $\Delta$ is effective. A log pair is called* log smooth *(or just smooth for simplicity) if $X$ is smooth and* Supp$\Delta$ *is simple normal crossing.*

**Definition 2.1.21** (Singularities of pairs)**.** *Let $(X, \Delta)$ be a log pair, then we define the singularities of the pair as follows: for a birational morphism $\pi : X' \to X$, we can write*

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta) + \sum a(E, X, \Delta)E,$$

*where $\Delta'$ is the proper transform of $\Delta$ on $X'$, and the summation runs over every $\pi$-exceptional divisor on $X'$, then we say:*

*(1)* $(X, \Delta)$ *is* terminal *if $a(E, X, \Delta) > 0$ for every birational morphism $\pi : X' \to X$ and every $\pi$-exceptional divisor $E$ on $X'$;*

*(2)* $(X, \Delta)$ *is* canonical *if $a(E, X, \Delta) \geq 0$ for every birational morphism $\pi : X' \to X$ and every $\pi$-exceptional divisor $E$ on $X'$;*

*(3)* $(X, \Delta)$ *is* Kawamata log terminal (klt for short) *if $\lfloor \Delta \rfloor = 0$ and $a(E, X, \Delta) > -1$ for every birational morphism $\pi : X' \to X$ and every $\pi$-exceptional divisor $E$ on $X'$;*

*(4)* $(X, \Delta)$ *is* purely log terminal (dlt for short) *if $a(E, X, \Delta) > -1$ for every birational morphism $\pi : X' \to X$ and every $\pi$-exceptional divisor $E$ on $X'$;*

*(5)* $(X, \Delta)$ *is* log canonical (lc for short) *if $a(E, X, \Delta) \geq -1$ for every birational morphism $\pi : X' \to X$ and every $\pi$-exceptional divisor $E$ on $X'$.*

*We say $X$ has terminal (resp. canonical, klt, lc) singularities if $K_X$ is $\mathbb{Q}$-Cartier and the pair $(X, 0)$ is terminal (resp. canonical, klt, lc). Note that if $(X, 0)$ is dlt, then it is klt.*


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## 2.2 Minimal Model Program and Abundance Conjecture

In this section, we will give a brief review of the development of the Minimal Model Program and the Abundance Conjecture in recent years.

### 2.2.1 Numerical dimension, abundant divisors, and the Abundance Conjecture

**Definition 2.2.1.** *Let $X$ be a projective variety, and $D$ be a pseudo-effective $\mathbb{Q}$-Cartier divisor on $X$. The numerical dimension of $D$, denoted by $\nu(X, D)$ (or by $\nu(D)$ if there is no ambiguity), is defined by:*

$$\nu(X, D) := \sup\{k \in \mathbb{N} | \limsup_{m\to\infty} \frac{\dim H^0(X, \lfloor mD + A \rfloor)}{m^k} > 0\},$$

*Where $A$ is a fixed ample Cartier divisor on $X$. In the case $D$ is not pseudoeffective, we define $\nu(D) = -\infty$.*

By definition, it is obviously that $\nu(X, D) \geq \kappa(X, D)$ for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$. Also, if $D$ is nef, then we have (cf. [Nak04, Proposition V.2.7(6)]):

$$\nu(X, D) = \sup\{k \in n | D^k \not\equiv 0\}.$$

Next, we recall the abundant divisor.

**Definition 2.2.2.** *Let $X$ be a normal projective variety and $L$ be a pseudo-effective $\mathbb{Q}$-Cartier divisor on $X$. We say $L$ is* abundant *(or good) if the equality $\kappa(L) = \nu(L)$ holds.*

One of the most important and difficult conjectures in birational geometry is the following conjecture, called the Abundance Conjecture:

**Conjecture 2.2.3.** *(Abundance Conjecture) Let $(X, \Delta)$ be a klt pair. If $(K_X + \Delta)$ is nef, then it is abundant.*


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The abundance conjecture is confirmed up to 3-dimensional ([Kaw92a]). Also, another important result about abundant (anti)canonical divisor is the following:

**Theorem 2.2.4.** *(cf. [Fuj11, Theorem 1.1]) Let $(X, \Delta)$ be a klt pair. If $\pm(K_X + \Delta)$ is nef and abundant, then it is semiample.*

### 2.2.2 The Cone Theorem and Fundamentals of the Minimal Model Program

The Minimal Model Program is one of the most important questions in birational geometry. The Minimal Model Conjecture states that every normal projective variety is either birational equivalent to a minimal model or birational equivalent to a Mori fiber space. The following are the definitions of these two notions:

**Definition 2.2.5.** *Let $X$ be a normal projective variety. A minimal model of $X$ is a variety $X'$ satisfies the followings:*

*(1) $X'$ is birationally equivalent to $X$;*

*(2) $X'$ is $\mathbb{Q}$-factorial and has at worst terminal singularities;*

*(3) $K_{X'}$ is nef.*

*If moreover, $K_{X'}$ is abundant, then we say $X'$ is a good minimal model of $X$. Therefore, the Abundance Conjecture implies every minimal model is good. Note that by [Fuj11, Theorem 1.1], a good minimal model has semiample canonical divisor.*

**Definition 2.2.6.** *A Mori fibre space is an algebraic fibre space $f : X \to Y$ that satisfies the following:*

*(1) $X$ is $\mathbb{Q}$ factorial and has at worst terminal singularities;*

*(2) $-K_F$ is ample, where $F$ is a general fibre of $f$;*

*(3) $f$ has relative Picard number 1.*


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The following important theorem, which is called the cone theorem, plays a key role in the minimal model program:

**Theorem 2.2.7.** *(Cone theorem, [KM98, Theorem 3.7]) Let $(X, \Delta)$ be a klt pair. Then*

*(1) There are at most countably infinitely many rational curves $\{C_i\}$ such that $-2\dim X < (K_X + \Delta).C_i < 0$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}C_i.$$

*(2) For every ample divisor $H$ and $\varepsilon > 0$, we have*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+\varepsilon H\geq 0} + \sum_{\text{finite}} \mathbb{R}C_i.$$

*(3) Let $R \subset \overline{NE}(X)$ be a $K_X + \Delta$-negative extremal face, then there exists a unique surjective morphism $c_R : X \to Y$ between projective varieties such that for every integral curve $C \subset X$, $c_R(C)$ is a point if and only if $[C] \in R$. The map $c_R$ is called the contraction of $R$.*

*(4) If $D$ is a Cartier divisor such that $D.C = 0$ for all $[C] \in R$, then there exists a Cartier divisor $D_Y$ on $Y$ such that $c_R^*(D_Y) \sim D$.*

Note that $c_R$ has connected fibre: if we take the Stein factorization $c'_R : X \to Z$ of $c_R$, then $c'_R$ also satisfies the condition that a curve is contracted by $c'_R$ if and only if the intersection number is zero, hence $c_R = c'_R$ by the uniqueness.

According to the cone theorem, for a normal projective variety with mild singularities $X$, either $K_X$ is nef, or there exists a contraction $X \to Y$ to a "smaller" variety $Y$. Naively speaking, the idea of the Minimal Model Program is that when $K_X$ is not nef, then we apply the Cone Theorem on $X$ to find a contraction $X \to Y$ to the smaller variety $Y$, and we hope either $Y$ is minimal, or the contraction is a fibre type, or we apply the Cone Theorem on $Y$ again to find a furthermore smaller variety. Before we explain the details, we give the following definitions:


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**Definition 2.2.8.** *Let $c_R : X \to Y$ be an extremal contraction as in the cone theorem.*

*(1) We say $c_R$ is of fibre type if* $\dim Y < \dim X$*;*

*(2) We say $c_R$ is a divisorial contraction if $c_R$ is a birational morphism with* $\mathrm{Exc}(c_R)$ *is of codimension 1;*

*(3) We say $c_R$ is a small contraction if $c_R$ is a birational morphism with* $\mathrm{Exc}(c_R)$ *is of codimension* $\geq 2$*;*

Note that by Kleinmen's Criterion, $-K_X$ is $c_R$-ample. Hence if $c_R$ is of fibre type, then it is a Mori fibre space. Also, by [KM98, Proposition 3.36 and Corollary 3.43], if either $\dim Y < \dim X$ or $c_R$ is a divisorial contraction, then $Y$ is $\mathbb{Q}$-factorial with $\rho(Y) = \rho(X) - 1$. Moreover, $Y$ is terminal if $c_R$ is a divisorial contraction and hence the Cone Theorem still holds on $Y$. However, by [KM98, Paragraph 2.6, Case 3], if $c_R$ is a small contraction, then $K_Y$ is never $\mathbb{Q}$-Gorenstein and then the Cone Theorem fails on $Y$. Therefore, we have to find another $\mathbb{Q}$-Gorenstein variety $X'$ to replace $Y$:

**Definition 2.2.9.** *(cf. [KM98, Definition 2.8]) Let $(X, \Delta)$ be a log pair, and $c_R : X \to Y$ be a $(K_X + \Delta)$-negative small contraction. A $(K_X + \Delta)$-flip is a $\mathbb{Q}$-Gorenstein variety $X'$ together with a proper birational morphism $c'_R : X' \to Y$ such that $(K_{X'} + \Delta')$ is $c'_R$-ample and* $\mathrm{Exc}(c'_R)$ *is of codimension at least 2, where $\Delta'$ is the proper transform of $\Delta$ on $X'$.*

By [KM98, Proposition 3.37], if the flip exists, then $X'$ is a terminal $\mathbb{Q}$-factorial variety with $\rho(X') = \rho(X)$. Therefore, either $X'$ itself is a minimal model, or we can apply the Cone Theorem on $X'$.

Now, we can introduce the details of the Minimal Model Program. The process of the Minimal Model Program is as follows: Let $X$ be a normal projective terminal $\mathbb{Q}$-factorial variety.

(1) If $K_X$ is nef, then stop;


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(2) If $K_X$ is not nef, then let $R$ be an extremal ray of $\overline{NE}(X)$ such that $K_X.R < 0$, and let $c_R : X \to X'$ be the contraction of $R$ as in the cone theorem;

(3) If $c_R$ is of fibre type, then stop;

(4) If $c_R$ is a divisorial contraction, then replace $X$ by $X'$ and return to (1);

(5) If $c_R$ is a small contraction, then let $X \dashrightarrow X^+$ be a flip, replace $X$ by $X^+$, and return to (1).

The Minimal Model Conjecture states that when we stare from a normal projective terminal $\mathbb{Q}$-factorial variety $X$, we can keep going with the above progress, and after finitely many steps it will end up with either a minimal model or a Mori fibre space.

The two key problems that need to be solved in this program are the existence and termination of flips: If we know the flips exist, then we can keep going with the progress. Moreover, assuming the termination of flips, then after at most finitely many flips, there must appear either a Mori fibre space or a divisorial contraction. In the first case, the progress is completed. In the second case, since flips will not change the Picard number, and a divisorial contraction will let the Picard number be strictly decreased, there can be at most $\rho(X)-1$ divisorial contraction in the progress of Minima Model Program. Therefore, this progress must end in finitely many steps if we know the termination of flips. At this moment, by [KM98], [Sho03], and [Fuj04], the existence and the termination of flips are known up to dimension 4. On the other hand, by [BCHM10], if $X$ is of general type, then the existence of flips is known for any dimension.

*Remark* 2.2.10. Since the cone theorem holds for klt pairs, by [KM98, Corollary 3.42 and 3.43], we also expect the MMP works for klt pairs $(X, \Delta)$. As in the standard MMP, the main problem is the existence and termination of klt flips. By [Kaw92b], [Sho03], and [BCHM10], the existence of klt flips is known if either $\dim X \leq 4$ or $\kappa(X, K_X + \Delta) = \dim X$, and the termination of klt flips is known if either $\dim X \leq 3$ or $\dim X = 4$ and $(X, \Delta)$ is canonical.


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## 2.3 Asymptotic base locus

In this section, we will introduce the basic properties of asymptotic invariants of $\mathbb{Q}$-divisor and the relation between these invariants and the positivity of $\mathbb{Q}$-Cartier divisors. Further details of this topic can be found in [BBP13], [BKK+15], and [ELM+06].

**Definition 2.3.1.** *Let $D$ be a Cartier divisor on a normal variety, the stable base locus of $D$, denoted by $\boldsymbol{B}(D)$, is defined as follows:*

$$\boldsymbol{B}(D) := \cap_{m\in\mathbb{Z}_{>0}}\boldsymbol{Bs}(mD).$$

*If $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, and $l$ is a natural number such that $lD$ is a Cartier divisor, then we define $\boldsymbol{B}(D)$ to be the stable base locus of $lD$.*

*Moreover, fix an ample $\mathbb{Q}$-divisor $A$, we can define the following sets*

$$\boldsymbol{B}_+(D) := \cap_{\varepsilon>0}\boldsymbol{B}(D-\varepsilon A);$$
$$\boldsymbol{B}_-(D) := \cup_{\varepsilon>0}\boldsymbol{B}(D+\varepsilon A);$$

*We call $\boldsymbol{B}_+(D)$ the augmented base locus of $D$, and $\boldsymbol{B}_-(D)$ the restricted base locus (or diminished base locus) of $D$.*

From the definition, we have

$$\mathbf{B}_-(D) \subset \mathbf{B}(D) \subset \mathbf{B}_+(D).$$

The reason is for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D'$, the inequality $\mathbf{B}(D+D') \subset \mathbf{B}(D)\cup\mathbf{B}(D')$ always holds, which implies $\mathbf{B}(D+\varepsilon A) \subset \mathbf{B}(D)$ and $\mathbf{B}(D-\varepsilon A) \supset \mathbf{B}(D)$ if $A$ is (semi)ample.

**Lemma 2.3.2.** *For a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$, both $\boldsymbol{B}_+(D)$ and $\boldsymbol{B}_-(D)$ are independent of the choice of $A$.*

*Proof.* Let $A, A'$ be 2 ample $\mathbb{Q}$-Cartier divisors. Let $0 < \delta \ll 1$ be a rational number such


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that $A - \delta A'$ is ample. Then for any $\varepsilon > 0$, we have

$$\mathbf{B}(D - \varepsilon\delta A') \subset \mathbf{B}(D - \varepsilon\delta A' - \varepsilon(A - \delta A')) = \mathbf{B}(D - \varepsilon A).$$

This implies

$$\cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon\delta A') \subset \cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon A).$$

Since it is obviously that

$$\cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon A) = \cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon\delta A),$$

we have

$$\cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon A') \subset \cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon A).$$

By choose $\delta'$ such that $A' - \delta A$ is ample, and do the same computation again, we can get the converse inclusion. Thus,

$$\cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon A') = \cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon A)$$

and hence $\mathbf{B}_+(D)$ is independent to the choice of $A$.

For $\mathbf{B}_-(D)$, observe that we have

$$\mathbf{B}(D + \varepsilon\delta A') \supset \mathbf{B}(D + \varepsilon\delta A' + \varepsilon(A - \delta A')) = \mathbf{B}(D + \varepsilon A).$$

Thus, we have

$$\cup_{\varepsilon>0}\mathbf{B}(D + \varepsilon A') \supset \cup_{\varepsilon>0}\mathbf{B}(D + \varepsilon A)$$

and hence by the same computation of above, we can conclude $\mathbf{B}_-(D)$ is also independent to the choice of $A$. □

*Remark* 2.3.3. By the definition, both $\mathbf{B}_+(D)$ and $\mathbf{B}(D)$ are Zariski closed subsets of $X$. But by [Les14], $\mathbf{B}_-(D)$ can be a *countably infinite union* of Zariski closed subsets.

**Lemma 2.3.4.** *Both $\boldsymbol{B}_+(D)$ and $\boldsymbol{B}_-(D)$ are numerical invariant, that is, if $D \equiv D'$ for*


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*another $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D'$, then $\boldsymbol{B}_{\pm}(D) = \boldsymbol{B}_{\pm}(D')$.*

*Proof.* Let $D' \equiv D$, then we can write $D' = D + L$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L \equiv 0$. Note that $L + \varepsilon A$ is ample for any ample divisor $A$ and positive rational number $\varepsilon$. Therefore, for all $\varepsilon > 0$, we have

$$\mathbf{B}(D' + \varepsilon A) = \mathbf{B}(D + (L + \varepsilon A)) \subset \cup_{\delta>0}\mathbf{B}(D + \delta(L + \varepsilon A)) = \mathbf{B}_{-}(D),$$

Which implies

$$\mathbf{B}_{-}(D') = \cup_{\varepsilon>0}\mathbf{B}(D' + \varepsilon A) \subset \mathbf{B}_{-}(D).$$

Since the converse inclusion can be proved in the same way, we conclude that $\mathbf{B}_{-}(D') = \mathbf{B}_{-}(D)$. Similarly, for $\mathbf{B}_{+}(-)$, we have

$$\mathbf{B}(D - \varepsilon A) = \mathbf{B}(D' - (L + \varepsilon A)) \supset \cap_{\delta>0}\mathbf{B}(D' - \delta(L + \varepsilon A)) = \mathbf{B}_{+}(D'),$$

which implies $\mathbf{B}_{+}(D) = \cap_{\varepsilon>0}\mathbf{B}(D - \varepsilon A) \supset \mathbf{B}_{+}(D')$, and the converse inclusion can be proved by the same way. □

*Remark* 2.3.5. Unlike $\mathbf{B}_{+}(D)$ and $\mathbf{B}_{-}(D)$, the stable base locus $\mathbf{B}(D)$ is *not* a numerical invariant. For example, Consider a smooth projective curve $X$ with $g(X) \geq 1$, there exists a divisor $D$ such that $\deg D = 0$ and $D$ is not a torsion divisor. Then we have $D \equiv 0$, but $\mathbf{B}(D) = X \neq \emptyset = \mathbf{B}(0)$.

In the last of this section, we will introduce the following beautiful correspondence between the positivities of $\mathbb{Q}$-Cartier divisors and the asymptotic base locus.

**Proposition 2.3.6.** *(cf. [BKK+15, section 4]) Let $D$ be a $\mathbb{Q}$-Cartier divisor on a normal projective variety $X$, then:*

*(1) $D$ is $\mathbb{Q}$-effective $\Leftrightarrow \boldsymbol{B}(D) \neq X$;*

*(2) $D$ is semiample $\Leftrightarrow \boldsymbol{B}(D) = \emptyset$;*

*(3) $D$ is pseudo-effective $\Leftrightarrow \boldsymbol{B}_{-}(D) \neq X$;*


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*(4) $D$ is nef $\Leftrightarrow$ $\boldsymbol{B}_-(D) = \emptyset$;*

*(5) $D$ is big $\Leftrightarrow$ $\boldsymbol{B}_+(D) \neq X$;*

*(6) $D$ is ample $\Leftrightarrow$ $\boldsymbol{B}_+(D) = \emptyset$.*



*Proof.* (1) and (2) are just by the definition of $\mathbb{Q}$-effective divisors and semiample divisors. For (3), note that $\mathrm{Big}(X)$ is big and $\mathrm{Pse}(X)$ is the closure of $\mathrm{Big}(X)$, so $D \in \mathrm{Pse}(X)$ if and only if $D + \varepsilon H$ is big for every big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H$ and positive rational number $\varepsilon$. In particular, $D \in \mathrm{Pse}(X)$ implies $D + \varepsilon A$ is effective for every ample divisor $A$ and $\varepsilon > 0$, hence $\mathbf{B}_-(D) \neq X$ because it is countably union of proper closed subset of $X$. Conversely, if $D$ is not pseudo-effective, then since $\mathrm{Pse}(X)$ is closed, for any ample divisor $A$, there is a sufficuently small positive rational number $\delta$ such that for all $0 < \varepsilon < \delta$, $D + \varepsilon A$ is not pseudo-effective (hence not $\mathbb{Q}$-effective), and hence $\mathbf{B}_-(D) = X$.

For (4), note that the nef cone is the closure of the ample cone. Thus, if $D$ is nef, then $D + \varepsilon A$ is ample for every ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H$ and positive rational number $\varepsilon$. Since ample divisor is semiample, this implies for every $\varepsilon > 0$, $\mathbf{B}(D + \varepsilon A) = \emptyset$, hence $\mathbf{B}_-(D) = \emptyset$. Conversely, if $D$ is not nef, then for any ample divisor $A$, there is a sufficiently small positive rational number $\delta$ such that for all $0 < \varepsilon < \delta$, $D + \varepsilon A$ is not nef (in particular, not semiample). Hence $D + \varepsilon A$ has non-empty stable base locus, which implies $\mathbf{B}_-(D) \neq \emptyset$.

For (5), if $D$ is big, then there is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A$ such that $D - A$ is an effective $\mathbb{Q}$-divisor. In particular, $\mathbf{B}_+(D) \subset \mathbf{B}(D - A)$ is a proper subset of $X$. Conversely, if $D$ is not big, then for every ample $\mathbb{Q}$-divisor $A$, $D - A$ is not $\mathbb{Q}$-effective. In particular, since for every positive rational number $\varepsilon$, $\varepsilon A$ is still an ample $\mathbb{Q}$-divisor, which implies $\mathbf{B}(D - \varepsilon A) = X$, and hence $\mathbf{B}_+(D) = X$.

Finally, for (6), if $D$ is ample, then since the ample cone is open, for any ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A$, there is a sufficiently small positive rational number $\delta$ such that for all $0 < \varepsilon < \delta$, $D - \varepsilon A$ is ample (hence semiample). In particular, $\mathbf{B}_+(D) \subset \mathbf{B}(D - \varepsilon A) = \emptyset$. Conversely, if $D$ is not ample, then for every ample $\mathbb{Q}$-divisor $A$ and positive rational number $\varepsilon$, $D - \varepsilon A$ is not nef (otherwise, $D = (D - \varepsilon A) + \varepsilon A$ is the sum of a nef divisor


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and an ample divisor, hence ample because the ample cone is the interior of the nef cone). In particular, $D - \varepsilon A$ is not semiample, hence $\mathbf{B}(D - \varepsilon A) \neq \emptyset$. Since $\{\mathbf{B}(D - \varepsilon A)\}_{\varepsilon}$ is decreasing as $\varepsilon \to 0$, by the noetherian induction, there is a positive rational number $\varepsilon$ such that $\mathbf{B}(D - \varepsilon A) = \mathbf{B}_{+}(D)$, hence $\mathbf{B}_{+}(D) \neq \emptyset$. □

## 2.4 The nef reduction map and nef dimension

In this subsection, we will introduce the notion of nef reduction and nef dimension, which is proved by Tsuji and Bauer et. al. in [Tsu00] and [BCE+00].

**Theorem 2.4.1.** *Let $X$ be a normal projective variety, and $L$ be a $\mathbb{Q}$-Cartier divisor on $X$. Then there exists an almost holomorphic dominant rational map $f : X \to Y$ with connected fibres, called the nef reduction of L, such that:*

*(1) $L$ is numerically trivial on all compact fibres $F$ of $f$ with $\dim F = \dim X - \dim Y$.*

*(2) For every very general point $x \in X$ and every irreducible curve $C$ on $X$ passing through $x$ and not contracted by $f$, we have $L.C > 0$. In other words, there exists a subset $Z \subset X$, which is a union of at most countably infinitely many proper closed subsets of $X$, such that if $C$ is a curve on $X$ with $L.C = 0$, then either $f(C)$ is a point or $C \subset Z$.*

*The map $f$ is unique up to birational equivalence of $Y$. The nef dimension of L, denoted by $n(X, L)$ (or $n(L)$ if there is no ambiguity), is defined by $n(L) := \dim Y$.*

Here we recall some important properties of nef dimension.

**Proposition 2.4.2.** *Let $X$ be a normal projective variety and $L$ be a nef $\mathbb{Q}$-Cartier divisor on $X$. Then:*

*(1) (cf. [BCE+00, Proposition 2.8]) $\kappa(L) \leq \nu(L) \leq n(L)$;*

*(2) (cf. [BP04, Proposition 2.1]) Let $f : Y \to X$ be a surjective morphism between normal projective varieties, then we have $n(f^*L) = n(L)$.*


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**Lemma 2.4.3.** *In the second case of the above proposition, let $\pi : X \dashrightarrow Z$ be a nef reduction map of $L$, then $g := \pi \circ f : Y \dashrightarrow Z$ is a nef reduction map of $f^*L$.*

*Proof.* Let $y \in Y$ be a very general point of $Y$, we need to show for an integral curve $C \subset Y$, $g(C)$ is a point if and only if $(f^*L).C = 0$. Suppose $g(C)$ is a point, then either $f(C)$ is a point, or $f(C)$ is a curve such that $\pi(f(C))$ is a point. In the first case, it is trivially that $(f^*D).C = 0$ for every $\mathbb{Q}$-Cartier divisor $D$ on $X$. In the second case, since $x := f(y)$ is a very general point, by the definition of nef reduction, $L.f(C) = 0$, hence $(f^*L).C = \deg(C \to f(C))(L.f(C)) = 0$. Conversely, if $\pi \circ f(C)$ is not a point, then $f(C)$ is a curve that is not contracted by $\pi$. Since $f(C)$ passing through the very general point $x := f(y)$, by the definition of nef reduction, we have $L.f(C) \neq 0$, hence $(f^*L).C = \deg(C \to f(C))(L.f(C)) \neq 0$, this completes our proof. □

Note that by [Amb04, Remark 2.3], if $L$ is nef and abundant, then $\kappa(L) = \nu(L) = n(L)$. Also, if $L$ is semiample, then the nef reduction map is just the semiample fibration of $L$, and hence $L$ is abundant. In general, it is difficult to deal with the nef divisor which is not abundant. However, for smooth projective surfaces, Ambro gives the following classification of nef and divisors with maximal nef dimension:

**Theorem 2.4.4.** *(cf. [Amb04, Theorem 0.3]) Let $X$ be a smooth projective surface, and $L$ be a nef divisor with $n(L) = 2$, then one of the following holds:*

*(1) $D$ itself is big ;*

*(2) $D$ is not big, but $K_X + tD$ is big for all $t > 2$;*

*(3) there is a birational morphism $\mu : X \to Y$ between smooth projective surfaces such that $sD \equiv \mu^*(-K_Y)$ for some positive rational number $0 < s \leq 2$, and $D$ is algebraically equivalent to an effective $\mathbb{Q}$-divisor $D_e$.*

At the end of this subsection, we prove the following properties about nef divisors.

**Lemma 2.4.5.** *Let $X$ be a normal projective variety, and $D$ be a nef and $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor. Suppose that $n(X, D) = \nu(X, D)$, then we have $n(D) = \nu(D) = \kappa(D)$, that is, $D$ is nef and good.*


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*Proof.* The proof almost directly follows from the argument of [BP04, Theorem 2.1] with a little modification. Let $f : X \dashrightarrow T$ be a nef reduction map of $D$, up to change a birational model of $T$ we may assume $T$ is smooth. Consider the graph closure $G_0 \subset X \times T$, $p_0 : G_0 \to X$ be the first projection, and $q_0 : G_0 \to T$ be the second projection. Then $p_0(q_0^{-1}(t))$ is a compact fiber $F_t$ of $f$ for general $t \in T$, and $p_0 : q_0^{-1}(t) \to F_t$ is an isomorphism. Since the nef reduction is almost holomorphic, let $G$ be a normalization of $G_0$, and $p : G \to X$, $q : G \to T$ be the maps induced by the normalization, we have $p$ is an isomorphism over general $F_t$ and it is still a general fibre of $q$.

Now, since $D$ is nef and $\mathbb{Q}$-effective, so does $p^*D$. Moreover, since by the definition of nef reduction map, $D|_{F_t}$ is numerically trivial, thus $D|_{F_t}$ is $\mathbb{Q}$-trivial since we assume $D$ is $\mathbb{Q}$-effective, and since $p$ is an isomorphism near the general $F_t$, we conclude $p^*D|_{F_t}$ is $\mathbb{Q}$-trivial. Replace $D$ by $mD$ for $m \gg 0$ we may assume $D$ is nef, effective, and not dominate $T$. Therefore, by applying [Nak04, Corollary III,5,9], let $\nu : T' \to T$ be a flattening of $q$, $G'$ be a resolution of singularities of the main component of $G \times_T T'$, and $\pi : G' \to G$, $q' : G' \to T'$ be the induced maps, then there exists an effective $\mathbb{Q}$-divisor $D'$ on $T'$ such that $q'^*(D') = \pi^*p^*D$.

Now, since we assumed that $k := \nu(X, D) = n(X, D)$, we have $D^k$ is not numerically trivial, this implies $(D')^k > 0$. So we have $D'$ is nef and big and hence $n(X, D) = \kappa(T', D') = \kappa(X, D)$. □

**Lemma 2.4.6.** *Let $(X, \Delta)$ be a normal projective variety, and $D = \pm(K_X + \Delta)$ is a nef with $n(X, D) = \nu(X, D)$. If either* $\dim X - n(D) \leq 3$ *or $X$ is smooth with $\Delta = 0$, then $D$ is $\mathbb{Q}$-efective with $n(D) = \nu(D) = \kappa(D)$, that is, $D$ is nef and good. In particular, $D$ is semiample by [Fuj11]*

*Proof.* Note that we do not assume $D$ is $\mathbb{Q}$-effective in our assumption. Now, repeat the proof above, we only have $D|_{F_t}$ is numerically trivial at first. In the case $\dim X - n(D) \leq 3$, since the abundance theorem holds for dimension $\leq 3$, we have $D|_{F_t}$ is $\mathbb{Q}$-trivial. In the case $X$ is smooth and $\Delta = 0$, we have $D$ is $\mathbb{Q}$-trivial by [CP11]. So we have the horizontal part of $D$ is $\mathbb{Q}$-effective (because $D|_{F_t}$ is $\mathbb{Q}$-trivial implies $f_*\mathcal{O}_X(mD)$ have non-zero stalk at $t$, thus $f_*\mathcal{O}_X(mD)$ is non-zero sheaf and hence the horizontal part of


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$mD$ is effective). Thus, by replacing $D$ with $mD$ for $m \gg 0$ we may assume $D$ is vertical. Let $H$ be a nef and effective divisor on $T$ such that $D + q^*H$ is effective. Then by [Nak04, Corollary III,5,9] again, let $\nu : T' \to T$ be a flattening of $q$, $G'$ be a resolution of singularities of the main component of $G \times_T T'$, and $\pi : G' \to G$, $q' : G' \to T'$ be the induced maps, then there exists an effective $\mathbb{Q}$-divisor $D'$ on $T'$ such that $q'^*(D') = \pi^*(p^*D + q^*H) = \pi^*p^*D + q'^*\nu^*H$, which implies $\pi^*p^*D = q'^*(D' - \nu^*H)$ and hence $D - \nu^*H$ is nef. Again, since $k := \nu(X, D) = n(X, D)$, we have $D^k$ is not numerically trivial, this implies $(D' - \nu^*H)^k > 0$. So we have $D' - \nu^*H$ is nef and big and hence $n(X, D) = \kappa(T', D' - \nu^*H) = \kappa(X, D)$. $\square$



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# Chapter 3 The geometry of algebraic fibre spaces

In this section, we will introduce some important properties of algebraic fibre spaces and give some important special classes of algebraic fibre spaces.

## 3.1 Basic properties and the Weakly Positive Theorem

First, we introduce the following two basic properties for algebraic fibre spaces.

**Theorem 3.1.1.** *(cf. [Laz04, Lemma 2.1.13]) Let $f : X \to Y$ be an algebraic fibre space, and $D$ be an Cartier divisor on $Y$, then for every positive integer $m$, we have*

$$H^0(X, mf^*D) = H^0(Y, mD).$$

*In particular, $\kappa(D) = \kappa(f^*D)$.*

**Lemma 3.1.2.** *Let $f : X \to Y$ be an algebraic fibre space between normal varieties. Let $E$ be an effective $f$-exceptional $\mathbb{Q}$-divisor on $X$, and $D$ be a $\mathbb{Q}$-divisor on $Y$. Suppose $f^*D + nE$ is $\mathbb{Q}$-effective for some $n \in \mathbb{N}$, then $D$ is $\mathbb{Q}$-effective.*

*Proof.* Let $m \in \mathbb{N}$ sufficiently divisible such that $mD$ and $mE$ has integer coefficients, and $m(f^*D + nE)$ is effective. Since $f$ is an algebraic fibre space and $f(\mathrm{Supp}E)$ has codimension at least 2 in $Y$, letting $Y_0 := Y - f(\mathrm{Supp}E)$, $X_0 := X - f^{-1}f(\mathrm{Supp}E)$, we


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have the following natural isomorphisms:

$$H^0(X_0, f^*(mD)|_{X_0}) \cong H^0(Y_0, (mD)|_{Y_0}) \cong H^0(Y, mD) \cong H^0(X, f^*(mD)).$$

In particular, if $(f^*(mD))|_{X_0}$ is effective, then so does $mD$. Thus, if $m(f^*D + nE)$ is effective, then $(f^*(mD))|_{X_0}$, and hence $mD$ itself is effective. □

In the following of this subsection, we will give a brief review of the definition and basic properties of weakly positive sheaves. We adopt the convention and results in [EG19, Section 2.2] which will be necessary. More details of weakly positive sheaves can be found in [Cam04, Section 4.2], [EG19, Section 2.2], and [Vie83, Section 1].

**Definition 3.1.3.** *Let $X$ be a normal quasi-projective variety, $\mathcal{G}$ be a coherent sheaf on $X$, and $A$ be a fixed ample divisor on $X$.*

*(1) We say that $\mathcal{G}$ is generically globally generated if the natural map $H^0(X, \mathcal{G}) \otimes \mathcal{O}_X \to \mathcal{G}$ is surjective over the generic point of $X$.*

*(2) We say that $\mathcal{G}$ is weakly positive if for any natural number $n$, there is a natural number $m$ such that the sheaf $(S^{nm}(\mathcal{G}))^{**} \otimes \mathcal{O}_X(mA)$ is generically globally generated, where $S^n(-)$ denote the $n$-th symmetric power, and $(-)^{**}$ denotes the double dual.*

Note that the definition is independent of the choice of $A$. Also, by [Vie83, Remark 1.3.iv], if $\mathcal{G}|_U$ is weakly positive for some open dense subset $U \subset X$ such that $X - U$ has codimension at least 2, then $\mathcal{G}$ is weakly positive on $X$. The following result for weakly positivity of divisorial sheaves is basic but important:

**Proposition 3.1.4.** *A divisorial sheaf $\mathcal{O}_X(D)$ on a normal quasi-projective variety $X$ is weakly positive if and only if for a fixed ample divisor $A$ and any positive integer $n$, $nD + A$ is $\mathbb{Q}$-effective. In particular, a line bundle $\mathcal{O}_X(D)$ on a normal projective variety $X$ is weakly positive if and only if $D$ is pseudo-effective.*

*Proof.* Since one can check the weakly positivity of a sheaf on an open subset $U$ with $X - U$ has codimension at least 2, by replacing $X$ with the smooth locus of $X$, we may


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assume $X$ is smooth. In this case, since $\mathcal{O}_X(D)$ is a line bundle, we have

$$(S^{nm}(\mathcal{O}_X(D)))^{**} \otimes \mathcal{O}_X(mA) = \mathcal{O}_X(nmD + mA).$$



Thus, from the definition, $\mathcal{O}_X(D)$ is weakly positive if and only if for any positive integer $n$, there is a positive integer $m$ such that $\mathcal{O}_X(nmD + mA)$ is generically globally generated. Since $\mathcal{O}_X(nmD + mA)$ is of rank 1, the map $H^0(X, \mathcal{O}_X(nmD + mA)) \otimes \mathcal{O}_X \to \mathcal{O}_X(nmD + mA)$ is either surjective over the generic point or identically zero. In particular, the map is surjective over the generic point if and only if $H^0(X, nmD + mA) \neq 0$. Therefore, $\mathcal{O}_X(D)$ is weakly positive if and only if for any positive integer $n$, there is a positive integer $m$ such that $nmD + mA$ is linear equivalent to an effective divisor, that is, $nD + A$ is $\mathbb{Q}$-effective, which completes our proof. □

Next, we will introduce the Weakly Positive Theorem. This theorem is one of the most important tools to study the relation of the positivity of (anti-)canonical divisors in algebraic fibre spaces.

**Theorem 3.1.5.** *(cf. [Cam04, Theorem 4.13], [Fuj17, Theorem 1.1]) Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties with general fibre $F$. Let $(X, \Delta)$ be a log pair such that $(F, \Delta_F)$ is log canonical, where $\Delta_F$ is defined by $(K_X + \Delta)|_F = K_F + \Delta_F$. Then for sufficiently divisible positive integer $m$, the sheaf $f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$ is weakly positive.*

*Proof.* When $Y$ is smooth, this theorem is just [Fuj17, Theorem 1.1]. In general, let $\mu : Y' \to Y$ be a resolution of singularities of $Y$, $X'$ be the normalization of the main component of $X \times_Y Y'$, and $\pi : X' \to X$, $f' : X' \to Y'$ be the induced morphisms.

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y. \end{array}$$

Note that $F$ is still a general fibre of $f'$. Let $\Delta'$ be the proper transform of $\Delta$ on $X'$, then we can write

$$K'_X + \Delta' + E^- = \pi^*(K_X + \Delta) + E^+$$


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for some effective $\pi$-exceptional $\mathbb{Q}$-divisors $E^+, E^-$. From our construction, we can see that $f'(\mathrm{Supp}E^{\pm}) \subset \mathrm{Exc}(\mu)$. This implies $\mathrm{Supp}E^{\pm} \cap F = \emptyset$ for general fibre $F$, and hence the pair $(F, (\Delta' + E^-)|_F) \cong (F, \Delta_F)$ is log canonical. Now, since the theorem holds if the base is smooth, we have the sheaf $f'_*\mathcal{O}_{X'}(m(K_{X'/Y'} + \Delta' + E^-))$ is weakly positive for sufficiently divisible positive integer $m$. Let $U' := Y' - \mathrm{Exc}(\mu)$, and $U := Y - \mu(\mathrm{Exc}(\mu)) \cong U'$. Then we have the sheff $(f'_*\mathcal{O}_{X'}(m(K_{X'/Y'} + \Delta' + E^-)))|_{U'}$ is also weakly positive. Hence

$$\begin{aligned}
(f'_*\mathcal{O}_{X'}(m(K_{X'/Y'} + \Delta' + E^-)))|_{U'} &\cong (\mu_* f'_*\mathcal{O}_{X'}(m(K_{X'/Y'} + \Delta' + E^-)))|_U \\
&\cong (f_*\pi_*\mathcal{O}_{X'}(m(K_{X'} - f'^*K_{Y'} + \Delta' + E^-)))|_U \\
&\cong (f_*\pi_*\mathcal{O}_{X'}(m(K_{X'} - \pi^*f^*K_Y + \Delta' + E^-)))|_U \\
&\cong (f_*\pi_*\mathcal{O}_{X'}(m(\pi^*(K_X + \Delta - f^*K_Y) + E^-)))|_U \\
&\cong (f_*\mathcal{O}_X(m(K_X + \Delta - f^*K_Y)))|_U
\end{aligned}$$

is also weakly positive, where the third isomorphism holds since $K_{Y'}|_{U'} = K_Y|_U$ if we identify $\mu|_{U'} : U' \to U$ be the identity map, the fourth isomorphism is because $f'^{-1}(U') \cong f^{-1}(U)$, and the last isomorphism is because $E^-|_{\pi^{-1}(f^{-1}(U))} = 0$. Since $Y - U$ has codimension at least 2 in $Y$, this implies $f_*\mathcal{O}_X(m(K_X + \Delta - f^*K_Y)) = f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$ is weakly positive, which completes our proof. $\square$

*Remark* 3.1.6. In general, the weakly positive theorem does not imply $K_{X/Y}$ is pseudo-effective. For example, let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $Y = \mathbb{P}^1$, $f$ be the first projection, and $g$ be the second projection. Then $K_X - f^*K_Y = -2g^*P$, which is not pseudo-effective. Here, $P$ is a point on the second factor. Note that in this case, $f_*\mathcal{O}_X(K_{X/Y})$ is the zero sheaf, which is weakly positive by definition. Thus, this example does not contradict the weakly positivity theorem.

At the end of this subsection, we recall two useful lemmas about the weakly positive sheaves in [EG19].

**Lemma 3.1.7.** *([EG19, Lemma 2.4]) Let $f : Y' \to Y$ be a surjective projective morphism between geometrically normal quasi-projective varieties over a field, let $\mathcal{G}$ be a torsion-*


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*free coherent sheaf on $Y$:*

*(1) If there is no $f$-exceptional divisor on $Y'$, and $\mathcal{G}$ is weakly positive, then $f^*\mathcal{G}$ is also weakly positive. Here a prime divisor $E$ on $Y'$ is called $f$-exceptional if $f(E)$ has codimension at least 2 in $Y$.*

*(2) If $f^*\mathcal{G} \otimes \mathcal{O}_{Y'}(E)$ is weakly positive for some effective $f$-exceptional divisor $E$ on $X$, then $\mathcal{G}$ is weakly positive.*

**Lemma 3.1.8.** *([EG19, Lemma 2.5])Let $\mathcal{F} \to \mathcal{G}$ be a generically surjective morphism between coherent sheaves on a normal quasi-projective variety, if $\mathcal{F}$ is weakly positive, so is $\mathcal{G}$.*

## 3.2 Canonical bundle formulas

In this section, we will introduce two canonical bundle formulas. One is Ambro's canonical bundle formula of klt-trivial fibration, and the other is the canonical bundle formula of Fujino-Mori. Ambro's formula is a very important tool for study varieties with positive anticanonical divisor, and Fujino-Mori's formula is deeply related to the study of varieties with positive canonical divisor.

### 3.2.1 The klt-trivial fibration and Ambro's Canonical bundle formula

At first, we will introduce the definition of **b**-divisors and the klt-trivial fibration.

**Definition 3.2.1.** *Let $D$ be a divisor on a normal variety $X$. A **b**-divisor $\mathbf{D}$ contains a family of divisors $\{\mathbf{D}_{X'}\}$, where $X'$ is taken over all higher birational models $\pi : X' \to X$ such that $\pi$ is a proper birational morphism, and $\mathbf{D}_{X'}$ is a divisor on $X'$ such that $\pi_*(\mathbf{D}_{X'}) = D$. If $D$ is a $\mathbb{Q}$-divisor, then the $\mathbb{Q}$-**b**-divisor is defined in the same way.*

**Definition 3.2.2.** *A klt-trivial fibration (which is equivalent to the lc-trivial fibration in the sense of [Amb05]) is an algebraic fibre space $f : (X, B) \to Y$ between normal varieties with a sub log pair $(X, B)$ such that*


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*(1)* $(X, B)$ *is sub-klt over the generic point of* $Y$*;*

*(2) rank* $f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = 1$*;*

*(3)* $K_X + B \sim_{\mathbb{Q}} f^*D$ *for some* $\mathbb{Q}$*-Cartier divisor* $D$ *on* $Y$*.*

*Where the discrepancy* $\mathbb{Q}$*-*$\mathbf{b}$*-divisor* $\mathbf{A}(X, B) = \{\mathbf{A}_{X'}\}$ *is defined by the formula*

$$K_{X'} = \pi^*(K_X + B) + \mathbf{A}_{X'}.$$

Next, we introduce the definition of the moduli $\mathbb{Q}$-$\mathbf{b}$-divisors and the discriminant $\mathbb{Q}$-$\mathbf{b}$-divisors.

**Definition 3.2.3.** *Let* $f : (X, B) \to Y$ *be a klt-trivial fibration, we define the discriminant* $\mathbb{Q}$*-divisor* $B_Y$ *of* $f : (X, B) \to Y$ *in the following way: Let* $P$ *be a prime divisor on* $Y$*, which is Cartier in a neighborhood of its generic point, then we define*

$$b_P := \max\{t \in \mathbb{Q} | (X, B + tf^*P) \text{ is sublc over the generic point of } P\},$$

*and set*

$$B_Y := \sum_P (1 - b_P)P,$$

*where* $P$ *runs over all prime divisor of* $Y$*. We set* $M_Y = D - K_Y - B_Y$ *and call* $M_Y$ *the moduli* $\mathbb{Q}$*-divisor of* $f : (X, B) \to Y$*.*

*The moduli* $\mathbb{Q}$*-*$\mathbf{b}$*-divisor* $\mathbf{M} = \{\mathbf{M}_{Y'}\}$ *and the discriminant* $\mathbb{Q}$*-*$\mathbf{b}$*-divisor* $\mathbf{B} = \{\mathbf{B}_{Y'}\}$ *is defined in the following way: For a proper birational morphism* $\mu : Y' \to Y$*, let* $X'$ *be a normalization of the main component of* $X \times_Y Y'$ *such that the induced morphism* $\pi : X' \to X$ *is proper and birational. Define* $B_{X'}$ *by*

$$K_{X'} + B_{X'} = \pi^*(K_X + B),$$

*then* $f' : (X', B_{X'}) \to Y'$ *is also a klt-trivial fibration. Thus, let* $M_{Y'}$ *and* $B_{Y'}$ *be the moduli* $\mathbb{Q}$*-divisor and discriminant* $\mathbb{Q}$*-divisor of* $f' : (X', B_{X'}) \to Y'$*, then we set* $\mathbf{M}_{Y'} =$


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$M_{Y'}$ *and* $\mathbf{B}_{Y'} = B_{Y'}$.

**Lemma 3.2.4.** *Let* $f : (X, B) \to Y$ *be a klt trivial fibration between normal varieties. By restricting* $f$ *on the preimage of the smooth locus of* $Y$*, the pullback of every Weil-divisors is well-defined. Write* $B = B^h + B^1 + (B^v)'$ *such that the image of every component of* $B^1$ *is a prime divisor, and* $(B^v)'$ *is exceptional, then* $f^*(B_Y) \geq B^1$*. In particular,*

*(1) If* $B^h$ *is effective, then the negative part of* $f^*(B_Y) - B^1$ *is* $f$*-exceptional;*

*(2) If* $B$ *is effective, then so does* $B_Y$*.*

*Proof.* Write $B^1 = \sum_{i,j} a_{ij} D_{ij}$ such that each $f(D_{ij}) = P_i$ for some prime divisor $P_i$ on $Y$. Then on the smooth locus of $Y$, for each $P_k$ we have

$$f^*(b_{P_k})P_k + \sum_k a_{kj} D_{kj} \leq \sum_j D_{kj},$$

which implies

$$f^*(1 - b_{P_k})P_k - \sum_k a_{kj} D_{kj} \geq f^*P_k - \sum_j D_{kj} \geq 0$$

since $f^*P_k \geq \sum_j D_{kj}$. Sum up over every irreducible component of $B_Y$, we conclude

$$B_Y = \sum f^*(1 - b_{P_i})P_i \geq \sum_{i,j} a_{ij} D_{ij} = B^1,$$

hence our conclusion follows. □

Now we can state the theorem of Ambro:

**Theorem 3.2.5.** *(cf. [Amb05, Theorem 3.3])Let* $f : (X, B) \to Y$ *be a klt trivial fibration. If* $B^h$ *is effective, then the moduli-***b***-divisor* $\mathbf{M}$ *is nef and abundant. In particular,* $M_Y$ *is effective.*

*Proof.* We only prove the effectiveness of $M_Y$ since the other parts directly follow the original theorem of Ambro ([Amb05, Theorem 3.3]). Since the moduli-**b**-divisor $\mathbf{M}$ is


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nef and abundant, there exists a birational morphism $\mu : Y' \to Y$ such that $Y'$ is smooth and $M_{Y'}$ is nef and abundant (hence $\mathbb{Q}$-effective). Note that in general $M_Y$ may not be $\mathbb{Q}$-Cartier, but it is $\mathbb{Q}$-Cartier over the smooth locus of $Y$. Denote the smooth locus of $Y$ by $Y_0$. Since $\mu_*(M_{Y'}) = M_Y$, over the open dense subset $\mu^{-1}(Y_0)$ of $Y'$ we have the equality $\mu^* M_Y = M_{Y'} + E^+ - E^-$ for $E^+, E^-$ be some effective $\mu$-exceptional $\mathbb{Q}$-divisors on $Y'$. Hence

$$\mu^* M_Y + E^- \geq M_{Y'}$$

is $\mathbb{Q}$-effective over $\mu^{-1}(Y_0)$. Thus, $\mu^* M_Y$ is $\mathbb{Q}$-linear equivalent to a divisor which is effective outside the exceptional set over $\mu^{-1}(Y_0)$. Therefore, there is a positive integer $m$ and a rational function $g \in K(Y) \cong K(Y_0) \cong K(Y')$ such that over the open set $\mu^{-1}(Y_0)$, we have

$$m\mu^* M_Y + \text{div}(g \circ \mu) = \mu^*(mM_Y + \text{div}(g)) \geq 0.$$

This implies on the open set $Y_0$, we have $mM_Y + \text{div}(g) \geq 0$. That is, $M_Y$ is $\mathbb{Q}$-effective over $Y_0$. Since $Y$ is normal, $Y - Y_0$ has codimension at least 2, which implies $M_Y$ is $\mathbb{Q}$-effective over $Y$. $\square$

### 3.2.2 The canonical bundle formula of Fujino-Mori

In this subsection, we will give a brief review of the canonical bundle formula derived by Fujino and Mori (cf.[FM00]).

**Theorem 3.2.6.** *(cf.[FM00, Theorem 4.5]) Let $(X, \Delta)$ be a klt pair, and $f : X \to Y$ be an algebraic fibre space with general fibre $F$ such that $Y$ is smooth and $\kappa(F, (K_X + \Delta)|_F) = 0$. Then for sufficiently high smooth projective birational models $\pi : X' \to X$, $\mu : Y' \to Y$ of $X$ and $Y$, we can consider the following commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\mu"] & Y. \end{tikzcd}$$

*Let $(X', \Delta')$ be a sub klt pair such that if we write $K_{X'} + D = \pi^*(K_X + \Delta)$, then $\Delta' - D$ is an effective $\pi$-exceptional $\mathbb{Q}$-divisor (such $\Delta'$ exists since $(X, \Delta)$ is klt). Then there exists*


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*$\mathbb{Q}$-divisors $B, L$ on $Y'$ and a $\mathbb{Q}$-divisor $R$ on $X'$ satisfying the following:*

*(1) $K_{X'} = f'^*(K_{Y'} + B + L) + R$;*

*(2) For any non-negative integer $n$, $f'_*\mathcal{O}_{X'}(\lfloor nR^+ \rfloor) \cong \mathcal{O}_{Y'}$;*

*(3) $R^-$ is both $\pi$-exceptional and $f'$-exceptional;*

*(4) $B$ is effective,* Supp$B$ *is contained in the branch locus of $f'$, and every coefficients of $B$ are less than 1. In particular, $(Y', B)$ is klt;*

*(5) $L$ is nef;*

*(6) For any non-negative integer $n$, $H^0(X', n(K_{X'} + \Delta')) = H^0(Y', n(K_{Y'} + B + L))$. In particular, $\kappa(X', K_{X'} + \Delta') = \kappa(Y', K_{Y'} + B + L)$;*

*Remark* 3.2.7. In the proof of [FM00], the explicit construction of $X'$ and $Y'$ is as follows: At first, let $\mu : Y' \to Y$ be the log resolution of Br$(f)$, the branch locus of $f$. Next, for the construction of $X'$, first we let $X_0$ be the normalization of the main component of $X \times_Y Y'$, $\pi_0 : X_0 \to X$ and $f_0 : X_0 \to Y'$ be the induced maps, and $X'$ is the log resolution of Sing$(X_0) \cup f'^{-1}(\text{Br}(f')) \cup \pi_0^{-1}(\text{Supp}\Delta)$.

*Remark* 3.2.8. In the above formula, assume $X$ is $\mathbb{Q}$-Gorenstein and has at worst canonical singularities. Then when $\Delta = 0$, we can also take $\Delta' = 0$.



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# Chapter 4 Positivity of anticanonical divisors in algebraic fibre spaces

The main purpose of this section is to study the positivity of the anti-canonical divisors in algebraic fibre spaces. The ingredients of this section have been published as the article [Cha23b].

The main theorem in this section is the following result, which can be thought of as an analog of the Iitaka conjecture of the anti-canonical Iitaka dimension.

**Theorem 4.0.1.** *[cf. Theorem 4.2.1] Let $f : X \to Y$ be an algebraic fibre space between normal projective $\mathbb{Q}$-Gorenstein varieties, and $F$ be a general fibre of $f$. Suppose $X$ has at worst klt singularities, and $-K_X$ is effective with stable base locus $\boldsymbol{B}(-K_X)$ which does not dominate $Y$. Then we have*

$$\kappa(X, -K_X) \leq \kappa(F, -K_F) + \kappa(Y, -K_Y).$$

Note that there are examples satisfying the strict inequality in Theorem 4.0.1:

*Example* 4.0.2. Let $X$ be an elliptic K3 surface, that is, $X$ is a K3 surface and there is a morphism $f : X \to \mathbb{P}^1$ such that general fibres of $f$ are smooth elliptic curves. Then we have

$$0 = \kappa(X, -K_X) < \kappa(F, -K_F) + \kappa(Y, -K_Y) = 1,$$


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which gives an example of strict inequality in Theorem 4.0.1.

A key ingredient of the proof of Theorem 4.0.1 is the following injectivity theorem, which is a variant of Ejiri-Gongyo's injectivity theorem (cf. [EG19, Theorem 4.2]):

**Theorem 4.0.3.** *Let $(X, \Delta)$ be a klt pair, and $f : X \to Y$ be an algebraic fibre space such that $Y$ is normal and $\mathbb{Q}$-Gorenstein. Let $D$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Y$ such that $\kappa(Y, -K_Y - D) = 0$. Suppose that $-(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective with the stable base locus $\boldsymbol{B}(-(K_X + \Delta) - f^*D)$ that does not surject to $Y$. Then for a general fibre $F$ of $f$, the following morphism between graded ring defined by restriction*

$$\bigoplus_{m \geq 0} H^0(X, \lfloor m(-(K_X + \Delta) - f^*D) \rfloor) \to \bigoplus_{m \geq 0} H^0(F, \lfloor -m(K_X + \Delta)|_F) \rfloor)$$

*is injective. In particular, we have $\kappa(X, -(K_X + \Delta) - f^*D) \leq \kappa(F, -(K_X + \Delta)|_F)$.*

The above injectivity theorem implies Theorem 4.0.1 holds under the assumption $\kappa(-K_Y) = 0$. On the other hand, the proof of the injectivity theorem also deeply relies on the following effectivity result of anti-canonical divisors.

**Theorem 4.0.4** (cf. Theorem 4.2.2)**.** *Let $(X, \Delta)$ be a klt pair and $f : X \to Y$ be an algebraic fibre space such that $Y$ is normal and $\mathbb{Q}$-Gorenstein. Let $D$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Y$. Suppose that $-(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective with the stable base locus $\boldsymbol{B}(-(K_X + \Delta) - f^*D)$ that does not surject to $Y$. Then $-K_Y - D$ is $\mathbb{Q}$-effective.*

The motivation of our investigation traces back to the following question asked by Demailly-Peternell-Schneider in [DPS01]:

**Question 4.0.5.** *Let $f : X \to Y$ be a surjective morphism between normal projective $\mathbb{Q}$-Gorenstein varieties. Suppose that $-K_X$ is pseudo-effective and the non-nef locus of $-K_X$ does not dominate $Y$. Is $-K_Y$ pseudo-effective?*

We recall some previous results about the above question:

- In [CZ13, Main Theorem], Chen and Zhang proved that if there is a lc pair $(X, \Delta)$ such that if $-(K_X + \Delta)$ is nef, then $-K_Y$ is pseudo-effective. Also, [CZ13, Exam-


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ple 1.5] shows that if the non-log canonical locus of $(X, \Delta)$ surjects onto $Y$, then $-(K_X + \Delta)$ being nef does not imply that $-K_Y$ is pseudo-effective.

- In [Den21, Theorem D], by using analytic methods, Deng proved that if there is a log canonical pair $(X, \Delta)$ such that $-(K_X+\Delta)$ is pseudo-effective and the non-nef locus of $-(K_X + \Delta)$ does not surject onto $Y$ via $f$, then $-K_Y$ is pseudo-effective.

- In [EG19, Theorem 3.1], Ejiri and Gongyo generalized Chen and Zhang's theorem, by showing that even if $(X, \Delta)$ is not log canonical, as long as $(F, \Delta|_F)$ is lc for general fibres $F$, then $-K_Y$ is still pseudo-effective. The proof is algebraic and it can be generalized to the positive characteristic.

Using the method in the proof of [EG19, Theorem 3.1], with additional ideas from [CZ13, Main Theorem], we can prove Theorem 4.1.1, which generalized [EG19, Theorem 3.1] in the case of characteristic zero. Furthermore, Theorem 4.1.1 simplifies in the following theorem which generalizes [CZ13, Main Theorem].

**Theorem 4.0.6.** *[cf. Theorem 4.1.1] Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties. Suppose the following conditions hold:*

*(1) There is a log pair $(X, \Delta)$ which is log canonical;*

*(2) $Y$ is $\mathbb{Q}$-Gorenstein and there is a $\mathbb{Q}$-Cartier divisor $D$ on $Y$ such that $L := -(K_X + \Delta) - f^*D$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor;*

*(3) The restricted base locus $\boldsymbol{B}_-(L)$ does not surject onto $Y$ via $f$.*

*Then $-K_Y - D$ is pseudo-effective.*

Note that our result does not completely cover [Den21, Theorem D(a)]. The reason is the equality $\mathrm{NNef}(-) = \mathbf{B}_-(-)$ is only confirmed for varieties $X$ such that there exists a klt pair $(X, \Delta)$ for some $\mathbb{Q}$-divisor $\Delta$ on $X$ (cf. [BBP13, Conjecture 1.7], [CB13, Theorem 1.2]). Using this theorem, we can give an algebraic proof of a bigness criterion of anti-canonical divisor:


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**Theorem 4.0.7** (cf. Theorem 4.1.7)**.** *Under the same notation and assumption of Theorem 4.0.6. Assume $L$ is big, and one of the following conditions holds:*

*(1)* $\boldsymbol{B}_+(L)$ *does not dominate* $Y$*;*

*(2)* $\boldsymbol{B}_-(L)$ *does not surject onto* $Y$*, and* $(X, \Delta)$ *is klt.*

*Then* $-K_Y - D$ *is big.*

## 4.1 Pseudoeffectiveness and bigness of anticanonical divisors

In this subsection, we will discuss the pseudoeffectiveness and bigness of anticanonical divisors. At first, we will prove the following theorem, which is a generalization of [EG19, Theorem 3.1], and our proof follows the original proof of Ejiri and Gongyo closely.

**Theorem 4.1.1.** *Let* $f : X \to Y$ *be an algebraic fibre space between normal projective varieties such that* $Y$ *is* $\mathbb{Q}$*-Gorenstein. Let* $\Delta = \Delta^+ - \Delta^-$ *be a* $\mathbb{Q}$*-Weil divisor on* $X$ *such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier,* $(F, \Delta^+|_F)$ *is log canonical for general fibre* $F$ *of* $f$*, and* $f(\mathrm{Supp}\Delta^-) \neq Y$*. Suppose that there is a* $\mathbb{Q}$*-Cartier divisor* $D$ *on* $Y$ *such that* $L := -(K_X + \Delta) - f^*D$ *is a pseudo-effective* $\mathbb{Q}$*-Cartier divisor such that* $\boldsymbol{B}_-(L)$ *does not surject onto* $Y$ *via* $f$*. Then, for* $l \in \mathbb{N}$ *such that* $l(K_X + \Delta), l(K_Y + D)$ *are Cartier and* $l\Delta^-$ *is integral, we have that*

$$\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^- + B))$$

*is weakly positive for some effective* $f$*-exceptional* $\mathbb{Q}$*-divisor* $B$*. Moreover, if* $Y$ *has at worst canonical singularities, then we can take* $B = 0$*.*

*In particular, if* $\Delta$ *is effective, then* $-K_Y - D$ *is pseudo-effective by Lemma 3.1.7(2).*


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*Proof.* The proof is based on the methods in the proof of [EG19, Theorem 3.1], modified with ideas in [CZ13, Main Theorem]. First, we prove the theorem under the stronger assumption that $f$ is equi-dimensional. Let

$$\mathcal{F} := \mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^-)),$$

and $A$ be an ample divisor on $X$. Since one can check the weakly positivity over an open set whose complement has codimension at least 2, it suffices to show that for all $n \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that the sheaf $(\mathcal{F}^{\otimes nm} \otimes \mathcal{O}_X(mlA))|_U$ is weakly positive on some Zariski open subset $U$ of $X$ with $\operatorname{codim}(X - U) \geq 2$.

Since $L$ is pseudo-effective, for any $n \in \mathbb{N}$, the $\mathbb{Q}$-Cartier divisor $G_n := L + \frac{1}{n}A$ is big with $\mathbf{B}(G_n) \subset \mathbf{B}_-(L)$. Therefore, $\mathbf{B}(G_n)$ is a proper Zariski closed subset that does not dominate $Y$. Let $\pi : X' \to X$ be a birational morphism such that $\pi^{-1}(\operatorname{Supp}\mathbf{B}(G_n) \cup \operatorname{Supp}\Delta) \cup \operatorname{Exc}(\pi)$ is a snc divisor. Let $f' := f \circ \pi : X' \to Y$ and write

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta) + E,$$

where $\Delta'$ is the proper transform of $\Delta$. The effective decomposition yields $(\Delta')^{\pm} = (\Delta^{\pm})'$ and we can write $E = E^+ - E^-$. Let $F$ be a general fiber of $f$ and $F' = \pi^{-1}(F)$, then every component of $E^-|_{F'}$ has coefficient at most 1 since $(F, \Delta|_F)$ is log canonical. Pick $\Gamma_n \sim_{\mathbb{Q}} \pi^* G_n$ be a general divisor such that $(F', (\Delta'^+ + \Gamma_n + E^-)|_{F'})$ is log canonical, then by Theorem 3.1.5, for $k \gg 0$, the sheaf

$$f'_* \mathcal{O}_{X'}(kl(K_{X'} + \Delta'^+ + \Gamma_n + E^-)) \otimes \mathcal{O}_Y(-klK_Y)$$

is weakly positive on $Y$. Therefore, for $m \gg 0$, we have the following generically sur-


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jective morphisms

$$
\begin{aligned}
&f^*(f'_*\mathcal{O}_{X'}(nml(K_{X'}+\Delta'^+ + G + E^-) \otimes \mathcal{O}_Y(-nmlK_Y))\\
\cong& f^*f_*\pi_*\mathcal{O}_{X'}(nml(K_{X'}+\Delta'^+ + G + E^-)) \otimes f^*\mathcal{O}_Y(-nmlK_Y)\\
\to& \pi_*\mathcal{O}_{X'}(nml(K_{X'}+\Delta'^+ + G + E^-)) \otimes f^*\mathcal{O}_Y(-nmlK_Y)\\
=& \pi_*\mathcal{O}_{X'}(nml(E+\Delta'^- - \pi^*f^*D + \frac{1}{n}\pi^*A + E^-)) \otimes f^*\mathcal{O}_Y(-nmlK_Y).\\
=& \pi_*\mathcal{O}_{X'}(nml(E^+ +\Delta'^- - \pi^*f^*D + \frac{1}{n}\pi^*A)) \otimes f^*\mathcal{O}_Y(-nmlK_Y) =: \mathcal{G}.
\end{aligned}
$$

By [Laz04, Theorem 1.7.6], the map $f^*f_*(-) \to (-)$ is generically surjective since on the open dense subset $X - (\mathrm{Supp}(\Delta^-) \cup \mathrm{Supp}(f^*D))$, the sheaf

$$\pi_*\mathcal{O}_{X'}(nml(K_{X'}+\Delta'^+ + \Gamma_n + E^-)) = \pi_*\mathcal{O}_{X'}(nml(E^+ + \Delta'^- - \pi^*f^*D + \frac{1}{n}\pi^*A))$$

is isomorphic to the ample line bundle $\mathcal{O}_X(mlA)$, and $\mathrm{Supp}(\Delta^-) \cup \mathrm{Supp}(f^*D)$ does not dominate $Y$. As $f$ is equidimensional, there is no $f$-exceptional divisor, thus $\mathcal{G}$ is weakly positive by Lemma 3.1.7(1) and Lemma 3.1.8.

Let $\mathcal{G}_1 := (\pi_*\mathcal{O}_{X'}(nml(\Delta'^- - \pi^*f^*D + \frac{1}{n}\pi^*A)) \otimes f^*\mathcal{O}_Y(-nmlK_Y))$. Since $E^+$ is effective and $\pi$-exceptional, there is a smooth open subvariety $U \subset X$ with $\mathrm{codim}(X - U, X) \geq 2$ such that $\mathcal{F}, \mathcal{G}$ are locally free on $U$ and $\mathcal{G}|_U \cong \mathcal{G}_1|_U$. Therefore, $\mathcal{G}_1$ is weakly positive since $\mathrm{codim}(X - U, X) \geq 2$. By the projection formula, we have

$$
\begin{aligned}
\mathcal{G}_1|_U \cong& (\mathcal{O}_X(nml(\Delta^- - f^*D + \frac{1}{n}A)) \otimes f^*\mathcal{O}_Y(-nmlK_Y))|_U\\
\cong& (\mathcal{O}_X(nml(\Delta^- - f^*(K_Y + D))) \otimes \mathcal{O}_X(mlA))|_U\\
\cong& (\mathcal{F}^{\otimes nm} \otimes \mathcal{O}_X(mlA))|_U,
\end{aligned}
$$

where the last isomorphism is due to $\mathcal{F}|_U$ being a line bundle by our choice of $U$. $(\mathcal{F}^{\otimes nm} \otimes \mathcal{O}_X(mlA))|_U$ is weakly positive on $U$. It follows that $\mathcal{F}^{\otimes nm} \otimes \mathcal{O}_X(mlA)$ is weakly positive on $X$ since $\mathrm{codim}(X - U, X) \geq 2$. Lastly, $n$ can be taken to be arbitrarily large, thus $\mathcal{F}$ is weakly positive.


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The proof for the general case follows verbatim to the argument of [EG19, Theorem 3.1]. By the flattening theorem in [AO00, 3.3, flattening lemma], there is a smooth birational modification $\mu : Y' \to Y$ such that let and $X'$ be the normalization of the main component of $X \times_Y Y'$, then $\pi : X' \to X$ is proper birational and the induced morphism $f' : X' \to Y'$ is equidimensional

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\mu"] & Y. \end{tikzcd}$$

Now we define $\Delta'$ by

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta),$$

and write $\Delta' = \Delta'^{+} - \Delta'^{-}$, then we have

$$-(K_{X'} + \Delta') - \pi^* f^* D = \pi^*(-(K_X + \Delta) - f^* D)$$

is pseudo-effective with the restricted base locus does not surject onto $Y'$. Therefore, $\mathcal{O}_{X'}(l(f^*(-K_Y' - \mu^* D) + \Delta'^{-}))$ is weakly positive since $f'$ is equidimensional. Now write $K_{Y'} = \mu^* K_Y + E$, and $E = E^+ - E^-$, then we have

$$\mu^*(-K_Y) + E^- = -K_{Y'} + E^+ \geq -K_{Y'}.$$

Thus $\mathcal{O}_{X'}(l(f'^*(\mu^*(-K_Y) + E^- - \mu^* D) + \Delta'^{-}))$ is also weakly positive, and so does $\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^- + \pi_*(f'^* E^-)))$. Now, define $B := \pi_*(f'^* E^-))$, then since $f'$ is equidimensional, we have $f_* B = \mu_* E^- = 0$. Also, if $Y$ has at worst canonical singularities, then $E^- = 0$. This completes the proof. $\square$

Using the above theorem, we can generalize Corollary 2.3 in [CZ13] in the following:

**Corollary 4.1.2.** *Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein. Suppose there exists a log pair $(X, \Delta)$ such that $-(K_X + \Delta)$ is pseudo-effective with the restricted base locus $\boldsymbol{B}_-(-(K_X + \Delta))$ not surjects*


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*onto $Y$, and for general fibre $F$ of $f$, $(F, \Delta|_F)$ is log canonical. Then either $Y$ is uniruled, or $K_Y \sim_{\mathbb{Q}} 0$. Moreover, in the case $Y$ is not uniruled, if $-(K_X+\Delta)$ is nef, then $Y$ has at worst canonical singularities.*

*Proof.* The proof follows from Theorem 4.1.1 by repeating the original proof in [CZ13, Corollary 2.3]. Suppose $Y$ is not uniruled, first, we show that $K_Y$ is pseudoeffective. Let $\mu : W \to Y$ be any resolution of singularities, then $W$ is also not uniruled. By [BDPP13], a smooth projective variety is uniruled iff its canonical divisor is not pseudo-effective. So $K_W$ is pseudo-effective, write $K_W = \mu^* K_Y + \sum a_i E_i$, then by Lemma 3.1.7(2), we see that $K_Y$ is pseudo effective. But by Theorem 4.1.1, we have $-K_Y$ is pseudo-effective. Therefore, both $K_Y$ and $-K_Y$ are pseudo-effective, hence $K_Y \sim_{\mathbb{Q}} 0$.

For the second part, consider the induced dominant rational map $\varphi : X \dashrightarrow W$. If $-(K_X+\Delta)$ is nef, then by [Zha05], we have $W$ has Kodaira dimension zero, so $K_W = \mu^* K_Y + \sum a_i E_i$ is $\mathbb{Q}$-effective. Thus, $\sum a_i E_i \sim_{\mathbb{Q}} K_W$ is $\mathbb{Q}$-effective, so $a_i \geq 0$ for all $i$ and hence $Y$ has at worst canonical singularities. □

**Corollary 4.1.3.** *Let $f : (X, \Delta) \to Y$ be an algebraic fibre space between normal projective varieties such that $(X, \Delta)$ is klt and $Y$ is $\mathbb{Q}$-Gorenstein. Suppose furthermore that $-(K_X+\Delta)$ is pseudo-effective with the non-nef locus $\mathrm{NNef}(-(K_X+\Delta))$ not surject onto $Y$, then $-K_Y$ is pseudo-effective.*

*Proof.* By [CB13], if $X$ has at worst klt singularities, then the non-nef locus is equivalent to the restricted base locus, hence the statement immediately follows from Theorem 4.1.1. □

**Corollary 4.1.4.** *The following theorems [EG19, Theorem 4.2, Proposition 4.4, Corollary 4.5, Corollary 4.7, and Corollary 5.1] still hold under the assumption $L$ is pseudo-effective with $\boldsymbol{B}_-(L)$ not surjects onto $Y$ (instead of being nef).*

*Proof.* Since by Lemma 2.3.4, $\mathbf{B}_-(-)$ only depends on the numerical equivalence classes of $\mathbb{Q}$-divisors. So, in all the above-mentioned theorems in [EG19], we can replace the nefness of $L$ in [EG19] by the weaker assumption that $L$ is pseudo-effective with $\mathbf{B}_-(L)$


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does not surject onto $Y$. Then by using Theorem 4.1.1 instead of [EG19, Theorem 3.1], all the original proof can work under this assumption. Hence the same conclusion holds. $\square$

As a corollary of Theorem 4.1.1, we have a bigness criterion for $-K_Y - D$, whose statement is very similar to [FG12, Theorem 3.1], but they cover different cases.

**Corollary 4.1.5.** *Keep the same notation as in Theorem 4.1.1. Suppose there is a big $\mathbb{Q}$-Cartier divisor $H$ on $Y$ such that $-(K_X + \Delta) - f^*(D + H)$ is pseudo-effective with $\mathbf{B}_-(-(K_X+\Delta)-f^*(D+H))$ does not surject onto $Y$, then for any $\mathbb{Q}$-Cartier divisor $D_0$ on $Y$, there is a rational number $\varepsilon > 0$ such that for sufficiently divisible positive integer $l$, the sheaf*

$$\mathcal{O}_X(l(f^*(-K_Y - D - \varepsilon D_0) + \Delta^- + B))$$

*is weakly positive for some effective $f$-exceptional $\mathbb{Q}$-divisor $B$. We can take $B = 0$ if $Y$ has at worst canonical singularities.*

*Proof.* Since $H$ is big, for $0 < \varepsilon \ll 1$, $H - \varepsilon E$ is still big. So we have

$$-(K_X + \Delta) - f^*(D + \varepsilon E) = (-(K_X + \Delta) - f^*(D + H)) + f^*(H - \varepsilon E)$$

is pseudo-effective with $\mathbf{B}_-(-(K_X + \Delta) - f^*(D + \varepsilon E))$ doesn't surject to $Y$, hence our result follows from Theorem 4.1.1. $\square$

*Remark* 4.1.6. In Corollary 4.1.5, if $\Delta \geq 0$, this means for any $\mathbb{Q}$-Cartier divisor $D_0$ on $Y$, there is a rational number $\varepsilon > 0$ such that $-K_Y - D - \varepsilon D_0$ is pseudo-effective, hence $-K_Y - D \in \mathrm{Int}(\mathrm{Eff}(Y)) = \mathrm{Big}(Y)$ is big.

As another application of Theorem 4.1.1, we have another bigness criterion of $-K_Y$ in the following. The reader can compare our results with [Den21, Theorem E] and [EIM20, Corollary 3.5 and Remark 3.6].

**Theorem 4.1.7.** *Under the same notation and assumption as in Theorem 4.1.1. Suppose furthermore that $L := -(K_X + \Delta) - f^*D$ is big, and one of the following holds:*

*(1) $\boldsymbol{B}_+(L)$ does not dominate $Y$;*


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*(2) $\mathbf{B}_-(L)$ does not surject onto $Y$, and $(F, \Delta_F^+)$ is klt.*

*Then for any $\mathbb{Q}$-Cartier divisor $D_0$ on $Y$, there is a rational number $\varepsilon > 0$ such that for sufficiently large and divisible integer $l$, the sheaf*

$$\mathcal{O}_X(l(f^*(-K_Y - D - \varepsilon D_0) + \Delta^- + B))$$

*is weakly positive for some effective $f$-exceptional $\mathbb{Q}$-divisor $B$. We can take $B = 0$ if $Y$ has at worst canonical singularities.*

*In particular, if $\Delta \geq 0$, then $-K_Y - D$ is big.*

*Proof.* For case (1), let $A$ be an ample divisor on $X$ such that $A - f^*D_0$ is ample and effective. Since $\mathbf{B}_+(L)$ does not surject onto $Y$, there exists $0 < \varepsilon \ll 1$ such that $\mathbf{B}(L - \varepsilon A)$ does not surject onto $Y$. We have

$$\mathbf{B}(L - \varepsilon A) \supseteq \mathbf{B}(L - \varepsilon A + \varepsilon(A - f^*D_0)) = \mathbf{B}(L - \varepsilon f^*D_0) \supseteq \mathbf{B}_-(L - \varepsilon f^*D_0).$$

Therefore, $\mathbf{B}_-(L - \varepsilon f^*D_0)$ does not surject onto $Y$. By Theorem 4.1.1, we conclude that the sheaf $\mathcal{O}_X(l(f^*(-K_Y - D - \varepsilon D_0) + \Delta^- + B))$ is weakly positive for sufficiently divisible integer $l$ and some effective $f$-exceptional $\mathbb{Q}$-divisor $B$.

For (2), it suffices to show that $\mathbf{B}_-(-(K_X + \Delta') - f^*D - \varepsilon f^*D_0)$ does not surject over $Y$ for some $\mathbb{Q}$-divisor $\Delta'$ where $(F, \Delta_F'^+)$ is log canonical.

We consider $L_n := L - \frac{1}{n}f^*D_0$. Note that $L$ is big and hence so is $L_n$ for $n \gg 0$. Let $G$ be an effective $\mathbb{Q}$-divisor such that $L_n - G$ is ample. Since we assume that $\mathbf{B}_-(L)$ does not surject onto $Y$, it follows that $\mathbf{B}(L + \delta A)$ does not surject onto $Y$ for any $\delta > 0$ and ample divisor $A$. Since $pL - \frac{1}{n}f^*D_0 + \delta A - G = (p-1)L + \delta A + L_n - G$ for any $p \in \mathbb{N}$, we have

$$\mathbf{B}(pL - \frac{1}{n}f^*D_0 + \delta A - G) = \mathbf{B}((p-1)L + \delta A + L_n - G) \subset \mathbf{B}((p-1)L + \delta A)$$

does not surject onto $Y$. As a result, neither does $\mathbf{B}_-(pL - \frac{1}{n}f^*D_0 - G) = \mathbf{B}_-(L_{pn} - \frac{1}{p}G)$.


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Now since $(F, \Delta_F^{+})$ is klt, by letting $\Delta' := \Delta + \frac{1}{p}G$, we have $(F, \Delta_F'^{+})$ is still klt for $p \gg 0$. Thus,

$$\mathbf{B}_{-}(L_{pn} - \frac{1}{p}G) = \mathbf{B}_{-}(-(K_X + \Delta') - f^*D - \frac{1}{np}f^*D_0)$$

does not surject onto $Y$. By Theorem 4.1.1, there is an effective $f$-exceptional $\mathbb{Q}$-divisor $B$ on $X$ such that $\mathcal{O}_X(l(f^*(-(K_Y + D) - \frac{1}{np}D_0) + \Delta^{-} + B))$ is weakly positive for sufficiently divisible integer $l$. □

By the above theorems in this section, we can give more properties of the relative anti-canonical divisor, generalizing [Deb01, Theorem 3.12], [EIM20, Corollary 3.7], and [CD13, Theorem 5.1].

**Corollary 4.1.8.** *Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein and is not a point. Let $(X, \Delta)$ be a log pair such that $(F, \Delta_F)$ is log canonical for general fibres $F$ of $f$, then:*

*(1) Let $D_0$ be a pseudo-effective $\mathbb{Q}$-Cartier divisor on $Y$ that is not numerically trivial. Then $\boldsymbol{B}_{-}(-(K_{X/Y} + \Delta + f^*D_0))$ must surjects onto $Y$;*

*(2) $\boldsymbol{B}_{+}(-(K_{X/Y} + \Delta))$ always surjects onto $Y$. Moreover, if $-(K_{X/Y} + \Delta)$ is big, and $(F, \Delta|_F)$ is klt for general fibres $F$, then $\boldsymbol{B}_{-}(-(K_{X/Y} + \Delta))$ surjects onto $Y$.*

*In particular:*

*(1) If $(X, \Delta)$ is log canonical, then $-(K_{X/Y} + \Delta)$ is not ample;*

*(2) If $(X, \Delta)$ is klt, then $-(K_{X/Y} + \Delta)$ is not nef and big.*

*Proof.* For (1), let $D = -K_Y + D_0$, then since $-K_Y - D = -D_0$ is not pseudo-effective, $-(K_X + \Delta) - f^*D = -(K_{X/Y} + \Delta + f^*D_0)$ must *not* satisfies the condition of Theorem 4.1.1. Hence $\mathbf{B}_{-}(-(K_{X/Y} + \Delta + f^*D_0))$ should surject onto $Y$. For (2), let $D = -K_Y$ in Theorem 4.1.7, then $-K_Y - D = 0$ is not big, so we can get the result by the same


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method. The last statement immediately follows from the fact a divisor $A$ is ample (resp. nef) if and only if $\mathbf{B}_+(A)$ (resp. $\mathbf{B}_-(A)$) is empty. $\square$



## 4.2 Effectiveness of anticanonical Iitaka dimension and the anticanonical Kodaira dimension

The goal of this subsection is to prove the following inequality about the Kodaira dimension of the anticanonical divisor, which can be thought of as the anti-canonical version of the Iitaka conjecture.

**Theorem 4.2.1.** *Let $f:(X,\Delta)\to Y$ be an algebraic fibre space between normal projective varieties such that $\Delta$ is effective, $K_X+\Delta$ is $\mathbb{Q}$-Cartier, and $Y$ is $\mathbb{Q}$-Gorenstein. Suppose there is a $\mathbb{Q}$-Cartier divisor $D$ on $Y$ such that $L:=-(K_X+\Delta)-f^*D$ is $\mathbb{Q}$-effective and $\boldsymbol{B}(L)$ does not dominate $Y$. Suppose furthermore that $(F,\Delta_F)$ is klt for general fibres $F$ of $f$, where $\Delta_F$ is defined by $-(K_F+\Delta_F)=-(K_X+\Delta)|_F=L_F$. Then we have*

$$\kappa(X,L)\leq\kappa(F,L_F)+\kappa(Y,-K_Y-D).$$

*In particular, if $X$ has at worst klt $\mathbb{Q}$-Gorenstein singularities, and $-K_X$ is effective with stable base locus $\boldsymbol{B}(-K_X)$ does not dominate $Y$, then we have*

$$\kappa(X,-K_X)\leq\kappa(F,-K_F)+\kappa(Y,-K_Y).$$

At first, we need to prove the effectiveness of $-K_Y-D$ under this assumption. By observing Theorem 4.1.1 and Theorem 4.1.7(1), the $\mathbb{Q}$-effectiveness of $-K_Y-D$ is an analogous statement of these two theorems under the assumption that $-K_X$ is $\mathbb{Q}$-effective.

**Theorem 4.2.2.** *Let $f:X\to Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein. Let $\Delta=\Delta^+-\Delta^-$ be a $\mathbb{Q}$-divisor on $X$ such that $(K_X+\Delta)$ is $\mathbb{Q}$-Cartier, $f(\mathrm{Supp}\Delta^-)\neq Y$ and $(F,\Delta_F^+)$ is klt for general fibres $F$ of $f$. Let*


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*$D$ be a $\mathbb{Q}$-Cartier divisor on $Y$ such that $L := -(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective with the stable base locus $\boldsymbol{B}(L)$ does not dominate $Y$, then we have that $f^*(-K_Y - D) + \Delta^- + E_X$ is $\mathbb{Q}$-effective for some effective $f$-exceptional $\mathbb{Q}$-divisor $E_X$. Moreover, we can take $E_X = 0$ if one of the following assumptions holds:*

*(1) $f$ is equidimensional;*

*(2) $\Delta$ is effective;*

*(3) $Y$ has at worst canonical singularities.*

*Proof.* The idea is similar to the proof of [Amb04, Theorem 4.1]. At first, we prove that $f : (X, B) \to Y$ is a klt-trivial fibration in the sense of [FG14], where $B := \Delta + L$. Since $L$ is $\mathbb{Q}$-effective with stable base locus $\mathbf{B}(L)$ which does not dominate $Y$, then up to $\mathbb{Q}$-linear equivalence, we may assume that $\Delta_F + L|_F$ is effective and klt for general fibres $F$. Let $B := \Delta + L$, then by the above discussion, we have Nklt$(\Delta^+ + L)$ does not surject onto $Y$, so $(X, B)$ is subklt over the generic point of $Y$.

Let $\pi : X' \to X$ be any birational morphism such that $X'$ is smooth. By the formula $K_{X'} = \pi^*(K_X + B) + \mathbf{A}_{X'}$, we have $\mathbf{A}_{X'} = -B' + E$, where $B'$ is the proper transform of $B$ on $X'$ and $E$ is a $\pi$-exceptional divisor. Since $(X, B)$ is subklt over the generic point of $Y$ and $B^h$ is effective by our assumption, we have $\lceil -B \rceil$ (resp. $\lceil -B' \rceil$) has no horizontal component for $f$ (resp. $f \circ \pi$), and the horizontal component of $\lceil E \rceil$ for $f \circ \pi$ is effective. This implies that $f_*\pi_*\mathcal{O}_{X'}(\mathbf{A}_{X'})$ is of rank at least 1. Moreover, $\lceil \mathbf{A}_{X'} \rceil = \lceil -B' + E \rceil = \lceil -B' \rceil + \lceil E \rceil$ is the sum of the vertical divisor $\lceil -B' \rceil$ and the $\pi$-exceptional divisor $\lceil E \rceil$. Now we let $A$ be a Cartier divisor on $Y$ such that $\pi^*f^*A \geq \lceil -B' \rceil$ (such $A$ exists since $\lceil -B' \rceil$ is vertical for $f \circ \pi$), then by the projection formula and [Deb01, Lemma 7.11 and 7.12], we have

$$\begin{aligned} f_*\pi_*\mathcal{O}_{X'}(\mathbf{A}_{X'}) &\leq f_*\pi_*\mathcal{O}_{X'}(\pi^*f^*A + \lceil E^+ \rceil) \\ &= f_*(\mathcal{O}_X(f^*A) \otimes \pi_*\mathcal{O}_{X'}(\lceil E^+ \rceil)) \\ &\cong f_*(\mathcal{O}_X(f^*A) \otimes \mathcal{O}_X) = \mathcal{O}_Y(A). \end{aligned}$$


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In particular, $f_*\pi_*\mathcal{O}_{X'}(\mathbf{A}_{X'})$ is of rank 1 for any birational morphism $\pi : X' \to X$ with $X'$ smooth, hence $f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil)$ is of rank 1. Moreover, we have

$$K_X + B = K_X + \Delta + L \sim_{\mathbb{Q}} f^*(-D).$$

Therefore, $f : (X, B) \to Y$ is a klt-trivial fibration in the sense of [FG14], which is also equivalent to the lc-trivial fibration in the sense of [Amb04], so we can apply the construction of moduli divisor on $f' : X' \to Y'$ to get

$$f^*(-D) \sim_{\mathbb{Q}} K_X + B \sim_{\mathbb{Q}} f^*(K_Y + M_Y + B_Y),$$

and hence $(-K_Y - D) \sim_{\mathbb{Q}} M_Y + B_Y$.

Thus, to show the $\mathbb{Q}$-effectiveness of $f^*(-K_Y - D) + \Delta^- + E$ for some $f$-exceptional divisor $E$, it remains to show that $f^*(M_Y + B_Y) + \Delta^- + E$ is $\mathbb{Q}$-effective. Similar to Theorem 4.1.1, we first prove the theorem under the assumption that $f$ is equi-dimensional. Since $B^h$ is effective, by Theorem 3.2.5, $M_Y$ is $\mathbb{Q}$-effective over $Y$. Next, let $B^v$ be the vertical part of $B$. By the definition of $B$, we have $B + \Delta^- = \Delta^+ + L$ is effective. Hence $B^v + \Delta^- = (\Delta^+ + L)^v$ is effective since $\Delta^-$ is vertical. Also, by Lemma 3.2.4, the negative part of $f^*B_Y - B^v$ is $f$-exceptional (note that the pullback of Weil divisors is well-defined for equi-dimensional morphism). Since $f$ is equidimensional, there is no $f$-exceptional divisor on $X$ and hence $f^*B_Y \geq B^v$. Thus, we have $f^*(M_Y + B_Y) \geq B^v$ since $M_Y$ is an effective $\mathbb{Q}$-divisor, which implies $f^*(M_Y + B_Y) + \Delta^- \geq B^v + \Delta^- \geq 0$.

The proof for the general case again follows verbatim to the argument of [EG19, Theorem 3.1]. By the flattening theorem, there is a normal birational modification $\mu : Y' \to Y$ such that let $X'$ be the normalization of the main component of $X \times_Y Y'$, then $\pi : X' \to X$ is proper birational and the induced morphism $f' : X' \to Y'$ is equidimensional.

$$\begin{tikzcd}
X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "\mu"] & Y
\end{tikzcd}$$


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Now we define $\Delta'$ by

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta),$$

and write $\Delta' = \Delta'^+ - \Delta'^-$, then we have

$$-(K_{X'} + \Delta') - \pi^* f^* D = \pi^*(-(K_X + \Delta) - f^* D)$$

is effective with the stable base locus not dominant over $Y'$. Therefore, $f^*(-K_{Y'} - \mu^* D) + \Delta'^-$ is effective since $f'$ is equidimensional. Now write $K_{Y'} = \mu^* K_Y + E$, and $E = E^+ - E^-$, then we have

$$\mu^*(-K_Y) + E^- = -K_{Y'} + E^+ \geq -K_{Y'}.$$

Thus $f'^*(\mu^*(-K_Y) + E^- - \mu^* D) + \Delta'^-$ is also $\mathbb{Q}$-effective, and so does

$$\pi_*(f'^*(\mu^*(-K_Y) + E^- - \mu^* D) + \Delta'^-) = f^*(-K_Y - D) + \Delta^- + \pi_*(f'^* E^-).$$

Now, define $E_X := \pi_*(f'^* E^-)$, then since $f'$ is equidimensional, we have $f_*(E_X) = \mu_* E^- = 0$, hence $E_X$ is $f$-exceptional. Also, if $Y$ has at worst canonical singularities, then $E^- = 0$. Finally, if $\Delta$ is effective, then $\Delta^- = 0$ and hence $f^*(-K_Y - D)$ is $\mathbb{Q}$-effective by Lemma 3.1.2. □

Next, we need the following proposition, which is a variant of [EG19, Proposition 4.4].

**Proposition 4.2.3.** *Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein. Let $\Delta = \Delta^+ - \Delta^-$ be a $\mathbb{Q}$-divisor on $X$ such that $(K_X + \Delta)$ is $\mathbb{Q}$-Cartier, $f(\mathrm{Supp}\Delta^-) \neq Y$ and $(F, \Delta_F^+)$ is klt for general fibres $F$ of $f$. Let $D$ and $E$ be $\mathbb{Q}$-Cartier divisors on $Y$ such that $L := -(K_X + \Delta) - f^* D$ is $\mathbb{Q}$-effective with the stable base locus $\boldsymbol{B}(L)$ does not dominate $Y$. Suppose furthermore that there is an effective $\mathbb{Q}$-divisor $\Gamma$ such that $L - g^* E \sim_{\mathbb{Q}} \Gamma \geq 0$. Moreover, assume one of the following three conditions holds:*


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*(1) $f$ is equidimensional;*

*(2) $\Delta$ is effective;*

*(3) $Y$ has at worst canonical singularities.*

*Then for $0 < \varepsilon \ll 1$, $f^*(-K_Y - D - \varepsilon E) + \Delta^-$ is $\mathbb{Q}$-effective.*

*Proof.* The proof follows the argument of [EG19, Proposition 4.4] closely. We consider

$$\Delta_\varepsilon := \Delta + \varepsilon\Gamma; \qquad D_\varepsilon := D + \varepsilon E; \qquad L_\varepsilon := -(K_X + \Delta_\varepsilon) - f^*D_\varepsilon.$$

Then $L_\varepsilon = L - \varepsilon(\Gamma + f^*E) \sim_{\mathbb{Q}} (1-\varepsilon)L$. Thus, for $\varepsilon < 1$, $L_\varepsilon$ is $\mathbb{Q}$-effective with the stable base locus $\mathbf{B}(L_\varepsilon)$ that does not dominate $Y$. For $0 < \varepsilon \ll 1$, $(F, (\Delta_\varepsilon)|_F)$ is still klt for general fibre $F$. Therefore, applying Theorem 4.2.2 on $L_\epsilon$, $f^*(-K_Y - D_\varepsilon) + \Delta_\varepsilon^- \leq f^*(-K_Y - D - \varepsilon E) + \Delta^-$ is $\mathbb{Q}$-effective. $\square$

The next theorem is a variant of Ejiri and Gongyo's injectivity theorem [EG19, Theorem 1.2]. This is an application of Proposition 4.2.3, and plays a key role in the proof of Theorem 4.2.1.

**Theorem 4.2.4.** *Use the notation and assumption in Proposition 4.2.3. Suppose there exists a $\mathbb{Q}$-Cartier divisor $P$ on $X$ such that $P \geq \Delta^-$, $f(\mathrm{Supp}P) \neq Y$, and $\kappa(X, f^*(-K_Y - D) + P) = 0$. Then for any general fibre $F$ of $f$, the following morphism between graded ring defined by restriction*

$$\bigoplus_{m\geq 0} H^0(X, \lfloor mL \rfloor) \to \bigoplus_{m\geq 0} H^0(F, \lfloor mL_F \rfloor)$$

*is injective. In particular, this implies $\kappa(X, L) \leq \kappa(F, L_F)$.*

*Proof.* We closely follow the proof of [EG19, Theorem 4.2, Corollary 4.7]. Consider the map between graded rings defined by restriction

$$\bigoplus_{m\geq 0} H^0(X, \lfloor mL \rfloor) \to \bigoplus_{m\geq 0} H^0(F, \lfloor mL_F \rfloor).$$


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As in the proof of [EG19, Corollary 4.7], the kernel of this map are the sections corresponding to the effective divisors $N \in |mL|$ such that Supp$N \supset F$. So it suffices to show that for every effective $\mathbb{Q}$-divisor $N \sim_{\mathbb{Q}} L$, Supp$N$ does not contains $F$.

Since $\kappa(X, f^*(-K_Y - D) + P) = 0$, there is a unique effective $\mathbb{Q}$-Cartier divisor $M \sim_{\mathbb{Q}} f^*(-K_Y - D) + P$. Let $F$ be a normal irreducible fibre of $f$ such that $y = f(F)$ is a smooth point on $Y$, $f$ is flat over an open dense neighborhood of $y$, $F \cap \mathrm{Supp}P = \emptyset$, and $F \not\subseteq \mathrm{Supp}M$. Now, let $\pi : X' \to X$ (resp. $\mu : Y' \to Y$) to be the blow-up of $X$ (resp. $Y$) with respect to $F$ (resp, $y$). Then as in the proof of [EG19, Theorem 4.2], since $f$ is flat over an open neighborhood of $y$, we have $X' = X \times_Y Y'$ by [Sta, Lemma 31.32.3] (If $Y$ is a curve, then $X' = X$, $Y' = Y$, and both $\pi$ and $\mu$ are identity maps). Also, since $y$ is a smooth point on $Y$, $Y'$ is still normal, and $\mathbb{Q}$-Gorenstein and $X'$ is still normal since $f'$ is flat with normal fibres over a neighborhood of exceptional locus, where $f'$ is the induced morphism $f' : X' \to Y'$. Now we have the following commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "\mu"] & Y.
\end{tikzcd}$$

Write $K_{Y'} = \mu^* K_Y + aE$, where $E$ is the exceptional divisor of $\mu$ on $Y'$, and $a = \dim Y - 1$. Let $G$ be the exceptional divisor of $\pi$ (where $G \cong F \times E$), then by the flatness over $F$, we have $G = f'^* E$. Also, since $F \cap \mathrm{Supp}\Delta^- = \emptyset$, we can write $-\pi^*(K_X + \Delta) = -(K_{X'} + \Delta') + bG$ for some $b \leq \mathrm{codim}(F, X) - 1 = \dim Y - 1 = a$, where $\Delta'$ is the proper transform of $\Delta$ on $X'$. Let

$$\begin{aligned}
L' &:= -(K_{X'} + \Delta') - f'^*(\mu^* D - aE) \\
&= -(K_{X'} + \Delta') - \pi^* f^* D + aG = \pi^* L + (a - b)G \geq \pi^* L,
\end{aligned}$$

which is also effective with the stable base locus that does not dominate $Y'$.

Now, suppose there exists an effective $\mathbb{Q}$-divisor $N \sim_{\mathbb{Q}} L$ with Supp$N \supset F$. Then


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we have $\mathrm{Supp}(\pi^* N) \supset \mathrm{Supp} G$. Now define

$$N' := \pi^* N + (a-b)G \sim_{\mathbb{Q}} \pi^* L + (a-b)G = L',$$

then $N'$ is effective with $\mathrm{Supp} N' \supset \mathrm{Supp} G$. Therefore, there exists a rational number $0 < \delta \ll 1$, such that $\Gamma := N' - \delta G \geq 0$, then

$$L' - \delta f'^* E = L' - \delta G \sim_{\mathbb{Q}} N' - \delta G = \Gamma \geq 0.$$

So we can apply Proposition 4.2.3 on $X', Y', \mu^* D - aE, L', \delta E$, and $\Gamma$ to conclude that $f'^*(-K_{Y'} - \mu^* D + aE - \varepsilon\delta E) + \Delta'^-$ is $\mathbb{Q}$-effective for $0 < \varepsilon \ll 1$, hence $f'^*(-K_{Y'} - \mu^* D + aE - \varepsilon\delta E) + \pi^* P$ is $\mathbb{Q}$-Cartier and $\mathbb{Q}$-effective. Note that

$$f'^*(-K_{Y'} - \mu^* D + aE - \varepsilon\delta E) + \pi^* P = \pi^*(f^*(-K_Y - D) + P) - \varepsilon\delta G.$$

Therefore, $G \subset \mathrm{Supp}(\pi^* M)$ by the uniqueness of $M$, thus $F \subset \mathrm{Supp} M$, but this is a contradiction to our choice of $F$, hence the proof is completed. □

*Proof. of Theorem 4.2.1* First, we prove the theorem under the assumption that $Y$ is smooth. By Theorem 4.2.2, we have $-K_Y - D$ is $\mathbb{Q}$-effective, and by Theorem 4.2.4, we only need to consider the case of $\kappa(Y, -K_Y - D) > 0$. Consider the following commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "\mu"] \arrow[d, "g'"] & Y \arrow[d, dashed, "g"] \\
Z' \arrow[r, dashed, "\nu"] & Z.
\end{tikzcd}$$

Where

(1) $g : Y \dashrightarrow Z$ is the rational map defined by $m(-K_Y - D)$ for some sufficiently divisible $m \gg 0$ such that $\dim(\overline{\mathrm{Im}(g)}) = \kappa(Y, -K_Y - D)$ and $\mathbf{Bs}(m(-K_Y - D)) = \mathbf{B}(-K_Y - D)$.


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(2) $g' : Y' \to Z'$ is a minimal resolution of Iitaka fibration induced by $m(-K_Y - D)$. More precisely, consider the following diagram

$$\begin{tikzcd} Y_0 \arrow[r, "\mu_0"] \arrow[d, "g_0"] & Y \arrow[d, dashed, "g"] \\ Z' \arrow[r, dashed, "\nu"] & Z. \end{tikzcd}$$

Here $g_0 : Y_0 \to Z'$ is the Iitaka fibration induced by $m(-K_Y - D)$. Note that $Y_0$ is possibly singular, but by the smoothness of $Y$, the singularity of $Y_0$ must contained in $\mathrm{Exc}(\mu_0)$. Now we let $\mu' : Y' \to Y_0$ be a minimal resolution of singularities.

(3) $\mu := \mu_0 \circ \mu' : Y' \to Y$ and $g' := g_0 \circ \mu' : Y' \dashrightarrow Z'$ are the induced maps, note that $\mu$ is birational and $g'$ is an algebraic fibre space.

(4) $X'$ is the normalization of the main component of $X \times_Y Y'$, and $\pi : X' \to X$ and $f' : X' \to Y'$ are the induced morphism. Note that we have $f'(\mathrm{Exc}(\pi)) \subset \mathrm{Exc}(\mu)$.

(5) $F$ is a general fibre of $f$ (by the construction of $X'$ and $f'$, $F$ is also a general fibre of $f'$).

(6) $G'$ is a general fibre of $g'$. Note that $G'$ is smooth by the smoothness of $Y'$

(7) $W' := f'^{-1}(G')$ is a general fibre of $g' \circ f'$.

Here $X'$ and $f'$ are constructed by the following way: Let $X_0$ be the main component of $X \times_Y Y'$ with projections $f_0 : X_0 \to Y'$ and $\pi_0 : X_0 \to X$. Then we have $\mathrm{Exc}(\pi_0) \subset f_0^{-1}(\mathrm{Exc}(\mu))$ and $\pi_0(\mathrm{Exc}(\pi_0)) \subset f^{-1}(\mu(\mathrm{Exc}(\mu)))$. Since $X$ is normal, $X_0 - \mathrm{Exc}(\pi_0) \cong X - \pi_0(\mathrm{Exc}(\pi_0))$ is normal. Hence for the normalization $\pi' : X' \to X_0$ of $X_0$, $\pi'$ is an isomorphism over $X_0 - \mathrm{Exc}(\pi_0)$. Thus, we have

$$\mathrm{Exc}(\pi) \subset \pi'^{-1}(\mathrm{Exc}(\pi_0)) \subset f'^{-1}(\mathrm{Exc}(\mu)),$$

where $f' = f_0 \circ \pi'$ and $\pi := \pi_0 \circ \pi'$.

By the easy addition formula [Mor85, Corollary 1.7], we have

$$\kappa(X', \pi^* L) \leq \kappa(W', (\pi^* L)|_{W'}) + \dim Z'.$$


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Since $\kappa(X', \pi^*L) = \kappa(X, L)$ and $\dim Z' = \kappa(Y, -K_Y - D)$, this implies

$$\kappa(X, L) \leq \kappa(W', (\pi^*L)|_{W'}) + \kappa(Y, -K_Y - D).$$

Thus, it remains to show that $\kappa(W', (\pi^*L)|_{W'}) \leq \kappa(F, L_F)$. To show this, let $B := \pi^*(K_{X/Y} + \Delta) - K_{X'/Y'}$, then

$$\mu^*(-K_Y - D) = -K_{Y'} - (\mu^*(K_Y + D) - K_{Y'}),$$
$$\pi^*L = -(K_{X'} + B) - f'^*(\mu^*(K_Y + D) - K_{Y'}).$$

Next, write

$$K_{Y'} = \mu^*K_Y + \sum a_i E_i,\ K_{X'} = \pi^*(K_X + \Delta) + \sum b_j P_j,$$

then $B = \sum a_i f'^* E_i - \sum b_j P_j$. Since $\Delta$ is effective, if $b_j > 0$ then $P_j$ must be $\pi$-exceptional, which implies $f'(P_j) \subsetneq \mathrm{Exc}(\mu)$. Therefore, $f'(\mathrm{Supp}B^-) \subset \mathrm{Exc}(\mu)$ and hence $\mathrm{Supp}(B^-) \subset f'^{-1}\mathrm{Exc}(\mu)$. Note that by the construction of the Iitaka fibration, we may assume $\mu_0(\mathrm{Exc}(\mu_0))$ is contained in $\mathbf{B}(-K_Y - D)$, which implies $\mathrm{Exc}(\mu_0) \subset \mu_0^{-1}\mathbf{B}(-K_Y - D)$. Note that by the smoothness of $Y$, we have the singularity of $Y_0$ must contained in $\mathrm{Exc}(\mu_0)$, which implies $\mathrm{Exc}(\mu') \subset \mu'^{-1}(\mathrm{Exc}(\mu_0))$ and hence

$$\mathrm{Exc}(\mu) \subset \mathrm{Exc}(\mu') \cup \mu'^{-1}(\mathrm{Exc}(\mu_0)) = \mu'^{-1}(\mathrm{Exc}(\mu_0)) \subset \mu^{-1}(\mathbf{B}(-K_Y - D)).$$

Thus, for sufficiently divisible $m \gg 0$, $\mathrm{Supp}(\mu^*(m(-K_Y - D))) \supset \mathrm{Exc}(\mu)$, which implies

$$\mathrm{Supp}(f'^*\mu^*(m(-K_Y - D))) \supset f'^{-1}(\mathrm{Exc}(\mu)) \supset \mathrm{Supp}(B^-).$$

Therefore, $P_{mn} := f'^*\mu^*(mn(-K_Y - D)) \geq B^-$ for $n \gg 0$. Consider the morphism $f'|_{W'} : W' \to G'$, whose general fibre is also a general fibre $F$ of $f$ by our construction.


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Then we have

$$\begin{aligned}
\kappa(W', (f'^*\mu^*(-K_Y - D) + P_{mn})|_{W'}) &= \kappa(W', f'^*\mu^*(-K_Y - D)|_{W'}) \\
&= \kappa(G', \mu^*(-K_Y - D)|_{G'}) \\
&= \kappa(G_0, \mu_0^*(-K_Y - D)|_{G_0}) = 0,
\end{aligned}$$

where the last equality is from the structure of Iitaka fibration.

Now, consider the morphism $f'|_{W'} : W' \to G'$ and the divisors

$$\begin{aligned}
&- K_{G'} - (\mu^*(K_Y + D)|_{G'} - K_{G'}) = \mu^*(-K_Y - D)|_{G'}, \\
&- (K_{W'} + B_{W'}) - f'^*(\mu^*(K_Y + D)|_{G'} - K_{G'}) = (\pi^* L)|_{W'}.
\end{aligned}$$

Here $B_{W'}$ is defined by $(K_{X'} + B)|_{W'} = K_{W'} + B_{W'}$. Note that we have

$$\begin{aligned}
(K_{W'} + B_{W'})|_F &= (K_{X'} + B)|_F \\
&= (\pi^*(K_X + \Delta) - f'^*(K_{Y'} - \mu^* K_Y))|_F \\
&= (\pi^*(K_X + \Delta))|_F \\
&\cong (K_X + \Delta)|_F \\
&= K_F + \Delta_F,
\end{aligned}$$

where the isomorphism follows from the fact that $\pi$ is an isomorphism over general fibres $F$. In particular, let $B_F$ defined by $(K_{W'} + B_{W'})|_F = K_F + B_F$, then $B_F = \Delta_F$ under the natural isomorphism $\pi^{-1}(F) \cong F$, hence $(F, B_F)$ is klt. Hence we can apply Theorem 4.2.4 (since $G'$ is smooth) to conclude that

$$\kappa(W', (\pi^* L)|_{W'}) \leq \kappa(F, -(K_{W'} + B_{W'})|_F) = \kappa(F, -(K_X + \Delta)|_F) = \kappa(F, L_F).$$

This completes our proof when $Y$ is smooth.

For the general case, consider the following diagram



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$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y. \end{array}$$



Here, $Y'$ is a resolution of singularities of $Y$, and $X'$ is a normalization of the main component of $X \times_Y Y'$. Similar to the above argument, we may assume $f'(\mathrm{Exc}(\pi)) \subset \mathrm{Exc}(\mu)$. Now we let $L = -(K_X + \Delta) - f^*D$, and write $\pi^*L = -(K_{X'} + \Delta') + E^+ - E^- - f'^*\mu^*D$, where $\Delta'$ is the proper transform of $\Delta$ on $X'$, and we can $K_{X'} + \Delta' = \pi^*(K_X + \Delta) + E^+ - E^-$ for some effective $\pi$-exceptional $\mathbb{Q}$-divisors. Note that by the construction of $X'$, both $E^+$ and $E^-$ are vertical with $f(\mathrm{Supp}E^+), f(\mathrm{Supp}E^-) \subset \mathrm{Exc}(\mu)$, hence there exists an effective $\mu$-exceptional divisor $N$ on $Y'$ such that $f'^*N \geq E^+ \geq E^+ - E^-$. Now, since the theorem holds if the base is smooth, we have

$$\kappa(L) \leq \kappa(-(K_{X'} + \Delta') - f^*\mu^*D + f'^*N) \leq \kappa(-(K_F + \Delta_F)) + \kappa(-K_{Y'} - \mu^*D + N).$$

Note that $-(K_{X'} + \Delta') - f^*\mu^*D + f'^*N = \pi^*L + (f^*N - (E^+ - E^-))$ is still effective with the stable base locus does not dominant $Y'$. Now, we write $K_{Y'} = \mu^*K_Y + B^+ - B^-$ for some $\mu$-exceptional divisors $B^+, B^-$. Then we have

$$\begin{aligned}\kappa(-K_{Y'} - \mu^*D + N) &= \kappa(\mu^*(-K_Y - D) + N + B^- - B^+) \\ &\leq \kappa(\mu^*(-K_Y - D) + N + B^-) = \kappa(\mu^*(-K_Y - D))\end{aligned}$$

since $N$ and $B$ are $\mu$-exceptional, hence the result holds for general case. □

The following corollaries directly follow from Theorem 4.2.1:

**Corollary 4.2.5.** *Let $(X, \Delta)$ be a log pair on $X$ and $f : X \to Y$ be an algebraic fibre space between normal projective varieties. Suppose $Y$ is $\mathbb{Q}$-Gorenstein and $(F, \Delta_F)$ is klt for general fibres $F$. Then:*

*(1) Let $D$ be a $\mathbb{Q}$-Cartier divisor on $X$ such that $-(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective and*



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*the stable base locus* $\boldsymbol{B}(-(K_X+\Delta)-f^*D)$ *does not dominate* $Y$*, then*

$$\dim Y - \kappa(Y, -K_Y - D) \leq \dim X - \kappa(X, -(K_X+\Delta) - f^*D).$$

*(2) Suppose* $-(K_{X/Y}+\Delta)$ *is* $\mathbb{Q}$*-effective with* $\mathbf{B}(-(K_{X/Y}+\Delta))$ *that does not dominate* $Y$*, then for any effective* $\mathbb{Q}$*-Cartier divisor* $D_0$ *on* $Y$*, we have*

$$\kappa(X, -(K_{X/Y}+\Delta) + f^*D_0) \leq \kappa(F, -(K_F+\Delta_F)) + \kappa(Y, D_0).$$

*Remark* 4.2.6. There is another way to prove Theorem 4.2.1: First, we prove the theorem under the assumption $f$ is equidimensional. Under this assumption, $f'$ is also equidimensional, hence so does $W' \to G'$. Therefore, we can prove the inequality holds for equidimensional morphism by the same steps of the above proof. Then for the general case, we can let $X' \to Y'$ be the normalization of the flattening $f$, and use the same argument to prove the inequality holds for $f$ by the fact the inequality holds for $f'$.

*Remark* 4.2.7. In a collaborative work with Marta Benozzo and Iacopo Brivio, we also proved there is a positive characteristic version of Theorem 4.2.1 (cf. [BBC23]). Before the collaborative work started, in [Ben22] M. Benozzo had already proved the positive characteristic version of this inequality holds if either $\dim X \leq 3$, or $\dim Y = 1$.

At the end of this subsection, we give two examples to show both the assumption on stable base locus and the klt assumption are needed.

*Example* 4.2.8. Let $Y$ be a smooth curve with genus $g \geq 2$, and consider the ruled surface $X = \mathbb{P}_Y(\mathcal{O} \oplus \mathcal{O}(-K_Y - D))$, where $D$ is an ample divisor on $Y$ with $\deg D = d > 2g-2$. Then $K_X = -2C_0 - f^*D$, where $f$ is the structure morphism $f : X \to Y$ and $C_0$ is the distinguished section. Note that $-K_X \geq f^*D + C_0 \geq 0$ and $(f^*D + C_0)^2 = d + 2 - 2g > 0$, which implies $-K_X$ is big, but $-K_Y$ is anti-ample. Hence the conclusions of the above theorems fail. Note that in this case, we have $\mathbf{B}_-(-K_X) = \mathbf{B}(-K_X) = \mathbf{B}_+(-K_X) = \mathrm{Supp} C_0$, which is surject onto $Y$.

*Example* 4.2.9. Let $C$ be an elliptic curve, $d$ be a divisor on $C$ with $\deg d > 0$, $X := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-d))$ and $\pi : X \to C$ be the structure map. Let $\Delta = C_0$, where $C_0$ is the


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distinguish section of $X$. Then we have $-(K_X + \Delta) = C_0 + dF$, where $F$ is the general fibre of $\pi$. In this case, $(X, \Delta)$ is lc but not klt, and we have

$$2 = \kappa(X, -(K_X + \Delta)) > \kappa(F, -(K_X + \Delta)|_F) + \kappa(C, -K_C) = 1.$$

Note that we can prove $\mathbf{B}(-(K_X + \Delta))$ does not dominant $C$ in the following way: Suppose $\mathbf{B}(-(K_X + \Delta))$ dominate $C$, then we must have $\mathbf{B}(-(K_X + \Delta)) \supseteq C_0$, and hence $\mathbf{Bs}(-(K_X + \Delta)) \supseteq C_0$. Note that $-(K_X + \Delta) - C_0 = dF$. This implies

$$h^0(X, -(K_X + \Delta)) = h^0(X, -(K_X + \Delta) - C_0) = h^0(X, dF) = h^0(C, d) = \deg d.$$

Also, by the Riemann-Roch theorem, we have $h^0(dF) - h^1(dF) + h^2(dF) = \deg d$. Since $h^2(dF) = h^0(K_X - dF) = 0$, which implies $h^1(dF) = 0$. Consider the exact sequence

$$0 \to \mathcal{O}_X(dF) \to \mathcal{O}_X(-(K_X + \Delta)) \to \mathcal{O}_{C_0}(-(K_X + \Delta)|_{C_0}) \to 0,$$

the facts $h^1(dF) = 0$ and $h^0(X, -(K_X + \Delta)) = h^0(X, dF)$ implies $h^0(C_0, -(K_X + \Delta)|_{C_0}) = 0$. However, $-(K_X + \Delta)|_{C_0} = (C_0 + dF)|_{C_0} = 0$ since $C_0|_{C_0} = -dF|_{C_0}$, which implies $h^0(C_0, -(K_X + \Delta)|_{C_0}) = h^0(C, 0) = 1$, a contradiction. Note that in this example, $-K_Y$ is effective, and $-(K_X + \Delta) - tF$ is also effective for $0 < t < 1$, but $-K_Y - \varepsilon P$ is not $\mathbb{Q}$-effective for any $\varepsilon > 0$ and any closed point $P \in C$, which shows both Proposition 4.2.3 and Theorem 4.2.4 fails for lc pairs. At this moment, we do not know whether Theorem 4.2.2 holds for lc pair or not.

## 4.3 Generalization to rational maps.

Comparing [EG19, Section 4 and Section 5] to Theorem 4.2.1, it seems natural to ask whether Theorem 4.2.1 holds if $f$ is an almost holomorphic fibration. Unfortunately, we have the following counterexample:

*Example* 4.3.1. Let $X := \mathbb{P}^2 \times \mathbb{P}^1$, and $p_1 : X \to \mathbb{P}^2$ be the first projection. Then let $\mu : Y \to \mathbb{P}^2$ be the blow-up of 13 general points $P_1, ..., P_{13}$ on $\mathbb{P}^2$, and denote $E_1, ..., E_{13}$



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be the corresponding exceptional divisors. Then we have $E_i^2 = -1$ and $(E_i.E_j) = 0$ if $i \neq j$. Since 14 general points define a quartic plane curve, for a general point $P$ on $\mathbb{P}^2$, there is a quartic $C_0$ passing $P$ and all $P_i$ with multiplicity 1. Now, let $C$ be the proper transform of $C_0$ on $Y$. Then we have $(C.E_i) = 1$ for all $i$, and

$$(-K_Y.C) = (-\pi^* K_{\mathbb{P}^2} - \sum E_i.\pi^* C_0 - \sum E_i) = -1.$$

Therefore, $-K_Y$ is not almost nef, hence not pseudo-effective by [BKK+15, Proposition 4.2 and 4.5]. Let $f : X \dashrightarrow Y$ be the induced rational map, which is almost holomorphic since the fibre is well-defined and closed over $Y - \mathrm{Exc}(\mu)$. Then $-K_X$ is ample, but $-K_Y$ is not effective, hence the inequality in Theorem 4.2.1 does not hold for the map $f$.

This example also shows that the last assertion of Theorem 4.1.1, Theorem 4.1.7, and Theorem 4.2.2 can not be generalized to the situation that $f$ is only an almost holomorphic fibration even if $D = 0$. However, by Lemma 3.1.7, we have the following result, which is a generalization of [Den21, Lemma 4.1]:

**Proposition 4.3.2.** *Let $f : X \dashrightarrow Y$ be a birational map between normal projective varieties. Let $X_0, Y_0$ be the maximal open sets of $X, Y$ such that $f|_{X_0} : X_0 \to Y_0$ is a morphism. Let $\Delta$ be a $\mathbb{Q}$-divisor such that both $K_X + \Delta$ and $K_Y + f_*\Delta$ are $\mathbb{Q}$-Cartier. Suppose that $\pm(K_X + \Delta)$ is pseudo-effective (resp. effective, big), and $Y - Y_0$ has codimension at least 2, then $\pm(K_Y + f_*\Delta)$ is pseudo-effective (resp. effective, big). In particular, this result holds if $f$ is either a divisorial contraction or a $(K_X + \Delta)$-flip.*

*Proof.* We work on the case of anti-canonical divisors, and the case of canonical divisors can be derived in the same way. Let $g : W \to Y$ be a resolution of indeterminacy of $f$ such that $W$ is normal and denote $\pi : W \to X$ be the corresponding birational morphism.

Suppose that $-(K_X + \Delta)$ is pseudo-effective, then we write

$$K_W + \Delta_W = \pi^*(K_X + \Delta) + E_X,$$

where $\Delta_W$ is the proper transform of $\Delta$ on $W$, and $E_X$ is $\pi$-exceptional. Then we have


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$-\pi^*(K_X + \Delta) = -K_W - \Delta_W + E_X$ is pseudo-effective. Next, let $(f_*\Delta)_W$ be the proper transform of $f_*\Delta$ on $W$. Then there is a $g$-exceptional divisor $E_Y$ such that $K_W + (f_*\Delta)_W = g^*(K_Y + f_*\Delta) + E_Y$. Therefore,

$$-g^*(K_Y + f_*\Delta) = (-K_W - \Delta_W + E_X) + (\Delta_W - (f_*\Delta)_W) + E_Y - E_X.$$

Note that $(\Delta_W - (f_*\Delta)_W)$ is $g$-exceptional, and since $Y - Y_0$ has codimension at least 2, every $\pi$-exceptional divisor is $g$-exceptional. So $-g^*(K_Y + f_*\Delta) = (-K_W - \Delta_W + E_X)$ + ($g$-exceptional divisors). Hence by Lemma 3.1.7(2), $-(K_Y + f_*\Delta)$ is pseudo-effective.

Suppose now that $-(K_X+\Delta)$ is effective. Given a birational morphism $\pi : X' \to X$, if $\pi^*D$ is effective outside the exceptional locus, then $D$ is effective itself. Thus, under the assumption that $-(K_X + \Delta)$ is effective, we can use the same argument of the pseudo-effective case to show that $-(K_Y + f_*\Delta)$ is effective, by using this fact where we use Lemma 3.1.7 in the pseudo-effective case.

Suppose now that $-(K_X+\Delta)$ is big. Since $Y-Y_0$ has codimension at least 2, for any $\mathbb{Q}$-divisor $G$ on $Y$, there is a $\mathbb{Q}$-divisor $G'$ on $X$ such that $f_*G' = G$. Since $-(K_X + \Delta)$ is big, we have $-(K_X + \Delta + \varepsilon G')$ is pseudo-effective for $\varepsilon$ sufficiently small. Thus, replacing $\Delta$ by $\Delta + \varepsilon G'$, the pseudo-effective case implies $-(K_Y + f_*(\Delta + \varepsilon G')) = -(K_Y + f_*\Delta) - \varepsilon G$ is pseudo-effective. Since $G$ is arbitrary, this implies $-(K_Y + f_*\Delta)$ is big. $\square$

By the above proposition, it is natural to ask whether we can generalize Theorem 4.1.1, Theorem 4.1.7, and Theorem 4.2.2 to almost holomorphic fibration with $Y - Y_0$ has codimension at least 2. To answer this question, first, we define the pullback of divisors under rational maps:

**Definition 4.3.3.** *(cf. [Mat14, Definition 1.2]) Let $f : X \dashrightarrow Y$ be a rational map. Then for a ($\mathbb{Q}$-)divisor $D$ on $Y$, we can define the pullback $f^*D$ in the following way: Let $\pi : X' \to X$ be a resolution of indeterminacy on $f$, and $f' : X' \to Y$ be the corresponding resolution. Then we define $f^*D := \pi_* f'^* D$. Note that this definition does not depend on the choice of resolution because the push-forward of exceptional divisors is zero.*


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Now we have the following generalization:

**Proposition 4.3.4.** *Let $f : X \dashrightarrow Y$ be an almost holomorphic fibration between normal projective varieties, with $X_0, Y_0$ be the maximal open subsets of $X$ and $Y$ such that $f|_{X_0} : X_0 \to Y_0$ is a morphism. Suppose that $Y - Y_0$ has codimension at least 2, then the conclusions of Theorem 4.1.1, 4.1.7, and 4.2.2 still hold by replacing the assumption on $\mathbf{B}_-(L)$ (resp. $\mathbf{B}_+(L), \mathbf{B}(L)$) with the one that $\mathbf{B}_-(L) \cap X_0$ (resp. $\mathbf{B}_+(L) \cap X_0, \mathbf{B}(L) \cap X_0$) does not surject onto $Y_0$.*

*Proof.* Let $\pi : X' \to X$ be a resolution of indeterminacy, $f' : X' \to Y$ be the corresponding resolution. We can write $-(K_{X'} + \Delta') - f'^*D = \pi^*L + E^+ - E^-$ for some effective $\pi$-exceptional divisor $E^+, E^-$, where $\Delta'$ is the proper transform of $\Delta$ on $X'$. Since $f$ is almost holomorphic, we may assume $f'(\mathrm{Exc}(\pi)) \subset Y - Y_0$. In particular, every $\pi$-exceptional divisors are $f'$-exceptional.

Now, suppose $L$ is pseudo-effective and $(\mathbf{B}_-(L) \cap X_0)$ does not surject onto $Y_0$. Then $\mathbf{B}_-(-(K_{X'} + \Delta' + E^+ - E^-) - f'^*D) = \mathbf{B}_-(\pi^*L)$ is not surject onto $Y$ since $\mathbf{B}_-(\pi^*L) \subset \pi^{-1}(\mathbf{B}_-(L))$. Indeed, let $H$ be an ample Cartier divisor on $X$ and $A$ be an ample Cartier divisor on $X'$ such that $A - \pi^*H$ is also ample. Then for any $0 < \varepsilon \ll 1$, we have

$$\begin{aligned}\mathbf{B}(\pi^*L + \varepsilon A) &= \mathbf{B}(\pi^*(L + \varepsilon H) + \varepsilon(A - \pi^*H)) \\ &\subset \mathbf{B}(\pi^*(L + \varepsilon H)) \subset \pi^{-1}(\mathbf{B}(L + \varepsilon H)) \subset \pi^{-1}(\mathbf{B}_-(L)).\end{aligned}$$

Hence $\mathbf{B}_-(\pi^*L) \subset \pi^{-1}(\mathbf{B}_-(L))$ follows from the definition of $\mathbf{B}_-(-)$. Since $f(\mathrm{Supp}E) \neq Y$, we can apply Theorem 4.1.1 to conclude that for sufficiently divisible positive integer $l$, $\mathcal{O}_{X'}(l(f'^*(-K_Y - D) + \Delta'^- + E^- + B))$ is weakly positive for some effective $f'$-exceptional divisor $B$, hence so is $\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^- + \pi_*B))$. In particular, since $E^-$ is also $f'$-exceptional, by the weakly positivity of $\mathcal{O}_{X'}(l(f'^*(-K_Y - D) + \Delta'^- + E^- + B))$ and Lemma 3.1.7(2), we conclude that if $\Delta$ is effective, then $-K_Y - D$ is pseudo-effective, which generalizes Theorem 4.1.1 (and hence Theorem 4.1.7). Using a similar method, and applying Lemma 3.1.2, we can also generalize Theorem 4.2.2. □


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*Remark* 4.3.5. *The above proof shows that even if we only assume $f$ is almost holomorphic, $\mathcal{O}_X(l(f^*(-K_Y-D)+\Delta^-+\pi_*B))$ is still weakly positive. But without the assumption of $Y-Y_0$ having codimension at least 2, then the pseudo-effectiveness (resp. effectiveness, bigness) of $\mathcal{O}_X(l(f^*(-K_Y-D)+\pi_*B))$ and $\mathcal{O}_{X'}(l(f'^*(-K_Y-D)+E^-+B))$ is not enough to imply the pseudo-effectiveness (resp. effectiveness, bigness) of $-K_Y-D$ even if $\Delta$ is effective. However, if $f$ is only a rational map, then* Supp$(E^-)$ *may map onto $Y$ and hence the weak positivity of $\mathcal{O}_X(l(f^*(-K_Y-D)+\Delta^-+\pi_*B))$ may not be true.*

**Proposition 4.3.6.** *The inequality of Theorem 4.2.1 holds if $f$ is an almost holomorphic fibration with $Y-Y_0$ has codimension at least 2.*

*Proof.* The proof is similar to the argument generalizing Theorem 4.2.1 from smooth cases to the general case. Consider the following diagram

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, dashed, "f"] \\ Y' \arrow[r, "\mu"] & Y. \end{tikzcd}$$

Here, $Y'$ is log resolution of $(Y, Y-Y_0)$, and $X'$ is a normalization of the resolution of indeterminacy of $X \to Y'$. We may assume for the induced morphisms $\pi : X' \to X$ and $f' : X' \to Y'$, Exc$(\pi)$ is purely codimension 1 and $f'(\text{Exc}(\pi)) \subset \text{Exc}(\mu)$. Now we let $L = -(K_X+\Delta)-f^*D$, and write $\pi^*L = -(K_{X'}+\Delta')+E^+-E^--f'^*\mu^*D$, where $K_{X'}+\Delta' = \pi^*(K_X+\Delta)+E^+-E^-$. Note that both $E^+$ and $E^-$ are vertical and there exists an effective $\mu$-exceptional divisor $N$ on $Y'$ such that $f'^*N \geq E^+ \geq E^+-E^-$ by the construction of $X'$. Now, since the theorem holds if $f$ is an algebraic fibre space, we have

$$\kappa(L) \leq \kappa(-(K_{X'}+\Delta')-f^*\mu^*D+f'^*N) \leq \kappa(-(K_F+\Delta_F))+\kappa(-K_{Y'}-\mu^*D+N).$$

Note that $-(K_{X'}+\Delta')-f^*\mu^*D+f'^*N = \pi^*L+(f^*N-(E^+-E^-))$ is still effective with the stable base locus does not dominant $Y'$. Now, we write $K_{Y'} = \mu^*K_Y+B^+-B^-$


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for some $\mu$-exceptional divisors $B^+, B^-$. Then we have

$$\begin{aligned}\kappa(-K_{Y'} - \mu^* D + N) &= \kappa(\mu^*(-K_Y - D) + N + B^- - B^+) \\ &\leq \kappa(\mu^*(-K_Y - D) + N + B^-) = \kappa(\mu^*(-K_Y - D))\end{aligned}$$

since $N$ and $B$ are $\mu$-exceptional, hence the proof completes. $\square$



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# Chapter 5 The Iitaka Conjecture and the Generalized Nonvanishing Conjecture

The main purpose of this section is to discuss the Generalized Nonvanishing Conjecture and the Iitaka Conjecture. The ingredients of this section have been published as the preprint [Cha23a].

Given an algebraic fibre space $f : X \to Y$ between (smooth) projective varieties over a field with characteristic 0. An important question in birational geometry is how to compare birational invariants of $X$, $Y$, and the general fibre $F$ of $f$. One of the most important birational invariants is the Kodaira dimension, which is the Iitaka dimension of the canonical divisor. For this question, a well-known classical conjecture, called the Iitaka Conjecture is stated as the following:

**Conjecture 5.0.1** (Iitaka Conjecture)**.** *Let $f : X \to Y$ be an algebraic fibre space between smooth projective varieties, and $F$ be a very general fibre of $f$.*

*(1) $(C_{n,m})$ If $\dim X = n$ and $\dim Y = m$, then we have*

$$\kappa(X) \geq \kappa(F) + \kappa(Y);$$


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*(2)* $(C^{-}_{n,m})$ *If* $\dim X = n$, $\dim Y = m$, *and* $\kappa(X) \geq 0$, *then we have*

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

Obviously, $C^{-}_{n,m}$ is a weaker version of $C_{n,m}$. Although $C_{n,m}$ is still an open problem in general, there are many special cases have been proven:

- $\dim Y \leq 2$ ([Kaw82], [Cao18]);
- $\dim X \leq 6$ ([Bir09]);
- $Y$ is of general type ([Fuj17, Theorem 1.7], [Vie83, Corollary IV]);
- $F$ is of general type ([Fuj20, Corollary 4.3.5]);
- $F$ has a good minimal model ([Kaw85], [Has20]). In particular, if $\dim F \leq 3$ ([Kaw92a]);
- $Y$ has maximal Albanese dimension ([CP17], [HPS18]);
- $f$ is a smooth fibration, and $F$ is either a curve or of general type ([PS22]).

Note that the case $Y$ has maximal Albanese dimension recovered the case $\dim Y = 1$ since every smooth projective curve with non-negative Kodaira dimension has maximal Albanese dimension (cf. [Fuj20, Section 4.5.1]). Moreover, as Birkar mentioned in [Bir09], the following conjecture, called the Ueno conjecture, is closely related to the Iitaka conjecture.

**Conjecture 5.0.2** (Ueno Conjecture)**.** *Let* $X$ *be a smooth projective variety with* $\kappa(X) = 0$, *and* $\alpha : X \to A$ *be the Albanese map of* $X$ *(which is an algebraic fibre space by [Kaw81]). Then we have:*

*(1)* $\kappa(F) = 0$ *for the general fibre* $F$ *of* $f$*;*

*(2) There is an étale cover* $A' \to A$ *such that* $X \times_A A'$ *is birational to* $F \times A'$ *over* $A$*.*


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It is obvious that the work of [CP17] and [HPS18] implies that the first part of the Ueno conjecture is true.

The goal of this section is to prove some special cases of Iitaka Conjecture. In particular, if the source space is seven-dimensional. We will prove our result by following the idea in the proof of [Bir09] with some modifications. The main ingredient of our modification are the following conjectures introduced by Lazić and Peternell in [LP20]:

**Conjecture 5.0.3** (Generalized Nonvanishing Conjecture)**.** *Let $(X, \Delta)$ be a klt pair of a normal projective variety $X$ such that $K_X + \Delta$ is pseudo-effective. Let $L$ be a nef $\mathbb{Q}$-divisor on $X$. Then for every $t \geq 0$, the numerical class of the divisor $K_X + \Delta + tL$ belongs to the effective cone, that is, there exists an effective $\mathbb{Q}$-Cartier divisor $D$ such that $K_X + \Delta + tL \equiv D$.*

**Conjecture 5.0.4** (Generalized Abundance Conjecture)**.** *Let $(X, \Delta)$ be a klt pair of a normal projective variety $X$ such that $K_X + \Delta$ is pseudo-effective. Let $L$ be a nef Cartier divisor on $X$ such that $K_X + \Delta + L$ is nef. Then there is a semiample $\mathbb{Q}$-Cartier divisor divisor $M$ satisfying $K_X + \Delta + L \equiv M$.*

In [LP20, Corollary C and D], Lazić and Peternell proved these two conjectures hold if either $\dim X \leq 2$, or $\dim X = 3$ with $\nu(K_X + \Delta) \geq 1$. On the other hand, in 2022, there is another more strong version of the Generalized Abundance Conjecture (cf. [LMP+22]).

**Conjecture 5.0.5** (Stronger version of Generalized Nonvanishing Conjecture)**.** *Let $(X, \Delta)$ be a log canonical pair of a normal projective variety $X$. Let $L$ be a nef $\mathbb{Q}$-divisor on $X$, and $t$ be a non-negative rational number. If $K_X + \Delta + tL$ is pseudoeffective, then the numerical class of the divisor $K_X + \Delta + tL$ belongs to the effective cone, that is, there exists an effective $\mathbb{Q}$-Cartier divisor $D$ such that $K_X + \Delta + tL \equiv D$.*

For the surface case, this stronger version is proved by Han-Liu in [HL20]. After the work of Han-Liu, this stronger version is formulated by Lazić-Matsumura-Peternell-Tsakanikas-Xie in [LMP+22], and in this article, they proved in dimension 3, when $L = -(K_X + \Delta)$ is nef, the stronger version is true if either $X$ is $\mathbb{Q}$-factorial, or $X$ has rational singularities.


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In the following of this thesis, we focus on the original version of the Generalized Nonvanishing Conjecture, that is, in the case $(X, \Delta)$ is klt and $K_X + \Delta$ is pseudo-effective.

## 5.1 The Generalized Nonvanishing Conjecture in dimension 3

In this section, we will discuss Conjecture 5.0.3, the original version of the Generalized Nonvanishing Conjecture. Here we will use the following notations:

- The Generalized Non-vanishing conjecture in dimension $d$ by $N_d$;
- The Generalized Non-vanishing conjecture in dimension $d$ with $\kappa(X, K_X + \Delta) = k$ by $N_{d,k}$;
- The Generalized Non-vanishing conjecture in dimension $d$ with $\kappa(X, K_X + \Delta) = k$ and $q(X) = q$ by $N_{d,k,q}$;
- The Generalized Non-vanishing conjecture in dimension $d$ with $\kappa(X, K_X + \Delta) = k$, $q(X) = q$, and nef divisor $L$ with nef dimension $n$ by $N_{d,k,q,n}$.

We will prove the following special cases of the Generalized Nonvanishing Conjecture:

**Theorem 5.1.1.** *We have the following:*

*(1)* $N_{d,k,q,1}$ *is true if* $k \geq 0$*;*

*(2)* $N_{d,k,q,2}$ *is true if* $k \geq 0$ *and* $d \leq 5$*;*

*(3)* $N_{3,k,q,3}$ *is true if* $q > 0$*;*

*(4)* $N_{d,0,q,n}$ *is true if* $n = d$ *and* $q \geq d - 2$.



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*In particular, by (1)-(3) and [LP20, Corollary D], the Generalized Nonvanishing Conjecture for a three dimensional klt pair $(X, \Delta)$ is true unless $\kappa(K_X + \Delta) = \nu(K_X + \Delta) = 0$, $n(L) = 3$, and $q(X) = 0$.*

At first, we study the Generalized Nonvanishing Conjecture for nef divisors has nef dimension 1. By [BCE+00, Proposition 2.11], a nef divisor $L$ on a smooth projective variety $X$ with $n(L) = 1$ is numerically equivalent to a semiample divisor. Here we generalize this result to varieties that have rational singularities, by using the ideas of birational descent trick of nef divisor in [LP20].

**Lemma 5.1.2.** *Let $X$ be a normal projective variety such that $X$ has rational singularities, and $L$ be a nef $\mathbb{Q}$-Cartier divisor on $X$. If $n(L) \leq 1$, then $L$ is numerically equivalent to a semiample $\mathbb{Q}$-divisor on $X$.*

*Proof.* By replacing $X$ and $L$ with a resolution $\phi : X' \to X$ and $\phi^* L$, we may assume $X$ is smooth in the following way: Since $f \circ \phi$ is also a nef reduction map of $\phi^* L$, where $f$ is a nef reduction map of $L$, and by [LP20, Lemma 2.14], if $\phi^* L$ is numerically equivalent to a semiample $\mathbb{Q}$-divisor, then $L$ is numerically equivalent to a semiample $\mathbb{Q}$-Cartier divisor. Hence it suffices to prove this theorem in the case $X$ is smooth. Moreover, by replacing $L$ with $mL$ for some positive integer $m$, we may assume $L$ is Cartier.

Now, if $n(L) = 0$ then $L \equiv 0$, and hence the result is trivial. If $n(L) = 1$, let $f : X \dashrightarrow Y$ be a nef reduction map of $L$ with general fiber $F$ such that $Y$ is smooth. Then we have $L|_F \equiv 0$. Let $\pi_0 : X_0 \to X$ be a resolution of indeterminacy of $f$ with $X_0$ smooth, and let $f_0 : X_0 \to Y$ be the induced map. Since the nef reduction is almost holomorphic, $F$ is still a general fiber of $f_0$. Therefore, by [LP20, Lemma 3.1], there is a diagram

$$\begin{tikzcd} X' \arrow[r, "\pi''"] & \tilde{X} \arrow[r, "\pi'"] \arrow[d, "f'"] & X_0 \arrow[r, "\pi_0"] \arrow[d, "f_0"] & X \\ & Y' \arrow[r, "\mu"] & Y & \end{tikzcd}$$

such that $\pi''$ $\pi'$ and $\mu$ are birational, $Y'$ is smooth, $\tilde{X}$ is the normalization of the main component of $X_0 \times_Y Y'$, $X'$ is a resolution of singularities of $\tilde{X}$, and there is a $\mathbb{Q}$-divisor


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$D'$ on $Y'$ such that $f'^*D' \equiv \pi'^*\pi_0^*L$ (in fact, since $\dim Y = 1$, $Y' = Y$ and $\mu$ is the identity map). Let $f'' := f' \circ \pi'' : X' \to Y'$ and $\pi'' := \pi'' \circ \pi' : X' \to X_0$, we still have $f''^*D' \equiv \pi''^*\pi'^*\pi_0^*L$. Therefore, by [LP20, Lemma 2.14], to show $L$ is numerically equivalent to a semiample $\mathbb{Q}$-divisor, it suffices to show that $D'$ is numerical equivalent to a semiample $\mathbb{Q}$-divisor. Note that $\dim(Y') = n(L) = 1$. Thus, any numerically non-trivial nef divisor on $Y'$ is ample. Hence $D'$ is ample and we are done. □

**Corollary 5.1.3.** *$N_{d,k,q,1}$ is true if $k \geq 0$.*

*Proof.* By [KM98, Thoerem 5.22], if $(X, \Delta)$ is a klt pair, then $X$ has rational singularities. Hence this result immediately follows from Lemma 5.1.2. □

Using the similar idea of the above proof, with Ambro's classification of nef divisor on smooth projective surfaces with maximal nef dimension([Amb05, Theorem 0.3]), and Hashizume's theorem for a generalization of Iitaka Conjecture under the assumption that general fibres have abundant canonical divisor ([Has20, Theorem 1.4]), we can prove $N_{d,k,q,2}$ is true if $d \leq 5$:

**Lemma 5.1.4.** *Let $(X, \Delta)$ be a klt pair of normal projective variety. Let $L$ be a nef $\mathbb{Q}$-Cartier divisor on $X$ with $n(L) \leq 2$ and $f : X \dashrightarrow Y$ be a nef reduction map with respect to $L$ with general fiber $F$. Assume $\dim F \leq 3$. Let $\Delta_F$ be the $\mathbb{Q}$-divisor on $F$ defined by $(K_X + \Delta)|_F = K_F + \Delta_F$. Suppose that $\kappa(F, K_F + \Delta_F) \geq 0$, then $K_X + \Delta + tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor for $t \gg 0$. In particular, if $\kappa(K_X + \Delta) \geq 0$, then for every $t \geq 0$ $K_X + \Delta + tL$ numerically equivalent to an effective $\mathbb{Q}$-divisor*

*Proof.* This statement is trivial if $n(L) = 0$, so we may assume $n(L) \geq 1$. As in the proof of Lemma 5.1.2, we may assume $L$ is Cartier and there is a diagram

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \\ Y' & \end{tikzcd}$$

such that $X'$ is and $Y'$ are smooth, $\pi$ is birational, $f'$ is an algebraic fiber space with general fibers whose general fibers $F'$ is birational to $F$, and there is a $\mathbb{Q}$-divisor $D'$ on $Y'$ such that


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$f'^*D' \equiv \pi^*L$. Moreover, by replacing $X'$ with a log resolution of $\mathrm{Exc}(\pi) \cup \pi^{-1}(\mathrm{Supp}(\Delta))$, we may assume $\mathrm{Supp}(\Delta') \cup \mathrm{Exc}(\pi)$ is simple normal crossing, where $\Delta'$ be the proper transform of $\Delta$ on $X'$.

Note that $D'$ is nef with $n(D') = n(L) = n(\pi^*L) \leq 2$. Now, we write

$$K_{X'} + \Delta' + E^- = \pi^*(K_X + \Delta) + E^+,$$

where both $E^+, E^-$ are effective $\pi$-exceptional $\mathbb{Q}$-divisors such that $\mathrm{Supp}E^+$ and $\mathrm{Supp}E^-$ has no common components. Note that $(X', \Delta' + E^-)$ is klt since $\Delta' + E^-$ has snc support and every coefficient of $\Delta' + E^-$ is less than 1 (because we had assume $(X, \Delta)$ is klt). Moreover, since $\dim F \leq 3$ and

$$(K_{X'} + \Delta' + E^-)|_{F'} = (\pi^*(K_X + \Delta) + E^+)|_{F'} = \pi^*(K_F + \Delta_F) + E^+|_{F'}$$

has non-negative Kodaira dimension, $(K_{X'} + \Delta' + E^-)|_{F'}$ is abundant. Therefore, by [Has20, Theorem 1.4], for any $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor $M$ on $Y'$, we have

$$\kappa(X', K_{X'} + \Delta' + E^- - f'^*K_{Y'} + f'^*M) \geq \kappa(F', (K_{X'} + \Delta' + E^-)|_{F'})) + \kappa(Y', M) \geq \kappa(Y', M).$$

Now, we consider the question case by case on $n(L) = \dim Y'$. In the case $n(L) = 1$, $Y'$ is a curve and hence $D'$ is ample. This shows that for $t \gg 0$

$$\kappa(X', K_{X'} + \Delta' + E^- - f'^*K_{Y'} + f'^*(K_{Y'} + tD')) \geq \kappa(Y', K_{Y'} + tD') = 1.$$

Therefore, $K_{X'} + \Delta' + E^- + t\pi^*L$ is numerically equivalent to the $\mathbb{Q}$-effective divisor $K_{X'} + \Delta' + E^- - f'^*K_{Y'} + f'^*(K_{Y'} + tD')$. Note that we have

$$K_{X'} + \Delta' + E^- + t\pi^*L + E_1 = \pi^*(K_X + \Delta + tL) + E_2$$

for some $\pi$-exceptional effective $\mathbb{Q}$-divisors $E_1$ and $E_2$. This implies $\pi^*(K_X + \Delta + tL) + E_2 + L'_0$ is effective for some numerically trivial $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L'_0$. By [KM98,


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Theorem 5.22], both $X$ and $X'$ has rational singularities, hence there exists a numerically trivial $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L_0$ on $X$ such that $\pi^* L_0 = L_0'$. This means $\pi^*(K_X + \Delta + tL + L_0) + E_2$ is effective, which implies $K_X + \Delta + tL + L_0$ is effective, hence we are done.

In the case $n(L) = 2$, since $Y'$ is smooth and $\dim Y' = n(D') = 2$, by [Amb05, Theorem 0.3], there are three possible cases:

(1) $D'$ itself is big (hence $K_{Y'} + tD'$ is big for $t \gg 0$);

(2) $D'$ is not big, but $K_{Y'} + tD'$ is big for all $t > 2$;

(3) there is a birational morphism $\mu : Y' \to Y_0$ between smooth projective surfaces such that $sD' \equiv \mu^*(-K_{Y_0})$ for some positive rational number $s$, and $D'$ is algebraically equivalent (hence numerically equivalent) to an effective $\mathbb{Q}$-divisor $D_e$.

Now, in the first two cases, $K_{Y'} + tD'$ is big for $t \gg 0$, hence we have

$$\kappa(X', K_{X'} + \Delta' + E^- - f'^* K_{Y'} + f'^*(K_{Y'} + tD')) \geq \kappa(Y', K_{Y'} + tD') = 2.$$

Therefore, $K_{X'} + \Delta' + E^- + t\pi^* L$ is numerically equivalent to the $\mathbb{Q}$-effective divisor $K_{X'} + \Delta' + E^- - f'^* K_{Y'} + f'^*(K_{Y'} + tD') = K_{X'} + \Delta' + E^- + tf'^* D'$. Hence by the same argument of above, we have $K_X + \Delta + tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor.

In the last case, we have $K_{Y'} + (-\pi^* K_{Y_0}) = G$ is effective because $Y_0$ is smooth. Moreover, we have

$$G + (t - s)D_e = K_{Y'} + (-\pi^* K_{Y_0}) + (t - s)D_e \equiv K_{Y'} + tD'.$$

Note that $M := G + (t - s)D_e = K_{Y'}$ is effective if $t \gg 0$. Thus, by [Has20, Theorem 1.4] again, we have

$$\kappa(X', K_{X'} + \Delta' + E^- - f'^* K_{Y'} + f'^* M) \geq \kappa(Y', M) \geq 0.$$

Since $f'^* M \equiv f'^*(K_{Y'} + tD') \equiv f'^* K_{Y'} + t\pi^* L$, $K_{X'} + \Delta' + E^- + t\pi^* L$ is numerically


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equivalent to the effective $\mathbb{Q}$-divisor $K_{X'}+\Delta'+E^- - f'^*K_{Y'}+f'^*M$, hence by the same argument of above, $K_X+\Delta+tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor.

For the last statement, if $t=0$ then the statement is trivial. If $t>0$, then pick $s\gg t>0$, we have

$$K_X+\Delta+tL=\frac{t}{s}((\frac{s}{t}-1)(K_X+\Delta)+K_X+\Delta+sL).$$

Since $K_X+\Delta$ is $\mathbb{Q}$-effective and $K_X+\Delta+sL$ is numerically equivalent to an effective divisor for $t\gg 0$, hence we are done. □

The following corollary is directly follows from Lemma 5.1.4.

**Corollary 5.1.5.** *Let $(X,\Delta)$ be a klt pair of normal projective variety such that $K_X+\Delta$ is pseudoeffective, and $L$ be a nef Cartier divisor on $X$.*

*(1) If $\dim=3$ and $0\le n(L)\le 2$, then for every $t\ge 0$, $K_X+\Delta+L$ is numerically equivalent to an effective $\mathbb{Q}$-divisor;*

*(2) If $\dim=4$ and $1\le n(L)\le 2$, then for $t\gg 0$, $K_X+\Delta+tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor;*

*(3) If $\dim=4$, $1\le n(L)\le 2$, and $\kappa(K_X+\Delta)\ge 0$, then for every $t\ge 0$, $K_X+\Delta+tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor;*

*(4) If $\dim=5$ and $n(L)=2$, then for $t\gg 0$, $K_X+\Delta+tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor;*

*(5) If $\dim=5$,$n(L)=2$, and $\kappa(K_X+\Delta)\ge 0$, then for every $t\ge 0$, $K_X+\Delta+tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor.*

*Remark* 5.1.6. Our proof of Lemma 5.1.4 is different from the proof in [Cha23c], although our proof is also obtained by the nef reduction. The difference is as the following: the proof in [Cha23c] is using the nef reduction of $K_X+B+L$, which implies $(K_X+B)|_F\equiv 0\equiv L|_F$ if $F$ is a general fibre of the nef reduction. Hence it can be shown that a good


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birational modification of the nef reduction with respect to $K_X + B + L$ is a klt-trivial fibration. Therefore, by using the theory of klt-trivial fibration and Zariski decomposition, one can deduce $K_X+B+L$ is numerically equivalent to a semiample $\mathbb{Q}$-divisor. However, in our proof $K_X+B+L$ may not be nef and we are using the nef reduction of $L$, hence we can not control the restriction of $K_X + B$ on the general fibre $F$, except we know $K_X + B$ is $\mathbb{Q}$-effective if $\dim F \leq 3$ by the usual Abundance conjecture in dimension 3. Also, although by [LP20, Theorem 4.1], one can take a birational modification $\pi : X \dashrightarrow X'$ to make $K_{X'} + \pi_*\Delta + t\pi_*L$ being nef for $t \gg 0$, but in this case, we can not control the nef dimension of $K_{X'} + \pi_*\Delta + t\pi_*L$.

In the following, we will prove the Generalized Nonvanishing Conjecture holds for nef divisor with maximal nef dimension if the relative dimension of the Albanese map is at most 2 over its image. In particular, if $\dim X = 3$ with $q > 0$. Before we start the proof, we need some lemmas obtained by standard methods in the minimal model program.

**Lemma 5.1.7.** *Let $(X, \Delta)$ be a klt pair of normal projective variety, $f : X' \to X$ be a log resolution of $(X, \Delta)$, and $\Delta'$ be the proper transform of $\Delta$ on $X'$. Write*

$$K_{X'} + \Delta' + E^- = f^*(K_X + \Delta) + E^+$$

*for some $f$-exceptional effective $\mathbb{Q}$-divisors $E^+, E^-$ such that* $\mathrm{Supp}E^+$ *and* $\mathrm{Supp}E^-$ *has no common components. Let $L$ be a $\mathbb{Q}$-Cartier divisor on $X$. Suppose $K_{X'}+\Delta'+E^-+f^*L$ is $\mathbb{Q}$-effective (resp. numerically equivalent to an effective $\mathbb{Q}$-divisor), then $K_X + \Delta + L$ is $\mathbb{Q}$-effective (resp. numerically equivalent to an effective $\mathbb{Q}$-divisor). In particular, since $(X', \Delta' + E^-)$ is a smooth klt pair with $\kappa(K_{X'} + \Delta' + E^-) = \kappa(K_X + \Delta)$, to prove $N_{d,k,q,n}$, it suffices to show $N_{d,k,q,n}$ holds for smooth pair $(X, \Delta)$.*

*Proof.* We have

$$K_{X'} + \Delta' + E^- + f^*L = f^*(K_X + \Delta + L) + E^+.$$

If $K_{X'}+\Delta'+E^-+f^*L = f^*(K_X+\Delta+L)+E^+$ is $\mathbb{Q}$-effective, then $K_X+\Delta+L$ is also $\mathbb{Q}$-effective since $E^+$ is $f$-exceptional. If $K_{X'}+\Delta'+E^-+f^*L$ is numerically equivalent


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to an effective $\mathbb{Q}$-divisor, there exists a numerically trivial $\mathbb{Q}$-divisor $L_0'$ on $X'$ such that $f^*(K_X+\Delta+L)+E^+ + L_0'$ is an effective $\mathbb{Q}$-Cartier divisor. Again, since both $X$ and $X'$ have rational singularities, there exists a numerically trivial $\mathbb{Q}$-divisor on $L_0$ on $X$ such that $f^*L_0 = L_0'$. Thus, $f^*(K_X+\Delta+L+L_0)+E^+ = f^*(K_X+\Delta+L)+E^+ + L_0'$ has non-negative Kodaira dimension, and so does $K_X+\Delta+L+L_0$. Hence $K_X+\Delta+L \equiv K_X+\Delta+L+L_0$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. ☐

**Lemma 5.1.8.** *Let $(X,\Delta)$ be a $\mathbb{Q}$-factorial klt pair, $R$ be a $(K_X+\Delta)$-negative extremal ray of $X$, such that the contraction with respect to $R$ is not of fiber type. Let $\pi : X \dashrightarrow Y$ be the corresponding divisorial contraction or $(K_X+\Delta)$-flip. Let $L$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ such that $L.R \leq 0$, and $t$ is a non-negative rational number, then we have*

$$\kappa(K_X+\Delta+tL) = \kappa(K_Y+\pi_*\Delta+t\pi_*L).$$

*In particular, if $K_Y+\pi_*\Delta+t\pi_*L$ is big, then so does $K_X+\Delta+tL$.*

*Proof.* Note that $(Y,\pi_*\Delta)$ is also klt $\mathbb{Q}$-factorial. If $\pi$ is a flip then $X$ and $Y$ are isomorphic in codimension 2, hence the equality is trivial. Therefore, we may assume $\pi$ is a divisorial contraction. In this case, we can write $\pi^*\pi_*L+E_L = L$ for some $\pi$-excpetional $\mathbb{Q}$-divisor $E_L$. Since $L.R \leq 0$, $E_L$ is effective by the negativity lemma. Let $\Delta = \Delta_1+\Delta_2$ such that both $\Delta_1,\Delta_2$ are effective, Supp$\Delta_1$ and Supp$\Delta_2$ have no common component, every component of $\Delta_1$ is not contracted by $\pi$, and $\Delta_2$ is $\pi$-exceptional. Then $\Delta_1$ is the proper transform of $\pi_*\Delta$ on $X$, and we have

$$\begin{aligned} K_X+\Delta+tL &= K_X+\Delta_1+tL+\Delta_2 \\ &= \pi^*(K_X+\pi_*\Delta)+E+tL+\Delta_2 \\ &= \pi^*(K_Y+\pi_*\Delta+t\pi_*L)+(E+tE_L+\Delta_2) \end{aligned}$$

for some $\pi$-exceptional $\mathbb{Q}$-divisor $E$. Note that $E$ itself may not be effective, but we have $(E+tE_L+\Delta_2).R = (K_X+\Delta+tL).R < 0$. Since $\Delta_2$ is also $\pi$-exceptional, by the


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negativity lemma again we conclude that $tE_L + E + \Delta_2$ is effective. Hence the equality

$$\kappa(K_X + \Delta + tL) = \kappa(K_Y + \pi_*\Delta + t\pi_*L)$$

holds. □

**Corollary 5.1.9.** *Let $(X, \Delta)$ be a 2-dimensional klt pair such that $\kappa(K_X + \Delta) \geq 0$, and $L$ be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ with $n(L) = 2$, then $K_X + \Delta + tL$ is big for $t \gg 0$.*

*Proof.* By replacing $L$ with $mL$ for some sufficiently divisible integer $m$, we may assume $L$ is Cartier. First, we consider the special case that $K_X + \Delta + sL$ is nef for some $s \geq 0$. Since $L$ is nef with $n(L) = 2$, for $t \gg 0$ we have $K_X + \Delta + tL$ is nef wit $n(K_X + \Delta + tL) = 2$, so by [LP20, Corollary C], $K_X + \Delta + tL \equiv M$ for some semiample $\mathbb{Q}$-divisor $M$ with $n(M) = 2$. Since the nef dimension of a semiample $\mathbb{Q}$-divisor is equivalent to the Kodaira dimension, which implies $M$ is big, hence so does $K_X + \Delta + tL$.

In general, by [LP20, Proposition 4.1], there is an $L$-trivial $(K_X + \Delta)$-MMP $\pi : X \to Y$ such that $K_Y + \pi_*\Delta_Y + s\pi_*L$ is nef for some $s > 4$. Moreover, since $\pi$ is $L$-trivial, which implies $\pi^*\pi_*L \equiv L$ and hence $\pi_*L$ is nef with $n(\pi_*L) = 2$. This implies for $t \gg 0$, $K_Y + \pi_*\Delta_Y + t\pi_*L$ is nef with $n(K_Y + \pi_*\Delta_Y + t\pi_*L) = 2$, hence by the same argument as above, $K_Y + \pi_*\Delta_Y + t\pi_*L$ is big. By Lemma 5.1.8, this implies $K_X + \Delta + tL$ is big, which completes our proof. □

Now we can prove the following proposition:

**Proposition 5.1.10.** *Let $(X, \Delta)$ be klt pair of a normal projective variety such that $K_X + \Delta$ is pseudoeffective and $q(X) > 0$. Let $\alpha : X \to A(X)$ be the Albanese map and $F$ be a general fibre of $\alpha$. If* $\dim F \leq 2$*, then for any nef $\mathbb{Q}$-Cartier divisor $L$ with maximal nef dimension, there is a numerically trivial $\mathbb{Q}$-Cartier divisor $D$ on $A$ such that $\kappa(K_X + \Delta + tL + \alpha^*D) \geq 0$ for $t \gg 0$. In particular, if $\kappa(K_X + \Delta) \geq 0$, then for every $t \geq 0$, $K_X + \Delta + tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor.*

*Proof.* Let $f : X' \to (X, \Delta)$ be a log resolution of $(X, \Delta)$, and write $K_{X'} + \Delta' + E^- = f^*(K_X + \Delta) + E^+$ as before (here $\Delta'$ is the proper transform of $\Delta$ on $X'$). Let $\alpha'$ :


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$X' \to A(X')$ be the Albanese map of $X'$. Since $X$ has rational singularities, by [BS95, Lemma 2.4.1] we have $A(X') = A(X)$ and $\alpha' = \alpha \circ f$. Assume our result holds for $(X', \Delta' + E^-)$ and $f^*L$, that is, $\kappa(K_{X'} + \Delta' + E^- + tf^*L + \alpha'^*D) \geq 0$ for $t \gg 0$. Since $K_{X'} + \Delta' + E^- + tf^*L + \alpha'^*D = K_{X'} + \Delta' + E^- + f^*(tL + \alpha^*D)$, then by Lemma 5.1.7, $K_X + \Delta + tL + \alpha^*D$ also has non-negative Kodaira dimension. Therefore, we can replace $(X, \Delta)$ and $L$ by $(X', \Delta' + E^-)$ and $f^*L$, hence we may assume $(X, \Delta)$ is smooth.

Let $F$ be a general non-empty fiber of $\alpha$. By the generic smoothness, we may assume every connected component of $F$ is smooth and irreducible. Also, by [BC15, Theorem 4.1], it suffices to show that $K_F + \Delta_F + tL_F$ is relatively big for $t \gg 0$, where $\Delta_F$ is defined by $(K_X + \Delta)|_F = K_F + \Delta_F$, and $L_F := L|_F$. Note that if $\kappa(K_X + \Delta) \geq 1$, then $\alpha$ may not be surjective and possibly has disconnected fibers, but [BC15, Theorem 4.1] does not require the morphism to be surjective or has connected fibers.

Since we assume $L$ is of maximal nef dimension, $L_F$ is also of maximal nef dimension on $F$, hence there is at least an irreducible component of $F$, say $Z$, satisfying that $\dim Z = \dim F$ and $L_Z := L|_Z$ is still of maximal nef dimension. Also, we have $K_F|_Z = K_Z$ since $Z$ is a smooth connected component of $F$. Defining $\Delta_Z := \Delta_F|_Z$, then we have $K_Z + \Delta_Z + tL_Z = (K_F + \Delta_F + tL_F)|_Z$.

Now, if $\dim Z = 0$, then $\alpha$ is a generically finite morphism over its image, and hence any divisors are always relatively big. If $\dim Z = 1$, then $Z$ is a smooth projective curve. Therefore, we have $K_Z + tL_Z$, and hence $K_Z + \Delta_Z + tL_Z$ is ample for $t \gg 0$, which implies $K_F + \Delta_F + tL_F$ is big. If $\dim Z = 2$, then $Z$ is a smooth projective surface with $\kappa(K_Z + \Delta_Z) \geq 0$ since $K_X + \Delta$ is pseudoeffecitve. Thus, by Corollary 5.1.9, $K_Z + \Delta_Z + tL_Z$ is big for $t \gg 0$, which also implies the bigness of $K_X + \Delta_F + tL_F$. Hence we proved the bigness of $K_F + \Delta_F + tL_F$. $\square$

**Corollary 5.1.11.** *Let $(X, \Delta)$ be a klt pair of normal projective variety such that $K_X + \Delta$ is pseudo-effective, and $L$ be a nef divisor on $X$ such that $n(L) = \dim X$. Then for every $t \geq 0$, $K_X + \Delta + tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor if one of the following conditions holds:*


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1. $\dim X \leq 3$ *and* $q(X) > 0$;

2. $\kappa(K_X + \Delta) = 0$ *and* $\dim X - q(X) \leq 2$.

*Proof.* The first case immediately follows from Proposition 5.1.10 by the usual Nonvanishing Theorem in dimension 3. For the second case, we only need to show the Albanese map is surjective (If $X$ is smooth and $\Delta = 0$, it is well-known that $\alpha$ is an algebraic fibre space by [Kaw81]). Let $\alpha : X \to A(X)$ be the Albanese morphism, $A_0$ be the normalization of $\mathrm{Im}\alpha$, and $X \to A'$ be the Stein factorization of $X \to A_0$ (this is a morphism since $X$ is normal). By [CP17, Theorem 4.22] and the last remark in [HPS18, Section 1], we can prove the inequality in [HPS18, Theorem 1.1] holds when $X$ replaced by a klt pair $(X, \Delta)$ (see Remark 5.1.12). This implies $\kappa(A') \leq 0$. On the other hand, $A'$ has maximal Albanese dimension, which implies $\kappa(A') \geq 0$. Hence $\kappa(A') = 0$ and the image of $A'$ in $A$, which is just $\mathrm{Im}\alpha$, is an abelian subvariety of $A$. By the universal property of Albanese, we conclude that $A' = \mathrm{Im}\alpha = A(X)$ and hence $\alpha$ is an algebraic fibre space over $A(X)$, in particular, $\alpha$ is surjective. □

*Proof of Theorem 5.1.1.* The four cases in Theorem 5.1.1 immediately follow from Corollary 5.1.3, Corollary 5.1.5 and Corollary 5.1.11. □

*Remark* 5.1.12. As mentioned in [HPS18], by using [CP17, Theorem 4.22], it is easy to show the inequality in [HPS18, Theorem 1.1] holds for klt pairs by making a very little modification of the proof of the last part of [HPS18, Lemma 5.1]. For the convenience of readers, we write down the details here: Let $(X, \Delta) \to Y$ be an algebraic fibre space between normal projective varieties with general fibre $F$ such that $(X, \Delta)$ is a klt pair of with $\kappa(K_X + \Delta) = 0$ and $Y$ has maximal Albanese dimension. Let $\pi : X' \to X$ be a log resolution of $(X, \Delta)$, write $K_{X'} + \Delta' = \pi^*(K_X + \Delta) + E^+ - E^-$ (where $\Delta'$ is the proper transform of $\Delta$ on $X'$). Replace $(X, \Delta)$ with $(X', \Delta' + E^-)$, and replace $F$ by $F'$ (here $F'$ is the general fibre of $X' \to Y$), we have $(X', \Delta' + E^-)$ is klt, the equality

$$\kappa(X', K_{X'} + \Delta' + E^-) = \kappa(X', \pi^*(K_X + \Delta) + E^+) = \kappa(X, K_X + \Delta)$$


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holds, and the inequality

$$\begin{aligned}\kappa(F', (K_{X'} + \Delta' + E^-)|_{F'}) &= \kappa(F', (\pi^*(K_X + \Delta) + E^+)|_{F'}) \\ &\geq \kappa(F', (\pi^*(K_X + \Delta))|_{F'}) = \kappa(F, (K_X + \Delta)|_F)\end{aligned}$$

also holds. Thus, if the inequality in [HPS18, Theorem 1.1] holds for $(X', \Delta'+E^-) \to Y$, then so does $(X, \Delta) \to Y$, hence we may assume the pair $(X, \Delta)$ is smooth.

Let $Y_0 := \mathrm{Im}(Y \to A(Y))$ and $Y \to Y'$ be the Stein factorization of $Y \to Y_0$. Since $Y \to Y_0$ is generically finite, we have $Y \to Y'$ is birational, $Y'$ is normal, $X \to Y'$ is an algebraic fibre space, and $Y' \to A(Y)$ is finite over its image (which is just $Y_0$). By [Kaw81, Theorem 13], there is an étale covering $Y'' \to Y'$ such that $Y'' \cong Z \times K$, where $Z$ is of general type with $\kappa(Z) = \kappa(Y') = \kappa(Y)$ and $K$ is an abelian variety. Let $X'' := X \times_{Y'} Y''$, since $Y'' \to Y'$ is étale, so does the induced may $g : X'' \to X$. Hence $X''$ is smooth since $X$ is. Let $\Delta'' := g^*(\Delta)$, then we have $K_{X''} + \Delta'' = g^*(K_X + \Delta)$ and $(X'', \Delta'')$ is klt since $g$ is étale. Consider the morphism $h : (X'', \Delta'') \to Z$ induced by $X'' \to Y'' = K \times Z$, and let $G$ be a general fibre of $h$. Then we have the induced morphism $G \to K$, which has general fibre $F$. Since $(G, \Delta''|_G)$ is klt, by [CP17, Theorem 4.22], we have

$$\kappa(G, (K_{X''} + \Delta'')|_G) = \kappa(G, K_G + \Delta''|_G) \geq \kappa(F, (K_{X''} + \Delta'')|_F) = \kappa(F, (K_X + \Delta)|_F).$$

Since $Z$ is of general type, by [Fuj17, Theorem 1.7] (or [Cam04, Section 4]), we have

$$\kappa(X'', K_{X''} + \Delta'') \geq \kappa(G, (K_{X''} + \Delta'')|_G) + \dim Z \geq \kappa(F, (K_X + \Delta)|_F) + \dim Z.$$

Since $\kappa(X'', K_{X''} + \Delta'') = \kappa(X, K_X + \Delta)$ and $\dim Z = \kappa(Y') = \kappa(Y)$, the proof is completed.

*Remark* 5.1.13. If $N_{d,k,q,n}$ is true, then we can show $K_X + \Delta + tL$ is numerically equivalent to an effective $\mathbb{Q}$-divisor for klt pair $(X, \Delta)$ with $\dim X = d$, $\kappa(K_X + \Delta) = k$, $n(L) = n$, and $\tilde{q}(X) = q$ in the following way: By Lemma 5.1.7, we may assume $(X, \Delta)$ is smooth. Let $\pi : (X', \Delta') \to (X, \Delta)$ be a quasi-étale covering (which is étale by the Nagata-


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Zariski purity theorem and the smoothness of $X$) with $q(X') = \tilde{q}(X)$. Then $(X', \Delta')$ is klt with $K_{X'} + \Delta' = \pi^*(K_X + \Delta)$, hence $\kappa(K_{X'} + \Delta') = \kappa(K_X + \Delta)$. By our assumption, $K_{X'}+\Delta'+t\pi^*L = \pi^*(K_X+\Delta+tL)$ is numerically equivalent to an effective divisor $D$. Now, follows the argument in [WZ21, Corllary 4.2], since $K_X + \Delta + tL = \frac{1}{\deg \pi}\pi_*(K_{X'} + \Delta' + t\pi^*L)$ is weak numerically equivalent $\frac{1}{\deg \pi}\pi_* D$, by the smoothness of $X$, they are numerically equivalent. Hence we have

$$K_X + \Delta + tL = \frac{1}{\deg \pi}\pi_*(K_{X'} + \Delta' + t\pi^*L) \equiv \frac{1}{\deg \pi}\pi_* D \geq 0.$$

## 5.2 The Iitaka Conjecture in dimension 7

Using the Generalized Nonvanishing Conjecture, we can prove the following result about algebraic fibre spaces with general fibres has Kodaira dimension zero, by making a little modification on the proof in [Bir09]:

**Theorem 5.2.1.** *Let $f : X \to Y$ be an algebraic fibre space between smooth projective varieties with general fibre $F$. Assume $\kappa(F) = 0$ and $N_{d,k,q}$ holds for $d = \dim Y$, $k = \kappa(Y)$, and $q = q(Y)$, then we have $\kappa(X) \geq \kappa(Y)$. In particular, $C_{n,3}$ holds if $\kappa(F) = 0$ and either $\kappa(Y) \geq 1$, or $\kappa(Y) = 0$ but $q(Y) > 0$.*

*Proof of Theorem 5.2.1.* Since this statement is trivial if $\kappa(Y) = -\infty$, we may assume $\kappa(Y) \geq 0$. By [FM00, Theorem 4.5], there is a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

such that $X'$ and $Y'$ are smooth, $\pi$ and $\mu$ are birational morphism, and $f'$ is an algebraic fibre space with SNC branch. Let $p > 1$ be a positive integer such that $f'_*\mathcal{O}_{X'}(pK_{X'}) \neq 0$, then there exists an effective $\mathbb{Q}$-divisor $B$ and a nef $\mathbb{Q}$-divisor $L$ on $Y'$ satisfies the followings:

(1) $pK_{X'} = pf'^*(K_{Y'} + B + L) + R$ for some $\mathbb{Q}$-divisor $R$ on $X'$.


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(2) If we write $R = R^+ - R^-$ with $R^+, R^-$ both effective and without common components, then $R^-$ is both $\pi$-exceptional and $f'$-exceptional.

Since we assume $N_{d,k,q}$ holds for $d = \dim Y = \dim Y'$, $k = \kappa(Y) = \kappa(Y')$, and $q = q(Y) = q(Y')$, for any non-negative rational number $j$, $K_{Y'} + jL$ lies in the effective cone $\mathrm{Eff}(Y')$. In particular, since $B$ is effective, for any $j \geq 0$, there is an effective $\mathbb{Q}$-divisor $D_j$ such that $D_j \equiv K_{Y'} + B + jL =: L_j$.

Let $i, j \gg 0$ be sufficiently divisible integers, and $M := \pi_* f'^*(D_j - L_j) \equiv 0$. Since $R^-$ is $\pi$-exceptional, there exist some effective $\pi$-exceptional divisors $E^+, E^-$ such that

$$i\pi^*(jpK_X + pM) + E^+ - E^- = i(jpK_{X'} + jR^- + pf'^*(D_j - L_j)).$$

Thus, we have

$$\begin{aligned}
H^0(i(jpK_X + pM)) &= H^0(i\pi^*(jpK_X + pM)) = H^0(i\pi^*(jpK_X + pM) + E^+) \\
&\geq H^0(i\pi^*(jpK_X + pM) + E^+ - E^-) \\
&= H^0(i(jpK_{X'} + jR^- + pf'^*(D_j - L_j))) \\
&= H^0(i(jpf'^*(K_{Y'} + B + L) + jR^+ + pf'^*(D_j - K_{Y'} - B - jL))) \\
&= H^0(i((j-1)p(K_{Y'} + B) + jR^+ + pD_j)) \\
&\geq H^0(i(j-1)pK_{Y'}).
\end{aligned}$$

By [CP11, Theorem 3.1], we have

$$H^0(ijpK_X) \geq H^0(ijpK_X + ipM) = H^0(i(jpK_X + pM)) \geq H^0(i(j-1)pK_{Y'}).$$

Since $i$ can be taken arbitrarily large and sufficiently divisible, this implies

$$\kappa(X) = \kappa(X, jpK_X) \geq \kappa(Y, (j-1)pK_Y) = \kappa(Y),$$

which completes the proof. □

*Remark* 5.2.2. The above proof also shows that even if we only have $K_Y + B + jL$ is


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numerically $\mathbb{Q}$-effective for some fixed $j \in \mathbb{Q}_{>1}$, the result of the theorem still holds.

As a corollary, we can prove $C_{7,m}$ is true unless $m = 3$ and the base has $\kappa = q = 0$.

**Theorem 5.2.3.** *Let $f : X \to Y$ be an algebraic fibre space between smooth projective varieties, and $F$ be a general fibre of $f$. Let $n = \dim X$ and $m = \dim Y$. If $n \leq 7$, then $C_{n,m}$ holds unless $n = 7$, $m = 3$, $\kappa(F) = \kappa(Y) = 0$, and $q(X) = q(F) = \tilde{q}(Y) = 0$ . In particular, $C^-_{n,m}$ holds if $n \leq 7$.*

*Proof.* We may assume $\kappa(F) \geq 0$ and $\kappa(Y) \geq 0$ (otherwise, this statement is trivial). By [Cao18] and [Kaw82] (or [CP17] and [HPS18]), $C_{n,m}$ holds if $m \leq 2$. By [Kaw85], $C_{n,m}$ holds if $F$ has a good minimal model. In particular, by [Lai11, Theorems 4.4 and 4.5], [Kaw81, Theorem 1], and the existence of good minimal model in dimension 3, $C_{n,m}$ holds if either $\dim F - \kappa(F) \leq 3$, or in the case $\kappa(F) = 0$ and $\dim F - q(F) \leq 3$. This implies $C_{n,m}$ holds if either $n - m \leq 3$, or $n - m = 4$ but eithre $\kappa(F) \geq 1$ or $\kappa(F) = 0$ with $q(F) \geq 1$. Therefore, when $n \leq 7$, $C_{n,m}$ is confirmed except for the case $n = 7, m = 3$, and $\kappa(F) = q(F) = 0$.

Now, by Theorem 5.2.1, if either $\kappa(Y) > 0$ or $q(Y) > 0$, then we have $\kappa(X) \geq \kappa(Y)$, hence we are done. If $\kappa(Y) = q(Y) = 0$ but $\tilde{q}(Y) > 0$, then there exists a quasi-étale covering $Y' \to Y$ (hence étale by the Nagata-Zariski purity theorem and the smoothness of $Y$) with $q(Y') = \tilde{q}(Y) > 0$. Let $X' := X \times_Y Y'$, then $X' \to X$ is also étale (hence $X'$ is smooth), and the morphism $X' \to Y'$ induced by the base change is an algebraic fiber space between smooth projective varieties with general fiber $F$. Since the Kodaira dimension is invariant under étale covering, by Theorem 5.2.1 again, we have the inequality

$$\kappa(X) = \kappa(X') \geq \kappa(F) + \kappa(Y') = \kappa(F) + \kappa(Y).$$

Finally, if $q(F) = \tilde{q}(Y) = 0$, then by [Fuj05, Theorem 1.6], $q(X) = 0$. This completes the proof. □

*Remark* 5.2.4. The remaining unknown case of $C_{7,m}$ satisfies the following conditions:

(1) $m = 3$;


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(2) $\kappa(F) = \kappa(Y) = q(X) = q(F) = \tilde{q}(Y) = 0$;

(3) In the decomposition $pK_{X'} = pf'^*(K_{Y'} + B + L) + R$ in the Fujino-Mori's canonical bundle formula, $\kappa(Y', K_{Y'} + B) = \nu(Y', K_{Y'} + B) = 0$ and $n(L) = 3$.

If one can prove $\kappa(X') \geq 0$ under the above assumptions, then the proof of $C_{7,m}$ is completed.


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# Chapter 6 Miscellanies

## 6.1 Computation of asymptotic base locus

In [EIM20, Corollary 3.5, Remark 3.6], Ejiri, Iwai, and Matsumura describe the asymptotic base locus of $-K_Y - D$ when $Y$ has at worst canonical singularities. In Theorem 4.1.1, Theorem 4.1.7, and Theorem 4.2.2, we proved the positivity of $-K_Y - D$ without assuming that $Y$ has at worst canonical singularities. Thus, it is natural to ask the following question:

**Question 6.1.1.** *Can we describe the asymptotic base locus of $-K_Y - D$ without assuming that $Y$ has at worst canonical singularities?*

It seems not easy to show that [EIM20, Corollary 3.5, Remark 3.6] is still true when $Y$ has klt singularities or worse (if it is true). Naively thinking, if $Y$ has at worst canonical singularities, then for any resolution $\mu : Y' \to Y$, we have $\mu^*(-K_Y) \geq -K_{Y'}$. Therefore, if we can compute the intersection number and/or asymptotic invariants of $-K_{Y'}$ on some birational model $Y'$ (for example, as the proof of [FG12, Theorem 4.1], which uses [Kaw98, Theorem 2] to compute $(-K_Y.C)$ on a higher birational model), then it will give a lot of information about asymptotic invariants of $-K_Y$. However, this approach does not work without assuming $Y$ has canonical singularities.

On the other hand, when the morphism is smooth, by using the similar idea of [FG12, Theorem 4.1], we have the following result:

**Proposition 6.1.2.** *Let $f : X \to Y$ be a smooth fibration between normal projective $\mathbb{Q}$-*


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*Gorenstein varieties. Assume $X$ has at worst klt singularities and $-K_X$ is semiample, then for any integral curve $C$ on $Y$, we have either $-K_Y.C \geq 0$, or $C \subset \mathrm{Sing}(Y)$.*

*Proof.* We use the same idea of proof in [FG12, Theorem 4.1] with a little modification. Consider the morphism $f : X \to Y$, the idea is the following: Let $C$ be a curve on $Y$ s,t, $C \not\subseteq \mathrm{Sing}(Y)$, we want to show $-K_Y.C \geq 0$. Let $\mu : Y' \to Y$ be a resolution of $Y$ s,t, $\mathrm{Exc}(\mu)$ is simple normal crossing and the center of $\mu$ not contains $C$. Write $K_{Y'} = \mu^*(K_Y) + E$, and let $C'$ be the strict transform of $C$ on $Y'$, we have $(-K_Y.C) = (\mu^*(-K_Y).C')$. So we can compute the intersection number on $Y'$. Then for some good birational modification $f' : X' \to Y'$ of $f$ s,t, there exists some $D \sim_{\mathbb{Q}} -K_{X'}$ satisfies Kawamata's positivity theorem ([Kaw98, Theorem 2]), by the theorem we have $0 \sim_{\mathbb{Q}} K_{X'} + D \sim f'^*(K_{Y'} + \Delta_0 + M)$ with $M$ is nef, $\Delta_0$ support on the image of vertical part of $D$. Thus $\mu^*(-K_Y) \sim_{\mathbb{Q}} M + \Delta_0 + E$. Hence if we can make good choice of resolution $f'$ and divisor $D$ s,t, $\Delta_0 + E$ is effective with $C' \not\subseteq \mathrm{Supp}(\Delta_0 + E)$, then we have

$$(-K_Y.C) = (\mu^*(-K_Y).C') = (M + \Delta_0 + E.C') \geq 0.$$

To do this, let $C \subseteq Y$ be an integral curve not contained in the singular locus. Now since $-K_X$ is semiample, there exist some general member $A \in |-mK_X|$ s,t, $\mathrm{Supp}(A) \cap f^{-1}(Y_{sm})$ is smooth and SNC, and $\mathrm{Supp}(f(A^v)) \not\supseteq C$. Let $U := Y_{sm} - \mathrm{Supp}(f(A^v))$, we have $C \cap U \neq \emptyset$ and $f|_{\mathrm{Supp}(A) \cap f^{-1}(U)} : \mathrm{Supp}(A) \cap f^{-1}(U) \to U$ is smooth. Note that since $X$ has klt singularities, so for $m >> 0$ by the generality we have $(X, \frac{1}{m}A)$ is also klt. Now let $\mu : Y' \to Y$ be a resolution of $Y$ s,t, $\mu$ is an isomorphism on $U$ and $\mu^{-1}(Y - U)$ is SNC. Denote $\mathrm{Exc}(\mu) = \cup E_i$ and write $K_{Y'} = \mu^* K_Y + E$. Let $\widetilde{X} := X \times_Y Y'$. Note that $\widetilde{f} : \widetilde{X} \to Y'$ is smooth since $f$ is, so since $Y'$ is smooth, so does $\widetilde{X}$. Consider the following diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & \widetilde{X} \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

Since $f$ and $\widetilde{f}$ are smooth, we have $K_{\widetilde{X}} = \varphi^*(K_X) + f^*E$. Now we have $-K_X \sim_{\mathbb{Q}}$


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$\frac{1}{m}A$, and we define $\widetilde{D}$ be a divisor on $\widetilde{X}$ by $K_{\widetilde{X}} + \widetilde{D} = \varphi^*(K_X + \frac{1}{m}A) \sim_{\mathbb{Q}} 0$, then we have $\widetilde{D} = \varphi^*\frac{1}{m}A - \widetilde{f}^*E$. Let $U' := \mu^{-1}U$, and let $\psi : X' \to \widetilde{X}$ be a resolution s,t, $\psi$ is isomorphism on $\widetilde{f}^{-1}(U')$ and $\mathrm{Supp}A' \cup \mathrm{Supp}f'^{-1}(Y' - U')$ is an SNC divisor, where $f' := \widetilde{f} \circ \psi : X' \to Y'$ and $A'$ is the strict transform of $A$ on $X'$. Now we define a divisor $D$ on $X'$ by $K_{X'} + D = \phi^*(K_X + \widetilde{D}) \sim_{\mathbb{Q}} 0$, then $D \sim_{\mathbb{Q}} -K_{X'}$.

Let $\mathrm{Exc}(\varphi) = \cup \widetilde{E}_i, \mathrm{Exc}(\psi) = \cup E'_i$, then

$$\begin{aligned} D &= \psi_*^{-1}\widetilde{D} - \sum_i a(E'_i, \widetilde{X}, \widetilde{D})E'_i \\ &= \psi_*^{-1}(\varphi^*\frac{1}{m}A - \widetilde{f}^*E) - \sum a(E'_i, \widetilde{X}, \widetilde{D})E'_i \\ &= \psi_*^{-1}(\varphi^*\frac{1}{m}A - \widetilde{f}^*E) - \sum a(E'_i, X, \frac{1}{m}A)E'_i \end{aligned}$$

Note that

$$\begin{aligned} \varphi^*\frac{1}{m}A &= K_{\widetilde{X}} - \varphi^*K_X + \varphi_*^{-1}\frac{1}{m}A - \sum a(\widetilde{E}_i, X, \frac{1}{m}A)\widetilde{E}_i \\ &= \widetilde{f}^*E + \varphi_*^{-1}\frac{1}{m}A - \sum a(\widetilde{E}_i, X, \frac{1}{m}A)\widetilde{E}_i \end{aligned}$$

So we have

$$\begin{aligned} D &= \psi_*^{-1}(\varphi^*\frac{1}{m}A - \widetilde{f}^*E) - \sum a(E'_i, X, \frac{1}{m}A)E'_i \\ &= \psi_*^{-1}(\widetilde{f}^*E + \varphi_*^{-1}\frac{1}{m}A - \sum a(\widetilde{E}_i, X, \frac{1}{m}A)\widetilde{E}_i - \widetilde{f}^*E) - \sum a(E'_i, X, \frac{1}{m}A)E'_i \\ &= \psi_*^{-1}(\varphi_*^{-1}\frac{1}{m}A - \sum a(\widetilde{E}_i, X, \frac{1}{m}A)\widetilde{E}_i) - \sum a(E'_i, X, \frac{1}{m}A)E'_i \\ &= \frac{1}{m}A' - \psi_*^{-1}(\sum a(\widetilde{E}_i, X, \frac{1}{m}A)\widetilde{E}_i) - \sum a(E'_i, X, \frac{1}{m}A)E'_i \end{aligned}$$

have all coefficient $< 1$ since $(X, \frac{1}{m}A)$ is klt. And by our construction $f'(D^v) \subset \mathrm{Exc}(\mu)$, thus $-\lceil D \rceil$ is effective and $\phi \circ \psi$-exceptional, hence $D$ satisfies condition 3 of Kawamata's Positivity theorem.

Now we have $0 \sim_{\mathbb{Q}} K_{X'} + D = f'^*(K_{Y'} + M + \Delta_0)$, and then $M + \Delta_0 \sim_{\mathbb{Q}} -K_{Y'} = \mu^*(-K_Y) - E$, thus $\mu^*(-K_Y) \sim_{\mathbb{Q}} M + \Delta_0 + E$. By the above discussion, since $M$ is


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nef, and by our construction $\mathrm{Supp}(\Delta_0 + E) \subseteq \mathrm{Exc}(\mu)$. So let $C'$ be the strict transform of $C$ on $Y'$, we have $C' \nsubseteq \mathrm{Supp}(\Delta_0 + E)$. Thus it remains to show that $\Delta_0 + E$ is effective. Write $E = \sum e_i E_i$ (here $e_i$ may be zero), we only need to check $\mathrm{mult}_{E_i}(E + \Delta_0) \geq 0$ for all $i$. By the Kawamata's Positivity theorem, the coefficient of $E_i$ in $\Delta_0$ is $1 - c_i$, where

$$c_i = \sup\{t \in \mathbb{Q} | D + tf'^* E_i \text{ is lc over the generic point of } E_i\}.$$

Since $K_{X'} + D = \phi^*(K_{\widetilde{X}} + \widetilde{D})$, we have

$$c_i = \sup\{t \in \mathbb{Q} | \widetilde{D} + t\widetilde{f}^* E_i \text{ is lc over the generic point of } E_i\}.$$

Thus $\mathrm{mult}_{E_i}(E + \Delta_0) = e_i + 1 - c_i$. Now we have $\widetilde{D} + t\widetilde{f}^* E_i = \varphi^* \frac{1}{m} A - \widetilde{f}^* E + t\widetilde{f}^* E_i$. Then since $\varphi^* \frac{1}{m} A$ is effective , so $c_i - e_i \leq 1$, hence $\mathrm{mult}_{E_i}(E + \Delta_0) = e_i + 1 - c_i \geq 0$. $\square$

However, a morphism being smooth is a very strong condition. Therefore, we may expect to generalize this theorem to general morphism. From the above result, we may expect the description of [EIM20] still holds in the general case. Nevertheless, we can not directly do the same computation in general cases. The reason is, in general, $X \times_Y Y'$ may not be normal, and even it can be reducible. Moreover, if we replace $X \times_Y Y'$ by the normalization of its main component, then we do not have the equality $K_{\widetilde{X}} = \varphi^*(K_X) + f^* E$, so we can not directly do the same computation as above.

## 6.2 Maximal rationally connected fibration

Another important property for varieties with positive anticanonical divisors is the maximal rationally (chain-)connected fibration. First, we give the following definition:

**Definition 6.2.1.** *Let $X$ be a variety.*

*(1) $X$ is called rationally connected (RC for short) if for any two very general points $x, y \in X$, there exists a rational curve $C \subset X$ such that $x, y \in C$.*


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*(2)* *$X$ is called rationally chain-connected (RCC for short) if for any two very general points $x, y \in X$, there exist finitely many rational curves $C_1, ..., C_n \subset X$ such that $C_i \cap C_{i+1} \neq \emptyset$, $x \in C_1$, and $y \in C_n$.*

From definition, rationally connectedness implies rationally chain-connectedness. Conversely, by [HM07, Corollary 1.8], if $(X, \Delta)$ is a dlt pair and $X$ is rationally chain-connected, then $X$ is rationally connected. In particular, these 2 notions are equivalent for varieties with at worst klt singularities.

Here we introduced the notion of the maximal rationally chain-connected fibration. The following theorem is proved in [KMM92b].

**Theorem 6.2.2.** *Let $X$ be a normal projective variety, and $L$ be a $\mathbb{Q}$-Cartier divisor on $X$. Then there exists an almost holomorphic dominant rational map $f : X \dashrightarrow Y$ with connected fibres, called the maximal rationally chain-connected fibration, such that:*

*(1)* *Every general fibres of $X$ are rationally chain-connected;*

*(2)* *For every very general point $y \in Y$ and every rational curve $C$, either $C \in f^{-1}(y)$ or $C \cap f^{-1}(y) = \emptyset$. That is, almost all rational curves are contracted by $f$.*

*The map $f$ is called the maximal rationally chain-connected fibration (MRCC fibration for short). The MRC fibration is unique up to birational equivalence of $Y$. The rational dimension of $X$, denoted by* $\mathrm{rd}(X)$ *is defined by the dimension of the general fibre of $f$, that is,* $\mathrm{rd}(X) := \dim X - \dim Y$.

*Remark* 6.2.3. By the results in [HM07], the rational dimension is a birational invariant for varieties that have at worst klt singularities. However, it is not birational invariant in general. For example, let $E \subset \mathbb{P}^3$ be an elliptic curve, and $C(E)$ be the cone of $E$ in $\mathbb{P}^3$. Note that $C(E)$ is birationally equivalent to $E \times \mathbb{P}^1$. However, in this case, $\mathrm{rd}(C(E)) = 2$ but $\mathrm{rd}(E \times \mathbb{P}^1) = 1$, which shows that the rational dimension is not a birational invariant in general. Note that in this case, $C(E)$ has a log canonical surface singularity at the vertex.

A natural question is to ask for an algebraic fibre space $f : X \to Y$ with general fibre


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$F$, how to relate the rational dimension of $X, Y$ and $F$. Unfortunately, by the following examples, at least we can not have a direct inequality between rd$(X)$ and rd$(F)$ + rd$(Y)$.

*Example* 6.2.4. Let $X$ be a rational elliptic surface obtained by blowing up the 9 intersection points of two smooth elliptic curves on $\mathbb{P}^2$. Then $-K_X$ is semiample, with the semiample fibration being an elliptic fibration $X \to \mathbb{P}^1$. Let $E$ be a general fibre, which is an elliptic curve, then we have

$$2 = \mathrm{rd}(X) > \mathrm{rd}(E) + \mathrm{rd}(\mathbb{P}^1) = 1.$$

Conversely, let $X'$ be an elliptic K3 surface, $X \to \mathbb{P}^1$ be the corresponding elliptic fibration, and $E'$ be a general fibre. Then

$$0 = \mathrm{rd}(X') < \mathrm{rd}(E') + \mathrm{rd}(\mathbb{P}^1) = 1,$$

hence both of the inequalities fail in general.



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# Chapter 7 Further questions and discussions

## 7.1 Generalized Nonvanishing Conjecture, Iitaka Conjecture, and Serrano Conjecture

As we mentioned before, the general case of the Generalized Nonvanishing Conjecture and the Iitaka conjecture is still unsolved. Therefore, it is natural to try to continue the works on these two conjectures. In the previous works, the main idea to solve the Generalized Nonvanishing Conjecture is the nef reduction map and the Albanese morphism. Recently, there is another new idea in birational geometry, called the generalized pair (cf. [Bir20, Definition 4.4]):

**Definition 7.1.1** (Generalized pair)**.** *Let $X$ be a normal projective variety. A generalized pair consists an effective $\mathbb{Q}$-divisor on $X$, a birational morphism $\pi : X' \to X$, and a nef divisor $M'$ on $X'$ such that $K_X + B + \pi_* M'$ is $\mathbb{Q}$-Cartier.*

The generalized pair is a new idea in birational geometry which has been studied in recent years. For example, in [Bir20, Section 5], Birkar explores various applications of generalized pairs in the study of birational geometry, particularly in the context of the minimal model program. Notably, the moduli divisor and the discriminant divisor in Fujino-Mori's Canonical bundle formula also form a generalized pair. Therefore, it is also can expect the possibility to solve some cases of Generalized Nonvanishing Conjecture and/or


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Iitaka Conjecture by using the generalized pairs, as its application on the minimal model program.

On the other hand, as we mentioned above, in our proof of Theorem 5.2.1, we only need $K_X + B + tL$ is numerically effective "for some $t > 1$" (not need for all $t \geq 0$), and the remained case of Generalized Nonvanishing Conjecture for threefolds is that $X$ has $q = \kappa = 0$ and $n(L) = 3$. In the case $L$ is strictly nef, which means $L.C > 0$ for any integral curve on $X$, there is another relative question, called the Serrano's Conjecture (cf. [Ser95], [WZ21]):

**Conjecture 7.1.2** (Serrano's Conjecture)**.** *Let $X$ be a smooth projective variety, and $L$ be a strictly nef Cartier divisor on $X$. Then for $t > \dim X + 1$, $K_X + tL$ is ample.*

**Conjecture 7.1.3** (Singular version of Serrano's Conjecture)**.** *Let $(X, \Delta)$ be a klt pair, and $L$ be a strictly nef Cartier divisor on $X$. Then for $t \gg 0$, $K_X + \Delta + tL$ is ample.*

For the original (smooth) version, when $\dim X = 3$, almost all cases of the Serrano Conjecture are proved by Campana-Chen-Peternell (cf. [CCP08]), except the special case that $X$ is a Calabi-Yau threefold with $L.c_2 = 0$. However, the remained case is still unsolved at this moment. Also, for the singular version, if $X$ is a threefold with at worst terminal singularities, it is proved by Wang-Zhong (cf. [WZ21, Theorem 1.3]) that the Serrano Conjecture holds if $X$ is not a Calabi-Yau threefold with $L.c_2 = 0$. For general three-dimensional klt pairs, they prove this Conjecture under the assumption $\kappa(X) \geq 1$. As our proof of the Generalized Nonvanishing Conjecture, the most difficult case of both the two conjectures is the Calabi-Yau threefolds. Therefore, to (partially) solve these conjectures, one possible situation is to work on higher-dimension varieties with either non-zero irregularity, or non-zero Kodaira dimension.

## 7.2 Iitaka fibration of nef anticanonical divisors

Let $X$ be a normal projective variety with at worst klt singularities and $-(K_X + \Delta)$ being nef and effective. A natural question is: can we describe the Iitaka fibration of


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$-(K_X+\Delta)$? In the case where $-(K_X+\Delta)$ is semiample, it is well-known that the fiber must have a numerically trivial canonical divisor. Furthermore, Fujino and Gongyo have proven that the image of the semiample fibration must be of Fano type ([FG12, Corollary 3.4]). Therefore, it is reasonable to expect that even under the weaker assumption that $-(K_X+\Delta)$ is nef and effective, there may still be a meaningful description for the Iitaka fibration. For example, it would be interesting to explore the structure of the fiber and base of the Iitaka fibration of nef and effective $-(K_X+\Delta)$.

For this question, by [Fuj11, Theorem 1.1], if $-(K_X+\Delta)$ is nef but not semiample, then it is not abundant, which implies for a threefold $X$ such that $-(K_X+\Delta)$ is nef with positive Iitaka dimension but is not semiample, one can show that it must satisfy $\kappa(X,-(K_X+\Delta)) = 1$ and $\nu(X,-(K_X+\Delta)) = 2$. Consequently, the image of the Iitaka fibration must be a rational curve. This discussion suggests the possibility of providing a more explicit structure for the Iitaka fibration of $-(K_X+\Delta)$ especially in three dimensions.

On the other hand, when $-(K_X+\Delta)$ is semiample, it becomes interesting to study the structure of the semiample fibration determined by $-(K_X+\Delta)$ in more detail. For example, one might inquire whether such a fibration is smooth or locally trivial.

Besides the structure of Iitaka fibration, it is also interesting to study the structure of section ring $R(X,-(K_X+\Delta))$. Note that $R(X,D)$ is finitely generated as $\mathbb{C}$-algebra if either $\kappa(D)\leq 1$, or $D$ is semiample. In particular, by the above discussion, we can easily see that $R(X,-(K_X+\Delta))$ is a finitely generated $\mathbb{C}$-algebra is dim $X\leq 3$. Therefore, a question is that: if $-(K_X+\Delta)$ is nef and effective, then whether $R(X,-(K_X+\Delta))$ is always a finitely generated $\mathbb{C}$-algebra (in any dimension).

## 7.3 Birationally (super)rigidity of Fano threefolds

In this thesis, we established an "anticanonical" version of the Iitaka Conjecture for varieties that have $\mathbb{Q}$-effective anticanonical divisors. In the context of the minimal model


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program, such varieties are expected to have a Mori fiber space structure. Another closely related topic in the study of Mori fiber spaces is birationally (super)rigidity. A Mori fibre space $f : X \to Z$ is called birationally (super)rigid if $X \to Z$ does not have a ”non-trivial” birational map to another Mori fiber space $f' : X' \to Z'$. one of its crucial properties is that every rational $n$-dimensional Fano variety $X$ with Picard number 1 and $n \geq 2$ is always non-rigid since $X$ will birational equivalent to $X' := \mathbb{P}^{n-1} \times \mathbb{P}^1$ and it has a Mori fibre space structure $X' \to \mathbb{P}^1$. Therefore, the notion of birational (super)rigidity is also highly related to the rationality problem of Fano varieties.

It is known that a $\mathbb{Q}$-Fano threefold $X$ with Picard number 1 is birationally super-rigid if and only if there is no maximal center on $X$ ([KOPP22, Definition 2.4], [Cor95, Proposition 2.10], and [CS08, Theorem 1.26]). Here, a maximal center is a closed subset which is the base locus of some special mobile linear systems. To obtain the excluded method of maximal center on smooth projective threefold, one important tool is Corti's $4\mu^2$ inequality ([Cor00, Theorem 3.12]). This inequality provides a lower bound for the intersection number of mobile linear systems that are not semiample. For example, in [CP06, Section 2 and 3], Cheltsov and Park use the $4\mu^2$ inequality to prove that every smooth sextic double solid does not have maximal centers (hence birationally superrigid and non-rational).

In the case where the variety $X$ is not smooth, Krylov-Okada-Paemurru-Park, in [KOPP22, Theorem 1.2], proved that the $2\mu^2$ inequality holds for a $cA_1$-type singularity. This is the first Corti-type inequality result on singular points, and they utilized this inequality to prove the birational superrigidity of a sextic double solid with only $A_1$-type singularities.

A natural question is: whether we can construct Corti-type inequalities for varieties with ”more singular” points and employ them to establish the rigidity of specific Fano threefolds. To derive such inequalities, one possible approach is to consider a resolution of singularities $X' \to X$ and use the fact that the $4\mu^2$ inequality holds for $X'$. This could allow us to compute a similar inequality by comparing the intersection numbers of $X'$ and $X$. Another possible way is to follow the strategy outlined in [KOPP22], which proved


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Corti-type inequality for $A_1$ points. We might attempt a similar computation for other singularities to derive analogous inequalities.



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