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碩士論文

Department of Mathematics

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Master's Thesis

法諾三維多樣體的導出範疇之研究

On the Derived Categories of Some Fano Threefolds

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摘要

本文探討兩個皮卡數為一的平滑法諾三維空間之導出範疇。第一個是指數為一、次數為十四的三維空間（下稱「第一空間」）；第二個是指數為二、次數為三的三維空間（下稱「第二空間」）。

對這兩個空間，我們各自選取其有界導出範疇中的一個特別子範疇。就第一空間而言，該子範疇 ── 常被稱作庫茲涅佐夫成分 ── 是由一對標準例外對象（包含一個秩二向量束與結構層）的右正交補所構成；對第二空間，則取由結構層與超平面類線束組成之例外對的右正交補。

二零零四年，庫茲涅佐夫建立了一個對應，聯繫這兩類法諾三維空間的模空間，並證明：對每一個平滑的第一空間，都存在一個平滑的第二空間，使得它們各自的庫茲涅佐夫成分彼此等價。

本論文有兩個主要目標：第一，證明上述兩個子範疇皆不含例外對象；第二，說明庫茲涅佐夫的等價可以實現為傅立葉ー穆凱變換。

關鍵字：法諾三維多樣體、導出範疇、右正交補範疇

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Abstract

In this article, we study the derived categories of two smooth Fano threefolds with Picard number one. The first is the threefold of index one and degree fourteen, which we call "X_{14}"; the second is the threefold of index two and degree three, referred to as "Y_3". For each variety we consider a distinguished subcategory of its bounded derived category of coherent sheaves. In the case of X_{14}, this subcategory—often called the Kuznetsov component—is defined as the right orthogonal to the standard exceptional pair consisting of a rank-two vector bundle and the structure sheaf. For Y_3, the analogous subcategory is the right orthogonal to the pair formed by the structure sheaf and the line bundle associated with the hyperplane class. In a 2004 paper, Alexander Kuznetsov constructed a correspondence between the moduli stacks that classify these two families of Fano threefolds. More precisely, for every smooth Fano threefold X_{14} there exists a smooth Fano threefold Y_3 such that their Kuznetsov components are equivalent. This thesis has two main goals: first, to prove that the two chosen subcategories contain no exceptional ob-

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jects; and second, to show that Kuznetsov's equivalence can be realized as a Fourier–Mukai transform.

Keywords: Fano Threefolds, Derived Category, Right Orthogonal Complement

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Chapter 1 Introduction

The bounded derived category of coherent sheaves on a smooth projective variety encodes subtle geometric information. A foundational example is the reconstruction theorem of Bondal and Orlov (see [BO95]): for a smooth projective variety X with ample canonical or anticanonical bundle, any equivalence $D^b(Y) \simeq D^b(X)$ with a smooth variety Y forces $Y \cong X$. In such cases, the derived category determines the variety up to isomorphism.

Kawamata extended this to the case where the canonical or anticanonical bundle is big, showing that $D^b(Y) \simeq D^b(X)$ implies Y birational to X. These results reveal the derived category as a powerful invariant reflecting birational geometry.

This categorical perspective naturally aligns with the minimal model program, where flops connect different minimal models. In the case of threefolds, Bridgeland (see [**?**]) showed that for two crepant resolutions $\pi_1 : Y_1 \to X$ and $\pi_2 : Y_2 \to X$ of a projective threefold X with at worst terminal singularities, the derived categories $D^b(Y_1)$ and $D^b(Y_2)$ are equivalent. Also, in [KPS18], the technical argument in [Kuz04] is actually based on Bridgeland's work, which establishes an equivalence between the derived categories of two specific projective bundles over two Fano threefolds via a particular Fourier–Mukai transform.

Moreover, two related but different geometric object may have equivalence derived category. A classical example is the equivalence between an abelian variety and its dual, realized via a Fourier – Mukai transform with the Poincaré bundle as kernel. This phenomenon occurs in particular when the canonical bundle is trivial. Such examples suggest that derived categories may capture hidden symmetries beyond classical birational geometry.

In the case of Fano threefolds, this idea becomes particularly powerful: their derived categories sometimes admit equivalences between seemingly very different varieties. For instance, as shown in [Kuz09], one may consider the so-called Kuznetsov component in the derived category of a Fano threefold, defined as the right orthogonal complement of an exceptional collection associated with natural geometric vector bundles.

For $d = 4, 5$, the Fano threefolds of index 1 and degree $4d + 2$ and the del Pezzo threefolds of degree d admit equivalent Kuznetsov components. For $d = 3$, each Fano threefold of index 1 and degree 14 is associated with a del Pezzo threefold of degree 3 whose derived category contains an equivalent component.

Moreover, as shown in [KPS18], for $d = 3, 4, 5$, the Hilbert scheme of conics on

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a Fano threefold of index 1 and degree $4d + 2$ is isomorphic to the Hilbert scheme of lines on the corresponding del Pezzo threefold of degree d, further confirming the deep relationship reflected by the equivalence of derived components.

This thesis introduces the relationship between two specific smooth Fano threefolds: the Fano threefold X_{14} of index 1 and degree 14, and the cubic threefold Y_3 of index 2 and degree 3. In [Kuz04], for any such X_{14}, Kuznetsov constructs an associated Y_3 and shows that a semiorthogonal component of $D^b(Y_3)$ matches a component of $D^b(X_{14})$. More precisely, under this correspondence, we have the equivalence

$$\mathcal{B}_{Y_3} \to \mathcal{A}_{X_{14}}$$

where each side denotes the right orthogonal complement of a natural exceptional collection in the derived category.

The subcategory $\mathcal{A}_{X_{14}} \subset D^b(X_{14})$ is defined as the right orthogonal to the exceptional pair $(\mathcal{U}_2, \mathcal{O}_{X_{14}})$, and the subcategory $\mathcal{B}_{Y_3} \subset D^b(Y_3)$ is the right orthogonal to the collection $(\mathcal{O}_{Y_3}, \mathcal{O}_{Y_3}(1))$. Both constructions arise naturally from geometric considerations: $\mathcal{U}_2$ is a stable vector bundle constructed via Mukai's method, which was used to classify Fano threefolds, while $\mathcal{O}$ and $\mathcal{O}(1)$ represent the simplest line bundles on a cubic hypersurface.

The goal of this thesis is to study the structure of $\mathcal{A}_{X_{14}}$ and $\mathcal{B}_{Y_3}$, and in particular, to show that it does not admit any exceptional objects and rewrite the equivalence in [Kuz04] as a Fourier Mukai transform $\phi_{\mathcal{K}} : \mathcal{B}_{Y_3} \to \mathcal{A}^*_{X_{14}}$ where $\mathcal{K} = \iota_{Z*}\big(\mathcal{O}_Y(2e)|_Z\big)$ and Z is a closed sub-variety of $X \times Y$. The dual of $A^*_{X_{14}}$ here is because of the different definition of $\mathcal{A}_{X_{14}}$ in [Kuz04], which is ${}^{\perp}\langle\mathcal{O}, \mathcal{U}^*\rangle$ and equals to $(\langle\mathcal{U}, \mathcal{O}\rangle^{\perp})^*$.

To achieve this, we provide the details of the computations which is omitted in [Kuz09], in which the numerical Grothendieck groups $K_0(X_{14})_{\text{num}}$ and $K_0(Y_3)_{\text{num}}$ are explicitly described. Using this framework, we reduce the problem to a matrix computation involving the Euler pairing. We then demonstrate that no class in $\mathrm{ch}(\mathcal{A}_{X_{14}})$ or $\mathrm{ch}(\mathcal{B}_{Y_3})$ satisfies the numerical conditions required to represent an exceptional object.

Regarding the Fourier - Mukai transform, we refer to the construction in [Kuz04], where the right mutation is interpreted as a Fourier–Mukai transform. We further compute the composition of two such transforms within this framework.

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Chapter 2 Preliminary on Fano threefolds

We begin by reviewing several basic definitions and facts about smooth Fano varieties. A smooth Fano variety is a projective variety V such that the anticanonical divisor $-K_V$ is ample. The following proposition summarizes some foundational properties (see [IP99]):

Proposition 2.1 ([IP99, Proposition 2.1.2])**.** Let V be a smooth Fano variety. Then:

1. $H^i(V, \mathcal{O}_V) = 0$ for all $i > 0$;

2. Pic$(V) \cong H^2(V, \mathcal{O}_V)$ is a finitely generated, torsion-free $\mathbb{Z}$-module.

Since Pic(V) is finitely generated and torsion-free, there exists a maximal integer $r > 0$ such that $-K_V = rH$ for some ample Cartier divisor H. This divisor H is called the fundamental divisor of V, and the integer r is referred to as the index of V. The quantity $d = H^{\dim V}$ is called the degree of V. Finally, the number $g = \frac{1}{2}(-K_V)^3 + 1$ is called the genus of V (see [IP99]).

Remark 2.2. We briefly describe the linear system $|-K_V|$. By [IP99, Corollary 2.4.6], $|-K_V|$ is base point free if $\rho(V) = 1$. In this case, the associated morphism $\varphi_{|-K_V|}$ is either a finite morphism of degree 2 or an embedding (see [IP99, Proposition 4.1.11]). The definition of genus is justified by the fact that when $\varphi_{|-K_V|}$ is an embedding, V can be realized as a threefold of degree $(-K_V)^3 = 2g-2$ in $\mathbb{P}^{g+1}$. Moreover, for a general smooth curve $C \subset V$ obtained as the complete intersection of two hyperplanes, the restriction $\varphi_{|-K_V|}|_C : C \to \mathbb{P}^{g-1}$ is the canonical map, and C has genus g.

In this article, we focus on the case where Pic$(V) \cong \mathbb{Z}$. We begin with smooth Fano threefolds V of index 1. We denote by V_{2g-2} a Fano threefold of degree $2g-2$ and genus g. In this case, the possible values of g satisfy $2 \leq g \leq 12$ with $g \neq 11$ (see [IP99, Proposition 5.2.3], also Mukai's work on Fano threefolds of genus g).

We now introduce the classification of smooth Fano threefolds of index 1 and Picard number 1. In the following theorem for $g \leq 5$, the morphism $\varphi_{|-K_V|}$ is either an embedding, in which case V_{2g-2} can be realized as a complete intersection in $\mathbb{P}^{g+1}$, or a finite morphism of degree 2.

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Theorem 2.3 (Iskovskikh, see [IP99, Proposition 4.1.12]). Let V_{2g-2} be a smooth Fano threefold of index 1 and genus g, where $2 \leq g \leq 5$. Then the morphism

$$\varphi_{|-K_V|} : V_{2g-2} \to \mathbb{P}^{g+1}$$

classifies V_{2g-2} as follows:

- $g = 2$: Sextic double solid
- $g = 3$: either $V_4 \to \mathbb{P}^4$ is a finite morphism of degree 2 onto a quadric in $\mathbb{P}^4$, ramified along a degree 8 surface, or $V_4 \hookrightarrow \mathbb{P}^4$ is a quartic hypersurface.
- $g = 4$: $V_6 \hookrightarrow \mathbb{P}^5$ is a complete intersection of a quadric and a cubic.
- $g = 5$: $V_8 \hookrightarrow \mathbb{P}^6$ is a complete intersection of three quadrics.

For $g \geq 6$, the image of the anticanonical morphism is no longer a complete intersection. In these cases, a method for biregular classification was developed by Mukai.

Theorem 2.4 (Mukai, see [IP99, Theorem 5.2.3] or [BKM24, Theorem 1.2]). Let V_{2g-2} be a smooth Fano threefold of index 1 and Picard number $\rho(V_{2g-2}) = 1$, with genus $g \geq 6$. Then V_{2g-2} admits the following classification:

- $g = 6$: a transverse linear section of a complete intersection of a quadric and the cone $\widetilde{\mathrm{Gr}}(2,5) \subset \mathbb{P}^{10}$ over $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$.
- $g = 7$: fix a nondegenerate symmetric bilinear form on $\mathbb{C}^9$; then V_{12} is a transverse linear section of the 10-dimensional variety

$$\{W \in \mathrm{Gr}(4, \mathbb{C}^9) \mid q(W, W) = 0\} \subset \mathrm{Gr}(4, \mathbb{C}^9).$$

- $g = 8$: a transverse linear section of $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$.
- $g = 9$: fix a nondegenerate skew-symmetric bilinear form q on $\mathbb{C}^6$; then V_{16} is a transverse linear section of the 6-dimensional variety

$$\{W \in \mathrm{Gr}(3, \mathbb{C}^6) \mid q(W, W) = 0\} \subset \mathrm{Gr}(3, \mathbb{C}^6) \subset \mathbb{P}^{19}.$$

- $g = 10$: fix a nondegenerate skew-symmetric 4-linear form q on $\mathbb{C}^7$; then V_{18} is a transverse linear section of the 5-dimensional variety

$$\{W \in \mathrm{Gr}(5, \mathbb{C}^7) \mid q(W, W, W, W) = 0\} \subset \mathrm{Gr}(5, \mathbb{C}^7) \subset \mathbb{P}^{20}.$$

- $g = 12$: fix three nondegenerate skew-symmetric bilinear forms q_1, q_2, q_3 on $\mathbb{C}^7$; then V_{22} is the variety

$$\left\{W \in \mathrm{Gr}(3, \mathbb{C}^7) \,\middle|\, q_1(W, W) = q_2(W, W) = q_3(W, W) = 0\right\} \subset \mathrm{Gr}(3, \mathbb{C}^7) \subset \mathbb{P}^{34}.$$

For $g = 8, 9, 10, 12$, the classification is based on the following construction involving the so-called Mukai bundle.

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Theorem 2.5 (Mukai bundle; see [BKM24] for details)**.** Let k be an algebraically closed field of characteristic zero, and let V be a smooth Fano threefold over k of genus $g = ts \geq 6$ with $t, s \geq 2$, and Picard number $\rho(V) = 1$. Then there exists a unique stable vector bundle $\mathcal{U}_t$ on V such that

$$\mathrm{rk}(\mathcal{U}_t) = t, \quad c_1(\mathcal{U}_t) = K_V, \quad H^\bullet(V, \mathcal{U}_t) = 0, \quad \text{and} \quad \mathrm{Ext}^\bullet(\mathcal{U}_t, \mathcal{U}_t) = \mathrm{k}.$$

Moreover, the dual bundle $\mathcal{U}_t^*$ is globally generated with

$$\dim H^0(V, \mathcal{U}_t^*) = t + s, \quad \text{and} \quad H^{>0}(V, \mathcal{U}_t^*) = 0.$$

Since $\mathcal{U}_r^*$ is globally generated, we may consider the evaluation map

$$H^0(V, \mathcal{U}_t^*) \otimes \mathcal{O}_V \twoheadrightarrow \mathcal{U}_t^*,$$

whose dual gives an injection

$$\mathcal{U}_t \hookrightarrow H^0(V, \mathcal{U}_t^*)^* \otimes \mathcal{O}_V.$$

This defines a morphism

$$V \longrightarrow \mathrm{Gr}(r, r+s), \quad x \mapsto \mathcal{U}_t|_x \subset H^0(V, \mathcal{U}_t^*)^*.$$

This morphism factors through the anticanonical embedding, and studying it allows one to classify such Fano threefolds up to biregular isomorphism.

For smooth Fano threefolds of Picard number 1, the index i_V satisfies $1 \leq i_V \leq 4$. The following theorems describe the classification in the cases $i_V = 2, 3, 4$.

Theorem 2.6. Let V_d be a Fano threefold of index 2 and Picard number 1 with degree d. Then V_d is classified as follows:

- $d = 5$: V_5 is a transverse linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$;
- $d = 4$: $V_4 \subset \mathbb{P}^5$ is a complete intersection of two quadric hypersurfaces;
- $d = 3$: $V_3 \subset \mathbb{P}^4$ is a cubic threefold;
- $d = 2$: $V_2 \to \mathbb{P}^3$ is a double cover ramified along a quartic surface;
- $d = 1$: $V_1 \subset \mathbb{P}(3,2,1,1,1)$ is a degree 6 hypersurface.

Theorem 2.7. Let V be a smooth Fano threefold of Picard number 1 and index i_V. Then:

- If $i_V = 3$, then $V \subset \mathbb{P}^4$ is a quadric hypersurface;
- If $i_V = 4$, then $V \cong \mathbb{P}^3$.

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Chapter 3 Preliminary on Derived category

In this section we briefly review the notions of derived category, including mutation, semiorthogonal decomposition, exceptional objects, and exceptional collections. (Our exposition follows [Huy23].)

Let $\mathcal{D}$ be a triangulated category and let $\mathcal{D}_0 \subset \mathcal{D}$ be an <u>admissible</u> full triangulated subcategory; that is, the inclusion functor $\iota\colon \mathcal{D}_0 \hookrightarrow \mathcal{D}$ admits both a left and a right adjoint, which we denote by ι_* and $\iota^!$, respectively. For any object $E \in \mathcal{D}$ one has canonical decompositions with respect to $\mathcal{D}_0$ and its orthogonals. More precisely, there exist two distinguished triangles (see [Huy23, Chap. 7, Ex. 1.2])

$$\begin{aligned} {}_EP \longrightarrow E \longrightarrow \iota^*E, &\qquad {}_EP \in {}^{\perp}\mathcal{D}_0, \\ \iota^!E \longrightarrow E \longrightarrow P_E, &\qquad P_E \in \mathcal{D}_0^{\perp}. \end{aligned}$$

Moreover, the choice of decomposition is unique up to isomorphism. More precisely, suppose we are given two distinguished triangles:

$$\begin{aligned} F \longrightarrow E \longrightarrow G, &\qquad F \in {}^{\perp}\mathcal{D}_0, \quad G \in \mathcal{D}_0, \\ F' \longrightarrow E \longrightarrow G', &\qquad F' \in \mathcal{D}_0, \quad G' \in \mathcal{D}_0^{\perp}. \end{aligned}$$

Then we have

$$F \cong {}_EP, \quad G \cong \iota_*E, \quad F' \cong \iota^!E, \quad G' \cong P_E.$$

${}^{\perp}\mathcal{D}_0$ is right-admissible and $\mathcal{D}_0^{\perp}$ is left-admissible. In this case, one has

$$({}^{\perp}\mathcal{D}_0)^{\perp} = \mathcal{D}_0 = {}^{\perp}(\mathcal{D}_0^{\perp}).$$

Write

$$k\colon {}^{\perp}\mathcal{D}_0 \hookrightarrow \mathcal{D}, \qquad j\colon \mathcal{D}_0^{\perp} \hookrightarrow \mathcal{D}$$

for the inclusions. Let $k\colon {}^{\perp}\mathcal{D}_0 \hookrightarrow \mathcal{D}$ and $j\colon \mathcal{D}_0^{\perp} \hookrightarrow \mathcal{D}$ be the inclusion functors. If we decompose E into two parts, then by the uniqueness discussed above, we obtain the

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following two distinguished triangles:

$$k^! E \longrightarrow E \longrightarrow \iota^* E, \qquad k^! E \in {}^\perp\mathcal{D}_0,\ \iota^* E \in \mathcal{D}_0,$$
$$\iota^! E \longrightarrow E \longrightarrow j^* E, \qquad \iota^! E \in \mathcal{D}_0,\ j^* E \in \mathcal{D}_0^\perp.$$

where $k^!$ and j^* are the right and left adjoints of k and j, respectively.

Definition 3.1. [Huy23, Chap. 7, Def. 1.5] The right mutation through $\mathcal{D}_0$ is the functor

$$\mathbf{R}_{\mathcal{D}_0} := k^! : \mathcal{D} \longrightarrow {}^\perp\mathcal{D}_0,$$

and the left mutation through $\mathcal{D}_0$ is the functor

$$\mathbf{L}_{\mathcal{D}_0} := j^* : \mathcal{D} \longrightarrow \mathcal{D}_0^\perp.$$

In some cases, we can describe ι^* and $\iota^!$ more explicitly. Consider the following definition.

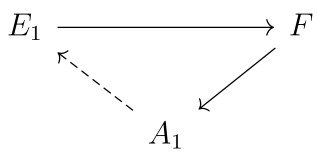
Definition 3.2 (Exceptional object, see [Huy23])**.** An object $E \in \mathcal{D}$ is called exceptional if it satisfies

$$\mathrm{Ext}^\bullet(E, E) = \mathrm{k}.$$

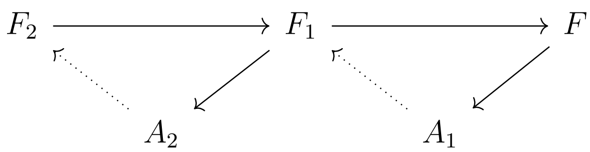
Remark 3.3. The category $\langle E \rangle \subset \mathcal{D}$ is admissible, and we have

$$\iota^* F \cong \bigoplus_m \mathrm{Hom}(F, E[m])^* \otimes E[m], \qquad \iota^! F \cong \bigoplus_m \mathrm{Hom}(E, F[m]) \otimes E[m].$$

So far, we have seen that if there is an admissible subcategory $\mathcal{D}_1 \subset \mathcal{D}$, then for any $F \in \mathcal{D}$, we have the following triangle:



where $A_1 \in \mathcal{D}_1$ and $E_1 \in {}^\perp\mathcal{D}_1$. If there exists another admissible subcategory $\mathcal{D}_2 \subset {}^\perp\mathcal{D}_1$, then we have



where $E_2 \in {}^\perp\mathcal{D}_1 \cap {}^\perp\mathcal{D}_2$ and $A_i \in \mathcal{D}_i$. By continuing to find admissible subcategories, we are able to decompose the triangulated category into several pieces. This leads us to the following definition of semiorthogonal decomposition.

Definition 3.4 (Semiorthogonal decomposition)**.** Let $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ be a sequence of admissible subcategories of $\mathcal{D}$ satisfying $\mathrm{Hom}(\mathcal{D}_j, \mathcal{D}_i) = 0$ for all $j > i$, and such that for

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every $F \in \mathcal{D}$, we have the following sequence of F_j and distinguished triangles:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{n-1} & \longrightarrow \cdots \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F \\ & \nwarrow & \swarrow & & & \nwarrow \quad \swarrow & & \nwarrow \quad \swarrow & \\ & & A_n & & & A_2 & & A_1 & \end{array}$$

with $A_i \in \mathcal{D}_i$. In this case, we write

$$\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n \rangle.$$

Definition 3.5. Given an ordered sequence of exceptional object $(E_1,\, E_2,\, \cdots,\, E_i)$, we say that it is an exceptional collection if $RHom(E_j,\, E_i) = 0$ for any $j > i$. In this case, we have following semi-orthogonal decomposotion

$$\langle \mathcal{D}',\, \langle E_1 \rangle, \langle E_2 \rangle, \cdots, \langle E_i \rangle \rangle$$

where $\mathcal{D}' = \langle E_1,\, E_2,\, \cdots,\, E_i \rangle^{\perp}$

Back to the case of Fano threefolds with Picard number 1. For such varieties, there are two known exceptional collections in their derived categories, each giving rise to a semiorthogonal decomposition. It is natural to explore the relation between the components defined by these collections.

The first exceptional collection comes from the cohomological vanishing conditions in the theorem above and consists of $\big(\mathcal{U}_r, \mathcal{O}_V\big)$. Therefore, for smooth Fano threefolds of Picard number 1, index 1, and genus $g \geq 6$, we consider the semiorthogonal decomposition:

$$\langle \mathcal{A}_V, \mathcal{U}_r, \mathcal{O}_V \rangle,$$

where

$$\mathcal{A}_V = \langle \mathcal{U}_r, \mathcal{O}_V \rangle^{\perp} = \left\{ \mathcal{F} \in \mathcal{D}^b(V) \;\middle|\; \mathrm{Ext}^{\bullet}(\mathcal{U}_r, \mathcal{F}) = 0,\ \mathrm{Ext}^{\bullet}(\mathcal{O}_V, \mathcal{F}) = H^{\bullet}(V, \mathcal{F}) = 0 \right\}.$$

On the other hand, for index i smooth Fano threefolds, there is another exceptional collection:

$$\mathcal{O}_V,\, \mathcal{O}_V(H),\, \ldots,\, \mathcal{O}_V((i-1)H)$$

which forms an exceptional collection in $\mathcal{D}^b(V)$.

In particular, for $i = 2$, we have the following semiorthogonal decomposition:

$$\langle \mathcal{B},\, \mathcal{O}_V,\, \mathcal{O}_V(H) \rangle.$$

Kuznetsov raised the following conjecture for $1 \leq d \leq 5$ in [Kuz09]:

Conjecture 3.6. Let $\mathcal{M}_d^i$ be the moduli stack of smooth Fano threefolds of index i and degree d. Then there exists a correspondence

$$Z_d \subseteq \mathcal{M}^1_{4d+2} \times \mathcal{M}^2_d$$

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which is dominant on each factor, such that for all $(X_{4d+2}, Y_d) \in \mathcal{M}^1_{4d+2} \times \mathcal{M}^2_d$, we have

$$\mathcal{A}_{X_{4d+2}} \cong \mathcal{B}_{Y_d}.$$

This conjecture was proved by Kuznetsov for $d = 3, 4, 5$, while counterexamples are now known for $d = 1, 2$. Moreover, Kuznetsov later modified the conjecture and proved the revised version; see [KS25].

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Chapter 4 Cohomology on Fano threefolds

On a smooth Fano threefold X, we have $K_0(X)_{\text{num}} \cong \mathbb{Z}^4$, generated by

$$\mathcal{O}_X, \quad \mathcal{O}_H, \quad \mathcal{O}_L, \quad \mathcal{O}_p.$$

It is therefore reasonable to compute $\operatorname{ch}(\mathcal{O}_X)$, $\operatorname{ch}(\mathcal{O}_H)$, $\operatorname{ch}(\mathcal{O}_L)$, and $\operatorname{ch}(\mathcal{O}_p)$, as well as the Euler pairing on $K_0(X)_{\text{num}} \cong \mathbb{Z}^4$. Most of these results are stated without proof in [Kuz09]; we provide full details below.

Lemma 4.1 (Generalizing Lemma from [Isk89])**.** Let X be a smooth Fano threefold of index r, and let $L \subset X$ be a line. If $|H|$ is very ample, then

$$N_{L/X} \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2), \quad \text{with } d_1 + d_2 = r - 2.$$

Remark 4.2. Regarding the existence of lines: for the index 1, Picard number 1 case with $-K_X$ very ample, see [Šo79]. For index 2, see [KPS18], where the Hilbert scheme of lines on index 2, Picard number 1 Fano threefolds of degrees $3, 4, 5$ is discussed.

Proof. We first argue that there exists a nonsingular hyperplane section $H \subset X$ containing $Z := L$. We modify the classical Bertini theorem to construct such a section.

Consider the incidence relation

$$R = \{(x, H) \in Z \times |H| \mid H \cap X \text{ is singular at } x\},$$

which is equal to

$$\{(x, H) \in Z \times |H| \mid T_x(X) \subset H\}.$$

If H is singular at x, then $Z \subset T_x(X) \subset H$. Thus, R is a closed subscheme of $Z \times |H - Z|$, where $|H - Z|$ is the linear system of hyperplane sections containing Z. More precisely, it is defined by

$$\mathbb{P}(H^0(\mathcal{I}_Z \otimes \mathcal{O}(H))) \subset |H|,$$

and has dimension $\dim |H| - 2$.

Now consider the projections:

1. $P_Z : R \to Z$,

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2. $P_{|H-Z|} : R \to |H - Z|$.

Over any point $x \in Z$, the fiber of P_Z consists of hyperplane sections singular at x, which is a linear subspace of dimension dim $|H| - 4$. Therefore, dim $R =$ dim $|H| - 3$, so the image Im$(P_{|H-Z|}) \subset |H - Z|$ has dimension at most dim $|H| - 3$. This implies that the set of hyperplanes singular along Z is a proper closed subset of $|H - Z|$.

By Bertini's theorem, the general member of $|H - Z|$ is smooth outside the base locus, which is Z. Hence, the general member in $|H - Z|$ is a smooth hyperplane section containing Z.

Choose such a smooth hyperplane section $H \subset X$ containing Z. Consider the exact sequence on X:

$$0 \longrightarrow \mathcal{I}_{H/X} \longrightarrow \mathcal{I}_{Z/X} \longrightarrow {\iota_{Z/X}}_* \mathcal{I}_{Z/H} \longrightarrow 0.$$

Pulling back to Z, we get:

$$\iota_{Z/H}{}^* \mathcal{N}_{H/X}{}^* \longrightarrow \mathcal{N}_{Z/X}{}^* \longrightarrow \mathcal{N}_{Z/H}{}^* \longrightarrow 0.$$

Taking duals gives:

$$0 \longrightarrow \mathcal{N}_{Z/H} \xrightarrow{f} \mathcal{N}_{Z/X} \longrightarrow \iota^*_{Z/H} \mathcal{N}_{H/X} \longrightarrow 0.$$

We now argue that the map f is injective: The middle term is a rank 2 bundle, and the third term is a line bundle, so the map to the line bundle must have nontrivial kernel. Thus, f is fiberwise injective, and hence globally injective.

From the geometry: $\iota_{Z/H}{}^* \mathcal{N}_{H/X} \cong \mathcal{O}_{\mathbb{P}^1}(1), \quad \mathcal{N}_{Z/H}{}^* \cong \mathcal{I}/\mathcal{I}^2 = \mathcal{O}_{\mathbb{P}^1}(-Z^2)$.

We compute:

$$Z^2 = 2g(Z) - 2 - Z \cdot K_H = -2 - Z \cdot ((K_X + H)|_H) = -2 - (1 - r) = -3 + r.$$

Hence, $c_1(\mathcal{N}_{Z/H}) = r - 2$. By Grothendieck's theorem, we conclude:

$$\mathcal{N}_{Z/X} \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2), \quad \text{with } d_1 + d_2 = r - 2. \qquad \square$$

Proposition 4.3. For a Fano threefolds V of Picard number 1, with index r, and H is very ample, we have :

$$\mathrm{ch}(\mathcal{O}_V) = 1, \quad \mathrm{ch}(\mathcal{O}_H) = H - \frac{d}{2}L + \frac{d}{6}P, \quad \mathrm{ch}(\mathcal{O}_L) = L + \frac{2-r}{2}P, \quad \mathrm{ch}(\mathcal{O}_P) = P.$$

Proof. Also, recall the Chern class for E,

$$\mathrm{ch}(E) = \text{rank} + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) \cdots$$

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Thus, $\mathrm{ch}(\mathcal{O}_V) = 1,$ and

$$\begin{aligned}\mathrm{ch}(\mathcal{O}_H) &:= \mathrm{ch}(\mathcal{O}_V) - \mathrm{ch}(\mathcal{O}_V(-H)) \\ &= 1 - (1 - H + \frac{H^2}{2} - \frac{H^3}{6}) \\ &= H - \frac{1}{2}H^2 + \frac{1}{6}H^3 \\ &= H - \frac{d}{2}L + \frac{d}{6}P\end{aligned}$$

where $d = H^3$ is the degree of V, and since $H \cdot L = P$, we have $H^2 = dL$.

Next, we see the calculation of $\mathrm{ch}(\mathcal{O}_L)$. Let $\iota_L : L \hookrightarrow V$ be the embedding. Then, by Grothendick Riemann Roch, we have

$$\begin{aligned}\mathrm{ch}_V(\iota_*\mathcal{O}_L) = \mathrm{ch}_V(\iota_!\mathcal{O}_L) &= \iota_{L*}(\mathrm{ch}_L(\mathcal{O}_L) \cdot \mathrm{Td}(T_\iota)) = \iota_{L*}\left(1 \cdot \mathrm{Td}(T_\iota)\right) \\ &= \iota_{L*}\left(1 + \frac{c_1(T_\iota)}{2}\right) \qquad (*)\end{aligned}$$

where $T_\iota = T_L - \iota_L{}^*T_V \in K_0(X)$. Due to $0 \to T_L \to \iota_L{}^*T_V \to N_{L/V} \to 0$, we get:

$$T_\iota = -N_{L/V} \in K_0(X) \quad \text{and } c_2(T_\iota) = 0,\ c_1(T_\iota) = -c_1(N_{L/V}) = r - 2$$

Plug into $(*)$, we then have

$$\mathrm{ch}(\mathcal{O}_L) = \iota_L{}^*\left(1 + \frac{(r-2)p}{2}\right) = L + \frac{(r-2)P}{2}$$

Last, about $\iota_p : p \to V$, we have $\mathrm{ch}(\mathcal{O}_P) = \iota_{p*}\left(\mathrm{Td}(N_{p/V})\right) = \iota_{p*}(1) = P$ □

Proposition 4.4. For a Fano threefolds V of Picard number 1, index r , degree d the Euler pairing is given by

$$\chi(E, F) := \sum_i (-1)^i \dim \mathrm{Hom}(E, F[i]) = \chi_0(\mathrm{ch}(E)^* \cdot \mathrm{ch}(F)),$$

where

$$\chi_0(x + yH + zL + wP) = x + \frac{r^3d + 24}{12r}y + \frac{rz}{2} + w$$

Proof. We're going to use Hirzebruch – Riemann – Roch theorem, which is following identity

$$\chi(E, F) = (\mathrm{ch}(E)^*.\, \mathrm{ch}(F).\, \mathrm{Td}(T_V))_3$$

Thus, our goal now is to compute $\mathrm{Td}(T_V)$ directly. We first recall the definition of Todd class, for vector bundle F, we have

$$\mathrm{Td}(F) = 1 + \frac{c_1(F)}{2} + \frac{c_1^2(F) + c_2(F)}{12} + \frac{c_1(F)c_2(F)}{24} + \cdots$$

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$$1 = \chi(\mathcal{O}_V) = \chi(\mathcal{O}_V, \mathcal{O}_V) = (\text{ch}(\mathcal{O}_V) \cdot \text{Td}(T_V))_3 = \frac{c_1 c_2}{24}$$

(Here by Kodaira vanishing, we have $1 = \chi(\mathcal{O}_V)$).

On the other hand, since the total chern polynomial $c_t(K_V) = 1 + (-r)Ht$, we have $1 + c_1(T_V)t = c_t(\bigwedge^3 T_V) = 1 + rHt$. Thus, $c_1(T_V) = rH$ and $c_2(T_V) = \frac{24}{r}L$

This implies

$$c_t(T_V) = 1 + rH + \frac{24}{r}L + c_3(T_V),$$

and

$$\text{Td}(T_V) = 1 + \frac{rH}{2} + \frac{r^3 d + 24}{12r}L + P$$

Thus,

$$\chi(E, F) = (\text{ch}(E)^* \cdot \text{ch}(F) \cdot \text{Td}(T_V))_3 = \chi_0 \left(\text{ch}(E)^* \cdot \text{ch}(F)\right)$$

where

$$\chi_0(x + yH + zL + wP) = x + \frac{r^3 d + 24}{12r}y + \frac{rz}{2} + w$$ □

4.1 On index 1 case

After above computational proposition, we have following related table. First, on index 1 case, except for the only two cases in $g = 2$ and $g = 3$, $|-K_V = H|$ is very ample. In such cases, for index 1, genus g smooth Fano threefolds, we see that

$$\text{ch}(\mathcal{O}_V) = 1,\ \text{ch}(\mathcal{O}_H) = H - (g-1)L + \frac{g-1}{3}p,\ \text{ch}(\mathcal{O}_L) = L + \frac{1}{2}p,\ \text{ch}(\mathcal{O}_p) = p$$

and have following tables.

$\text{ch}(E)^* \cdot \text{ch}(F)$	$E = \mathcal{O}_V$	$E = \mathcal{O}_H$	$E = \mathcal{O}_L$	$E = \mathcal{O}_P$
$F = \mathcal{O}_V$	$(1,0,0,0)$	$(0,-1,1-g,\frac{1-g}{3})$	$(0,0,1,-\frac{1}{2})$	$(0,0,0,1)$
$F = \mathcal{O}_H$	$(0,1,1-g,\frac{g-1}{3})$	$(0,0,2-2g,0)$	$(0,0,0,1)$	$(0,0,0,0)$
$F = \mathcal{O}_L$	$(0,0,1,\frac{1}{2})$	$(0,0,0,-1)$	$(0,0,0,0)$	$(0,0,0,0)$
$F = \mathcal{O}_P$	$(0,0,0,1)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$

where (a,b,c,d) means $a + bH + cL + dP$.

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$\chi(E,F)$	$E=\mathcal{O}_V$	$E=\mathcal{O}_H$	$E=\mathcal{O}_L$	$E=\mathcal{O}_P$
$F=\mathcal{O}_V$	1	$-g-1$	0	-1
$F=\mathcal{O}_H$	2	$1-g$	1	0
$F=\mathcal{O}_L$	1	-1	0	0
$F=\mathcal{O}_P$	1	0	0	0

which comes from plugging the above table into
$\chi_0(x+yH+zL+wP) = x + \frac{g+11}{6}y + \frac{z}{2} + w.$

Also, we can extend the Euler form on $\mathrm{ch}(K_0(X)_{\text{num}})$ by $\mathbb{Q}$-linearity to $\mathrm{ch}(K_0(X)_{\text{num}})\otimes\mathbb{Q}$. Therefore, if we take $\{1,H,L,P\}$ to be a $\mathbb{Q}$ basis of $K_0(X)_{\text{num}}\otimes\mathbb{Q}$, then we get following table.

$\chi(u,v)$	$u=1$	$u=H$	$u=L$	$u=P$
$v=1$	1	$-\dfrac{g+11}{6}$	$\dfrac{1}{2}$	-1
$v=H$	$\dfrac{g+11}{6}$	$1-g$	1	0
$v=L$	$\dfrac{1}{2}$	-1	0	0
$v=P$	1	0	0	0

Corollary 4.5. There is no exceptional object in $\mathcal{A}_{X_{14}}$

Proof. Since $K_0(X)_{\text{num}} = \langle[\mathcal{O}_V],[\mathcal{O}_H],[\mathcal{O}_L],[\mathcal{O}_P]\rangle_{\mathbb{Z}}$, as shown in the appendix of [Kuz09], it follows that $\mathrm{ch}(\mathcal{A}_X)\subset K_0(X)_{\text{num}}$ is also a free abelian group.

Moreover, since

$$\langle \mathcal{A}_X,\, \mathcal{U}_2,\, \mathcal{O}_X\rangle$$

is a semi-orthogonal-decomposition, we have $\mathrm{ch}(\mathcal{A}_X)\oplus\mathrm{ch}(\mathcal{U}_2)\oplus\mathrm{ch}(\mathcal{O}_{X_{14}}) = K_0(X)_{\text{num}}$, and

$$\begin{aligned}\mathrm{ch}(\mathcal{A}_X)_{\mathbb{Q}} &= \left\{E\in K_0(X)_{\text{num}}\otimes\mathbb{Q}\,\middle|\; \chi(\mathcal{U}_2,E)=0,\,\chi(\mathcal{O}_X,E)=0\;\right\}\\ &\subset K_0(X)_{\text{num}}\otimes\mathbb{Q} = \langle 1,H,L,P\rangle_{\mathbb{Q}}\end{aligned}$$

Since $\mathrm{ch}(\mathcal{U}_2) = 2 - H + \frac{g-4}{2}L - \frac{g-10}{12}p$, $\mathrm{ch}(\mathcal{O}_{X_{14}}) = 1$, we then have

$$\mathrm{ch}(\mathcal{A}_X)_{\mathbb{Q}} = \langle 1-\frac{g}{2}L+\frac{g-4}{4}P,\; H-\frac{3g-6}{2}L+\frac{7g-40}{12}P\rangle_{\mathbb{Q}} \cong \mathbb{Q}^2$$

Under the isomorphsim to $\mathbb{Q}^2$, the Euler form will become

$$\chi_{\mathcal{A}_{14}} = \begin{pmatrix}-3 & -4\\ -1 & -7\end{pmatrix}$$

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Since if $E \in \mathcal{A}_{X_{14}}$ is an exceptional object, then $\chi(E, E) = 1$. But

$$(a, b)\begin{pmatrix} -3 & -4 \\ -1 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1$$

have no real solution. □

4.2 On index 2 case

Let d be degree of index 2 Fano threefolds, then $1 \leq d \leq 5$, and for $d = 3,\ 4,\ 5$, $|H|$ is very ample. Similarly, we have

$$\mathrm{ch}(\mathcal{O}_V) = 1,\ \mathrm{ch}(\mathcal{O}_H) = H - \frac{d}{2}L + \frac{d}{6}P,\ \mathrm{ch}(\mathcal{O}_L) = L,\ \mathrm{ch}(\mathcal{O}_p) = p$$

and have following two tables

$\chi(E,F)$	$E = \mathcal{O}_V$	$E = \mathcal{O}_H$	$E = \mathcal{O}_L$	$E = \mathcal{O}_P$
$F = \mathcal{O}_V$	1	$-d-1$	1	-1
$F = \mathcal{O}_H$	2	$-d$	1	0
$F = \mathcal{O}_L$	1	-1	0	0
$F = \mathcal{O}_P$	1	0	0	0

$\chi(u,v)$	$u = 1$	$u = H$	$u = L$	$u = P$
$v = 1$	1	$-\dfrac{d+3}{3}$	1	-1
$v = H$	$\dfrac{d+3}{3}$	$-d$	1	0
$v = L$	1	-1	0	0
$v = P$	1	0	0	0

Similar to the proof in index 1, we have following corollary.

Corollary 4.6. There is no exceptional object in $\mathcal{B}_{Y_3}$

Proof. since $\langle \mathcal{B}_Y,\ \mathcal{O},\ \mathcal{O}(H)\rangle$is a semi-orthogonal-decomposition, we have $\mathrm{ch}(\mathcal{B}) \oplus \mathrm{ch}(\mathcal{O}) \oplus \mathrm{ch}(\mathcal{O}(H)) = K_0(X)_{\text{num}}$, and

$$\begin{aligned}\mathrm{ch}(\mathcal{B})_{\mathbb{Q}} &= \left\{E \in K_0(X)_{\text{num}} \otimes \mathbb{Q} \,\middle|\, \chi(\mathcal{O}, E) = 0,\ \chi(\mathcal{O}(H), E) = 0\ \right\} \\ &\subset K_0(X)_{\text{num}} \otimes \mathbb{Q} = \langle 1, H, L, P\rangle_{\mathbb{Q}}\end{aligned}$$

Since $\mathrm{ch}(\mathcal{O}(H)) = H - \frac{d}{2}L - \frac{d}{6}P$, $ch(\mathcal{O}_{X_{14}}) = 1$, we then have

$$\mathrm{ch}(\mathcal{B}_Y)_{\mathbb{Q}} = \langle 1 - L, H - \frac{d}{2}L + \frac{d-6}{6}P\rangle_{\mathbb{Q}} \cong \mathbb{Q}^2$$

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Under the isomorphsim to $\mathbb{Q}^2$, the Euler form will become

$$\chi_{\mathcal{A}_{14}} = \begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix}$$

Also,

$$(a, b) \begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1$$

have no real solution. □

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Chapter 5 Relation between X_{14} and Y_3

Our first goal is to relate the data of X_{14} and its embedding into $\mathrm{Gr}(2,6)$, with a cubic threefold Y_3 together with a rank-2 vector bundle E. In Mukai's classification of smooth Fano threefolds of index 1, $d = 14$, which I denote it X_{14} here, it is five hyperplane intersect $Gr(2,6)$ in $\mathbb{P}^{14}$ (One can refer the statement in [IP99] for classification of smooth Fano n fold of index $n-2$ and $g \geq 6$).

More precisely, consider the Plücker embedding

$$Gr(2,6) \hookrightarrow \mathbb{P}(\bigwedge^2(\mathbb{C}^6))$$

given by

$$\text{Span}(u,v) \mapsto [u \wedge v],$$

where u,v is linear independent. Then X_{14} is isomorphic to 5 hyperplane cut $Gr(2,6)$ in $\mathbb{P}(\bigwedge^2(\mathbb{C}^6))$

For index 2, degree 3 smooth Fano threefolds, it must isomorphic to a cubic hypersurace in $\mathbb{P}^4$, which I denote it as Y_3 . In [Kuz04], X_{14} will correspond to a pair $(Y_3, \mathcal{E})$, where $\mathcal{E}$ is an instanton bundle of charge 2 on Y_3. Under this correspondence, the $\mathcal{A}_{X_{14}}$ will isomorphic to $\mathcal{B}_{Y_3}$.

Before we introduce the instanton bundle, we first introduce the correspondence between X_{14} and (Y_3, E), where E is the theta bundle on Y_3, and then we introduce the correspondence between theta bundle and instanton bundle of charge 2.

5.1 Correspondence beteen X_{14} and (Y_3, E)

We first construct an associated Y_3 for a given X_{14}. Let $A \cong \mathbb{C}^5$ and $V \cong \mathbb{C}^6$ be two vector space. Since X_{14} can be embedded into $Gr(2,V)$ with image cut by five hyperplane sections. Thus, we consider a $\mathbb{C}$ linear map $f : A \to (\bigwedge^2(V))^*$, and denote $X_{14} = V_+(f(A)) \cap Gr(2,V) \subseteq \mathbb{P}^{14}$. We then have following lemma

Lemma 5.1. If X_f is smooth then rank$f(a) \geq 4$ ($i.e.$ rank$f(a) = 4$ or 6) for all $a \neq 0 \in A$. For such f, we call it regular, and we will always require our f to be regular in the following article.

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Also, for given X_f we can recover f up to $GL(A) \times GL(V)$ action. For $GL(V)$ action, it depends on the embedding $X_f \hookrightarrow Gr(2, V)$. For $GL(A)$ action, it depends on different choice of the isomorphism

$$A \to \operatorname{Ker}(\bigwedge^2 V^* \to H^0(X_f, \mathcal{O}(1)))$$

We now construct the associate cubic threefolds from the regular f. For a regular f, it induce an morphsim

$$V \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{\tilde{f}} V^* \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}(A)}$$

For example, in local chart $D_+(x_0) \subset \mathbb{P}(A)$, the morphism can be written as

$$\sum_{i=1}^{5} [e_i \otimes \frac{g_i}{x_0}] \mapsto \sum_{1 \leq i,j \leq 5} B_j(e_i, -) \otimes \frac{x_j}{x_0} g_i,$$

where $g_i \in \mathcal{O}_{\mathbb{P}(A)}(D_+(x_0))$, $\{e_i\}$ is a basis of V, and $B_i(-,-) = f(e_i) \in \bigwedge^2 V^*$. Let $q = [x_0 : \cdots x_5] \in D_+(x_0)$, then it's isomorphic on the stalk of q if and only if it is isomorphic on the fiber of q. On the fiber the morphism is

$$\sum_{i=1}^{5} [e_i \otimes a_i] \mapsto \sum_{1 \leq i,j \leq 5} B_j(e_i, -) \otimes \frac{x_j}{x_0} a_i,$$

where $a_i \in \mathbb{C}$. Thus, $\tilde{f}|_q$ is not isomorphism iff $f(q) = f([x_0 : \cdots x_5]) = \sum_{1 \leq j \leq 5} x_j B_j$, is not full rank, which is equivalence to say $q \in V_+(Pf \circ f) \in \mathbb{P}(A)$. Here $Pf(M)^2 = det(M)$ for skew symmetric M. This $V_+(Pf \circ f)$ is the associate cubic threefolds in $\mathbb{P}(A)$. We denote it by Y_f. So far, we construct how to correspondence a Fano threefolds of index 1, degree 14, X_f, to a Fano threefolds of index 2, degree 3. In [Kuz04], Kuznetsov shows that there is an equivalence $\mathcal{B}_{Y_f} \to \mathcal{A}_{X_f}$.

Next, we construction the associated theta bundle E on Y_f for a given X_f From above argument, we see that

$$V \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{\tilde{f}} V^* \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}(A)}$$

is isomorphic on the open set $D_+(Pf \circ f)$. We first show that $\tilde{f}$ is injective. Consider $\operatorname{Ker}(\tilde{f})$. It have support on Y_f, which means any local section of $operatorname Ker(\tilde{f})$ is annihilate by local section of ideal sheaf of Y_f, this imply $\operatorname{Ker}(\tilde{f}) = 0$, as subsheaf of locally free is torsion free. For Coker $\tilde{f}$, since $\tilde{f}$ is isomophism on $D_+(Pf \circ f)$, $\operatorname{Supp}(\operatorname{Coker}(f)) = Y_f$. Also, as skew symmetric form have even rank, $\tilde{f}$ is always rank 4 on fiber of $q \in Y_f$. This imples $\operatorname{Coker}(\tilde{f}) = \iota_*(E_f)$, for some rank 2 vector bundle on $Y_f \overset{\iota}{\hookrightarrow} \mathbb{P}(A)$. We then have following exact sequene

$$0 \longrightarrow V \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{\hat{f}} V^* \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}(A)} \xrightarrow{\iota_*} \iota_* E_f \longrightarrow 0$$

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E_f is called the theta bundle of f. The above exact sequence give the isomorphsim $\gamma_f : V^* = H^0(V^* \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}(A)}) \cong H^0(E_f)$. The main theorem about the correspondence is following, see [Kuz04]:

Theorem 5.2. Associating the regular f gives an $GL(A) \times GL(V)$ equivariant correspondence between following

1. Regular f in $\mathbb{P}(A^* \otimes V^* \otimes V^*)$ that correspondence to smooth X_f.

2. The triple $(Y,\, E,\, \gamma)$, where Y is a cubic threefolds in $\mathbb{P}(A)$, E is a bundle of rank 2 on Y, and $\gamma : V^* \to H^0(Y,\, E)$ is an isomorphism with following conditions

$$c_1(E) = 2[H],\, c_2(E) = 5[l],\, H^\bullet(Y, E(t)) = 0, \text{ for } 1 \le t \le 3.$$

5.2 Correspondence between the theta bundle and instanton bundle of charge 2

Next we introduce the relation between theta bundle on Y and the related instanton bundle.

Definition 5.3. Let Y_3 be a cubic threefolds in $\mathbb{P}^4$, then $\mathcal{E}$ is an instanton bundle if $c_1(\mathcal{E}) = 0$, $H^1(Y,\, \mathcal{E}(-1)) = 0$. It's called instanton bundle of topological charge $k \in \mathbb{Z}$ if, additionally, $c_2(\mathcal{E}) = k[l]$.

From following proposition, we can see the correspondence between instanton bundle of charge 2 and the theta bundle

Proposition 5.4. The following is equivalence

1. $\mathcal{E}$ is an instanton bundle of charge 2

2. $\mathcal{E}(1)$ satisfies the conditions in Theorem 2.5, i.e.

$$c_1(E) = 2[H],\, c_2(E) = 5[l],\, H^\bullet(Y, E(t)) = 0, \text{ for } 1 \le t \le 3.$$

3. $\mathcal{E}(1)$ is a theta bundle

4. $\mathcal{E}$ is an instanton bundle of charge satisfies

$$H^1(Y, \mathcal{E}(-1)) = H^1(Y, \mathcal{E}(1)) = H^2(Y, \mathcal{E}(1)) = H^2(Y, \mathcal{E} \otimes \mathcal{E}) = 0$$

From Theorem 2.5, we see that the isomorphism class of X_f will correspond to the isomorphism class of the the pair (Y_f, E), where E is a theta bundle. Also, from Proposition 2.6, we see the correspond between theta bundle and instanton bundle of charge 2. Thus, there should be isomorphism between the moduli stack of X_{14}, denoted as $\mathcal{M}^1_{14}$, and the moduli stack of pair $(Y_3, \mathcal{E})$, denoted as $\mathcal{M}^2_3$. In [Kuz04], Kuznetsov also shows that under this isomorphism,

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$i.e.$ the isomorphic class of $X_f \mapsto$ isomorphic class of Y_f , we have $\mathcal{A}_{X_{14}} \cong \mathcal{B}_{Y_3}$. Moreover, the fiber of this correspond is isomorphic to the moduli stack of instanton bundle of charge on Y_3, which is the bundle $\mathcal{E}$ satisfy the condition 4 in Proposition 2.6. Here we denote it $\mathcal{M}_0(Y)$. In [MT01], they show that $\mathcal{M}_0(Y)$ is smooth of dimension 5.

5.3 Equivalence Between $\mathcal{A}_{X_{14}}$ and $\mathcal{B}_{Y_3}$

Here we briefly introduce the main result in [Kuz04].

Theorem 5.5. (Kuznetsov) For any X_{14}, the associated Y_3 satisfies the fact that $\mathcal{B}_{Y_3}$ is equivalence to $\mathcal{A}_{X_{14}}$

Consider following diagram

$$\begin{array}{ccccc} \mathbb{P}_Y(E^*) & & & & \mathbb{P}_X(\mathcal{U}) \\ & \searrow^{\psi_1} & & \swarrow^{\psi_2} & \\ \downarrow p_Y & & Q & & \downarrow p_X \\ & & \downarrow \iota_Q & & \\ Y & & \mathbb{P}(V) & & X \end{array}$$

where $\mathbb{P}_Y(E^*) := \mathbb{P}(\bigoplus_i S^i(E))$ and $\mathbb{P}_X(\mathcal{U}) := \mathbb{P}(\bigoplus_i S^i(\mathcal{U}^*))$. p_X and p_Y denote the natural projections. The morphism ψ_1 is defined as the composition

$$\mathbb{P}_Y(E^*) \longrightarrow \mathrm{Fl}(1,2;V) \longrightarrow \mathbb{P}(V),$$

and similarly for ψ_2. Both morphisms have the same image $Q \subset \mathbb{P}(V)$ (see [Kuz04]).

Consider the fiber product

$$W = \mathbb{P}_Y(E^*) \times_Q \mathbb{P}_X(\mathcal{U}).$$

It can be regarded as a closed subvariety of $\mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U})$. More precisely,

$$W = \{((L_a, a), (L_U, U)) \in \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U}) \mid L_a = L_U\},$$

where $a \in Y$ is a skew form, and $U \in X \subset \mathrm{Gr}(2, V)$ is a two-dimensional subspace of V. The lines $L_a \subset E^*|_a$ and $L_U \subset \mathcal{U}|_U$ represent the points in the projective bundles.

Let K denote the pushforward of the structure sheaf $\mathcal{O}_W$ along the natural embedding $W \hookrightarrow \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U})$. We denote by $\mathcal{O}_Y(e)$ and $\mathcal{O}_X(e)$ the pullbacks of $\mathcal{O}(1)$ via ψ_1 and ψ_2, respectively. These are also the relative ample line bundles on $\mathbb{P}_Y(E^*)$ and $\mathbb{P}_X(\mathcal{U})$. When no confusion arises, we will simply write $\mathcal{O}(e)$. We also set:

$$\mathcal{O}(y) := p_Y^*\mathcal{O}(1), \qquad \mathcal{O}(x) := p_X^*\mathcal{O}(1).$$

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In [Kuz04], for any Fano threefold X_{14}, the associated cubic threefold Y_3 satisfies

$$\mathcal{B}_{Y_3} \simeq \mathcal{A}_{X_{14}}.$$

Moreover, for any $A \in \mathcal{B}_{Y_3}$, the equivalence is given explicitly by

$$A \mapsto p_{X*} \circ \Phi_K \circ \mathbf{R}_{\mathcal{O}(3e-y)} \left(p_Y^* A \otimes \mathcal{O}_Y(2e)\right),$$

where: Φ_K is the Fourier-Mukai transform from $D^b(\mathbb{P}_Y(E^*))$ to $D^b(\mathbb{P}_X(\mathcal{U}))$ with kernel K, $\mathbf{R}_{\mathcal{O}(3e-y)}$ denotes the right mutation functor through the line bundle $\mathcal{O}(3e-y)$.

5.4 Description of the Equivalence by a Fourier Mukai transform

In Appendix B of [KPS18], Kuznetsov gives a geometric proof that the Hilbert scheme of conics on X_{14} is isomorphic to the Hilbert scheme of lines on the associated cubic threefold Y_3. At the end of the paper, they also mention that the equivalence between $\mathcal{A}_{X_{14}}$ and $\mathcal{B}_{Y_3}$ can be expressed as a single Fourier-Mukai transform, and under this equivalence, the ideal sheaf of a conic on X_{14} corresponds to the ideal sheaf of a line on Y_3. Since the primary goal in Appendix B of [KPS18] is to establish the isomorphism between these Hilbert schemes, this result is stated without proof.

In what follows, we imitate the proof from [Kuz04] and rewrite the equivalence

$$A \mapsto p_{X*} \circ \Phi_K \circ \mathbf{R}_{\mathcal{O}(3e-y)} \left(p_Y^* A \otimes \mathcal{O}_Y(2e)\right)$$

from $\mathcal{B}_{Y_3} \to {\mathcal{A}_{X_{14}}}^* = {}^{\perp}\langle \mathcal{O}, \mathcal{U}^* \rangle$ as a Fourier−Mukai transform

$$\Phi_{\mathcal{K}} : D^b(Y) \longrightarrow D^b(X),$$

where

$$Z = \{(a, U) \in Y \times X \mid \ker(a) \cap U \neq 0\}, \text{ and } \mathcal{K} = \iota_{Z*}\big(\mathcal{O}_Y(2e)|_Z\big)$$

Note that Z is actually the image of following embedding (see [Kuz04] about the composition of following map is an embedding):

$$W \xhookrightarrow{\iota_W} \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U}) \xrightarrow{p_Y \times p_X} Y \times X$$

This identification follows from the exact sequence:

$$0 \longrightarrow E^* \longrightarrow V \otimes \mathcal{O}_Y \longrightarrow V^* \otimes \mathcal{O}_Y(1) \longrightarrow 0$$

In this exact sequence, for each fiber at a point a, we see that $E^*|_a$ is the kernel of the skew-symmetric form a. Here, $a \in Y \subset \mathbb{P}(V)$ is the class of a skew form. Hence, we have $E^*|_a \supset L_a = L_U \subset U$ for some line L_a, L_U if and only if $\ker(a) \cap U \neq 0$. This

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implies that the image of the composition $\iota_W \circ (p_Y \times p_X)$ is precisely Z.

First, since right mutation commutes with autoequivalences, we have:

$$\mathbf{R}_{\mathcal{O}(3e-y)}\big((p_Y^*A) \otimes \mathcal{O}_Y(2e)\big) = \big(\mathbf{R}_{\mathcal{O}(e-y)}(p_Y^*A)\big) \otimes \mathcal{O}_Y(2e), \quad \text{in } D^b(\mathbb{P}_Y(E^*)).$$

Moreover, since Fourier–Mukai transforms behave naturally under tensoring with line bundles on the source, we also have

$$\Phi_K(- \otimes \mathcal{O}_Y(2e)) = \Phi_{K\otimes\mathcal{O}_Y(2e)}(-),$$

and hence,

$$p_{X*} \circ \Phi_K \circ \mathbf{R}_{\mathcal{O}(3e-y)}\big((p_Y^*A) \otimes \mathcal{O}_Y(2e)\big) = p_{X*} \circ \Phi_{K(2e)} \circ \mathbf{R}_{\mathcal{O}(e-y)}(p_Y^*A).$$

Our plan is to rephrase the right mutation as a Fourier–Mukai transform. For this, we recall the following proposition from [Kuz04], whose proof was omitted; we provide the details here in the case of a right mutation:

Proposition 5.6. Let M be a smooth projective variety and let $E \in D^b(M)$. Then the right mutation through E can be expressed as a Fourier–Mukai transform:

$$\mathbf{R}_E \cong \Phi_{\mathcal{K}_E},$$

where $\mathcal{K}_E$ is described by the following exact triangle in $D^b(M \times M)$:

$$\mathcal{K}_E \longrightarrow \Delta_*\mathcal{O}_M \xrightarrow{\text{ev}^*} RHom(E, \omega_M[\dim M]) \boxtimes E.$$

In the case we need later, $\omega_{\mathbb{P}_{Y_3}(E^*)} = \mathcal{O}(-2e)$ and $E = \mathcal{O}(e-y)$, the third term in the triangle becomes

$$\mathcal{O}(y-3e) \boxtimes \mathcal{O}(e-y).$$

Proof. Let $p_1, p_2 : M \times M \to M$ be the projections onto the first and second factors, respectively. Since the Fourier–Mukai transform is a composition of exact functors in the derived sense, we have the following exact triangle for any $F \in D^b(M)$:

$$\Phi_{\mathcal{K}_E}(F) \longrightarrow F \xrightarrow{\text{ev}^*} \mathbf{R}p_{2*}\big(R\mathcal{H}om(E, \omega_M[\dim M]) \boxtimes E \otimes p_1^*F\big).$$

We now compute the third term of the triangle:

$$\begin{aligned} Rp_{2*}\big(R\mathcal{H}om(E, \omega_M[\dim M]) \boxtimes E \otimes p_1^*F\big) &= Rp_{2*}\big(R\mathcal{H}om(E, \omega_M[\dim M] \otimes F) \boxtimes E\big) \\ &= Rp_{2*}\big(R\mathcal{H}om(E, S(F)) \boxtimes E\big) \\ &\qquad \text{(by definition of Serre functor)} \\ &\cong R\Gamma\big(R\mathcal{H}om(E, S(F))\big) \otimes E \quad \text{(by flat base change)} \\ &\cong RHom(F, E)^* \otimes E. \end{aligned}$$

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On the other hand, the right mutation functor $\mathbf{R}_E$ is defined by the exact triangle:

$$\mathbf{R}_E(F) \longrightarrow F \xrightarrow{\text{ev}^*} RHom(F, E)^* \otimes E.$$

Thus, we conclude that $\Phi_{\mathcal{K}_E}(F) \cong \mathbf{R}_E(F)$, as required. □

Back to the reduction of

$$p_{X*}\phi_K \mathbf{R}_{\mathcal{O}(3e-y)}\big((p_Y^* A) \otimes \mathcal{O}_Y(2e)\big).$$

As shown earlier, this is equal to

$$p_{X*}\phi_{K(2e)}\mathbf{R}_{\mathcal{O}(e-y)}\big(p_Y^* A\big),$$

and by the previous proposition, it can also be written as

$$p_{X*}\phi_{K(2e)}\phi_{\mathcal{K}_{\mathcal{O}(e-y)}}(p_Y^* A).$$

Hence, we are led to consider the composition of two Fourier–Mukai transforms. That is, we aim to compute the kernel K_1 such that

$$\phi_{K_1} = \phi_{K(2e)} \circ \phi_{\mathcal{K}_{\mathcal{O}(e-y)}},$$

and we denote this convolution by

$$K_1 := K(2e) * \mathcal{K}_{\mathcal{O}(e-y)}.$$

Since the operation $K(2e) * (-)$ is a composition of pullbacks and pushforwards, it is again an exact functor. Therefore, we obtain the following exact triangle:

$$K_1 \longrightarrow K(2e) \longrightarrow K(2e) * \big(\mathcal{O}(y-3e) \boxtimes \mathcal{O}(e-y)\big).$$

To compute the last term, observe:

$$\begin{aligned} K(2e) * \big(\mathcal{O}(y-3e) \boxtimes \mathcal{O}(e-y)\big) &= \mathcal{O}(y-3e) \boxtimes \phi_{K(2e)}\big(\mathcal{O}(e-y)\big) \\ &\quad (\phi_A \circ \phi_{B\boxtimes C} = \phi_{B\boxtimes \phi_A(C)}) \\ &= \mathcal{O}(y-3e) \boxtimes \phi_K\big(\mathcal{O}(3e-y)\big) \\ &\quad (\text{since } \phi_{K(2e)}(\mathcal{F}) = \phi_K(\mathcal{F}(2e))) \\ &= \mathcal{O}(y-3e) \boxtimes \mathcal{O}(x-e) \\ &\quad (\text{by Proposition 3.7 in [Kuz04]}) \end{aligned}$$

So far, we have shown that

$$p_{X*}\phi_K \mathbf{R}_{\mathcal{O}(3e-y)}\big((p_Y^* A) \otimes \mathcal{O}_Y(2e)\big) = p_{X*}\phi_{K_1}(p_Y^* A),$$

where the kernel K_1 is given by the exact triangle described above.

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Next, we introduce the Grothendieck–Verdier duality theorem, which we will later apply to understand dual behavior under pushforward:

Theorem 5.7 (Grothendieck–Verdier duality)**.** Let $f : M \to N$ be a morphism between smooth projective varieties. Define the duality functor

$$\mathbb{D}_M(-) := R\mathcal{H}om(-, \omega_M[\dim M]),$$

and similarly for $\mathbb{D}_N$. Then, we have the natural equivalence

$$\mathbf{R}f_* \circ \mathbb{D}_M \cong \mathbb{D}_N \circ \mathbf{R}f_*.$$

Remark 5.8. The situation we need is the following identity:

$$Rp_{Y*}\big(\mathcal{O}(y-3e)\big) \cong E^*(-y)[-1].$$

Proof. Note that we have the following canonical isomorphisms of canonical bundles:

$$\omega_{\mathbb{P}_Y(E^*)} \cong \mathcal{O}(-2e), \qquad \omega_{\mathbb{P}_X(\mathcal{U})} \cong \mathcal{O}(-2e), \qquad \omega_{Y_3} \cong \mathcal{O}(-2y), \qquad \omega_{X_{14}} \cong \mathcal{O}(-x).$$

Applying Grothendieck–Verdier duality, we obtain:

$$\begin{aligned}\big(Rp_{Y*}\mathcal{O}(y-3e)\big)^* \otimes \mathcal{O}(-2y)[3] &\cong Rp_{Y*}\big(\mathcal{O}(y-3e)^* \otimes \mathcal{O}(-2e)\big)[4] \\ &= Rp_{Y*}\big(\mathcal{O}(3e-y) \otimes \mathcal{O}(-2e)\big)[4] \\ &= Rp_{Y*}\big(\mathcal{O}(e-y)\big)[4].\end{aligned}$$

Rearranging the identity, we get:

$$Rp_{Y*}\mathcal{O}(y-3e) \cong E^*(-y)[-1] \qquad \square$$

Back to the proof. Since $\phi_{K_1} \circ p_Y^* = \phi_{(p_Y \times \mathrm{id})_* K_1}$, and we have the exact triangle

$$(p_Y \times \mathrm{id})_* K_1 \longrightarrow (p_Y \times \mathrm{id})_* K(2e) \longrightarrow E^*(-y) \boxtimes \mathcal{O}(x-e)[-1],$$

let us write $K_2 := (p_Y \times \mathrm{id})_* K_1$. So far, we obtain

$$p_{X*}\phi_K \mathbf{R}_{\mathcal{O}(3e-y)}((p_Y^* A) \otimes \mathcal{O}_Y(2e)) = p_{X*}\phi_{K_2}.$$

Similarly, since $p_{X*}\phi_{K_2} = \phi_{(\mathrm{id}\times p_X)_* K_2}$, we conclude:

$$p_{X*}\phi_K \mathbf{R}_{\mathcal{O}(3e-y)}((p_Y^* A) \otimes \mathcal{O}_Y(2e)) = \phi_{K_3},$$

where $K_3 := (\mathrm{id} \times p_X)_* K_2 \in D^b(X \times Y)$, and K_3 is defined by the following exact triangle:

$$K_3 \longrightarrow (p_Y \times p_X)_* K(2e) \longrightarrow E^*(-y) \boxtimes Rp_{X*}(\mathcal{O}(x-e))[-1].$$

(Note that the e in $K(2e)$ refers to the relative $\mathcal{O}(1)$ on $\mathbb{P}_Y(E^*)$.)

Our goal is now to show that $\phi_{K_3} = \phi_{(p_Y \times p_X)_* K(2e)}$ on $\mathcal{A}_Y$. Take $F \in \mathcal{A}_Y$, then it

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suffices to prove when $\mathcal{K}' = E^*(-y) \boxtimes Rp_{X*}(\mathcal{O}(x-e))[-1]$, we have $\phi_{\mathcal{K}'}(F) = 0$.

Proof. We compute:

$$\phi_{\mathcal{K}'}(F) = R\Gamma\left((E^*(-y) \otimes F) \boxtimes Rp_{X*}(\mathcal{O}(x-e))\right)[-1].$$

By flat base change, this is:

$$R\Gamma(Y, E^*(-y) \otimes F) \otimes Rp_{X*}(\mathcal{O}(x-e))[-1].$$

Now consider the exact sequence

$$0 \to E^*(-1) \to V \otimes \mathcal{O}_Y(-1) \to V^* \otimes \mathcal{O}_Y \to 0.$$

By the definition of $F \in \mathcal{A}_Y$, we know:

$$R\Gamma(Y, V \otimes \mathcal{O}_Y(-y) \otimes F) = RHom(V \otimes \mathcal{O}_Y(y), F) = 0,$$

and similarly

$$R\Gamma(Y, V^* \otimes \mathcal{O}_Y \otimes F) = RHom(V \otimes \mathcal{O}_Y, F) = 0.$$

Therefore,

$$R\Gamma(Y, E^*(-y) \otimes F) = 0.$$

Hence the whole expression vanishes. □

With the above argument, we have shown that the equivalence given in [Kuz04],

$$p_{X*}\phi_K \mathbf{R}_{\mathcal{O}(3e-y)}((p_Y^* A) \otimes \mathcal{O}_Y(2e)),$$

is equal to $\phi_{(p_Y \times p_X)_* K(2e)}$ on $\mathcal{A}_Y$, which is in turn equal to $\phi_{\mathcal{K}}$, where $\mathcal{K} = \iota_{Z*}\big(\mathcal{O}_Y(2e)|_Z\big)$.

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