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對於動機版山本積分的餘積公式

Coproduct Formula for Motivic Version of Yamamoto's Integral

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口試委員會審定書

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Coproduct Formula for Motivic Version of Yamamoto's Integral

本論文係范谷瑜君（R10221014）在國立臺灣大學數學系完成之碩士學位論文，於民國 113 年 5 月 31 日承下列考試委員審查通過及口試及格，特此證明

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所　　長：

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摘要

亞歷山大·岡察洛夫證明了對於動機疊代積分的霍普夫代數中餘積之具體公式。山本積分是一種疊代積分的推廣，山本修司引入山本積分並給予多重 ζ 星值一種新的積分表示式。在本篇論文中，我們考慮動機版山本積分並推廣岡察洛夫的餘積公式至那些動機積分。作為範例，我們將計算一些特定形式的舒爾多重 ζ 值之餘積。

關鍵字：動機疊代積分、山本積分、岡察洛夫的餘積公式

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Abstract

Goncharov proved an explicit formula for the coproduct in the Hopf algebra of motivic iterated integrals. Yamamoto introduced Yamamoto's integral which generalizes iterated integrals and gave a new integral expression for multiple zeta star values using Yamamoto's integral. In this paper, we consider the motivic version of Yamamoto's integral and generalize Goncharov's coproduct formula to those motivic integrals. As an example, we will compute the coproduct of a certain type of Schur multiple zeta values.

Keywords: motivic iterated integral, Yamamoto's integral, Goncharov's coproduct formula

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Contents

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Notation

$\zeta(k_1, \ldots, k_d)$	Multiple zeta value, 2
γ	Piecewisely smooth path, 2
$I_\gamma(a_0; a_1, \ldots, a_k; a_{k+1})$	Iterated integral, 2
$(X, \preceq, \delta)$	Labeled partially ordered set, 3
$\preceq$	Partial order, 3
δ	Labeling map, 3
$\mathsf{Tot}(X)$	Collection of fine totally ordered sets of X, 5
$I_\gamma(X)$	Yamamoto's integral, 5
$\mathcal{P}$	Commutative ring of periods, 8
$I^{\mathfrak{m}}_\gamma(a_0; a_1, \ldots, a_k; a_{k+1})$	Motivic iterated integral, 8
$\zeta^{\mathfrak{m}}(k_1, \ldots, k_d)$	Motivic multiple zeta value, 9
$I^{\mathfrak{m}}_\gamma(X)$	Motivic version of Yamamoto's integral, 9

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μ	Motivic version of $2\pi i$, 10
$\mathcal{A}$	Quotient ring $\mathcal{P}/\mu\mathcal{P}$, 10
$\pi : \mathcal{P} \to \mathcal{A}$	Natural projection, 10
$\Delta : \mathcal{P} \to \mathcal{A} \otimes \mathcal{P}$	Motivic coaction, 10
$\mathcal{H}$	$\mathbb{Q}$-algebra $\langle I^{\mathfrak{m}}_{\gamma}(a_0; a_1, \ldots, a_k; a_{k+1})\rangle_{\mathbb{Q}}$, 10
$I^{\mathfrak{a}}$	Projection of $I^{\mathfrak{m}}$ under π, 10
$\bigoplus_{k\geq 0} \mathcal{A}_k$	Graded structure of $\mathcal{A}$, 11
$\mathfrak{L}$	Quotient ring $\mathcal{A}/\mathcal{A}^2_{>0}$, 11
$\pi' : \mathcal{A} \to \mathfrak{L}$	Natural projection, 11
$I^{\mathfrak{l}}$	Projection of $I^{\mathfrak{a}}$ under π', 11
D_r	Infinitesimal coaction, 11
$\widetilde{X}_{\gamma}$	Extension of X with respect to γ, 13
$Y^{\prec X}$ (resp. $Y^{\succ X}$)	Subset of X (lower (resp. upper) set of Y in X), 14
$Y^{\rightarrow X}$ (resp. $Y^{\leftarrow X}$)	Subset of X (set of maximal (resp. minimal) elements in $Y^{\prec X}$ (resp. $Y^{\succ X}$)), 14
$Y^{\preceq X}$ (resp. $Y^{\succeq X}$)	Subset of X (lower (resp. upper) closure of Y in X), 15
X_r	Collection of some partially ordered set of X, 16
$X_{(p,\widehat{Y},q)}$	Labeled subposet of X, 16

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$\mathsf{Tot}_{\overline{Y}}(X)$	Subset of $\mathsf{Tot}(X)$, 17
$P_Y(X)$	Labeled subposet of X, 17
$T_{Y_1,Y_2}(\overline{X})$	The triple $(X, (\preceq_{\overline{X}} \setminus \preceq_{Y_1}) \cup \preceq_{Y_2}, \delta)$, 17
$x \leftrightarrow x'$	Adjacecy of elements in X, 23
$C_{X,Y}(y)$	Connected component of y, 23
$C_X(Y)$	Component set of Y, 23
Δ_Y	Map from $\mathcal{H}$ to $\mathcal{A} \otimes \mathcal{H}$, 26

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Chapter 1 Introduction

1.1 Multiple zeta values

Multiple zeta values are a special family of real numbers that appear in various mathematical fields such as the theory of mixed Tate motives, knot invariant, or evaluation of Feynman diagrams. In recent decades, numerous studies have been conducted on multiple zeta values. The history of multiple zeta values goes back to the time of Leonhard Euler who discovered the famous evaluation formula

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

More generally, he showed that the values of the Riemann zeta function at positive even integers are expressed as

$$\zeta(2k) = \sum_{0<n} \frac{1}{n^{2k}} = \frac{B_{2k}}{2(2k)!}(2\pi i)^{2k}$$

where B_{2k} is the $2k$-th Bernoulli number. Euler also tried to evaluate the values of the Riemann zeta function at a positive odd integer in terms of π, and although Euler failed to achieve the original objective, he found different types of evaluation

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formulas. Let

$$\zeta(k_1, k_2) = \sum_{0<n_1<n_2} \frac{1}{n_1^{k_1} n_2^{k_2}}$$

be a double sum analog of Riemann zate values. Euler showed that

$$\zeta(3) = \zeta(1, 2)$$

and analogous expressions for other Riemann zeta values. Motivated by Euler's work Hoffman defined multiple zeta values (MZVs in short) as follows.

Definition 1.1 (multiple zeta value).

$$\zeta(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} \qquad (k_1, \ldots, k_{d-1} \in \mathbb{Z}_{>0}, k_d \in \mathbb{Z}_{>1})$$

Hoffman started to investigate linear/algebraic relations with rational coefficients among MZVs and proved several families of MZV relations including a special case of the duality relation. A few years later, Kontsevich found an iterated integral expression for MZVs. For example,

$$\zeta(3) = \int_{0<t_1<t_2<t_3<1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3}.$$

In general, we define the iterated integral as follows.

Definition 1.2 (iterated integral). For $a_0, \ldots, a_{k+1} \in \mathbb{C}$ with $a_0 \neq a_1, a_k \neq a_{k+1}$, and a piecewisely smooth path $\gamma : [0, 1] \to \mathbb{C}$ from a_0 to a_{k+1} such that $\gamma((0, 1)) \subset \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ (that is, γ does not pass through any points of $\{a_1, \ldots, a_k\}$), we define

$$I_\gamma(a_0; a_1, \ldots, a_k; a_{k+1})$$

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by the iterated integral

$$\int_{0<t_1<\cdots<t_k<1} \prod_{j=1}^{k} \frac{d\gamma(t_j)}{\gamma(t_j)-a_j} \in \mathbb{C}.$$

Remark 1.3. When the γ is not explicitly explained, γ is a piecewisely smooth path from a_0 to a_{k+1} which does not pass through any poles of differential forms.

Remark 1.4. When the $\gamma(t) = a_0(1-t) + a_{k+1}t$, we denote $I_\gamma(a_0; a_1, \ldots, a_k; a_{k+1})$ simply by $I(a_0; a_1, \ldots, a_k; a_{k+1})$.

Remark 1.5. MZVs are iterated integrals with $a_i \in \{0,1\}$, $a_0 = 0, a_1 = 1, a_k = 0, a_{k+1} = 1$ and $\gamma(t) = t$. More precisely,

$$\zeta(k_1, \ldots, k_d) = (-1)^d I(0; 1, \{0\}^{k_1-1}, \ldots, 1, \{0\}^{k_d-1}; 1).$$

1.2 Yamamoto's integral

Yamamoto defined Yamamoto's integral associated with 2-posets, which generalizes the iterated integral and provides a simple representation for multiple zeta star values ([Yam14]). Later, Hirose, Murahara, and Onozuka proved that Yamamoto's integral expression can also be used to express Schur multiple zeta values with constant entries on the diagonals ([HMO23]). Here, we define Yamamoto's integral in a generalized setting and state its basic property ([Yam14], Proposition 2.3).

Definition 1.6 (labeled poset). $X = (X, \preceq, \delta)$ is called a labeled poset if X is a finite set,

$$\preceq \subset X \times X$$

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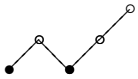
is a partial order, and

$$\delta : X \to \mathbb{C}$$

is a map (this map is called a labeling map of X).

We denote the corresponding strict partial order of $\preceq$ by $\prec$.

A labeled poset X is represented by a Hasse diagram with colored vertices. For example, the diagram



represents

$$X := \{x_1, x_2, x_3, x_4, x_5\}$$

$$\preceq := \{(x_1, x_2), (x_3, x_2), (x_3, x_4), (x_4, x_5), (x_3, x_5)\}$$

$$\delta(x) := \begin{cases} 0 & \text{if } x = x_2, x_4, x_5 \\ 1 & \text{if } x = x_1, x_3. \end{cases}$$

Definition 1.7. (1) For $(X, \preceq, \delta)$ and $Y \subset X$, we define labeled subposet as

$$Y = (Y, \preceq', \delta') := (Y, \preceq \cap Y^2, \delta|_Y).$$

(2) For $X = (X, \preceq_X, \delta_X)$ and $Y = (Y, \preceq_Y, \delta_Y)$, we can define the direct sum

$$X \sqcup Y = (X \sqcup Y, \preceq_{X \sqcup Y}, \delta_{X \sqcup Y}),$$

where

$$a \preceq_{X \sqcup Y} b \iff \begin{cases} a \preceq_X b \text{ and } a, b \in X \\ a \preceq_Y b \text{ and } a, b \in Y, \end{cases}$$

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and

$$\delta_{X\sqcup Y}(a) = \begin{cases} \delta_X(a) \text{ if } a \in X \\ \\ \delta_Y(a) \text{ if } a \in Y. \end{cases}$$

(3) A labeled poset $(X, \preceq, \delta)$ is said to be irreducible if

$$X \neq Y \sqcup X_{\widehat{Y}},$$

as a direct sum of posets for all non-empty proper subset Y of X ($\emptyset \subsetneq Y \subsetneq X$), where $X_{\widehat{Y}}$ is the subposet of X restricted to $X \setminus Y$.

(4) For $(X, \preceq, \delta)$ and incomparable elements $a, b \in X$, we define

$$X_a^b \coloneqq (X, \preceq \cup \{(x, y) |\ x \preceq a \text{ and } b \preceq y\}, \delta).$$

Notice that X_a^b becomes a labeled poset.

(5) For $(X, \preceq, \delta)$, we define

$$\mathsf{Tot}(X) \coloneqq \{(X, \preceq', \delta) \mid \ \preceq \subset \preceq' \text{ and } \preceq' \text{ is a total order}\}.$$

Definition 1.8 (admissible labeled poset). For a path $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma((0, 1)) \subseteq \mathbb{C} \setminus \delta(X)$, we say that X is admissible (with respect to γ) if $\delta(x) \neq \gamma(1)$ for any maximal elements of X, and $\delta(x) \neq \gamma(0)$ for any minimal elements of X.

Definition 1.9 (Yamamoto's integral). Let γ be a path. For an admissible labeled poset X, Yamamoto's integral is defined by

$$I_\gamma(X) \coloneqq \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta(x)}(\gamma(t_x)),$$

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where

$$\Delta(X) := \{\mathbf{t} = (t_x)_{x\in X} \in (0,1)^X \mid t_x < t_y \text{ if } x \prec y\},$$

$$\omega_a(t) := \frac{dt}{t-a}.$$

Remark 1.10. The definition 1.9 slightly differs from that in the original article ([Yam14]), where the labeling map is

$$\delta : X \to \{0,1\},$$

the associated 1-forms are

$$\omega_0(t) = \frac{dt}{t} \text{ and } \omega_1(t) = \frac{dt}{1-t},$$

and the path $\gamma : [0,1] \to \mathbb{C}$ is defined by

$$\gamma(t) = t.$$

Proposition 1.11. Let X, Y be two labeled posets. For a path $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma((0,1)) \subseteq \mathbb{C} \setminus (\delta_X(X) \cup \delta_Y(Y))$ and incomparable elements $a, b \in X$, we have

(1) $I_\gamma(X)I_\gamma(Y) = I_\gamma(X \sqcup Y)$.

(2) $I_\gamma(X) = I_\gamma(X_a^b) + I_\gamma(X_b^a)$.

Proof. (1) By Definition 1.9, we have

$$\begin{aligned} I_\gamma(X)I_\gamma(Y) &= \int_{\Delta(X)} \prod_{x\in X} \omega_{\delta_X(x)}(\gamma(t_x)) \int_{\Delta(Y)} \prod_{y\in Y} \omega_{\delta_Y(y)}(\gamma(t_y)) \\ &= \int_{\Delta(X)\times\Delta(Y)} \prod_{x\in X} \omega_{\delta_X(x)}(\gamma(t_x)) \prod_{y\in Y} \omega_{\delta_Y(y)}(\gamma(t_y)). \end{aligned}$$

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By Definition 1.7, we have

$$\Delta(X) \times \Delta(Y) = \Delta(X \sqcup Y).$$

Hence, we get

$$I_\gamma(X) I_\gamma(Y) = \int_{\Delta(X \sqcup Y)} \prod_{x \in X \sqcup Y} \omega_{\delta_{X \sqcup Y}(x)}(\gamma(t_x)) = I_\gamma(X \sqcup Y).$$

(2) By Definition 1.9, we have

$$\begin{aligned} I_\gamma(X_a^b) + I_\gamma(X_b^a) &= \int_{\Delta(X_a^b)} \prod_{x \in X_a^b} \omega_{\delta_{X_a^b}(x)}(\gamma(t_x)) + \int_{\Delta(X_b^a)} \prod_{x' \in X_b^a} \omega_{\delta_{X_b^a}(x')}(\gamma(t_{x'})) \\ &= \int_{\Delta(X_a^b)} \prod_{x \in X} \omega_{\delta_X(x)}(\gamma(t_x)) + \int_{\Delta(X_b^a)} \prod_{x' \in X} \omega_{\delta_X(x')}(\gamma(t_{x'})) \\ &= \int_{\Delta(X_a^b) \sqcup \Delta(X_b^a)} \prod_{x \in X} \omega_{\delta_X(x)}(\gamma(t_x)). \end{aligned}$$

By Definition 1.7, we have

$$\Delta(X_a^b) \sqcup \Delta(X_b^a) = \Delta(X).$$

Hence, we get

$$I_\gamma(X_a^b) + I_\gamma(X_b^a) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta_X(x)}(\gamma(t_x)) = I_\gamma(X).$$

□

Repeated use of (2) of Proposition 1.11 yields an expression for Yamamoto's

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integral as a sum of iterated integrals:

$$I_\gamma(X) = \sum_{Y \in \mathrm{Tot}(X)} I_\gamma(Y).$$

1.3 Motivic iterated integral

Motive is a concept in algebraic geometry proposed by Alexander Grothendieck in the 1960s. In particular, the theory of mixed Tate motives has intricate and multifaceted relationships with MZVs. Deligne and Goncharov defined the motivic iterated integrals, which are elements in the commutative ring

$$\mathcal{P} := \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{MT}(\mathbb{Q})}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}))$$

of periods obtained from the comparison isomorphism between the fiber functors ω_{dR} and ω_{B} of the Tannakian category $\mathcal{MT}(\mathbb{Q})$ of mixed Tate motives over $\mathbb{Q}$ (here, $\mathrm{Isom}^{\otimes}_{\mathcal{MT}(\mathbb{Q})}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})$ denotes the affine scheme of tensor isomorphisms from ω_{dR} to ω_{B}). It is known that there is a ring homomorphism

$$\mathrm{per} : \mathcal{P} \to \mathbb{C}$$

called period map, and period map per is speculated to be injective. For $k \geq 0$, $a_0, \ldots, a_{k+1} \in \mathbb{Q}$ and a piecewisely smooth path $\gamma : [0,1] \to \mathbb{C}$ from a_0 to a_{k+1} such that $\gamma((0,1)) \subset \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ and $\gamma'(0), \gamma'(1) \in \mathbb{Q} \setminus \{0\}$, one can define the motivic iterated integral

$$I^{\mathfrak{m}}_\gamma(a_0; a_1, \ldots, a_k; a_{k+1})$$

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which is mapped to $I_\gamma(a_0; a_1, \ldots, a_k; a_{k+1})$ under the period map, and thus the motivic multiple zeta values are defined by

$$\zeta^{\mathfrak{m}}(k_1, \ldots, k_d) := (-1)^d I^{\mathfrak{m}}_\gamma(0; 1, \{0\}^{k_1-1}, \ldots, 1, \{0\}^{k_d-1}; 1).$$

Motivic iterated integral I^m_γ satisfy various properties. Here, we list some of them which we will use in later arguments.

Proposition 1.12. (1) $I^{\mathfrak{m}}_\gamma(a_0; a_1) = 1$.

(2) $I^{\mathfrak{m}}_\gamma(a_0; a_1, \ldots, a_k; a_{k+1}) = (-1)^k I^{\mathfrak{m}}_{\gamma^{-1}}(a_{k+1}; a_k, \ldots, a_1; a_0)$.

(3) Motivic iterated integrals $I^{\mathfrak{m}}_\gamma(a; \alpha; b)$ satisfy shuffle relation i.e.

$$I^{\mathfrak{m}}_\gamma(a; \alpha; b) \cdot I^{\mathfrak{m}}_\gamma(a; \beta; b) = I^{\mathfrak{m}}_\gamma(a; \alpha \sqcup\!\sqcup \beta; b).$$

(4) Motivic iterated integrals $I^{\mathfrak{m}}_\gamma(a; \omega; b)$ satisfy path composition formula i.e.

$$I^{\mathfrak{m}}_{\gamma\gamma'}(a; \omega; b) = \sum_{\alpha\beta=\omega} I^{\mathfrak{m}}_\gamma(a; \alpha; c) \cdot I^{\mathfrak{m}}_{\gamma'}(c; \beta; b),$$

where $\gamma\gamma'$ is the composition of γ and γ'.

Now, we consider $(X, \preceq, \delta)$ with the labeling map $\delta : X \to \mathbb{Q}$, and define the motivic version of Yamamoto's integral as follows.

Definition 1.13. Let X be a labeled poset with the labeling map

$$\delta : X \to \mathbb{Q}.$$

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Motivic version of Yamamoto's integral is defined by

$$I_\gamma^{\mathfrak{m}}(X) := \sum_{Y \in \mathrm{Tot}(X)} I_\gamma^{\mathfrak{m}}(Y),$$

where $I_\gamma^{\mathfrak{m}}(Y)$ is the motivic iterated integral, which corresponds to the iterated integral $I_\gamma(Y)$.

1.4 Coaction formula of motivic iterated integrals

Let $\mathcal{A}$ be the quotient $\mathcal{P}/\mu\mathcal{P}$ and $\pi : \mathcal{P} \to \mathcal{A}$ the natural projection to the quotient, where μ is the motivic version of $2\pi i$. By the theory of mixed Tate motives, there is a natural motivic coaction

$$\Delta : \mathcal{P} \to \mathcal{A} \otimes \mathcal{P}.$$

Let

$$\mathcal{H} := \langle I_\gamma^{\mathfrak{m}}(a_0; a_1, \ldots, a_k; a_{k+1}) \rangle_{\mathbb{Q}} \subset \mathcal{P}$$

be the subspace spanned by motivic iterated integrals over $\mathbb{Q}$. Then, $\mathcal{H}$ becomes a $\mathbb{Q}$-subalgebra by the shuffle product formula. Notice that the image $\pi(I_\gamma^{\mathfrak{m}}(a_0; a_1, \ldots, a_k; a_{k+1}))$ depends only on the endpoints a_0 and a_{k+1} of the path, so we drop γ from the notation and denoted it as $I^{\mathfrak{a}}(a_0; a_1, \ldots, a_k; a_{k+1})$. The coaction of $I_\gamma^{\mathfrak{m}}(a_0; a_1, \ldots, a_k; a_{k+1})$ is given by the following formula.

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Theorem 1.14 ([Bro12], Theorem 2.4;[Gon05], Theorem 1.2). We have

$$\begin{aligned}&\Delta(I^{\mathfrak{m}}_{\gamma}(a_0;a_1,\dots,a_k;a_{k+1}))\\&=\sum_{s=0}^{k}\sum_{\substack{i_0<\cdots<i_{s+1}\\ i_0=0,i_{s+1}=k+1}}\prod_{p=0}^{s}I^{\mathfrak{a}}(a_{i_p};a_{i_p+1},\dots,a_{i_{p+1}-1};a_{i_{p+1}}))\otimes I^{\mathfrak{m}}_{\gamma}(a_{i_0};a_{i_1},\dots,a_{i_s};a_{i_{s+1}}),\end{aligned}$$

It is known that $\mathcal{A}$ has a structure of graded algebra i.e., $\mathcal{A}=\bigoplus_{k\geq 0}\mathcal{A}_k$, and we let $\mathcal{A}_{>0}\coloneqq\bigoplus_{k>0}\mathcal{A}_k$ and $\mathfrak{L}\coloneqq\mathcal{A}/\mathcal{A}^2_{>0}$. We denote by π' the natural projection from $\mathcal{A}$ to $\mathfrak{L}$ and define

$$I^{\mathfrak{l}}(a_{i_p};a_{i_p+1},\dots,a_{i_{p+1}-1};a_{i_{p+1}})\coloneqq\pi'(I^{\mathfrak{a}}(a_{i_p};a_{i_p+1},\dots,a_{i_{p+1}-1};a_{i_{p+1}})).$$

Definition 1.15 ([Bro12]). For $r\in\mathbb{N}$, the infinitesimal coaction $D_r:\mathcal{H}\to\mathfrak{L}\otimes\mathcal{H}$ is defined by

$$\begin{aligned}&D_r(I^{\mathfrak{m}}_{\gamma}(a_0;a_1,\dots,a_k;a_{k+1}))\\&=\sum_{s=0}^{k-r}I^{\mathfrak{l}}(a_s;a_{s+1},\dots,a_{s+r};a_{s+r+1}))\otimes I^{\mathfrak{m}}_{\gamma}(a_0;a_1,\dots,a_s,a_{s+r+1},\dots,a_k;a_{k+1}),\end{aligned}$$

In this article, we also consider the following.

Definition 1.16. For $r\in\mathbb{N}$, $D'_r:\mathcal{H}\to\mathcal{A}\otimes\mathcal{H}$ is defined by

$$\begin{aligned}&D'_r(I^{\mathfrak{m}}_{\gamma}(a_0;a_1,\dots,a_k;a_{k+1}))\\&=\sum_{s=0}^{k-r}I^{\mathfrak{a}}(a_s;a_{s+1},\dots,a_{s+r};a_{s+r+1}))\otimes I^{\mathfrak{m}}_{\gamma}(a_0;a_1,\dots,a_s,a_{s+r+1},\dots,a_k;a_{k+1}).\end{aligned}$$

Remark 1.17. By viewing Δ and D'_r as endomorphisms on $\mathcal{A}\otimes\mathcal{H}$ that map $a\otimes h\in$

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$\mathcal{A} \otimes \mathcal{H}$ to $(a \otimes 1) \cdot \Delta(h)$ and $(a \otimes 1) \cdot D'_r(h)$, they are related by

$$\exp\left(\sum_{r=1}^{\infty} D'_r\right) = \Delta.$$

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Chapter 2 Main theorems

2.1 Notations

In order to state our main theorem, we will define some symbols. To describe the endpoints of the integrals, we define the extension of X as follow.

Definition 2.1. For a labeled poset $X = (X, \preceq, \delta)$ and a path γ, we define the extension of X with respect to γ as

$$\widetilde{X}_\gamma := (\widetilde{X}, \widetilde{\preceq}, \widetilde{\delta}),$$

where

$$\begin{aligned}
\widetilde{X} &:= X \cup \{x_0, x_1\} \\
\widetilde{\preceq} &:= \preceq \cup \{(x_0, x) | x \in \widetilde{X}\} \cup \{(x, x_1) | x \in \widetilde{X}\} \\
\widetilde{\delta}(x) &:= \begin{cases} \delta(x) & \text{if } x \in X \\ \gamma(0) & \text{if } x = x_0 \\ \gamma(1) & \text{if } x = x_1. \end{cases}
\end{aligned}$$

When the path γ is clear from the context, we drop the γ from the notation.

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Example 2.2. Let $X = (X, \preceq, \delta)$ be the labeled poset

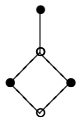
$$X := \{x_1, x_2, x_3\}$$

$$\preceq := \{(x_1, x_2), (x_3, x_2)\}$$

$$\delta(x) := \begin{cases} 0 & \text{if } x = x_2 \\ 1 & \text{if } x = x_1, x_3 \end{cases}$$

and $\gamma : [0, 1] \to \mathbb{C}$ be the path $\gamma(t) = t$. Then, X and $\widetilde{X}$ are depicted as

and



respectively.

To describe the endpoints of the integrals, we define the following subsets.

Definition 2.3. For posets $Y \subset X$, we define the subsets

$$Y^{\prec X} := Y^{\downarrow X} \setminus Y = \bigcup_{y \in Y} y^{\downarrow X} \setminus Y = \bigcup_{y \in Y} \{x \in X \mid x \preceq y\} \setminus Y$$

$$Y^{\succ X} := Y^{\uparrow X} \setminus Y = \bigcup_{y \in Y} y^{\uparrow X} \setminus Y = \bigcup_{y \in Y} \{x \in X \mid x \succeq y\} \setminus Y,$$

and their subsets

$$Y^{\rightarrow X} := \{p \in Y^{\prec X} \mid \nexists\, l \in Y^{\prec X} \text{ s.t. } p \prec l\}$$

$$Y^{\leftarrow X} := \{q \in Y^{\succ X} \mid \nexists\, u \in Y^{\succ X} \text{ s.t. } u \prec q\},$$

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where $y^{\uparrow X}$ and $y^{\downarrow X}$ (resp. $Y^{\uparrow X}$ and $Y^{\downarrow X}$) denote the upper and lower closures of y (resp. Y).

In the following, we will use $y^{\succeq X}$ and $y^{\preceq X}$ (resp. $Y^{\succeq X}$ and $Y^{\preceq X}$) instead of $y^{\uparrow X}$ and $y^{\downarrow X}$ (resp. $Y^{\uparrow X}$ and $Y^{\downarrow X}$). We will also use the following notations.

(1) We denote $\{y\}^{\prec X}$ (resp. $\{y\}^{\succ X}, \{y\}^{\rightarrow X}, \{y\}^{\leftarrow X}$) by $y^{\prec X}$ (resp. $y^{\succ X}, y^{\rightarrow X}, y^{\leftarrow X}$).

(2) We denote $x \in Y^{\rightarrow X}$ (resp. $x \in Y^{\leftarrow X}$) by $x \rightarrow_X Y$ or $Y \leftarrow_X x$ (resp. $x \leftarrow_X Y$ or $Y \rightarrow_X x$).

(3) When $Y^{\rightarrow X}$ (resp. $Y^{\leftarrow X}$) is a singleton we denote its unique element by $\rightarrow_X Y$ (resp. $\leftarrow_X Y$).

(4) When the set X is clear, we denote $\rightarrow_X$ and $\leftarrow_X$ simply by $\rightarrow$ and $\leftarrow$.

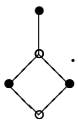
Example 2.4. Let $X = (X, \preceq, \delta)$ be the labeled poset

$$X \coloneqq \{x_1, x_2, x_3, x_4, x_5\}$$

$$\preceq \coloneqq \{(x_1, x_2), (x_2, x_4), (x_4, x_5), (x_1, x_3), (x_3, x_4), (x_1, x_4), (x_2, x_5), (x_3, x_5), (x_1, x_5)\}$$

$$\delta(x) \coloneqq \begin{cases} 0 & \text{if } x = x_1, x_4 \\ 1 & \text{if } x = x_2, x_3, x_5, \end{cases}$$

i.e.,



Let Y be the subset $\{x_2, x_4\}$ of X. Then, we have $Y^{\prec X} = \{x_1, x_3\}$, $Y^{\succ X} = \{x_5\}$, $Y^{\rightarrow X} = \{x_3\}$ and $Y^{\leftarrow X} = \{x_5\}$.

To state the formula for D_r, we define a set X_r as follow.

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Definition 2.5. Let $X = (X, \preceq, \delta)$ be a labeled poset. For $r \in \mathbb{N}$, we define

$$X_r := \{Y = (Y, \preceq_Y, \delta_Y) \subset X \mid |Y| = r,\ \forall\, x \in (X \setminus Y)\ \nexists\, y, y' \in Y \text{ s.t. } y \preceq x \preceq y'\}.$$

To state the right hand side of tensor product of formula, we define a set $X_{(p,\widehat{Y},q)}$ as follow.

Definition 2.6. Let $X = (X, \preceq, \delta)$ be a labeled poset. Fix $Y \in X_r$. For $p \to Y$ and $q \leftarrow Y$, we define

$$X_{(p,\widehat{Y},q)} := (X_{\widehat{Y}}, \preceq_{(p,\widehat{Y},q)}, \delta_{\widehat{Y}})$$

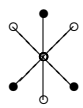
where

$$\begin{aligned}
X_{\widehat{Y}} :=& X \setminus Y \\
\preceq_{(p,\widehat{Y},q)} :=& \preceq \setminus \{(a,b) \in \preceq \mid a \in Y \text{ or } b \in Y\} \\
&\cup \{(a,b) \in X_{\widehat{Y}} \times X_{\widehat{Y}} \mid a \in Y^{\prec X},\ p \preceq b\} \\
&\cup \{(a,b) \in X_{\widehat{Y}} \times X_{\widehat{Y}} \mid b \in Y^{\succ X},\ a \preceq q\} \\
\delta_{\widehat{Y}} :=& \delta|_{X_{\widehat{Y}}}
\end{aligned}$$

When $|Y^{\to X}| = |Y^{\leftarrow X}| = 1$, we denote $X_{(p,\widehat{Y},q)}$ simply by $X_{\widehat{Y}}$.

We denote $X_{(p,\widehat{\{y\}},q)}$ (resp. $X_{\widehat{\{y\}}}$) simply by $X_{(p,\widehat{y},q)}$ (resp. $X_{\widehat{y}}$).

Example 2.7. Let X be the labeled poset



and Y be the center of the diagram. For the bottom white vertex p and the top

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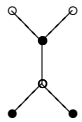
black vertex q, $X_{(p,\widehat{Y},q)}$ is depicted as follows:



Definition 2.8. Let X be a labeled poset. For $Y \in X_r$ and $\overline{Y} = (\overline{Y}, \preceq_{\overline{Y}}, \delta_{\overline{Y}}) \in \mathsf{Tot}(Y)$, we define

$$\mathsf{Tot}_{\overline{Y}}(X) := \{\overline{X} = (\overline{X}, \preceq_{\overline{X}}, \delta_{\overline{X}}) \in \mathsf{Tot}(X) \,|\, \overline{Y} \in \overline{X}_r, \preceq_{\overline{Y}} \subset \preceq_{\overline{X}}\}$$

Notice that when $r = 1$, we have $\mathsf{Tot}_{\overline{Y}}(X) = \mathsf{Tot}(X)$.

Further for $p \to Y$ and $q \leftarrow Y$, we define

$$\mathsf{Tot}_{p,\overline{Y},q}(X) := \{\overline{X} \in \mathsf{Tot}_{\overline{Y}}(X) \,|\, \preceq_{(p,\widehat{Y},q)} \subset \preceq_{\overline{X}}\}$$

Definition 2.9. Let X be a labeled poset. For $Y \subset X$, we define $P_Y(X) = X_{\widehat{Y}}$.

Definition 2.10. For $Y \subset X$, $Y_1, Y_2 \in \mathsf{Tot}(Y)$ and $\overline{X} \in \mathsf{Tot}_{Y_1}(X)$, we define

$$T_{Y_1,Y_2}(\overline{X}) = (X, (\preceq_{\overline{X}} \setminus \preceq_{Y_1}) \cup \preceq_{Y_2}, \delta).$$

Notice that in general $T_{Y_1,Y_2}(\overline{X})$ is not a poset.

2.2 A formula for D_r

Lemma 2.11. Let X be a labeled poset. For $Y \in X_r$ and $\overline{Y} \in \mathsf{Tot}(Y)$, $\{\mathsf{Tot}_{p,\overline{Y},q}(X)\}_{\substack{p \to Y \\ q \leftarrow Y}}$ forms a partition of $\mathsf{Tot}_{\overline{Y}}(X)$

Proof. We check the definition of a partition. Since $Y \in X_r$, $\mathsf{Tot}_{p,\overline{Y},q}(X)$ is nonempty.

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For all $\overline{X} \in \mathsf{Tot}_{\overline{Y}}(X)$, we have $\overline{X} \in \mathsf{Tot}_{p,\overline{Y},q}(X)$, where

p is the unique element determined by $\overline{X}$ s.t. $p \to_X Y$ and $\forall\, l \in Y^{\prec X}\ l \preceq_{\overline{X}} p$

q is the unique element determined by $\overline{X}$ s.t. $q \leftarrow_X Y$ and $\forall\, u \in Y^{\succ X}\ q \preceq_{\overline{X}} u$

By above, since p, q are unique, if $p \neq p'$ or $q \neq q'$, then $\mathsf{Tot}_{p,\overline{Y},q}(X) \cap \mathsf{Tot}_{p',\overline{Y},q'}(X) = \emptyset$. $\square$

Lemma 2.12. Let X be a labeled poset. For $Y \in X_r$, $\overline{Y} \in \mathsf{Tot}(Y)$ and $\overline{X} \in \mathsf{Tot}_{p,\overline{Y},q}(X)$, we have

(1) $P_{\overline{Y}}$ is a function from $\mathsf{Tot}_{p,\overline{Y},q}(X)$ to $\mathsf{Tot}(X_{(p,\widehat{Y},q)})$.

(2) $P_{\overline{Y}}$ is a surjection, and for $W = \overline{X_{(p,\widehat{Y},q)}} \in \mathsf{Tot}(X_{(p,\widehat{Y},q)})$ we have

$$P_{\overline{Y}}^{-1}(W) = \{\overline{X}_{\overline{Y},x} \,|\, x \in p^{\succeq W} \cap q^{\prec W}\},$$

where

$$\overline{X}_{\overline{Y},x} \coloneqq (X, \preceq_W \cup \{(x', y) \,|\, x' \in x^{\preceq W},\ y \in Y\} \cup \{(y, x') \,|\, x' \in x^{\succ W},\ y \in Y\} \cup \preceq_{\overline{Y}}, \delta).$$

Proof. (1) By definition of $\mathsf{Tot}_{p,\overline{Y},q}(X)$,

$$\forall\, \overline{X} \in \mathsf{Tot}_{p,\overline{Y},q}(X),\ \preceq_{X_{(p,\widehat{Y},q)}} \subset \preceq_{\overline{X}} \Rightarrow \preceq_{X_{(p,\widehat{Y},q)}} \subset \preceq_{\overline{X}_{\widehat{Y}}}$$

i.e. $\overline{X}_{\widehat{Y}} \in \mathsf{Tot}(X_{(p,\widehat{Y},q)})$, so $P_{\overline{Y}}$ is a function from $\mathsf{Tot}_{p,\overline{Y},q}(X)$ to $\mathsf{Tot}(X_{(p,\widehat{Y},q)})$.

(2) Let $W = \overline{X_{(p,\widehat{Y},q)}} \in \mathsf{Tot}(X_{(p,\widehat{Y},q)})$. Then there exists

$$\overline{X}_{\overline{Y},p} \coloneqq ((X, \preceq_W \cup \{(x', y) \,|\, x' \in p^{\preceq W},\ y \in Y\} \cup \{(y, x') \,|\, x' \in p^{\succ W},\ y \in Y\} \cup \preceq_{\overline{Y}}), \delta_X)$$

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s.t. $P_{\overline{Y}}(\overline{X}_{\overline{Y},p}) = W$. Moreover, we have

$$P_{\overline{Y}}^{-1}(W) = \{\overline{X}_{\overline{Y},x} \mid x \in p^{\succeq W} \cap q^{\prec W}\}.$$

□

We rewrite the definition of D_r and D'_r by our notation.

Definition 2.13. Let X be a labeled poset and $\widetilde{X}$ be its extension with respect to some path. For $p, q \in \widetilde{X}$ and $Y \subset X$, we define

$$I^{\mathfrak{a}}_{(p;q)}(Y) \coloneqq \pi(I^{\mathfrak{m}}_{\gamma}(Y)) \text{ with an arbitrary path } \gamma \text{ from } \widetilde{\delta}_{\widetilde{X}}(p) \text{ to } \widetilde{\delta}_{\widetilde{X}}(q)$$

and

$$I^{\mathfrak{l}}_{(p;q)}(Y) \coloneqq \pi'(I^{\mathfrak{a}}_{(p;q)}(Y)).$$

Definition 2.14. Let X be a totally ordered set. We define

$$D'_r : \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$$

and

$$D_r = (\pi' \otimes id) \circ D'_r : \mathcal{H} \to \mathfrak{L} \otimes \mathcal{H}.$$

as

$$D'_r(I^{\mathfrak{m}}_{\gamma}(X)) \coloneqq \sum_{Y \in X_r} I^{\mathfrak{a}}_{(\to_{\widetilde{X}} Y; Y \to_{\widetilde{X}})}(Y) \otimes I^{\mathfrak{m}}_{\gamma}(X_{\widehat{Y}})$$

and

$$D_r(I^{\mathfrak{m}}_{\gamma}(X)) \coloneqq \sum_{Y \in X_r} I^{\mathfrak{l}}_{(\to_{\widetilde{X}} Y; Y \to_{\widetilde{X}})}(Y) \otimes I^{\mathfrak{m}}_{\gamma}(X_{\widehat{Y}})$$

respectively.

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Theorem 2.15. Let X be a labeled poset and γ be a path. Then,

$$D_r(I_\gamma^{\mathfrak{m}}(X)) = \sum_{\substack{Y\in X_r\\ Y \text{ is irr.}}} \sum_{\substack{p\to_{\tilde{X}} Y\\ q\leftarrow_{\tilde{X}} Y}} I^{\mathfrak{l}}_{(p;q)}(Y)\otimes I_\gamma^{\mathfrak{m}}(X_{(p,\hat{Y},q)}).$$

Proof. First, by Definition 1.13 and 2.14, we have

$$D_r(I_\gamma^{\mathfrak{m}}(X)) = \sum_{\overline{X}\in \mathrm{Tot}(X)} D_r(I_\gamma^{\mathfrak{m}}(\overline{X})) = \sum_{\overline{X}\in \mathrm{Tot}(X)} \sum_{Z\in \overline{X}_r} I^{\mathfrak{l}}_{(\to_{\tilde{\overline{X}}} Z; Z\to_{\tilde{\overline{X}}})}(Z)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}}).$$

Since the summation is finite, we can sum over $\overline{X}$ first to get

$$D_r(I_\gamma^{\mathfrak{m}}(X)) = \sum_{Y\in X_r} \sum_{Z\in \mathrm{Tot}(Y)} \sum_{\overline{X}\in \mathrm{Tot}_Z(X)} I^{\mathfrak{l}}_{(\to Z; Z\to)}(Z)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}}).$$

By Lemma 2.11, we have

$$D_r(I_\gamma^{\mathfrak{m}}(X)) = \sum_{Y\in X_r} \sum_{Z\in \mathrm{Tot}(Y)} \sum_{\substack{p\to_{\tilde{X}} Y\\ q\leftarrow_{\tilde{X}} Y}} \sum_{\overline{X}\in \mathrm{Tot}_{p,Z,q}(X)} I^{\mathfrak{l}}_{(\to Z; Z\to)}(Z)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}}).$$

By Lemma 2.12, we have

$$D_r(I_\gamma^{\mathfrak{m}}(X)) = \sum_{Y\in X_r} \sum_{Z\in \mathrm{Tot}(Y)} \sum_{\substack{p\to Y\\ q\leftarrow Y}} \sum_{\overline{X}_{\widehat{Z}}\in \mathrm{Tot}(X_{(p,\widehat{Z},q)})} \sum_{\overline{X}\in P_Z^{-1}(\overline{X}_{\widehat{Z}})} I^{\mathfrak{l}}_{(\to Z; Z\to)}(Z)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}}).$$

For fixed p, q, we use (5) of Proposition 1.12 and the definition of D_r to get

$$\begin{aligned}
\sum_{\overline{X}\in P_Z^{-1}(\overline{X}_{\widehat{Z}})} I^{\mathfrak{l}}_{(\to Z; Z\to)}(Z)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}}) &= \left(\sum_{\overline{X}\in P_Z^{-1}(\overline{X}_{\widehat{Z}})} I^{\mathfrak{l}}_{(\to Z; Z\to)}(Z)\right)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}})\\
&= \left(I^{\mathfrak{l}}_{(p;p\to_{\widetilde{\overline{X}}_{\widehat{Z}}})}(Z) + \cdots + I^{\mathfrak{l}}_{(\to_{\widetilde{\overline{X}}_{\widehat{Z}}} q;q)}(Z)\right)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}})\\
&= I^{\mathfrak{l}}_{(p;q)}(Z)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}}).
\end{aligned}$$

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Hence, we have

$$\begin{aligned}
D_r(I_\gamma^{\mathfrak{m}}(X)) &= \sum_{Y\in X_r}\sum_{Z\in\mathrm{Tot}(Y)}\sum_{\substack{p\to Y\\ q\leftarrow Y}}\sum_{\overline{X}_{\widehat{Z}}\in\mathrm{Tot}(X_{(p,\widehat{Z},q)})} I^{\mathfrak{l}}_{(p;q)}(Z)\otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}})\\
&= \sum_{Y\in X_r}\sum_{Z\in\mathrm{Tot}(Y)}\sum_{\substack{p\to Y\\ q\leftarrow Y}} I^{\mathfrak{l}}_{(p;q)}(Z)\otimes\left(\sum_{\overline{X}_{\widehat{Z}}\in\mathrm{Tot}(X_{(p,\widehat{Z},q)})} I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Z}})\right)\\
&= \sum_{Y\in X_r}\sum_{Z\in\mathrm{Tot}(Y)}\sum_{\substack{p\to Y\\ q\leftarrow Y}} I^{\mathfrak{l}}_{(p;q)}(Z)\otimes I_\gamma^{\mathfrak{m}}(X_{(p,\widehat{Z},q)}).
\end{aligned}$$

Since the underlying set of Y and Z are the same and p, q depend only on Y, we have

$$D_r(I_\gamma^{\mathfrak{m}}(X)) = \sum_{Y\in X_r}\sum_{\substack{p\to Y\\ q\leftarrow Y}}\sum_{Z\in\mathrm{Tot}(Y)} I^{\mathfrak{l}}_{(p;q)}(Z)\otimes I_\gamma^{\mathfrak{m}}(X_{(p,\widehat{Y},q)}).$$

Notice that

$$\sum_{Z\in\mathrm{Tot}(Y)} I^{\mathfrak{l}}_{(p;q)}(Z) = \prod_{\substack{Y_i\subset Y\\ Y_i \text{ is irr.}}} I^{\mathfrak{l}}_{(p;q)}(Y_i).$$

If Y is reducible, then

$$\pi'\left(\prod_{\substack{Y_i\subset Y\\ Y_i \text{ is irr.}}} I^{\mathfrak{l}}_{(p;q)}(Y_i)\right) = 0,$$

hence

$$D_r(I_\gamma^{\mathfrak{m}}(X)) = \sum_{\substack{Y\in X_r\\ Y \text{ is irr.}}}\sum_{\substack{p\to Y\\ q\leftarrow Y}} I^{\mathfrak{l}}_{(p;q)}(Y)\otimes I_\gamma^{\mathfrak{m}}(X_{(p,\widehat{Y},q)}).$$

□

Corollary 2.16. Let X be a labeled poset and γ be a path. Then,

$$D_1(I_\gamma^{\mathfrak{m}}(X)) := \sum_{x\in X}\sum_{\substack{p\to_{\widetilde{X}} x\\ q\leftarrow_{\widetilde{X}} x}} I^{\mathfrak{a}}_{(p;q)}(x)\otimes I_\gamma^{\mathfrak{m}}(X_{\widehat{x}}).$$

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Lemma 2.17. If $Y \in X_r$ for some r, then for $Y_1, Y_2 \in \mathsf{Tot}(Y)$,

$$T_{Y_1,Y_2} : \mathsf{Tot}_{Y_1}(X) \to \mathsf{Tot}_{Y_2}(X)$$

is a bijection.

Proof. Let $\overline{X} \in \mathsf{Tot}_{Y_1}(X)$. Then, we have $\preceq_{Y_2} \subset ((\preceq_{\overline{X}} \setminus \preceq_{Y_1}) \cup \preceq_{Y_2})$. Since as a set $Y_1 = Y_2$ and

$$\{(y,x) \in \preceq_{\overline{X}} \mid x \in (X \backslash Y_2),\ y \in Y_2\} \cup \{(x,y) \in \preceq_{\overline{X}} \mid x \in (X \backslash Y_2),\ y \in Y_2\} \subset ((\preceq_{\overline{X}} \setminus \preceq_{Y_1}) \cup \preceq_{Y_2}),$$

$Y_2 \in X_r$ under the order $((\preceq_{\overline{X}} \setminus \preceq_{Y_1}) \cup \preceq_{Y_2})$. Hence, $T_{Y_1,Y_2}(\overline{X}) \in \mathsf{Tot}_{Y_2}(X)$.

Now, we swap Y_1 and Y_2. Then $T_{Y_2,Y_1} : \mathsf{Tot}(Y_2) \to \mathsf{Tot}(Y_1)$ is the inverse of T_{Y_1,Y_2}, i.e.,

$$T_{Y_2,Y_1} \circ T_{Y_1,Y_2} = id_{\mathsf{Tot}_{Y_1}(X)} \text{ and } T_{Y_1,Y_2} \circ T_{Y_2,Y_1} = id_{\mathsf{Tot}_{Y_2}(X)}.$$

Hence, T_{Y_1,Y_2} is a bijection from $\mathsf{Tot}_{Y_1}(X)$ to $\mathsf{Tot}_{Y_2}(X)$. $\square$

Proposition 2.18. Let X be a labeled poset and γ be a path. Fix a totally ordered set $Z \in \mathsf{Tot}(Y)$. Then,

$$D'_r(I^{\mathfrak{m}}_{\gamma}(X)) = \sum_{Y \in X_r} \sum_{\overline{X} \in \mathsf{Tot}_Z(X)} \prod_{i=1}^{n_Y} I^{\mathfrak{a}}_{(\to_{\widetilde{\overline{X}}} Z; Z \to_{\widetilde{\overline{X}}})}(Y_i) \otimes I^{\mathfrak{m}}_{\gamma}(\overline{X}_{\widehat{Z}}).$$

Proof. From the proof of Theorem 2.15 and by Definition 1.13 and 2.14, we have

$$D'_r(I^{\mathfrak{m}}_{\gamma}(X)) = \sum_{Y \in X_r} \sum_{Z \in \mathsf{Tot}(Y)} \sum_{\overline{X} \in \mathsf{Tot}_Z(X)} I^{\mathfrak{a}}_{(\to_{\widetilde{\overline{X}}} Z; Z \to_{\widetilde{\overline{X}}})}(Z) \otimes I^{\mathfrak{m}}_{\gamma}(\overline{X}_{\widehat{Z}}).$$

By Lemma 2.17, we can fix a totally ordered set $Z \in \mathsf{Tot}(Y)$ first and change the

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order of the summation to get

$$D'_r(I^{\mathfrak{m}}_{\gamma}(X)) = \sum_{Y\in X_r} \sum_{\overline{X}\in \mathrm{Tot}_Z(X)} \sum_{\overline{Y}\in \mathrm{Tot}(Y)} I^{\mathfrak{a}}_{(\to Z;Z\to)}(\overline{Y}) \otimes I^{\mathfrak{m}}_{\gamma}(\overline{X}_{\widehat{Z}}).$$

Let $\{Y_i\}_i$ be the set of irreducible components of Y. Then,

$$D'_r(I^{\mathfrak{m}}_{\gamma}(X)) = \sum_{Y\in X_r} \sum_{\overline{X}\in \mathrm{Tot}_Z(X)} \prod_{\substack{Y_i\subset Y \\ Y_i \text{ is irr.}}} I^{\mathfrak{a}}_{(\to Z;Z\to)}(Y_i) \otimes I^{\mathfrak{m}}_{\gamma}(\overline{X}_{\widehat{Z}}).$$

□

2.3 A formula for Δ

Definition 2.19. Let X be a labeled poset. For $x, x' \in X$, we say x' is adjacent to x if

$$x \to x' \text{ or } x' \to x,$$

and denote it as $x \leftrightarrow x'$. For $y \in Y \subset X$, we define the connected component of y in Y as

$$C_{X,Y}(y) \coloneqq \{y' \in Y \mid \exists\, y_1, \ldots, y_n \in Y \text{ s.t. } y = y_1 \leftrightarrow \cdots \leftrightarrow y_n = y'\}$$

and the component set of Y as

$$C_X(Y) \coloneqq \{C_{X,Y}(y) \mid y \in Y\}.$$

Notice that $C_X(Y)$ forms a partition of Y.

Lemma 2.20. Let X be a labeled poset. For $Y \subset X$,

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(1) P_Y is a function from $\mathsf{Tot}(X)$ to $\mathsf{Tot}(X_{\widehat{Y}})$.

(2) P_Y is a surjection.

Proof. (1) By the definition of $\mathsf{Tot}(X)$, $\preceq_{\overline{X}_{\widehat{Y}}}$ is a total order and

$$\preceq_{X_{\widehat{Y}}} \subset \preceq_{\overline{X}_{\widehat{Y}}} \quad \text{for } \overline{X} \in \mathsf{Tot}(X)$$

we have $P_Y(\overline{X}) \in \mathsf{Tot}(X_{\widehat{Y}})$ i.e. P_Y is a function from $\mathsf{Tot}(X)$ to $\mathsf{Tot}(X_{\widehat{Y}})$.

(2) For $W = \overline{X_{\widehat{Y}}} \in \mathsf{Tot}(X_{\widehat{Y}})$, we will construct $\overline{X} \in \mathsf{Tot}(X)$ s.t. $P_Y(\overline{X}) = W$. Now for each $y \in Y$, we pick a minimal element of $(y^{\succ X} \setminus Y, \preceq_W)$ and denote it by m_y. Let $m = \{m_y \,|\, y \in Y\}$. For $x \in m$, we define $Y_x = \{y \in Y \,|\, m_y = x\}$. For each $x \in m$, we pick a $Z_x \in \mathsf{Tot}(Y_x)$ and define

$$\preceq_{\overline{X}} = \preceq_W \cup \bigcup_{x \in m} \overline{\preceq_{Z_x}} \cup \bigcup_{\substack{x, x' \in m \\ x \prec_W x'}} \preceq_{Y_{x,x'}},$$

where

$$\overline{\preceq_{Z_x}} = \preceq_{Z_x} \cup \{(a,b) \,|\, a \in Z_x,\ b \in x^{\succeq W}\} \cup \{(a,b) \,|\, a \in x^{\prec W}, b \in Z_x\}$$

$$\preceq_{Y_{x,x'}} = \{(a,b) \,|\, a \in Y_x,\ b \in Y_{x'}\}.$$

Now for each $y \in Y$, we need to check $\{(p,y) \,|\, p \in y^{\preceq X}\} \subset \preceq_{\overline{X}}$ and $\{(y,q) \,|\, q \in y^{\succeq X}\} \subset \preceq_{\overline{X}}$.

For $p \in y^{\preceq X} \setminus Z_{m_y}$, since $y^{\preceq X} \subset m_y^{\preceq X}$, we have

$$\{(p,y) \,|\, p \in y^{\preceq X} \setminus Z_{m_y}\} \subset \preceq_{\overline{X}},$$

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and for $p \in y^{\preceq X} \cap Z_{m_y}$, since $Z_{m_y} \in \mathsf{Tot}(Y_{m_y})$, we have

$$\{(p, y) \mid p \in y^{\preceq X} \cap Z_{m_y}\} \subset \preceq_{\overline{X}} .$$

Similarly, since $Z_{m_y} \in \mathsf{Tot}(Y_{m_y})$, by definition of m_y, we have

$$\{(y, q) \mid q \in y^{\succeq X}\} \subset \preceq_{\overline{X}} .$$

Hence, $(X, \preceq_{\overline{X}}, \delta) \in \mathsf{Tot}(X)$. By definition of $\preceq_{\overline{X}}$, we have $P_Y(\overline{X}) = W$ i.e., P_Y is a surjection.

□

Definition 2.21. Let X be a labeled poset. For any $Y \subset X$ and $\overline{X} \in \mathsf{Tot}(X)$, we define a relation on $P_Y^{-1}(\overline{X_{\widehat{Y}}})$ by

$$\overline{X} \sim_T \overline{X}' \iff \overline{X} = T_{Y_1,Y_2}(\overline{X}')$$

for some $Y_1, Y_2 \in \mathsf{Tot}(Y)$.

Lemma 2.22. Let X be a labeled poset. For any $Y \subset X$ and $\overline{X} \in \mathsf{Tot}(X)$, $\sim_T$ is an equivalence relation on $P_Y^{-1}(\overline{X_{\widehat{Y}}})$.

Proof. (1) Reflexivity: For all $\overline{X} \in \mathsf{Tot}(X)$ we have $(Y, \preceq_{\overline{X}}) \in \mathsf{Tot}(Y)$. This imply $\overline{X} = T_{Y,Y}(\overline{X})$.

(2) Symmetry: If $\overline{X} \sim_T \overline{X}'$, then $\overline{X} = T_{Y_1,Y_2}(\overline{X}')$. This implies $(Y, \preceq_{\overline{X}}) = Y_2$. Similarly, we take $(Y, \preceq_{\overline{X}'}) \in \mathsf{Tot}(Y)$. Then, we have $T_{Y_2,Y}(\overline{X}) = \overline{X}'$.

(3) Transitivity: If $\overline{X} \sim_T \overline{X}'$ and $\overline{X}' \sim_T \overline{X}''$, then $\overline{X} = T_{Y_1,Y_2}(\overline{X}')$ and $\overline{X}' =$

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$T_{Y_1',Y_2'}(\overline{X}'')$. Similarly, we take $(Y, \preceq_{\overline{X}''}) \in \mathsf{Tot}(Y)$. Then, we have $T_{Y,Y_2}(\overline{X}'') = \overline{X}$. $\square$

Definition 2.23. By Lemma 2.22, for $W \in \mathsf{Tot}(X_{\widehat{Y}})$ and $\overline{X} \in P_Y^{-1}(W)$, we define the equivalence class of $\overline{X}$ as

$$[\overline{X}] \coloneqq \{\overline{X}' \in P_Y^{-1}(W) \,|\, \overline{X}' \sim_T \overline{X}\}$$

and the corresponding partition of $P_Y^{-1}(W)$ as

$$E_W = P_Y^{-1}(W)/\sim_T \coloneqq \{[\overline{X}] \,|\, \overline{X} \in P_Y^{-1}(W)\}.$$

With our notations, Goncharov's formula can be rewritten as follows:

Theorem 2.24 ([Bro12], Theorem 2.4;[Gon05], Theorem 1.2). Let X be a totally ordered set. Then, we have

$$\Delta(I_\gamma^{\mathfrak{m}}(X)) \coloneqq \sum_{Y \subset X} \prod_{Z \in C_X(Y)} I^{\mathfrak{a}}_{(\to_{\tilde{X}} Z; Z \to_{\tilde{X}})}(Z) \otimes I_\gamma^{\mathfrak{m}}(X_{\widehat{Y}})$$

with our notations.

Definition 2.25. Let X be a labeled poset and γ be a path. For $Y \subset X$, we define

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) \coloneqq \sum_{\overline{X} \in \mathsf{Tot}(X)} \prod_{Z \in C_{\overline{X}}(Y)} I^{\mathfrak{a}}_{(\to_{\tilde{\overline{X}}} Z; Z \to_{\tilde{\overline{X}}})}(Z) \otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Y}}).$$

Remark 2.26. Let X be a labeled poset and γ be a path. Then we have the following relation.

$$\Delta(I_\gamma^{\mathfrak{m}}(X)) = \sum_{Y \subset X} \Delta_Y(I_\gamma^{\mathfrak{m}}(X)).$$

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Theorem 2.27. Let X be a labeled poset, γ be a path and $Y \subset X$. If for all $Z \in C_X(Y)$, $Z \in X_{|Z|}$ and $|Z^{\to \widetilde{X}}| = |Z^{\leftarrow \widetilde{X}}| = 1$, then one has

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \prod_{Z \in C_X(Y)} I^{\mathfrak{a}}_{(p_Z;q_Z)}(Z) \otimes I_\gamma^{\mathfrak{m}}(X_{\widehat{Y}}),$$

where $p_Z := \to_{\widetilde{X}} Z$ and $q_Z := \leftarrow_{\widetilde{X}} Z$.

Proof. First, by definition of $\Delta_Y(I_\gamma^{\mathfrak{m}}(X))$ and Lemma 2.20 we have

$$\begin{aligned}\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) &= \sum_{\overline{X} \in \mathrm{Tot}(X)} \prod_{Z \in C_{\overline{X}}(Y)} I^{\mathfrak{a}}_{(\to_{\widetilde{\overline{X}}} Z; Z \to_{\widetilde{\overline{X}}})}(Z) \otimes I_\gamma^{\mathfrak{m}}(\overline{X}_{\widehat{Y}}) \\ &= \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \sum_{\overline{X} \in P_Y^{-1}(W)} \prod_{Z \in C_{\overline{X}}(Y)} I^{\mathfrak{a}}_{(\to Z; Z \to)}(Z) \otimes I_\gamma^{\mathfrak{m}}(W).\end{aligned}$$

By Lemma 2.22, we have

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \sum_{e \in E_W} \sum_{\overline{X} \in e} \prod_{Z \in C_{\overline{X}}(Y)} I^{\mathfrak{a}}_{(\to Z; Z \to)}(Z) \otimes I_\gamma^{\mathfrak{m}}(W).$$

Note that $\overline{X} \in [\overline{X}']$ implies that $C_{\overline{X}}(Y) = C_{\overline{X}'}(Y)$, $\to_{\widetilde{\overline{X}}} Z = \to_{\widetilde{\overline{X}'}} Z$, and $\leftarrow_{\widetilde{\overline{X}}} Z = \leftarrow_{\widetilde{\overline{X}'}} Z$, so we may use the notation $C_e(Y) := C_{\overline{X}}(Y)$ for $\overline{X} \in e$. Thus, we may swap the order of the summation and the product to find

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \sum_{e \in E_W} \prod_{Z \in C_e(Y)} \left(\sum_{Z' \in [Z]} I^{\mathfrak{a}}_{(\to_{\tilde{e}} Z; Z \to_{\tilde{e}})}(Z') \right) \otimes I_\gamma^{\mathfrak{m}}(W),$$

where $Z' \in [Z]$ means that Z' is the set Z with the new order determined by $\overline{X} \in e$ and the same labeling map i.e. $Z' = T_{Y_1,Y_2}(Z)$ for some $Y_1, Y_2 \in \mathrm{Tot}(Y)$. Since the order of $Z' \in [Z]$ is contains the order of Y, we have

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \sum_{e \in E_W} \prod_{Z \in C_e(Y)} \left(\sum_{Z' \in \mathrm{Tot}(Z, \preceq_Y)} I^{\mathfrak{a}}_{(\to Z; Z \to)}(Z') \right) \otimes I_\gamma^{\mathfrak{m}}(W).$$

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Note that $\{V \cap Z \mid V \in C_X(Y)\} = C_Y(Z)$ and $Z = \bigsqcup_{S \in C_Y(Z)} S$. Thus, by proposition 1.11, we have

$$\begin{aligned}\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) &= \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \sum_{e \in E_W} \prod_{Z \in C_e(Y)} \left(I^{\mathfrak{a}}_{(\to Z; Z \to)} \left(\bigsqcup_{V \in C_X(Y)} (V \cap Z) \right) \right) \otimes I_\gamma^{\mathfrak{m}}(W) \\ &= \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \sum_{e \in E_W} \prod_{Z \in C_e(Y)} \prod_{V \in C_X(Y)} I^{\mathfrak{a}}_{(\to Z; Z \to)}(V \cap Z) \otimes I_\gamma^{\mathfrak{m}}(W).\end{aligned}$$

The products are finite products, so we can change the order of the products to get

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \sum_{e \in E_W} \prod_{V \in C_X(Y)} \prod_{Z \in C_e(Y)} I^{\mathfrak{a}}_{(\to Z; Z \to)}(V \cap Z) \otimes I_\gamma^{\mathfrak{m}}(W). \tag{2.1}$$

For $W \in \mathrm{Tot}(X_{\widehat{Y}})$ and $V \in C_X(Y)$, since $|V^{\to \widetilde{X}}| = |V^{\leftarrow \widetilde{X}}| = 1$, we may let $\to_{\widetilde{X}} V = p_V$ and $\leftarrow_{\widetilde{X}} V = q_V$ and define

$$B_{W,V} = \{x \in W \mid p_V \preceq_W x \prec_W q_V\}$$

and

$$V_W := \left\{ (V_x)_{x \in B_{W,V}} \;\middle|\; \begin{array}{c} \bigcup_{x \in B_{W,V}} V_x = V \text{ and} \\ \left(\bigcup_{x \in B_{W,V}} \preceq_{V_x}\right) \cup \left(\bigcup_{\substack{x, x' \in B_{W,V} \\ x \prec x'}} \{(a,b) \mid a \in V_x,\ b \in V_{x'}\} \right) = \preceq_V \end{array} \right\}.$$

Now, note that in the right hand side of (2.1), $C_X(Y)$ is independent of e. Also, since $V \in X_{|V|}$, E_W has a 1-1 correspondence with $\prod_{V \in C_X(Y)} V_W$. Thus, we can change the order of summation $\sum_{e \in E_W}$ and the product $\prod_{V \in C_X(Y)}$ to get

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \prod_{V \in C_X(Y)} \sum_{(V_x)_x \in V_W} \prod_{x \in B_{W,V}} I^{\mathfrak{a}}_{(x; x \to_{\widetilde{W}})}(V_x) \otimes I_\gamma^{\mathfrak{m}}(W).$$

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By (5) of Proposition 1.12, we have

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \sum_{W \in \mathrm{Tot}(X_{\widehat{Y}})} \prod_{V \in C_X(Y)} I^{\mathfrak{a}}_{(p_V;q_V)}(V) \otimes I_\gamma^{\mathfrak{m}}(W)$$

and by Definition 1.13, we get

$$\Delta_Y(I_\gamma^{\mathfrak{m}}(X)) = \prod_{V \in C_X(Y)} I^{\mathfrak{a}}_{(p_V;q_V)}(V) \otimes I_\gamma^{\mathfrak{m}}(X_{\widehat{Y}}),$$

which completes the proof. □

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Chapter 3 Examples

In this chapter, we will explicitly compute a few examples of Theorems 2.15 and 2.27. Our first two examples are also special cases of (the motivic version of) Schur multiple zeta values (hereafter abbreviated as SMZV) due to a theorem by Hirose, Murahara and Onozuka. Thus, let us first give a definition of Schur multiple zeta values before going into the examples.

Definition 3.1 (Young diagram). Let λ be a finite subset of $\mathbb{N}^2$. λ is called a Young diagram if λ satisfying

$$(i+1,j) \in \lambda \text{ or } (i,j+1) \in \lambda \implies (i,j) \in \lambda \text{ for } (i,j) \in \mathbb{N}^2.$$

Definition 3.2 (skew Young diagram). $\lambda = \lambda' \setminus \lambda''$ is called a skew Young diagram, if there exist Young diagrams λ' and λ'' such that $\lambda' \subset \lambda''$.

Definition 3.3 (semi-standard Young tableaux). For a fixed skew Young diagram λ, we define the set of semi-standard Young tableaux $\mathrm{SSYT}(\lambda)$ as the set of maps f from λ to $\mathbb{N}$ satisfying the conditions:

- If $(i,j),(i,j+1) \in \lambda$ then $f(i,j) \leq f(i,j+1)$,
- If $(i,j),(i+1,j) \in \lambda$ then $f(i,j) < f(i+1,j)$.

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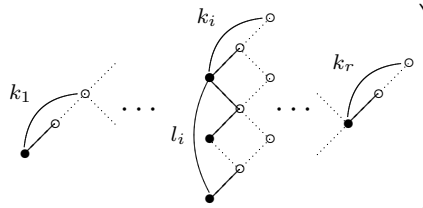
We call a map $\mathbf{k} : \lambda \to \mathbb{N}$ an index for λ. Furthermore, we say that $\mathbf{k}$ is admissible if $\mathbf{k}(i,j) \geq 2$ for any $(i,j) \in \lambda$ such that $(i+1,j) \notin \lambda$ and $(i,j+1) \notin \lambda$.

Definition 3.4 (Schur multiple zeta values). For an admissible index $\mathbf{k}$, the Schur multiple zeta value is defined by

$$\zeta(\mathbf{k}) = \sum_{f \in \mathrm{SSYT}(\lambda)} \frac{1}{\prod_{(i,j)\in\lambda} f(i,j)^{\mathbf{k}(i,j)}}.$$

Hirose, Murahara, and Onozuka proved that Yamamoto's integral expression can also be used to represent SMZVs with constant entries on the diagonals.

Theorem 3.5 ([HMO23], Theorem 1.2). If $\mathbf{k}$ is an admissible index such that $\mathbf{k}(i,j) = \mathbf{k}(i+1,j+1)$ for all $(i,j),(i+1,j+1) \in \lambda$, then the SMZV $\zeta(\mathbf{k})$ has Yamamoto's integral expression, i.e. we have

$$\zeta\left(\begin{array}{ccccc} & & & & \boxed{k_1} \\ & \boxed{k_i} & & \iddots & \\ & & \ddots & & \\ & \iddots & & \boxed{k_i} & \\ \boxed{k_r} & & & & \end{array}\right) = I\left(\begin{array}{c}\text{  }\end{array}\right),$$

where l_i is the number of k_i's.

Let us define the motivic version of the SMZV with constant entries on the diagonal by Yamamoto's integral on the right-hand side of this theorem. In the following examples, we will make use of the facts

$$I^{\mathfrak{m}}(0;0;1) = I^{\mathfrak{m}}(0;1;1) = I^{\mathfrak{m}}(a;a_1,\ldots,a_k;a) = 0 \quad (k \geq 1). \tag{3.1}$$

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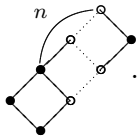
Example 3.6. Let us consider the case

$$(X, \preceq, \delta) = \quad .$$



In other words,

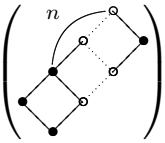
$$X \coloneqq \{x_1, \ldots, x_n, x'_1, \ldots, x'_n\}$$

$$\preceq \coloneqq \{(x_a, x_b) \mid a \leq b\} \cup \{(x'_a, x'_b) \mid a \leq b\} \cup \{(x_a, x'_b) \mid a \leq b\}$$

$$\delta(x) \coloneqq \begin{cases} 1 \text{ if } x \in \{x'_1, x'_2, x_1, x_n\} \\ \\ 0 \text{ others.} \end{cases}$$

By Theorem 3.5, we have

$$\zeta^{\mathfrak{m}} \left(\begin{array}{|c|c|} \hline n & 1 \\ \hline 1 & n \\ \hline \end{array} \right) = I^{\mathfrak{m}} \left(\quad \right) .$$



Let us calculate $D'_3(I^{\mathfrak{m}}(X))$ using Theorem 2.15. For convenience, we assume $n \geq 5$ and set $\delta(x_0) = 0, \delta(x'_{n+1}) = 1$. First, notice that the irreducible elements in X_3 are either of the four types

Type 1. $Y_{1,m} = \{x_m, x_{m+1}, x_{m+2}\}$.

Type 2. $Y_{2,m} = \{x'_m, x_m, x_{m+1}\}$.

Type 3. $Y_{3,m} = \{x'_m, x'_{m+1}, x_{m+1}\}$.

Type 4. $Y_{4,m} = \{x'_m, x'_{m+1}, x'_{m+2}\}$.

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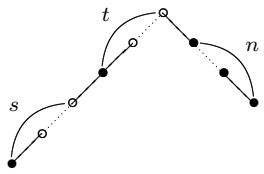
Thus, by Theorem 2.15 and property of Motivic iterated integrals we have

$$
\begin{aligned}
D_3'(I(X)) =& I^{\mathfrak{l}}_{(x_0;x_1')}(Y_{1,1}) \otimes I^{\mathfrak{m}}(X_{(x_0,\widehat{Y_{1,1}},x_1')}) + I^{\mathfrak{l}}_{(x_0;x_2')}(Y_{2,1}) \otimes I^{\mathfrak{m}}(X_{(x_0,\widehat{Y_{2,1}},x_2')}) \\
&+ I^{\mathfrak{l}}_{(x_1';x_3')}(Y_{2,2}) \otimes I^{\mathfrak{m}}(X_{(x_1',\widehat{Y_{2,2}},x_3')}) + I^{\mathfrak{l}}_{(x_1';x_4)}(Y_{2,2}) \otimes I^{\mathfrak{m}}(X_{(x_1',\widehat{Y_{2,2}},x_4)}) \\
&+ I^{\mathfrak{l}}_{(x_1;x_3)}(Y_{3,1}) \otimes I^{\mathfrak{m}}(X_{(x_1,\widehat{Y_{3,1}},x_3)}) + I^{\mathfrak{l}}_{(x_1';x_4)}(Y_{3,2}) \otimes I^{\mathfrak{m}}(X_{(x_1',\widehat{Y_{3,2}},x_4)}) \\
&+ I^{\mathfrak{l}}_{(x_{n-1};x_{n+1}')}(Y_{3,n-1}) \otimes I^{\mathfrak{m}}(X_{(x_{n-1},\widehat{Y_{3,n-1}},x_{n+1}')}) \\
&+ I^{\mathfrak{l}}_{(x_{n-2}';x_{n+1}')}(Y_{3,n-1}) \otimes I^{\mathfrak{m}}(X_{(x_{n-2}',\widehat{Y_{3,n-1}},x_{n+1}')}).
\end{aligned}
$$

Using Theorem 2.15 or Proposition 2.18 is a easy way to find formula of D_r or D_r' for some type of SMZVs.

Example 3.7. Let us consider the case

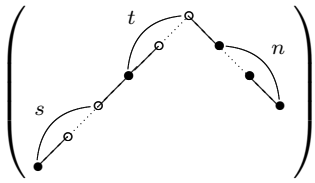
$(X, \preceq, \delta) =$ 

.

In other words,

$$
\begin{aligned}
X &:= \{x_1, \dots, x_s, x_{s+1}, \dots, x_{s+t}, x_1', \dots, x_n'\} \\
\preceq &:= \{(x_a, x_b) \mid a \le b\} \cup \{(x_a', x_b') \mid a \le b\} \cup \{(x_a', x_{s+t}) \mid a \in \{1, \dots, n\}\} \\
\delta(x) &:= \begin{cases} 1 \text{ if } x \in \{x_1', \dots, x_n', x_1, x_{s+1}\} \\ 0 \text{ others.} \end{cases}
\end{aligned}
$$

By Theorem 3.5, we have

$I^{\mathfrak{m}}\Bigg($ 

$\Bigg) = \zeta^{\mathfrak{m}}\left(\begin{array}{|c|c|c|c|c|}\hline & & & & s \\ \hline 1 & \cdots & 1 & & t \\ \hline\end{array}\right).$

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Let us calculate $\Delta(I^{\mathfrak{m}}(X))$ using Theorem 2.27. For convenience, we set $\delta(x_0) = 0, \delta(x_{s+t+1}) = 1$. First, using Remark 2.26 we get

$$\Delta(I^{\mathfrak{m}}(X)) = \sum_{Y \subset X} \Delta_Y(I^{\mathfrak{m}}(X)).$$

Next, using Theorem 2.27, Proposition 1.11, and (3.1), we see that Δ_Y vanishes except for the following three types of Y's.

Type 1. $Y_{1,m} = \{x_m, \ldots, x_{s+t}, x'_1, \ldots, x'_n\}$, where $1 \leq m \leq s+t$.

Type 2. $Y_{2,m} = \{x_1, \ldots, x_s, x_{s+m}, \ldots, x_{s+t}, x'_1, \ldots, x'_n\}$, where $2 \leq m \leq t$.

Type 3. $Y_{3,m_1,m_2} = \{x_2, \ldots, x_{s+m_1}, x_{s+m_2}, \ldots, x_{s+t}, x'_1, \ldots, x'_n\}$,

where $1 \leq m_1 \leq t-2$ and $m_1 + 1 < m_2 \leq t$.

Hence, we can write the formula of $\Delta(I^{\mathfrak{m}}(X))$ by three types of Y

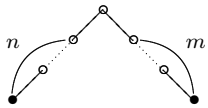
$$\Delta(I^{\mathfrak{m}}(X)) = \sum_{m=1}^{s+t} I^{\mathfrak{a}}_{(x_{m-1};x_{s+t+1})}(Y_{1,m}) \otimes I^{\mathfrak{m}}(X_{\widehat{Y_{1,m}}})$$

$$+ \sum_{m=2}^{t} I^{\mathfrak{a}}_{(x_0;x_s)}(X_{1,s}) I^{\mathfrak{a}}_{(x_{s+m-1};x_{s+t+1})}(Y_{2,m} \setminus X_{1,s}) \otimes I^{\mathfrak{m}}(X_{\widehat{Y_{2,m}}})$$

$$+ \sum_{m_1=1}^{t-2} \sum_{m_2=m_1+2}^{t} I^{\mathfrak{a}}_{(x_1;x_{s+m_1+1})}(X_{2,s+m_1}) I^{\mathfrak{a}}_{(x_{s+m_2-1};x_{s+t+1})}(Y_{3,m_1,m_2} \setminus X_{2,s+m_1}) \otimes I^{\mathfrak{m}}(X_{\widehat{Y_{3,m}}}).$$

where $X_{a,b} = \{x_a, x_{a+1}, \ldots, x_{b-1}, x_b\}$. By Theorem 3.5, we can compute the motivic coaction of some types of SMZVs using Theorem 2.27.

Example 3.8. Let us consider the case

$X =$ 

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In other words,

$$X \coloneqq \{x_1, \dots, x_n, x_{n+1}, x'_1, \dots, x'_m\}$$

$$\preceq \coloneqq \{(x_a, x_b) \mid a \le b\} \cup \{(x'_a, x'_b) \mid a \le b\} \cup \{(x'_a, x_{n+1}) \mid a \in \{1, \dots, m\}\}$$

$$\delta(x) \coloneqq \begin{cases} 1 \text{ if } x \in \{x_1, x'_1\} \\ \\ 0 \text{ others.} \end{cases}$$

By Definition 1.13, we have

$$I^{\mathfrak{m}}(X) = \sum_{i=1}^{n} \binom{n-i+m}{n-i} \zeta^{\mathfrak{m}}(i, n-i+m+1) + \sum_{j=1}^{m} \binom{n+m-j}{m-j} \zeta^{\mathfrak{m}}(j, n+m-j+1).$$

Let us calculate $\Delta(I^{\mathfrak{m}}(X))$ using Theorem 2.27. First, using Proposition 1.12 it is easy to show the formula

$$I^{\mathfrak{m}}(0; \{0\}^a, 1, \{0\}^b; 1) = (-1)^{a+1} \binom{a+b}{a} \zeta^{\mathfrak{m}}(a+b+1),$$

and using $\binom{i}{k} = \binom{i-1}{k} + \binom{i-1}{k-1}$, it is easy to show the formula

$$\sum_{k=0}^{i} (-1)^{i-k+1} \frac{m}{m+k} \binom{i}{k} = \frac{(-1)^{i+1}}{\binom{i+m}{m}}.$$

Next, using the fact in Remark 2.26, we get

$$\Delta(I^{\mathfrak{m}}(X)) = \sum_{Y \subset X} \Delta_Y(I^{\mathfrak{m}}(X)).$$

Finally, using Theorem 2.27, Proposition 1.11 and (3.1), we see that Δ_Y vanishes except for the following two types of Y's.

Type 1. $Y_{1,i} = \{x_i, \dots, x_{n+1}, x'_1, \dots, x'_m\}$.

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Type 2. $Y_{2,j} = \{x_1, \ldots, x_{n+1}, x'_j, \ldots, x'_m\}$.

Since $I^{\mathfrak{m}}(X_{\widehat{Y_{1,i}}})$ and $I^{\mathfrak{m}}(X_{\widehat{Y_{2,j}}})$ are single zeta values, hence we can express the formula of $\Delta(I^{\mathfrak{m}}(X))$ in terms of two types of Y and single zeta values as

$$\Delta(I^{\mathfrak{m}}(X)) = 1\otimes I^{\mathfrak{m}}(X)+I^{\mathfrak{a}}(X)\otimes 1+\sum_{i=3}^{n+1} I^{\mathfrak{a}}(Y_{1,i})\otimes(-\zeta^{\mathfrak{m}}(i-1))+\sum_{j=3}^{m+1} I^{\mathfrak{a}}(Y_{2,j})\otimes(-\zeta^{\mathfrak{m}}(j-1)).$$

By Definition 1.13 and the formula above we have

$$\begin{aligned}
I^{\mathfrak{a}}(Y_{1,n+1-i}) &= \sum_{k=0}^{i} \binom{m-1+k}{k} I^{\mathfrak{a}}(0;\{0\}^{i-k},1,\{0\}^{m+k};1) \\
&= \sum_{k=0}^{i} \frac{m}{m+k}\binom{m+k}{k}(-1)^{i-k+1}\binom{m+i}{i-k}\zeta^{\mathfrak{a}}(m+i+1) \\
&= \binom{m+i}{m}\left(\sum_{k=0}^{i}(-1)^{i-k+1}\frac{m}{m+k}\binom{i}{k}\right)\zeta^{\mathfrak{a}}(m+i+1) \\
&= \binom{m+i}{m}\frac{(-1)^{i+1}}{\binom{i+m}{m}}\zeta^{\mathfrak{a}}(m+i+1) \\
&= (-1)^{i+1}\zeta^{\mathfrak{a}}(m+i+1).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\Delta(I^{\mathfrak{m}}(X)) =& 1\otimes I^{\mathfrak{m}}(X) + I^{\mathfrak{a}}(X)\otimes 1 \\
&+ \sum_{i=0}^{n-2}(-1)^i\zeta^{\mathfrak{a}}(m+i+1)\otimes\zeta^{\mathfrak{m}}(n-i) \\
&+ \sum_{j=0}^{m-2}(-1)^j\zeta^{\mathfrak{a}}(n+j+1)\otimes\zeta^{\mathfrak{m}}(m-j).
\end{aligned}$$

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