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馬克斯偉-陳-西蒙斯：陳數與晶格規范場論

Maxwell Chern-Simons: Chern Number and Lattice Gauge Theory

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本論文係甘佳盛（R10222085）在國立臺灣大學物理系完成之碩士學位論文，於民國 114 年 06 月 04 日承下列考試委員審查通過及口試及格，特此證明。

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摘要

在本論文中，我們介紹了陳-西蒙斯（Chern-Simons, CS）理論與麥克斯韋-陳-西蒙斯（Maxwell-Chern-Simons, MCS）理論，並展示了它們所具有的一些有趣特性，例如能級量子化、分数量子統計等。我們接著在平面和環面上對 MCS 理論進行量子化。同時，我們也簡要介紹了陳數（Chern number）。特別地，計算定義在空間環面上的 U(1) MCS 理論的平直（零）模的陳數，是本論文的主要目標。我們發現，為了實現這一目標，有必要引入扭曲邊界條件。最後，我們採用了修改後的 Villain 表達式與轉移矩陣方法推導了晶格 MCS 理論的哈密頓量。我們發現該晶格哈密頓量在結構上與連續情況相似，但由於引入了杯積（cup product）而產生了一些有趣的修正。最終，在晶格上計算陳數的過程，與連續情況頗為相似。

關鍵字：拓撲場論，陳數，晶格規范場論，馬克斯偉陳西蒙斯

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Abstract

In this thesis, we introduce Chern-Simons (CS) theory and Maxwell-Chern-Simons (MCS) and demonstrate they exhibits some interesting features, such as level quanitzation, fractional statistics, etc. We proceed by quantizing the MCS on a plane and torus. We also give a brief introduction to the Chern number. In particular, the calculation of the Chern number of the flat (zero) modes of $U(1)$ MCS on a spatial torus comprises our primary goal. We find that, to achieve this aim, it is necessary to employ the twisted boundary conditions. Finally, we use the modified Villain formulation and transfer matrix methods to derive the Hamiltonian of lattice MCS theory. We see that the lattice Hamiltonian has similar structure as in the continuum, except some interesting modifications coming from the use of cup product. Finally, the computation of the Chern number on the lattice turns out to be reminiscent of the case in the continuum.

Keywords: Topological Field Theory, Chern Number, Lattice Gauge Theory, Maxwell-Chern-Simons

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Chapter 1 Introduction

The Chern-Simons theory comprises one of the beautiful examples of topological quantum field theory (TQFT). From the mathematical side, it comes from the notion of Chern-Simons (CS) forms. This CS form provide a natural candidate of Lagrangian to gauge theories. It could only be defined on an odd-dimensional spacetime M and may exhibits non-trivial topological features, depending on the topology of M. As a topological field theory, Chern-Simons theory has zero Hamiltonian. As a result, the naive educated guess would be that it has no local propagating degree of freedoms, unlike the familiar Maxwell or Yang-Mills theories. However, it is shown that it is possible to construct and even compute important topological invariant, with some subtleties regarding the gravitational anomaly. One such invariant is the expectation value of non-local observables called Wilson loops in CS theory, as shown in the monumental paper by Witten [53].

On the other hand, the possibility of the inclusion of Maxwell or Yang-Mills terms to Chern-Simons theory has been pondered by the physics community. Such theories, named topologically massive gauge theory, possess massive photon excitations. When quantized on a torus with $U(1)$ gauge group (so that it is Maxwell-Chern-Simons (MCS) theory) in 3.3, it shows some topological features like level quantization as in CS theory. One of our main interest is to compute the Chern number of $U(1)$ MCS theory on a torus. As we shall see, when calculating the Cern number for such theory, it is performed on a

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gauge field torus arising from the zero modes of $U(1)$ MCS theory. By zero mode, we mean zero momentum modes which are compatible with the topological structure of torus. The ground states of the $U(1)$ CS as well as MCS theories are degenerate. As such, the usual way of computing the Chern number by counting zero's of the wavefunction [32] on regions where it is defined is not applicable. Therefore, our goal is to calculate the Chern number of such zero modes of the $U(1)$ theory.

Henceforth, we will turn to the computation of Chern number by the so-called generalized periodic boundary condition method. The idea is to apply a $U(1)$ twist (which is defined modulo 2π) to the boundary condition of the theory. By varying the twist, i,e, let say one cycle, the wavefunctions within the degenerate Hilbert space will evolve from one to the another. Repeating this process until reaching the initial wavefunction gives us the desired Chern number. In the physical context, the motivation of proposing this approach lies in the consideration of many-particle interactions [37], since the aforementioned counting zero method only applies to non-degenerate case. We emphasize that this generalized method of Chern number calculation actually applies to both non-degenerate and degenerate cases.

Our final point of focus is the lattice construction of $U(1)$ MCS theory. As it turns out, the naive substitution of continuum derivative by lattice derivative does not make sense for Chern-Simons action. This will be elaborated in 5.2.1. The end result is that it requires the language of cup product to describe the theory. Indeed, by this modification, we could show that for $U(1)$ case, the lattice theory is able to produce desirable features, of which the most notable one is the level quantization.

This thesis is outlined as the following. In Chapter 2, we introduce some basic fea-

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tures of the Chern-Simons theory, such as fractional statistics, expectation values of Wilson loops and present a brief section on level quantization. We then introduce the MCS theory, followed by quantization on the plane and torus. The ground state wavefunctional is unique on the plane whereas it is degenerate on the torus. We shall see that in order for our theory to be interesting by any means, it is necessary to consider $U(1)$ gauge group, since the topology of $U(1)$ group is compatible with the topology of torus. The consequence is that the wavefunctions are required to be invariant under large gauge transformation, the level k is required to be quantized. This is followed by the introduction of the notion of Chern number in Chapter 4. There, we shall see that twisted boundary conditions is employed to compute the Chern number of the MCS theory on a torus. We finally address the lattice gauge theory of $U(1)$ CS and MCS theory on a periodic cubic lattice in Chapter 5. The modified Villain formulation and its motivation are explained. By the modification of cup product as well as this formulation, one may construct the lattice gauge theory of $U(1)$ Chern-Simons and Maxwell-Chern-Simons. Then, in constructing the Hamiltonian, we introduce the concept of transfer matrix and use it to derive the lattice Hamiltonian of MCS theory.

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Chapter 2 Chern-Simons Theory

2.1 Chern-Simons coupled to matter fields

The minimal coupling of the Chern-Simons gauge field to the matter field is described by the following action [12]

$$S = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \int d^3x \, J^\mu a_\mu, \tag{2.1}$$

where k is called the level of the CS action. The equations of motion from (2.1) are

$$\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = J^\mu. \tag{2.2}$$

If we write out (2.2) in component form, it gives

zeroth component:

$$\frac{k}{2\pi} B = \rho, \tag{2.3}$$

spatial component:

$$\frac{k}{2\pi} \epsilon^{ij} E_j = J^i, \tag{2.4}$$

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with $J^\mu = (\rho, J^i)$. With point charges as our matter source,

$$\rho = \sum_a q_a \delta(\vec{x} - \vec{x}_a(t)), \tag{2.5}$$

What (2.3) tells us is that the Chern-Simons term attaches a delta function flux tube to every point charges q_a with flux magnitude $\frac{2\pi q_a}{k}$. On the other hand, Eq.(2.4) is reminiscent of the quantum Hall effect, in which an transverse current is induced by an electric field. Indeed, if the gauge field a_μ is regarded as just background field, rather than dynamical field, then Eq.(2.4) is the encapsulation of integer quantum Hall effect.

To see how this charge-flux binding is manifested more concretely, it is instructive to solve for the gauge potential a_μ. We first impose the condition $a_0 = 0$ to solve the constraint (2.3). Assuming the charge density is given by (2.5) and using the fact that (for $z = x + iy$)

$$\frac{1}{2\pi}(\partial_x \partial_y - \partial_y \partial_x)\text{arg}(z) = \delta(|z|), \tag{2.6}$$

and

$$\frac{1}{2\pi}(\partial_x^2 + \partial_y^2)\text{arg}(z) = 0, \tag{2.7}$$

we can solve the constraint (2.3) to obtain

$$a_i(\vec{x}, t) = \frac{1}{k}\sum_{a=1}^{N} q_a\, \partial_i \text{arg}(\vec{x} - \vec{x}_a(t)), \tag{2.8}$$

and show that (2.8) satisfies the Coulomb gauge

$$\nabla \cdot a = 0. \tag{2.9}$$

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Since for $\mathbb{C}\backslash\{0\}$, the function $\arg(z)$ and $\ln|z|$ satisfy the Cauchy-Riemann conditions

$$(\partial_x \arg)(z) = -\frac{y}{r^2} = -(\partial_y \ln|z|), \tag{2.10}$$

and

$$(\partial_y \arg)(z) = \frac{x}{r^2} = (\partial_x \ln|z|), \tag{2.11}$$

(2.8) can also be written as

$$a_i(\vec{x},t) = \frac{1}{k}\sum_{a=1}^{N} q_a\, \epsilon_{ij} \frac{(x_a^j(t) - x^j)}{|\vec{x} - \vec{x}_a(t)|^2}. \tag{2.12}$$

Although the gauge field of (2.8) looks like pure gauge configuration and could be removed by the following unitary operator

$$U = \exp\left(i\frac{1}{k}\sum_{a=1}^{N} q_a \arg(\vec{x} - \vec{x}_a) \right), \tag{2.13}$$

it in turn makes the originally single-valued wavefunction to be multi-valued. Namely there is a corresponding transformation onto the wavefunctions

$$\psi(\vec{x}) \longrightarrow \psi'(\vec{x}) = e^{\frac{-i}{k}\sum_{a=1}^{N} q_a\, \theta_a(\vec{x})}\psi(\vec{x}), \tag{2.14}$$

where $\theta_a(\vec{x}) = \arg(\vec{x} - \vec{x}_a)$. Now the explicit gauge potential has been 'gauged' away but the nontrivial statistical information lies inside the boundary condition of the wavefunction.

Consider now two point charges, both with unit charge for simplicity and one of them is encircling around the remaining stationary charge. After the process, by (2.14),

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the wavefunction now acquires a phase factor of

$$\psi' \longrightarrow \exp\left(\frac{2\pi i}{k}\right)\psi'. \tag{2.15}$$

We can view the encircling process as a double exchange process, so the exchange phase factor is $\frac{\pi}{k}$. This kind of gauge is called *anyon gauge*[25].

Another gauge is the so-called *CS gauge*, in which the wavefunction obeys bosonic statistics, but the Hamiltonian is

$$H = \frac{-1}{2m}\sum_{a=1}^{N}\nabla_a^2, \quad \nabla_a = \partial_a - iq_a a_a. \tag{2.16}$$

In this gauge, we can also obtain the same result through the Aharonov-Bohm effect. Since the gauge potential is now present, when one particle moves around another, the phase factor is produced from the holonomy of the gauge potential

$$\exp\left(iq_a \oint_C a\right) = \exp\left(q_a q_b \frac{2\pi i}{k}\right), \tag{2.17}$$

which is the same result as (2.15) with $q_a, q_b = 1$.

We see now that, in the Chern-Simons theory, the exchange phase factor is

$$\Delta\theta_{\text{exc}} = \frac{\pi}{k}. \tag{2.18}$$

For $k = 1$, a bosonic wavefunction becomes fermionic in nature and vice versa. For general k, the particles can have any statistics valued in $U(1)$, hence the name *anyons*, which was coined by [52] and is only special to 2 spatial dimensions.

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2.2 Wilson line

Let R be the irreducible representation of the gauge group G. Consider computing the holonomy of a_i around a loop $\mathcal{C}$, and taking the trace of this holonomy in the representation R. The definition of the Wilson loop, in the continuum, is given by

$$W_R[\mathcal{C}] = \mathrm{Tr}_R\, P \exp\left(i \oint_{\mathcal{C}} a_i dx^i\right), \tag{2.19}$$

where the symbol P refers to the path-ordering. For $G = U(1)$, there is no need for path ordering and we have

$$W_{q_e}[\mathcal{C}] = \exp\left(iq_e \oint_{\mathcal{C}} a_i dx^i\right). \tag{2.20}$$

Because of the compactness of $U(1)$, we have $q_e \in \mathbb{Z}$. The physical interpretation of the Wilson line could be given as follows. By introducing a conserved current associated with the closed particle current loop $\mathcal{C}$ parametrized by $\vec{z}(x^0)$

$$J^\mu(x) = q_e \delta^{(d)}(\vec{x} - \vec{z}(x_0)) \frac{dz^\mu}{dx_0}, \tag{2.21}$$

with $z^0 = x^0$, we can rewrite (2.20) as

$$\begin{aligned} \exp\left(iq_e \oint_{\mathcal{C}} a_\mu dz^\mu\right) &= \exp\left(i \int d^d x \delta^{(d)}(\vec{x} - \vec{z}(x_0)) \int dx^0 \frac{dz^\mu}{dx_0} a_\mu\right) \\ &= \exp\left(i \int d^D x J^\mu a_\mu\right). \end{aligned} \tag{2.22}$$

Hence the expectation value of the Wilson loop operator can be viewed as the ratio of the partition function of the system, with external current source inserted, to that without the

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source

$$
\begin{aligned}
\left\langle \exp\left(iq_e \oint_{\mathcal{C}} a_\mu dz^\mu \right) \right\rangle &= \frac{1}{Z} \int \mathcal{D}a \, e^{iS_{CS} + i\int d^D x \, J^\mu A_\mu} \\
&= \frac{Z[J]}{Z[0]}.
\end{aligned} \tag{2.23}
$$

In general, there are two ways to think about Wilson loop operator [23]

a) rectangular timelike loops: creation, propagation and annihilation of two static sources of particle and antiparticle fixed at two spatial locations respectively, where the propagation is along time direction. The creation takes place at an initial time ($t = t_i$) and the annihilation at final time ($t = t_f$).

b) spacelike loop: vacuum fluctuation in the absence of external sources.

Indeed, there is no any intrinsic differences between the above two cases. They are related to each other by Lorentz rotation (or by Euclidean rotation in imaginary time)[23]. We will mostly be interested in the first case.

To make the connection to case (a) more established, pick a timelike rectangular contour which has time span T and spatial length L, in which the bottom left corner point is at $x = 0$. This loop is counterclockwise oriented. Assuming $T \gg L$, we see that the time slice of the contour at $x = L$ and $x = 0$ has $J_0 = 1$ and $J_0 = -1$ respectively. This implies there are two oppositely charged static particles separated by a distance L all along the time span. The timelike Wilson line (in the $T \gg L$ limit) then represents the process of creation of static particle and antiparticle pair, propagate the pair by time T and annihilates it.

In the $T \to \infty$ limit, we know that $Z[0]$ is given by $Z[0] = e^{-TE_0}$, where E_0 is the

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vacuum energy. Then, from (2.23), we have

$$\left\langle \exp\left(iq_e \oint_{\mathcal{C}} a_\mu dz^\mu \right) \right\rangle = e^{-(E_0(J)-E_0)T}. \tag{2.24}$$

Since the charges are static, the energy term in the exponent of the RHS of (2.24) is the interaction energy, or potential, between the static source pair [17, 31]. Therefore, we conclude that the potential $V(R)$ is related to the vacuum expectation value of the Wilson line

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln\langle W[\mathcal{C}]\rangle. \tag{2.25}$$

The Wilson line expectation value is typically characterized by its geometric properties of the loop and these are the 3 generic cases:

a)*Area law*:

$$\langle W[\mathcal{C}]\rangle = e^{-\sigma RT}, \tag{2.26}$$

where from (2.25), it give rises to a linear potential

$$V(R) = \sigma R. \tag{2.27}$$

As a result, the energy cost required to separate a pair of source charges increases linearly with the distance R and the charges are confined.

b) *Perimeter law*:

$$\langle W[\mathcal{C}]\rangle = e^{-\rho(R+T)}. \tag{2.28}$$

This leads to, according to (2.25), a constant potential which remains even at large distance

$$V(R) = \rho. \tag{2.29}$$

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Therefore, the perimeter law corresponds to the deconfined phase since the charges are not confined. Despite of this, there is no massless gauge boson (they are all massive) and this is regarded as the Higgs phase, which bears close resemblance to a superconductor[Fradkin].

c) *Scale-invariant law*:

$$\langle W[\mathcal{C}]\rangle = e^{-(\alpha\frac{T}{R}+\beta\frac{R}{T})}. \tag{2.30}$$

This is also known as the Coulomb phase, as the corresponding potential has the familiar form of

$$V(R) = \frac{\alpha}{R}. \tag{2.31}$$

This is a deconfined phase with massless gauge bosons.

2.3 Path Integral Quantization: Linking of knots

The exchange phase appeared below the discussion of (2.17) can also be obtained in the path integral quantization. Namely, we compute the expectation value of a product of Wilson loops

$$\langle W(L)\rangle = \left\langle \prod_{i=1}^{N} \exp\Big(iq_i \oint_{\mathcal{C}_i} a\Big)\right\rangle_{\text{CS}}, \tag{2.32}$$

where the subscript CS denotes the expectation value evaluated with respect to Gaussian measure by Chern-Simons action. This product of loops characterizes in total N loop trajectories of charged particles, each with charge q_i and traversing loop $\mathcal{C}_i$. As shown by Polyakov [40], the result is given by

$$\langle W(L)\rangle = \exp\Big(\frac{i\pi}{k}\sum_{i,j} q_i q_j L(\mathcal{C}_i, \mathcal{C}_j)\Big), \tag{2.33}$$

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where $L(\mathcal{C}_i, \mathcal{C}_j$ is the integral (Gauss) linking number given by

$$L(\mathcal{C}_i, \mathcal{C}_j) = \frac{1}{4\pi} \oint_{C_i} dx^\mu \oint_{C_j} dy^\nu \, \epsilon_{\mu\nu\lambda} \frac{(x^\lambda - y^\lambda)}{|x-y|^3}. \tag{2.34}$$

This expression (2.34) is actually pretty common in the literature of higher-form symmetries [21] and TQFT [4].

In the following, we give a detailed direct computation of (2.32). Because of (2.22), we can use the theorem of Gaussian integral to find

$$\langle W(L) \rangle = exp\Big(- \frac{i}{2} \sum_{i,j} \int dx^3 \int dy^3 J_i^\mu(x) G_{\mu\nu}(x-y) J_j^\nu(y) \Big), \tag{2.35}$$

where $G^{\mu\nu}(x-y)$ is the Green function of the Chern-Simons action, also known as the propagator of the gauge field $\langle a_\mu(x) a_\nu(y) \rangle$. The Latin indexes i and j parameterise the particle current loops and Greek indices are the spacetime coordinates. Then we only need to compute $G_{\mu\nu}(x-y)$. We proceed as usual by adding a covariant gauge fixing term, $L_{gf} = -\frac{1}{2\alpha}(\partial_\mu a^\mu)^2$. For $\alpha \to 0$, we recover the Lorentz gauge.

With the gauge fixing term, our Lagrangian now becomes

$$L_{CS,gf} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\alpha} (\partial_\mu a^\mu)^2 + J^\mu a_\mu \tag{2.36}$$

where the current $J^\mu = \sum_i J_i^\mu$ is the sum of N particle current loops. It can be rewritten as, up to surface term,

$$L_{CS,gf} = \frac{1}{2} a_\mu \left[\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu - \frac{1}{\alpha} (\partial^\mu \partial^\lambda) \right] a_\lambda + J^\mu a_\mu. \tag{2.37}$$

Using the theorem of Gaussian integral, and comparing to (2.35), we see that the propa-

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gator is

$$G_{\mu\nu}(x-y) = \langle x|\left(\frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu - \frac{1}{\alpha}\partial^\mu\partial^\lambda\right)^{-1}|y\rangle. \tag{2.38}$$

Transforming (2.38) to momentum space, the momentum propagator satisfies the following equation

$$\left(-\frac{ik}{2\pi}\epsilon^{\mu\nu\lambda}p_\nu + \frac{1}{\alpha}p^\mu p^\lambda\right)G_{\lambda\sigma}(p) = g^\mu_\sigma. \tag{2.39}$$

Contracting (2.39) with p_μ, we get

$$-\frac{p^2p^\lambda}{\alpha}G_{\lambda\sigma}(p) = p_\sigma \quad\Rightarrow\quad p^\lambda G_{\lambda\sigma}(p) = -\frac{\alpha}{p^2}p_\sigma. \tag{2.40}$$

Substituting (2.40) into (2.39), contracting it with $\epsilon_{\mu^\nu\mu}$, and using the identity $\epsilon_{\mu^\nu\mu}\epsilon^{\mu\nu\lambda} = \delta^\nu_\mu\delta^\lambda_\nu - \delta^\lambda_\mu\delta^\nu_\nu$, we have

$$-\frac{ik}{2\pi}\left(p_\mu G_{\nu\sigma} - p_\nu G_{\mu\sigma}\right) = \epsilon_{\mu\nu\sigma} - \frac{\epsilon_{\mu\nu\mu}p^\mu p_\sigma}{p^2}. \tag{2.41}$$

Contracting with p^μ further gives

$$p^2G_{\nu\sigma} - p_\nu p^\mu G_{\mu\sigma} = \frac{2\pi i}{k}\epsilon_{\mu\nu\sigma}p^\mu. \tag{2.42}$$

Then using (2.40) and relabeling the indices gives the desired propagator

$$G_{\mu\nu}(p) = \frac{2\pi i}{k}\frac{\epsilon_{\mu\nu\lambda}p^\lambda}{p^2} + \frac{\alpha p_\mu p_\nu}{p^4}. \tag{2.43}$$

In the Lorentz gauge, we have the desired propagator in real space-(time)

$$G_{\mu\nu}(p) = \frac{2\pi}{k}\frac{\epsilon_{\mu\nu\lambda}\partial^\lambda}{\nabla^2} \tag{2.44}$$

By the fact that $\nabla^2\frac{1}{|x-y|} = -4\pi\delta^3(x-y)$ and $\partial^\lambda\frac{1}{|x-y|} = -\frac{(x^\lambda-y^\lambda)}{|x-y|^3}$, it is straightforward

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that

$$\langle W(L)\rangle = exp\bigg(-\frac{i\pi}{k}\frac{1}{4\pi}\sum_{i,j}\int dx^3\int dy^3 J_i^{\mu}(x)\epsilon_{\mu\nu\lambda}\frac{(x^{\lambda}-y^{\lambda})}{|x-y|^3}J_j^{\nu}(y)\bigg), \tag{2.45}$$

Then rewriting the current term as a closed loop integral as did in (2.22) and using (2.34) yields the result (2.33). The Gauss linking number $L(C_i, C_j)$ is an integer when the loops C_i and C_j do not intersect.

For self-intersecting terms in (2.33), according to [53], point-splitting regularization is adopted to make them well-defined. According to this prescription, every loop C_i is assigned a framing, which is a normal vector field along C_i. Then we displace the loop along the direction appointed by this vector field to obtain a new loop C^{i}. The self-intersecting terms can then be interpreted as the linking of C_i with C^{i}, which is a ribbon bounded by them. A physical interpretation of the framing of the loops is given by [53]. If we apply a 2π twist to a framed knot, the self-linking number changes by one unit. It follows that the expectation value of the loop shifts by an amount of

$$\langle W(L)\rangle \to exp\bigg(\frac{i\pi q_i^2}{k}\bigg)\langle W(L)\rangle. \tag{2.46}$$

This additional phase corresponds to the exchange phase of a particle with spin identified with

$$s = \frac{q_i^2}{2k}, \tag{2.47}$$

which agrees with (2.18). Hence, we have arrived at the same result by using path integral formalism in evaluating the expectation value of Wilson loops, which shows that the sources of $U(1)$ Chern-Simons theory possess fractional statistics.

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2.4 Level quantization

In all the previous sections, we have actually not assumed compact gauge group. As a result, the level k could be arbitrary values and there is no quantization of it. The level quantization actually depends on the topology of the gauge group G or the spacetime M. More precisely, for $U(1)$ CS, we also need to have spacetime M with non-trivial topology to have quantized k. One such option would be a spatial torus, where one could consider large gauge transformation compatible with the $U(1)$ nature of the gauge group. Another choice, which is also quite common in the physics literature, is to pick Euclidean periodic time. Since a transformation that winds around S^1 (periodic time) an integer number of times is now possible [42], it could be shown that k must be quantized so that the exponentiated action is invariant under such transformation.

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Chapter 3 Maxwell-Chern-Simons Theory

3.1 Introduction to MCS

In [10], it is shown that in 2+1 dimensions, it is possible to have massive gauge field without violating gauge invariance. By considering the usual Maxwell term together with the Chern-Simons term, they showed that the theory possesses massive excitations with spin 1. This is somehow the generalization of the fact that $U(1)$ gauge field in (2+1)D is equivalent to a compact scalar [34].

With the Lagrangian density

$$L = -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_{\mu\nu}a_{\lambda} \tag{3.1}$$

the equation of motion follows as

$${}_{\mu}F^{\mu\nu} + \frac{ke^2}{4\pi}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = 0 \tag{3.2}$$

The equation of motion is gauge invariant, hence there is single degree of freedom, in contrast to the Proca mass term which incorporates mass into gauge field at the expense

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of gauge invariance. To see the massive nature of the theory, it is instructive to write the field equation (3.2) as

$$\partial_\mu F_\nu - \partial_\nu F_\mu - \frac{k}{2\pi} F_{\mu\nu} = 0 \tag{3.3}$$

Taking the divergence and using (3.2) give

$$\begin{aligned} (^{\mu}_{\mu} + m^2) F^\nu &= 0 \\ m &= \frac{|k|e^2}{2\pi} \end{aligned} \tag{3.4}$$

where $F^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}$ is the dual field strength. This clearly demonstrates the massive character of the gauge field excitations. Because of the topological nature of the mass term, (3.1) is also called *topologically massive gauge theory*. However, this does not tell us the information about the spin of the excitations.

There are at least two ways to determine the spin of the excitations. The first is to solve the theory in terms of a free massive scalar field. The first way involves the scalar field construction of the MCS theory. The second way comes to inspecting the one-particle states in the representation of the (2+1)-dimensional Poincaré algebra [12, 50]. There are two Casimirs that carry unitary irreducible representation of the universal covering of the Poincaré group

$$P^2 = P^\mu P_\mu \qquad W = P_\mu J^\mu = \frac{1}{2} P_\mu \epsilon^{\mu\nu\rho} L_{\nu\rho} \tag{3.5}$$

The first is the square of the momentum operator while the second is the Pauli-Lubanski pseudoscalar, the (2+1)d analogue of the Pauli-Lubanski pseudovector in (3+1)d. Their respective eigenvalues m^2 and $-sm$ determine the mass m and spin s of the representation of the states. We first express the field equation (3.2) in terms of F and Fourier

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transform to get

$$-i\epsilon^{\mu\nu\rho}p_\nu F_\rho + \frac{ke^2}{2\pi}F^\mu = 0 \tag{3.6}$$

By taking the adjoint(vector) representation of the $SO(2+1)$ Lorentz algebra

$$(J^\mu)_{\alpha\beta} = i\epsilon^\mu_{\alpha\beta} \tag{3.7}$$

equation (3.7) becomes

$$(p\cdot J)^{\mu\nu}F_\nu = -\frac{k}{|k|}mF^\mu \tag{3.8}$$

This is just the Pauli-Lubanski eigenvalue condition with spin given by

$$s = \frac{k}{|k|} = \pm 1 \tag{3.9}$$

with sign determined by the sign of the Chern-Simons level.

Another canonical way to analyze the theory is to compute the propagator of (3.1). Inserting a gauge fixing term with covariant gauge $L_{gf} = -\frac{1}{2\alpha}(\partial_\mu a^\mu)^2$ as in section, the propagator is found to be

$$G_{\mu\nu}(p) = \frac{e^2}{p^2 - e^4\kappa^2}\left(g_{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) - \frac{i\kappa e^2\epsilon_{\mu\nu\lambda}p^\lambda}{p^2(p^2 - e^4\kappa^2)} + \frac{\alpha p_\mu p_\nu}{p^4} \tag{3.10}$$

$$\kappa = k/2\pi \tag{3.11}$$

The propagator above has a pole at $p^2 = (\frac{ke^2}{2\pi})^2$, implying the gauge field is massive.

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3.2 Canonical Quantization in Plane

The canonical quantization of the MCS theory can provide us interesting information to the Hilbert space of the wavefunctionals of the gauge field. Since we are discussing gauge theory, we have additional constraint to care about: Gauss's law. We adopt the quantization first, followed by imposing the constraint on the space of states approach. This is what we mean by canonical quantization of gauge theory. The mathematical framework underlying this quantization is called *geometric quantization*. See more about this framework on [5, 34, 35].

On the other hand, we could also have solved the constraint first, then followed by quantizing the resulting unconstrained theory. This had been analysed in [41]. Note that the results of the two approaches do not always coincide. But for gauge theory, we expect they should yield the same results. We begin with the analysis of MCS on the plane and perform the quantization in the Weyl or temporal $(A_0 = 0)$ gauge.

Our A_i are the canonical variables while the canonical momentum gets a contribution from Chern-Simons term

$$\Pi^i = \frac{L}{\dot{A}_i} = \frac{\dot{A}^i}{e^2} + \frac{k}{4\pi}\epsilon^{ij}A_j \tag{3.12}$$

The Hamiltonian density, by a Legendre transformation, is

$$H = \Pi^i \dot{A}_i - L = \frac{e^2}{2}\left(\Pi^i - \frac{k}{4\pi}\epsilon^{ij}A_j\right)^2 + \frac{1}{2e^2}(\epsilon_i^{ij}A_j)^2 \tag{3.13}$$

where we have omitted the Lagrange multiplier term $A_0({}_i\Pi^i + \frac{k}{4\pi}\epsilon_i^{ij}A_j)$ corresponding to the Gauss law.

The Hamiltonian $H = \int d^2x\, H$, when expressed in terms of the electric and magnetic

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fields,

$$E^i = -\dot{A}^i B = \epsilon_i^{ij} A_j \tag{3.14}$$

respectively, is given by

$$H = \frac{1}{2e^2} \int d^2x \, (E^2 + B^2) \tag{3.15}$$

At the quantum level, we have the following commutation relations

$$[A^i(\vec{x}), \Pi^j(\vec{y})] = i\delta(\vec{x} - \vec{y}) \tag{3.16}$$

$$[E^i(\vec{x}), E^j(\vec{y})] = -i\kappa e^4 \epsilon^{ij} \delta(\vec{x} - \vec{y}) \tag{3.17}$$

$$[E^i(\vec{x}), B(\vec{y})] = ie^2 \epsilon_j^{ij} \delta(\vec{x} - \vec{y}) \tag{3.18}$$

$$[B(\vec{x}), B(\vec{y})] = 0 \tag{3.19}$$

where κ as in (3.11). The Hamiltonian equation $\dot{A}^i = i[H, A^i]$ and $\dot{\Pi}^i = i[H, \Pi^i]$ give the spatial components of the equation of motion. The time component

$$G =_i \Pi^i + \frac{k}{4\pi} B = 0 \tag{3.20}$$

which is exactly the Gauss law, is imposed as a selection rule for physical states

$$GPhys = 0 \tag{3.21}$$

That is, we do not require $G = 0$ at the operator level. Rather it is a constraint that restricts the Hilbert space to be the space of gauge-invariant states, as (3.21) implies that only states that are invariant by local gauge transformation are the physical states.

The theory can be solved in a transparent way in the Schrödinger representation Here

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we have quantum states which are functionals $\Psi(A)$ of dynamical variables A on which the gauge group action acts

$$\Psi \leftrightarrow \Psi(A) \tag{3.22}$$

The operator A acts on the functionals by mutiplication

$$A(\mathbf{r}) \leftrightarrow A(\mathbf{r})\Psi(A) \tag{3.23}$$

while the canonical momentum operator by functional differentiation

$$\square(\mathbf{r}) \leftrightarrow \frac{1}{i}\frac{\delta}{\delta A(\mathbf{r})}\Psi(A). \tag{3.24}$$

Then the energy eigenstates satisfy the Schrödinger equation

$$\frac{1}{2}\int d\mathbf{r}\left[\left(\frac{1}{i}\frac{\delta}{\delta A^i(\mathbf{r})} - \frac{k}{4\pi}\epsilon^{ij}A^j(\mathbf{r})\right)^2 e^4 + B^2(\mathbf{r})\right]\Psi(A) = E\Psi(A) \tag{3.25}$$

as well as the subsidiary constraint (3.21)

$$\left[\frac{1}{i}\partial_i\frac{\delta}{\delta A_i(\mathbf{r})} + \frac{k}{4\pi}B(\mathbf{r})\right]\Psi(A) = 0. \tag{3.26}$$

The most general wavefunctionals satisfying (3.21) has the following form

$$\Psi(A) = e^{i\chi(A)}\Phi(A_T) \tag{3.27}$$

where

$$\chi(A) = \frac{k}{4\pi}\int dr\, B\boldsymbol{\nabla}^{-1}\cdot A \tag{3.28}$$

$$\boldsymbol{\nabla}^{-1} = \frac{\boldsymbol{\nabla}}{\nabla^2} \tag{3.29}$$

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The presence of the phase $\chi(A)$ implies that the wavefunctionals are not gauge invariant. Rather they transform under the gauge transformation via a 1-cocyle [26]

$$\Psi(A + \boldsymbol{\nabla}\Lambda) = e^{\frac{ik}{4\pi}\int dr\, B\Lambda}\Psi(A) \tag{3.30}$$

We made use of the fact that local gauge transformation does not affect the transverse part. If there is no Chern-Simons term $(k = 0)$, the Gauss law statement would be $\Psi(A^g) = \Psi(A)$, where A^g is the gauge transform of A by a group element g. This is what happen in Maxwell theory, where the ground state wavefunctional can be written in a gauge invariant form [17, 49]

$$\Psi_{0,Maxwell}(A) = \mathcal{N}exp\Big(-\frac{1}{4\pi^2}\int dr\int dr' \frac{B(r)\cdot B(r')}{|r-r'|^2}\Big) \tag{3.31}$$

which is a functional of magnetic field configurations.

There is another perspective to see the non-gauge invariance of the MCS wavefunctional. Apart from a total derivative, the Chern-Simons term can be written in a gauge-invariant, albeit spatially nonlocal form

$$\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}A_{\mu\nu}A_{\lambda} = \frac{k}{4\pi}B\,\boldsymbol{\nabla}^{-1}\cdot E - \frac{k}{4\pi}E\cdot\boldsymbol{\nabla}^{-1}B - \frac{k}{8\pi}\epsilon^{\mu\nu\lambda}_{\lambda}(F_{\mu\nu}\boldsymbol{\nabla}^{-1}\cdot A) \tag{3.32}$$

After considering the space integral of (3.32), under a gauge transformation $A \to A + \nabla\Lambda$, (3.32) changes up to a total time derivative $-\frac{k}{8\pi}\frac{d}{dt}\big(\epsilon^{0ij}F_{ij}\Lambda\big) = -\frac{k}{4\pi}\frac{d}{dt}\big(B\Lambda\big)$. The changes of Lagrangian under a gauge transformation by a total time derivative of a function is crucial to understand the phases appearing in (3.30). If such changes of a total time derivative of a function is present, that function will appear in the transformation law for the quantum states (our wavefunctional) [26, 49]. Indeed, we see that this is exactly the

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phase in (3.30).

This exemplifies a rather general relation between a 1-cocyle appearing in the action of a gauge transformation on the quantum states as well as the non-gauge invariance of the Lagrangian. In fact, we can even say more in generalizing the gauge transformation to symmetry transformation. For example, 1-cocyle may happen in quantum mechanics whenever one has a symmetry transformation that leaves the action S invariant, but Lagrangian by a total time derivative. This kind of phenomenon in quantum mechanics as well as quantum field theory may be actually unified in the language of cocycles. For more details, see section 6.2 of "Topological Investigation of Quantized Gauge Theories" in [49].

To see the contrast of MCS thory with Maxwell theory, one may solve the MCS ground state. The lowest energy eigenstate is shown to be [10]

$$\Psi_{0,MCS}(A) = e^{i\chi(A)} exp\Big(-\frac{1}{2e^2}\int dr \int dr'\, A_i(r)G^{ij}(r-r')A_j(r')\Big) \tag{3.33}$$

$$G^{ij}(r-r') = \int \frac{dk}{(2\pi)^2} e^{-ik\cdot(r-r')}(\delta^{ij} - \hat{k}^i\hat{k}^j)\sqrt{\kappa^2 e^4 + k^2} \tag{3.34}$$

In the Helmholtz-Hodge decomposition in R^2

$$A = \boldsymbol{\nabla}\lambda + A_T \tag{3.35}$$

the ground state (3.33) is expressed by the following illuminating form [13]

$$\Psi_{0,MCS}(A) = \Big(exp\, i\frac{k}{4\pi}\int dr\, B\lambda\Big)\Big(exp - \frac{1}{2e^2}\int dr\, A_T^i\sqrt{-\nabla^2 + \kappa^2 e^4} A_T^i\Big) \tag{3.36}$$

The kernel accompanying the transverse degree of freedom reflects the massive character

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of the excitations since

$$\left(\sqrt{-\nabla^2+\kappa^2 e^4}\right)(x,y)=\int\frac{dk}{(2\pi)^2}e^{-ik\cdot(x-y)}\sqrt{k^2+\kappa^2 e^4} \tag{3.37}$$

Also, in (3.34), the function $\delta^{ij}-\hat{k}^i\hat{k}^j$ is the so-called transverse delta function we usually encountered in Coulomb gauge quantization of gauge theory, especially Maxwell theory. This is the reason why A_T appears in (3.36).

Let us now outline how to derive (3.33) as the algebra turns out to be lengthy. Since there is only one degree of freedom for the MCS gauge field in 2+1d, we expect all the energy content is contained in the transverse parts. It is also reasonable to assume $\Phi(A_T)$ in (3.27) is a general bilinear form K^{ij} of A_T and expect the transverse delta function to be the natural candidate. With this assumption, putting the ansatz (3.27) in (3.25) gives something like

$$\frac{1}{2}Tr\,G+(k^2+\kappa^2e^4)A_T^2-e^4A_i(k)G^{ij}(k)A_j(-k)=E \tag{3.38}$$

where E is the energy density. The ground state energy, obtained by setting all the quadratic terms equal to 0, is therefore

$$E_0=\frac{1}{2}Tr\,G=\int dr\int\frac{dk}{(2\pi)^2}\sqrt{k^2+\kappa^2e^4} \tag{3.39}$$

This is indeed the infinite vacuum energy of the ground state as expected. In this way, it is evident that G^{ij} is indeed given by (3.34).

We now briefly mention the scalar field construction. This is a natural construction since there is only one degree of freedom. One classic example of this is the free Maxwell theory in the same spacetime dimensions. There, the $U(1)$ theory is dual to a

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free massless compact scalar field and have zero spin, as is reflected by the fact that massless representations of the three-dimensional Poincaré group describe spinless particles. The commutation relations (3.16) and the equations of motion (3.2) can be solved by the following identification

$$E^i = -\epsilon^{ij}\hat{\ }_j\dot{\phi} - \kappa e^{2\hat{\ }i}\phi \tag{3.40a}$$

$$B = e^2\sqrt{-\nabla^2}\phi \tag{3.40b}$$

$$\hat{\ }_i =_i /\sqrt{-\nabla^2} \tag{3.40c}$$

One can check that they give rise to the desired commutation relations

$$({}^{\mu}_{\mu} + \kappa^2 e^4)\phi = 0 \tag{3.41a}$$

$$[\phi(r), \dot{\phi}(r')] = i\delta \tag{3.41b}$$

The Hamiltonian, expressed in terms of the scalar field form, is

$$H = \frac{1}{2e^2}\int d^2x\,[E^2 + B^2] = \frac{e^2}{2}\int d^2x\,[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + \kappa^2 e^4\phi^2] \tag{3.42}$$

This allows us to see the theory indeed contains excitations of mass κ. The momentum operator and the rotation generator on the other hand are

$$P^i = \int d^2x\,(E \times B)^i = \int d^2x\,\epsilon^{ij}E^jB = e^2\int d^2x\,\dot{\phi}^j\phi \tag{3.43a}$$

$$M = \int d^2x\,r \times (E \times B) = -\int d^2x\,r \cdot EB = -e^2\int d^2x\,\dot{\phi}r^i\epsilon^{ij}_j\phi \tag{3.43b}$$

From the above form for P^i and M, we see that the mass term does not contribute to them, as is expected from the topological nature of Chern-Simons term.

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Thus we show that there is a unique groundstate wavefunctional for MCS in a plane, which are the harmonic oscillator groundstates of the oscillators of the massive scalar (3.42). Besides, its form and properties like mass and spin can be determined exactly.

3.3 Canonical Quantization on the Torus

In this section, we are going to determine the ground state wavefunctions of the MCS theory on a torus, which is required to determine the Chern number. On a Riemann surface Σ of genus g, there are nontrivial $2g$ homology cycles associated to those noncontractible loops of Σ. The decomposition (3.35) is not sufficient, as there exists topological modes which does not depend on the momentum. Hence such modes are also called zero modes [38].

They correspond to the global flat sector ($F = 0$, with $\Pi^i = 0$ in (3.20)) of the theory while the non-flat sectors are responsible for the massive photonic excitations. We emphasize that such topological modes necessarily possess zero momentum, but the converse is not true. Part of zero momentum modes are comprised of the massive photon harmonic oscillator at rest, with $k = 0$ in (3.39). Hence, there are two parts comprising the zero-momentum sector of the theory: topological modes and static massive photon harmonic oscillators [41]. Since we will only focus on the topological modes, we will use zero modes to mean topological modes.

The Hilbert space can be seen as a product

$$H = H_{nf} \otimes H_f \tag{3.44}$$

since from (3.15), these two modes clearly decouple from each other. For the flat modes, B

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vanishes and we only take account of the electric field term in (3.15). In fact, by inspecting (3.12), we see that the form of electric field has the analog of a charged particle moving in a magnetic field by the following correspondence

$$E \quad \longleftrightarrow \quad p + eA = \Pi^i - \frac{k}{4\pi}\epsilon^{ij}A_j \tag{3.45}$$

Then the whole discussion can be translated to the language of a charged particle in magnetic field, although now the space is the gauge field space. Since we are considering $U(1)$ theory, our gauge field is periodic up to 2π and the space would be gauge field torus, which can be seen as a square of length 2π with periodic boundary condition.

The discussion above might seem a bit too abstract. It is instructive for us to outline our setup to see how all these arise. Our spatial manifold is a torus with spatial length $L_x \times L_y$. We also adopt the natural unit $e = 1$. In the $A_0 = 0$ gauge, we decompose the gauge field A_i in spatial modes $A_i = \frac{a_i}{L_i}(t)e^{ip_i x_i}$ [41]. Here our sole focus is the zero mode with $p_i = 0$. Substituting these zero modes into (3.1), after integrating over space, yields the following Lagrangian

$$L = \frac{1}{2}\dot{a}_i^2 + \frac{k}{4\pi}(a_2\dot{a}_1 - a_1\dot{a}_2) \tag{3.46}$$

We add a total derivative term $\frac{k}{4\pi}\frac{d}{dt}(a_1 a_2)$. This results in phase space like form $p\dot{q}$ for the CS term and the resulting Lagrangian is just

$$L = \frac{1}{2}\dot{a}_i^2 + \frac{k}{2\pi}a_2\dot{a}_1 \tag{3.47}$$

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The canonical momenta for the modes a_i are respectively

$$b_1 = \dot{a}_1 + \frac{k}{2\pi} a_2 \tag{3.48a}$$

$$b_2 = \dot{a}_2 \tag{3.48b}$$

In terms of the canonical variables, the Hamiltonian is

$$H = \frac{1}{2}\left[\left(b_1 - \frac{k}{2\pi} a_2\right)^2 + b_2^2\right] \tag{3.49}$$

We recognize that the is just the Hamiltonian for a charged particle in magnetic field in the Landau gauge. Now the form (3.49) makes the correspondence (3.45) evident, although now the gauge field A in (3.49) is different. Recall that we have added a total time derivative term in converting one form of CS term to another form. This may be thought of as a gauge transformation, as the form $v \cdot A$ is invariant up to a total time derivative term under gauge transformation. For instance, in (3.47), v stands for $\dot{a}_1$ whereas A for $\frac{k}{2\pi} a_2$ (Landau gauge).

Since we are considering $U(1)$ gauge field, there are large gauge transformation to worry about. These are the set of gauge transformations $g(x)$ which is not connected to the identity at spatial infinity, or in other words, it cannot be written as $g(x) = e^{i\epsilon(x)}$ for a globally well-defined function $\epsilon(x)$ [34]. Their effect is to shift the gauge field by $2\pi Z$, that is

$$a \to a + \omega \qquad \omega \in \Omega^1_{2\pi Z}(\Sigma) \tag{3.50}$$

where $\Omega^1_{2\pi Z}(\Sigma)$ denotes the space of closed one forms with periods of $2\pi Z$. For the torus

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in consideration, one example of large gauge transformation could be given

$$\xi(x_i) = \frac{2\pi N_{x_i} x_i}{L_{x_i}} \tag{3.51}$$

where $N_{x_i} \in Z$ are the winding numbers N_x and N_y around the two nontrivial cycles of the spatial torus. It is periodic up to $2\pi Z$ on the spatial torus where $x_i \sim x_i + L_{x_i}$. This compatibility of the transformation (3.51) to the periodicity of the spatial torus makes it necessary to take them into account in the full theory.

Since these are gauge transformations, the states should be invariant under their action. The shift (3.50) are simply translations in a_i. However, the generator of such translation are not the canonical momenta as usual. This is due to the fact that in writing down the Hamiltonian (3.49), we have made a particular gauge choice in the sense of the analogy mentioned above and this gauge choice breaks translational invariance in the y-direction, as can be seen as $A_{a_1}(a_2) = \frac{k}{2\pi}a_2$. Under a translation $a_2 \to a_2 + \lambda_2$, the 'gauge field' A_{a_1} transforms by

$$\begin{aligned} A_{a_1}(a_2 + \lambda_2) &= A_{a_1}(a_2) + \frac{k}{2\pi}\lambda_2 \\ &= A_{a_1}(a_2) + \nabla_{a_1}\left(\frac{k}{2\pi}\lambda_2 a_1\right) \end{aligned} \tag{3.52}$$

Hence translation of a_2 is equivalent to a gauge transformation. The reason underlying this phenomenon is that the magnetic field (which is a pseudoscalar in 2 spatial dimensions) $B = \boldsymbol{\nabla} \times A$ in our problem is uniform. The consequence of this is a translate in a_2 always has to be accompanied by a gauge transformation.

Since the Hamiltonian is not invariant under translation, the canonical momenta b_i does not commute with the Hamiltonian H. By the reasoning above, a translation followed

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by an appropriate gauge transformation is the symmetry generators. It constitutes the appropriate translation generator T_i for our problem. In the physics literature of Quantum Hall effect, this generalized translation operator is called *magnetic translation operator* [16, 29]. For our case, the generator for $T_i = e^{iD_i}$ is given by

$$D_1 = b_1 \tag{3.53a}$$

$$D_2 = b_2 + \frac{k}{2\pi} a_1 \tag{3.53b}$$

It is straightforward to check that D_i commute with the Hamiltonian. From (3.53a) and (3.53b), it is evident that periodic boundary conditions for the wavefunctions $\Psi(a_1, a_2)$ is not possible as long as there is 'magnetic field'. The global gauge invariance for the wavefunctions can be written as the following boundary conditions

$$T_1^{2\pi}\Psi(a_1, a_2) = \Psi(a_1, a_2) \tag{3.54a}$$

$$T_2^{2\pi}\Psi(a_1, a_2) = \Psi(a_1, a_2) \tag{3.54b}$$

Explicitly these boundary conditions are

$$\Psi(a_1 + 2\pi, a_2) = \Psi(a_1, a_2) \tag{3.55a}$$

$$\Psi(a_1, a_2 + 2\pi) = e^{-ika_1}\Psi(a_1, a_2) \tag{3.55b}$$

Although $T_1^{2\pi}$ and $T_2^{2\pi}$ commute with the Hamiltonian, they do not commute with each other. Instead they obey the following commutation relation

$$T_1^{2\pi} T_2^{2\pi} = e^{i2\pi k} T_2^{2\pi} T_1^{2\pi} \tag{3.56}$$

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In order to have gauge invariant states, we must have $k \in Z$. The magnetic field in our case is $\frac{k}{2\pi}$ while the area of the torus is $(2\pi)^2$. Then the phase $2\pi k$ appearing in (3.56) is simply the magnetic flux through the a_i-torus. The quantization of the level k can thus be interpreted as the magnetic flux through a finite volume space is quantized as an integer units of magnetic flux quantum (in our unit, magnetic flux quantum ϕ_0 is just 2π).

We are only interested in the ground states of the problem at hand, which is lowest Landau level. Recall that all the discussions have been carried out in the Landau gauge. Interestingly, there are two forms of solution in this gauge satisfying the boundary conditions (3.55). The first solution is [2, 44]

$$\Psi_\ell(a_1, a_2) = \sum_{m \in Z} exp\left(i(\ell + km)a_1 - \frac{k}{4\pi}\left(a_2 + \frac{2\pi\ell}{k} + 2\pi m\right)^2\right) \tag{3.57}$$

with $\ell = 0, 1, \ldots, k$. In complex coordinates, we can write in the following form

$$\Psi_\ell(a_1, a_2) = \theta \begin{bmatrix} \ell/k \\ 0 \end{bmatrix} (ka_z|ik) e^{-\frac{ka_2^2}{4\pi}} \tag{3.58}$$

From (3.58), we see that apart from a Gaussian term in a_2, the solution is Jacobi theta function, which is reminiscent of the ground states of pure Chern-Simons theory.

The second solution is found to be [2, 24]

$$\Psi_\ell(a_1, a_2) = \sum_{m \in Z} exp\left(i\left(\ell + km - \frac{ka_1}{2\pi}\right)a_2 - \frac{k}{4\pi}\left(a_1 - \frac{2\pi\ell}{k} - 2\pi m\right)^2\right) \tag{3.59}$$

whose complex form is

$$\Psi_\ell(a_z, a_{\bar{z}}) = e^{-\frac{\pi k}{2} a_z a_{\bar{z}}} e^{\frac{\pi k}{2} a_z^2} \theta \begin{bmatrix} \ell/k \\ 0 \end{bmatrix} (ka_z|ik) \tag{3.60}$$

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We recognize that the solution above could be written in the general form

$$\Psi_\ell(a_z, a_{\bar{z}}) = e^{-\frac{\pi k}{2} a_z a_{\bar{z}}} \psi(a_z) \tag{3.61}$$

This form reminds us of the solution of pure Chern-Simons theory, which also has the similar form.

The reason for these 2 seemingly different solutions lies in two facts:

i) as shown in [2, 51], the set of translation operators which leaves the boundary conditions unchanged are given by

$$t_1 \equiv T_1^{2\pi/k} \tag{3.62a}$$

$$t_2 \equiv T_2^{2\pi/k} \tag{3.62b}$$

ii) Even though t_1 and t_2 commute with the Hamiltonian, they do not commute with each other.

Therefore, it is only possible to diagonalize one of them along with the Hamiltonian. Then the solution given by (3.57) is just the eigenstate of t_1 with eigenvalue $exp(i2\pi\ell/k)$. On the other hand, the solution (3.59) is the eigenstate of t_2 with similar eigenvalue.

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Chapter 4 Chern Number Calculation

4.1 Introduction to Chern Number

One of the most important topological invariants signaling the route to topological materials is the first Chern number, which we will call Chern number from now on. The Chern number has been well studied as a mathematical object long before its extensive applications in condensed matter physics. In mathematics, the first Chern class is closely related to the first Chern number and exemplifies the notion of characteristic class [36].

The definition of the Chern number, denoted by c_1, is given by the following: it is the integral of the Berry curvature F (a 2-form curvature) over a 2-dimensional closed manifold M

$$c_1 = \frac{1}{2\pi}\int_M F \tag{4.1}$$

The Berry curvature F serves as an example of the first Chern class C_2 in condensed matter physics. The important mathematical fact is that the first Chern class C_2 (actually all Chern classes C_{2j}) is cohomology class with integral coefficients [33], that is $C_2 \in H^2(M, Z)$. We note that the factor 2π is essential for this property. Therefore, the integral of C_2 with closed 2-dimensional manifolds gives an integer, which according to (4.1), implies $c_1 \in Z$.

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In general, the Berry curvature F may be expressed locally as $F = dA$, where A is the Berry connection (a *local* differential one-form)

$$A = i\psi(R)d\psi(R) = i\psi(R)\frac{}{R^i}\psi(R)dR^i \tag{4.2}$$

and d is the exterior derivative. The set $R = \{R_1, R_2, \ldots, R_n\}$ stands for the multi-dimensional parameters constituting a n-dimensional parameter space N. The parameters control the Hamiltonian $H(R)$ as well as other operators if any. The Hamiltonian eigenstates $\psi(R)$ are in general not single-valued in the whole parameter space.

In this case, at least two patches are required to cover M. Within these different patches, the eigenstates $\psi(R)$ are different functions of the parameters. Let say we have two patches O_1, O_2 and assume their intersection is non-zero. If we consider now the overlap region of any two patches $O \in O_1 \cap O_2$, the eigenstates are related by a gauge transformation of the form

$$\psi(R)_1 = e^{if(R)}\psi(R)_2 \tag{4.3}$$

where $f(R)$ is single-valued in O.

In condensed matter physics, the parameters R are the d-dimensional Bloch momentum, each defined modulo 2π. For our purpose, we only consider the case $d = 2$. The parameter space and hence the manifold M in (4.1) is the Brillouin zone (BZ). Due to the periodicity of BZ, it possesses the topology of a 2-dimensional torus T^2, the Brillouin torus (BT)[19]. Since the eigenstates $\psi(k)$ in the usual discussion of Chern number is assumed to be nondegenerate, we have abelian ($U(1)$) connection A. Then we may compute the connection A using the eigenstates, followed by the Berry curvature F and finally c_1. In this way, since the Chern number c_1 is a topological invariant, it characterizes the

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topological property of the complex vector bundle on the base torus T^2, with fibers being the Hilbert space H acted by the structure group $U(1)$. For more details on the complex vector bundle structure on Bloch bundle, see [19].

Let us discuss briefly the interpretation of Chern number, which makes its topological nature more manifest. Let us restrict to the case $d = 2$, which is our object of interest. From (4.1), Chern number can be seen as the flux of the Berry curvature over the Brillouin zone. This is analogous to the quantized magnetic flux of a magnetic charge residing at the origin over a sphere, although our manifold here is a torus. Suppose now that we could find a global gauge such that $F = dA$ is valid globally. In this case, we only need one coordinate patch to parametrize the manifold M. The Berry connection A is therefore a continuous and single-valued function over the whole BZ.

Since the BZ is a torus (the BT), it has no boundary. If such a global gauge exist, from (4.1) we see that c_1 vanishes. This leads to an interesting meaning of Chern number: an obstruction to find a global gauge over BZ. Such an obstruction is absent (the existence of global gauge) if and only if the Chern number is zero. Whenever the Chern number is nonzero, there will be singular points on which the wavefunction on the BZ has ill-defined phase. This could only happen when the wavefunction vanishes on those points. [32] showed that the Chern number is equal to the number of such zeros of the eigenstate on the BZ, if the eigenstate is not degenerate. As a result, one could find the Chern number by counting the zero of the wavefunction on the BZ [16, 32]. This obstruction interpretation is particularly instructive since it also generalizes to other case of topological invariants. One instance is the Z_2 invariant, where the obstruction happens to be over half the BZ [3].

For our case of MCS, we have degenerate Hilbert space. As such, the counting zero

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method is not applicable. In principle, when one has a degenerate Hilbert space and wishes to compute c_1 using the eigenstates, one has to resort to the non-abelian Berry coneection with a little generalization to (4.2). We show that there is another useful way to achieve this aim and it will be the subject of the rest of this chapter.

4.2 Chern Number by Generalized Toroidal B.C. (Twisted B.C.)

A famous paper by Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) [48] in 1982 showed that for integer quantum Hall effect (IQHE), the Hall conductance is proportional to a topological invariant, which turns out to be the first Chern number. This first Chern number, as mentioned in the last section, is found by integrating over the BZ. So the expression for the Hall conductance σ_{xy} is simply

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\int_{BZ} F \tag{4.4}$$

The Berry curvature F is constructed using the ground state (Bloch wave function) of the quantum hall system. (4.4) is conceptually linked to the band theory, so it only applies to non-interacting cases. This is indeed a shortcoming since we have also interacting cases to deal with, for instance the fractional quantum hall effect (FQHE).

To overcome this, we introduce the computation of Chern number by generalized toroidal boundary condition or generalized periodic boundary condition (GPBC). We know that the BZ has the topology of T^2. Therefore any wavefunction $\psi(k)$ defined on the BZ

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has to be also periodic, that is

$$\begin{aligned}\psi(k_x+2\pi,k_y)&=\psi(k_x,k_y)\\ \psi(k_x,k_y+2\pi)&=\psi(k_x,k_y)\end{aligned}\tag{4.5}$$

The idea of GPBC is to give a twist (a phase) to the periodic boundary conditions so that (4.5) becomes

$$\begin{aligned}\psi(k_x+2\pi,k_y)&=e^{i\theta_x}\psi(k_x,k_y)\\ \psi(k_x,k_y+2\pi)&=e^{i\theta_y}\psi(k_x,k_y)\end{aligned}\tag{4.6}$$

The terminology twist comes from the language of fiber bundle, where the angles (θ_x,θ_y) apply a $U(1)$ twist to the vector bundle. Hence GPBS ic also called twisted boundary conditions in many physics literature.

In both cases of the IQHE and FQHE, instead of BZ, we consider the wavefunction on the *spatial* torus with uniform magnetic field *going through* it. As discussed in the last chapter, the existence of a magnetic field makes it necessary to consider magnetic translation as the symmetry of the theory on the spatial torus, instead of translation symmetry. In effect, the twisted boundary conditions are [16, 37]

$$\begin{aligned}T_x^{2\pi}\psi(x+L_x,y)&=e^{i\alpha_xL_x}\psi(x,y)\\ T_y^{2\pi}(x,y+L_y)&=e^{i\alpha_yL_y}\psi(x,y)\end{aligned}\tag{4.7}$$

In [16, 37], they consider the GPBC for the case of N-particles system, so there are extra indices for the coordinates to label the particles. Besides, from (4.7), the parameters α_x and α_y have unit of inverse length while in (4.6), θ_x and θ_y are clearly dimensionless.

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The twist parameters $\theta_x = \alpha_x L_x$ and $\theta_y = \alpha_y L_y$ are defined modulo 2π. Then we have a parameter space of torus and denote it as B. Turning on θ_x and θ_y has the consequence of shifting the momenta of the wavefunction by $\frac{\theta_x}{L_x}$ and $\frac{\theta_y}{L_y}$. This may be achieved by picking the following choice of background vector potential

$$A_x = \frac{\theta_x}{L_x}, \qquad A_y = \frac{\theta_y}{L_y} \tag{4.8}$$

Because of the choice (4.8) and GPBC (4.7), the wavefunction is now function of θ_x, θ_y. That is, we could write $\psi(x, y; \theta_x, \theta_y)$. Therefore, we may determine a connection A' over B and finally the curvature F'.

As the twists θ_x, θ_y are varied from 0 to 2π, the connection A' is used to transport the wavefunction. In the end, the wavefunction has to be back to itself after the variation since 2π twists is amounts to nothing effectively. This result only applies to nondegenerate cases. For the degenerate cases such as MCS theory or FQHE, we will see that 2π variations in θ_x and θ_y shift one wavefunction to another wavefunction in the degenerate subspace. Then one needs to enlarge the domain of the variation of $\vec{\theta}$ so that the starting wavefunction is finally back to itself, as shown in the next section.

In short, we have the first Chern number found by integrating the curvature over the Brillouin zone

$$c_1 = \frac{1}{2\pi} \int_{BZ} F \tag{4.9}$$

On the other hand, we have another integer defined by

$$c_1' = \frac{1}{2\pi} \int_{B} F' \tag{4.10}$$

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In [54], it is shown that these two integers are actually equivalent

$$c_1 = c_1' \tag{4.11}$$

Yet, the RHS of (4.11) is more general. Its definition is beyond band theory, so it is also valid to interacting as well as degenerate cases, which is indeed the case of FQHE. This method is thus applicable to our case of MCS. However, as noted above, some modifications are required to compute the Chern number in both cases. [?]

4.3 Chern Number for Degenerate Cases: MCS on torus

Now we are ready to compute the Chern number of Maxwell-Chern-Simons theory on the torus. As did in section 4.2, we generalize the periodic boundary conditions in the sense of magnetic translation (3.54a),(3.54b) to

$$T_1^{2\pi}\Psi(a_1, a_2) = e^{i\theta_1}\Psi(a_1, a_2) \tag{4.12a}$$

$$T_2^{2\pi}\Psi(a_1, a_2) = e^{i\theta_2}\Psi(a_1, a_2) \tag{4.12b}$$

so that we now have twisted boundary conditions for the MCS ground state wavefunctions on torus. Explicitly these twisted boundary conditions are

$$\Psi(a_1 + 2\pi, a_2) = e^{i\theta_1}\Psi(a_1, a_2) \tag{4.13a}$$

$$\Psi(a_1, a_2 + 2\pi) = e^{i\theta_2}e^{-ika_1}\Psi(a_1, a_2) \tag{4.13b}$$

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One could show that the wavefunction satisfying the boundary conditions (4.13a), (4.13b) is given by

$$\Psi_\ell(a_1, a_2; \theta_1, \theta_2) = \sum_{m \in Z} exp\Big(i\Big(\ell + km + \frac{\theta_1}{2\pi}\Big)a_1 - \frac{k}{4\pi}\Big(a_2 + \frac{2\pi\ell}{k} + 2\pi m + \frac{\theta_1}{k}\Big)^2 + i\Big(m + \Big(\ell + \frac{\theta_1}{2\pi}\Big)/k\Big)\theta_2\Big) \tag{4.14}$$

where $\ell = 0, 1, \ldots, k-1$ and $\mathcal{N}$ is the normalization factor. Then we carry out the unitary transformation as in Niu-Thouless-Wu's paper

$$\Psi_\ell(a_1, a_2; \theta_1, \theta_2) \to e^{-i\frac{\phi_1}{2\pi}a_1} e^{-i\frac{\phi_2}{2\pi}a_2} \Psi_\ell(a_1, a_2; \theta_1, \theta_2) \tag{4.15}$$

From now on, we denote the wavefunction as $\Psi_\ell(\theta_1, \theta_2)$.

In order to compute the Chern number of MCS, let us look at the effect of varying the twist parameters θ_1,θ_2 from 0 to 2π. If we do so to θ_1 and start from one of the ground states available, there is a shift of that ground state to the next one (up to a phase)

$$\Psi_\ell(\theta_1 + 2\pi, \theta_2) = e^{-ia_1} \Psi_{\ell+1}(\theta_1, \theta_2) \tag{4.16}$$

Because of this shift, we have the enlarged θ_1 domain $[0 \leq \theta_1 \leq 2\pi k]$. In the end, the ground state will return to itself (up to a phase) when θ_1 reaches $2\pi k$ since $\Psi_{\ell+k} = \Psi_\ell$. Now we do the same to θ_2 and obtain

$$\Psi_\ell(\theta_1, \theta_2 + 2\pi) = e^{-ia_2} e^{i\frac{2\pi\ell}{k}} e^{i\frac{\theta_1}{k}} \Psi_\ell(\theta_1, \theta_2) \tag{4.17}$$

In this case, the ground state maps back to itself (up to a phase). This phase can be eliminated by a gauge transformation since the functions in the phase are all single-valued. Hence the domain of ϕ_2 integration can be regarded as $[0 \leq \theta_2 \leq 2\pi]$ as usual.

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The Chern number is computed by using the usual expression

$$c_1' = \frac{1}{2\pi}\int_0^{2\pi k} d\theta_1 \int_0^{2\pi} d\theta_2 \left(\frac{\partial A_2'}{\partial \theta_1} - \frac{\partial A_1'}{\partial \theta_2}\right), \tag{4.18}$$

where $A_i' = i\langle \Psi_\ell | \frac{\partial \Psi_\ell}{\partial \theta_i}\rangle$. Then, it can be written as

$$c_1' = \frac{1}{2\pi}\int_0^{2\pi} d\theta_2 \Big[A_2'(2\pi k, \theta_2) - A_2'(0, \theta_2)\Big] - \int_0^{2\pi k} d\theta_1 \Big[A_1'(\theta_1, 2\pi) - A_1'(\theta_1, 0)\Big]. \tag{4.19}$$

The first term in the parenthesis of (4.19) vanishes owing to the absence of θ_2 term in (4.16). In (4.17), there is $e^{i\frac{\theta_1}{k}}$ term, therefore

$$A_1'(\theta_1, 2\pi) - A_1'(\theta_1, 0) = \frac{1}{k}, \tag{4.20}$$

and its integral gives 2π. The final result is

$$c_1' = 1 \tag{4.21}$$

By inspecting the behaviour of wavefunctions under the 2π changes of fluxes and identifying the correct range of the twist parameters, we are able to compute the Chern number of MCS, which is unity. In fact, this discussion has been carried out in some physical contexts. For instance, in [Fractional quantum Hall effect in the absence of Landau levels], the effect of FQHE in an interacting fermionic model is investigated. They call the degenerate subspace by ground state manifold (GSM). Using the twisted paramters as we did here, the total Chern number of GSM is determined to be 1, similar to our result.

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Chapter 5 Lattice MCS Theory

5.1 Introduction: Modified Villain Formulation

In this chapter, we are going to construct the lattice version of $U(1)$ Maxwell-Chern-Simons theory. There are some issues one needs to take into account before considering the lattice theory of $U(1)$ (or compact) gauge fields. The first issue is that, as we will show in Sec. 5.2, naive substitution of lattice form of the continuum Chern-Simons action is not feasible as it leads to non-gauge invariant action, even for boundaryless spacetime. This renders a modified lattice Chern-Simons action necessary.

Another issue lies at the heart of monopoles, or in general, instantons. As shown in [23, 39], when considering compact QED ($U(1)$ gauge fields a_μ) on the lattice in 3D Euclidean dimensions, we have the following partition function

$$Z = \prod_{r,\mu} \int_{-\pi}^{\pi} \frac{da_\mu(r)}{2\pi}\, e^{-S}, \tag{5.1}$$

with the action

$$S = \frac{1}{2e^2} \sum_{r,\mu\nu} \big[1 - \cos f_{\mu\nu}(r)\big]. \tag{5.2}$$

In the expressions above, we label lattice sites by r. Then $a_\mu(r)$ means the gauge field link at lattice site r with direction labelled by $\hat{\mu}$. Then $f_{\mu\nu}(r)$ are plaquette terms with two

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directions $\hat{\mu}$ and $\hat{\nu}$. The variable $f_{\mu\nu}$ in (5.2) is the lattice discretization of the electromagnetic field strength tensor. It represents the oriented sum of the gauge field links around a plaquette

$$f_{\mu\nu}(r) = \Delta_\mu a_\nu(r) - \Delta_\nu a_\mu(r) = \frac{a_\nu(r+\hat{\mu}) - a_\nu(r)}{\eta_\mu} - \frac{a_\mu(r+\hat{\nu}) + a_\mu(r)}{\eta_\nu}, \tag{5.3}$$

where Δ_μ is the forward lattice derivative and η_μ is the lattice spacing in the $\hat{\mu}$ direction. In the literature, other notations are also used and we choose freely between both of them. For instance, we also have

$$Z = \prod_{r,\mu} \int_{-\pi}^{\pi} \frac{da_{r,\mu}}{2\pi} e^{-S}, \tag{5.4}$$

and

$$S = \frac{1}{2e^2} \sum_{r,\mu\nu} \left[1 - \cos f_{r,\mu\nu}\right], \tag{5.5}$$

where the meaning of all indices remain unchanged.

Clearly, the action (5.2) has periodicity of 2π in a_μ, so that $a_\mu \in [-\pi, \pi]$. We can see that the action is minimized when $f_{\mu\nu}(r) = 0$ or $f_{\mu\nu}(r) = 2\pi n$ for $n \in Z$. However, the non-linearity brought by the cosine term in (5.2) makes it difficult to be solved analytically. In order to remedy this problem while still preserving the periodicity property, it is useful to adopt the Villain formulation or periodic Gaussian formulation [31, 45]. In the Villain formulation, we substitute (5.1) with

$$Z = \prod_{r,\mu} \int_{-\pi}^{\pi} \frac{da_\mu(r)}{2\pi} \sum_{n_{\mu\nu}=-\infty}^{\infty} exp\left[-\frac{1}{4e^2} \sum_p \Big(f_{\mu\nu}(r) - 2\pi n_{\mu\nu}(r)\Big)^2\right], \tag{5.6}$$

where the summation over p denotes summation over plaquettes, $\sum_p \equiv \sum_{r,\mu\nu}$. In later sections, we will do the same for summation over links l and cubes c as well, i.e. $\sum_l = \sum_{r,\mu}$. The integer-valued fields $n_{\mu\nu}$ are defined on the plaquettes. While the field strength

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$f_{\mu\nu}$ satisfies the Bianchi identity, $n_{\mu\nu}$ on the other hand does not. In fact, the following quantity

$$Q(\tilde{r}) = \frac{1}{2}\epsilon_{\mu\nu\lambda}\Delta_\mu n_{\nu\lambda} \in Z\,, \tag{5.7}$$

characterizes the number of monopoles residing at the dual lattice sites $\tilde{r}$, or the centers of the cubes, and thus the violation of the Bianchi identity. In differential form notation [47], (5.7) has the compact form

$$Q_c = (dm)_c \in Z\,, \tag{5.8}$$

where the index c implies the quantity Q is defined on the cubes.

The Villain Maxwell action (5.6) obviously is periodic with respect to the shift $f_{\mu\nu} \to f_{\mu\nu} + 2\pi$. This enables us to take into account of the monopoles while solving the theory analytically. These topological excitations may have significant effects on the phase diagram of the theory. In 3D compact QED, using Villain formulation, it is shown that the proliferation of monopoles causes the confinement of electric charge at all temperature [23, 39]. It should be noted that the Villain approach is still just an approximation to the original action (5.2) and approaches to it in the limit $e^2 \to 0$. However, we will assume the Villain formulation to the action lie in the same universality class as the original action and hence share the same phase diagram, as was already verified in some well-known examples. Some such instances are the XY model in 2D & 3D, and compact QED in 3D [44].

A crucial point is in order. First, instead of (5.6), we write it as

$$Z = \prod_{r,\mu} \int_{-L}^{L} \frac{da_\mu(r)}{2\pi} \sum_{n_{\mu\nu}=-\infty}^{\infty} exp\Big[-\frac{1}{4e^2}\sum_p \Big(f_{\mu\nu}(r) - 2\pi n_{\mu\nu}(r)\Big)^2\Big], \tag{5.9}$$

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where now $a_\mu \in [-L, L]$. As noted in [45], there are in general two approaches (or interpretations) to the Villain formulation. The first choice is to take $L = \pi$. Then we have to sum independently over the integer-valued fields $n_{\mu\nu}$'s and our action is just (5.6). The second option is to take $L = \infty$, so now we have R-valued gauge field a_μ. However, we need to make a restriction on the summation over the sets of fields $n_{\mu\nu}$ to avoid over-counting of configurations. Clearly, a shift of $n_{\mu\nu} \to n_{\mu\nu} + \Delta_\mu m_\nu - \Delta_\nu m_\mu$ is equivalent to a redefinition of $f_{\mu\nu}$ and counts again configurations which are already being counted as we have R-valued fields a_μ now.

Our theory now consists of the action

$$S = \frac{1}{4e^2} \sum_p \Big(f_{\mu\nu} - 2\pi n_{\mu\nu} \Big)^2, \tag{5.10}$$

with the following gauge symmetry

$$\begin{aligned} a_\mu &\to a_\mu + \Delta_\mu \Lambda + 2\pi m_\mu, \\ f_{\mu\nu} &\to f_{\mu\nu} + 2\pi \big(\Delta_\mu m_\nu - \Delta_\nu m_\mu \big), \\ n_{\mu\nu} &\to n_{\mu\nu} + \Delta_\mu m_\nu - \Delta_\nu m_\mu, \end{aligned} \tag{5.11}$$

where Λ is a real-valued field on the sites and m_μ is integer-valued on the links. In virtue of (5.11), we have restored the periodicity of a_μ. In the end, our partition function becomes

$$Z = \prod_{r,\mu} \int_{-\infty}^{\infty} \frac{da_\mu(r)}{2\pi} \sum_{n_{\mu\nu}=-\infty}^{\infty}{}' \, exp\Big[-\frac{1}{4e^2} \sum_p \Big(f_{\mu\nu}(r) - 2\pi n_{\mu\nu}(r) \Big)^2 \Big], \tag{5.12}$$

where the primed summation means no counting over those $n_{\mu\nu}$'s differing by a curl as in (5.11).

We now use a somewhat different language to describe the second approach, which

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is quite common in the modern literature [8, 15]. Our starting point is

$$Z = \prod_{r,\mu} \int_{-\infty}^{\infty} \frac{da_\mu(r)}{2\pi} \sum_{n_{\mu\nu}=-\infty}^{\infty} exp\Big[-\frac{1}{4e^2}\sum_p \Big(f_{\mu\nu}(r)\Big)^2\Big], \tag{5.13}$$

which arises from the expansion of the cosine term in (5.2). This theory has a 1-form R global shift symmetry:

$$a_l \to a_l + \xi, \quad \xi \in R, \tag{5.14}$$

where we label the link field by index l. In order to obtain a $U(1)$ gauge theory, we gauge the $2\pi Z$ subgroup of the R symmetry by introducing dynamical integer-valued gauge fields $n_{\mu\nu}$ (or n_p) on the plaquettes as in (5.12). In doing this, we impose the same gauge symmetry as in (5.11). Then the shift $a_l \to a_l + 2\pi$ amounts to a gauge transformation and we have a $R/2\pi Z = U(1)$ gauge theory. According to [47, 55], if we start with the partition function (5.12), we can use the 1-form gauge redundancy (5.11) to gauge fix the a_l so that $a_l \in [-\pi, \pi]$. In this way, we say we have chosen the gauge-fixing condition $a_l \in [\pi, \pi]$. But this is essentially the first approach to the Villain formulation (5.9). Then we see that these two approaches can be interchanged to each other, hence equivalent. In later sections, we will adopt the second approach since it is more convenient to perform the Gaussian integration in a_l by using R-valued fields.

Since the periodicity is keep intact in the Villain formulation, the action (5.6) also takes the effect of monopoles into consideration. However, it is well-known that in the continuum, the monopole configurations are not gauge invariant in the presence of Chern-Simons term [1, 34]. Therefore, we are required to sought a way to eliminate the monopoles to obtain a fully gauge-invariant action. We solved this problem by imposing a constraint

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which forbids the monopoles from the theory. Such a constraint is expressed by [47]

$$(dn)_c = 0 \quad \forall c. \tag{5.15}$$

For this purpose, the following term will be introduced to the action

$$S_{nomonopole} \equiv -i \sum_c \lambda_c (dn)_c. \tag{5.16}$$

Integrating over $\lambda_c \in [-\pi, \pi]$ at each cube produces the desirable constraint (5.15). The conventional Villain formulation combined with the constraint suppressing the instantons is called the modified Villain formulation (MVF). It has been shown to reproduce many desirable features of the corresponding continuum theories [8]. Since our primary interest is the $U(1)$ Maxwell-Chern-Simons theory, we will adopt the MVF to construct its lattice theory. Before delving into this endeavor, we start with the construction of the lattice CS theory using MVF. We shall see that it is a quite non-trivial task.

5.2 Lattice U(1) Chern-Simons Theory

In this section, we study the lattice $U(1)$ Chern-Simons theory and will see that it reproduces many features of its continuum counterpart.

5.2.1 Failure of naive CS action

As noted earlier, the naive substitution of continuum derivative in the CS term by forward lattice derivative Δ_μ leads to non-gauge-invariant CS action. To see this, we

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write down the naive lattice CS term

$$S = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\Delta_\nu a_\lambda. \tag{5.17}$$

Under a gauge transformation,

$$a_\mu \to a_\mu + \Delta_\mu\Lambda, \tag{5.18}$$

where Λ is the gauge function, (5.17) changes by

$$S \to S + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\Delta_\mu\Lambda\Delta_\nu a_\lambda + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\Delta_\nu\Delta_\lambda\Lambda + \frac{k}{4\pi}\Delta_\mu\Lambda\Delta_\nu\Delta_\lambda\Lambda. \tag{5.19}$$

Clearly, the second and third terms in (5.19) vanish. Using the summation by parts on a lattice (assuming lattice without boundary) [14],

$$\sum_r g(r)\Delta_\mu h(r) = -\sum_r \hat{\Delta}_\mu f(r)h(r), \tag{5.20}$$

where $\hat{\Delta}_\mu$ denote the backward lattice derivative, the first term in (5.19) becomes

$$-\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\Lambda\hat{\Delta}_\mu\Delta_\nu a_\lambda, \tag{5.21}$$

which does not vanish in general.

This non-vanishing term arising from gauge transformation, even for boundaryless lattice, is undesirable for CS action, since it violates the well-known lore that the continuum CS action is gauge invariant for manifold without boundary. This inconsistency requires a modification of the CS term on a lattice. In this connection, the pioneering work in the lattice Chern-Simons term is [20]. There, they proposed a local lattice Chern-Simons term to study the quantum field theory of vortices in 3D.

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The form of the action they used is (written in modern notation)

$$S = \frac{k}{4\pi}\sum_r a_\mu(r) S_\mu \tilde{f}^\mu(r). \tag{5.22}$$

S_μ is the forward shift operator whose action on a function $g(r)$ is $S_\mu g(r) = g(r+\hat{\mu})$ while $\tilde{f}^\mu = \epsilon^{\mu\nu\lambda} f_{\nu\lambda}$ is the dual field strength.. The action (5.22), albeit rather strange looking, is gauge-invariant since $\hat{\Delta}_\mu S_\mu = \Delta_\mu$ and $\Delta_\mu \tilde{f}^\mu = 0$. For interested reader, this CS action, although written in different form, are also used in [11, 50]. Our starting point will be the gauge-invariant action (5.22).

However, in these aforementioned earlier works, they assume non-compact gauge group and hence only the local aspects of the theory are studied, i.e. anyonic statistics [14]. Another consequence is the absence of level k quantization as there is no consideration of large gauge transformation. On the other hand, our main goal is to study $U(1)$ lattice CS theory. As we will see in the next section, the consideration of compact gauge group leads to extra terms to (5.22).

5.2.2 Construction of Lattice CS action: Lagrangian Formulation

Recently, there have been a few of literature devoted to studying the quantization of $U(1)$ lattice CS action, especially the global features [6, 27, 28]. Following them, our spacetime setting will be the 3D Euclidean cubic lattice with periodic boundary condition. We will use the language of cup product [27] and differential forms on the hypercubic lattice [47] to describe the theory. (See Appendix **??** for a definition of the cup product.) For the generalizations of cup product, i.e. higher cup products on hypercubic lattice, see [7]. The cup product is the lattice analogue of the wedge product used in the description

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of continuum CS theory. It satisfies the Leibniz rule, that is for a p-form f and q-form g,

$$d(f \cup g) = df \cup g + (-1)^p f \cup dg. \tag{5.23}$$

Now we start the construction of the lattice CS action. In the cup product notation, the starting action (5.22) is written as

$$S = -\frac{ik}{4\pi} \sum_c (a \cup da)_c, \tag{5.24}$$

where we have summation over the cubes. Note that the gauge field a_l is real-valued, $a_l \in R$ on any links. In other words, we start with non-compact fields and later on introduce the Villain fields to obtain a compact gauge theory. As noted before, the action (5.24) is gauge-invariant under local gauge transformation

$$a_l \to a_l + (d\Lambda)_l. \tag{5.25}$$

To see this, the gauge variation of (5.24) under (5.25) is

$$-\frac{ik}{4\pi} \sum_c (d\Lambda \cup da)_c, \tag{5.26}$$

which vanishes by using the Leibniz rule (5.23), the property $d^2 = 0$ and the fact that there is no boundary. However, it fails to preserve gauge-invariant property under large gauge transformation, as it should be,

$$\begin{aligned} a_l \to a_l + 2\pi m_l, &\quad m_l \in Z \ onlinks \\ n_p \to n_p + (dm)_p, &\quad n_p \in Z \ onplaquettes \end{aligned} \tag{5.27}$$

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where n_p is the discrete Villain fields. Under the transformation (5.27), the extra terms arising from (5.24) are

$$-\frac{ik}{4\pi}\sum_c \left[(2\pi m \cup da)_c + (a \cup 2\pi dm)_c + (2\pi m \cup dm)_c\right]. \tag{5.28}$$

The first two terms can be eliminated by the inclusion of the extra terms to the action so that it becomes

$$-\frac{ik}{4\pi}\sum_c \left[(a \cup da)_c - (a \cup 2\pi n)_c - (2\pi n \cup a)_c\right]. \tag{5.29}$$

The inclusion of the Villain fields n_p makes the theory compact . The remaining third term in (5.19) leads to extra term which cannot be eliminated by any means. However, in the path integral formalism, the least requirement we have is the exponentiated action to be invariant, instead of the action itself. Having this in mind, since $(m \cup dm)_c \in Z$, we see that the level k must be even integer, $k \in 2Z$. In fact, apart from (5.28), there are extra terms arising from (5.29) when we consider (5.27). As we will show shortly, the same quantization condition is derived.

On the other hand, the extra terms we introduce (5.29) are also not gauge invariant under local gauge transformation (5.25). Instead, its gauge variation is given by

$$\frac{ik}{4\pi}\sum_c \left[(d\Lambda \cup 2\pi n)_c + (2\pi n \cup d\Lambda)_c\right], \tag{5.30}$$

which may be written as, using (5.23),

$$-\frac{ik}{4\pi}\sum_c \left[(\Lambda \cup 2\pi dn)_c + (2\pi dn \cup \Lambda)_c\right]. \tag{5.31}$$

This forces us to impose the condition $(dn)_c = 0$ for arbitrary gauge function Λ. This

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implies that in order to have a fully gauge-invariant action, dynamical monopoles must be eliminated. This is accomplished by introducing a Lagrange multiplier term to the action

$$S_{nomonopole} \equiv -\frac{i}{2\pi}\sum_{c}(\lambda \cup dn)_c, \tag{5.32}$$

where λ is an angular field on the sites. We now explain the seemingly discrepancy between the expressions (5.16) and (5.32). It is well known that, by Poincaré duality, there is an isomorphism between a p-form on the lattice and a $(d-p)$-form on the dual lattice. In (hyper-)cubic lattice, there is a one-to one correspondence between a p-form defined on the dual lattice and the same form on the lattice [15, 47]. From (5.16), we have that by Poincaré duality, λ_c can be viewed as the $\star$-dual (Hodge dual) of a 0-form $\tilde{\lambda}_{\tilde{r}}$ on the dual lattice. By the reasoning above, we may make the identification $\tilde{\lambda}_{\tilde{r}} \leftrightarrow \lambda_r$. In this way, we recover the Lagrange multiplier form (5.32). There is another way to derive (5.29) [6]. Considering our $3D$ CS action is the boundary of a $4D$ action, using Stokes's theorem, we can express this $4D$ action as a sum of terms. The terms expressible as total derivative will be (5.24) and (5.29).

In the end, our lattice $U(1)$ CS action is given by

$$S = -\frac{ik}{4\pi}\sum_{c}\left[(a \cup da)_c - (a \cup 2\pi n)_c - (2\pi n \cup a)_c\right] - i\sum_{c}(\lambda \cup dn)_c, \tag{5.33}$$

with the following gauge symmetry

$$\begin{aligned} a_l &\to a_l + (d\Lambda)_l + 2\pi m_l, \\ n_p &\to n_p + (dm)_p, \\ \lambda_r &\to \lambda_r + 2\pi s_r. \end{aligned} \tag{5.34}$$

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Compare (5.11). Looking again the large gauge transformation of (5.33) which gives a shift of

$$
\begin{aligned}
&S(a+2\pi m, n+dm, \lambda) - S(a, n, \lambda) \\
&= \frac{ik}{4\pi} \sum_c \big[(2\pi m \cup da)_c - (2\pi m \cup 2\pi n)_c - (2\pi n \cup 2\pi m) - (2\pi dm \cup a)_c - (2\pi dm \cup 2\pi m)_c, \\
&= -\frac{ik}{2} \sum_c \big[(m \cup n)_c + (n \cup m)_c + (dm \cup m)_c\big]. \qquad (5.35)
\end{aligned}
$$

In going to the last line, we have used $d(m \cup a) = dm \cup a - m \cup da$. The bracketed terms in the last line of (5.35) are all integers-valued. Then we obtain the same quantization condition $k \in 2Z$ so that e^{-S} is invariant under this 1-form Z gauge transformation. The evenness of the level k implies that only bosonic CS theory can be proved directly using our action. The CS theory with odd k is fermionic CS theory. As such, additional data, like spin structure, is required to specify the theory. For our interest, we will only focus on the $k \in 2Z$ case.

5.3 Hamiltonian formulation of lattice MCS

5.3.1 Transfer Matrix

In lattice gauge theory, one of the common methods to find the Hamiltonian is via the transfer matrix method [9, 18, 31, 43, 46]. In this section, we implement this method to construct the Hamiltonian of the U(1) MCS theory. More specifically, it is the Hamiltonian of the modified Villain formulation of MCS theory that we are going to construct. The origin of the transfer matrix lies in the path integral formulation of quantum mechanics. In the Euclidean path integral formulation, the transition amplitude for a particle propagating

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from position x to x' is given by

$$x' e^{-H(\tau-\tau')} x = \int_x^{x'} Dx''\, e^{-S_E[x'']}, \tag{5.36}$$

where S_E is the Euclidean action of a particle with potential V.

Consider discretizing the time into N small intervals, $\epsilon = (\tau - \tau')/N$. Then we can write (5.36) in the following suggestive form (with H being the Hamiltonian of the system)

$$x' e^{-H(\tau-\tau')} x = \prod_{\ell=0}^{N-1} \left[\int dx^{(\ell)}\, T(x^{(\ell+1)}, x^{(\ell)}) \right], \tag{5.37}$$

where the transfer matrix $T(x^{(\ell+1)}, x^{(\ell)})$ is given by

$$T(x^{(\ell+1)}, x^{(\ell)}) = \left(\frac{1}{2\pi\epsilon}\right)^{1/2} exp\left\{ -\epsilon \left[\frac{1}{2} \left(\frac{x^{(\ell+1)} - x^{(\ell)}}{\epsilon} \right)^2 + V(x^{(\ell)}) \right] \right\}. \tag{5.38}$$

We have used $x^{(\ell)}$ to denote the position at time τ_ℓ. The expressions (5.37) and (5.38) imply the connection between the transfer matrix and the imaginary time evolution operator

$$T(x^{(\ell+1)}, x^{(\ell)}) = x^{(\ell+1))} e^{-\epsilon H} x^{(\ell)}. \tag{5.39}$$

Given the knowledge of the transfer matrix, it is possible to extract the Hamiltonian. We follow the method outline in [43]. Since the position eigenstates $x^{(\ell)}$ is the simultaneous eigenstates of the position operator $\hat{X}$

$$X x^{(\ell)} = x^{(\ell)} x^{(\ell)}, \tag{5.40}$$

we can introduce its canonical conjugate operator, the momentum operator P. They satisfy

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the commutation relation

$$[X, P] = i. \tag{5.41}$$

We then define the operator, which depends on the real parameter α,

$$L(\alpha) = exp(-i\alpha P). \tag{5.42}$$

Using the commutation relation (5.41), the following useful commutation relation may be derived [COHEN]

$$[\hat{X}, L(\alpha)] = \alpha L(\alpha). \tag{5.43}$$

Given an eigenstate x of X, (5.43) implies $L(\alpha)$ generates translation

$$L(\alpha)x = x + \alpha. \tag{5.44}$$

That is, $L(\alpha)x$ is the eigenstate of X with eigenvalue $x+\alpha$. The position eigenstates form a complete, orthonormal basis

$$x'x = \delta(x' - x), \tag{5.45}$$

which has the following consequence

$$x^{(\ell+1)}e^{-i\alpha P}x^{(\ell)} = \delta\big(x^{(\ell+1)} - x^{(\ell)} - \alpha\big). \tag{5.46}$$

Now we rewrite (5.38) in the form

$$T(x^{(\ell+1)}, x^{(\ell)}) = \int \frac{d\alpha}{\sqrt{2\pi\epsilon}} \Big[\delta\big(x^{(\ell+1)} - x^{(\ell)} - \alpha\big) exp\Big(-\frac{\alpha^2}{2\epsilon}\Big)\Big] exp\Big(-\epsilon V(x^{(\ell)})\Big) \tag{5.47}$$

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Using (5.46), the above expression becomes

$$T(x^{(\ell+1)}, x^{(\ell)}) = x^{(\ell+1)}\Big[\int d\alpha\, exp\Big(-\frac{\alpha^2}{2\epsilon} - i\alpha P\Big)\Big]exp\Big(-\epsilon V(X)\Big)x^{(\ell)}. \tag{5.48}$$

Two important steps are in order. We perform the Gaussian integration over α and obtain

$$T(x^{(\ell+1)}, x^{(\ell)}) = x^{(\ell+1)}exp\Big(-\epsilon\frac{P^2}{2}\Big)exp\Big(-\epsilon V(X)\Big)x^{(\ell)}. \tag{5.49}$$

In the limit $\epsilon \to 0$, making use of the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$, we may make the approximation

$$exp\Big(-\epsilon\frac{P^2}{2}\Big)exp\Big(-\epsilon V(X)\Big) \approx exp\Big\{-\epsilon\Big(\frac{P^2}{2} + V(X)\Big)\Big\}, \tag{5.50}$$

since we could ignore the commutator terms as they contain higher order terms $O(\epsilon^2)$. The final result is then

$$T(x^{(\ell+1)}, x^{(\ell)}) = x^{(\ell+1)}exp\Big\{-\epsilon\Big(\frac{P^2}{2} + V(X)\Big)\Big\}x^{(\ell)}. \tag{5.51}$$

Comparing (5.51) to (5.39), it is clear that the Hamiltonian is given by

$$H = \frac{P^2}{2} + V(X), \tag{5.52}$$

which is the familiar expression for a particle in potential V. Therefore, the above procedures demonstrate that the Hamiltonian may be extracted if one could find out the expression of transfer matrix. From (5.37), the role of the transfer matrix has the following interpretation: given an initial position configuration at a certain time slice (a Hilbert space of position states), it generates the position configuration at the next time slice. This is

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made manifest in the expression assumed by T in (5.38).

In this way, closer inspection shows that the steps in (5.47) and (5.48) is equivalent to taking the inverse Fourier transform of the Gaussian function with coordinate $\alpha = x^{(\ell+1)} - x^{(\ell)}$. This feature will be common in the Hamiltonian formulation of lattice field theory. The extension to field theory is straightforward with some generalizations. Now we have field configurations in the Hilbert space defined on a hypersurface at a fixed time slice, instead of position configuration. There is at least one additional crucial difference between the example we gave and lattice gauge theory: the imposition of the constraint that physical states have to obey Gauss law, i.e. the states allowed in the Hilbert space have to be gauge-invariant.

5.3.2 Lattice Maxwell Terms

In the Hamiltonian approach, we keep time as continuous variable while 2-dimensional space as discrete. For this reason, it is more convenient to have asymmetric space-time lattice, For this reason, we denote the lattice spacing in the time direction as a_t while the spatial counterpart as a. The lattice Maxwell terms in 3D and 4D have been well studied and understood [18, 31, 43]. Here, we only state out the final result of the lattice Maxwell action in 3D:

$$S_{Maxwell} = \frac{1}{2e^2 a_t} \sum_{timelike\, p} (1 - \cos a_t a f_{0i}) + \frac{a_t}{2e^2 a^2} \sum_{spatial\, p} (1 - \cos a^2 f_{12}). \tag{5.53}$$

The above expressions are adopted so that it has the desired form in the continuum limit

$$S_{Maxwell} = \frac{1}{4e^2} \int d^3x \, f_{\mu\nu} f^{\mu\nu}, \tag{5.54}$$

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where $\int d^3x \to a_t a^2 \sum_p$.

5.3.3 Lattice MCS Hamiltonian

In this section, we embark on the main task: determine the Hamiltonian of the lattice MCS theory and its global features. Our lattice MCS action is given by the sum of (5.33) and (5.55), which we rewrite here for convenience (retrieving lattice spacing):

$$\begin{aligned} S =& \frac{1}{2e^2 a_t} \sum_{timelike\, p} \left(a_t a f_{0i} - 2\pi n_{0i}\right)^2 + \frac{a_t}{2e^2 a^2} \sum_{spatial\, p} \left(a^2 f_{12} - 2\pi n_{12}\right)^2 \\ &- \frac{ika_t a^2}{4\pi} \sum_c \left[(a \cup da)_c - \eta_c (a \cup 2\pi n)_c - \eta_c (2\pi n \cup a)_c\right] - i \sum_c (\lambda \cup dn)_c, \end{aligned} \tag{5.55}$$

where $\eta_c = 1/(a_t a)$ for timelike plaquette Villain fields, i.e. n_{01}, n_{20}, and $\eta_c = 1/(a^2)$ for n_{12} so that the Villain fields n are dimensionless. We have the following partition function and path integral measure

$$Z = \left(\prod_\ell \int_{-\infty}^{\infty} \frac{da_\ell}{2\pi}\right) \left(\prod_p \sum_{n_p \in Z}\right) \left(\prod_c \int_{-\pi}^{\pi} \frac{d\lambda_c}{2\pi}\right) e^{-S}. \tag{5.56}$$

We have normalized with factors $1/2\pi$ since we could always gauge fix $a_\ell \in [-\pi, \pi)$ [47]. Following the previous subsection, we wish to find the transfer matrix so as to determine the Hamiltonian. Before doing this, there is a few technical points.

The first point is there is a convenient gauge choice in the modified Villain formulation, which demands $dn = 0$ [22]. First, we integrate out λ_c, which imposes the flatness condition on n_p. Then, we may choose to gauge fix $n_{01} = 0$, $n_{20} = 0$ and $n_{12} = 0$ except for $n_{01}(L_0 - 1, L_1 - 1, y)$, $n_{20}(L_0 - 1, x, L_2 - 1)$ and $n_{12}(\tau, L_1 - 1, L_2 - 1)$, where L_μ denotes the total number of lattice sites for direction μ. These are the remaining gauge in-

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variant holonomies that we may thread along each of the three non-contractible directions. In particular, we further denote

$$
\begin{aligned}
n_{01}(L_0-1, L_1-1, y) &= n_1, \\
n_{20}(L_0-1, x, L_2-1) &= n_2, \\
n_{12}(\tau, L_1-1, L_2-1) &= n_0.
\end{aligned}
\tag{5.57}
$$

The second point is that the conjugate momentum of a_0, n_1 and n_2 are zero, since their time derivatives are absent in the action (5.55). As such, they play the role of the Lagrange multiplier and give rise to constraints acting on the Hilbert space. In finding the Hamiltonian, they may be set to be zero,

$$
A_0 = n_1 = n_2 = 0. \tag{5.58}
$$

The $A_0 = 0$ is called the temporal gauge and is used frequently in the Hamiltonian analysis of lattice gauge theory [9, 18, 30, 31].

Following the notation in section 5.3.1, we denote the gauge field configurations at a given time slice and its counterparts at the next time slice as $a^{(\ell)}$ and $a^{(\ell+1)}$ respectively. After the gauge fixing (5.57) and the prescription (5.58), the action (5.55) becomes

$$
\begin{aligned}
S = \sum_{\ell} \Bigg[& \frac{a^2}{2e^2 a_t} \sum_{n \in links} \left(a_n^{(\ell+1)} - a_n^{(\ell)}\right)^2 + \frac{a_t a^2}{2e^2} \sum_{r \neq (L_1-1, L_2-1)} \left(f_{r,12}^{(\ell)}\right)^2 \\
& + \frac{a_t a^2}{2e^2} \left(a^2 f_{12}^{(\ell)} - 2\pi n_0\right)^2_{(L_1-1, L_2-2)} \\
& + \frac{ika_t a^2}{4\pi} \sum_{r \in sites} a_{r-\hat{1},1}^{(\ell)} \left(a_{r,2}^{(\ell+1)} - a_{r,2}^{(\ell)}\right) - a_{r-\hat{2},2}^{(\ell)} \left(a_{r,1}^{(\ell+1)} - a_{r,1}^{(\ell)}\right) \Bigg],
\end{aligned}
\tag{5.59}
$$

where the index r now denotes the two-dimensional spatial lattice whereas $n = (r, i)$

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denotes the spatial link pointing in direction i at spatial sites r. In deriving (5.59), we have used the definition of $f_{\mu\nu}$ given by (5.3) and cup product in Appendix **??**. Note that the Lagrange multiplier term is absent since we have already integrated it out.

As before, we introduce the sets of states of field configurations a^{ℓ}, which are simultaneous eigenstates of the commuting operators $\hat{a}_n$:

$$\hat{a}_n a^{(\ell)} = a_n^{(\ell)} a^{(\ell)}. \tag{5.60}$$

Then we have also the momentum operators $\hat{p}_n$ canonically conjugate to $\hat{a}_n$, obeying the commutation relation similar to (5.41). From (5.37), (5.38) and (5.59), the transfer matrix may be written down

$$\begin{aligned}
T(a_n^{(\ell+1)}, a_n^{(\ell)}) = exp\Bigg[&- \frac{a^2}{2e^2 a_t} \sum_{n \in links} \left(a_n^{(\ell+1)} - a_n^{(\ell)}\right)^2 \\
&+ \frac{ika_t a^2}{4\pi} \sum_{r \in sites} a_{r-\hat{2},2}^{(\ell)} \left(a_{r,1}^{(\ell+1)} - a_{r,1}^{(\ell)}\right) - a_{r-\hat{1},1}^{(\ell)} \left(a_{r,2}^{(\ell+1)} - a_{r,2}^{(\ell)}\right) \\
&- \frac{a_t a^2}{2e^2} \sum_{r \neq (L_1-1, L_2-1)} \left(f_{r,12}^{(\ell)}\right)^2 - \frac{a_t a^2}{2e^2} \left(a^2 f_{12}^{(\ell)} - 2\pi n_0\right)^2_{(L_1-1, L_2-2)} \Bigg].
\end{aligned} \tag{5.61}$$

As explained in section 5.3.1, applying the inverse Fourier transform to the 'coordinates' $a_n^{(\ell+1)} - a_n^{(\ell)}$ and using the Gaussian integral formula give the Hamiltonian

$$\begin{aligned}
H = &\frac{e^2}{2a^2} \sum_{r \in sites} \left(\hat{p}_{r,1} - \frac{ka^2}{4\pi} \hat{a}_{r-\hat{2},2}\right)^2 + \frac{e^2}{2a^2} \sum_{r \in sites} \left(\hat{p}_{r,2} + \frac{ka^2}{4\pi} \hat{a}_{r-\hat{1},1}\right)^2 \\
&+ \frac{a_t a^2}{2e^2} \sum_{r \neq (L_1-1, L_2-1)} \left(\hat{f}_{r,12}\right)^2 + \frac{a_t a^2}{2e^2} \left(a^2 \hat{f}_{12} - 2\pi n_0\right)^2_{(L_1-1, L_2-2)}.
\end{aligned} \tag{5.62}$$

To find the constraints arising from a_0, n_1 and n_2, there are two approaches to do it, which are actually closely related. The first way consists of the straightforward recovery

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of a_0, n_1 and n_2 to the action (5.59), which is just (5.55). Upon this recovery, we immediately see that there are quadratic terms in (5.55) which couples these three terms to the 'coordinates' $a_n^{(\ell+1)} - a_n^{(\ell)}$. As such, after performing the inverse Fourier transform of $a_n^{(\ell+1)} - a_n^{(\ell)}$, we get some extra terms which involve a_0, n_1 and n_2, apart from the original term with a_0, n_1 and n_2 in (5.55) (those terms with these three terms that do not couple with $a_n^{(\ell+1)} - a_n^{(\ell)}$). Recollecting these terms gives us the desired constraints. The computational aspect of this approach turns out to be tedious, so we turn to the second approach. For the details, we refer to [38].

The idea of the second approach is that, instead of inverse Fourier transform $a_n^{(\ell+1)} - a_n^{(\ell)}$, we inverse Fourier transform the field strength $f'_{0i} = a_t a f_{0i} - 2\pi n_i$. Written in terms of f'_{01} and f'_{20}, the action (5.55) becomes

$$\begin{aligned} S =& \frac{1}{2e^2 a_t} \sum_{timelike\, p} \left[\left(f'_{01}\right)^2 + \left(f'_{20}\right)^2 \right] + \frac{ika}{4\pi} \sum_r \left[a_{r-\hat{2},2} f'_{01} - a_{r-\hat{1},1} f'_{02} \right] \\ &+ \frac{a_t}{2e^2 a^2} \sum_{spatial\, p} \left(f'_{12}\right)^2 - \frac{ika_t}{4\pi} \sum_c \left[a_0 \cup f'_{12} - \eta_c (2\pi n \cup a) \right], \end{aligned} \tag{5.63}$$

where $f'_{12} = a^2 f_{12} - 2\pi n_{12}$. It is clear that performing the inverse Fourier transform of f'_{0i} with the Fourier factor $exp(-ip_i f'_{0i})$ and a field rescaling $p_i \to p_i/a$ produce the Hamiltonian (5.62) as well as the terms along the second line of (5.63). Conversely, we may kind of reverse the argument above. Suppose we start with the partition function

$$Z = \left(\prod_\ell \int_{-\infty}^{\infty} \frac{da_\ell}{2\pi} \right) \left(\sum_{n_0 \in Z} \sum_{n_1 \in Z} \sum_{n_2 \in Z} \right) e^{-S}, \tag{5.64}$$

with the action (5.63).

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By the argument above, we can rewrite (5.64) as

$$Z = \left(\prod_{\ell} \int_{-\infty}^{\infty} \frac{da_\ell}{2\pi} \right) \left(\sum_{n_0 \in Z} \sum_{n_1 \in Z} \sum_{n_2 \in Z} \right) \left(\int_{\infty}^{\infty} dp_i \right)$$
$$exp \left(- a_t H + \frac{ip_i f'_{0i}}{a} + \frac{ika_t}{4\pi} \sum_c \left[a_0 \cup f'_{12} - \eta_c (2\pi n \cup a) \right] \right). \tag{5.65}$$

From now on, we let all the lattice spacings to be unity, i.e. $a_t = a = 1$ to simplify the notation. Then $f'_{\mu\nu} = f_{\mu\nu} - 2\pi n_{\mu\nu}$ with the gauge fixing prescription (5.57) and $\eta_c = 1$. We see that in (5.65), apart from the Hamiltonian, all the other terms in the exponential is related to a_0, n_1 and n_2 which serve as Lagrange multiplier term in the Hamiltonian analysis. The first constraint, which is the Gauss law, results from integrating out a_0 in (5.65) at each lattice site, is

$$\nabla_1 \hat{p}_{r,1} + \nabla_2 \hat{p}_{r,2} + \frac{k}{4\pi} \hat{f}_{r,12} = 0 \quad for \quad r \neq (L_1 - 1, L_2 - 2), \tag{5.66}$$
$$\nabla_1 \hat{p}_{r,1} + \nabla_2 \hat{p}_{r,2} + \frac{k}{4\pi} \hat{f}_{r,12} - \frac{k}{2} n_0 = 0 \quad for \quad r = (L_1 - 1, L_2 - 2)\, or\, (0,0). \tag{5.67}$$

Suppose we now impose periodic boundary condition on the spatial lattice, so that it is a lattice torus. Then we have the relation

$$\sum_r \left(\nabla_1 \hat{p}_{r,1} + \nabla_2 \hat{p}_{r,2} + \frac{k}{4\pi} \hat{f}_{r,12} \right) = 0., \tag{5.68}$$

where we have summed all the lattice sites. Combining (5.66), (5.67) and (5.3.3) , we obtain the condition $kn_0 = 0$. Since the level k is non-vanishing, it is immediate that

$$n_0 = 0. \tag{5.69}$$

As a result, there is no summation over n_0 in (5.64). This implies, for our MCS

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theory, the spatial Villain fields has to be zero when our spatial lattice is a torus. It further simplifies our Hamiltonian

$$H =\frac{e^2}{2}\sum_r \left(\hat{p}_{r,1} - \frac{k}{4\pi}\hat{a}_{r-\hat{2},2}\right)^2 + \frac{e^2}{2}\sum_{r\in sites}\left(\hat{p}_{r,2} + \frac{k}{4\pi}\hat{a}_{r-\hat{1},1}\right)^2 + \frac{1}{2e^2}\sum_r \left(\hat{f}_{r,12}\right)^2. \tag{5.70}$$

The Hamiltonian (5.70) bears resemblance to its continuum counterpart, except for some displacement of site for $\hat{a}$ in the quadratic term in (5.70). This may be attributed to the presence of cup product in the definition of lattice MCS action. Despite the form of (5.66) and (5.67), it should be understood that they are constraints imposed on the states of the Hilbert space:

$$e^{i\hat{G}}phys = phys, \tag{5.71}$$

where $\hat{G} = \nabla_i \hat{p}_{r,i} + \frac{k}{4\pi}\hat{f}_{r,12}$ with implicit sum over $i = 1, 2$ and for sites r.

Then we turn to the constraint arising from time-like Villain fields n_1 and n_2. Note that since we have already imposed the flatness condition on n, they are not changing once they are fixed, although we allow different possible values for them in the partition function (5.64). Summing out them in (5.65) produces the following constraints

$$exp\left(i\sum_{r_1=0}^{L_1-1}\left(2\pi\hat{p}_{(r_1,r_2),2} - \frac{k}{2}\hat{a}_{(r_1,r_2+1),1}\right)\right)phys = phys, \tag{5.72}$$

$$exp\left(i\sum_{r_2=0}^{L_2-1}\left(2\pi\hat{p}_{(r_1,r_2),1} + \frac{k}{2}\hat{a}_{(r_1+1,r_2),2}\right)\right)phys = phys. \tag{5.73}$$

Both constraints above are the lattice analogue of the statement that physical states be invariant under large gauge transformation along either of the two non-contractible cycles of the spatial torus. The consequence of this is we have actually a compact theory, i.e. $U(1)$

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theory. This should be so since the modified Villain formulation of a theory preserves the periodicity of the action. The generators of such large gauge transformation are those in the exponentials in (5.72) and (5.73). Closer inspection shows that if we divide these operators by 2π which turn out to be

$$\hat{D}_1 = \sum_{r_1=0}^{L_1-1} \left(\hat{p}_{(r_1,r_2),2} - \frac{k}{4\pi} \hat{a}_{(r_1,r_2+1),1} \right), \tag{5.74}$$

$$\hat{D}_2 = \sum_{r_2=0}^{L_2-1} \left(\hat{p}_{(r_1,r_2),1} + \frac{k}{4\pi} \hat{a}_{(r_1+1,r_2),2} \right), \tag{5.75}$$

we see that they are reminiscent of the magnetic translation operator for the action (3.46). Hence they can be regarded as the lattice analogue of magnetic translation operators. In this sense, they may be seen as 1-form symmetry generator as in the continuum theory where we know that magnetic translation operators leave the Hamiltonian invariant.

It is straightforward to verify the following commutation relations

$$[H, \hat{G}] = 0, \tag{5.76}$$

$$[H, \hat{D}_i] = 0. \tag{5.77}$$

The more interesting one comes from considering the commutator between $\hat{D}_1$ and $\hat{D}_2$:

$$[\hat{D}_1, \hat{D}_2] = -\frac{ik}{2\pi}, \tag{5.78}$$

which leads to

$$e^{i2\pi \hat{D}_1} e^{i2\pi \hat{D}_2} = e^{i2\pi k} e^{i2\pi \hat{D}_2} e^{i2\pi \hat{D}_1}, \tag{5.79}$$

which is identical to (3.56). By the similar reasoning, the level must be an integer, $k \in Z$. The result above might be seemingly inconsistent with the fact the we could only

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focusing on even k (bosonic) CS theory in the previous section. As demonstrated in [38], the fermionic nature of the odd k theory could still be probed by the fact that one is not able to impose identity constraint at the same time on the following large gauge generators, $e^{i\hat{D}_1}$, $e^{i2\pi\hat{D}_2}$ and $e^{i2\pi(\hat{D}_1+\hat{D}_2)}$.

By the discussion around (3.62a) and (3.62b), the Hilbert space of the lattice MCS theory is also k-degenerate. The relevant generators in this case are $e^{i2\pi\hat{D}_1/k}$ and $e^{i2\pi\hat{D}_2/k}$, which are indeed identical to (3.62a) and (3.62b). We may choose to diagonalize $\hat{D}_1$ or $\hat{D}_2$ and pick the remaining other to act as ladder operator using (5.78). This completes our construction of lattice MCS Hamiltonian and some discussion of its topological features when our space happens to be a torus. For more details, see [38].

5.3.4 Chern Number on lattice (Zero modes on lattice)

To determine the Chern number of the zero modes on lattice just as we did in the continuum, we need to consider the Fourier transform of the Hamiltonian. Looking at (5.70), the displacement in lattice positions of the fields a implies there will be factor of e^{ip_j} if we Fourier transform the Hamiltonian, where $j = 1, 2$. However, zero modes have no momentum in their Fourier modes. Therefore, the Hamiltonian for the zero modes have becomes

$$H = \frac{e^2 S}{2}\left[\left(p_1 - \frac{k}{4\pi}a_2\right)^2 + \frac{e^2}{2}\left(p_2 + \frac{k}{4\pi}a_1\right)^2\right]. \tag{5.80}$$

This is the Hamiltonian for (3.46), with p_j being the canonical momenta of the zero modes. Then we may add a total time derivative term to (5.80) as did in Section 3.3. The rest of the discussion on the Chern number follows in Chapter 3.

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Chapter 6 Epilogue

Our journey comes to an end. We have witnessed the different aspects of Maxwell-Chern-Simons theory, which arises from one of the archetypical examples of topological field theory, Chern-Simons theory. The consideration of Maxwell action together with that of CS certainly still preserves some of the global features of pure CS action. For example, the degeneracy of the ground state Hilbert space is unaltered by this addition of Maxwell term. Equipped with the tool of flux insertion, one could obtain the twisted boundary conditions to compute the Chern number of Maxwell-Chern-Simons theory on a torus, which is shown to be one. We see that the consistent translation operator of the MCS theory in this case is the magnetic translation operator, which is closely related to the notion of 1-cocyle. Lat but not least, we have also finished the non-trivial task of the construction of lattice Maxwell-Chern-Simons Hamiltonian. To this end, the transfer matrix and modified Villain formulation have been fruitful. This is followed by the derivation of possible constraints on the theory. Apart from the familiar Gauss's law, we saw there are so-called large gauge constraints on the states of Hilbert space, showing that we have indeed a $U(1)$ theory.

There are some other questions waiting to be explored. One prim examples would be the boundary theory of the lattice Chern-Simons as well as Maxwell-Chern-Simons theory. Besides, as to now, there is no clear prescription on the construction of non-abelian

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lattice Chern-Simons theory. There are also many well-known facts about Chern-Simons in the continuum but their lattice counterparts are not clear yet. Having able to answer one of these questions will be instrumental to our understanding of the properties of the topological field theory in the lattice settings. The author hopes to venture into this vast, beautiful directions in the future.

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Appendix A — Quick Review: Cup Product on Cubic Lattice in 3D

This appendix only contains the definition of cup product on the cubic lattice in 3D for completeness. For more cases, like in 4D or other generalizations, see [7, 27]. The only cup product we will be using in 3D is either of the form

$$\alpha^{(1)} \cup \beta^{(2)}, \quad \beta^{(2)} \cup \alpha^{(1)} \quad or \quad \alpha^{(0)} \cup \beta^{(3)}, \tag{A.1}$$

where $\alpha^{(p)}$ denotes a p-form and similar for $\beta^{(q)}$. Here are the definitions of the expressions in (A.1)

$$(\alpha^{(1)} \cup \beta^{(2)})_{r,012} = \alpha_{r,0}\beta_{r+\hat{0},12} + \alpha_{r,1}\beta_{r+\hat{1},20} + \alpha_{r,2}\beta_{r+\hat{2},01}, \tag{A.2a}$$

$$(\beta^{(2)} \cup \alpha^{(1)}))_{r,012} = \beta_{r,01}\alpha_{r+\hat{1}+\hat{3},2} - \beta_{r,02}\alpha_{r+\hat{0}+\hat{2},1} + \beta_{r,12}\alpha_{r+\hat{1}+\hat{2},0}, \tag{A.2b}$$

$$(\alpha^{(0)} \cup \beta^{(3)})_{r,012} = \alpha_r\beta_{r,012}. \tag{A.2c}$$

In the above expressions, whenever we cyclic permutations of the index of the 1-form α and 2-form β, we get positive sign. There is one minus sign exemplifying this idea. The index in this case is 021, which is clearly odd permutation of 012 and gives the minus sign. In fact, expression (A.2a) is just (5.22) in disguise. The cube term in (A.2c) has no

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orientation. As such For pictorial illustrations of (A.2a), (A.2b) and (A.2c), see [27].

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