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碩士論文

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Master's Thesis

動機化 DT/PT 對應

Motivic DT/PT Correspondence

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摘要

本論文證明了在假設半穩定物件的模疊具有好模空間的情況下，卡拉比丘三維複流形上 Donaldson–Thomas／Pandharipande–Thomas（DT/PT）對應的動機版本。我們採用 Toda 所構造的 t-結構心臟與弱穩定條件，在此架構下定義動機 DT 與 PT 不變量。本文的主要結果為一組動機等式，揭示兩類不變量之間的關係。證明方法延續 Bridgeland 與 Toda 的穿牆架構，並依賴卜辰璟近期證明的動機積分恆等式。本研究為數值 DT/PT 對應提供了動機上的提升。

關鍵字：動機 Donaldson–Thomas 不變量、好模空間、卡拉比丘流形、動機積分恆等式

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Abstract

This thesis proves a motivic version of the Donaldson–Thomas/Pandharipande–Thomas (DT/PT) correspondence on Calabi–Yau threefolds under the assumption on the existence of some good moduli spaces. Working with the heart constructed and a weak stability condition constructed by Toda, we define motivic DT and PT invariants via vanishing cycles. The main result establishes a motivic identity between these invariants, following Bridgeland and Toda's wall-crossing approach. The argument relies on the motivic integral identity recently proved by Bu. This provides a refinement of the numerical DT/PT correspondence and supports further development of categorified curve-counting theories.

Keywords: Motivic Donaldson-Thomas invariants, Good moduli space, Calabi-Yau threefolds, Motivic integral identity

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Chapter 1 Introduction

1.1 Curve counting theories on Calabi–Yau threefolds

Enumerative geometry seeks to count geometric objects that satisfy specified conditions. In the setting of Calabi–Yau threefolds, various curve-counting theories have emerged, each offering different perspectives and techniques. Among these, Gromov–Witten (GW) theory counts curves via intersection theory on moduli spaces of stable maps, whereas Donaldson–Thomas (DT) theory counts ideal sheaves or, more generally, coherent sheaves on the threefold.

Let X be a Calabi–Yau threefold. For a non-negative integer $g \geq 0$ and a homology class $\beta \in H_2(X, \mathbb{Z})$, we consider the moduli stack of stable maps $\overline{\mathcal{M}}_g(X, \beta)$, i.e., the moduli stack of maps $f\colon C \to X$ where C is a nodal curve with finite automorphism group and $f_*[C] = \beta$. This moduli stack carries a zero-dimensional virtual fundamental class, and we define its Gromov–Witten invariant by

$$\mathrm{GW}_{g,\beta} := \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

On the other hand, we may consider $I_n(X, \beta)$, the moduli space of ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ such that the Chern character of $\mathcal{I}$ is $(1, 0, -\beta, -n)$. This moduli space also carries a zero-dimensional virtual fundamental class, and we define its Donaldson–Thomas invariant by

$$\mathrm{DT}_{n,\beta} := \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Fixing a Calabi–Yau threefold X and a class $\beta \in H_2(X, \mathbb{Z})$, we consider the following generating functions:

$$Z_\beta^{\mathrm{DT}}(q) := \sum_{n \in \mathbb{Z}} \mathrm{DT}_{n,\beta} q^n,$$

$$Z_\beta^{\mathrm{red}}(q) := \frac{Z_\beta^{\mathrm{DT}}(q)}{Z_0^{\mathrm{DT}}(q)}.$$

The MNOP conjecture, formulated by Maulik–Nekrasov–Okounkov–Pandharipande in their two foundational papers [MNOP06a, MNOP06b], predicts a deep equivalence

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between the generating functions of GW and DT invariants.

Conjecture 1 (MNOP conjecture)**.** We have an identity of power series:

$$\exp\left(\sum_{g,\beta} \mathrm{GW}_{g,\beta} u^{2g-2} v^{\beta}\right) = \sum_{\beta} Z^{\mathrm{red}}_{\beta}(-e^{iu}) \cdot v^{\beta}.$$

Later developments introduced alternative curve-counting frameworks. In particular, Pandharipande–Thomas (PT) theory counts stable pairs [PT09]. A stable pair is a pure one-dimensional coherent sheaf $\mathcal{F}$ together with a section $s\colon \mathcal{O}_X \to \mathcal{F}$ such that s has zero-dimensional support. The moduli space of stable pairs $P_n(X,\beta)$ parametrizing such pairs $(\mathcal{F}, s)$ with $[\mathcal{F}] = \beta$ and $\chi(\mathcal{F}) = n$ for each $n \in \mathbb{Z}$ and $\beta \in H_2(X,\mathbb{Z})$. This moduli space also carries a zero-dimensional virtual fundamental class, producing integer-valued invariants

$$P_{n,\beta} := \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z},$$

and a generating series

$$Z^{\mathrm{PT}}_{\beta}(q) := \sum_{n\in\mathbb{Z}} P_{n,\beta} q^n.$$

The DT/PT correspondence, first conjectured by Pandharipande and Thomas, was established in significant generality by Toda [Tod10] and Bridgeland [Bri11], using wall-crossing in derived categories and Hall algebra techniques. Their work shows that DT and PT invariants satisfy a remarkable identity:

Theorem 1.1.1 (DT/PT correspondence [Bri11, Tod10])**.**

$$Z^{\mathrm{PT}}_{\beta}(q) = Z^{\mathrm{red}}_{\beta}(q).$$

More recently, Pardon [Par23] introduced a powerful and universal framework for curve counting. His method rigorously demonstrates that these curve-counting theories can be reduced to local models, such as the local curve or local surface cases. This reduction yields a proof of the MNOP conjecture, thereby solving one of the central problems in the enumerative geometry of Calabi−Yau threefolds.

Theorem 1.1.2 ([Par23])**.**

$$Z^{\mathrm{PT}}_{\beta}(-e^{iu}) = Z^{\mathrm{GW}}_{\beta}(u).$$

1.2 Categorification of Donaldson–Thomas Theory

Given the success of numerical curve-counting theories, a natural question arises: can these invariants be enhanced to capture more refined geometric or topological information? One promising direction is to develop motivic or cohomological refinements, which aim to lift integer-valued invariants to classes in the Grothendieck ring of varieties or to mixed Hodge structures, such that their Euler characteristics recover the classical numerical invariants.

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Among the known curve-counting theories, DT theory appears particularly well-suited for such enhancements. The works of Kontsevich–Soibelman [KS08] and Joyce–Song [JS12] envisioned the possibility of defining motivic DT invariants, but their realization required additional data such as orientation and d-critical structures. The existence of global orientation data was unclear at the time, limiting the scope of these theories.

Recent breakthroughs, however, have clarified many of these technical issues. The theory of shifted symplectic structures developed by [PTVV13] provides a derived geometric framework for moduli stacks. Building on this, Joyce and collaborators—including Bussi, Brav, Schürg, Meinhardt, Szendrői, Upmeier, Ben-Bassat, and Dupont—developed a rigorous theory of d-critical structures and orientation data on moduli stacks [Joy07b, Joy15, BBBBJ15, BBD+15, BBJ19, JU21].

Building on these foundational developments, we may define the motivic Donaldson–-Thomas and Pandharipande–Thomas invariants as elements in the ring of monodromic motives on algebraic stacks, using motivic Behrend function constructions and motivic integration:

Definition (5.1)**.** We define

$$\mathrm{DT}^{\mathrm{mot}}_{n,\beta}(X) = \int_{\mathcal{M}^{n,\beta}_{\mathrm{DT}}} (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}})\nu^{\mathrm{mot}}_{\mathcal{M}^{n,\beta}_{\mathrm{DT}}} \in \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C}),$$
$$\mathrm{PT}^{\mathrm{mot}}_{n,\beta}(X) = \int_{\mathcal{M}^{n,\beta}_{\mathrm{PT}}} (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}})\nu^{\mathrm{mot}}_{\mathcal{M}^{n,\beta}_{\mathrm{PT}}} \in \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C}).$$

This thesis aims to refine the DT/PT correspondence to the motivic level. We follow Toda's argument in [Tod10, Tod20], which realizes the DT/PT correspondence as a wall-crossing phenomenon of weak stability conditions in a triangulated category. The crucial point is to upgrade the integration map to the motivic level, this relies on a recent result on motivic integral identity proved by Bu [Bu24]. The following is the main result of this thesis, establishing an identity between motivic DT and PT invariants provided the existence of the good moduli spaces for the stack of semistable objects:

Main Theorem. (= Theorem 5.1.1). Under the assumption that the stack of semistable objects admits good moduli spaces. We have the following identity:

$$\sum_{n,\beta} \mathrm{DT}^{\mathrm{mot}}_{n,\beta}(X) q^n x^\beta = \left(\sum_{n \geq 0} \mathrm{DT}^{\mathrm{mot}}_{n,0}(X) q^n \right) \left(\sum_{n,\beta} \mathrm{PT}^{\mathrm{mot}}_{n,\beta}(X) q^n x^\beta \right).$$

Organization of the thesis. Chapter 2 recalls the necessary geometry preliminaries on the moduli stacks, including the construction of the heart $\mathcal{A}_X$, weak stability conditions, and the existence problem of good moduli spaces for semistable objects. Chapter 3 introduces the theory of shifted symplectic structures and orientation data, which play a key role in defining motivic Behrend functions. In Chapter 4, we define the motivic DT and PT invariants and state the motivic integral identity. Finally, Chapter 5 presents the motivic DT/PT correspondence and proves the main identity.

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Notations and Conventions

- We work over the field of complex numbers $\mathbb{C}$ throughout the paper.
- By an algebraic variety, we mean a finite type separated $\mathbb{C}$-scheme.
- Throughout this paper we work with schemes, algebraic spaces, and algebraic stacks that are quasi-separated and locally of finite type; in addition, every algebraic stack is assumed to have a separated diagonal.
- We denote by $\mathbb{S}$ the ∞-category of spaces; see [Lur09].
- We follow Toen's definition of derived algebraic stacks. A derived algebraic stack is a derived stack that is n-geometric for some n and locally of finite presentation (see [TV08]); in our convention, its cotangent complex is perfect, and the rank of that complex is called the virtual dimension.
- For a morphism $X \to Y$ in a triangulated category, we denote Y/X a cone of $X \to Y$, i.e., Y/X fit into an exact triangle

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Y/X \longrightarrow 0$$

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Chapter 2 Stability conditions and moduli spaces

In this chapter, we recall some basic facts about DT/PT moduli spaces. We fix a smooth projective Calabi-Yau threefold X, and consider its bounded derived category of coherent sheaves $D^b(X) := D^b(\mathrm{Coh}(X))$. As [PT09] suggests, the DT/PT correspondence may be realized as a wall crossing in the derived category $D^b(X)$. However, the existence of a Bridgeland stability condition on the entire derived category $D^{\mathrm{b}}(X)$ is not known for a general Calabi–Yau threefold. Therefore, following Toda's approach [Tod10], we restrict to a triangulated subcategory $\mathcal{D}_X \subset D^{\mathrm{b}}(X)$ and work with a weak stability condition on it. In the first section we review weak stability conditions in the sense of [Tod10].

2.1 Weak stability conditions on triangulated categories

There are generally two approaches to constructing weak stability conditions: one via slicings and the other via hearts of t-structures. In our case, we adopt only the heart-based approach and thus do not define slicings in this paper. For details, we refer the reader to [Tod10]. Given a triangulated category $\mathcal{D}$ and its Grothendieck group $K(\mathcal{D})$ together with a finitely generated $\mathbb{Z}$-module Γ and a $\mathbb{Z}$-linear homomorphism,

$$\mathrm{cl} : K(\mathcal{D}) \to \Gamma$$

and also a filtration on Γ,

$$\Gamma_\bullet : 0 \subsetneq \Gamma_0 \subsetneq \cdots \subsetneq \Gamma_N = \Gamma,$$

such that each quotient

$$\mathbb{H}_i := \Gamma_i/\Gamma_{i-1}$$

is a free. For each i, we set $\mathbb{H}_i^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{H}_i, \mathbb{C})$ and fix norms $\| * \|_i$ on $\mathbb{H}_i \otimes_{\mathbb{Z}} \mathbb{R}$. Let $Z = \{Z_i\}_{i=0}^N \in \prod_{i=0}^N \mathbb{H}_i^\vee$ be a set of functions. We define a map $Z : \Gamma \to \mathbb{C}$ as follows. For $v \in \Gamma$, we take the unique integer $i \in \{0, \ldots, N\}$ such that $v \in \Gamma_i$, but $v \notin \Gamma_{i-1}$, define $Z(v) = Z_i([v])$ where $[v]$ denote the class of v in $\mathbb{H}_i$, and denote $\|E\| := \|[\mathrm{cl}(E)]\|_i$.

The function Z can be seen as a function on $K(\mathcal{D}) \to \mathbb{C}$ by composing $\mathrm{cl} : K(\mathcal{D}) \to \Gamma$. We will write $Z(E) = Z(\mathrm{cl}(E))$ for short.

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Definition. Let Slice($\mathcal{D}$) be the set of slicing on $\mathcal{D}$ that are locally finite [Tod10]. The space of weak stability conditions $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ is defined to be the collection of pairs $(Z, \mathcal{P}) \in \prod_{i=0}^{N} \mathbb{H}_i^\vee \times \mathrm{Slice}(\mathcal{D})$, subject to the following conditions.

(i) For any non-zero $E \in \mathcal{P}(\phi)$, we have

$$Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi).$$

(ii) There is a constant $C > 0$ such athat for any non-zero $E \in \bigcup_{\phi\in\mathbb{R}} \mathcal{P}(\phi)$, we have

$$\|E\| \leq C|Z(E)|.$$

2.1.1 Constructions of weak stability conditions

Definition. Fix a bounded t-structure on $\mathcal{D}$ and let $\mathcal{A}$ be its heart. A weak stability function on $\mathcal{A}$ is a function $Z \in \prod_{i=0}^{N} \mathbb{H}_i^\vee$ such that for any non-zero $E \in \mathcal{A}$, we have

$$Z(E) \in \{r \exp(i\pi\phi) : r > 0, 0 < \phi \leq 1\}.$$

By definition, for any $0 \neq E \in \mathcal{A}$ there is a well-defined slope, $\arg Z(E) \in (0, \pi]$. We have the following easy observation.

Lemma 2.1.1. Let Z be a weak stability function on $\mathcal{A}$. Consider an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

in $\mathcal{A}$. Then one of the following chains of inequalities for the arguments holds:

$$\arg Z(F) \leq \arg Z(E) \leq \arg Z(G),$$
$$\arg Z(F) \geq \arg Z(E) \geq \arg Z(G).$$

Proof. Indeed, let i be the unique integer such that $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$. If $\mathrm{cl}(F) \in \Gamma_{i-1}$, then we must have $\mathrm{cl}(G) \in \Gamma_i \setminus \Gamma_{i-1}$ and

$$Z(E) = Z(\mathrm{cl}(E)) = Z_i([\mathrm{cl}(E)]) = Z_i([\mathrm{cl}(F)]) = Z(F).$$

In this case, obviously one of the inequality holds. Similarly, if $\mathrm{cl}(G) \in \Gamma_{i-1}$ and $\mathrm{cl}(F) \in \Gamma_i \setminus \Gamma_{i-1}$ one of the above inequality holds. So let us assume that both $\mathrm{cl}(F)$ and $\mathrm{cl}(G)$ are in $\Gamma_i \setminus \Gamma_{i-1}$. In this case we have

$$Z(E) = Z(G) + Z(F).$$

This equality can be written as

$$r \exp(i\pi\phi) = r_1 \exp(i\pi\phi_1) + r_2 \exp(i\pi\phi_2)$$

for some $r, r_1, r_2 > 0$ and $0 < \phi, \phi_1, \phi_2 \leq 1$. It is then clear that one of the inequality holds. □

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Definition. Given a weak stability function $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ on $\mathcal{A}$. An object $0 \neq E \in \mathcal{A}$ is Z-semistable (resp. stable) if it satisfies the following condition.

- For any exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

in $\mathcal{A}$, we have

$$\arg Z(F) \leq \arg Z(G) \quad (\text{resp.} \arg Z(F) < \arg Z(G)).$$

Definition (H-N property)**.** Given a weak stability function $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ on $\mathcal{A}$. We say that Z has Harder-Narasimhan property if for any object $E \in \mathcal{A}$ there is a $\mathbb{Z}$-filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that each subquotient $F_j = E_j/E_{j-1}$ is Z-semistable with

$$\arg Z(F_1) > \arg Z(F_2) > \cdots > \arg Z(F_n).$$

It is not easy to construct a slicing on a triangulated category. Nevertheless, it is often more natural to consider the heart of a t-structure. The following proposition shows that one can construct a stability condition using a heart and a stability function defined on it.

Proposition 2.1.2 ([Tod10])**.** Giving a heart of a bounded t-structure on $\mathcal{D}$ and a weak stability function on it with the H-N property is the same as giving a pair $(Z, \mathcal{P}) \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee} \times \text{Slice}(\mathcal{D})$ satisfying the condition (i) in 2.1.

Denotes $\text{Coh}_{\leq 1}(X)$ the abelian subcategory of coherent sheaves with support dimension less or equal to 1 on X. We consider the triangulated subcategory

$$\mathcal{D}_X := \langle \mathcal{O}_X, \text{Coh}_{\leq 1}(X) \rangle_{\text{tr}} \subset D^b(\text{Coh}(X)).$$

This will be the main triangulated category we consider.

2.2 Weak stability condition on $\mathcal{D}_X$

We begin by fixing notation and definitions. Denote by $N_1(X)$ the abelian group of numerical curve classes on X and by $A(X)$ its ample cone. Set

$$N_{\leq 1}(X) = \mathbb{Z} \oplus N_1(X), \qquad \Gamma := N_{\leq 1}(X) \oplus \mathbb{Z}.$$

Finally, let $K(\mathcal{D}_X)$ be the Grothendieck group of the triangulated category $\mathcal{D}_X$.

Lemma 2.2.1. The homomorphism $\text{cl} : K(\mathcal{D}_X) \to \Gamma$ given by

$$E \mapsto (\text{ch}_3(E), \text{ch}_2(E), \text{ch}_0(E))$$

is well-defined.

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Proof. We need to show that ch_* has integer coefficients for $* = 0, 2, 3$ on $\mathcal{D}_X$. By definition of $\mathcal{D}_X$, its grothendieck group is generated by $\mathcal{O}_X$ and $\mathrm{Coh}_{\leq 1}(X)$, we only need to show that $\mathrm{ch}_3(E)$, $\mathrm{ch}_2(E)$ has integer coefficient for $E \in \mathrm{Coh}_{\leq 1}(X)$. By HRR, we have

$$\mathrm{ch}_3(E) = \int_X (\mathrm{ch}_2(E) + \mathrm{ch}_3(E)) \cdot \left(1 + \frac{\mathrm{c}_2(X)}{12}\right) = \chi(E) \in \mathbb{Z}$$

On the other hand, since E has support dimension ≤ 1, we know that $\mathrm{c}_1(E) = 0$. Hence

$$\mathrm{ch}_2(E) = \frac{\mathrm{c}_1(E)^2 - 2\mathrm{c}_2(E)}{2} = -\mathrm{c}_2(E).$$

□

2.2.1 Construction

We now recall the construction of weak stability condition on $\mathcal{D}_X$. Consider the following filtration on Γ

$$\Gamma_\bullet : 0 \subsetneq \mathbb{Z} \subsetneq N_{\leq 1} \subsetneq \Gamma$$

where each inclusion is the natural inclusion of direct summand. Then we have $\mathbb{H}_0 = \mathbb{Z}$, $\mathbb{H}_1 = N_1(X)$ and $\mathbb{H}_2 = \mathbb{Z}$. . Consider

$$\xi = (-z_0, -i\omega, z_1)$$

where $z_0, z_1 \in \{r \exp(i\pi\theta) \,|\, r > 0, \theta \in (\pi/2, \pi)\}$ and $\omega \in A(X)$.

For $(s, l, r) \in \mathbb{Z} \oplus N_1(X) \oplus \mathbb{Z}$ we define $Z_\xi = \{Z_{\xi,i}\}_{i=0}^2 \in \mathbb{H}_0^\vee \times \mathbb{H}_1^\vee \times \mathbb{H}_2^\vee$ by

$$Z_{\xi,0}(s) = -z_0 \cdot s, Z_{\xi,1}(l) = -i\omega \cdot l, Z_{\xi,2}(r) = z_1 \cdot r.$$

Thus, we have a family of functions Z_ξ.

Proposition 2.2.2 ([Tod10, Lemma 3.5])**.** Let

$$\mathcal{A}_X := \left\langle \mathcal{O}_X,\ \mathrm{Coh}_{\leq 1}(X)[-1] \right\rangle_{\mathrm{ex}} \subset \mathcal{D}_X$$

be the smallest extension-closed subcategory of $\mathcal{D}_X$ containing $\mathcal{O}_X$ and $\mathrm{Coh}_{\leq 1}(X)[-1]$. Then $\mathcal{A}_X$ is the heart of a bounded t-structure on $\mathcal{D}_X$.

Lemma 2.2.3 ([Tod10, Lemma 2.17 + 3.8])**.** The association $\xi \mapsto (Z_\xi, \mathcal{A}_X)$ determines a continuous family in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$. We denote this continuous family $\mathcal{V}_X$.

2.3 Moduli stacks of semistable objects

We now collect some foundational properties of the moduli stack of objects in $\mathcal{A}_X$. As shown in [Lie06], there exists an algebraic stack $\mathcal{M}$, which parametrizes objects $E \in$

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$D^b(X)$ satisfying

$$\mathrm{Ext}^i(E,E) = 0 \quad \text{for all } i < 0.$$

Consider the determinant map

$$\det : \mathcal{M} \to \mathrm{Pic}(X), E \mapsto \det(E)$$

The objects in $\mathcal{D}_X$ are contained in the fiber at $[0] = \mathcal{O}_X \in \mathrm{Pic}(X)$. We denote $\mathcal{M}_0$ the fiber of det at $[0]$. Then the moduli stack of objects $\mathcal{O}bj(\mathcal{A}_X)$, as an abstract stack, is a substack of $\mathcal{M}_0$. The stack $\mathcal{O}bj(\mathcal{A}_X)$ admits a graded decomposition

$$\coprod_{v\in\Gamma} \mathcal{O}bj^v(\mathcal{A}_X) = \mathcal{O}bj(\mathcal{A}_X)$$

where $\mathcal{O}bj^v(\mathcal{A}_X)$ is the moduli stack of objects $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = v \in \Gamma$. For a weak stability condition of the form $\sigma_\xi = (Z_\xi, \mathcal{A}_X)$ and a numerical type v we may consider its moduli stack of semistable objects $\mathcal{M}^v(\sigma_\xi) \subset \mathcal{O}bj^v(\mathcal{A}_X)$. We would like to focus on objects of rank one or has dimension zero. Define

$$\Gamma^1 = \{v \in \Gamma \,|\, v = (-n, -\beta, 1), \beta : \text{effective}\}$$
$$\Gamma^0 = \{v \in \Gamma \,|\, v = (-n, 0, 0), n \geq 0\}$$

Theorem 2.3.1 ([Tod10])**.** Let $v \in \Gamma^0 \cup \Gamma^1$ be a numerical type.

1. the moduli stack of objects
$$\mathcal{O}bj^v(\mathcal{A}_X) \subset \mathcal{M}_0$$
is open in $\mathcal{M}_0$. In particular, $\mathcal{O}bj^v(\mathcal{A}_X)$ is an algebraic stack locally of finite type over $\mathbb{C}$.

2. For any weak stability condition of the form σ_ξ, the substack of σ_ξ-semistable objects
$$\mathcal{M}^v(\sigma_\xi) \subset \mathcal{O}bj^v(\mathcal{A}_X)$$
is open in $\mathcal{O}bj^v(\mathcal{A}_X)$ of finite type over $\mathbb{C}$.

Our next goal is to identify the stack of σ_ξ-semistable objects for each ξ.

Lemma 2.3.2 ([Tod10], Lemma 3.11)**.** Let $\sigma_\xi \in \mathcal{V}_X$ be a weak stability condition. If $E \in \mathcal{A}_X$ is a σ_ξ-semistable object of rank 1. Then E can be expressed as an extension in $\mathcal{A}_X$,

$$0 \longrightarrow I_C \longrightarrow E \longrightarrow Q[-1] \longrightarrow 0$$

where I_C is the ideal sheaf of a 1-dimensional subscheme $C \subset X$ and Q is a coherent sheaf of zero-dimensional.

Proof. First note that since $E \in \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X)[-1]\rangle_{\mathrm{ex}}$ and of rank 1. There is a filtration

$$0 = E_{-1} \subset E_0 \subset E_1 \subset E_2 = E$$

such that $F_i = E_i/E_{i-1}$, we have

$$F_0, F_2 \in \mathrm{Coh}_{\leq 1}(X)[-1], F_1 = \mathcal{O}_X.$$

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If F_2 is a shift of 1-dimensional sheaf. We take a further quotient of $F_2 \twoheadrightarrow \widetilde{F}_2$ such that $\widetilde{F}_2$ is pure of 1-dimensional up to a shift. Then we have a surjection $E \twoheadrightarrow \widetilde{F}_2$ in $\mathcal{A}_X$. Note that pure 1-dimension sheaves are automatically semistable of phase $1/2$ and an object of rank 1 always has phase $> 1/2$. Thus there should be no morphism from E to a pure 1-dimensional sheaf. Hence F_2 is a shift of a 0-dimensional sheaf Q. On the other hand, we have a distinguished triangle

$$F_0 \to E_1 \to \mathcal{O}_X \to F_0[1]$$

after a shift we get a distinguished triangle

$$E_1 \to \mathcal{O}_X \to F_0[1] \to E_1[1].$$

Since $F_0[1]$ is a 1-dimensional sheaf. The object E_1 is then an ideal sheaf I_C of a 1-dimensional closed subscheme C. $\square$

We consider the following important stacks of semistable objects.

Theorem 2.3.3 ([Tod10, Proposition 3.12])**.** For a stability condition of the form σ_ξ where $\xi = (-z_0, -i\omega, z_1)$. For $v = (-n, -\beta, 1) \in \Gamma^1$ we have the following

1. Suppose that arg $z_0 <$ arg z_1. We have

$$\mathcal{M}^v(\sigma_\xi) = [I_n(X,\beta)/\mathbb{G}_m]$$

where $I_n(X,\beta)$ is the moduli space of ideal sheaves $I_C \subset \mathcal{O}_X$ with numerical type $(-n, -\beta, 1)$.

2. Suppose that arg $z_0 >$ arg z_1. We have

$$\mathcal{M}^v(\sigma_\xi) = [P_n(X,\beta)/\mathbb{G}_m]$$

where $P_n(X,\beta)$ is the moduli space of two term complexes $(\mathcal{O}_X \xrightarrow{s} \mathcal{F})$ for stable pairs with numerical type $(-n, -\beta, 1)$.

In both situations, the $\mathbb{G}_m$-action is trivial.

We denote the above two moduli stacks as $\mathcal{M}^v_{\mathrm{DT}}$ and $\mathcal{M}^v_{\mathrm{PT}}$ when $v = (-n, -\beta, 1)$. For a stability condition σ_ξ on the wall (i.e. $\arg(z_0) = \arg(z_1)$), we denote $\mathcal{M}^{ss,v} = \mathcal{M}^v(\sigma_\xi)$ for $v \in \Gamma^1 \cup \Gamma^0$ and

$$\mathcal{M}^{ss} = \coprod_{v \in \Gamma^1 \cup \Gamma^0} \mathcal{M}^{ss,v}.$$

2.4 The Good moduli space

In [Alp13], Alper introduced the notion of a good moduli space as a fundamental concept in intrinsic GIT. Subsequently, the work of Alper–Halpern-Leistner–Heinloth

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[AHLH18] formulated sufficient criteria for good moduli spaces to exist, while Alper–Hall–Rydh [AHR19] established a structural theorem. For our purposes, the structural theorem guarantees the existence of a Nisnevich local structure, which allows motivic properties to be reduced to local situations. This local structure is an essential ingredient for the motivic integral identity [Bu24]. In this chapter, we will study the existence problem of good moduli spaces for semistable objects.

Definition. We say that an algebraic stack $\mathcal{X}$ admits a good moduli space if there is a morphism $\phi : \mathcal{X} \to Y$, where Y is an algebraic space satisfied the following conditions

1. The pushforward $\phi_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(Y)$ is an exact functor.
2. The natural map $\mathcal{O}_Y \to \phi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

2.4.1 AHLH's Existence Theorem

Let $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$. For any scheme S we denote Θ_S the base change $\Theta \times S$. For a DVR R with fraction field K and a uniformizer π, let 0 be the unique closed point of Θ_R.

Definition. An algebraic stacks $\mathcal{X}$ is said to be Θ-reductive if for every DVR R, any commutative diagram of solid arrows

$$\begin{tikzcd} \Theta_R \setminus \{0\} \arrow[r] \arrow[d] & \mathcal{X} \\ \Theta_R \arrow[ur, dashed] & \end{tikzcd}$$

can be completed in a unique way.

Remark 2.4.1. We cover $\Theta \setminus \{0\}$ by the two open subsets

$$\Theta_K \quad \text{and} \quad \big[(\mathbb{A}^1 \setminus \{0\})/\mathbb{G}_m\big] \times \mathrm{Spec}R \;\cong\; \mathrm{Spec}R.$$

Accordingly, specifying a morphism $\Theta_R \setminus \{0\} \to \mathcal{X}$ is equivalent to giving morphisms $\Theta_K \to \mathcal{X}$ and $\mathrm{Spec}R \to \mathcal{X}$ together with an identification of their restrictions over $\mathrm{Spec}K$.

Set

$$\overline{\mathrm{ST}}_R \;:=\; \big[\mathrm{Spec}\big(R[s,t]/(st-\pi)\big)/\mathbb{G}_m\big],$$

where s and t are given $\mathbb{G}_m$-weights 1 and -1, respectively. The equations $s = t = 0$ define a closed point, which we denote by 0.

Definition. An algebraic stack $\mathcal{X}$ is called **S**-complete if, for every DVR R and every commutative diagram of solid arrows

$$\begin{tikzcd} \overline{\mathrm{ST}}_R \setminus \{0\} \arrow[r] \arrow[d] & \mathcal{X} \\ \overline{\mathrm{ST}}_R \arrow[ur, dashed] & \end{tikzcd}$$

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there exists a unique dotted arrow completing the diagram.

Remark 2.4.2. The open substack $\overline{\mathrm{ST}}_R \setminus \{0\}$ is covered by the loci $\{s \neq 0\}$ and $\{t \neq 0\}$. Observe that

$$\mathrm{Spec}\big(R[s,t]_s/(st-\pi)\big)/\mathbb{G}_m \cong \mathrm{Spec}\big(R[s,t]_s/(t-\tfrac{\pi}{s})\big)/\mathbb{G}_m \cong \mathrm{Spec}\big(R[s]_s\big)/\mathbb{G}_m \cong \mathrm{Spec}R,$$

and similarly

$$\mathrm{Spec}\big(R[s,t]_t/(st-\pi)\big)/\mathbb{G}_m \cong \mathrm{Spec}\big(R[t]_t\big)/\mathbb{G}_m \cong \mathrm{Spec}R.$$

Hence specifying a morphism $\overline{\mathrm{ST}}_R \setminus \{0\} \to \mathcal{X}$ is equivalent to giving two morphisms $\mathrm{Spec}R \to \mathcal{X}$ together with an isomorphism of their restrictions over $\mathrm{Spec}K$.

We can now state the existence result.

Theorem 2.4.3 (J. Alper, D. Halpern Leistner and J. Heinloth [AHLH18])**.** An algebraic stack $\mathcal{X}$ of finite type with affine stabilizers admits a good moduli space if and only if $\mathcal{X}$ is Θ-reductive and **S**-complete.

Remark 2.4.4. It was shown in [AHLH18, Section 7] that if $\mathcal{A} \subset D^b(X)$ is the heart of a Noetherian bounded t-structure, then the substack of semistable objects with respect to a Bridgeland stability condition on $D^b(X)$ admits a good moduli space, provided it is of finite type. The proof relies on the following three key steps:

(1) First, one uses the construction of the moduli stack of objects in a noetherian abelian category due to Artin and Zhang [AZ01]. They showed that, under suitable finiteness conditions, this stack is **S**-complete and Θ-reductive.

(2) Second, Halpern-Leistner [HL14, Proposition 6.2.7] identifies this moduli stack with the open substack of Lieblich's moduli stack of complexes which parametrizes flat families whose fibers lie in the heart of a t-structure. This shows that the stack of objects in the heart of a t-structure inherits **S**-completeness and Θ-reductivity.

(3) Finally, it is shown that the open substack of semistable objects—defined via a Bridgeland stability condition—preserves **S**-completeness and Θ-reductivity and hence admits a good moduli space by the main theorem of [AHLH18].

However, two technical issues arise in our setting. First, the heart $\mathcal{A}_X$ under consideration is not the heart of a t-structure on the entire derived category $D^b(X)$, but only on a full triangulated subcategory $\mathcal{D}_X \subset D^b(X)$. Second, the stability condition we employ is not a Bridgeland stability condition in the usual sense, but rather a weak stability condition. As a result, it is unclear how the second and third steps in the argument described above can be adapted to our setting.

2.4.2 Good moduli space for Semistable objects on the wall

We fix a stability condition $\sigma_0 \in \mathcal{V}_X$ on the wall i.e. $\arg(z_0) = \arg(z_1) =: \eta$. Note that $\arg \eta > 1/2$. So an object E with slope greater than $1/2$ must be either of positive rank or has 0-dimensional support. We ask the following question.

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Conjecture 2. For numerical types $v \in \Gamma^0 \cup \Gamma^1$. The stack $\mathcal{M}^{ss,v}$ admits a separated good moduli space.

As seen in Remark 2.4.4(2), we hope the stack $\mathcal{O}bj(\mathcal{A}_X)$ is **S**-complete and Θ-reductive.

Conjecture 3. The open substack $\mathcal{O}bj(\mathcal{A}_X) \subset \mathcal{M}_0$ is **S**-complete and Θ-reductive.

Corollary 2.4.5. Assuming the existence of a good moduli space for $\mathcal{M}^{ss,v}$ for $v \in \Gamma^0 \cup \Gamma^1$. The stack $\mathcal{M}^{ss}$ is Nisnevich locally fundamental, i.e. There exists a Nisnevich cover $\mathcal{X}_i \to \mathcal{M}^{ss}$ such that $\mathcal{X}_i = [U/\mathrm{GL}(n)]$ for some n and some affine scheme U.

Proof. By [AHR19, Theorem 6.1], there is a Nisnevich cover by linear fundamental stack for each v. Thus the stack $\mathcal{M}^{ss}$ is Nisnevich locally fundamental. □

2.4.3 Some properties for semistable objects

We start from the following criterion

Proposition 2.4.6. Assuming **S**-completeness and Θ-reductivity for $\mathcal{O}bj(\mathcal{A}_X)$. An open substack $\mathcal{U} \subset \mathcal{O}bj(\mathcal{A}_X)$ of finite type admits a separated good moduli space if the following statements holds.

(a) (Θ-reductive) For any DVR R with fraction field K and residue field κ, for every R-point E of $\mathcal{O}bj(\mathcal{A}_X)$ and every $\mathbb{Z}$-graded filtration

$$0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E$$

satisfying $E_i = 0$ for $i << 0$ and $E_i = E$ for $i >> 0$, such that E_i/E_{i+1} are R-point of $\mathcal{O}bj(\mathcal{A}_X)$. If E and $\mathrm{gr}(E_\bullet|_K)$ are both in $\mathcal{U}$. Then $\mathrm{gr}(E_\bullet|_\kappa)$ is also in $\mathcal{U}$.

(b) (**S**-completeness) For any two κ-points E, F of $\mathcal{U}$, if there are $\mathbb{Z}$-graded filtrations in $\mathcal{O}bj(\mathcal{A}_X)(\mathrm{Spec}(\kappa))$

$$\begin{gathered} 0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E \\ F \supset \cdots \supset F^{i-1} \supset F^i \supset F^{i+1} \supset \cdots \supset 0 \end{gathered}$$

satisfying $E_i = 0$, $F^i = F$ for $i \ll 0$ and $E_i = E$, $F^i = 0$ for $i \gg 0$ and $E_i/E_{i-1} \cong F^i/F^{i+1}$ for all i. Then $\mathrm{gr}(E_\bullet)$ is also in $\mathcal{U}(\kappa)$.

Proof. By Theorem 2.4.3, it suffices to verify Θ-reductivity and **S**-completeness for $\mathcal{U}$. These conditions can be translated into conditions (a) and (b) via Remark 2.4.1 and Remark 2.4.2. The proof follows essentially the same strategy as in [Alp25, Proposition 8.3.6]. □

This proposition originally appears in Alper's lecture notes [Alp25], where it is used in the proof that the moduli stack of semistable vector bundles admits a good moduli space.

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In the original setting, the abelian category under consideration is that of coherent sheaves. Alper applies this proposition in combination with certain properties of semistable vector bundles to establish the existence of a good moduli space. Motivated by this strategy, we aim to first prove that certain semistable objects in our setting exhibit analogous properties.

Lemma 2.4.7. Let E be a σ_0-semistable object of rank 1. Then any subobject of E with slope $> 1/2$ is also σ_0-semistable.

Proof. Let F be a subobject of E. If F has rank 0 then F must be a shift of 0-dimensional sheaf which is obviously semistable. So we may assume F has rank 1. If F is not semistable, then there is an exact sequence in $\mathcal{A}_X$ that destabilize F

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0$$

where F/G has slope $= 1/2$. Now consider the cokernel of $G \to F \to E$ we get an exact sequence

$$0 \longrightarrow F/G \longrightarrow E/G \longrightarrow E/F \longrightarrow 0\ .$$

Since F/G has slope $= 1/2$, the support of F/G has dimension > 0. So G must has rank 1. Thus E/G is a rank 0 object which contain F/G, so the support of F/G also has dimension > 0. Therefore, E/G has slope $1/2$ which is a contradiction since E is semistable. $\square$

Lemma 2.4.8. Let E be a σ_0-semistable object of rank 1 of slope $> 1/2$. Then any quotient of E is also σ_0-semistable of slope $> 1/2$. In particular, if the quotient has rank 0 then it has 0-dimensional support.

Proof. Let F be a subobject of E. If F has rank 1, then it has slope $> 1/2$. By the semistability of E, the quotient E/F must has slope $> 1/2$. Hence support of E/F is 0-dimensional, which is σ_0-semistable. So we may assume F has rank 0. Now if there is an exact sequence destabilize E/F,

$$0 \longrightarrow G \longrightarrow E/F \longrightarrow H \longrightarrow 0\ .$$

Then there is a surjective morphism $E \to E/F \to H$, where H has slope $1/2$. Taking kernel of the morphism $E \to H$ then this H also destabilize E which is a contradiction. $\square$

Lemma 2.4.9. The category of σ_0-semistable objects with slope $> 1/2$ is closed under extension.

Proof. Let F be an extension of σ_0-semistable objects E and G in $\mathcal{A}_X$ satisfying

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0\ .$$

If there is an exact sequence destabilizes F, say

$$0 \longrightarrow X \longrightarrow F \longrightarrow Y \longrightarrow 0\ .$$

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where X has slope $> 1/2$ and Y has slope $= 1/2$. We may assume that Y is semistable of slope $= 1/2$. Then the composition $E \to F \to Y$ is zero. So $E \to F$ factor through X. Hence we obtain an exact sequence

$$0 \longrightarrow X/E \longrightarrow G \longrightarrow Y \longrightarrow 0 \ ,$$

which violates the semistability of G. $\square$

Using the above lemmata, we now prove the following proposition.

Proposition 2.4.10. If the stack $\mathcal{O}bj(\mathcal{A}_X)$ is **S**-complete and Θ-reductive, then $\mathcal{M}^{ss,v}$ admits a good moduli space for all $v \in \Gamma^0 \cup \Gamma^1$.

Proof. The result for $v \in \Gamma^0$ is well-known. We apply Proposition 2.4.6 on $\mathcal{M}^{ss,v} \subset \mathcal{O}bj(\mathcal{A}_X)$ for $v \in \Gamma^1$. It suffices to show (a) and (b).

(a) (Θ-reductive) For any DVR R with fraction field K and residue field κ, for every R-point E of $\mathcal{O}bj(\mathcal{A}_X)$ and every $\mathbb{Z}$-graded filtration

$$0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E$$

satisfying $E_i = 0$ for $i \ll 0$ and $E_i = E$ for $i \gg 0$, such that E_i/E_{i+1} are R-point of $\mathcal{O}bj(\mathcal{A}_X)$. If E and gr$(E_\bullet|_K)$ are both in $\mathcal{M}^{ss,v}$. Then gr$(E_\bullet|_\kappa)$ is also in $\mathcal{M}^{ss,v}$.

Proof. First note that $(E_i/E_{i-1})|_K$ cannot have slope $1/2$ for any i, otherwise the projection gr$(E_\bullet|_K) \to (E_i/E_{i-1})|_K$ would destabilize gr$(E_\bullet|_K)$. By the openness of $\mathcal{O}bj^v(\mathcal{A}_X)$ for each v, the $(E_i/E_{i-1})|_\kappa$ cannot have slope $1/2$ for any i. Let i be the unique index such that $(E_i/E_{i-1})|_\kappa$ has positive rank. Then $E_i|_\kappa$ has rank 1 and hence has slope $> 1/2$. By Lemma 2.4.7, $E_i|_\kappa$ is semistable. Applying Lemma 2.4.8, we deduces that $(E_i/E_{i-1})|_\kappa$ is semistable. Finally, Lemma 2.4.9 implies that gr$(E_\bullet|_\kappa)$ is semistable. $\square$

(b) (**S**-completeness) For any two κ-points E, F of $\mathcal{M}^{ss,v}$, if there are $\mathbb{Z}$-graded filtrations in $\mathcal{O}bj(\mathcal{A}_X)(\kappa)$

$$\begin{gathered} 0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E \\ F \supset \cdots \supset F^{i-1} \supset F^i \supset F^{i+1} \supset \cdots \supset 0 \end{gathered}$$

satisfying $E_i = 0$, $F^i = F$ for $i \ll 0$ and $E_i = E$, $F^i = 0$ for $i \gg 0$ and $E_i/E_{i-1} \cong F^i/F^{i+1}$ for all i. Then gr$(E_\bullet)$ is also in $\mathcal{M}^{ss,v}$.

Proof. Since E is an object of rank 1. There is a unique i such that E_i/E_{i-1} has rank 1. Thus by Lemma 2.4.9, it suffices to show E_j/E_{j-1} has 0-dimensional support for all $j \neq i$ and E_i/E_{i-1} is semistable. Using Lemma 2.4.7, we see that E_j is semistable of rank 1 for all $j \geq i$ and F_j is semistable of rank 1 for all $j \leq i$. Now applying Lemma 2.4.8 we get E_j/E_{j-1} has 0-dimensional support for all $j \neq i$ and E_i/E_{i-1} is a rank 1 semistable object. $\square$

The existence of a good moduli space for $\mathcal{M}^{ss,v}$ then follows from Proposition 2.4.6. $\square$

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Chapter 3 Shifted symplectic structure and orientation

Compared to numerical Donaldson-Thomas invariants, defining motivic or cohomological Donaldson-Thomas invariants requires more information. Therefore, we need to introduce shifted symplectic structures and orientations.

3.1 Shifted symplectic structure

We begin by briefly recalling some definitions and key existence results from the theory of shifted symplectic structures, as introduced in [PTVV13] via derived algebraic geometry; see also [Par24] for a nice introduction.

Definition ([PTVV13, Definition 1.8])**.** For an affine derived scheme $\mathfrak{X} = \mathrm{Spec} A$, where $A \in$ **cdga**, we denote by $\mathcal{A}^p(\mathfrak{X}, n)$ (resp. $\mathcal{A}^{p,\mathrm{cl}}(\mathfrak{X}, n)$) the space of (resp. closed) p-forms of degree n on $\mathfrak{X}$.

Remark 3.1.1. In contrast to the classical setting, where closed forms are defined as a subset of all forms (namely, those annihilated by the de Rham differential), in the derived framework a closed p-form of degree n is not merely a form satisfying a condition. Rather, it consists of a p-form together with a specified ”closing structure”, a homotopy-theoretic datum encoding the vanishing of the differential in a coherent way. There is a forgetting morphism $\mathcal{A}^{p,\mathrm{cl}}(\mathfrak{X}, n) \to \mathcal{A}^p(\mathfrak{X}, n)$ that forget the closing structure.

We consider these two functors $\mathcal{A}^p(-, n)$ and $\mathcal{A}^{p,\mathrm{cl}}(-, n)$ as derived prestacks. Actually, one can show that they are derived stacks for the étale topology.

Proposition 3.1.2. The derived prestacks $\mathcal{A}^{p,cl}(-, n), \mathcal{A}^p(-, n) :$ **cdga** $\to \mathbb{S}$ are derived stacks for the étale topology.

Thus, we may globalize the definitions of p-forms to arbitrary derived stacks using mapping stacks.

Definition. Given a derived stack $\mathfrak{X}$. We define the space of p-forms, closed p-forms by

$$\mathcal{A}^p(\mathfrak{X}, n) := \mathrm{Map}_{\mathbf{dSt}}(\mathfrak{X}, \mathcal{A}^p(-, n)),\ \mathcal{A}^{p,cl}(\mathfrak{X}, n) := \mathrm{Map}_{\mathbf{dSt}}(\mathfrak{X}, \mathcal{A}^{p,cl}(-, n)).$$

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The space of closed p-forms is not easy to describe. However, for p-forms we have the following proposition.

Proposition 3.1.3. Given a derived algebraic stack $\mathfrak{X}$, and its cotangent complex $\mathbb{L}_{\mathfrak{X}}$. Then we have an equivalence of simplicial sets

$$\mathcal{A}^p(\mathfrak{X}, n) \cong \mathrm{Map}(\mathcal{O}_{\mathfrak{X}}, \wedge^p \mathbb{L}_{\mathfrak{X}}[n]).$$

In particular, we have

$$\pi_0(\mathcal{A}^p(\mathfrak{X}, n)) \cong H^n(\mathfrak{X}, \wedge^p \mathbb{L}_{\mathfrak{X}})$$

Given a morphism of derived stacks $f : \mathfrak{X} \to \mathfrak{Y}$ there is a pullback map of (closed) n-shifted p-form

$$f^\star : \mathcal{A}^p(\mathfrak{Y}, n) \to \mathcal{A}^p(\mathfrak{X}, n), f^\star : \mathcal{A}^{p,cl}(\mathfrak{Y}, n) \to \mathcal{A}^{p,cl}(\mathfrak{X}, n)$$

3.1.1 Shifted symplectic structure

By **Proposition** 3.1.3, any 2-form ω of degree n on a derived algebraic stack $\mathfrak{X}$ corresponds to a morphism of quasi-coherent complex

$$\mathcal{O}_{\mathfrak{X}} \to \wedge^2 \mathbb{L}_{\mathfrak{X}}[n]$$

which is equivalence to a morphism from tangent complex to n-shifted shifted cotangent complex

$$\Theta_\omega : \mathbb{T}_{\mathfrak{X}} \to \mathbb{L}_{\mathfrak{X}}[n].$$

Definition. Let $\mathfrak{X}$ be a derived algebraic stack. An n-shifted symplectic structure on $\mathfrak{X}$ is a closed 2-form $\omega \in \mathcal{A}^{p,cl}(\mathfrak{X}, n)$ of degree n such that the underlying 2-form induces an equivalence

$$\Theta_\omega : \mathbb{T}_{\mathfrak{X}} \xrightarrow{\sim} \mathbb{L}_{\mathfrak{X}}[n]$$

A derived algebraic stack $\mathfrak{X}$ with an n-shifted symplectic structure $\omega_{\mathfrak{X}}$ is called an n-shifted symplectic stack and we denote it by $(\mathfrak{X}, \omega_{\mathfrak{X}})$. Given a pair of n-shifted symplectic stacks $(\mathfrak{X}, \omega_{\mathfrak{X}})$, $(\mathfrak{Y}, \omega_{\mathfrak{Y}})$, there is an induced n-shifted symplectic structure on $\mathfrak{X} \times \mathfrak{Y}$ given by

$$\omega_{\mathfrak{X}} \boxplus \omega_{\mathfrak{Y}} = \mathrm{pr}_1^\star \omega_{\mathfrak{X}} + \mathrm{pr}_2^\star \omega_{\mathfrak{Y}}$$

where $\mathrm{pr}_1 : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X}$, $\mathrm{pr}_2 : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Y}$ are the projections. We now state the main existence theorem for shifted symplectic structures.

Theorem 3.1.4 ([PTVV13, Theorem 2.5])**.** Let X be a smooth projective Calabi-Yau n-fold and let $\mathfrak{Y}$ be a d-shifted symplectic derived Artin stack. Then the derived mapping stack

$$\underline{\mathrm{dMap}}(X, \mathfrak{Y}) : (\mathrm{dSch}/\mathbb{C}) \to \mathbb{S}, T \mapsto \mathrm{Map}(X \times T, \mathfrak{Y})$$

carries a $(d-n)$-shifted symplectic structure from $\mathcal{Y}$.

Corollary 3.1.5. The derived moduli stack $\mathrm{Perf}(X)$ of perfect complexes on a Calabi–Yau threefold X carries a (-1)-shifted symplectic structure.

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Proof. This is because the derived stack of perfect complexes on X can be expressed as the derived mapping stack

$$\mathrm{Perf}(X) \simeq \underline{\mathrm{dMap}}(X, \mathrm{Perf}),$$

where Perf denotes the derived moduli stack of perfect complexes. It is shown in [PTVV13] that Perf carries a canonical 2-shifted symplectic structure, so the claim follows from the previous theorem. $\square$

3.1.2 d-critical structures

The notion of a d-critical structure on an algebraic stack was introduced in [BBBBJ15]. They defined a canonical étale sheaf $\mathcal{S}_X$ for each algebraic space X and showed that $\mathcal{S}_X$ can be decompose into $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{C}$. For a morphism $f : X \to Y$ there is a pullback map $f^\star : f^{-1}\mathcal{S}_Y^0 \to \mathcal{S}_X^0$.

Definition. Let X be an algebraic space and $s \in \Gamma(X, \mathcal{S}_X^0)$ a section. An étale d-critical chart for (X, s) is an étale morphism $\eta : R \to X$, a smooth scheme U, a closed embedding $i : R \hookrightarrow U$ with ideal sheaf I of $i^{-1}\mathcal{O}_U$ defining R and a morphism $f : U \to \mathbb{A}^1$ on U with $f|_{i(R)^{red}} = 0$, $i(R) = \mathrm{Crit}(f)$ and $f + I^2 = s|_R$. We denote a d-critical chart as above by (R, η, U, f, i).

Definition ([Joy15, Definition 2.5])**.** Let X be an algebraic space and $s \in \Gamma(X, \mathcal{S}_X^0)$ a section. The section s is called a d-critical structure on X if for each point $x \in X$ there exists a d-critical chart (R, η, U, f, i) such that the image of η contains x.

Given a smooth morphism $f : X \to Y$ of algebraic space and a d-critical structure $s \in \Gamma(Y, \mathcal{S}_Y^0)$. There is a pullback d-critical structure $f^\star s$ on X [Joy15, Proposition 2.8]. Using this notion we may extend the definition of d-critical structure on algebraic stacks.

Definition. Given an algebraic stack $\mathcal{X}$. The sheaf $\mathcal{S}_{\mathcal{X}}^0$ is defined to be the assignment

$$T \mapsto \Gamma(T, \mathcal{S}_T^0)$$

in the lisse-étale topos on $\mathcal{X}$. A d-critical structure on $\mathcal{X}$ is a section $s \in \Gamma(\mathcal{X}, \mathcal{S}_{\mathcal{X}}^0)$ such that $s|_T \in \Gamma(T, \mathcal{S}_T^0)$ is a d-critical structure for each smooth morphism $T \to \mathcal{X}$.

For an algebraic stack $\mathcal{X}$ equipped with a d-critical structure $s \in \Gamma(\mathcal{X}, \mathcal{S}_{\mathcal{X}}^0)$, we refer to the pair $(\mathcal{X}, s)$ as a d-critical stack. A morphism of d-critical stacks $f : (X, s) \to (Y, t)$ is a morphism f such that $f^\star t = s$.

Proposition 3.1.6 ([Joy15, Proposition 2.11])**.** Let $(\mathcal{X}, s)$, $(\mathcal{Y}, t)$ be d-critical stacks, $\mathrm{pr}_1 : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\mathrm{pr}_2 : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ be projections, set $s \boxplus t := \mathrm{pr}_1^\star s + \mathrm{pr}_2^\star t$. Then the product $(\mathcal{X} \times \mathcal{Y}, s \boxplus t)$ is a d-critical stack.

Let $\mathfrak{X}$ be a (-1)-shifted symplectic stack with classical truncation $\mathcal{X} = \mathfrak{X}_{\mathrm{cl}}$ an algebraic stack. There is an induced d-critical structure on the classical truncation $\mathcal{X}$. More precisely, the following theorem was proved by [BBBBJ15]

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Theorem 3.1.7 ([BBBBJ15, Theorem 3.18])**.** There is a truncation functor

$$F : \left\{ \begin{array}{l} \infty\text{-category of } (-1)\text{-shifted} \\ \text{symplectic stacks } (\mathfrak{X}, \omega_{\mathfrak{X}}) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} 2\text{-category of} \\ \text{d-critical stacks } (\mathcal{X}, s) \end{array} \right\}.$$

3.2 Orientation

Definition. Given a (-1)-shifted symplectic stack $(\mathfrak{X}, \omega_{\mathfrak{X}})$ over $\mathbb{C}$. An orientation of $\mathfrak{X}$ is a line bundle $K_{\mathfrak{X}}^{\frac{1}{2}} \to \mathfrak{X}$, together with an isomorphism $o_{\mathfrak{X}} : (K_{\mathfrak{X}}^{\frac{1}{2}})^{\otimes 2} \cong K_{\mathfrak{X}}$, where $K_{\mathfrak{X}}$ is defined to be the determinant line bundle of the cotangent complex of $\mathfrak{X}$, which is called the canonical bundle of $\mathfrak{X}$. We called an (-1)-shifted symplectic stack with an orientation an oriented (-1)-shifted symplectic stack and denote it by $(\mathfrak{X}, \omega_{\mathfrak{X}}, o_{\mathfrak{X}})$.

Given a pair of oriented (-1)-shifted symplectic stacks $(\mathfrak{X}, \omega_{\mathfrak{X}}, o_{\mathfrak{X}}), (\mathfrak{Y}, \omega_{\mathfrak{Y}}, o_{\mathfrak{Y}})$, there is an induced orientation $\mathfrak{X} \times \mathfrak{Y}$ given by the isomorphism

$$o_{\mathfrak{X}\times\mathfrak{Y}} : (K_{\mathfrak{X}}^{\frac{1}{2}} \boxtimes K_{\mathfrak{Y}}^{\frac{1}{2}})^{\otimes 2} \cong K_{\mathfrak{X}} \boxtimes K_{\mathfrak{Y}} \cong K_{\mathfrak{X}\times\mathfrak{Y}}.$$

Remark 3.2.1. In [BBBBJ15], the notion of the canonical line bundle $K_{\mathcal{X}}$ for a d-critical stack $(\mathcal{X}, s)$ is also defined. It is further shown that if $(\mathcal{X}, s)$ arises as the classical truncation of a (-1)-shifted symplectic stack $\mathfrak{X}$, there is a canonical isomorphism

$$K_{\mathfrak{X}}|_{\mathcal{X}^{\text{red}}} \cong K_{\mathcal{X}}|_{\mathcal{X}^{\text{red}}}$$

on the reduced underlying stack. For simplicity, we write $K_{\mathcal{X}}$ to denote this restriction, even though it is technically a line bundle on $\mathcal{X}^{\text{red}}$.

Definition. The orientation on a d-critical stacks $(\mathcal{X}, s)$ is defined to be a line bundle $K_{\mathcal{X}}^{\frac{1}{2}}$ on $\mathcal{X}^{\text{red}}$ together with an isomorphism $o : (K_{\mathcal{X}}^{\frac{1}{2}})^{\otimes 2} \cong K_{\mathcal{X}}$. An oriented d-critical stack is a d-critical stack $(\mathcal{X}, s)$ together with an orientation o, we denote it by $(\mathcal{X}, s, o)$.

Lemma 3.2.2 ([Joy15, Lemma 2.58])**.** Given a smooth morphism $g : \mathcal{Y} \to \mathcal{X}$ of d-critical stacks. If g is compatible with the d-critical structures. Then an orientation $K_{\mathcal{X}}^{\frac{1}{2}}$ of $\mathcal{X}$ induces an orientation of $\mathcal{Y}$ given by

$$K_{\mathcal{Y}}^{\frac{1}{2}} = g^*(K_{\mathcal{X}}^{\frac{1}{2}}) \otimes \det(\mathbb{L}_{\mathcal{Y}/\mathcal{X}})|_{\mathcal{Y}^{\text{red}}}, g^\star o : (K_{\mathcal{Y}}^{\frac{1}{2}})^{\otimes 2} \cong K_{\mathcal{Y}}$$

Proposition 3.2.3 ([KPS24, Proposition 3.34], [Joy15, Theorem 2.56])**.** Let $(\mathcal{X}, s, o_{\mathcal{X}})$, $(\mathcal{Y}, t, o_{\mathcal{Y}})$ be oriented d-critical stacks. Then

(i) There is a natural isomorphism of canonical bundles

$$K_{\mathcal{X}} \boxtimes K_{\mathcal{Y}} \cong K_{\mathcal{X}\times\mathcal{Y}}$$

where the d-critical structures on $\mathcal{X} \times Y$ is given by $s \boxplus t$.

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(ii) There is an induced orientation on $\mathcal{X} \times \mathcal{Y}$ given by

$$o_{\mathcal{X}} \boxtimes o_{\mathcal{Y}} : (K_{\mathcal{X}}^{\frac{1}{2}} \boxtimes K_{\mathcal{Y}}^{\frac{1}{2}})^{\otimes 2} \cong K_{\mathcal{X}} \boxtimes K_{\mathcal{Y}} \cong K_{\mathcal{X}\times\mathcal{Y}}.$$

3.3 Results on the moduli stack of objects on a Calabi Yau threefold

In this section we focus on the moduli stack of perfect complex on a Calabi-Yau threefold. Let $\mathfrak{X}$ be the derived moduli stack of perfect complexes on a Calabi–Yau 3-fold equipped with a (-1)-shifted symplectic structure $\omega_{\mathfrak{X}}$ given by Corollary 3.1.5. The existence of an orientation on the moduli stack of perfect complexes on a Calabi - Yau threefold is ensured by the following theorem.

Theorem 3.3.1 ([JU21, Theorem 3.6])**.** There is an orientation $o : (K_{\mathfrak{X}}^{\frac{1}{2}})^{\otimes 2} \cong K_{\mathfrak{X}}$ on $\mathfrak{X}$ such that it is compatible with direct sums in the following sense: Let $\Phi_2 : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be the morphism corresponded to the direct sum of perfect complexes on X. Then there is an isomorphism

$$\lambda : \Phi_2^{\star} o \cong o \boxtimes o.$$

Theorem 3.3.2 ([KPS24, Corollary 8.19])**.** The morphism $\Phi_2 : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is compatible with (-1)-shifted symplectic structures, i.e. we have an equivalence

$$\Phi_2^{\star}\omega_{\mathfrak{X}} \sim \omega_{\mathfrak{X}} \boxplus \omega_{\mathfrak{X}}$$

The following proposition gives open substack of $\mathfrak{X}_{\mathrm{cl}}$ a natural derived structure.

Proposition 3.3.3 ([STV15, Proposition 2.1])**.** Let $\mathfrak{X}$ be a derived stack and let $\mathcal{X} := \mathfrak{X}_{\mathrm{cl}}$ denote its classical truncation. The truncation functor induces a one-to-one correspondence

$$\phi_{\mathrm{cl}} \ : \ \{\text{Zariski-open derived substacks of } \mathfrak{X}\} \xrightarrow{\sim} \{\text{Zariski-open substacks of } \mathcal{X}\}.$$

In other words, every Zariski-open substack of $\mathcal{X}$ carries a unique derived enhancement coming from $\mathfrak{X}$.

By this proposition, all of our moduli stacks carries a natural derived structure from $\mathfrak{X}$. Thus an oriented (-1)-shifted symplectic structure on $\mathfrak{X}$ induced an oriented (-1)-shifted symplectic structure on $\mathcal{O}bj(\mathcal{A}_X)$ that is compatible with direct sum.

Corollary 3.3.4. Let $\mathcal{O}bj(\mathcal{A}_X)$ be the moduli stack of objects in $\mathcal{A}_X$. There is an induced orientation on $\mathcal{O}bj(\mathcal{A}_X)$ that is compatible with direct sum and a d-critical structure that is compatible with the morphism $\mathcal{O}bj(\mathcal{A}_X) \times \mathcal{O}bj(\mathcal{A}_X) \to \mathcal{O}bj(\mathcal{A}_X)$.

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Chapter 4 Motivic invariants and Integral identity

In this chapter we recall motivic invariants for algebraic stacks, motivic Behrend functions and Bu's Motivic integral identity for (-1)-shifted symplectic stacks [Bu24].

4.1 Rings of motives

We begin by reviewing the fundamental notion of motive rings over an algebraic stack, following [Bu24, BNK25]; for the original reference, see [Joy07b].

4.1.1 Rings of Motives for stacks

We first define the Grothendieck ring of varieties over an algebraic stack $\mathcal{X}$. Recall that, by convention, the stack $\mathcal{X}$ is locally of finite type with affine stabilizers.

Definition. The Grothendieck ring of varieties over $\mathcal{X}$ is defined to be the abelian group

$$K_{\text{var}}(\mathcal{X}) = \hat{\bigoplus_{Z \to \mathcal{X}}} \mathbb{Q} \cdot [Z \to \mathcal{X}] / \sim,$$

where we sum over all isomorphism classes of morphisms $Z \to \mathcal{X}$, where Z are varieties and $\hat{\bigoplus}$ means we allow infinite sum $\sum_Z n_Z [Z \to \mathcal{X}]$ but for any quasi-compact open substack $\mathcal{U} \subset \mathcal{X}$, there are only finitely many Z such that $n_Z \neq 0$ and $Z \times_{\mathcal{X}} \mathcal{U} \neq \varnothing$. The relation is generated by $[Z] \sim [Z'] + [Z \setminus Z']$ for closed subschemes $Z' \subset Z$.

The Grothendieck ring of varieties carries a ring structure by taking fibre product over $\mathcal{X}$ on generators. It is also a commutative $K_{\text{var}}(\mathbb{C})$-algebra, where $K_{\text{var}}(\mathbb{C}) := K_{\text{var}}(\text{Spec}(\mathbb{C}))$, with the action induced by the homomorphism

$$K_{\text{var}}(\mathbb{C}) \to K_{\text{var}}(\mathcal{X}), [Z] \mapsto [Z \times \mathcal{X} \to \mathcal{X}].$$

We denote $\mathbb{L} = [\mathbb{A}^1] \in K_{\text{var}}(\mathbb{C})$.

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Definition. We consider the following localization

$$\mathbb{M}(\mathcal{X}) = K_{\text{var}}(\mathcal{X})\hat{\otimes}_{K_{\text{var}}(\mathbb{C})}K_{\text{var}}(\mathbb{C})[\mathbb{L}^{-1}]/(\mathbb{L}-1)\text{-torsion}$$
$$\hat{\mathbb{M}}(\mathcal{X}) = K_{\text{var}}(\mathcal{X})\hat{\otimes}_{K_{\text{var}}(\mathbb{C})}K_{\text{var}}(\mathbb{C})[\mathbb{L}^{-1}, (\mathbb{L}^k-1)^{-1}],$$

where we invert $\mathbb{L}^k - 1$ for all $k \geq 1$ and $\hat{\otimes}$ means we allow infinite sum $\sum_{Z\to\mathcal{X}}[Z \to \mathcal{X}] \otimes n_Z$, but for any quasi-compact open substack $\mathcal{U} \subset \mathcal{X}$, there are only finitely many Z such a that $n_Z \neq 0$ and $Z \times_{\mathcal{X}} \mathcal{U} \neq \varnothing$.

For a finite type morphism $\mathcal{Y} \to \mathcal{X}$ of algebraic stack, where $\mathcal{Y}$ has affine stabilizers one can associate a class $[\mathcal{Y} \to \mathcal{X}] \in \hat{\mathbb{M}}(\mathcal{X})$, which agrees with the usual one when $\mathcal{Y}$ is just a variety. The construction is as follows: for such $\mathcal{Y}$ there is a stratification of $\mathcal{Y}$ by locally closed substack $\mathcal{Y} = \coprod_i \mathcal{Y}_i$ such that $\mathcal{Y}_i = [U_i/\mathrm{GL}(n_i)]$, where U_i is a quasi affine scheme with a $\mathrm{GL}(n_i)$ action, thus we define

$$[\mathcal{Y}] := \sum_i \frac{1}{[\mathrm{GL}(n_i)]} \cdot [U_i] \in \hat{\mathbb{M}}(\mathcal{X}),$$

where $[\mathrm{GL}(n_i)] = \prod_{k=0}^{n_i-1}(\mathbb{L}^{n_i} - \mathbb{L}^k)$. It is easy to see that this definition does not depend on the choice of the stratification by passing through a common refinement.

Definition (Pullback)**.** For a morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks with affine stabilizers. We define a pullback map on generators

$$f^* : \hat{\mathbb{M}}(\mathcal{X}) \to \hat{\mathbb{M}}(\mathcal{Y}), \qquad [Z \to \mathcal{X}] \mapsto [Z \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}],$$

which is a $\hat{\mathbb{M}}(\mathbb{C})$-algebra homomorphism.

Definition (Pushforward)**.** Let $f : \mathcal{Y} \to \mathcal{X}$ be a finite type morphism, then we may also define an $\hat{\mathbb{M}}(\mathbb{C})$-algebra homomorphism

$$f_! : \hat{\mathbb{M}}(\mathcal{Y}) \to \hat{\mathbb{M}}(\mathcal{X}), \qquad [Z \to \mathcal{Y}] \mapsto [Z \to \mathcal{Y} \to \mathcal{X}].$$

In particular, if $\mathcal{X}$ itself is of finite type over Spec($\mathbb{C}$), we define the motivic integration to be the pushforward alone structure morphism $\mathcal{X} \to$ Spec($\mathbb{C}$), and denoted by

$$\int_{\mathcal{X}} : \hat{\mathbb{M}}(\mathcal{X}) \to \hat{\mathbb{M}}(\mathbb{C}).$$

We have Base change theorem and projection formula.

Proposition 4.1.1. Suppose we have a pullback diagram of algebraic stacks with affine stabilizers

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}' \arrow[d, "g"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

where f is of finite type. Then

(a) (Base change formula) We have $g^* \circ f_! = f'_! \circ g'^*$,

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(b) (Projection formula) For any $a \in \hat{\mathbb{M}}(\mathcal{Y})$ and $b \in \hat{\mathbb{M}}(\mathcal{X})$ we have

$$f_!(a \cdot f^*(b)) = f_!(a) \cdot b.$$

Proof. These can be verified on generators. We first prove (a). Let $[Z \to \mathcal{Y}] \in \hat{\mathbb{M}}(\mathcal{Y})$. Then

$$\begin{aligned}(g^* \circ f_!)([Z \to \mathcal{Y}]) &= g^*([Z \to \mathcal{Y} \to \mathcal{X}]) \\ &= [Z \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{X}'] \\ &= [Z \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}' \to \mathcal{X}'] \\ &= (f'_! \circ g'^*)([Z \to \mathcal{Y}]).\end{aligned}$$

Now we prove (b). Let $a = [Z \to \mathcal{Y}] \in \hat{\mathbb{M}}(\mathcal{Y})$, $b = [W \to \mathcal{X}] \in \hat{\mathbb{M}}(\mathcal{X})$.

$$\begin{aligned}f_!(a \cdot f^*(b)) &= f_!([Z \times_{\mathcal{Y}} W \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}]) \\ &= [Z \times_{\mathcal{Y}} W \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y} \to \mathcal{X}] \\ &= [Z \times_{\mathcal{X}} W \to \mathcal{X}] = f_!(a) \cdot b.\end{aligned}$$

□

4.1.2 Motives with monodromic action

Let $\hat{\mu} = \lim \mu_n$ be the projective limit of the groups μ_n for roots of unity. We consider the ring $\hat{\mathbb{M}}^{\text{mon}}(\mathcal{X})$ of motives with monodromic action on $\mathcal{X}$ in the sense of [Bu24]. For a variety Z, a good $\hat{\mu}$-action is a $\hat{\mu}$-action on Z that factors through a μ_n-action for some n and such that each orbit is contained in an open affine subscheme.

Definition. The monodromic Grothendieck ring of varieties of $\mathcal{X}$ is defined to be the abelian group

$$K_{\text{var}}^{\text{mon}}(\mathcal{X}) = \hat{\bigoplus_{Z \to \mathcal{X}}} \mathbb{Q} \cdot [Z \to \mathcal{X}]^{\hat{\mu}} / \sim,$$

where we run through all morphisms $Z \to \mathcal{X}$ with Z a variety with a good $\hat{\mu}$-action that is compatible with the trivial $\hat{\mu}$-action on $\mathcal{X}$.

The ring $K_{\text{var}}^{\text{mon}}(\mathcal{X})$ has a multiplication structure different from the one given by fiber product. Nevertheless, there is a natural ring homomorphism

$$\iota : K_{\text{var}}(\mathcal{X}) \to K_{\text{var}}^{\text{mon}}(\mathcal{X})$$

given by equipping the varieties with trivial $\hat{\mu}$-action. Moreover, there is an element

$$\mathbb{L}^{\frac{1}{2}} := 1 - [\mu_2]^{\hat{\mu}} \in K_{\text{var}}^{\text{mon}}(\mathbb{C})$$

with the nontrivial μ_2-action. It satisfies $(\mathbb{L}^{\frac{1}{2}})^2 = \mathbb{L}$.

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Definition. The monodromic ring of motives on $\mathcal{X}$ is defined to be the localization

$$\mathbb{M}^{\mathrm{mon}}(\mathcal{X}) = K^{\mathrm{mon}}_{\mathrm{var}}(\mathcal{X})\hat{\otimes}_{K_{\mathrm{var}}(\mathbb{C})} K_{\mathrm{var}}(\mathbb{C})[\mathbb{L}^{-1}]/(\mathbb{L}-1)\text{-torsion},$$
$$\hat{\mathbb{M}}^{\mathrm{mon}}(\mathcal{X}) = K^{\mathrm{mon}}_{\mathrm{var}}(\mathcal{X})\hat{\otimes}_{K_{\mathrm{var}}(\mathbb{C})} K_{\mathrm{var}}(\mathbb{C})[\mathbb{L}^{-1}, (\mathbb{L}^k-1)^{-1}]/\approx,$$

where the relation $\approx$ is defined in [BBBBJ15, Definition 5.5].

Now we define the motives of double covers.

Definition. Let $\mathcal{P} \to \mathcal{X}$ be a principal μ_2-bundle, there is a class

$$\Upsilon(\mathcal{P}) = \mathbb{L}^{-\frac{1}{2}}([\mathcal{X}] - [\mathcal{P}]^{\hat{\mu}}) \in \hat{\mathbb{M}}^{\mathrm{mon}}(\mathcal{X}),$$

where $\hat{\mu}$ acts on $\mathcal{P}$ via μ_2. See [BBBBJ15, Definition 5.5, 5.13] for details.

Note that Υ commute with pullback and satisfies the relation

$$\Upsilon(P \otimes_{\mathbb{Z}/2\mathbb{Z}} Q) = \Upsilon(P)\cdot\Upsilon(Q),$$

for principal $\mathbb{Z}/2\mathbb{Z}$-bundles P and Q.

4.2 Motivic Behrend Functions

The Behrend function $\nu_X \colon X \to \mathbb{Z}$ plays a pivotal role in the categorification of Donaldson–Thomas (DT) theory. In his work [Beh09], Behrend proved that if a scheme X is equipped with a symmetric obstruction theory, then the degree of its virtual fundamental class can be expressed as the weighted Euler characteristic

$$\int_{[X]^{\mathrm{vir}}} 1 \; = \; \chi\big(X, \nu_X\big) := \sum_{n\in\mathbb{Z}} n\cdot\chi(\nu_X^{-1}(n)).$$

From the perspective of DT theory, this result upgrades the theory from counting with a single number to counting with a constructible function: the Behrend function itself already constitutes a first-level categorification of DT invariants. Consequently, in the motivic and cohomological refinements of DT theory, the guiding principle is to further categorify the Behrend function, replacing ν_X by perverse sheaves, mixed Hodge modules, or motivic classes so that the resulting invariants capture increasingly deep geometric information. We now explain how this categorification is implemented in the motivic setting.

Let (X, s, o) be an oriented d-critical scheme over $\mathbb{C}$. Attached to the tuple is the motivic Behrend function $\nu^{\mathrm{mot}}_{X,s,o}$ defined by Bussi-Joyce-Meinhardt [BJM19] which we recall below.

Definition. Let (X, s, o) be an oriented d-critical scheme. Its motivic Behrend function $\nu^{\mathrm{mot}}_X \in \hat{\mathbb{M}}^{\mathrm{mon}}(X)$ is defined uniquely by the following property:

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- For any critical chart $i : \mathrm{Crit}(f) \hookrightarrow X$ where $f : U \to \mathbb{A}^1$ is a regular function on a smooth scheme U, there is an identity

$$i^*\nu_X^{\mathrm{mot}} = -\mathbb{L}^{-\frac{\dim U}{2}} \cdot \Phi_f([U]) \cdot \Upsilon(i^*(K_X^{\frac{1}{2}}) \otimes K_U^{-1}|_{\mathrm{Crit}(f)^{\mathrm{red}}}).$$

Here Φ_f is the motivic vanishing cycle and we identify principal μ_2-bundles with line bundles which square to trivial; see [Bu24, Definition 2.5.3] and [BBBBJ15, Theorem 5.7] for details.

Roughly speaking, the motivic Behrend function is defined locally via vanishing cycles together with local isomorphism data determined by the chosen orientation.

Note that the motivic Behrend function depends on the orientation and the d-critical structure. We will simply denote it by ν_X^{mot} if there is no ambiguity. The definition can be extend to algebraic stacks with orientation and d-critical structure.

Theorem 4.2.1 ([Bu24, Theorem 2.5.4])**.** Let $(\mathcal{X}, s, o)$ be an oriented d-critical stack that is Nisnevich-locally a quotient stack. Then there exists a unique class

$$\nu_{\mathcal{X}}^{\mathrm{mot}} = \nu_{\mathcal{X},s,o}^{\mathrm{mot}} \in \widehat{\mathbb{M}}^{\mathrm{mon}}(\mathcal{X}),$$

called the motivic Behrend function of $\mathcal{X}$, characterized by the following property: for any variety Y and any smooth morphism $f : Y \to \mathcal{X}$ of relative dimension d, we have

$$f^*\big(\nu_{\mathcal{X}}^{\mathrm{mot}}\big) = \mathbb{L}^{d/2}\,\nu_Y^{\mathrm{mot}} \in \widehat{\mathbb{M}}^{\mathrm{mon}}(Y),$$

where Y is equipped with the induced oriented d-critical structure.

For an oriented (-1)-shifted symplectic stack $(\mathfrak{X}, \omega_{\mathfrak{X}}, o)$, we write $\nu_{\mathfrak{X}}^{\mathrm{mot}} := \nu_{\mathcal{X}}^{\mathrm{mot}}$, where $\mathcal{X} = \mathfrak{X}_{\mathrm{cl}}$ is its classical truncation equipped with the induced orientation and d-critical structure.

Theorem 4.2.2 ([Bu24, Theorem 2.5.5], [BBBBJ15, Theorem 5.14])**.** Let $\mathcal{X}$, $\mathcal{Y}$ be oriented d-critical stacks that are Nisnevich locally quotient stacks, and let $f : \mathcal{Y} \to \mathcal{X}$ be a smooth morphism of relative dimension d which is compatible with the orientation and d-critical structures. Then there is an identity

$$f^*(\nu_{\mathcal{X}}^{\mathrm{mot}}) = \mathbb{L}^{d/2} \cdot \nu_{\mathcal{Y}}^{\mathrm{mot}} \in \hat{\mathbb{M}}^{\mathrm{mon}}(\mathcal{Y}).$$

The following theorem gives a formula for motivic Behrend function on the product, though it is a direct consequence of the Thom-Sebastiani Theorem for motivic vanishing cycle, we provide a proof here. In the proof we will use the Thom-Sebastiani Theorem for motivic vanishing cycle [BJM19, Theorem 2.4].

Theorem 4.2.3 (Thom-Sebastiani Theorem for Motivic Behrend function)**.**

Let $(\mathcal{X}, s, o_{\mathcal{X}})$, $(\mathcal{Y}, t, o_{\mathcal{Y}})$ be oriented d-critical stacks. Consider the induced oriented d-critical structure on the product $(\mathcal{X} \times \mathcal{Y}, s \boxplus t, o_{\mathcal{X}} \boxtimes o_{\mathcal{Y}})$. Then we have

$$\nu_{\mathcal{X}}^{\mathrm{mot}} \boxtimes \nu_{\mathcal{Y}}^{\mathrm{mot}} = \nu_{\mathcal{X}\times\mathcal{Y}}^{\mathrm{mot}} \in \hat{\mathbb{M}}^{\mathrm{mon}}(\mathcal{X} \times \mathcal{Y}).$$

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Proof. We first prove the theorem when $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces. Since algebraic spaces are Nisnevich locally covered by schemes and our motives satisfy Nisnevich descent [Bu24, Theorem 2.2.3], we may consider the case when $\mathcal{X} = X$, $\mathcal{Y} = Y$ are schemes.

We now verify this on d-critical charts. Let $i : \mathrm{Crit}(f) \hookrightarrow X$, $j : \mathrm{Crit}(g) \hookrightarrow Y$ be d-critical chart, where $f : U \to \mathbb{A}^1$, $g : V \to \mathbb{A}^1$ are regular functions. Then we consider the d-critical chart $i \times j : \mathrm{Crit}(f) \times \mathrm{Crit}(g) \hookrightarrow X \times Y$.

By the definition of motivic Behrend function (4.2) we have

$$(i\times j)^*\nu^{\mathrm{mot}}_{X\times Y} = \mathbb{L}^{-(\dim U\times V)/2}\Phi_{f\boxplus g}([U\times V])\Upsilon((i\times j)^*(K^{\frac{1}{2}}_{X\times Y})\otimes K^{-1}_{U\times V}|_{\mathrm{Crit}(f\boxplus g)^{\mathrm{red}}})$$

The Thom-Sebastiani Theorem for motivic vanishing cycles [BJM19, Theorem 2.4] yields

$$\Phi_f(U)\boxtimes\Phi_g(V) = \Phi_{f\boxplus g}(U\times V)$$

and the product orientation on $X\times Y$ is defined by $K^{\frac{1}{2}}_{X\times Y} = K^{\frac{1}{2}}_X \boxtimes K^{\frac{1}{2}}_Y$. Thus

$$\begin{aligned}
&\mathbb{L}^{-(\dim U\times V)/2}\Phi_{f\boxplus g}([U\times V])\Upsilon((i\times j)^*(K^{\frac{1}{2}}_{X\times Y})\otimes K^{-1}_{U\times V}|_{\mathrm{Crit}(f\boxplus g)^{\mathrm{red}}})\\
=&\mathbb{L}^{-\dim U/2}\Phi_f([U])\Upsilon(i^*(K^{\frac{1}{2}}_X)\otimes K^{-1}_X|_{\mathrm{Crit}(f)^{\mathrm{red}}})\boxtimes\mathbb{L}^{-\dim V/2}\Phi_g([V])\Upsilon(j^*(K^{\frac{1}{2}}_Y)\otimes K^{-1}_Y|_{\mathrm{Crit}(g)^{\mathrm{red}}})\\
=&i^*\nu^{\mathrm{mot}}_X\boxtimes j^*\nu^{\mathrm{mot}}_Y.
\end{aligned}$$

Now we consider the general case, by assumption the stacks $\mathcal{X}$ and $\mathcal{Y}$ are Nisnevich locally quotient stacks. We consider the case $\mathcal{X} = [X/\mathrm{GL}(n)]$ and $\mathcal{Y} = [Y/\mathrm{GL}(m)]$ when X and Y are algebraic space with orientations and d-critical structures. Let $p : X \to [X/\mathrm{GL}(n)]$, $q : Y \to [Y/\mathrm{GL}(m)]$ be quotient maps. By the construction of motivic Behrend function on stacks,

$$\nu^{\mathrm{mot}}_{\mathcal{X}} = [\mathrm{GL}(n)]^{-1}p_!(\mathbb{L}^{n^2/2}\cdot\nu^{\mathrm{mot}}_X), \nu^{\mathrm{mot}}_{\mathcal{Y}} = [\mathrm{GL}(m)]^{-1}q_!(\mathbb{L}^{m^2/2}\cdot\nu^{\mathrm{mot}}_Y)$$

We finally get

$$\begin{aligned}
\nu^{\mathrm{mot}}_{\mathcal{X}}\boxtimes\nu^{\mathrm{mot}}_{\mathcal{Y}} &= [\mathrm{GL}(n)\times\mathrm{GL}(m)]^{-1}(p\times q)_!(\mathbb{L}^{m^2+n^2/2}\cdot\nu^{\mathrm{mot}}_X\boxtimes\nu^{\mathrm{mot}}_Y)\\
&= [\mathrm{GL}(n+m)]^{-1}\cdot(p\times q)_!(\mathbb{L}^{(m+n)^2/2}\nu^{\mathrm{mot}}_{X\times Y}) = \nu^{\mathrm{mot}}_{\mathcal{X}\times\mathcal{Y}}.
\end{aligned}$$

□

4.3 Graded and Filtered objects

We introduce the notion of Graded objects and Filtered objects following [HL14].

Definition (Mapping stack)**.** Let $\mathcal{X}, \mathcal{Y}$ be stacks over some site, we define the mapping stack

$$\begin{aligned}
\underline{\mathrm{Map}}(\mathcal{Y},\mathcal{X}) :&(\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Gpd})\\
&T \mapsto \mathrm{Map}(\mathcal{Y}\times T,\mathcal{X})
\end{aligned}$$

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Definition. Let $\mathcal{X}$ be an algebraic stack over $\mathbb{C}$. The stack of graded objects in $\mathcal{X}$ is defined as the mapping stack over $\operatorname{Spec}(\mathbb{C})$

$$\begin{aligned}\operatorname{Grad}(\mathcal{X}) &:= \underline{\operatorname{Map}}([*/\mathbb{G}_m], \mathcal{X}) \\ \operatorname{Filt}(\mathcal{X}) &:= \underline{\operatorname{Map}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X})\end{aligned}$$

where, $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by weight -1.

These stacks are again algebraic stack by the following porposition.

Proposition 4.3.1 ([HL14, Proposition 1.1.2])**.** Let $\mathcal{X}$ be an algebraic stack over $\mathbb{C}$. Then $\operatorname{Grad}(\mathcal{X})$ and $\operatorname{Filt}(\mathcal{X})$ are also algebraic. Moreover, if $\mathcal{X}$ has affine stabilizers, then so do $\operatorname{Grad}(\mathcal{X})$ and $\operatorname{Filt}(\mathcal{X})$.

We now consider the following diagram

$$\begin{tikzcd} {[*/\mathbb{G}_m]} \arrow[r, shift left, leftarrow, "\mathrm{pr}"] \arrow[r, shift right, "\iota"'] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[r, shift left, leftarrow, "0"] \arrow[r, shift right, leftarrow, "1"'] & * \arrow[ll, bend right, "q"'] \end{tikzcd}$$

where q is the quotient map, $0, 1$ are the compositions $0, 1 : * \to \mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m$, ι is the inclusion map and pr is the projection map. These morphisms induced morphisms between $\operatorname{Grad}(\mathcal{X})$, $\operatorname{Filt}(\mathcal{X})$ and $\mathcal{X}$ via pullback

$$\begin{tikzcd} \operatorname{Grad}(\mathcal{X}) \arrow[r, shift left, "\mathrm{sf}"] \arrow[rr, bend left, "\mathrm{tot}"] & \operatorname{Filt}(\mathcal{X}) \arrow[l, shift left, "\mathrm{gr}"] \arrow[r, shift left, "\mathrm{ev}_0"] \arrow[r, shift right, "\mathrm{ev}_1"'] & \mathcal{X} \end{tikzcd}$$

The stack of graded objects $\operatorname{Grad}(\mathfrak{X})$ inherits a natural orientation and a (-1)-shifted symplectic structure from $\mathfrak{X}$.

Theorem 4.3.2 ([Bu24, Theorem 3.1.6])**.** The derived stack $\operatorname{Grad}(\mathfrak{X})$ has an (-1) shifted symplectic structure given by $\mathrm{tot}^\star \omega_{\mathfrak{X}}$ and an orientation $o_{\operatorname{Grad}(\mathfrak{X})}$ induced from $\mathfrak{X}$.

4.3.1 Morphisms of stacks of graded and filtered objects

In this subsection, we recall several foundational results from [HL14, Section 1] and [Bu24, Section 3] about stacks of graded and filtered objects.

Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks, we may consider the induced morphism on their stacks of graded and filtered objects via composition with f.

$$\begin{aligned}\operatorname{Grad}(f) &:= (f \circ -) : \operatorname{Grad}(\mathcal{Y}) \longrightarrow \operatorname{Grad}(\mathcal{X}) \\ \operatorname{Filt}(f) &:= (f \circ -) : \operatorname{Filt}(\mathcal{Y}) \longrightarrow \operatorname{Filt}(\mathcal{X}).\end{aligned}$$

These maps inherit properties from the map f. Below, we state several results that will be needed later in this paper.

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Proposition 4.3.3 ([HL14, Corollary 1.1.7+1.1.8])**.** If $f : \mathcal{Y} \to \mathcal{X}$ is an open immersion, then $\mathrm{Grad}(\mathcal{Y}) \cong \mathcal{Y} \times_{\mathcal{X}} \mathrm{Grad}(\mathcal{X})$. In particular, $\mathrm{Grad}(f)$ is an open immersion.

Proposition 4.3.4 ([HL14], Proposition 1.3.1)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks and $f : \mathcal{Y} \to \mathcal{X}$ be a representable morphism. Then the induced morphism $\mathrm{Filt}(f) : \mathrm{Filt}(\mathcal{Y}) \to \mathrm{Filt}(\mathcal{X})$ is also representable. Furthermore:

(1) If f is an open immersion, then so is $\mathrm{Filt}(f)$, and $\mathrm{Filt}(f)$ identifies $\mathrm{Filt}(\mathcal{Y})$ with the preimage of $\mathcal{Y} \hookrightarrow \mathcal{X}$ under the composition

$$\mathrm{Filt}(\mathcal{X}) \xrightarrow{\mathrm{gr}} \mathrm{Grad}(\mathcal{X}) \xrightarrow{\mathrm{tot}} \mathcal{X}$$

(2) If f is smooth (respectively, étale), then so are $\mathrm{Filt}(f)$ and $\mathrm{Grad}(f)$, and this also holds even if f is not representable.

Proposition 4.3.5 ([Bu24], Lemma 3.2.4.)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks and $f : \mathcal{Y} \to \mathcal{X}$ be an étale morphism. Then there is an pullback diagram

$$\begin{tikzcd} \mathrm{Filt}(\mathcal{Y}) \arrow[r] \arrow[d, "\mathrm{gr}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Filt}(\mathcal{X}) \arrow[d, "\mathrm{gr}"] \\ \mathrm{Grad}(\mathcal{Y}) \arrow[r] & \mathrm{Grad}(\mathcal{X}) \end{tikzcd},$$

where both horizontal morphisms are induced from f.

Sketch of proof. It suffices to show the morphism

$$\mathrm{Filt}(\mathcal{Y}) \longrightarrow \mathrm{Filt}(\mathcal{X}) \times_{\mathrm{Grad}(\mathcal{X})} \mathrm{Grad}(\mathcal{Y})$$

is an isomorphism. By **Proposition** 4.3.4 the morphisms

$$\mathrm{Filt}(\mathcal{Y}) \longrightarrow \mathrm{Filt}(\mathcal{X}),\ \mathrm{Filt}(\mathcal{X}) \times_{\mathrm{Grad}(\mathcal{X})} \mathrm{Grad}(\mathcal{Y}) \longrightarrow \mathrm{Filt}(\mathcal{X})$$

are étale. so we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Filt}(\mathcal{Y}) \arrow[rr, "f"] \arrow[dr] & & \mathrm{Filt}(\mathcal{X}) \times_{\mathrm{Grad}(\mathcal{X})} \mathrm{Grad}(\mathcal{Y}) \arrow[dl] \\ & \mathrm{Filt}(\mathcal{X}) & \end{tikzcd}$$

where the horizontal morphism is also étale. Then the result follows from the fact that the vertical arrows are $\mathbb{A}^1$-action retracts and f is $\mathbb{A}^1$-equivariant and étale so f is an isomorphism [Bu24, Lemma 3.2.5]. $\square$

Corollary 4.3.6. Let $f : \mathcal{Y} \to \mathcal{X}$ be an open immersion. We have the following commu-

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tative diagram:

$$\begin{tikzcd}
\mathrm{Grad}(\mathcal{Y}) \arrow[d] & \mathrm{Filt}(\mathcal{Y}) \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}_1"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & \mathcal{Y} \arrow[d] \\
\mathrm{Grad}(\mathcal{X}) & \mathrm{Filt}(\mathcal{X}) \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}_1"] & \mathcal{X}
\end{tikzcd}$$

where all the vertical maps are open immersions.

4.3.2 Derived version

The stacks of grade objects and filtered objects has a natural derived enhancement. For derived stacks $\mathfrak{X}, \mathfrak{Y}$, we defined derived mapping stacks

$$\begin{aligned}\underline{\mathrm{dMap}}(\mathfrak{Y}, \mathfrak{X}) : (\mathrm{dSch}/\mathbb{C}) &\to \mathbb{S} \\ T &\mapsto \mathrm{Map}(\mathfrak{Y} \times T, \mathfrak{X})\end{aligned}$$

from the ∞-category of derived shcemes to the ∞-category of spaces.

For a derived stack $\mathfrak{X}$, we define

$$\mathrm{dFilt}(\mathfrak{X}) := \underline{\mathrm{dMap}}([\mathbb{A}^1/\mathbb{G}_m], \mathfrak{X}),\ \mathrm{dGrad}(\mathfrak{X}) := \underline{\mathrm{dMap}}([*/\mathbb{G}_m], \mathfrak{X}).$$

Note that given dervied stacks $\mathfrak{X}, \mathfrak{Y}$. The derived mapping stack $\underline{\mathrm{dMap}}(\mathfrak{Y}, \mathfrak{X})_{\mathrm{cl}}$ is not necessary the same as the mapping stack $\underline{\mathrm{Map}}(\mathfrak{Y}_{\mathrm{cl}}, \mathfrak{X}_{\mathrm{cl}})$. However, we have the following result

Proposition 4.3.7 ([HL14, Theorem 1.2.1])**.** Let $\mathfrak{X}$ be a derived algebraic stack, then

$$\mathrm{dFilt}(\mathfrak{X})_{\mathrm{cl}} = \mathrm{Filt}(\mathfrak{X}_{\mathrm{cl}}), \mathrm{dGrad}(\mathfrak{X})_{\mathrm{cl}} = \mathrm{Grad}(\mathfrak{X}_{\mathrm{cl}})$$

If there is no ambiguity we will just write $\mathrm{Filt}(\mathfrak{X})$ and $\mathrm{Grad}(\mathfrak{X})$ for $\mathrm{dFilt}(\mathfrak{X})$ and $\mathrm{dGrad}(\mathfrak{X})$ for a derived stack $\mathfrak{X}$.

4.4 Motivic Integral Identity

The motivic integral identity was first conjectured by Kontsevich and Soibelman in [KS08, Conjecture 4.] as a key ingredient in the formulation of their motivic wall-crossing formula. The version stated below is a generalization that applies to oriented (-1)-shifted symplectic stacks, formulated in terms of the stacks of graded and filtered objects.

Theorem 4.4.1 ([Bu24, Theorem 4.2.2])**.** Let $\mathfrak{X}$ be an oriented (-1)-shifted symplectic stack over $\mathbb{C}$ with classical truncation $\mathcal{X} = \mathfrak{X}_{\mathrm{cl}}$. Assume that $\mathcal{X}$ is Nisnevich locally fundamental. Consider the (-1)-shifted Lagrangian correspondence

$$\mathrm{Grad}(\mathfrak{X}) \stackrel{\mathrm{gr}}{\leftarrow} \mathrm{Filt}(\mathfrak{X}) \stackrel{\mathrm{ev}_1}{\rightarrow} \mathfrak{X}$$

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Then we have the identity

$$\mathrm{gr}_! \circ \mathrm{ev}_1^*(\nu_{\mathfrak{X}}^{\mathrm{mot}}) = \mathbb{L}^{\mathrm{vdimFilt}(\mathfrak{X})/2} \nu_{\mathrm{Grad}(\mathfrak{X})}^{\mathrm{mot}}$$

in $\hat{\mathbb{M}}^{\mathrm{mon}}(\mathrm{Grad}(\mathcal{X}))$, where the $\mathrm{vdimFilt}(\mathfrak{X})$ denote the virtual dimension of $\mathrm{Filt}(\mathfrak{X})$, seen as a function $\pi_0(\mathrm{Filt}(\mathfrak{X})) \cong \pi_0(\mathrm{Grad}(\mathfrak{X})) \to \mathbb{Z}$.

Remark 4.4.2. The proof of the motivic integral identity relies on explicit local computations and a gluing argument that uses Nisnevich-local fundamental covers. Consequently, the existence of such covers is essential. In our setting, we invoke Corollary 2.4.5 to guarantee the required Nisnevich-local structure.

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Chapter 5 Motivic Donaldson Thomas invariants

5.1 Donaldson Thomas type invariants

Fix a numerical type $v \in \Gamma^1$. We now define motivic DT and PT invariants, respectively. Let $\mathcal{O}bj^v(\mathcal{A}_X)$ be the algebraic stack that parametrizes objects E in $\mathcal{A}_X$ with $\mathrm{cl}(E) = v$. Consider open substacks $\mathcal{M}^v_{\mathrm{DT}}, \mathcal{M}^v_{\mathrm{PT}}, \mathcal{M}^{ss,v} \subset \mathcal{O}bj^v(\mathcal{A}_X)$ each equipped with the induced orientation and (-1)-shifted symplectic structure from the derived stack of perfect complexes on X given in Theorem 3.3.1 and Theorem 3.3.2.

Definition. We define

$$\mathrm{DT}^{\mathrm{mot}}_{n,\beta}(X) = \int_{\mathcal{M}^v_{\mathrm{DT}}} (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}})\nu^{\mathrm{mot}}_{\mathcal{M}^v_{\mathrm{DT}}} \in \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C})$$

$$\mathrm{PT}^{\mathrm{mot}}_{n,\beta}(X) = \int_{\mathcal{M}^v_{\mathrm{PT}}} (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}})\nu^{\mathrm{mot}}_{\mathcal{M}^v_{\mathrm{PT}}} \in \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C}).$$

Theorem 5.1.1. Assuming the good moduli space for $\mathcal{M}^{ss,v}$ exists for each v, we have the following identity:

$$\sum_{n,\beta} \mathrm{DT}^{\mathrm{mot}}_{n,\beta}(X) q^n x^\beta = \left(\sum_{n\geq 0} \mathrm{DT}^{\mathrm{mot}}_{n,0}(X) q^n\right) \left(\sum_{n,\beta} \mathrm{PT}^{\mathrm{mot}}_{n,\beta}(X) q^n x^\beta\right)$$

in $\hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C})[[\Gamma^1]]$.

5.2 Motivic Hall Algebra

We recall the construction of the motivic Hall algebra, originally defined by Joyce in [Joy06, Joy07a], and further developed in recent formulations by Bu, Kinjo, and Núñez [Bu25, BNK25]. See also [Bri10] for a nice introduction. Following [Tod10, Section 4], since the stack $\mathcal{O}bj(\mathcal{A}_X)$ is not known to be algebraic, we restrict ourselves to defining the following Hall module. Define

$$\mathcal{H}(\mathcal{A}_X) = \bigoplus_{v \in \Gamma^1 \cup \Gamma^0} \mathbb{M}(\mathcal{O}bj^v(\mathcal{A}_X)).,$$

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as a $\mathbb{M}(\mathbb{C})$-module. We consider the $\mathbb{M}(\mathbb{C})$-submodule

$$\mathcal{H}^0 := \mathcal{H}^0(\mathcal{A}_X) := \bigoplus_{v \in \Gamma^0} \mathbb{M}(\mathcal{O}bj^v(\mathcal{A}_X)).$$

The $\mathbb{M}(\mathbb{C})$-module $\mathcal{H}(\mathcal{A}_X)$ carries a $\mathcal{H}^0$-bimodule structure defined as follows:

Given elements$a = [\mathcal{X} \xrightarrow{f} \mathcal{O}bj^v(\mathcal{A}_X)] \in \mathcal{H}(\mathcal{A}_X)$ and $b = [\mathcal{Y} \xrightarrow{g} \mathcal{O}bj^w(\mathcal{A}_X)] \in \mathcal{H}^0$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] & \mathcal{E}x^{(v,w)}(\mathcal{A}_X) \arrow[r] \arrow[d] & \mathcal{O}bj^{v+w}(\mathcal{A}_X) \\
\mathcal{X} \times \mathcal{Y} \arrow[r, "(f,g)"] & \mathcal{O}bj^v(\mathcal{A}_X) \times \mathcal{O}bj^w(\mathcal{A}_X) &
\end{tikzcd} \tag{$*$}$$

where $\mathcal{E}x^{(v,w)}(\mathcal{A}_X)$ is the stack which parametrizes exact sequences

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

where E, G are objects in $\mathcal{A}_X$ such that cl$(E) \in \Gamma^1$, cl$(G) \in \Gamma^0$. The morphism

$$\mathcal{E}x^{(v,w)}(\mathcal{A}_X) \to \mathcal{O}bj^v(\mathcal{A}_X) \times \mathcal{O}bj^w(\mathcal{A}_X)$$

is given by sending an exact sequence as above to the pair (E, G) and the morphism $\mathcal{E}x^{(v,w)}(\mathcal{A}_X) \to \mathcal{O}bj^{v+w}(\mathcal{A}_X)$ sends the above exact sequence to F. We define the product as

$$a * b = [\mathcal{X} \xrightarrow{f} \mathcal{O}bj^v(\mathcal{A}_X)] * [\mathcal{Y} \xrightarrow{g} \mathcal{O}bj^w(\mathcal{A}_X)] := [\mathcal{Z} \xrightarrow{h} \mathcal{O}bj^{v+w}(\mathcal{A}_X)].$$

The product $b * a$ is defined in the same way. We call this the Hall algebra product. It has been shown that this process can be formulated using stacks of graded and filtered objects via the following pullback diagram.

Lemma 5.2.1 ([KPS24, Proposition 8.26])**.** There is a pullback diagram such that the horizontal morphisms are open immersions

$$\begin{tikzcd}
\mathcal{E}x^{(v,w)}(\mathcal{O}bj(\mathcal{A}_X)) \arrow[r] \arrow[d] & \mathrm{Filt}(\mathcal{O}bj(\mathcal{A}_X)) \arrow[d] \\
\mathcal{O}bj^v(\mathcal{A}_X) \times \mathcal{O}bj^w(\mathcal{A}_X) \arrow[r, "\widetilde{\Phi}_2"] & \mathrm{Grad}(\mathcal{O}bj(\mathcal{A}_X))
\end{tikzcd},$$

where the morphism $\widetilde{\Phi}_2$ is given by inclusion of graded objects in weights 0 and 1.

Proof. Let $\mathcal{C}$ be the perfect complexes on X with trivial determinant and take the poset $P = \{0 < 1 < 2\}$. First apply [KPS24, Proposition 8.26] to $\mathcal{C}$ and P, , and then restrict the resulting diagram to the open substack $\mathcal{M}_0$ of universally gluable objects , this gives

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an open immersion of diagrams.

$$\begin{tikzcd} \mathcal{M}_0^{2-\text{filt}} \arrow[r] \arrow[d] & \text{Filt}(\mathcal{M}_0) \arrow[d] \\ \mathcal{M}_0 \times \mathcal{M}_0 \arrow[r, "\widetilde{\Phi}_2"] & \text{Grad}(\mathcal{O}bj(\mathcal{M}_0)) \end{tikzcd},$$

where $\mathcal{M}_0^{2-\text{filt}}$ denotes the moduli stack parametrizes pairs $E \subset F \in \mathcal{M}_0$. Further restricting on the open substack $\mathcal{O}bj(\mathcal{A}_X)$ we obtain

$$\begin{tikzcd} \mathcal{E}x(\mathcal{O}bj(\mathcal{A}_X)) \arrow[r] \arrow[d] & \text{Filt}(\mathcal{O}bj(\mathcal{A}_X)) \arrow[d] \\ \mathcal{O}bj(\mathcal{A}_X) \times \mathcal{O}bj(\mathcal{A}_X) \arrow[r, "\widetilde{\Phi}_2"] & \text{Grad}(\mathcal{O}bj(\mathcal{A}_X)) \end{tikzcd}.$$

The results then follows from the restriction

$$\begin{tikzcd} \mathcal{E}x^{(v,w)}(\mathcal{O}bj(\mathcal{A}_X)) \arrow[r] \arrow[d] & \mathcal{E}x(\mathcal{O}bj(\mathcal{A}_X)) \arrow[d] \\ \mathcal{O}bj^v(\mathcal{A}_X) \times \mathcal{O}bj^w(\mathcal{A}_X) \arrow[r] & \mathcal{O}bj(\mathcal{A}_X) \times \mathcal{O}bj(\mathcal{A}_X) \end{tikzcd}.$$

$\square$

On the other hand, we consider the following abelian group

$$\hat{\mathbb{M}}^{\text{mon}}(\mathbb{C})[\Gamma] := \bigoplus_{v \in \Gamma} \hat{\mathbb{M}}^{\text{mon}}(\mathbb{C}) \cdot x^v$$

where x^vs are formal variables. We define $*$-product on this abelian group as follows, for x^v and x^w we define

$$x^v * x^w := \mathbb{L}^{\chi(v,w)/2} x^{v+w}.$$

Extend $\hat{\mathbb{M}}^{\text{mon}}$-linearly for general elements, i.e.

$$\sum_v a_v x^v * \sum_w b_w x^w := \sum_u \left(\sum_{v+w=u} \mathbb{L}^{\chi(v,w)/2} a_v b_w \right) x^u$$

We will also consider the usual product $\cdot$ on $\hat{\mathbb{M}}^{\text{mon}}(\mathbb{C})[\Gamma]$ i.e. $x^v \cdot x^w := x^{v+w}$.

5.3 Motivic Integration Map

The integration map was used in the proof of the numerical DT/PT correspondence, as in [Bri11]. We now aim to upgrade this integration map to the motivic level. We define

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the following map:

$$
\begin{aligned}
I : \mathcal{H}(\mathcal{A}_X) &\longrightarrow \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C})[\Gamma],\\
[\mathcal{X} \xrightarrow{f} \mathcal{O}bj^v(\mathcal{A}_X)] &\mapsto \left(\int_{\mathcal{X}} f^*\nu^{\mathrm{mot}}_{\mathcal{O}bj(\mathcal{A}_X)}\right) x^v,
\end{aligned}
$$

and extended $\hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C})$-linearly. As in the proof of DT/PT correspondence [Bri11], [Tod10], we wish the integration map to respect the product structure. So we hope the same thing happen for this motivic integration map.

Conjecture 4. The map I is compatible with the $*$-product, i.e. for $a \in \mathcal{H}(\mathcal{A}_X)$, $b \in \mathcal{H}^0$, we have

$$I(a * b) = I(a) * I(b).$$

The key to proving this conjecture lies in the motivic integral identity, which requires the stack $\mathcal{O}bj(\mathcal{A}_X)$ to be Nisnevich locally fundamental. However, this assumption is rather strong and difficult to verify in our setting.

Instead, we make a weaker and more natural assumption: that the semistable locus $\mathcal{M}^{ss,v}$ is Nisnevich locally fundamental. This shift in perspective is motivated by the expectation that $\mathcal{M}^{ss,v}$ admits a good moduli space, a property that is both more believable and, in practice, more accessible to verification. Under this assumption, we can still prove the following weaker version of the conjecture.

Theorem 5.3.1. Assuming the good moduli space for $\mathcal{M}^{ss,v}$ exists for each v. We have the following result. Let $a = [\mathcal{X} \xrightarrow{f} \mathcal{O}bj^v(\mathcal{A}_X)] \in \mathcal{H}(\mathcal{A}_X)$ such that f factor through $\mathcal{M}^{ss,v}$ for some $v = (-n, -\beta, 1)$. Then for any $b = [\mathcal{Y} \xrightarrow{g} \mathcal{O}bj^{v_0}(\mathcal{A}_X)] \in \mathcal{H}^0(\mathcal{A}_X)$,

$$I(a) * I(b) = I(a * b), I(b) * I(a) = I(b * a)$$

Proof. To show $I(a) * I(b) = I(a * b)$, for each diagram $(*)$ it suffices to show the following:

$$\int_{\mathcal{Z}} h^*\nu^{\mathrm{mot}}_{\mathcal{O}bj(\mathcal{A}_X)} = \mathbb{L}^{-\chi(v,v_0)/2} \cdot \left(\int_{\mathcal{X}} f^*\nu^{\mathrm{mot}}_{\mathcal{O}bj(\mathcal{A}_X)}\right) \cdot \left(\int_{\mathcal{Y}} g^*\nu^{\mathrm{mot}}_{\mathcal{O}bj(\mathcal{A}_X)}\right).$$

First we consider the following diagram

$$
\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] \arrow[rrr, bend left, "h"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}x^{(v,v_0)}(\mathcal{A}_X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Filt}(\mathcal{O}bj(\mathcal{A}_X)) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{gr}"] & \mathcal{O}bj(\mathcal{A}_X) \\
\mathcal{X} \times \mathcal{Y} \arrow[r, "(f,g)"] & \mathcal{O}bj^v(\mathcal{A}_X) \times \mathcal{O}bj^{v_0}(\mathcal{A}_X) \arrow[r, "\widetilde{\Phi}_2"] & \mathrm{Grad}(\mathcal{O}bj(\mathcal{A}_X)) &
\end{tikzcd}
$$

By Corollary 4.3.6, we have the following diagram

$$
\begin{tikzcd}
\mathrm{Grad}(\mathcal{M}^{ss}) \arrow[d] & \mathrm{Filt}(\mathcal{M}^{ss}) \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}_1"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & \mathcal{M}^{ss} \arrow[d] \\
\mathrm{Grad}(\mathcal{O}bj(\mathcal{A}_X)) & \mathrm{Filt}(\mathcal{O}bj(\mathcal{A}_X)) \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}_1"] & \mathcal{O}bj(\mathcal{A}_X)
\end{tikzcd}
$$

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and note that h, $\widetilde{\Phi}_2 \circ (f,g)$ factor through Filt$(\mathcal{M}^{ss})$ and Grad$(\mathcal{M}^{ss})$ by Lemma 2.4.9. We again denote $h : \mathcal{Z} \to$ Filt$(\mathcal{M}^{ss}) \to \mathcal{M}^{ss}$ and $(f,g) : \mathcal{X} \times \mathcal{Y} \to$ Grad$(\mathcal{M}^{ss})$. So we rewrite

$$\int_{\mathcal{Z}} h^* \nu^{\mathrm{mot}}_{\mathcal{O}bj(\mathcal{A}_X)} = \int_{\mathcal{Z}} h^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}}$$

$$\left(\int_{\mathcal{X}} f^* \nu^{\mathrm{mot}}_{\mathcal{O}bj(\mathcal{A}_X)}\right) \cdot \left(\int_{\mathcal{Y}} g^* \nu^{\mathrm{mot}}_{\mathcal{O}bj(\mathcal{A}_X)}\right) = \left(\int_{\mathcal{X}} f^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}}\right) \cdot \left(\int_{\mathcal{Y}} g^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}}\right).$$

By our assumption, $\mathcal{M}^{ss,v}$ admits a good moduli space for all $v \in \Gamma^1 \cup \Gamma^0$ and thus $\mathcal{M}^{ss}$ is Nisnevich locally fundamental by Corollary 2.4.5. We now apply motivic integral identity,

$$\int_{\mathcal{Z}} h^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}} = \int_{\mathcal{X}\times\mathcal{Y}} (f,g)^* \widetilde{\Phi}_2^* \mathrm{gr}_! \mathrm{ev}_1^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}} = \mathbb{L}^{d/2} \int_{\mathcal{X}\times\mathcal{Y}} (f,g)^* \widetilde{\Phi}_2^* \nu_{\mathrm{Grad}(\mathcal{M}^{ss})},$$

where d is the virtual dimension of Filt$(\mathcal{O}bj(\mathcal{A}_X))$ at the $\mathbb{C}$-point sf$(\widetilde{\Phi}_2(E,F))$, where

$$(E,F) \in \mathcal{M}^{ss,v} \times \mathcal{M}^{ss,v_0}.$$

We use Lemma 5.3.2 to compute this number d:

$$d = \chi(\mathbb{L}_{\mathrm{Filt}(\mathcal{M}_0)}|_{\mathrm{sf}(E\oplus F)}) = \chi((\mathbb{L}_{\mathcal{M}_0}|_{E\oplus F})^{\leq 0}) = \chi(R\,\mathrm{Hom}(E\oplus F, E\oplus F)_0[1]^{\geq 0}),$$

where we use the fact that the tangent complex of a point P at $\mathcal{M}_0$ is describe as the traceless derived endomorphism [STV15, Section 5]. The traceless part fits into the exact triangle

$$R\,\mathrm{Hom}(E\oplus F, E\oplus F)_0[1] \to R\,\mathrm{Hom}(E\oplus F, E\oplus F)[1] \xrightarrow{\mathrm{tr}} R\Gamma(X, \mathcal{O}_X)[1] \to$$

The Calabi-Yau condition implies that

$$\chi(R\,\mathrm{Hom}(E\oplus F, E\oplus F)_0[1]) = \chi(R\,\mathrm{Hom}(E\oplus F, E\oplus F)[1]) + \chi(\mathcal{O}_X) = 0.$$

Then we get

$$\begin{aligned}\chi(R\,\mathrm{Hom}(E\oplus F, E\oplus F)_0[1]^{\geq 0}) &= -\chi(R\,\mathrm{Hom}(E\oplus F, E\oplus F)_0[1]^{<0}) \\ &= \chi(R\,\mathrm{Hom}(F,E)_0) \\ &= \chi(F,E) = -\chi(E,F) = -\chi(v,v_0)\end{aligned}$$

Thus, it remains to show that

$$\int_{\mathcal{X}\times\mathcal{Y}} (f,g)^* \widetilde{\Phi}_2^* \nu_{\mathrm{Grad}(\mathcal{M}^{ss})} = \left(\int_{\mathcal{X}} f^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}}\right) \cdot \left(\int_{\mathcal{Y}} g^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}}\right).$$

First note that the orientation we chooce (Theorem 3.3.1) ensure that the induced orientation on Grad$(\mathcal{M}^{ss})$ pullback to the product orientation on $\mathcal{M}^{ss} \times \mathcal{M}^{ss}$ induced from each factor $\mathcal{M}^{ss}$. On the other hand the compatibility of d-critical structures is ensured by

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Theorem 3.3.2. Therefore, we have

$$\widetilde{\Phi}_2^* \nu^{\mathrm{mot}}_{\mathrm{Grad}(\mathcal{M}^{ss})} = \Phi_2^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}} = \nu^{\mathrm{mot}}_{\mathcal{M}^{ss} \times \mathcal{M}^{ss}}.$$

By Thom-Sebastiani Theorem (4.2.3), we have

$$\nu^{\mathrm{mot}}_{\mathcal{M}^{ss} \times \mathcal{M}^{ss}} = \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}} \boxtimes \nu^{\mathrm{mot}}_{\mathcal{M}^{ss}}.$$

Combine everything together, we obtain

$$\begin{aligned} \int_{\mathcal{X} \times \mathcal{Y}} (f, g)^* \widetilde{\Phi}_2^* \nu^{\mathrm{mot}}_{\mathrm{Grad}(\mathcal{M}^{ss})} &= \int_{\mathcal{X} \times \mathcal{Y}} (f, g)^* \left(\nu^{\mathrm{mot}}_{\mathcal{M}^{ss,v}} \boxtimes \nu^{\mathrm{mot}}_{\mathcal{M}^{ss,v_0}} \right) \\ &= \left(\int_{\mathcal{X}} f^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss,v}} \right) \cdot \left(\int_{\mathcal{Y}} g^* \nu^{\mathrm{mot}}_{\mathcal{M}^{ss,v_0}} \right). \end{aligned}$$

Hence $I(a * b) = I(a) * I(b)$. The same argument works for $I(b * a) = I(b) * I(a)$, which complete the proof. □

Lemma 5.3.2 ([HL14, Lemma 1.2.3])**.** Let $\mathfrak{X}$ be a derived algebraic stack locally of finite presentation over $\mathbb{C}$. We have the following description of tangent complex of $\mathrm{Filt}(\mathfrak{X})$ in terms of $\mathbb{T}_{\mathfrak{X}}$

$$\mathrm{sf}^*(\mathbb{T}_{\mathrm{Filt}(\mathfrak{X})}) \cong \mathrm{tot}^*(\mathbb{T}_{\mathfrak{X}})^{\geq 0},$$

where ≤ 0 indicate the nonnegative weighted part with respect to the natural $\mathbb{G}_m$-action.

5.4 DT/PT correspondence

In order to make the statement of the DT/PT correspondence precise. We need to consider a completion of the Hall module $\mathcal{H}(\mathcal{A}_X)$.

Lemma 5.4.1 ([Tod20, Lemma])**.** We have the following observations.

1. For any $v \in \Gamma^0$, there is only finitely many ways to write $v = v_1 + \cdots + v_\ell$ for $v_i \in \Gamma^0 \setminus \{0\}$.
2. For any $v \in \Gamma^1$ there is only finitely many ways to write $v = v_1 \cdots + v_\ell + v_{\ell+1}$ for $v_1, \ldots, v_\ell \in \Gamma^0 \setminus \{0\}$ and $v_{\ell+1} \in \Gamma^1$.

For $* = 0$ or 1, we set

$$\widehat{\mathcal{H}}^* = \prod_{v \in \Gamma^*} \hat{\mathbb{M}}^{\mathrm{mon}}(\mathcal{O}bj^v(\mathcal{A}_X)), \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C})[[\Gamma^*]] = \prod_{v \in \Gamma^*} \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C}) \cdot x^v$$

The $*$-product is well-defined on $\widehat{\mathcal{H}}^*$ by the above lemma, which defines an $\hat{\mathcal{H}}^0$-bimodule structure on $\hat{\mathcal{H}}^1$ and the integration map also extends to this completion ring

$$\widehat{I} : \widehat{\mathcal{H}}^* \to \hat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C})[[\Gamma^*]], (a^v)_v \mapsto \sum I(a^v).$$

Moreover, we may extend Theorem 5.3.1 to this completion ring

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Theorem 5.4.2. Let $\hat{a} = (a^v)_v \in \widehat{\mathcal{H}}^1$ be an element such that a^v factor through $\mathcal{M}^{ss,v}$ for all $v \in \Gamma^1$. Then

$$\hat{I}(\hat{a}) * \hat{I}(\hat{b}) = \hat{I}(\hat{a} * \hat{b}), \hat{I}(\hat{b}) * \hat{I}(\hat{a}) = \hat{I}(\hat{b} * \hat{a})$$

for all $\hat{b} \in \hat{\mathcal{H}}^0$

Proof. This follows directly from 5.3.1. □

We consider the following two elements in $\widehat{\mathcal{H}}^1$.

$$\delta_* = \left([\mathcal{M}_*^v \to \mathcal{O}bj^v(\mathcal{A}_X)]\right)_v \in \widehat{\mathcal{H}}^1$$

for $* = \mathrm{DT},\ \mathrm{PT}$. By definition we get

$$\widehat{I}(\delta_{\mathrm{DT}}) = \frac{1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \sum_{n,\beta} \mathrm{DT}_{n,\beta}^{\mathrm{mot}} x^{(-n,-\beta,1)}$$

$$\widehat{I}(\delta_{\mathrm{PT}}) = \frac{1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \sum_{n,\beta} \mathrm{PT}_{n,\beta}^{\mathrm{mot}} x^{(-n,-\beta,1)}$$

We now introduce another element

$$\delta_\infty = \left([\mathcal{O}bj^{v_0}(\mathcal{A}_X) \xrightarrow{\mathrm{id}} \mathcal{O}bj^{v_0}(\mathcal{A}_X)]\right)_{v_0} \in \widehat{\mathcal{H}}^0$$

We formally consider the element

$$\epsilon_\infty := \log(\delta_\infty) = \delta_\infty - \frac{1}{2}\delta_\infty^2 + \frac{1}{3}\delta_\infty^3 + \cdots$$

so that $\exp(\epsilon_\infty) = \delta_\infty$. Applying the integration map on this element we obtain some motivic invariants which we denoted by N_n^{mot}.

$$\widehat{I}(\epsilon_\infty) =: \frac{1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \sum_{n \geq 0} N_n^{\mathrm{mot}} \cdot x^{(-n,0,0)}.$$

These elements satisfy the following identity in $\widehat{\mathcal{H}}^1$

Lemma 5.4.3 ([Tod20, Lemma 3.16])**.** We have the following identity

$$\delta_{\mathrm{DT}} * \delta_\infty = \delta_\infty * \delta_{\mathrm{PT}} \in \widehat{\mathcal{H}}^1.$$

Remark 5.4.4. The factor $(\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})$ appearing in the definition of motivic DT/PT invariants carries two important meanings. First, it may be written as $\mathbb{L}^{-1/2}(\mathbb{L}-1)$, where the $\mathbb{L}^{-1/2}$ factor arises from the fact that we are integrating over the quotient stack $\mathcal{M}_{\mathrm{DT}}^v = [M_{\mathrm{DT}}^v/\mathbb{G}_m]$. The motivic Behrend function behaves compatibly with smooth morphisms, and in particular, the pullback of the motivic Behrend function along the smooth projection $M_{\mathrm{DT}}^v \to \mathcal{M}_{\mathrm{DT}}^v$ introduces a factor of $\mathbb{L}^{1/2}$. Hence, the integration over the quotient stack must be corrected by multiplying with $\mathbb{L}^{1/2}$.

Second, the factor $(\mathbb{L}-1)$ ensures that the Euler characteristic of the resulting motivic generating series is well-defined. This is a key point in the so-called no pole theorem due

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to Joyce, which guarantees that the naive Euler characteristic of motivic DT invariants would otherwise diverge. The original proof of the no pole theorem is highly technical, but recent work by Kinjo, Bu, and Núñez [BNK25] has significantly simplified the argument and provided a more conceptual explanation of this phenomenon.

We are now ready to prove the DT/PT correspondence.

*Proof of **Theorem 5.1.1**.* By Lemma 5.4.3 we have the following identity

$$\delta_{\mathrm{DT}} = \delta_\infty * \delta_{\mathrm{PT}} * \delta_\infty^{-1}$$

Now we define the operator $\{\epsilon, x\} := \epsilon * x - x * \epsilon$. Then we have

$$\delta_{\mathrm{DT}} = \exp(\{\epsilon_\infty, -\})(\delta_{\mathrm{PT}}) = \delta_{\mathrm{PT}} + \{\epsilon_\infty, \delta_{\mathrm{PT}}\} + \tfrac{1}{2!}\{\epsilon_\infty, \{\epsilon_\infty, \delta_{\mathrm{PT}}\}\} + \tfrac{1}{3!}\cdots$$

Observed that for all $n \geq 0$ and $v \in \Gamma^1$ we have $\chi((-n,0,0), v) = n$, thus

$$x^{(-n,0,0)} * x^v - x^v * x^{(-n,0,0)} = \left(\left(\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}\right) x^{(-n,0,0)}\right) \cdot x^v$$

Thus, multiple both side by $\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}$ and apply Theorem 5.3.1 we obtained

$$\sum_{n,\beta} \mathrm{DT}^{\mathrm{mot}}_{n,\beta} x^{(-n,-\beta,1)} = \exp\left(\sum_{n\geq 0}\left(\frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right) N_n^{\mathrm{mot}} x^{(-n,0,0)}\right) \cdot \left(\sum_{n,\beta} \mathrm{PT}^{\mathrm{mot}}_{n,\beta} x^{(-n,-\beta,1)}\right)$$

denote $q = x^{-1,0,0}$ and $x^\beta = x^{(0,-\beta,1)}$ we may rewrite the above equation

$$\sum_{n,\beta} \mathrm{DT}^{\mathrm{mot}}_{n,\beta} q^n x^\beta = \exp\left(\sum_{n\geq 0}\left(\frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right) N_n^{\mathrm{mot}} q^n\right)\left(\sum_{n,\beta} \mathrm{PT}^{\mathrm{mot}}_{n,\beta} q^n x^\beta\right)$$

Since $\exp\left(\sum_{n\geq 0}\left(\frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right) N_n^{\mathrm{mot}} q^n\right)$ is independent of β so comparing the coefficient we finally get

$$\sum_{n,\beta} \mathrm{DT}^{\mathrm{mot}}_{n,\beta} q^n x^\beta = \left(\sum_{n\geq 0} \mathrm{DT}^{\mathrm{mot}}_{n,0} q^n\right) \cdot \left(\sum_{n,\beta} \mathrm{PT}^{\mathrm{mot}}_{n,\beta} q^n x^\beta\right)$$

□

5.4.1 The invariants N_n^{mot}.

The invariants $N_n := \chi(N_n^{\mathrm{mot}})$ are known as the generalized Donaldson-Thomas invariants [JS12]. It was shown to satisfies the following identity [JS12, Section 6.3]:

$$N_n = \sum_{k|n} \frac{-\chi(X)}{k^2}.$$

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We now compute these invariants at the motivic level using the ideas of [DM15, Section 6.7]. In the context of the motivic DT/PT correspondence, the invariants N_n^{mot} satisfy the identity

$$\exp\left(\sum_{n\geq 0}\left(\frac{\mathbb{L}^{\frac{n}{2}}-\mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\right)N_n^{\mathrm{mot}}q^n\right)=\sum_{n\geq 0}\mathrm{DT}_{n,0}^{\mathrm{mot}}q^n.$$

The motivic DT invariants for curve class $\beta = 0$ have already been studied in [BBS13]. More precisely, they proved the following result:

Theorem 5.4.5 ([BBS13, Theorem 3.3])**.** Let X be a smooth projective Calabi-Yau threefold, we have

$$\sum_{n\geq 0}\mathrm{DT}_{n,0}^{\mathrm{mot}}(-q)^n=\mathrm{Exp}\left(\frac{-\mathbb{L}^{-\frac{3}{2}}[X]q}{(1+\mathbb{L}^{\frac{1}{2}}q)(1+\mathbb{L}^{-\frac{1}{2}}q)}\right).$$

Here, Exp denotes the power structure exponential, defined by

$$\sum_{n\geq 1}a_nq^n\mapsto\prod_{n\geq 1}(1-q^n)^{-a_n},$$

where the identity $(1-q)^{-[X]}=\sum_{n\geq 0}[\mathrm{Sym}^nX]q^n$ holds in $\widehat{\mathbb{M}}^{\mathrm{mon}}(\mathbb{C})$. We refer to [BBS13, GZLMH06] for precise definitions of this operator. We now consider the Adams operations ψ^k as described in [DM15, Appendix B], defined by the generating series

$$\psi_q(a):=\sum_{k\geq 1}\psi^k(a)q^k=\frac{q\cdot\frac{d}{dq}(1-q)^{-[X]}}{(1-q)^{-[X]}}.$$

Then we get

$$(1-q)^{-[X]}=\exp\left(\int\psi_q([X])\frac{dq}{q}\right)=\exp\left(\sum_{k\geq 1}\frac{1}{k}\psi^k([X])q^k\right).$$

The operators ψ^k satisfies some basic properties.

Lemma 5.4.6. For a variety X, we have

$$\psi^k((-\mathbb{L}^{\frac{1}{2}})^n[X])=(-\mathbb{L}^{\frac{1}{2}})^{nk}\psi^k([X])$$

Proof. We use the following fact (see [BBS13, Section 1])

$$(1-q)^{-(-\mathbb{L}^{\frac{1}{2}})^n[X]}=(1-((-\mathbb{L}^{\frac{1}{2}})^nq))^{[X]}.$$

Then we get

$$\exp\left(\sum_{k\geq 1}\frac{1}{k}\psi^k((-\mathbb{L}^{\frac{1}{2}})^n[X])q^k\right)=\exp\left(\sum_{k\geq 1}\frac{1}{k}\psi^k([X])((-\mathbb{L}^{\frac{1}{2}})^nq)^k\right).$$

Taking the logarithm of both sides yields the desired formula. □

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Lemma 5.4.7. For a variety X, we have

$$\chi(\psi^k([X])) = \chi(X),$$

where $\chi(X)$ is the topological Euler characteristic of X.

Proof. This follows from the fact that

$$\chi((1-q)^{-[X]}) = (1-q)^{-\chi(X)}.$$

Indeed, taking logarithm of both side and use the fact that taking Euler characteristic is a ring homomorphism we get

$$\sum_{k\geq 1} \frac{1}{k}\chi(\psi^k([X]))q^k = -\chi(X)\log(1-q) = \sum_{k\geq 1}\frac{1}{k}\chi(X)q^k.$$

Comparing the coefficient yields the desired formula. □

Lemma 5.4.8. If we have an identity

$$\exp\left(\sum_{n\geq 1} a_n q^n\right) = \mathrm{Exp}\left(\sum_{m\geq 1} b_m q^m\right),$$

then

$$a_n = \sum_{k|n}\frac{1}{k}\psi^k(b_{\frac{n}{k}}).$$

Proof. By definition, the left hand side may be rewritten as

$$\begin{aligned}
\mathrm{Exp}\left(\sum_{m\geq 1} b_m q^m\right) &= \prod_{m\geq 1}(1-q^m)^{-b_m} \\
&= \prod_{m\geq 1}\exp\left(\sum_{k\geq 1}\frac{1}{k}\psi^k(b_m)q^{km}\right) \\
&= \exp\left(\sum_{m\geq 1}\sum_{k\geq 1}\frac{1}{k}\psi^k(b_m)q^{km}\right) \\
&= \exp\left(\sum_{n\geq 1}\sum_{k|n}\frac{1}{k}\psi^k(b_{\frac{n}{k}})q^n\right)
\end{aligned}$$

Taking the logarithm of both sides yields the desired formula

$$a_n = \sum_{k|n}\frac{1}{k}\psi^k(b_{\frac{n}{k}}).$$

□

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Using Theorem 5.4.5 and Lemma 5.4.8, 5.4.6, we may now derive an explicit formula for N_n^{mot}.

Theorem 5.4.9. For any $n \geq 0$ we have

$$N_n^{\mathrm{mot}} = \sum_{k|n} \frac{(-1)^{k-1}}{k} \cdot \frac{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{k}{2}} - \mathbb{L}^{-\frac{k}{2}}} \cdot \psi^k(\mathbb{L}^{-\frac{3}{2}}[X]).$$

Proof. We first calculate the power series

$$\frac{-\mathbb{L}^{-\frac{3}{2}}[X]q}{(1+\mathbb{L}^{\frac{1}{2}}q)(1+\mathbb{L}^{-\frac{1}{2}}q)}$$

Let $a := -\mathbb{L}^{\frac{1}{2}}$. Then

$$\begin{aligned}
\frac{-\mathbb{L}^{-\frac{3}{2}}[X]q}{(1+\mathbb{L}^{\frac{1}{2}}q)(1+\mathbb{L}^{-\frac{1}{2}}q)} &= \frac{a^{-3}[X]q}{(1-aq)(1-a^{-1}q)} \\
&= a^{-3}[X]q(1+aq+a^2q^2+\cdots)(1+a^{-1}q+a^{-2}q^2+\cdots) \\
&= a^{-3}[X]q(1+(a^{-1})(1+a^2)q+(a^{-1})^2(1+a^2+a^4)q^2+\cdots) \\
&= \sum_{n\geq 1} a^{-n-2}[X]\left(\sum_{k=0}^{n-1}(a^2)^k\right)q^n \\
&= \sum_{n\geq 1} a^{-n-2}[X]\frac{1-a^{2n}}{1-a^2}q^n \\
&= \sum_{n\geq 1} a^{-3}[X]\frac{a^n-a^{-n}}{a-a^{-1}}q^n
\end{aligned}$$

By Theorem 5.4.5, we have

$$\exp\left(\sum_{n\geq 0}\left(\frac{(-a)^n-(-a)^{-n}}{(-a)^-(-a)^{-1}}\right)N_n^{\mathrm{mot}}(-q)^n\right) = \operatorname{Exp}\left(\sum_{n\geq 1} a^{-3}[X]\frac{a^n-a^{-n}}{a-a^{-1}}q^n\right)$$

Thus by Lemma 5.4.8,

$$\begin{aligned}
N_n^{\mathrm{mot}} &= -\frac{a-a^{-1}}{a^n-a^{-n}}\sum_{k|n}\frac{1}{k}\psi^k\left(a^{-3}[X]\frac{a^{\frac{n}{k}}-a^{-\frac{n}{k}}}{a-a^{-1}}\right) \\
&= \sum_{k|n}\frac{1}{k}\cdot\frac{a-a^{-1}}{a^k-a^{-k}}\cdot\psi^k(-a^{-3}[X]) \\
&= \sum_{k|n}\frac{(-1)^{k-1}}{k}\cdot\frac{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{k}{2}}-\mathbb{L}^{-\frac{k}{2}}}\psi^k(\mathbb{L}^{-\frac{3}{2}}[X]).
\end{aligned}$$

□

Remark 5.4.10. Taking the Euler characteristic of both sides, using Lemma 5.4.7 we

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recover the following identity from [JS12, Section 6.3]:

$$N_n = \sum_{k|n} \frac{-\chi(X)}{k^2}.$$

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