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諾特線上 $(2,4)$-型之三維代數多樣體

Threefolds on the Noether Line of Type-$(2,4)$

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諾特線上(2,4)-型之三維代數多樣體

Threefolds on the Noether Line of Type-(2,4)

本論文係蘇品丞君（學號 R11221009）在國立臺灣大學數學研究所完成之碩士學位論文，於民國 113 年 6 月 13 日承下列考試委員審查通過及口試及格，特此證明。

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摘要

本文具體地構造出數個 $(2,4)$-型的三維代數多樣體的例子，並描述它們的極小模型 (minimal model) 與正則模型 (canonical model)；這些代數多樣體落在諾特線 (Noether line) 上。對於每個例子 X，我們也計算了上同調空間 $H^1(X,\mathscr{T}_X)$ 的維度；該空間描述了 X 的一階形變。這推廣了堀川穎二 (E. Horikawa) 在曲面 (即二維情形) 上的一部份工作。

關鍵字：雙有理幾何、模空間、一般型代數多樣體、諾特不等式、環面多樣體、歐拉示性數

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Abstract

We construct explicit examples of algebraic threefolds of general type with invariants $(\mathrm{vol}(X), p_g(X)) = (2, 4)$ and describe their minimal and canonical model(s); in particular, such threefolds lie on the Noether line. For each of the examples X, we compute the dimension of $H^1(X, \mathscr{T}_X)$, the space of first-order infinitesimal deformations of X; this partially generalizes E. Horikawa's work on surfaces.

Keywords: birational geometry, moduli space, varieties of general type, Noether inequality, toric varieties, Euler characteristic

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Introduction

Birational geometry is a branch of algebraic geometry that studies algebraic varieties and birational maps between them. Under such maps, there are two numerical invariants of pivotal importance: for a projective variety X of dimension n, we define its *geometric genus* to be

$$p_g(X) \coloneqq h^0(X, mK_X)$$

and its *canonical volume* to be

$$\mathrm{vol}(X) \coloneqq \lim_{m\to\infty} \frac{h^0(X, mK_X)}{m^n/n!}.$$

A variety is said to be of *general type* if $\mathrm{vol}(X) > 0$; it is *minimal* if K_X is nef. When X is nef and normal with at worst canonical singularities, it is known that $\mathrm{vol}(X)$ equals K_X^n, the top self-intersection of the canonical divisor.

Fix an integer n, we can represent a variety X of dimension n by a point on the xy-plane with coordinates being the pair of invariants $(\mathrm{vol}(X), p_g(X))$. Notice that the point is essentially in the first quadrant if X is of general type; also, $\mathrm{vol}(X)$ is conventionally drawn on the y-axis. This way, we obtain a "map" of varieties, so it is natural to ask whether for each point $(p, q) \in \mathbb{Z}^2_{>0}$ on the map we can find some variety represented by that point. More generally, we can consider the moduli space of varieties with a fixed pair of invariants $(\mathrm{vol}(X), p_g(X))$, then we are interested in questions such as:

- For what values of $(p, q) = (\mathrm{vol}(X), p_g(X))$ is the moduli space non-empty?
- What is the dimension of the moduli space?

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- How many connected components or deformation classes are there in the moduli space?

The study of these questions is often dubbed the "geography" of varieties of general type, possibly due to its resemblance with taking a population census, where one investigates the population and the relation among residents in a certain area on a map.

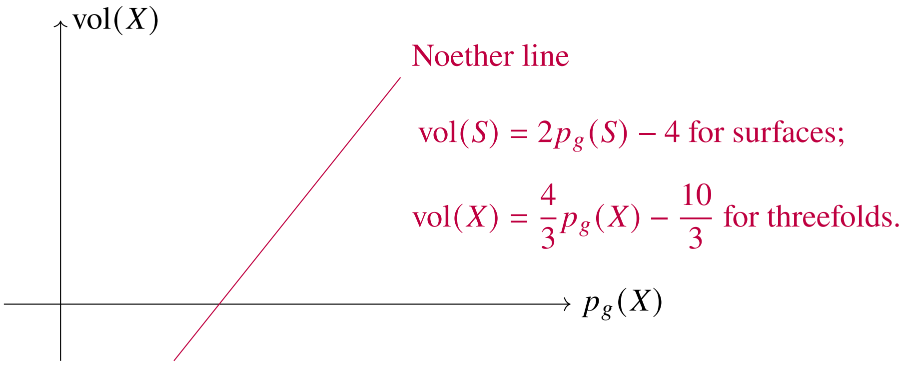
In dimension $n = 1$, the classical Riemann–Roch Theorem gives a relation between the two invariants: let C be a complete curve, then

$$\mathrm{vol}(C) = \deg K_C = 2p_g(C) - 2.$$

In dimension $n = 2$, a relation was first obtained by Max Noether [Noe75]: if S is a projective surface of general type, then

$$\mathrm{vol}(S) \geq 2p_g(S) - 4.$$

This means that the point representing S on the map must lie above or on the *Noether line*, as illustrated in the figure below.



The 3-dimensional analogue of Noether's inequality was first suggested by M. Kobayashi

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[Kob92] to be of the form

$$\mathrm{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}. \tag{N}$$

After nearly three decades of development, Chen, Chen, and Jiang [CCJ20] have proven this inequality to be true for *most* projective threefolds X of general type, with the only exceptions being the cases where $5 \leq p_g(X) \leq 10$. A detailed survey of the development history is also provided loc. cit. Throughout this paper, we will simply refer to (N) as the *Noether inequality*.

We are interested in the varieties that lie on the Noether line. In this regard, the surface case was studied thoroughly in the pioneering work of E. Horikawa [Hor76], where he classified all minimal surfaces lying on the Noether line and described their deformation types. Our aim is to generalize Horikawa's results to the threefold case. In Kobayashi's work, he already provided examples of threefolds to show that the Noether inequality is sharp; these examples were later generalized by Chen and Hu [CH17]. Furthermore, threefolds on the Noether line with $p_g(X) \geq 11$ have been well-studied; see for example [CP23].

In this thesis, we study carefully the smallest possible pair of invariants on the Noether line, namely the case where $(\mathrm{vol}(X), p_g(X)) = (2, 4)$, as a first step towards understanding threefolds on the Noether line with small invariants. Threefolds with this pair of invariants are said to be *of type*-$(2, 4)$.

In Chapter 1, we collect some facts scattered throughout the literature that will be used in the succeeding discussions. Chapter 2 is devoted to the study of three families of examples: in §2.1 we will carry out the explicit constructions of these threefolds and prove that they are smooth with the asserted pair of invariants $(\mathrm{vol}(X), p_g(X)) = (2, 4)$; in §2.2 we will investigate their minimal and canonical models. These examples are special

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cases of the Kobayashi–Chen–Hu constructions, and our discussion is along the same lines as [CH17]. §2.3 investigates the canonical images of the examples, which are used to tell apart different birational classes. A summary of the results will be given in §2.4 to conclude the chapter.

The moduli aspect of these threefolds is mainly elaborated in §3.1, where we compute, for each example X, the dimension of $H^1(X, \mathscr{T}_X)$, namely the space parametrizing first-order infinitesimal deformations of X. This computation is mostly a generalization of Horikawa's method in his work on surfaces; however, some parts of it were only achievable due to the utilization of toric methods, which was a theory still in its infancy during Horikawa's era. In §3.2, we compute the holomorphic Euler characteristic $\chi(\mathscr{T}_X)$ using the Hirzebruch–Riemann–Roch Theorem. This was the original method Horikawa used in [Hor76]; it not only serves as a double-check of the results in the previous section but also yields the topological Euler characteristic $\chi_{\mathrm{top}}(X)$ as a byproduct. Finally, in Chapter 4 we pose some problems for possible future work.

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Chapter 1 Preliminaries

Conventions. Unless otherwise specified, notations follow [Har77].

We work over base field $\mathbb{C}$ throughout this thesis. Since we are mainly concerned with varieties, the word *smooth* is synonymous with *regular* or *nonsingular*.

We write $\mathcal{O}_X(D)$ instead of $\mathcal{L}(D)$ for the line bundle corresponding to the Cartier divisor D on X. By abuse of notation, for a line bundle $\mathcal{L}$ we sometimes just use the same symbol $\mathcal{L}$ to mean its corresponding divisor $c_1(\mathcal{L})$.

If the space $\mathbb{P}^n$ is clear from the context, we omit the subscript in the twisting sheaves $\mathcal{O}_{\mathbb{P}^n}(m)$ and simply write $\mathcal{O}(m)$.

We use $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ to denote the *Hirzebruch surface* of degree e. We write ℓ for its fiber and δ for its distinguished section such that $\delta^2 = -e$. Since $\operatorname{Pic} \mathbb{F}_e$ is generated by ℓ and δ, we abbreviate $\mathcal{O}_{\mathbb{F}_e}(a\delta + b\ell) =: \mathcal{O}_{\mathbb{F}_e}(a, b)$.

We write $D \sim D'$ if two divisors D and D' are linearly equivalent, and we write $C \equiv C'$ if two cycles C and C' are numerically equivalent.

Branched Cyclic Coverings. Our main examples are all constructed by taking a double cover X over some other variety Y with designated branch locus B in Y, so we shall carry out the general construction of cyclic covers here.

Let Y be a variety and B be a divisor on Y such that $\mathcal{O}_Y(B) = \mathcal{L}^{\otimes m}$ for some $m \in \mathbb{Z}_{>0}$ and some line bundle $\mathcal{L}$ on Y, then we can take a section $s \in \Gamma(Y, \mathcal{O}_Y(B))$ so that

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$\operatorname{div}(s) = B$. Recall that we have a bundle projection $p : \mathbb{V}(\mathscr{L}) \to Y$ from the total space $\mathbb{V}(\mathscr{L})$ of the line bundle $\mathscr{L}$ to the base variety Y, along with a tautological section $t \in \Gamma(\mathbb{V}(\mathscr{L}), p^*\mathscr{L})$. We set X to be the subvariety in $\mathbb{V}(\mathscr{L})$ defined by the zero divisor of $(t^m - p^*s)$, so that it inherits a projection $\pi : X \to Y$ from p:

$$\begin{array}{ccc} X = \{t^m - p^*s = 0\} & \subset & \mathbb{V}(\mathscr{L}) \\ & \underset{m:1}{\overset{\pi}{\searrow}} & \downarrow p \\ & & Y. \end{array}$$

This X is called an *m-to-1 cyclic cover of Y branched at B* because intuitively we are taking t to be the m-th root of the section s, so each fiber of π has exactly m point except at the zeros of s, where the m sheets merge. The important properties of X and π are shown in the following proposition.

Proposition 1.1. Keep the notations as above.

(a) π is a finite morphism of degree m; moreover, π is flat.

(b) π maps $B' = \operatorname{div}(t)$ isomorphically to B.

(c) $\pi_*\mathscr{O}_X = \mathscr{O}_Y \oplus \mathscr{L}^{\otimes -1} \oplus \cdots \mathscr{L}^{\otimes -(m-1)}$, where $\mathscr{L}^{\otimes -r}$ means $(\mathscr{L}^\vee)^{\otimes r}$.

(d) If Y and B are smooth, then so are X and B'.

(e) (Riemann–Hurwitz formula) If both X and Y are smooth, then we have $\omega_X = \pi^*\left(\omega_Y \otimes \mathscr{L}^{\otimes m-1}\right)$, or in terms of divisors, $K_X = \pi^*\left(K_Y + \frac{m-1}{m}B\right)$.

(f) $\pi_*\omega_X = \omega_Y \otimes \left(\mathscr{O}_Y \oplus \mathscr{L} \oplus \cdots \mathscr{L}^{\otimes(m-1)}\right)$.

Proof. See [Laz04, §4.1.B] as well as [BHPV04, §I.16–17]. □

Projective Bundles. Given a locally free sheaf $\mathscr{E}$ of rank $r+1$ on W, we can form a variety $Y \coloneqq \mathbb{P}(\mathscr{E})$ that has a $\mathbb{P}^r$-bundle structure over W. This is another key ingredient in

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our construction, so we list some of its properties below.

Proposition 1.2. Keep the notations as above and let $p : Y \to W$ be the bundle projection, then

(a) $p_*\mathscr{O}_Y = \mathscr{O}_W$ and $p_*\mathscr{O}_Y(1) = \mathscr{E}$. More generally,

$$p_*\mathscr{O}_Y(m) = \begin{cases} \mathrm{Sym}^m(\mathscr{E}), & \text{if } m \geq 0; \\ 0, & \text{if } m < 0. \end{cases}$$

Furthermore,

$$R^i p_*\mathscr{O}_Y(m) = \begin{cases} 0, & \text{if } 0 < i < r; \\ (p_*\mathscr{O}_Y(-(m+r+1)))^\vee \otimes (\det \mathscr{E})^\vee, & \text{if } i = r. \end{cases}$$

(b) $\omega_Y = \mathscr{O}_Y(-(r+1)) \otimes p^*(\omega_W \otimes \det \mathscr{E})$.

(c) $\mathrm{Pic}\,\mathbb{P}(\mathscr{E}) = p^* \mathrm{Pic}\, W \times \mathbb{Z}$, where the $\mathbb{Z}$ factor is generated by the class $[\mathscr{O}_Y(1)]$.

(d) There is a 1-to-1 correspondence

$$\begin{array}{ccc} \{\text{sections } \sigma : W \to Y\} & \xleftrightarrow{1:1} & \{\text{quotient line bundles } \mathscr{E} \to \mathscr{L} \to 0\} \\ \sigma & \longmapsto & \sigma^*\mathscr{O}_Y(1). \end{array}$$

(e) In the case $r = 1$, a section σ gives rise to a divisor $D \coloneqq \sigma(W) \subset Y$; if $\mathscr{L}$ is its corresponding line bundle according to (d), then

$$\mathscr{O}_Y(D) \simeq \mathscr{O}_Y(1) \otimes (p^* \ker(\mathscr{E} \to \mathscr{L}))^\vee.$$

Proof. See [Har77, Exercises III.8.4, II.7.9, II.7.8, and Proposition V.2.6]. □

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Tools for Computing Cohomology.

Proposition 1.3 (Leray spectral sequence)**.** If $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ a morphism of ringed spaces and $\mathscr{F}$ is an $\mathscr{O}_X$-module, then there exists a spectral sequence with second page $E_2^{p,q} = H^p(Y, R^q f_* \mathscr{F})$ that converges to $E_\infty = H^*(X, \mathscr{F})$.

Proof. See, for example, [Vak24, Theorem 23.4.5]. □

Corollary 1.4. Let $p : Y = \mathbb{P}(\mathscr{E}) \to W$ be the projective bundle with rank $\mathscr{E} = r + 1$ as in Proposition 1.2 and $\mathscr{F}$ be a locally free $\mathscr{O}_W$-module of finite rank, then

$$H^n(Y, \mathscr{O}_Y(m) \otimes p^*\mathscr{F}) \simeq \begin{cases} H^n(W, p_*\mathscr{O}_Y(m) \otimes \mathscr{F}), & \text{if } m \geq 0; \\ H^{n-r}(W, R^r p_*\mathscr{O}_Y(m) \otimes \mathscr{F}), & \text{if } m \leq -(r+1); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First, using the projection formula we have

$$R^i p_*(\mathscr{O}_Y(m) \otimes p^*\mathscr{F}) \simeq R^i p_*\mathscr{O}_Y(m) \otimes p^*\mathscr{F}.$$

From the formulas in Proposition 1.2 (a), we know that the sheaf $R^i p_*\mathscr{O}(m)$ is only nonzero if either $i = 0$ and $m \geq 0$ or $i = r$ and $m \leq -(r+1)$. This means that the E_2 page of the Leray spectral sequence always has only one nonzero row, so each differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ is the zero map and the spectral sequence degenerates at $E_2 \simeq E_\infty \simeq H^*(Y, \mathscr{O}_Y(m) \otimes p^*\mathscr{F})$. The description of E_2 page in Proposition 1.3 above then gives the desired formula. □

Proposition 1.5 (Künneth formula)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be quasi-coherent sheaves on (separated) varieties X and Y respectively. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the

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natural projections and define $\mathscr{F} \boxtimes \mathscr{G} \coloneqq \pi_1^*\mathscr{F} \otimes \pi_2^*\mathscr{G}$, then

$$H^n(X \times Y, \mathscr{F} \boxtimes \mathscr{G}) = \bigoplus_{p+q=n} H^p(X, \mathscr{F}) \otimes H^q(Y, \mathscr{G}).$$

Proof. See [Kem93, Proposition 9.2.4]. □

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Chapter 2 Examples of Threefolds of Type-$(2,4)$

We construct and study the birational geometry of some threefolds of type-$(2,4)$ in this chapter. [CH17] is the inspiration and main reference for this chapter.

2.1 The Constructions

In this section, we construct three families of smooth threefolds of type-$(2,4)$. They will be referred to as type I, II, and III and sometimes denoted X_{I}, X_{II}, and X_{III} respectively in the remainder of this paper.

Construction–Proposition I. Fix a smooth surface B of degree 10 in $Y := \mathbb{P}^3$. Define X to be a double covering of $\mathbb{P}^3$ branched along B:

$$X \xrightarrow[2:1]{\tau} \mathbb{P}^3.$$

Then X is a minimal smooth threefold with $(\mathrm{vol}(X), p_g(X)) = (2,4)$.

Proof. In the language of Proposition 1.1, $\mathscr{L} = \mathscr{O}(5)$ and $\mathscr{L}^{\otimes 2} = \mathscr{O}_Y(B) = \mathscr{O}(10)$ in $\mathbb{P}^3$, so the Riemann–Hurwitz formula reads

$$\begin{aligned} \omega_X &= \tau^*(\omega_{\mathbb{P}^3} \otimes \mathscr{L}) \\ &= \tau^*(\mathscr{O}(-4) \otimes \mathscr{O}(5)) = \tau^*\mathscr{O}(1). \end{aligned} \tag{2.1}$$

This shows that K_X is ample and in particular nef, so X is minimal.

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Both $Y = \mathbb{P}^3$ and B are smooth, so X is smooth by part (d) of Proposition 1.1.

As X is smooth and K_X is nef, the volume can be computed via

$$\mathrm{vol}(X) = K_X^3 = [\tau^*\mathscr{O}(1)]^3 = \deg\tau \cdot [\mathscr{O}(1)]^3 = 2 \cdot 1 = 2.$$

On the other hand, the geometric genus is

$$p_g(X) = h^0(X, \omega_X) = h^0(\mathbb{P}^3, \tau_*\omega_X)$$

by the definition of pushforward, and part (f) of Proposition 1.1 gives

$$\tau_*\omega_X = \mathscr{O}(-4) \otimes (\mathscr{O} \oplus \mathscr{O}(5)) = \mathscr{O}(-4) \oplus \mathscr{O}(1), \tag{2.2}$$

so its global section is

$$p_g(X) = h^0(\mathbb{P}^3, \mathscr{O}(-4)) + h^0(\mathbb{P}^3, \mathscr{O}(1)) = 0 + 4 = 4. \qquad \square$$

Construction–Proposition II. Let $W = \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, over which we consider a $\mathbb{P}^1$-bundle $p : Y = \mathbb{P}(\mathscr{O}_W \oplus \mathscr{O}_W(-2,-2)) \to W$. Set $\Sigma_0 = p^*\ell$, $\Sigma_1 = p^*\delta$ and Σ_2 to be the divisor on Y corresponding to the quotient line bundle

$$\mathscr{E} = \mathscr{O}_W \oplus \mathscr{O}_W(-2,-2) \to \mathscr{O}_W(-2,-2) \to 0$$

via Proposition 1.2 (d).

Then there exists a smooth $B_0 \in |5\Sigma_2 + 10\Sigma_1 + 10\Sigma_0|$ such that $B_0 \cap \Sigma_2 = \varnothing$. Moreover,

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we define X to be a double covering of Y branched along $B \coloneqq B_0 + \Sigma_2$:

$$X \xrightarrow[2:1]{\tau} Y \xrightarrow[\mathbb{P}^1\text{-bundle}]{p} W \simeq \mathbb{P}^1 \times \mathbb{P}^1,$$

then X is a smooth threefold with $(\mathrm{vol}(X), p_g(X)) = (2, 4)$, but not minimal.

Proof. First, Proposition 1.2 (e) tells us that $\mathscr{O}_Y(\Sigma_2) = \mathscr{O}_Y(1)$. Moreover, since Σ_2 is a section on Y, the map p restricts to an isomorphism $\Sigma_2 \xrightarrow{\sim} W$, under which we can identify $\mathrm{Pic}\,\Sigma_2$ with $\mathrm{Pic}\,W = \mathbb{Z}\delta \oplus \mathbb{Z}\ell$.

Write $D = 5\Sigma_2 + 10\Sigma_1 + 10\Sigma_0$.

Claim. A general element of the linear system $|D|$ does not intersect Σ_2.

This claim would imply that $|D|$ is base-point free; indeed, if there were a base point, it must lie on the union $\Sigma_2 \cup \Sigma_1 \cup \Sigma_0$, but it cannot lie on Σ_2 by the claim, nor can it lie on Σ_0 or Σ_1 because $|\delta + \ell|$ is already base-point free on W (cf. [Har77, Theorem V.2.17]). This would also guarantee the existence of the desired divisor B_0, whose smoothness follows from Bertini's theorem.

Now we prove the claim. Recall that $K_W = -2\delta - 2\ell$, so by Proposition 1.2 (b),

$$\begin{aligned} \omega_Y &= \mathscr{O}_Y(-2) \otimes p^*(\omega_W \otimes \det(\mathscr{O}_W \oplus \mathscr{O}_W(-2,-2))) \\ &= \mathscr{O}_Y(-2) \otimes p^*(\mathscr{O}_W(-2,-2) \otimes \mathscr{O}_W(-2,-2)) \\ &= \mathscr{O}_Y(-2) \otimes p^*\mathscr{O}_W(-4,-4) \end{aligned} \tag{2.3}$$

$$K_Y = -2\Sigma_2 - 4\Sigma_1 - 4\Sigma_0,$$

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and the adjunction formula gives

$$\begin{aligned}(K_Y + \Sigma_2)|_{\Sigma_2} &= K_{\Sigma_2} = K_W \\ -\Sigma_2|_{\Sigma_2} - 4\delta - 4\ell &= -2\delta - 2\ell \quad \text{(since } \Sigma_1|_{\Sigma_2} = \delta \text{ and } \Sigma_0|_{\Sigma_2} = \ell) \\ \Sigma_2|_{\Sigma_2} &= -2\delta - 2\ell.\end{aligned} \tag{2.4}$$

In particular, $D|_{\Sigma_2} = 5 \cdot (-2\delta - 2\ell) + 10\delta + 10\ell = 0$, so in the short exact sequence

$$0 \longrightarrow \mathscr{O}_Y(D - \Sigma_2) \longrightarrow \mathscr{O}_Y(D) \longrightarrow \iota_* \mathscr{O}_{\Sigma_2} \otimes \mathscr{O}_Y(D) \longrightarrow 0,$$

the term on the right can be reduced to

$$\iota_* \mathscr{O}_{\Sigma_2} \otimes \mathscr{O}_Y(D) = \iota_* \left(\mathscr{O}_{\Sigma_2} \otimes \mathscr{O}_Y(D)|_{\Sigma_2} \right) = \iota_* \mathscr{O}_{\Sigma_2}$$

by the projection formula; here, $\iota : \Sigma_2 \hookrightarrow Y$ is the closed immersion. Moreover, in its associated long exact sequence

$$\cdots \longrightarrow H^0(Y, D) \longrightarrow H^0(\Sigma_2, \mathscr{O}_{\Sigma_2}) \longrightarrow H^1(Y, D - \Sigma_2) \longrightarrow \cdots,$$

one observes that

$$\begin{aligned} h^1(Y, D - \Sigma_2) &= h^1(Y, 4\Sigma_2 + 10\Sigma_1 + 10\Sigma_0) \\ &= h^1(Y, \mathscr{O}_Y(4) \otimes p^* \mathscr{O}_W(10, 10)) \\ &= h^1(W, p_* \mathscr{O}_Y(4) \otimes \mathscr{O}_W(10, 10)) && \text{Corollary 1.4} \\ &= h^1(W, \mathrm{Sym}^4(\mathscr{O}_W \oplus \mathscr{O}_W(-2, -2)) \otimes \mathscr{O}_W(10, 10)) && \text{Proposition 1.2 (b)} \\ &= h^1\left(W, \bigoplus_{k=1}^{5} \mathscr{O}_W(2k, 2k)\right) = 0, && \text{use Künneth formula} \end{aligned}$$

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so we have a surjection

$$\begin{array}{ccc} H^0(Y,D) & \twoheadrightarrow & H^0(\Sigma_2, \mathscr{O}_{\Sigma_2}) \simeq \mathbb{C}. \\ \Psi & & \Psi \\ s & \longmapsto & s|_{\Sigma_2} \end{array}$$

In particular, any nonzero section in $H^0(\Sigma_2, \mathscr{O}_{\Sigma_2})$ lifts to a section in $H^0(Y,D)$ which is non-vanishing on Σ_2, so its corresponding zero divisor is a general element in $|D|$ that avoids Σ_2 completely. This completes the proof of the claim.

By Proposition 1.1 (d), X is smooth because Y and B are.

Next, we compute the invariants of X. Since $B \sim 6\Sigma_2 + 10\Sigma_1 + 10\Sigma_0$, plugging (2.3) into the Riemann–Hurwitz formula yields

$$\begin{aligned} K_X &= \tau^*\left(K_Y + \frac{1}{2}B\right) \\ &= \tau^*(\Sigma_2 + \Sigma_1 + \Sigma_0) \\ \text{or } \omega_X &= \tau^*(\mathscr{O}_Y(1) \otimes p^*\mathscr{O}_W(1,1)). \end{aligned} \tag{2.5}$$

For the geometric genus, notice that

$$\mathscr{L} = \mathscr{O}_Y(3\Sigma_2 + 5\Sigma_1 + 5\Sigma_0), \tag{2.6}$$

so

$$\begin{aligned} h^0(X,\omega_X) = h^0(Y,\tau_*\omega_X) &= h^0(Y,\omega_Y) + h^0(Y,\omega_Y \otimes \mathscr{L}) \quad \text{by Proposition 1.2 (b)} \\ &= h^0(Y,-2\Sigma_2 - 4\Sigma_1 - 4\Sigma_0) + h^0(Y,\Sigma_2 + \Sigma_1 + \Sigma_0). \end{aligned}$$

The first term vanishes because it is the global section of the negative of an effective divi-

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sor; the second term is calculated via

$$
\begin{aligned}
h^0(Y, \Sigma_2 + \Sigma_1 + \Sigma_0) &= h^0(Y, \mathscr{O}_Y(1) \otimes p^*\mathscr{O}_W(1,1)) && \text{by definition} \\
&= h^0(W, p_*\mathscr{O}_Y(1) \otimes \mathscr{O}_W(1,1)) && \text{projection formula} \\
&= h^0(W, (\mathscr{O}_W \oplus \mathscr{O}_W(-2,-2)) \otimes \mathscr{O}_W(1,1)) && \text{Proposition 1.2 (a)} \\
&= h^0(W, \mathscr{O}_W(1,1)) + h^0(W, \mathscr{O}_W(-1,-1)) \\
&= 2 \times 2 + 0 \times 0 && \text{Künneth formula} \\
&= 4.
\end{aligned}
$$

Thus, $p_g(X) = 4$.

On the other hand, since K_X is not nef, we will compute the volume from the definition. To this end, we need to compute

$$
h^0(X, mK_X) = h^0\left(Y, \tau_*(\omega_X^{\otimes m})\right),
$$

where

$$
\begin{aligned}
\tau_*(\omega_X^{\otimes m}) &= \tau_*\tau^*(\omega_Y \otimes \mathscr{L})^{\otimes m} && \text{Riemann–Hurwitz formula} \\
&= \omega_Y^{\otimes m} \otimes \mathscr{L}^{\otimes m} \ \oplus \ \omega_Y^{\otimes m} \otimes \mathscr{L}^{\otimes m-1} && \text{Proposition 1.1 (c)} \\
&= \mathscr{O}_Y(m) \otimes p^*\mathscr{O}_W(m,m) \ \oplus \ \mathscr{O}_Y(m-3) \otimes p^*\mathscr{O}_W(m-5, m-5).
\end{aligned}
$$

The first term gives

$$
\begin{aligned}
h^0(Y, \mathscr{O}_Y(m) \otimes p^*\mathscr{O}_W(m,m)) &= h^0(W, p_*\mathscr{O}_Y(m) \otimes \mathscr{O}_W(m,m)) \\
&= h^0(W, \mathrm{Sym}^m(\mathscr{O}_W \oplus \mathscr{O}_W(-2,-2)) \otimes \mathscr{O}_W(m,m)) \\
&= h^0\left(W, \bigoplus_{k=0}^{m} \mathscr{O}_W(m-2k, m-2k)\right);
\end{aligned}
$$

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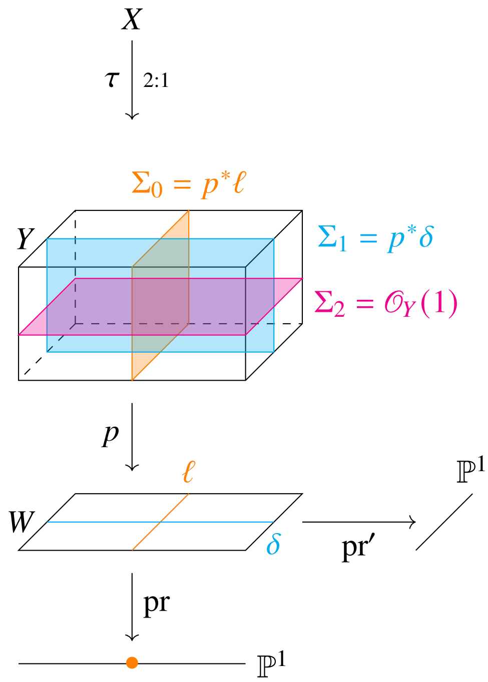

Figure 2.1: A schematic illustration of the constructions II and III. Notice that the projection pr′ onto the second $\mathbb{P}^1$ is only present in case II.

similarly, the second term gives

$$h^0\left(W, \bigoplus_{k=0}^{m-3} \mathscr{O}_W(m-5-2k, m-5-2k)\right).$$

Notice that $h^0(W, \mathscr{O}_W(k,k)) = (k+1)^2$ if $k \geq 0$ and zero otherwise, so both terms above are sums of squares of the first odd or even numbers up to m or $m-5$, depending on whether m is odd or even; in either case, the formula for such a sum has a leading coefficient of $1/6$ on the m^3 term, so

$$\mathrm{vol}(X) = \lim_{m\to\infty} \frac{h^0(X, mK_X)}{m^3/6} = \lim_{m\to\infty} \frac{m^3/6 + (\text{l.o.t}) + m^3/6 + (\text{l.o.t})}{m^3/6} = 2,$$

as desired.

The K_X-negative curves on X are described in Lemma 2.2, which show that X is not minimal. □

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Remark 2.1. As is already manifest in the preceding proof, the cohomology of line bundles on Y of the form

$$\mathscr{L} \coloneqq \mathscr{O}_Y(a\Sigma_2 + b\Sigma_1 + c\Sigma_0) = \mathscr{O}_Y(a) \otimes p^*\mathscr{O}_W(b,c) \quad \text{with } a,b,c \in \mathbb{Z}$$

can all be computed. More precisely, one "pushes downstairs" using Corollary 1.4, which reads

$$H^n(Y, \mathscr{O}_Y(a) \otimes p^*\mathscr{O}_W(b,c)) \simeq \begin{cases} H^n(W, p_*\mathscr{O}_Y(a) \otimes \mathscr{O}_W(b,c)), & \text{if } m \geq 0; \\ H^{n-1}(W, R^1p_*\mathscr{O}_Y(a) \otimes \mathscr{O}_W(b,c)), & \text{if } m \leq -2; \\ 0, & \text{otherwise.} \end{cases}$$

where $p_*\mathscr{O}_Y(a)$ and $R^1p_*\mathscr{O}_Y(a)$ are given by Proposition 1.2 (a). Now the cohomology groups of line bundles on W are easily obtained using the Künneth formula.

Construction–Proposition III. Let $W = \mathbb{F}_2$, over which we consider a $\mathbb{P}^1$-bundle $p : Y = \mathbb{P}(\mathscr{O}_W \oplus \mathscr{O}_W(-2,-4)) \to W$. Set $\Sigma_0 = p^*\ell$, $\Sigma_1 = p^*\delta$ and Σ_2 be the divisor on Y corresponding to the quotient line bundle

$$\mathscr{E} = \mathscr{O}_W \oplus \mathscr{O}_W(-2,-4) \to \mathscr{O}_W(-2,-4) \to 0$$

via Proposition 1.2 (d).

Then there exists a smooth $B_0 \in |5\Sigma_2 + 10\Sigma_1 + 20\Sigma_0|$ such that $B_0 \cap \Sigma_2 = \varnothing$. Moreover, we define X to be a double covering of Y branched along $B \coloneqq B_0 + \Sigma_2$:

$$X \xrightarrow[2:1]{\tau} Y \xrightarrow[\mathbb{P}^1\text{-bundle}]{p} W = \mathbb{F}_2,$$

then X is a smooth threefold with $(\mathrm{vol}(X), p_g(X)) = (2,4)$, but not minimal.

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Proof. The proof of Construction-Proposition II applies almost verbatim here as well. For future reference, we give some formulas used in the proof here:

$$\begin{aligned} B &\sim 6\Sigma_2 + 10\Sigma_1 + 20\Sigma_0 \\ \mathscr{L} &= \mathscr{O}_Y(3\Sigma_2 + 5\Sigma_1 + 10\Sigma_0) \\ \Sigma_2|_{\Sigma_2} &= K_W = -2\delta - 4\ell \\ K_Y &= -2\Sigma_2 - 4\Sigma_1 - 8\Sigma_0 \\ K_X &= \tau^*(\Sigma_2 + \Sigma_1 + 2\Sigma_0). \end{aligned} \tag{2.7}$$

The only difference with II is that the Künneth formula is no longer applicable to compute $h^i(W, \mathscr{O}_W(a,b))$; instead, notice that $W = \mathbb{F}_2 = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2)) \to \mathbb{P}^1$ is also a projective bundle, so $h^i(W, \mathscr{O}_W(a,b))$ can be computed by applying Corollary 1.4 in a similar fashion to Remark 2.1. See Appendix A for details. □

2.2 Minimal and Canonical Models

In this section, we explore the minimal and canonical models of the threefolds constructed in §2.1.

Models of Threefolds of Type I. The threefolds X of type I are already smooth with ample canonical bundle, so they are both their own minimal model and their own canonical model.

Models of Threefolds of Type II. Let X be a threefold of type II. Recall that τ restricts to an isomorphism on the branch locus, which contains Σ_2. We set $E_2 = \tau^{-1}(\Sigma_2) \subset X$ so that $E_2 \simeq \Sigma_2$ as varieties, and we can identify $\operatorname{Pic} E_2$ with $\operatorname{Pic} \Sigma_2 \simeq \operatorname{Pic} W = \mathbb{Z}\delta \oplus \mathbb{Z}\ell$ as

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before; on the other hand, $\tau^*\Sigma_2 = 2E_2$ as divisors, so we can write (2.5) as

$$K_X = 2E_2 + \tau^*(\Sigma_1 + \Sigma_0).$$

We also set

$$M = \tau^*(\Sigma_1 + \Sigma_0) \quad \text{and} \quad H = E_2 + M.$$

Lemma 2.2. Let X be a threefold of type II and E_2, M, and H be as above. Let $C \subset X$ be any irreducible curve. Then,

(a) $E_2|_{E_2} = -\delta - \ell$ and $H|_{E_2} = 0$.

(b) M is nef; moreover, $M.C > 0$ if and only if $p\tau(C)$ is a curve in $W = \mathbb{F}_0$.

(c) H is nef and big with $H^3 = 2$; moreover, $H.C = 0$ if and only if $C \subset E_2$.

(d) $K_X.C < 0$ if and only if $C \subset E_2$; moreover, if $C \equiv a\delta + b\ell$, then $K_X.C = -a - b$.

(e) There is no irreducible curve $C \subset X$ such that $K_X.C = 0$.

Proof. Since $(\tau^*\Sigma_1)|_{E_2} = \delta$ and $(\tau^*\Sigma_0)|_{E_2} = \ell$, by (2.5) the adjunction formula reads

$$(K_X + E_2)|_{E_2} = K_{E_2}$$

$$3E_2|_{E_2} + \delta + \ell = -2\delta - 2\ell,$$

which gives $E_2|_{E_2} = -\delta - \ell$ and $H|_{E_2} = [E_2 + \tau^*(\Sigma_1 + \Sigma_0)]|_{E_2} = 0$, the desired formulas in (a).

To prove (b), one first observes that $\delta + \ell$ is very ample on W by [Har77, Theorem V.2.17]. It then follows from the projection formula that

$$M.C = (p\tau)_*(M.C) = (p\tau)_*((p\tau)^*(\delta + \ell).C) = (\delta + \ell).(p\tau)_*C.$$

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This is zero if $p\tau(C)$ is a point and positive if $p\tau(C)$ is a curve, proving (b).

Suppose $C \subset X$ is an irreducible curve such that $H.C \le 0$. If C is contained in a fiber of $p\tau$, then $E_2.C > 0$, so $H.C = (E_2 + M).C > 0$, a contradiction. This shows that $p\tau(C)$ is not a point, so it must be a curve, and thus $M.C > 0$ by (b). Then,

$$E_2.C = H.C - M.C < 0,$$

so C must lie in E_2. But $H|_{E_2} = 0$ by (a), so $H.C = H|_{E_2}.C = 0$ for any irreducible curve C contained in E_2. This shows that H is nef.

Let us denote $P \coloneqq \Sigma_1 + \Sigma_0$ for brevity. We infer from (2.4) the following intersection numbers:

$$\begin{aligned} \Sigma_2^3 &= (\Sigma_2|_{\Sigma_2})^2 = (-2\delta - 2\ell)^2 &&= 8; \\ \Sigma_2^2.P &= (\Sigma_2|_{\Sigma_2}).(P|_{\Sigma_2}) = (-2\delta - 2\ell).(\delta + \ell) &&= -4; \\ \Sigma_2.P^2 &= (P|_{\Sigma_2})^2 = (\delta + \ell)^2 &&= 2; \\ P^3 & &&= 0, \end{aligned} \tag{2.8}$$

so

$$\begin{aligned} H^3 = \left[\tau^*\left(\frac{1}{2}\Sigma_2 + \Sigma_1 + \Sigma_0\right)\right]^3 &= \deg\tau \cdot \left(\frac{1}{2}\Sigma_2 + P\right)^3 \\ &= 2 \cdot \left(\frac{1}{8}\cdot 8 + 3\cdot\frac{1}{4}\cdot(-4) + 3\cdot\frac{1}{2}\cdot 2 + 0\right) = 2. \end{aligned}$$

This implies that H is big by the characterization [Laz04, Theorem 2.2.16] and completes the proof of (c).

For (d), notice that $K_X = E_2 + H$ with H nef, so if $K_X.C < 0$, then we must have

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$E_2.C < 0$ and thus $C \subset E_2$. In this case $H.C = 0$, so $K_X.C = E_2.C$ and we have

$$\left.\begin{aligned} E_2.\delta &= (-\delta - \ell).\delta = -1 \\ E_2.\ell &= (-\delta - \ell).\ell = -1 \end{aligned}\right\} \implies E_2.(a\delta + b\ell) = -a - b,$$

which is negative by [Har77, Corollary V.2.18]. Conversely, if $C \not\subset E_2$, then $E_2.C \geq 0$ and $H.C > 0$ by (c), so $K_X.C > 0$. This proves (d) and (e). □

Proposition 2.3. Let X be a threefold of type II. Denote by R_1 and R_0 the rays in $\overline{NE}(X)$ generated by the classes of $\delta \subset E_2$ and $\ell \subset E_2$ respectively. Then R_1 and R_0 are the only K_X-negative extremal rays and their corresponding contractions give rise to two smooth minimal models of X:

$$\phi_1 \coloneqq \mathrm{cont}_{R_1} : X \to X_1^{\min} \quad \text{and} \quad \phi_0 \coloneqq \mathrm{cont}_{R_0} : X \to X_0^{\min}.$$

Proof. We temporarily denote the images of ϕ_1 and ϕ_0 by X_1 and X_0 respectively. We will show that they are the minimal models of X.

The description of $\overline{NE}(X)$ and K_X-negative extremal rays follow from the previous Lemma 2.2.

The existence of contraction morphisms ϕ_1 and ϕ_0 are guaranteed by the Contraction theorem [KM98, Theorem 3.7(3)]. In fact, we can determine their corresponding supporting divisors: since $\tau^*\Sigma_1$ and $\tau^*\Sigma_0$ are nef divisors such that

$$\begin{aligned} \tau^*\Sigma_1.\delta &= 0 \\ \tau^*\Sigma_1.\ell &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} \tau^*\Sigma_0.\delta &= 1 \\ \tau^*\Sigma_0.\ell &= 0, \end{aligned} \tag{2.9}$$

we can use

$$H + \varepsilon\tau^*\Sigma_1 \quad \text{and} \quad H + \varepsilon\tau^*\Sigma_0 \quad \text{with } \varepsilon > 0 \text{ small}$$

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as the supporting divisors defining ϕ_1 and ϕ_0 respectively.

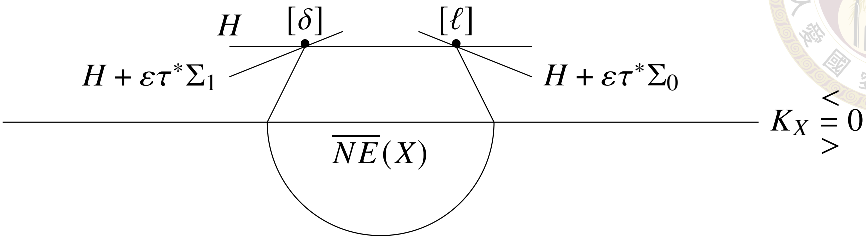

Figure 2.2: Cone of curve of a type II threefold.

From (2.9), one also sees that δ and ℓ are not numerically equivalent, so $R_1 \neq R_0$ and we know that $\phi_1(E_2) = \phi_1(\ell) =: \gamma_1$ and $\phi_0(E_2) = \phi_0(\delta) =: \gamma_0$ are curves. Since X is smooth, Mori's classification of extremal contractions is applicable; in particular, we find that ϕ_1 and ϕ_0 both belong to the case (3.3.1) *"extremal contraction to curve"* in [Mor82, Theorem 3.3], which shows that X_1 and X_0 are both smooth.

Now it suffices to show that X_1 and X_0 are actually minimal, i.e., they have nef canonical divisors. To avoid cumbersome notations, we shall write $\phi := \phi_0 : X \to X_0$ for now. Since ϕ is a smooth blowup that contracts the exceptional divisor E_2 to a curve γ_0, by [Har77, Exercise II.8.5(b)] we have

$$K_X = \phi^* K_{X_0} + E_2, \quad \text{or } \phi^* K_{X_0} = K_X - E_2 = H.$$

Let $C \subset X_0$ be an irreducible curve.

Case 1. $C = \gamma_0$. Then by projection formula,

$$K_{X_0}.\gamma_0 = K_{X_0}.\phi_*\delta = \phi_*(\phi^* K_{X_0}.\delta) = \phi_*(H.\delta) = 0.$$

Case 2. $C \neq \gamma_0$. Notice that ϕ is an isomorphism outside E_2, so

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- either $C \cap \gamma_0 = \{\text{points}\}$, in which case we set $C' = \overline{\phi^{-1}(C - \gamma_0)}$;
- or $C \cap \gamma_0 = \emptyset$, in which case we set $C' = \phi^{-1}(C)$.

In either case, we would have $\phi_* C' = C$, so by projection formula again we obtain

$$K_{X_0}.C = K_{X_0}.\phi_* C' = \phi_*(\phi^* K_{X_0}.C') = \phi_*(H.C') \geq 0.$$

This shows that K_{X_0} is nef. Since δ and ℓ are completely symmetric in this construction, the same argument also shows that K_{X_1} is nef. □

This Proposition 2.3 characterizes the minimal models of a threefold X of type II, namely $X_1^{\min} := X_1$ and $X_0^{\min} := X_0$. Next, we show that the contraction corresponding to H produces the (unique) canonical model of X.

Lemma 2.4. Let X be a threefold of type II and E_2, M, and H be as before. Then,

(a) $3H - K_X = H + M$ is nef and big.

(b) The linear system $|mH|$ is base-point free for $m \gg 0$, i.e., H is semiample.

(c) The morphisms $\psi_m := \phi_{|mH|} : X \longrightarrow X'_m \subset \mathbb{P}^N$ stabilize, that is, there exists a morphism $\psi : X \longrightarrow X'$ such that $X'_m \simeq X'$ and $\psi_m = \psi$ for sufficiently large and divisible m.

(d) The morphism ψ in (c) is birational onto its image.

Proof. (a) is evident from Lemma 2.2: both H and M are nef, and $(H+M)^3 \geq H^3 = 2 > 0$ so $H + M$ is big. Then (b) follows from (a) immediately by the Base-point Free theorem [KM98, Theorem 3.3].

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From (b) we know that ψ_m are indeed morphisms, and the assertion (c) follows from [Laz04, Theorem 2.1.27]. Notice that, by abuse of notation, we use the same symbol ψ_m for the map $X \to \mathbb{P}^N$ and its restriction onto image $X \to X'_m$. Finally, (d) is a consequence of the Iitaka fibration theorem [Laz04, Theorem 2.1.33]; see also the paragraph after [Laz04, Definition 2.2.1]. □

Proposition 2.5. Let X be a threefold of type II. Then the threefold X' in the previous Lemma 2.4 is the (unique) canonical model X^{can} of X, where ψ contracts $E_2 \subset X$ to a point.

Proof. From [KM98, Theorem 3.52(1)], we know that X^{can} is uniquely specified by

$$X^{\mathrm{can}} = \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, mK_X),$$

so it suffices to show that this is isomorphic to X'.

Consider the exact sequence associated with the closed subvariety E_2

$$0 \longrightarrow \mathscr{O}_X(-E_2) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{E_2} \longrightarrow 0.$$

For any pair of $m, m' \in \mathbb{Z}_{>0}$, we can tensor the above sequence with $\mathscr{O}_X(mH + m'E_2)$ to obtain

$$\begin{aligned} 0 \longrightarrow \mathscr{O}_X(mH + (m'-1)E_2) \longrightarrow \mathscr{O}_X(mH + m'E_2) \\ \longrightarrow \mathscr{O}_X(mH + m'E_2)|_{E_2} \longrightarrow 0. \end{aligned}$$

This induces a long exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow H^0(X, mH + (m'-1)E_2) \longrightarrow H^0(X, mH + m'E_2) \\ \longrightarrow H^0(E_2, (mH + m'E_2)|_{E_2}) \longrightarrow \cdots. \end{aligned}$$

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By Lemma 2.2 (a), $H^0(E_2, (mH + m'E_2)|_{E_2}) = H^0(E_2, m'(-\delta - \ell)) = 0$, so we have an isomorphism $H^0(X, mH + (m' - 1)E_2) \simeq H^0(X, mH + m'E_2)$, or in terms of linear systems, $|mH + m'E_2| = |mH + (m' - 1)E_2| + E_2$. Iterating this for each m' from m to 1, we eventually arrive at

$$|mK_X| = |mH + mE_2| = |mH| + mE_2 \text{ holds for all } m \in \mathbb{Z}_{>0}; \tag{2.10}$$

therefore,

$$\operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, mK_X) = \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, mH)$$

Now if we choose any m_0 such that $\psi_{m_0} = \psi$, then we see that $m_0 H = \psi^* \mathcal{O}_{X'}(1)$; since X' is projective, it follows that

$$\begin{aligned} X' &= \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X', \mathcal{O}_{X'}(m)) \\ &= \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, \psi^* \mathcal{O}_{X'}(m)) \quad \text{since } \psi \text{ is birational} \\ &= \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, m m_0 H) \\ &= \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, mH) = \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, mK_X) = X^{\text{can}}, \end{aligned}$$

as desired.

The description of ψ follows from the following claim.

Claim. For any sufficiently large and divisible $m \in \mathbb{Z}_{>0}$ and any irreducible curve $C \subset X$,

$$H.C = 0 \quad \text{if and only if} \quad \phi_{|mH|} \text{ contracts } C.$$

Indeed, since $H.\delta = H.\ell = 0$ and the rulings by δ and ℓ covers the whole E_2, this

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claim along with Lemma 2.2 (c) would imply that $\phi_{|mH|}$ contracts E_2 to a point.

Now we prove the claim. Suppose $H.C = 0$, then $mH|_C = 0$ on C since it is an effective divisor of degree 0; hence, $H^0(C, mH|_C) = H^0(\mathscr{O}_C) = \mathbb{C}$, so every section of mH restricts to constant on C, to wit, $\phi_{|mH|}(C) = \{\text{point}\}$. Conversely, if $H.C > 0$, then we can find a section $s \in H^0(X, mH)$ that vanishes only on H, so $s|_C$ vanishes only on $H \cap C$ but nowhere else on C, which shows that $\phi_{|mH|}$ is non-constant on C. □

Remark 2.6. In the last paragraph of the proof above, the restriction map $H^0(X, mH) \to H^0(C, mH|_C)$ must be surjective in the case $H.C = 0$, otherwise C would be a base curve, contradicting the fact that mH is base-point free.

Remark 2.7. To sum up, the maps ϕ_1, ϕ_2, and ψ give different contractions of the exceptional divisor E_2. In fact, one can see that the two minimal models of X are related by an *Atiyah flop*.

Remark 2.8. The observation (2.10) provides an alternative way of finding $\mathrm{vol}(X)$: since $h^0(mK_X) = h^0(mH)$, we have $\mathrm{vol}(X) = \mathrm{vol}(H) = H^3 = 2$ because H is nef and big. The same method applies for type-III examples below.

Models of Threefolds of Type III. Let X be a threefold of type III. Again, since τ restricts to an isomorphism on the branch locus containing Σ_2, we set $E_2 = \tau^{-1}(\Sigma_2) \subset X$ so that $E_2 \simeq \Sigma_2$ as varieties, and we can identify $\mathrm{Pic}\, E_2$ with $\mathrm{Pic}\, \Sigma_2 \simeq \mathrm{Pic}\, W = \mathbb{Z}\delta \oplus \mathbb{Z}\ell$ as before; on the other hand, $\tau^*\Sigma_2 = 2E_2$ as divisors, so we can write (2.7) as

$$K_X = 2E_2 + \tau^*(\Sigma_1 + 2\Sigma_0).$$

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We also set

$$M = \tau^*(\Sigma_1 + 2\Sigma_0) \quad \text{and} \quad H = E_2 + M.$$

Lemma 2.9. Let X be a threefold of type III and E_2, M, and H be as above. Let $C \subset X$ be any irreducible curve. Then,

(a) $E_2|_{E_2} = -\delta - 2\ell$ and $H|_{E_2} = 0$.

(b) M is nef; moreover, $M.C = 0$ if and only if $p\tau(C) = \delta$ in $W = \mathbb{F}_2$.

(c) H is nef and big with $H^3 = 2$; moreover, $H.C = 0$ if and only if $C \subset E_2$ or $p\tau(C) = \delta$ in W with $C \cap E_2 = \varnothing$.

(d) $K_X.\delta = 0$ and $K_X.\ell = -1$.

(e) $K_X.C < 0$ if and only if $C = \ell \subset E_2$ or $C \in |a\delta + b\ell|$ with $b \geq 2a > 0$.

(f) $K_X.C = 0$ if and only if $C = \delta \subset E_2$ or $p\tau(C) = \delta \subset W$ with $C \cap E_2 = \varnothing$.

Proof. The proof is roughly parallel to the proof of Lemma 2.2. Since $(\tau^*\Sigma_1)|_{E_2} = \delta$ and $(\tau^*\Sigma_0)|_{E_2} = \ell$, by (2.7) the adjunction formula reads

$$(K_X + E_2)|_{E_2} = K_{E_2}$$
$$3E_2|_{E_2} + \delta + 2\ell = -2\delta - 4\ell,$$

which gives $E_2|_{E_2} = -\delta - 2\ell$ and $H|_{E_2} = [E_2 + \tau^*(\Sigma_1 + 2\Sigma_0)]|_{E_2} = 0$, the desired formulas in (a).

In contrary to Lemma 2.2, this time $\delta + 2\ell$ is not very ample but only nef on W:

$$(\delta + 2\ell).(a\delta + b\ell) = -2a + 2a + b + 0 = b;$$

if the linear system $|a\delta + b\ell|$ on W contains an irreducible curve, then $b \geq 0$ by [Har77,

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Corollary V.2.18]. Nevertheless, it still follows from the projection formula that

$$M.C = (p\tau)_*(M.C) = (p\tau)_*((p\tau)^*(\delta + 2\ell).C) = (\delta + 2\ell).(p\tau)_*C,$$

which is non-negative. More precisely, $M.C = 0$ when $p\tau(C)$ is a point; if $p\tau(C)$ is a curve, then by [Har77, Corollary V.2.18] again, either $b > 0$ or $(a, b) = (1, 0)$. In the former case, $M.C > 0$; in the latter case, since $h^0(W, \delta) = 1$ implies that $|\delta|$ has only one irreducible curve δ, $p\tau(C)$ is necessarily δ. This proves (b).

Next, we show that H is nef. Suppose contrarily that $C \subset X$ is an irreducible curve such that $H.C < 0$. If C is contained in a fiber of $p\tau$, then $E_2.C > 0$, so $H.C = (E_2 + M).C > 0$, a contradiction. On the other hand, if $p\tau(C)$ is not a point, it must be a curve, then the hypothesis and nefness of M from (b) imply that

$$E_2.C = H.C - M.C < 0,$$

so C must lie in E_2. But $H|_{E_2} = 0$ by (a), so $H.C = H|_{E_2}.C = 0$, a contradiction again. This proves that H is nef.

Let us denote $P \coloneqq \Sigma_1 + 2\Sigma_0$ for brevity. We infer from (2.7) the following intersection numbers (be careful that $\delta^2 = -2$ this time):

$$\begin{aligned} \Sigma_2^3 &= (\Sigma_2|_{\Sigma_2})^2 = (-2\delta - 4\ell)^2 && = 8; \\ \Sigma_2^2.P &= (\Sigma_2|_{\Sigma_2}).(P|_{\Sigma_2}) = (-2\delta - 4\ell).(\delta + 2\ell) && = -4; \\ \Sigma_2.P^2 &= (P|_{\Sigma_2})^2 = (\delta + 2\ell)^2 && = 2; \\ P^3 & && = 0, \end{aligned} \tag{2.11}$$

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so

$$H^3 = \left[\tau^*\left(\frac{1}{2}\Sigma_2 + \Sigma_1 + 2\Sigma_0\right)\right]^3 = \deg\tau\cdot\left(\frac{1}{2}\Sigma_2 + P\right)^3$$
$$= 2\cdot\left(\frac{1}{8}\cdot 8 + 3\cdot\frac{1}{4}\cdot(-4) + 3\cdot\frac{1}{2}\cdot 2 + 0\right) = 2.$$

This proves that H is big by the characterization [Laz04, Theorem 2.2.16].

Let us now characterize irreducible curves C such that $H.C = 0$. For such C, the previous paragraph shows that $p\tau(C)$ is a curve and that either $E_2.C < 0$ or $E_2.C = M.C = 0$. In the former case, $C \subset E_2$ and we indeed have $H.C = 0$; in the latter case, $M.C = 0$ implies that $p\tau(C) = \delta$ by (b), and $E_2.C = 0$ implies that either $C \cap E_2 = \varnothing$ or $C = \delta \subset E_2$. Now (c) is proven.

Notice that $K_X = E_2 + H$ with H nef, so if $K_X.C < 0$, then we must have $E_2.C < 0$ and thus $C \subset E_2$. In this case $H.C = 0$, so $K_X.C = E_2.C$ and we have

$$\left.\begin{aligned} E_2.\delta &= (-\delta - 2\ell).\delta = 0 \\ E_2.\ell &= (-\delta - 2\ell).\ell = -1 \end{aligned}\right\} \implies E_2.(a\delta + b\ell) = -b, \tag{2.12}$$

proving (d) and (e). The condition $b \geq 2a$ is imposed to guarantee that $|a\delta + b\ell|$ contains irreducible curves.

Similarly, if $K_X.C = 0$, then either $E_2.C = -H.C < 0$ or $E_2.C = H.C = 0$. The first case is impossible since $C \subset E_2$ implies $H.C = 0$; now the assertion (f) follows from (c) and the calculation (2.12). □

Proposition 2.10. Let X be a threefold of type III. The ray R in $\overline{NE}(X)$ generated by the class of $\ell \subset E_2$ is the only K_X-negative extremal ray, and its corresponding contraction

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gives rise to a smooth minimal model of X:

$$\phi \coloneqq \text{cont}_R : X \to X^{\min}.$$

Proof. We temporarily denote the image of ϕ by X_0. We will show that it is a minimal model of X.

From previous Lemma 2.2 (d), we see that a K_X-negative curve must belong to some numerical class $a[\delta] + b[\ell]$ with $b > 0$. Since $[\delta]$ is a K_X-trivial class, we see that the ray $R = \mathbb{R}_{>0}[\ell]$ is the only K_X-negative extremal ray in $\overline{NE}(X)$. The situation is illustrated in the schematic Figure 2.3 below.

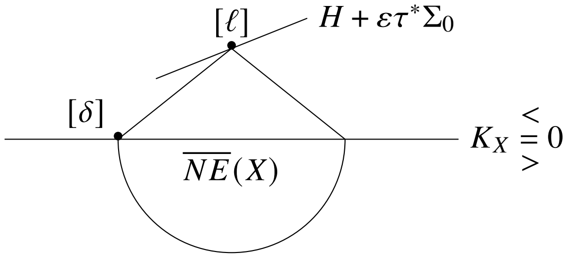

Figure 2.3: Cone of curve of a type III threefold.

The existence of a contraction morphism ϕ is then guaranteed by the Contraction theorem [KM98, Theorem 3.7(3)] again, with $H + \varepsilon\tau^*\Sigma_0$ being the supporting divisor.

The rest of the argument goes verbatim as Proposition 2.3. □

This way, we obtain the minimal model $X^{\min} \coloneqq X_0$ of a threefold X of type III. Next, we show that the contraction corresponding to H again produces the (unique) canonical model of X.

Lemma 2.11. Let X be a threefold of type III and E_2, M, and H be as before. Then,

(a) $3H - K_X = H + M$ is nef and big.

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(b) The linear system $|mH|$ is base-point free for $m \gg 0$, i.e., H is semiample.

(c) The morphisms $\psi_m := \phi_{|mH|} : X \to X'_m \subset \mathbb{P}^N$ stabilize, that is, there exists a morphism $\psi : X \to X'$ such that $X'_m \simeq X'$ and $\psi_m = \psi$ for sufficiently large and divisible m.

(d) The morphism ψ in (c) is birational onto its image.

Proposition 2.12. Let X be a threefold of type III. Then the threefold X' in the previous Lemma 2.11 is the (unique) canonical model X^{can} of X, where ψ contracts $E_2 \cup \tau^{-1}(\Sigma_1) \subset X$. More precisely, $\psi(E_2)$ is a point and $\psi(\tau^{-1}(\Sigma_1))$ is a curve passing through that point.

Proof of Lemma 2.11 and Proposition 2.12. The astute readers must have already noticed that the assertions are nearly verbatim as Lemma 2.4 and Proposition 2.5 except for the last one, so the proof works almost word for word as well. It suffices to specify which curves are contracted by ψ and what their images are.

Recall that an irreducible curve C is contracted by ψ if and only if $H.C = 0$; such curves are described in Lemma 2.9 (c). Since $H.\delta = H.\ell = 0$ and δ connects the ruling on E_2 by ℓ, we see that ψ contracts E_2 to a point just as in Proposition 2.5. To deal with $(p\tau)^{-1}\delta = \tau^{-1}(\Sigma_1)$, we first need a lemma

Lemma 2.13. The projection $p : Y \to W$ restricted onto Σ_1 gives a structure of ruled surface $\Sigma_1 \to \mathbb{P}^1$; in fact, $\Sigma_1 \simeq \mathbb{F}_0$. Denote by δ_1 and ℓ_1 the section and the fiber on Σ_1 respectively, then

$$\Sigma_0|_{\Sigma_1} = \ell_1, \quad \Sigma_2|_{\Sigma_1} = \delta_1, \quad \text{and} \quad \Sigma_1|_{\Sigma_1} = -2\ell_1.$$

In view of this Lemma,

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- The (connected components of) the preimages of δ_1 form a ruling on $\tau^{-1}(\Sigma_1)$.

- The curve $\ell_1' \coloneqq \tau^{-1}\ell_1 \subset \tau^{-1}(\Sigma_1)$ is pushed to $\tau_*\ell_1' = 2\ell_1$ as τ is a double cover, so

$$\tau_*(H.\ell_1') = (\frac{1}{2}\Sigma_2 + \Sigma_1 + 2\Sigma_0).2\ell_1 = (\frac{1}{2}\delta_1 - 2\ell_1 + 2\ell_1).2\ell_1 = 1 > 0.$$

  In particular, ℓ_1' is not contracted by ψ.

This implies that $\psi(\tau^{-1}(\Sigma_1)) = \psi(\ell_1')$ is a curve on X'.

It remains to prove Lemma 2.13. On a ruled surface $\mathbb{F}_e$, if there exists an effective irreducible curve $C \in |a\delta + b\ell|$ such that $C^2 = -a^2e + 2ab = 0$, then e must be zero because when $b = ae/2 \neq 0$ there is no irreducible curve in the linear system $|a\delta + b\ell|$ by [Har77, Corollary V.2.18]. In our case, $\Sigma_2|_{\Sigma_1}$ is such a curve:

$$(\Sigma_2|_{\Sigma_1})^2 = \Sigma_2.\Sigma_2.\Sigma_1 = (\Sigma_1|_{\Sigma_2}).(\Sigma_2|_{\Sigma_2}) = \delta.(-2\delta - 4\ell) = 0,$$

so $\Sigma_1 \simeq \mathbb{F}_0$. The formulas $\Sigma_0|_{\Sigma_1} = \ell_1$ and $\Sigma_2|_{\Sigma_1} = \delta_1$ are evident, and using the adjunction formula gives the declared formula for $\Sigma_1|_{\Sigma_1}$. □

2.3 Canonical Images

Recall that the *canonical image* of a projective variety is the closure of the image of the canonical map $\phi_{|K_X|}$. This is a birational invariant because $H^0(X, K_X)$ is. In this section, we will compute the canonical image of the threefolds of type I, II, and III; in particular, the upshot is that these three families I, II, and III have non-isomorphic canonical images, so they are genuinely different birational classes.

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Canonical Image of Threefolds of Type I.

Proposition 2.14. The canonical image of a threefold X of type I is $\mathbb{P}^3$.

Proof. Recall from 2.1 that

$$\begin{aligned} H^0(X, K_X) &= H^0(X, \tau^*\mathcal{O}_{\mathbb{P}^3}(1)) \\ &= H^0(Y, \mathcal{O}_{\mathbb{P}^3}(1)) \oplus H^0(Y, \mathcal{O}_{\mathbb{P}^3}(-4)) \quad \text{by (2.2)} \\ &= H^0(Y, \mathcal{O}_{\mathbb{P}^3}(1)), \end{aligned}$$

so the image of X under $\phi_{|K_X|}$ is equal to the image of $\mathbb{P}^3$ under $\phi_{|\mathcal{O}(1)|}$, which is simply $\mathbb{P}^3$. □

Canonical Image of Threefolds of Type II.

Lemma 2.15. Let X be a threefold of type II. Then M and $2E_2$ are respectively the moving part and the fixed part of the linear system of K_X, in other words,

$$|K_X| = |M| + 2E_2.$$

Proof. The inclusion $|M| + 2E_2 \subset |K_X|$ is evident, so it suffices to show that $|M|$ and $|K_X|$ have the same dimensions. We already know that $h^0(X, K_X) = p_g(X) = 4$; on the other hand,

$$\begin{aligned} h^0(X, M) &= h^0(X, \tau^*(\Sigma_1 + \Sigma_0)) \\ &= h^0(X, \tau^*(\Sigma_1 + \Sigma_0)) + h^0(X, \tau^*(-3\Sigma_2 - 4\Sigma_1 - 4\Sigma_0)) \quad \text{by (2.6)} \\ &= h^0(W, \mathcal{O}_W(1,1)) + 0 = 4, \end{aligned}$$

so $\dim|M| = 4 - 1 = \dim|K_X|$. This completes the proof. □

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Proposition 2.16. The canonical image of a threefold X of type II is the image of $\mathbb{F}_0$ embedded into $\mathbb{P}^3$ via the linear system $|\delta + \ell|$.

Proof. By Lemma 2.15,

$$\begin{aligned}\overline{\phi_{|K_X|}(X)} &= \overline{\phi_{|M|+2E_2}(X)} \\ &= \overline{\phi_{|M|}(X)} \\ &= \overline{\phi_{|(p\tau)^*(\delta+\ell)|}(X)} \\ &= \overline{\phi_{|\delta+\ell|}(W)},\end{aligned} \tag{2.13}$$

so the canonical image of X is precisely the image of $W = \mathbb{F}_0$ under the morphism given by $|\delta + \ell|$, which is an embedding into $\mathbb{P}^3$ because $\delta+\ell$ is very ample and $h^0(W, \delta+\ell) = 4$. □

Canonical Image of Threefolds of Type III.

Lemma 2.17. Let X be a threefold of type III. Then again, M and $2E_2$ are respectively the moving part and the fixed part of the linear system of K_X, in other words,

$$|K_X| = |M| + 2E_2.$$

Proof. The argument is essentially the same as Lemma 2.15. We already have $h^0(X, K_X) = p_g(X) = 4$, and since $\mathscr{L}^\vee = -(3\Sigma_2 + 5\Sigma_1 + 10\Sigma_0)$,

$$\begin{aligned}h^0(X, M) &= h^0(X, \tau^*(\Sigma_1 + 2\Sigma_0)) \\ &= h^0(X, \tau^*(\Sigma_1 + 2\Sigma_0)) + h^0(X, \tau^*(-3\Sigma_2 - 4\Sigma_1 - 8\Sigma_0)) \\ &= h^0(W, \mathscr{O}_W(1, 2)) + 0 = 4,\end{aligned}$$

so $\dim|M| = 4 - 1 = \dim|K_X|$. This completes the proof. □

Proposition 2.18. The canonical image of a threefold X of type III is the singular quadric

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defined by $xz = y^2$ in $\mathbb{P}^3$, which is isomorphic to the weighted projective space $\mathbb{P}(1,1,2)$.

Proof. Here, $M = (p\tau)^*(\delta + 2\ell)$, so the same reasoning as (2.13) shows that the canonical image of X is the image of morphism $\varphi : \mathbb{F}_2 \to \mathbb{P}^3$ defined by the linear system $\mathfrak{d} := |\delta + 2\ell|$. Notice that $\mathfrak{d}$ is only base-point free but not very ample in this case, so φ is not an embedding.

It suffices to show that the image of φ is the asserted singular quadric. Since $H^1(\mathbb{F}_2, 2\ell) = H^1(\mathbb{P}^1, \mathcal{O}(2)) = 0$, we have a short exact sequence

$$0 \longrightarrow H^0(\mathbb{F}_2, 2\ell) \longrightarrow H^0(\mathbb{F}_2, \delta + 2\ell) \xrightarrow{\text{restr.}} H^0(\delta, (\delta + 2\ell)|_\delta) \longrightarrow 0,$$

where

- $H^0(\mathbb{F}_2, 2\ell) = H^0(\mathbb{P}^1, \mathcal{O}(2)) = 3$. We choose basis elements x, y, z to be the pullback of monomials x_0^2, x_0x_1, x_1^2 on $\mathbb{P}^1$ with coordinates x_0 and x_1, so there is a relation $xz = y^2$ among the generators.
- Identifying $\delta \simeq \mathbb{P}^1$, we have $\ell|_\delta = \mathcal{O}_{\mathbb{P}^1}(1)$ and the adjunction formula gives

$$\begin{aligned} (K_{\mathbb{F}_2} + \delta)|_\delta &= K_\delta \\ -\delta|_\delta - 4\ell|_\delta &= \mathcal{O}_{\mathbb{P}^1}(-2) \end{aligned} \implies (\delta + 2\ell)|_\delta = 0,$$

  so $H^0(\delta, (\delta + 2\ell)|_\delta) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$. We choose a generator $w \neq 0$.
- By abuse of notation, we denote by the same symbols x, y, z for their image and w for its lifting in $H^0(\mathbb{F}_2, \delta + 2\ell)$. Since x, y, z restrict to 0 on δ by exactness, $\{x, y, z, w\}$ form a basis of $H^0(\mathbb{F}_2, \delta + 2\ell)$ and we can use this basis to define the map $\varphi : \mathbb{F}_2 \to \mathbb{P}^3$ by $\mathfrak{p} \to [x(\mathfrak{p}) : y(\mathfrak{p}) : z(\mathfrak{p}) : w(\mathfrak{p})]$.

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The relation among generators indicates that the image of φ is $V(xz - y^2) \subset \mathbb{P}^3$, which is well-known to be isomorphic to $\mathbb{P}(1,1,2)$, cf. [CLS11, Example 2.0.5]. □

Since the canonical image is birationally invariant, we obtain the following corollary.

Corollary 2.19. Threefolds of types I, II, and III belong to different birational classes.

2.4 Summary

In conclusion, we have the following "zoo" of threefolds of type-$(2,4)$.

Models :

$$\begin{array}{c} X_{\mathrm{I}} \\ \| \\ X_{\mathrm{I}}^{\mathrm{min}} \\ \| \\ X_{\mathrm{I}}^{\mathrm{can}} \end{array} \qquad \begin{array}{ccccc} & & X_{\mathrm{II}} & & \\ & \overset{\phi_1}{\swarrow} & \Big\downarrow \psi_{\mathrm{II}} & \overset{\phi_0}{\searrow} & \\ X_{\mathrm{II},1}^{\mathrm{min}} & & & & X_{\mathrm{II},0}^{\mathrm{min}} \\ & \searrow & & \swarrow & \\ & & X_{\mathrm{II}}^{\mathrm{can}} & & \end{array} \qquad \begin{array}{c} X_{\mathrm{III}} \\ \downarrow \phi_{\mathrm{III}} \\ X_{\mathrm{III}}^{\mathrm{min}} \\ \downarrow \\ X_{\mathrm{III}}^{\mathrm{can}} \end{array} \quad \psi_{\mathrm{III}} : X_{\mathrm{III}} \to X_{\mathrm{III}}^{\mathrm{can}}$$

Canonical image : $\mathbb{P}^3$ $\qquad$ $\mathbb{F}_0$ $\qquad$ $\mathbb{P}(1,1,2)$

Here,

- Every model is smooth except for $X_{\mathrm{II}}^{\mathrm{can}}$ and $X_{\mathrm{III}}^{\mathrm{can}}$.
- ϕ_1 and ϕ_0 are the extremal contractions corresponding to δ and ℓ respectively; they both contract E_2 to a curve.
- ψ_{II} contracts E_2 to a point.
- ϕ_{III} is the extremal contraction corresponding to ℓ, which contracts E_2 to a curve.
- ψ_{III} contracts E_2 to a point and $\tau^{-1}(\Sigma_1)$ to a curve through that point.

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Chapter 3 Moduli Aspects of Threefolds of Type-$(2,4)$

3.1 Sizes of Space of Deformations

The main result of this section is the following.

Theorem 3.1. Let $\mathscr{T}_X$ denote the *tangent sheaf* of X. Then the dimensions of the cohomology $H^i(X, \mathscr{T}_X)$ are as listed in Table 3.1 below, where x is either 2 or 3.

Type of X	Type I	Type II	Type III
$h^0(X, \mathscr{T}_X)$	0	0	0
$h^1(X, \mathscr{T}_X)$	270	269	270
$h^2(X, \mathscr{T}_X)$	0	0	x
$h^3(X, \mathscr{T}_X)$	0	0	$x-1$

Table 3.1: Cohomology of the tangent sheaf.

Remark 3.2. The space $H^1(X, \mathscr{T}_X)$ is interesting because it parametrizes infinitesimal deformations of X. See [Har77, Example III.9.13.2].

Remark 3.3. One should compare this Theorem 3.1 for types I and II with the analogous result [Hor76, Theorem 2.1] in the surface case, where the "Hirzebruch-like" case also has dimension 1 less than the "$\mathbb{P}^n$-like" case.

The rest of this section is devoted to the proof of Theorem 3.1. The general strategy to compute the cohomologies of $\mathscr{T}_X$ is to exploit the following exact sequence [Ser06,

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Definition 3.4.5]

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \tau^*\mathscr{T}_Y \longrightarrow \mathscr{T}_{X/Y} \longrightarrow 0, \tag{3.1}$$

which is applicable for any cyclic cover $\tau : X \to Y$. For each type of example, the computation will be organized into 4 steps:

- *Step 0.* Show that $h^0(X, \mathscr{T}_X) = 0$. Suppose contrarily that $H^0(X, \mathscr{T}_X)$ has a nonzero section v. Pick some nonzero $s \in H^0(X, K_X)$, then $s(v, -, -)$ defines a nonzero global 2-form, i.e. a nonzero element in $H^0(X, \bigwedge^2 \Omega_X)$. By Hodge symmetry, we will show that $H^2(X, \mathscr{O}_X) = 0$ to derive a contradiction.

  Alternatively, one may observe that $h^0(X, \mathscr{T}_X) = 0$ is in fact true for any variety X of general type. Indeed, $h^0(X, \mathscr{T}_X)$ is the dimension of $\mathrm{Aut}(X)$ as a complex Lie group, but in this case $\mathrm{Aut}(X)$ is finite by a finiteness theorem of Kobayashi and Ochiai [KO75].

- *Step 1.* Compute the cohomologies of $\mathscr{T}_{X/Y}$. By [Hor75, Lemma 10], for a double cover $\tau : X \to Y$, we know that

$$\mathscr{T}_{X/Y} \simeq \tau^*\mathscr{N}_B,$$

  where $\mathscr{N}_B \simeq \iota_*(\mathscr{O}_Y(B)|_B) = \iota_*\iota^*\mathscr{O}_Y(B)$ is the normal sheaf of the branch locus B and $\iota : B \hookrightarrow Y$ is the closed immersion. Notice that $\mathscr{N}_B$ is supported on the branch locus, on which τ is an isomorphism, so the cohomology of $\tau^*\mathscr{N}_B$ is the same as that of $\mathscr{N}_B$. To compute it, we take the exact sequence associated with a closed subvariety

$$0 \longrightarrow \mathscr{O}_Y(-B) \longrightarrow \mathscr{O}_Y \longrightarrow \iota_*\mathscr{O}_B \longrightarrow 0$$

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and tensor it with $\mathscr{O}_Y(B)$ to get

$$0 \longrightarrow \mathscr{O}_Y \longrightarrow \mathscr{O}_Y(B) \longrightarrow \iota_*\iota^*\mathscr{O}_Y(B) \simeq \mathscr{N}_B \longrightarrow 0 \tag{3.2}$$

by the projection formula. Then the cohomology of $\mathscr{N}_B$ can be extracted from those of $\mathscr{O}_Y$ and $\mathscr{O}_Y(B)$.

- *Steps 2 and 3.* Compute the cohomologies of $\tau^*\mathscr{T}_Y$. Since τ is a finite morphism, it has no higher direct image, so the Leray spectral sequence degenerates at page E_2 and gives

$$\begin{aligned} h^i(X, \tau^*\mathscr{T}_Y) &= h^i(Y, \tau_*\tau^*\mathscr{T}_Y) \\ &= h^i(Y, \tau_*\mathscr{O}_X \otimes \mathscr{T}_Y) && \text{projection formula} \\ &= h^i(Y, \mathscr{T}_Y) + h^i(Y, \mathscr{T}_Y \otimes \mathscr{L}^\vee) && \text{Proposition 1.1 (c).} \end{aligned} \tag{3.3}$$

Since Y is a toric variety in all cases, the first term can be computed by the toric Euler sequence [CLS11, Theorem 8.1.6]

$$0 \longrightarrow \Omega^1_Y \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_Y(-D_\rho) \longrightarrow \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathscr{O}_Y \longrightarrow 0,$$

where Σ denotes the fan of Y. If $\mathscr{E}xt(\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathscr{O}_Y, \mathscr{O}_Y) = 0$, we can take its dual to obtain

$$0 \longrightarrow (\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathscr{O}_Y)^\vee \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_Y(D_\rho) \longrightarrow \mathscr{T}_Y \longrightarrow 0, \tag{3.4}$$

from which we deduce the cohomology of $\mathscr{T}_Y$ as well as $\mathscr{T}_Y \otimes \mathscr{L}^\vee$ after tensoring (3.4) with $\mathscr{L}^\vee$.

Alternatively, sometimes the cohomology of $\mathscr{T}_Y \otimes \mathscr{L}^\vee$ can be obtained using the

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exact sequence from [Ser06, Definition 3.4.5] again

$$0 \longrightarrow \Theta_{Y/W} \longrightarrow \mathscr{T}_Y \longrightarrow p^*\mathscr{T}_W \longrightarrow 0 \tag{3.5}$$

and tensoring with $\mathscr{L}^\vee$. This is more efficient than the Euler sequence when eligible, but sadly it cannot be applied to compute $h^i(Y, \mathscr{T}_Y)$ because there would be too many non-zero terms in the long exact sequence induced by (3.5).

Computation for Threefolds of Type I.

Step 0.

$$\begin{aligned} h^2(X, \mathscr{O}_X) &= h^2(\mathbb{P}^3, \tau_*\mathscr{O}_X) \\ &= h^2(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}) + h^2(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(-5)) = 0, \end{aligned}$$

so $h^0(X, \mathscr{T}_X) = 0$ by previous discussion.

Step 1. Rewriting the terms in (3.2) allows us to fill in the cohomology:

	0	$\mathscr{O}_{\mathbb{P}^3}$	$\mathscr{O}_{\mathbb{P}^3}(10)$	$\mathscr{N}_B$	0
h^0 :	0	1	$\binom{13}{3} = 286$	**285**	
h^1 :		0	0	**0**	
h^2 :		0	0	**0**	
h^3 :		0	0	**0**	0.

Step 2. In view of (3.3),

$$h^i(X, \tau^*\mathscr{T}_Y) = h^i(Y, \mathscr{T}_Y) + h^i(Y, \mathscr{T}_Y \otimes \mathscr{O}(-5)).$$

The first term can be computed using the (dual of the) Euler sequence for projective spaces

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[Har77, Example II.8.20.1]:

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & \mathscr{O}_{\mathbb{P}^3} & \longrightarrow & \mathscr{O}_{\mathbb{P}^3}(1)^{\oplus 4} & \longrightarrow & \mathscr{T}_{\mathbb{P}^3} & \longrightarrow & 0 \\
h^0: & 0 & & 1 & & 16 & & \mathbf{15} & & \\
h^1: & & & 0 & & 0 & & \mathbf{0} & & \\
h^2: & & & 0 & & 0 & & \mathbf{0} & & \\
h^3: & & & 0 & & 0 & & \mathbf{0} & & 0.
\end{array}$$

Tensoring this sequence with $\mathscr{O}(-5)$, we can also compute the second term:

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & \mathscr{O}_{\mathbb{P}^3}(-5) & \longrightarrow & \mathscr{O}_{\mathbb{P}^3}(-4)^{\oplus 4} & \longrightarrow & \mathscr{T}_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(-5) & \longrightarrow & 0 \\
h^0: & 0 & & 0 & & 0 & & \mathbf{0} & & \\
h^1: & & & 0 & & 0 & & \mathbf{0} & & \\
h^2: & & & 0 & & 0 & & \mathbf{0} & & \\
h^3: & & & 4 & & 4 & & \mathbf{0} & & 0.
\end{array}$$

Therefore,

$$h^i(X, \tau^*\mathscr{T}_Y) = \begin{cases} 15, & \text{if } i = 0; \\ 0, & \text{if } i \geq 1. \end{cases}$$

Combining the results from Step 1 and Step 2, we conclude from the sequence (3.1) that

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & \mathscr{T}_X & \longrightarrow & \tau^*\mathscr{T}_Y & \longrightarrow & \mathscr{T}_{X/Y} & \longrightarrow & 0 \\
h^0: & 0 & & 0 & & 15 & & 285 & & \\
h^1: & & & \mathbf{270} & & 0 & & 0 & & \\
h^2: & & & \mathbf{0} & & 0 & & 0 & & \\
h^3: & & & \mathbf{0} & & 0 & & 0 & & 0.
\end{array}$$

In particular, $h^1(X, \mathscr{T}_X) = 270$ and $h^2(X, \mathscr{T}_X) = h^3(X, \mathscr{T}_X) = 0$.

Remark 3.4. This agrees with the general formula for $h^1(X, \mathscr{T}_X)$ given by Wavrik [Wav68, Theorem 7.2].

Computation for Threefolds of Type II.

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Step 0.

$$h^2(X,\mathscr{O}_X) = h^2(Y,\tau_*\mathscr{O}_X)$$
$$= h^2(Y,\mathscr{O}_Y) + h^2(Y,-3\Sigma_2 - 5\Sigma_1 - 5\Sigma_0) = 0,$$

so $h^0(X,\mathscr{T}_X) = 0$ by previous discussion.

Step 1. The first two terms of (3.2) can be readily computed by Corollary 1.4. For the first term,

$$h^i(Y,\mathscr{O}_Y) = h^i(W,\mathscr{O}_W) = \begin{cases} 1, & \text{if } i = 0; \\ 0, & \text{if } i = 1,2,3. \end{cases}$$

For the second term, since $B \sim 6\Sigma_2 + 10\Sigma_1 + 10\Sigma_0$, we have

$$h^i(Y,\mathscr{O}_Y(B)) = h^i\big(Y,\mathscr{O}_Y(6)\otimes p^*\mathscr{O}_W(10,10)\big)$$
$$= h^i\big(W,\mathrm{Sym}^6(\mathscr{O}_W\oplus\mathscr{O}_W(-2,-2))\otimes\mathscr{O}_W(10,10)\big)$$
$$= h^i\big(W,\mathscr{O}_W(-2,-2)\oplus\mathscr{O}_W\oplus\mathscr{O}_W(2,2)\oplus\cdots\oplus\mathscr{O}_W(10,10)\big)$$
$$= \begin{cases} 1^2+3^2+\cdots+11^2 = 286, & \text{if } i = 0 \text{ by Künneth formula}; \\ 0, & \text{if } i = 1; \\ 1, & \text{if } i = 2 \text{ by Serre duality.} \end{cases}$$

This yields

$$0 \longrightarrow \mathscr{O}_Y \longrightarrow \mathscr{O}_Y(B) \longrightarrow \iota_*\iota^*\mathscr{O}_Y(B) \simeq \mathscr{N}_B \longrightarrow 0$$

h^0 :	0	1	286	**285**	
h^1 :		0	0	**0**	
h^2 :		0	1	**1**	
h^3 :		0	0	**0**	0.

Step 2. We first show that the second term $h^1(Y,\mathscr{T}_Y\otimes\mathscr{L}^\vee)$ in (3.3) is zero. To do so, we

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tensor (3.5) with $\mathscr{L}^\vee$ to get

$$0 \longrightarrow \Theta_{Y/W} \otimes \mathscr{L}^\vee \longrightarrow \mathscr{T}_Y \otimes \mathscr{L}^\vee \longrightarrow p^*\mathscr{T}_W \otimes \mathscr{L}^\vee \longrightarrow 0. \tag{3.6}$$

Let us calculate the terms on the left and the right.

- By rank consideration, $\Theta_{Y/W}$ is a line bundle. Its corresponding divisor is

$$\begin{aligned} c_1(\Theta_{Y/W}) &= c_1(\mathscr{T}_Y) - c_1(p^*\mathscr{T}_W) \\ &= -K_Y + p^*K_W = 2\Sigma_2 + 2\Sigma_1 + 2\Sigma_0, \end{aligned}$$

so

$$\Theta_{Y/W} \otimes \mathscr{L}^\vee = \mathscr{O}_Y(-\Sigma_2 - 3\Sigma_1 - 3\Sigma_0).$$

Therefore, $h^i(Y, \Theta_{Y/W} \otimes \mathscr{L}^\vee) = 0$ for all $i = 0, 1, 2, 3$ by Corollary 1.4.

- For the term on the right, let $\pi : W \to \mathbb{P}^1$ be the projection and consider the following exact sequence coming from [Ser06, Definition 3.4.5] again:

$$0 \longrightarrow \Theta_{W/\mathbb{P}^1} \longrightarrow \mathscr{T}_W \longrightarrow \pi^*\mathscr{T}_{\mathbb{P}^1} \longrightarrow 0, \tag{3.7}$$

where

$$c_1(\Theta_{W/\mathbb{P}^1}) = c_1(\mathscr{T}_W) - \pi^*c_1(\mathscr{T}_{\mathbb{P}^1}) = 2\delta$$

and

$$\pi^*\mathscr{T}_{\mathbb{P}^1} = \pi^*\mathscr{O}_W(2\ell) = \mathscr{O}_Y(2\Sigma_0).$$

Pulling back via p yields

$$0 \longrightarrow \mathscr{O}_Y(2\Sigma_1) \longrightarrow p^*\mathscr{T}_W \longrightarrow \mathscr{O}_Y(2\Sigma_0) \longrightarrow 0,$$

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and tensoring with $\mathscr{L}^\vee$ gives

$$0 \to \mathscr{O}_Y(-3\Sigma_2 - 3\Sigma_1 - 5\Sigma_0) \to p^*\mathscr{T}_W \otimes \mathscr{L}^\vee \to \mathscr{O}_Y(-3\Sigma_2 - 5\Sigma_1 - 3\Sigma_0) \to 0.$$

Using Corollary 1.4 or Remark 2.1 again, the cohomologies of the two sides turn out to be all zero. Therefore, $h^i(p^*\mathscr{T}_W \otimes \mathscr{L}^\vee) = 0$ for all $i = 0, 1, 2, 3$.

Hence, the claim $h^i(Y, \mathscr{T}_Y \otimes \mathscr{L}^\vee) = 0$ for all $i = 0, 1, 2, 3$ now follows from the exact sequence (3.6). This way, we find that

$$h^i(X, \tau^*\mathscr{T}_Y) = h^i(Y, \mathscr{T}_Y)$$

and we only need to compute the latter.

Step 3. We shall describe Y as a toric variety so that the toric Euler sequence (3.4) is applicable and reads

$$0 \longrightarrow \mathscr{O}_Y^{\oplus 3} \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_Y(D_\rho) \longrightarrow \mathscr{T}_Y \longrightarrow 0, \tag{3.8}$$

where we used the fact that $\mathrm{Pic}(Y) \simeq \mathbb{Z}^3$ is generated by $\Sigma_2, \Sigma_1, \Sigma_0$ and that $\mathscr{E}xt^1\left(\mathscr{O}_Y^{\oplus 3}, \mathscr{O}_Y\right) = 0$. The problem then boils down to describing the rays ρ in the fan of Y and computing the cohomologies of their corresponding divisors D_ρ.

The fan of Y can be constructed using the toric description of projective bundles in [CLS11, p.338]. In favor of the convention in [CLS11], in this subsection we describe Y as

$$Y = \mathbb{P}(\mathscr{E}'), \quad \text{where } \mathscr{E}' = \mathscr{O}_W \oplus \mathscr{O}_W(2\delta + 2\ell),$$

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which is isomorphic to our original definition because

$$\mathbb{P}(\mathscr{E}') = \mathbb{P}(\mathscr{E} \otimes \mathscr{O}_W(2\delta + 2\ell)) \simeq \mathbb{P}(\mathscr{E}) = Y \quad \text{by [CLS11, Lemma 7.0.8(b)].}$$

In this case, $\operatorname{rank} \mathscr{E}' = 2$, so we have $e \coloneqq e_1 = (1, 0)$ and $e_0 = -e_1$, which gives $F_0 = \mathrm{Cone}(e)$ and $F_1 = \mathrm{Cone}(-e)$. We denote the minimal ray generators in the fan of $W = \mathbb{P}^1 \times \mathbb{P}^1$ by

$$u_1 = (1, 0), \quad u_2 = (0, 1), \quad u_3 = (-1, 0), \quad \text{and} \quad u_4 = (0, -1),$$

their respective rays by $\rho_i = \mathbb{R}_{\geq 0} u_i$ and the corresponding divisors by D_{ρ_i} $(i = 1, 2, 3, 4)$; see Figure 3.1 (a). This way, we can write $\mathscr{E}' = \mathscr{O}_W(D_0) \oplus \mathscr{O}_W(D_1)$ with

$$\begin{aligned} D_0 &= 0 && = 0D_{\rho_1} + 0D_{\rho_2} + 0D_{\rho_3} + 0D_{\rho_4}; \\ D_1 &= 2\delta + 2\ell && = 2D_{\rho_1} + 2D_{\rho_2} + 0D_{\rho_3} + 0D_{\rho_4}. \end{aligned}$$

Then,

$$\begin{aligned}
\tilde{\sigma}_{++,0} &= \mathrm{Cone}(u_1 + (2-0)e, u_2 + (2-0)e) + F_0 &&= \mathrm{Cone}(u_1 + 2e, \ u_2 + 2e, \ e) \\
\tilde{\sigma}_{++,1} &= \mathrm{Cone}(u_1 + (2-0)e, u_2 + (2-0)e) + F_1 &&= \mathrm{Cone}(u_1 + 2e, \ u_2 + 2e, \ -e) \\
\tilde{\sigma}_{+-,0} &= \mathrm{Cone}(u_1 + (2-0)e, u_4 + (0-0)e) + F_0 &&= \mathrm{Cone}(u_1 + 2e, \ u_4, \ e) \\
\tilde{\sigma}_{+-,1} &= \mathrm{Cone}(u_1 + (2-0)e, u_4 + (0-0)e) + F_1 &&= \mathrm{Cone}(u_1 + 2e, \ u_4, \ -e) \\
\tilde{\sigma}_{-+,0} &= \mathrm{Cone}(u_3 + (0-0)e, u_2 + (2-0)e) + F_0 &&= \mathrm{Cone}(u_3, \ u_2 + 2e, \ e) \\
\tilde{\sigma}_{-+,1} &= \mathrm{Cone}(u_3 + (0-0)e, u_2 + (2-0)e) + F_1 &&= \mathrm{Cone}(u_3, \ u_2 + 2e, \ -e) \\
\tilde{\sigma}_{--,0} &= \mathrm{Cone}(u_3 + (0-0)e, u_4 + (0-0)e) + F_0 &&= \mathrm{Cone}(u_3, \ u_4, \ e) \\
\tilde{\sigma}_{--,1} &= \mathrm{Cone}(u_3 + (0-0)e, u_4 + (0-0)e) + F_1 &&= \mathrm{Cone}(u_3, \ u_4, \ -e).
\end{aligned}$$

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Therefore, the rays in $\Sigma(1)$ have minimal generators

$$u_1 + 2e, \quad u_2 + 2e, \quad u_3, \quad u_4, \quad e, \quad \text{and} \quad -e;$$

or, in terms of coordinates in $N_{\mathbb{R}} \times \mathbb{R}^1 = \mathbb{R}^3$,

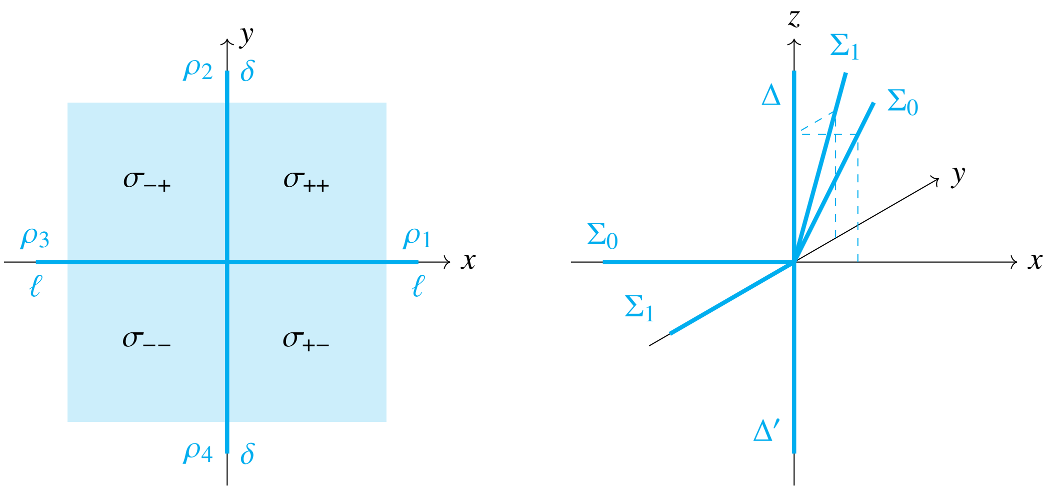
$$(1,0,2), \quad (0,1,2), \quad (-1,0,0), \quad (0,-1,0), \quad (0,0,1), \quad \text{and} \quad (0,0,-1).$$

From the description of pullback divisors [CLS11, Theorem. 4.1.12(b); Proposition 6.2.7], we see that

the rays generated by $(1,0,2)$ and $(-1,0,0)$ correspends to Σ_0 and

the rays generated by $(0,1,2)$ and $(0,-1,0)$ correspends to Σ_1,

so it suffices to specify which divisors Δ and Δ' correspond to the rays generated by $(0,0,\pm 1)$ respectively; see Figure 3.1 (b) below.



(a) Fan of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. (b) Rays in the fan of Y.

Figure 3.1: Relevant fans for threefolds of type II.

To this end, we suppose $\Delta = a\Sigma_2 + b\Sigma_1 + c\Sigma_0$. On the one hand, a similar calculation

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to (2.8) shows that

$$\Delta.\Sigma_0.\Sigma_1 = a$$

$$\Delta^2.\Sigma_0 = -2 + 2b$$

$$\Delta^2.\Sigma_1 = -2 + 2c.$$

On the other hand, these numbers can also be computed via intersecting Δ with the cycles corresponding to $\Sigma_0.\Sigma_1$, $\Delta.\Sigma_0$, and $\Delta.\Sigma_1$, which can be found using [CLS11, Proposition 6.4.4, Lemma 12.5.2] as follows. The intersection $\Delta.\Sigma_0$ is the 1-cycle $\frac{1}{\text{mult}(\tau)}[V(\tau)]$ in the Chow ring, where τ is the 2-dimensional facet span by the edges $(0,0,1)$ and $(1,0,2)$, which has multiplicity 1. If we denote σ to be the full-dimensional cone spanned by τ and $(0,1,2)$, then the intersection of $V(\tau)$ with Σ_1 is the ratio of multiplicities

$$\Delta.\Sigma_0.\Sigma_1 = \frac{\text{mult}(\tau)}{\text{mult}(\sigma)} = 1 = a.$$

Moreover, the ratio of coefficients in the wall relation

$$\mathbf{1}(0,1,2) + (\mathbf{-2})(0,0,1) + \mathbf{0}(1,0,2) + \mathbf{1}(0,-1,0) = 0$$

is precisely the ratio of the intersection numbers of $V(\tau)$ with its neighboring rays' corresponding diviors:

$$\Delta.\Sigma_0.\Sigma_1 : \Delta^2.\Sigma_0 : \Delta.\Sigma_0^2 : \Delta.\Sigma_0.\Sigma_1 = 1 : (-2) : 0 : 1,$$

so $\Delta^2.\Sigma_0 = -2$ and thus $b = 0$. A similar argument shows that $c = 0$, so we obtain that

$$\Delta = \Sigma_2.$$

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The same method also computes Δ'; alternatively, one can also deduce it from

$$0 = \operatorname{div}(\chi^{(0,0,1)}) = 2\Sigma_0 + 2\Sigma_1 + \Delta - \Delta'.$$

Either way, the result is

$$\Delta' = \Sigma_2 + 2\Sigma_1 + 2\Sigma_0.$$

In conclusion, the middle term in the Euler sequence (3.8) is

$$\bigoplus_{\rho \in \Sigma(1)} \mathscr{O}_Y(D_\rho) = \mathscr{O}_Y(\Sigma_0)^{\oplus 2} \oplus \mathscr{O}_Y(\Sigma_1)^{\oplus 2} \oplus \mathscr{O}_Y(\Sigma_2) \oplus \mathscr{O}_Y(\Sigma_2 + 2\Sigma_1 + 2\Sigma_0),$$

and the cohomologies can now be readily computed

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & \mathscr{O}_Y^{\oplus 3} & \longrightarrow & \bigoplus_{\rho \in \Sigma(1)} \mathscr{O}_Y(D_\rho) & \longrightarrow & \mathscr{T}_Y & \longrightarrow & 0 \\
h^0: & 0 & & 3 & & 19 & & \mathbf{16} & & \\
h^1: & & & 0 & & 0 & & \mathbf{0} & & \\
h^2: & & & 0 & & 1 & & \mathbf{1} & & \\
h^3: & & & 0 & & 0 & & \mathbf{0} & & 0.
\end{array}$$

Piecing everything together into the exact sequence (3.1), we get

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & \mathscr{T}_X & \longrightarrow & \tau^*\mathscr{T}_Y & \longrightarrow & \mathscr{T}_{X/Y} & \longrightarrow & 0 \\
h^0: & 0 & & 0 & & 16 & & 285 & & \\
h^1: & & & \mathbf{269} & & 0 & & 0 & & \\
h^2: & & & \mathbf{0} & & 1 & & 1 & & \\
h^3: & & & \mathbf{0} & & 0 & & 0 & & 0.
\end{array}$$

In particular, $h^1(X, \mathscr{T}_X) = 269$ and $h^2(X, \mathscr{T}_X) = h^3(X, \mathscr{T}_X) = 0$ as claimed.

Computation for Threefolds of Type III. The steps are largely the same as II, and the computation of cohomologies is explained in the proof of Construction–Proposition III,

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so we will just write down the results.

Step 0.

$$\begin{aligned} h^2(X,\mathscr{O}_X) &= h^2(Y,\tau_*\mathscr{O}_X) \\ &= h^2(Y,\mathscr{O}_Y) + h^2(Y,-3\Sigma_2-5\Sigma_1-10\Sigma_0) = 0, \end{aligned}$$

so $h^0(X,\mathscr{T}_X)=0$ by previous discussion.

Step 1. Using (3.2), computation shows that

$$\begin{array}{lcccccccc} & 0 & \longrightarrow & \mathscr{O}_Y & \longrightarrow & \mathscr{O}_Y(B) & \longrightarrow & \iota_*\iota^*\mathscr{O}_Y(B)\simeq\mathscr{N}_B & \longrightarrow 0 \\ h^0: & 0 & & 1 & & 286 & & \mathbf{285} & \\ h^1: & & & 0 & & 0 & & \mathbf{0} & \\ h^2: & & & 0 & & 1 & & \mathbf{1} & \\ h^3: & & & 0 & & 0 & & \mathbf{0} & 0. \end{array}$$

Remarkably, the numbers are exactly the same as those in type II.

Step 2. We use (3.6) and (3.7) again for $\mathscr{T}_Y\otimes\mathscr{L}^\vee$; however, we have more non-zero terms this time: $\Theta_{W/\mathbb{P}^1}=2\delta+2\ell$, so

$$\begin{aligned} p^*\Theta_{W/\mathbb{P}^1}\otimes\mathscr{L}^\vee &= \mathscr{O}_Y(-3\Sigma_2-3\Sigma_1-8\Sigma_0) \\ p^*\pi^*\mathscr{T}_{\mathbb{P}^1}\otimes\mathscr{L}^\vee &= \mathscr{O}_Y(-3\Sigma_2-5\Sigma_1-8\Sigma_0) \end{aligned}$$

and p^*(3.7) $\otimes\,\mathscr{L}^\vee$ reads

$$\begin{array}{lcccccccc} & 0 & \longrightarrow & p^*\Theta_{W/\mathbb{P}^1}\otimes\mathscr{L}^\vee & \longrightarrow & p^*\mathscr{T}_W\otimes\mathscr{L}^\vee & \longrightarrow & p^*\pi^*\mathscr{T}_{\mathbb{P}^1}\otimes\mathscr{L}^\vee & \longrightarrow 0. \\ h^0: & 0 & & 0 & & \mathbf{0} & & 0 & \\ h^1: & & & 1 & & \mathbf{1} & & 0 & \\ h^2: & & & 1 & & \mathbf{2} & & 1 & \\ h^3: & & & 0 & & \mathbf{1} & & 1 & 0; \end{array}$$

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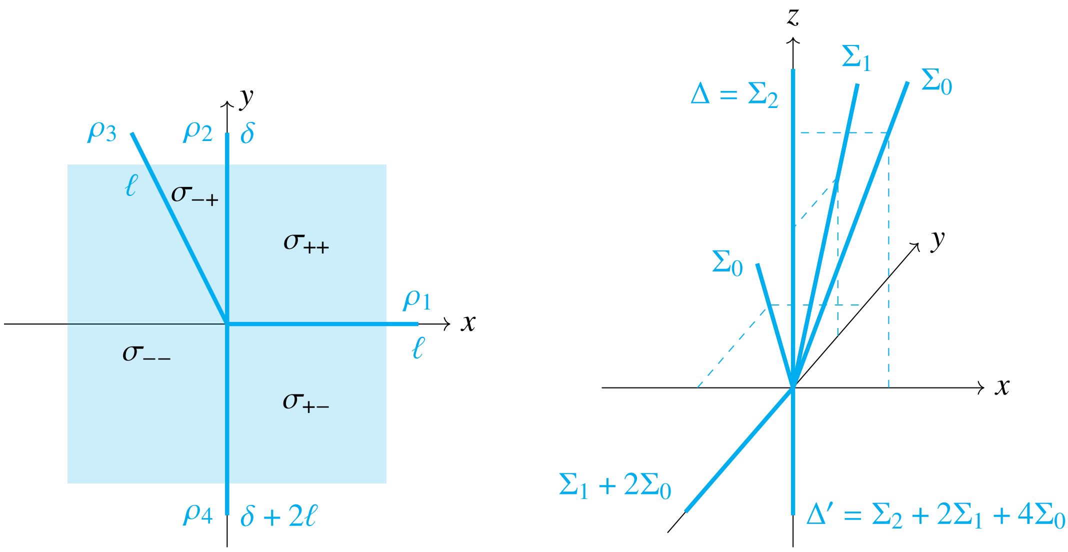
therefore, from $\Theta_{Y/W} \otimes \mathscr{L}^\vee = \mathscr{O}_Y(-\Sigma_2 - 3\Sigma_1 - 6\Sigma_0)$ and (3.6) we infer

$$\begin{array}{rccccccccc}
 & 0 & \longrightarrow & \Theta_{Y/W} \otimes \mathscr{L}^\vee & \longrightarrow & \mathscr{T}_Y \otimes \mathscr{L}^\vee & \longrightarrow & p^*\mathscr{T}_W \otimes \mathscr{L}^\vee & \longrightarrow & 0. \\
h^0: & 0 & & 0 & & \mathbf{0} & & 0 & & \\
h^1: & & & 0 & & \mathbf{1} & & 1 & & \\
h^2: & & & 0 & & \mathbf{2} & & 2 & & \\
h^3: & & & 0 & & \mathbf{1} & & 1 & & 0.
\end{array}$$

Step 3. The fans of W and Y in this case are shown in Figure 3.2 below.



(a) Fan of $W = \mathbb{F}_2$.

(b) Rays in the fan of Y.

Figure 3.2: Relevant fans for threefolds of type III.

In particular, the ray generators and their corresponding divisors are

ray gen.	$(1,0,4)$	$(0,1,2)$	$(-1,2,0)$	$(0,-1,0)$	$(0,0,1)$	$(0,0,-1)$
divisor	Σ_0	Σ_1	Σ_0	$\Sigma_1 + 2\Sigma_0$	Σ_2	$\Sigma_2 + 2\Sigma_1 + 4\Sigma_0$

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So, the middle term in the toric Euler sequence is known, and it yields

$$
\begin{array}{lccccc}
 & 0 \longrightarrow & \mathscr{O}_Y^{\oplus 3} \longrightarrow & \bigoplus_{\rho \in \Sigma(1)} \mathscr{O}_Y(D_\rho) \longrightarrow & \mathscr{T}_Y \longrightarrow & 0 \\
h^0 : & 0 & 3 & 20 & \mathbf{17} & \\
h^1 : & & 0 & 1 & \mathbf{1} & \\
h^2 : & & 0 & 1 & \mathbf{1} & \\
h^3 : & & 0 & 0 & \mathbf{0} & 0.
\end{array}
$$

Piecing everything together into the exact sequence (3.1), we get

$$
\begin{array}{lccccc}
 & 0 \longrightarrow & \mathscr{T}_X \longrightarrow & \tau^* \mathscr{T}_Y \longrightarrow & \mathscr{T}_{X/Y} \longrightarrow & 0 \\
h^0 : & 0 & 0 & 17 & 285 & \\
h^1 : & & \mathbf{270} & 2 & 0 & \\
h^2 : & & \mathbf{x} & 3 & 1 & \\
h^3 : & & \mathbf{x-1} & 1 & 0 & 0.
\end{array}
$$

Here, by dimension consideration, x is either 2 or 3; in particular, $h^1(X, \mathscr{T}_X) = 270$, $h^2(X, \mathscr{T}_X) = x$, and $h^3(X, \mathscr{T}_X) = x - 1$ as claimed.

Remark 3.5. Using the toric method, we can also easily obtain that

$$
h^i(\mathbb{F}_e, \mathscr{T}_{\mathbb{F}_e}) = \begin{cases} e+5, & \text{for } i = 0; \\ e-1, & \text{for } i = 1; \\ 0, & \text{for } i = 2. \end{cases}
$$

In [Hor76], Horikawa only needed the fact that $\chi(\mathbb{F}_e, \mathscr{T}_{\mathbb{F}_e}) = 6$, but the exact numbers for each h^i were not present.

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3.2 Euler Characteristics

Let X be a smooth threefold. Applying the Hirzebruch–Riemann–Roch theorem to the tangent sheaf $\mathscr{T}_X$ produces the formula

$$\chi(\mathscr{T}_X) = \frac{1}{2}c_1^3 - \frac{19}{24}c_1.c_2 + \frac{1}{2}c_3,$$

where we abbreviated $c_i \coloneqq c_i(\mathscr{T}_X)$ $(i = 1, 2, 3)$ because only the Chern classes of the tangent sheaf $\mathscr{T}_X$ are involved. In this section, we will compute $\chi(\mathscr{T}_X)$ for threefolds of type I, II, and III. The purpose is twofold: on the one hand, this serves as a double-check for Theorem 3.1; on the other hand, computing the last term $c_3(\mathscr{T}_X) = \chi_{\text{top}}(X)$ yields topological information of X.

General Strategies for Computation.

- Notice that

  $$K_X = c_1\left(\bigwedge^3 \mathscr{T}_X^\vee\right) = c_1(\mathscr{T}_X^\vee) = -c_1(\mathscr{T}_X),$$

  so

  $$\text{First term} = \frac{1}{2}c_1^3 = -\frac{1}{2}K_X^3.$$

- Since $\text{ch}(\mathscr{O}_X) = 1$, applying the Hirzebruch–Riemann–Roch theorem to $\mathscr{O}_X$ gives $\chi(\mathscr{O}_X) = \frac{1}{24}c_1.c_2$, so

  $$\text{Second term} = -\frac{19}{24}c_1.c_2 = -19\chi(\mathscr{O}_X).$$

- As noted above, the top Chern class $c_3(\mathscr{T}_X)$ is just the topological Euler charac-

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teristic $\chi_{\text{top}}(X)$ of X, it suffices to compute the latter. In all cases I, II and III, $X - B' \to Y - B$ is a 2-to-1 topological cover and B' is isomorphic to B, so

$$\chi_{\text{top}}(X - B') = 2\chi_{\text{top}}(Y - B)$$
$$\chi_{\text{top}}(X) - \chi_{\text{top}}(B) = 2(\chi_{\text{top}}(Y) - \chi_{\text{top}}(B))$$
$$\chi_{\text{top}}(X) = 2\chi_{\text{top}}(Y) - \chi_{\text{top}}(B)$$

and thus

$$\text{Third term} = \frac{1}{2}\chi_{\text{top}}(X) = \chi_{\text{top}}(Y) - \frac{1}{2}\chi_{\text{top}}(B).$$

The problem reduces to computing $\chi_{\text{top}}(Y)$ and $\chi_{\text{top}}(B)$. An important tool to do so is the Noether formula: for a smooth surface S,

$$\chi_{\text{top}}(S) = 12\chi(\mathcal{O}_S) - K_S^2,$$

which is, again, just the Hirzebruch–Riemann–Roch theorem applied to $\mathcal{O}_S$.

Computation for Threefolds of Type I.

First term. As computed before, $K_X^3 = \text{vol}(X) = 2$.

Second term. Recall that τ has no higher direct image, so for all $i = 0, 1, 2, 3$,

$$h^i(X, \mathcal{O}_X) = h^i(Y, \tau_*\mathcal{O}_X) = h^i(\mathbb{P}^3, \mathcal{O}) + h^i(\mathbb{P}^3, \mathcal{O}(-5)).$$

We have already calculated these in the proof of Theorem 3.1, so

$$\chi(X, \mathcal{O}_X) = \chi(\mathbb{P}^3, \mathcal{O}) + \chi(\mathbb{P}^3, \mathcal{O}(-5)) = 1 + (-4) = -3.$$

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Third term. It is a standard result that $\chi_{\mathrm{top}}(\mathbb{CP}^n) = n+1$. As for $\chi_{\mathrm{top}}(B)$, since B is a smooth surface of degree 10 in $\mathbb{P}^3$, the Noether formula reads

$$\chi_{\mathrm{top}}(B) = 12\chi(\mathscr{O}_B) - K_B^2.$$

From the standard exact sequence associated with a closed subvariety,

$$\begin{array}{lcccccccc}
 & 0 & \longrightarrow & \mathscr{O}_{\mathbb{P}^3}(-10) & \longrightarrow & \mathscr{O}_{\mathbb{P}^3} & \longrightarrow & \mathscr{O}_B & \longrightarrow & 0 \\
h^0: & 0 & & 0 & & 1 & & \mathbf{1} & & \\
h^1: & & & 0 & & 0 & & \mathbf{0} & & \\
h^2: & & & 0 & & 0 & & \mathbf{84} & & \\
h^3: & & & \binom{9}{3} = 84 & & 0 & & \mathbf{0} & & 0,
\end{array}$$

so $\chi(\mathscr{O}_B) = 1 - 0 + 84 - 0 = 85$. On the other hand, the adjunction formula gives $K_B = (K_X + B)|_B$, so

$$K_B^2 = [(K_X+B)|_B]^2 = B.(K_X+B)^2 = \mathscr{O}(10).\mathscr{O}(6)^2 = 360.$$

Therefore,

$$\chi_{\mathrm{top}}(X) = 2 \cdot 4 - (12 \cdot 85 - 360) = -652.$$

In conclusion,

$$\chi(\mathscr{T}_X) = -\frac{1}{2} \cdot 1 - 19 \cdot (-3) + \frac{1}{2} \cdot (-652) = -270,$$

which is consistent with Theorem 3.1 (I). ◊

Computation for Threefolds of Type II. Denote $P = \Sigma_1 + \Sigma_0$ as in the proof of Lemma 2.2 and recall the intersection numbers (2.8).

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First term. Since $K_X = \tau^*(\Sigma_2 + P)$,

$$K_X^3 = \deg \tau \cdot (\Sigma_2 + P)^3 = 2 \cdot (8 + 3 \cdot (-4) + 3 \cdot 2 + 0) = 4.$$

Second term. Again, τ has no higher direct image, so

$$\chi(X, \mathscr{O}_X) = \chi(Y, \mathscr{O}_Y) + \chi(Y, \mathscr{L}^\vee) = 1 + (-4) = -3.$$

Third term. In this case Y is a $\mathbb{P}^1$-bundle over $W = \mathbb{P}^1 \times \mathbb{P}^1$, so $\chi_{\text{top}}(Y) = \chi_{\text{top}}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 2^3 = 8$.

As for $B = \Sigma_2 \amalg B_0$, we have $\chi_{\text{top}}(B) = \chi_{\text{top}}(\Sigma_2) + \chi_{\text{top}}(B_0)$. Moreover, since $\Sigma_2 \simeq W = \mathbb{P}^1 \times \mathbb{P}^1$, we have $\chi_{\text{top}}(\Sigma_2) = 4$. Thus, it suffices to compute $\chi_{\text{top}}(B_0)$, which is equal to $12\chi(B_0) - K_{B_0}^2$ by Noether's formula.

On the one hand, $K_{B_0}^2$ can be calculated by the adjunction formula:

$$K_{B_0} = (K_Y + B_0)|_{B_0}, \quad \text{where} \begin{cases} K_Y = -2(\Sigma_2 + 2P) \\ B_0 = 5(\Sigma_2 + 2P), \end{cases} \tag{3.9}$$

so

$$\begin{aligned} K_{B_0}^2 &= (K_Y + B_0)|_{B_0}.(K_Y + B_0)|_{B_0} \\ &= B_0.(K_Y + B_0).(K_Y + B_0) \\ &= 45(\Sigma_2 + 2P)^3 && \text{by (3.9)} \\ &= 45(\Sigma_2^3 + 6\Sigma_2^2.P + 12\Sigma_2.P^2 + 8P^3) \\ &= 45(8 + 6 \cdot (-4) + 12 \cdot 2 + 0) \\ &= 360. \end{aligned}$$

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On the other hand, to compute $\chi(B_0)$, the standard exact sequence associated with a closed subvariety along with Corollary 1.4 gives

$$\begin{array}{lccccccccc} & 0 & \longrightarrow & \mathscr{O}_Y(-B_0) & \longrightarrow & \mathscr{O}_Y & \longrightarrow & \mathscr{O}_{B_0} & \longrightarrow & 0 \\ h^0: & 0 & & 0 & & 1 & & \mathbf{1} & & \\ h^1: & & & 0 & & 0 & & \mathbf{0} & & \\ h^2: & & & 0 & & 0 & & \mathbf{84} & & \\ h^3: & & & 84 & & 0 & & \mathbf{0} & & 0. \end{array}$$

Therefore,

$$\chi(B_0) = 84 - (-1) = 85;$$

$$\chi_{\text{top}}(B_0) = 12\chi(B_0) - K_{B_0}^2 = 12 \cdot 85 - 360 = 660;$$

$$\chi_{\text{top}}(B) = \chi_{\text{top}}(\Sigma_2) + \chi_{\text{top}}(B_0) = 4 + 660 = 664.$$

and we arrive at

$$\chi_{\text{top}}(X) = 2\chi_{\text{top}}(Y) - \chi_{\text{top}}(B) = 2 \cdot 8 - 664 = -648.$$

Putting everything together,

$$\chi(\mathscr{T}_X) = -\frac{1}{2} \cdot 4 - 19 \cdot (-3) + \frac{1}{2} \cdot (-648) = -269.$$

which is consistent with Theorem 3.1 (II). ◊

Computation for Threefolds of Type III. Curiously, the computation for type III is almost identical to type II. This is mostly due to the fact that the intersection numbers (2.11) of type III are the same as those (2.8) of type II, and that $\chi_{\text{top}}(\mathbb{F}_2) = \chi_{\text{top}}(\mathbb{P}^1) \cdot \chi_{\text{top}}(\mathbb{P}^1) = 4$

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because $\mathbb{F}_2$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. The upshot is that

$$\chi(\mathscr{T}_X) = -269,$$

which is again consistent with Theorem 3.1 (III). ◊

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Chapter 4 Discussion and Future Work

Classification of Threefolds of Type-$(2,4)$. It is natural to ask the following question.

Question 4.1. Are the families I, II, and III all the threefolds of type-$(2,4)$? To be more precise, given a threefold X of type-$(2,4)$, it is true that X must belong to one of the birational classes of I, II, or III?

Proposition 4.2. Let X be a minimal threefold of type-$(2,4)$. Then the canonical image of X is either $\mathbb{P}^3$, $\mathbb{F}_0$ or $\mathbb{P}(1,1,2)$.

Proof. Denote the canonical map by $\varphi := \phi_{|K_X|}$ and the canonical image by $W \subset \mathbb{P}^3$. By [Che07, 3.3], we know that if $\dim W = 1$ then $\mathrm{vol}(X) \geq 3$, which is impossible. Hence, $\dim W \geq 2$. On top of that, if $\dim W = 3$ then $W = \mathbb{P}^3$, so in the following we assume that $\dim W = 2$.

Now consider $\pi : X' \to X$ a resolution of indeterminacy of $|K_X|$, then we can decompose the linear system

$$|\pi^* K_X| = |M| + Z.$$

with a base-point free moving part $|M|$ and the fixed part Z. This way, we get a commutative diagram

$$\begin{array}{ccccc} X' & & & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\phi} & & & \\ X & \underset{\varphi}{\dashrightarrow} & W & \subset & \mathbb{P}^3, \end{array}$$

where ϕ is the map defined by the linear system $|\pi^* K_X|$.

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Let H be the divisor corresponding to $\mathcal{O}_W(1) = \mathcal{O}_{\mathbb{P}^3}(1)|_W$, then

$$d := \deg W = W.\mathcal{O}_{\mathbb{P}^3}(1).\mathcal{O}_{\mathbb{P}^3}(1) = H^2,$$

so

$$M^2 = \phi^* H^2 = dC, \quad \text{where } C \text{ is a fiber curve of } \phi.$$

Now,

$$\begin{aligned} 2 = \mathrm{vol}(X) = (\pi^* K_X)^3 &\geq (\pi^* K_X).M^2 \quad && X \text{ is minimal, so } \pi^* K_X \text{ is nef} \\ &= d(\pi^* K_X).C \\ &\geq d \geq 2 && W \text{ is non-degenerate in } \mathbb{P}^3. \end{aligned}$$

This implies that $d = \deg W = 2$, i.e., W is defined by a quadric in $\mathbb{P}^3$. By the theory of quadratic forms, under a suitable coordinate change, the equation of W is

$$\text{either } x^2 + y^2 + z^2 + w^2 = 0, \quad x^2 + y^2 + z^2 = 0, \quad x^2 + y^2 = 0, \quad \text{or} \quad x^2 = 0.$$

The latter two are not integral, thus impossible. The first one can be rewritten as

$$xy + zw = 0,$$

which is the image of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ via the Segre embedding; the second one can be rewritten as

$$xy + z^2 = 0,$$

which is isomorphic to $\mathbb{P}(1, 1, 2)$, as seen before. We remark that this result is in concordance with [Che07, Theorem 1.5(5)]. □

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This proposition gives evidence for the conjecture that the answer to Question 4.1 is affirmative.

Description of Birational Models as Weighted Hypersurfaces; Explicit Deformations. In Coughlan and Pignatelli's work [CP23], some of the models in §2.2 are described even more explicitly as weighted hypersurfaces in a weighted projective bundle. Under this description, it should be possible to either construct explicit deformations among the examples or prove that no such deformation exists.

Our examples are all constructed via taking a branched double cover over a base. The deformation of the branch locus should be closely related to the deformation of the cover. It shall be an interesting problem to investigate this relation.

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Appendix A C++ Code for Computing Cohomology on Hirzebruch Surfaces

In this appendix, we explain how to compute the dimension

$$h^k(\mathbb{F}_e, a\delta + b\ell) = h^k\left(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}}(a) \otimes \pi^*\mathcal{O}_{\mathbb{P}}(b)\right), \tag{$\bigstar$}$$

and provide a C++ code that creates a function to do this calculation.

As seen in Remark 2.1, the general strategy is to use the projective bundle structure $\pi : \mathbb{F}_e \to \mathbb{P}^1$ to "push downstairs."

Case $k = 0$.

$$\begin{aligned}(\bigstar) &= h^0\Big(\mathbb{P}^1, \pi_*\mathcal{O}_{\mathbb{F}}(a) \otimes \mathcal{O}_{\mathbb{P}}(b)\Big) && \text{projection formula}\\ &= h^0\Big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(b) \oplus \mathcal{O}_{\mathbb{P}}(b-e) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}}(b-ae)\Big). && \text{Proposition 1.2 (a)}\end{aligned}$$

Case $k = 1$.

Subcase $a \geq 0$. $R^1\pi_*\mathcal{O}_{\mathbb{F}}(a) = 0$, so the Leray spectral sequence degenerates at page 2 and

$$\begin{aligned}(\bigstar) &= h^1\Big(\mathbb{P}^1, \pi_*\mathcal{O}_{\mathbb{F}}(a) \otimes \mathcal{O}_{\mathbb{P}}(b)\Big)\\ &= h^1\Big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(b) \oplus \mathcal{O}_{\mathbb{P}}(b-e) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}}(b-ae)\Big)\\ &= h^0\Big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-b-2) \oplus \mathcal{O}_{\mathbb{P}}(-b-2+e) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}}(-b-2+ae)\Big)\end{aligned}$$

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by Serre duality.

Subcase $a = -1$. $R^1\pi_*\mathcal{O}_{\mathbb{F}}(a) = \pi_*\mathcal{O}_{\mathbb{F}}(a) = 0$, so the Leray spectral sequence is identically zero at page 2; consequently, $(\bigstar) = 0$.

Subcase $a \leq -2$. $\pi_*\mathcal{O}_{\mathbb{F}}(a) = 0$, so the Leray spectral sequence degenerates at page 2 again, and

$$\begin{aligned} R^1\pi_*\mathcal{O}_{\mathbb{F}}(a) &= [\pi_*\mathcal{O}_{\mathbb{F}}(-a-2)]^\vee \otimes \mathcal{O}_{\mathbb{P}}(e) \\ &= [\mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(e) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}}((-a-2)e)] \otimes \mathcal{O}_{\mathbb{P}}(e) \\ &= \mathcal{O}_{\mathbb{P}}(e) \oplus \mathcal{O}_{\mathbb{P}}(2e) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}}((-a-1)e), \end{aligned}$$

so

$$\begin{aligned} (\bigstar) &= h^0\Big(\mathbb{P}^1, R^1\pi_*\mathcal{O}_{\mathbb{F}}(a) \otimes \mathcal{O}_{\mathbb{P}}(b)\Big) \\ &= h^0\Big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(b+e) \oplus \mathcal{O}_{\mathbb{P}}(b+2e) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}}(b+(-a-1)e)\Big). \end{aligned}$$

Case $k = 2$. This can be reduced to the case $k = 0$ by Serre duality: since $K_{\mathbb{F}} = -2\delta - (e+2)\ell$, we have

$$(\bigstar) = h^0(\mathbb{F}_e, (-2-a)\delta + (-2-e-b)\ell).$$

We see that in each case the summands are

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(m)) = \begin{cases} m+1, & \text{if } m \geq 0; \\ 0, & \text{if } m < 0, \end{cases}$$

so the result is an arithmetic series, but gets cut off as soon as the summand becomes negative. To this end, we first create a function that calculates such arithmetic series with initial term `initial`, common difference `difference`, and number of terms `number`:

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```
int nonNegArithSeries(int initial, int difference, int number){
    int sum = 0;
    int termNow;
    for(int i = 0; i < number; i++){
        termNow = initial + i*difference;
        if (termNow >= 0)
            sum += termNow;
    }
    return sum;
}
```

Then create a function `hirzebruchCohomology` (see next page) whose return value is (★) by plugging in the formulas obtained above. Here, we use the return value `-1` to indicate an error.

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```
int hirzebruchCohomology(int k, int e, int a, int b){
    if (e < 0){
        return -1;
    }
    if (k == 0){
        return nonNegArithSeries(b+1, -e, a+1);
    }
    else if (k == 1){
        if (a >= 0){
            return nonNegArithSeries(-b-1, e, a+1);
        }
        if (a == -1){
            return 0;
        }
        if (a <= -2){
            return nonNegArithSeries(b+e+1, e, -a-1);
        }
    }
    else if (k == 2){
        a = -2-a;
        b = -2-e-b;
        return nonNegArithSeries(b+1, -e, a+1);
    }
    else if (k >= 3){
        return 0;
    }
    return -1;
}
```


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