國立臺灣大學理學院數學系

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master’s Thesis

關於射影對偶的退化

On Degenerations of Projective Dualities

黃建順

Jian-Shun Huang

指導教授: 林學庸 博士

Advisor: Hsueh-Yung Lin Ph.D.

中華民國 114 年 2 月

February, 2025

doi:10.6342/NTU202500514

doi:10.6342/NTU202500514

致謝

首先，我衷心感謝我的指導教授林學庸教授對我的指導與教誨。從大四一直到碩士這三年半的時間，教授耐心地引領我進入代數幾何的世界，幾乎隨時都願意放下手上的工作回答我的問題，他常能在抽象的敘述中給出精闢的解釋並舉出簡單的例子幫助我理解，在我完成碩士論文的過程中也不厭其煩地和我討論，鼓勵我也給我很大的包容，讓我能順利完成我的碩士學位。我也要特別感謝教授帶著我參加好幾次的研討會，讓我有機會認識很多學者並和他們交流。

我要特別感謝陳榮凱教授，教授在這幾年辦了好幾個學生讀書會，帶著我們接觸了很多代數幾何的領域，在讀書會過程也解答了很多我的疑惑。我也想感謝陳正傑教授和李庭諭教授，在我學習代數幾何的過程幫助我許多。此外，我也要感謝 Evgeny Shinder 教授在 2024 年暑假來臺訪問時和我討論。我也要感謝我的口試委員賴冠文教授和王賜聖教授給我的論文提出的疑問和建議。

感謝我的同學陳毅鴻、蘇品丞、張宏彬與張志煥，他們平時與我的討論與交流讓我獲益良多。我也想感謝那些在天文數學館的走廊上願意停下腳步回答我的問題的同學們。最後，我想感謝我的父母對我無限的支持和鼓勵。

doi:10.6342/NTU202500514

doi:10.6342/NTU202500514

摘要

這篇論文的主要目標是研究射影對偶的退化。我們證明在特定條件下，一個向量叢截面的零點集構成的光滑族的平坦極限可以描述爲額外法叢截面的零點集。作爲應用，我們探討 Shinder-Zhang 的五次橢圓曲線，並證明它們會退化爲 Mori-Mukai 連結中，三維二次曲面中的一條橢圓曲線。

關鍵字：退化、額外法叢、相交、齊性多樣體、代數幾何

doi:10.6342/NTU202500514

doi:10.6342/NTU202500514

Abstract

The primary aim of this note is to initiate the study of the degenerations of projective dualities. We prove that, under certain conditions, the flat limit of a smooth family of zero loci of general sections of a vector bundle can be described as the zero locus of a section of the excess normal bundle. As an application, we examine the case of Shinder-Zhang's degree-five elliptic curves and show that they degenerate to the elliptic curve on the quadric 3-fold, appearing in the Mori-Mukai link of Fano 3-fold.

Keywords: Degeneration, Excess Normal Bundle, Intersection, Homogeneous Variety, Algebraic Geometry

doi:10.6342/NTU202500514

doi:10.6342/NTU202500514

Contents

doi:10.6342/NTU202500514

doi:10.6342/NTU202500514

Chapter 1 Introduction

Let X be a smooth projective variety. We denote by $\mathbf{D}^b(X) := \mathbf{D}^b(\mathbf{Coh}(X))$ the bounded derived category of coherent sheaves of X. We say that a pair of smooth projective varieties X and Y are **D-equivalent** if $\mathbf{D}^b(X) \simeq \mathbf{D}^b(Y)$.

For example, let X be a 3-dimensional smooth linear section of the Grassmannian

$$X = \mathsf{Gr}(2,7) \cap \mathbb{P}^{13} \subseteq \mathbb{P}^{20}$$

in the Plücker embedding. It is a Calabi-Yau 3-fold. The classical projective dual of $\mathsf{Gr}(2,7)$ is the Pfaffian variety $\mathsf{Pf}(4,7)$. Let Y the intersection of the projective dual of $\mathsf{Gr}(2,7)$ and $\mathbb{P}^{13}$ in $\mathbb{P}^{20}$, which is smooth. Y is also a Calabi-Yai 3-fold. It has been proved by L. Borisov and A. Căldăraru that X and Y are D-equivalent [BC09]. We usually call (X, Y) a Pfaffian-Grassmannian CY3 pair.

Recently, L. Borisov first solved the problem of whether the class $\mathbb{L}$ of affine line is a zero-divisor in the Grothendieck ring of varieties using the Pfaffian-Grassmannian CY3 pair [Bor18]. The result was refined by N. Martin. He proved, in [Mar16], that for a Pfaffian-Grassmannian CY3 pair (X, Y),

$$\mathbb{L}^6 \cdot ([X] - [Y]) = 0.$$

doi:10.6342/NTU202500514

We say a pair of smooth projective varieties X and X are **L-equivalent** if

$$\mathbb{L}^k \cdot ([X] - [Y]) = 0$$

in the Grothendieck ring of variety $K_0(\mathbf{Var}/\mathbb{C})$ for some $k \geq 0$. The Pfaffian-Grassmannian CY3 pair (X, Y) gives an example of an L-equivalent pair.

In [KS18], A. Kuznetsov and E. Shinder conjectured that for a pair of smooth projective simply connected varieties, D-equivalence implies L-equivalence. The assumption of simple connectedness is necessary, as A. Efimov constructed a D-equivalent pair of abelian varieties that is not L-equivalent [Efi18, Theorem 3.1].

The pair (X, Y) above is an example of a D-equivalent pair that is also L-equivalent. In [Ito+19], A. Ito, M. Miura, S. Okawa, and K. Ueda constructed a pair of L-equivalent Calabi-Yau 3-fold (X', Y'), which is the zero locus of a general section of vector bundles over the G_2-Grassmannian pair. They showed that

$$\mathbb{L} \cdot ([X'] - [Y']) = 0.$$

Later, A. Kuznetsov proved that the pair (X', Y') is also D-equivalent [Kuz18].

In [Ito+19] it was mentioned that the pair (X', Y') is a degeneration of the Pfaffian-Grassmannian CY3 pair (X, Y). It is natural to ask whether there are other examples of D-equivalent and L-equivalent pairs (X, Y) that degenerate to another D-equivalent and L-equivalent pair.

In [SZ20], E. Shinder and Z. Zhang proved that if (C_1, C_2) is a pair of genus 1 curves of degree 5, which are the linear sections of the Grassmannian $\mathsf{Gr}(2, 5)$ and its projective

doi:10.6342/NTU202500514

dual (also isomorphic to $\mathsf{Gr}(2,5)$), then

$$\mathbb{L}^4 \cdot ([C_1] - [C_2]) = 0.$$

The pair (C_1, C_2) is also D-equivalent (see, e.g. [Kuz06, Section 6.1]). On the other hand, we have a pair of genus 1 curves (C_1', C_2') in the Mori-Mukai link between $\mathbb{P}^3$ and Q^3 [LS24, Proposition 3.6]. H.-Y. Lin and E. Shinder proved that the pair (C_1', C_2') is L-equivalent and satisfies

$$\mathbb{L} \cdot ([C_1'] - [C_2']) = 0.$$

In Corollary 4.4.2, we will prove that one of the genus 1 curves C_2 degenerate to the genus 1 curve C_2' in the quadric 3-fold.

Moreover, M. Rampazzo proved that the pair (C_1', C_2') is D-equivalent [Ram21, Lemma A.1]. We expect that the pairs (C_1, C_2) and (C_1', C_2') give another example of degeneration of DL-equivalent pairs. This thesis aims to establish some first steps toward this statement.

The paper is organized as follows. We review some necessary backgrounds in Section 2. We prove a general theorem about degeneration (Theorem 3.2.1) in Section 3. Finally, we prove the degeneration of the elliptic curve for the Q^3 side in Section 4.

We work over the complex numbers $\mathbb{C}$.

doi:10.6342/NTU202500514

doi:10.6342/NTU202500514

Chapter 2 Preliminaries

2.1 Algebraic Groups

We begin by reviewing some basic facts about algebraic groups. The details can be found in most classical textbooks, such as [Bor66]. We also use some results from a note written by Ottaviani on rational homogeneous varieties [Ott95].

Let G be a semisimple, simply connected, connected algebraic group. *Semisimple* means that G has no nontrivial (closed) normal connected solvable subgroups. It is known that any semisimple algebraic group is a direct sum of simple algebraic groups [Ott95, Theorem 6.13]. An algebraic group is *simple* if it has no nontrivial (closed) normal connected subgroups. The assumption of simple connectedness is natural, as the Lie algebra of an algebraic group is isomorphic to the Lie algebra of its universal cover.

By the classification of semisimple connected algebraic groups, there exists a bijection between simply connected semisimple algebraic groups and Dynkin diagrams. We typically use the Dynkin diagram to name the algebraic group. For instance, for a simply connected algebraic group of type C_2, we mean the algebraic group whose associated Dynkin diagram is of type C_2.

A subgroup B is called a **Borel** subgroup if it is maximal among all connected solv-

doi:10.6342/NTU202500514

able subgroups. A subgroup P is called a **parabolic** subgroup if it is closed and G/P is projective. A closed subgroup P is parabolic if and only if P contains a Borel subgroup [Bor66, 11.2, Corollary]. By definition, a Borel subgroup is parabolic.

For a simple, simply-connected, connected algebraic group G with Dynkin diagram Δ, there is a bijection between non-empty finite subsets $\Sigma \subseteq \Delta$ and the parabolic subgroups $P(\Sigma)$ of G, up to conjugation [Ott95, Theorem 7.8].

2.2 Homogeneous Varieties

Definition 2.2.1. A variety X is called a **homogeneous variety** if there exists a transitive algebraic group action on X.

Many varieties are homogeneous. For instance, the projective spaces $\mathbb{P}^n$, Grassmannians $\mathsf{Gr}(k, n)$, abelian varieties, quadric hypersurfaces, etc. A well-known decomposition theorem, given by Borel and Remmert [Ott95, Theorem 1.5], states that any projective homogeneous variety can be decomposed into a product of an abelian variety and a rational homogeneous variety. Therefore, it is natural to consider rational homogeneous varieties.

Moreover, another theorem by Borel and Remmert [Ott95, Theorem 1.6] states that any rational homogeneous variety X can be decomposed as

$$X \simeq G_1/P_1 \times \cdots \times G_n/P_n,$$

where G_i are simple algebraic groups and P_i are parabolic subgroups. Hence, we can focus on rational homogeneous varieties of the form G/P.

In the previous section, we mentioned that parabolic subgroups of a given simple,

doi:10.6342/NTU202500514

simply connected, connected algebraic group G with the Dynkin diagram Δ are classified. As a consequence, rational homogeneous varieties of the form $G/P(\Sigma)$ are in one-to-one correspondence with the **crossed Dynkin diagrams**, where $\Sigma \subseteq \Delta$ is the set of crossed nodes.

For example, a Borel subgroup of G is the minimal parabolic subgroup, so the corresponding crossed Dynkin diagram is fully crossed out. More explicitly, let G be of type A_n. Then $G \simeq \mathrm{SL}(n, \mathbb{C})$, and all Borel subgroups B are conjugate to the group of upper triangular matrices. The corresponding homogeneous variety G/B is the complete flag variety.

Let $X = G/P(\Sigma)$ be the homogeneous variety corresponding to the crossed nodes $\Sigma = \{i_1, \ldots, i_k\}$ of Δ. By [Ott95, Proposition 10.4], the Picard group $\mathsf{Pic}(X) \simeq \mathbb{Z}^{\oplus k}$. In particular, X has Picard rank 1 if and only if $|\Sigma| = 1$.

2.3 Flat Limit

Definition 2.3.1. A **flat family** is a flat surjective morphism $\pi\colon X \to C$ of schemes. We denote the fiber over $t \in C$ by $X_t \subseteq X$.

Obviously, not all families are flat. Let $\pi\colon Y \to C$ be a family over a curve, and let $0 \in C$ be a regular point. Suppose π is not flat over 0. In some situations, we can replace the fiber Y_0 so that the resulting family becomes flat. The intuition is to take the complement of Y_0 and then take the closure. We recall the definition of flat limit following [Vak, Section 24.3.13]. One can also find it in the classical textbook [Har77, III Section 9]

doi:10.6342/NTU202500514

Proposition-Definition 2.3.2. Let C be a 1-dimensional Noetherian scheme, 0 a regular closed point of C, X a locally Noetherian scheme, and $\pi\colon X \to C$ a morphism. Suppose Y is a closed subscheme of $\pi^{-1}(C \setminus \{0\})$ and is flat over $C \setminus \{0\}$. If Y' is the scheme-theoretic closure of Y in X, then Y' is flat over C. The fiber of 0 of Y', denoted Y'_0, is called the **flat limit** of Y.

We first prove the following lemma.

Lemma 2.3.3 ([Vak, Exercise 24.3.J],[Har77, III Proposition 9.7])**.** Let $\pi\colon X \to C$ be a morphism from a locally Noetherian scheme to a regular curve. Then π is flat if and only if π sends all the associated points of X to a generic point of C.

Proof. Suppose π is flat. Let $x \in X$ be such that $\pi(x) = y$ is a closed point. Since C is a regular curve, $\mathscr{O}_{C,y}$ is a DVR with maximal ideal $\mathfrak{m}_y = (t)$ and uniformizer t. t is not a zerodivisor. Since π is flat, the pullback of t to the maximal ideal $\mathfrak{m}_x$ of $\mathscr{O}_{X,x}$ is not a zerodivisor. Hence, x is not an associated point.

Conversely, suppose that π is not flat. That is, there exists $x \in X$ such that $y = \pi(x)$ is a closed point and $\mathscr{O}_{X,x}$ is not a flat $\mathscr{O}_{C,y}$-module. We still denote t as the uniformizer for the DVR $\mathscr{O}_{C,y}$. Since $\mathscr{O}_{C,y}$ is a DVR, it is, in particular, a PID. It is known that over a PID, a module is flat if and only if it is torsion-free. Therefore, $\mathscr{O}_{X,x}$ is not a torsion-free $\mathscr{O}_{C,y}$-module, i.e., the pullback of the uniformizer π^*t to $\mathscr{O}_{X,x}$ is a zerodivisor, so π^*t must be contained in an associated point. Hence, we found an associated point that is sent to a generic point. This completes the proof. $\square$

Proof of Proposition-Definition 2.3.2. We may assume $C = \operatorname{Spec} A$ where A is a discrete valuation ring. Let $0 \in \operatorname{Spec} A$ be the closed point and $\eta \in \operatorname{Spec} A$ be the generic point,

doi:10.6342/NTU202500514

X be a locally Noetherian scheme over A, and $Y \subseteq X_\eta$ is a closed subscheme that is flat over A. Finally, let Y' be the scheme-theoretic closure of Y in X.

By our assumption that $Y \subseteq X_\eta$ is flat over A, Y has no associated point. It is known that the associated points of the scheme-theoretic closure Y' are the associated points of Y, so Y' has no associated point. Therefore, by Lemma 2.3.3, Y' is flat over A. □

2.4 Zero Locus of a General Section

Let X be a smooth variety, $\mathscr{E}$ be a rank r vector bundle, and $\sigma \in H^0(X, \mathscr{E})$.

Definition 2.4.1. The **zero locus** of σ is defined as

$$Z(\sigma) = \{x \in X \mid \sigma(x) = 0\}.$$

Let $\mathscr{O}_X \to \mathscr{E}$ be the morphism of multiplying by σ. The ideal sheaf of $Z := Z(\sigma)$ is the image of the morphism $\mathscr{E}^\vee \to \mathscr{O}_X$. It is a closed subscheme of X with the expected codimension $r = \operatorname{rank}(\mathscr{E})$.

Lemma 2.4.2 ([EH16, Proposition-Definition 6.15(c)])**.** Let X be a smooth variety and $\mathscr{E}$ be a vector bundle of rank r. Suppose $Z = Z(\sigma)$ is the zero locus of a section of $\mathscr{E}$ of codimension r, which we assume to be smooth. Then

$$\mathscr{N}_{Z/X} \simeq \mathscr{E}|_Z,$$

where $\mathscr{N}_{Z/X} = (\mathscr{I}_Z / \mathscr{I}_Z^2)^\vee$ is the normal bundle of Z in X.

Proof. Let E be the total space of $\mathscr{E}$. We may view σ as a morphism $\sigma\colon X \to E$. The

doi:10.6342/NTU202500514

tangent bundle of E restricting to the zero section $X \subseteq E$ has the splitting $\mathscr{T}_E|_X \simeq \mathscr{T}_X \oplus \mathscr{E}$. The derivative $D\sigma$ restricting to the zero locus $Z \subseteq \sigma(X)$ is zero, so we get

$$\mathscr{T}_Z \to \mathscr{T}_X|_Z \xrightarrow{D\sigma} \mathscr{T}_X|_Z \oplus \mathscr{E}|_Z \to \mathscr{E}|_Z$$

is zero. For any $z \in Z$, the image of $\mathscr{T}_{X,z}$ inside $\mathscr{T}_{X,z} \oplus \mathscr{E}_z$ is the tangent space $\sigma(X)$.

Since Z is smooth of codimension r, $\sigma(X) \cap X$ transversely. That is, the projection $\mathscr{T}_{X,z} \to \mathscr{E}_z$ is surjective. Therefore, the composition

$$\mathscr{T}_X|_Z \xrightarrow{D\sigma} \mathscr{T}_X|_Z \oplus \mathscr{E}|_Z \to \mathscr{E}|_Z$$

is surjective. Computing rank, the sequence

$$0 \to \mathscr{T}_Z \to \mathscr{T}_X|_Z \to \mathscr{E}|_Z \to 0$$

is exact. Thus, $\mathscr{N}_{Z/X} \simeq \mathscr{E}|_Z$. $\square$

doi:10.6342/NTU202500514

Chapter 3 Main Result

3.1 Setup for the Main Theorem

Let $i\colon X \hookrightarrow Y$ be a closed embedding between smooth varieties with $\dim Y = n$. Let $\mathscr{G}$ be a vector bundle of rank r on Y which is a direct sum of r very ample line bundles. Let $\sigma \in H^0(Y, \mathscr{G})$ be a general regular section. Let $Z = Z(\sigma) \subseteq Y$ be the zero locus of σ. Define $W = X \cap Z$, which we also assume to be smooth as a scheme. We do not assume the intersection is transverse (i.e., the intersection W may have higher dimension). The following Cartesian diagram represents the setup:

$$\begin{tikzcd} W \arrow[r, hook, "f"] \arrow[d, hook, "j"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook, "i"] \\ Z \arrow[r, hook, "g"] & Y \end{tikzcd} \tag{3.1}$$

Note that $W = Z(i^*\sigma)$. We have morphism of exact sequences given by tangent bundles of the diagram (3.1)

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{T}_W \arrow[r, "f_*"] \arrow[d, "j_*"] & \mathscr{T}_X|_W \arrow[r, "p"] \arrow[d, "i_*|_W"] & \mathscr{N}_{W/X} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{T}_Z|_W \arrow[r, "g_*|_W"] & \mathscr{T}_Y|_W \arrow[r, "q"] & \mathscr{N}_{Z/Y}|_W \arrow[r] & 0. \end{tikzcd} \tag{3.2}$$

The vertical arrow on the right of the diagram (3.2) is defined by the composition $q \circ i_*|_W \circ p^{-1}$ and it is well-defined because the square on the left is commutative.

doi:10.6342/NTU202500514

Definition 3.1.1 ([Ful98, Section 6.3])**.** Let the notation be as above. We define

$$\mathscr{E}_W := \frac{j^*\mathscr{N}_{Z/Y}}{\mathscr{N}_{W/X}}$$

to be the **excess normal bundle** of the diagram (3.1).

One may view $\mathscr{E}_W$ as a way of measuring the non-transversality of the diagram (3.1). When the intersection is transverse, for any $w \in W$, we have

$$\mathscr{T}_{W,w} = \mathscr{T}_{X,w} \cap \mathscr{T}_{Z,w}, \quad \text{and} \quad \mathscr{T}_{Y,w} = \mathscr{T}_{X,w} + \mathscr{T}_{Z,w}.$$

Using the isomorphism theorem,

$$\frac{\mathscr{T}_{X,w}}{\mathscr{T}_{W,w}} \simeq \frac{\mathscr{T}_{X,w}}{\mathscr{T}_{X,w} \cap \mathscr{T}_{Z,w}} \simeq \frac{\mathscr{T}_{X,w} + \mathscr{T}_{Z,w}}{\mathscr{T}_{Z,w}} \simeq \frac{\mathscr{T}_{Y,w}}{\mathscr{T}_{Z,w}}.$$

Thus, the excess normal bundle is trivial.

3.2 Main Theorem

Let the notation be as in Section 3.1. Let $\tau \in H^0(Y, \mathscr{G})$ be another general section. Let $C := \mathbb{A}^1_{\mathbb{C}}$ denote the affine line, and set $C^\times := \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Let $\pi_1 \colon X \times C \to X$ be the first projection. Define the section $s \in H^0(X \times C, \pi_1^* i^* \mathscr{G})$ by

$$s(x,t) := t(\pi_1^* i^* \tau)(x) + (\pi_1^* i^* \sigma)(x).$$

We assume that, in a neighborhood D of $0 \in C$ with $t \neq 0$, $s(x,t)$ is a regular section for every t. Let $\mathcal{M} := Z(s(x,t)) \subseteq X \times C$ be the zero locus of s. By construction, $\mathcal{M}$ is a closed subscheme of $X \times C$. Under the assumption that s is a regular section, the fiber

doi:10.6342/NTU202500514

$Z(s(x,t))$ over $t \in D \setminus \{0\}$ has the expected codimension.

Let $\pi\colon \mathcal{M} \to C$ be the restriction of the second projection $\pi_2\colon X \times C \to C$ to $\mathcal{M}$. We use the notation $\mathcal{M}_t := \pi^{-1}(t) \subseteq \mathcal{M}$ for the fiber over t. Note that $\pi\colon \pi^{-1}(C^\times) \to C^\times$ is a submersion, so π is smooth over $C^\times$. In particular, π is flat over $C^\times$, and by Proposition-Definition 2.3.2, we can define

$$\mathcal{M}' := \overline{\mathcal{M} \setminus \mathcal{M}_0} \subseteq X \times C,$$

and $\mathcal{M}'$ is flat over C. Let $\pi'\colon \mathcal{M}' \to C$ be the restriction of π to $\mathcal{M}'$. By definition, the fiber $\mathcal{M}'_0 := \pi'^{-1}(0)$ is the flat limit of $\mathcal{M} \setminus \mathcal{M}_0$.

The image of $\tau \in H^0(Y, \mathscr{G})$ under the composition

$$H^0(Y, \mathscr{G}) \to H^0(W, \mathscr{G}|_W) \to H^0(W, \mathscr{E}_W) \tag{3.3}$$

is denoted by τ'. The zero locus of τ' in W is denoted by $Z(\tau')$. We now prove the following result, which is the main theorem of this note.

Theorem 3.2.1. Let the notation be the same as above. Then $\mathcal{M}'_0 = Z(\tau')$. In other words, the family $\mathcal{M}_t$ degenerate to $Z(\tau')$.

Proof. Using implicit function theorem, for any $p \in W$, there exists an analytic neighborhood of p on which the diagram (3.1) is of the form

$$\begin{tikzcd} V \times \{p\} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & V \times X' \arrow[d, hook] \\ V \times Z' \arrow[r, hook] & V \times Y' \end{tikzcd} \tag{3.4}$$

with all the morphisms restricting to V are identity.

doi:10.6342/NTU202500514

Therefore, we may assume that in (3.1), W is a point $\{p\}$, and let

$$\begin{tikzcd} \{p\} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d, hook] \\ Z' \arrow[r, hook] & Y' \end{tikzcd} \tag{3.5}$$

be the corresponding commutative diagram. Here, X', Y', and Z' are smooth. We assume that $\dim Y' = n$, $\dim X' = m$ and $\dim Z' = \ell$. Recall that Z' is the zero locus of a general section of a rank r vector bundle $\mathscr{G}$, so $\ell = n - r$.

Since Z' and Y' are smooth, using implicit function theorem, we can choose $U \subseteq Y'$ a local trivialization of $\mathscr{G}$ such that $\mathscr{G}|_U \simeq U \times \mathbb{C}^r$ and for $y \in U$, the section $\sigma(y) = (y_{\ell+1}, \ldots, y_n)$, the projection of the last r-components of y. That is, viewing $Z' = Z(\sigma)$ as a closed subvariety of Y', the coordinates of $z \in Z' \cap U$ is given by $(z_1, \ldots, z_\ell, 0, \ldots, 0)$.

Note that we have the closed embeddings $X' \hookrightarrow Y'$. We choose the coordinates near $p \hookrightarrow X'$ such that p maps to 0, and $X' \hookrightarrow Y'$ is given by

$$(x_1, \ldots, x_m) \mapsto (\underbrace{0, \ldots, 0}_{\ell}, x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-\ell-m})$$

In conclusion, on U, the coordinates are given by

$$(z_1, \ldots, z_\ell, x_1, \ldots, x_m, y_{\ell+m+1}, \ldots, y_n).$$

In this case,

$$\sigma|_{X' \cap U}(x) = (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-\ell-m}).$$

Hence, $\{p\} = Z(\sigma|_{X' \cap U}) = \{x_1 = \cdots = x_m = 0\}$.

By assumption, $\tau \in H^0(Y, \mathscr{G})$ is a regular section. We write $\tau(y) = (\tau_{\ell+1}(y), \ldots, \tau_n(y))$

doi:10.6342/NTU202500514

to match up the index of y. On $(X' \cap U) \times C$, the section $s(x,t)$ is given by

$$s(x,t) = (x_1 + t\tau_{\ell+1}(x), \dots, x_m + t\tau_{\ell+m}(x), t\tau_{\ell+m+1}(x), \dots, t\tau_n(x)).$$

When $t = 0$, we have $s(x,0) = (x_1, \dots, x_m, 0, \dots, 0)$ and $Z(s(x,0)) = \{x_1 = \cdots = x_m = 0\} = \{p\}$.

We apply the coordinates that we choose to the diagram (3.4) and we claim that on $V \times \{p\}$,

$$Z(\tau') = \{\tau_{\ell+m+1} = \cdots = \tau_n = 0\}$$

where τ' is the image of τ under the composition (3.3). Indeed, the fiber of the normal bundle $\mathscr{N}_{V\times\{p\}/V\times X'}$, which is actually the tangent bundle $\mathscr{T}_{X'}$, at any $(v,p) \in V \times \{p\}$ has basis

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_{(v,p)}, \dots, \left.\frac{\partial}{\partial x_m}\right|_{(v,p)} \right\}.$$

Also, the fiber of the normal bundle $\mathscr{N}_{V\times Z'/V\times Y'}|_{V\times\{p\}}$ at (v,p) has basis

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_{(v,p)}, \dots, \left.\frac{\partial}{\partial x_m}\right|_{(v,p)}, \left.\frac{\partial}{\partial y_{\ell+m+1}}\right|_{(v,p)}, \dots, \left.\frac{\partial}{\partial y_n}\right|_{(v,p)} \right\}.$$

Hence, the fiber of the excess bundle $\mathscr{E}_{V\times\{p\}} = \frac{\mathscr{N}_{V\times Z'/V\times Y'}|_{V\times\{p\}}}{\mathscr{N}_{V\times\{p\}/V\times X'}}$ at (v,p) has basis

$$\left\{ \left.\frac{\partial}{\partial y_{\ell+m+1}}\right|_{(v,p)}, \dots, \left.\frac{\partial}{\partial y_n}\right|_{(v,p)} \right\}.$$

The trivialization on $\mathscr{E}$ is induced by the trivialization U on $\mathscr{G}$. Therefore, the section τ' under the coordinates is of the form $\tau' = (\tau_{\ell+m+1}, \dots, \tau_n)$ and the result follows.

Finally, we show that on $W \cap U$, $Z(\tau') = \mathcal{M}'_0$. On one hand, for t near but not 0, if

doi:10.6342/NTU202500514

$(x,t) \in \mathcal{M} \setminus \mathcal{M}_0$, then

$$(x,t) \in \{(x,t) \in (X \cap U) \times C \mid t \neq 0,\ \tau_{\ell+m-1}(x) = \cdots = \tau_n(x) = 0\}.$$

by the coordinates we choose for $s(x,t)$. Then by taking closure and restricting to $t = 0$, we have $\mathcal{M}'_0 \subseteq Z(\tau')$.

On the other hand, the fiber $\mathcal{M}'_0$ and $Z(\tau')$ are both irreducible. By flatness of the family $\mathcal{M}' \to C$, the dimension of $\mathcal{M}'_0$ is the same as the dimension of $Z(\tau')$. Hence, we obtain the equality $\mathcal{M}'_0 = Z(\tau')$.

$\square$

doi:10.6342/NTU202500514

Chapter 4 Application

The main goal of this section is to apply Theorem 3.2.1 to the family of Shinder-Zhang's elliptic curves (Corollary 4.4.2).

4.1 Canonical bundle of $\mathsf{Gr}(k, n)$

We begin with the computation of the canonical bundle of the Grassmannain. We will constantly use it in the following subsections.

Let V be a vector space of dimension n. Let $\mathsf{Gr}(k, V)$ be the Grassmannian, i.e., the set of k-dimensional vector subspaces of V. Sometimes, we write $\mathsf{Gr}(k, n)$ if the vector space V is clear. It is well-known that the Grassmannian is a projective variety and we have the Plücker embedding $\mathsf{Gr}(k, n) \hookrightarrow \mathbb{P}$ where $\mathbb{P} = \mathbb{P}(\wedge^k V)$.

Let $\mathscr{U}$ and $\mathscr{Q}$ be the tautological subbundle and tautological quotient bundle of $\mathsf{Gr}(k, V)$, respectively. They satisfy the tautological exact sequence

$$0 \to \mathscr{U} \to \mathscr{O}_{\mathsf{Gr}(k,V)}^{\oplus n} \to \mathscr{Q} \to 0. \tag{4.1}$$

From the exact sequence (4.1), we see that $\det \mathscr{O}_{\mathsf{Gr}(k,V)}^{\oplus n} \simeq \det(\mathscr{U}) \otimes \det(\mathscr{Q})$. As $\det \mathscr{O}_{\mathsf{Gr}(k,V)}^{\oplus n} \simeq \mathscr{O}_{\mathsf{Gr}(k,n)}$, we know that $\det(\mathscr{U}) \simeq \det(\mathscr{Q})^\vee$.

doi:10.6342/NTU202500514

It is well-known that the tangent bundle of $\mathsf{Gr}(k,V)$ is $\mathscr{T}_{\mathsf{Gr}(k,V)} \simeq \mathscr{H}om(\mathscr{U},\mathscr{Q}) \simeq \mathscr{U}^\vee \otimes \mathscr{Q}$. The cotangent bundle $\Omega^1_{\mathsf{Gr}(k,V)} \simeq \mathscr{U} \otimes \mathscr{Q}^\vee$. The canonical bundle is

$$\omega_{\mathsf{Gr}(k,V)} = \det(\Omega^1_{\mathsf{Gr}(k,V)}) \simeq \wedge^k(\mathscr{U} \otimes \mathscr{Q}^\vee) \simeq \det(\mathscr{U})^{\otimes(n-k)} \otimes \det(\mathscr{Q})^{\otimes k} \simeq \det(\mathscr{U})^{\otimes n}.$$

As $\det(\mathscr{U}) \simeq \mathscr{O}_{\mathsf{Gr}(k,V)}(-1) := \mathscr{O}_{\mathbb{P}}(-1)|_{\mathsf{Gr}(k,V)}$ from the Plücker embedding, we conclude that

$$\omega_{\mathsf{Gr}(k,V)} \simeq \mathscr{O}_{\mathsf{Gr}(k,V)}(-n) \tag{4.2}$$

4.2 Shinder-Zhang's Elliptic Curves of Degree 5

In [SZ20], they constructed a pair of elliptic curves of degree 5 as the linear section of the Grassmannian $\mathsf{Gr}(2,5) \cap \mathbb{P}^4 \subseteq \mathbb{P}^9$ in the Plücker embedding. We review the construction.

Let V be a vector space of dimension 5. Consider the Grassmannian $\mathsf{Gr}(2,V)$. We have the Plücker embedding $\mathsf{Gr}(2,V) \to \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^9$. Let $A \subseteq \wedge^2 V^\vee$ be a general 5-dimensional subspace. By generality, the intersection $C := \mathsf{Gr}(2,V) \cap \mathbb{P}(A^\perp)$ in $\mathbb{P}^9$ is transverse, which is an elliptic curve of degree 5. That is, it is a smooth projective curve of genus 1 equipped with a line bundle $\mathscr{L}$ of degree 5. Indeed, by adjunction formula,

$$\omega_C = \omega_{\mathsf{Gr}(2,V)}|_C \otimes \det(\mathscr{N}_{C/\mathsf{Gr}(2,V)})$$

Here, $\omega_{\mathsf{Gr}(2,V)} \simeq \mathscr{O}(-5)$ by (4.2). For $\mathscr{N}_{C/\mathsf{Gr}(2,V)}$, we may view C as a zero locus of general section of $\mathscr{O}(1)^{\oplus 5}$ on $\mathsf{Gr}(2,V)$. Then $\mathscr{N}_{C/\mathsf{Gr}(2,V)} = \mathscr{O}(1)^{\oplus 5}|_C$. It follows that

doi:10.6342/NTU202500514

$\det(\mathscr{O}(1)^{\oplus 5}) = \mathscr{O}(5)|_C$. Therefore, by the previous computation,

$$\omega_C \simeq \mathscr{O}_{\mathsf{Gr}(2,V)}(-5)|_C \otimes \mathscr{O}_{\mathsf{Gr}(2,V)}(5)|_C \simeq \mathscr{O}_C$$

is trivial. The degree 5 line bundle $\mathscr{L}$ is simply the determinant of the normal bundle $\mathscr{N}_{C/\mathsf{Gr}(2,V)}$.

In fact, according to [SZ20, Lemma 2.6], the converse also holds, i.e., every elliptic quintic is a transverse intersection of $\mathsf{Gr}(2,5)$ with a linear subspace of dimension 4 in $\mathbb{P}^9$.

The projective dual of $\mathsf{Gr}(2,V)$ is non-canonically isomorphic to itself. Indeed, the projective dual of $\mathsf{Gr}(2,V)$ is $\mathsf{Pf}(2,V^\vee) = \mathbb{P}\{\omega \in \wedge^2 V^\vee \mid \mathrm{rank}(\omega) \leq 2\} = \mathsf{Gr}(2,V^\vee)$.

Let A be as above. By [Kuz06, Proposition 2.24], $\mathsf{Gr}(2,V) \cap \mathbb{P}(A^\perp)$ is smooth if and only if $\mathsf{Gr}(2,V^\vee) \cap \mathbb{P}(A)$ is. Using similar arguments, we know that the transverse intersection $C_2 := \mathsf{Gr}(2,V^\vee) \cap \mathbb{P}(A)$ is also an elliptic curve of degree 5.

4.3 Non-transverse Intersection

The intersection will become non-transverse if we choose the linear subspace $\mathbb{P}(A^\perp)$ in a special position. In fact, we can find linear subspace $\mathbb{P}(A^\perp)$ such that the linear section is the isotropic Grassmannian, which is also a quadric 3-fold.

4.3.1 Isotropic Grassmannian

Let W be a vector space of dimension $2m$ and $\omega \in \wedge^2 W^\vee$ be a symplectic form on W. The **isotropic Grassmannian** $\mathsf{IGr}(k,V)$ parametrizes the k-dimensional isotropic subspace of W, i.e., subspace of W on which ω vanishes. Note that if $k = 1$, then

doi:10.6342/NTU202500514

$\mathsf{IGr}(1, W) = \mathsf{Gr}(1, W) \simeq \mathbb{P}(W)$ since every line is isotropic.

We can describe $\mathsf{IGr}(k, W)$ as the zero locus of a general global section (induced by ω) of $\wedge^2 \mathscr{U}^\vee \simeq \mathscr{O}_{\mathsf{Gr}(k,W)}(1)$ in $\mathsf{Gr}(k, W)$, where $\mathscr{U}$ is the tautological subbundle of $\mathsf{Gr}(k, W)$. More explicitly, since we have the natural inclusion of vector bundles $\mathscr{U} \hookrightarrow W \otimes \mathscr{O}_{\mathsf{Gr}(k,W)}$, by taking the dual and wedge power, we get $(\wedge^2 W^\vee) \otimes \mathscr{O}_{\mathsf{Gr}(k,W)} \to \wedge^2 \mathscr{U}^\vee$. We can view ω as a global section of $(\wedge^2 W^\vee) \otimes \mathscr{O}_{\mathsf{Gr}(k,W)}$. By pushing ω to $\wedge^2 \mathscr{U}^\vee$, we get a section whose zero locus is $\mathsf{IGr}(k, W)$.

4.3.2 Non-transverse intersection

Recall that V is a 5-dimensional vector space. Fix a decomposition $V = W \oplus L$ with $\dim W = 4$ and $\dim L = 1$. We also fix a symplectic form $\omega \in \bigwedge^2 W^\vee$ on W, i.e., a non-degenerate alternating bilinear form.

We try to find a subspace $A \subseteq \wedge^2 V$ such that $\mathsf{IGr}(2, W) = \mathbb{P}(A) \cap \mathsf{Gr}(2, V)$ in the projective space $\mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^9$. Notice that we have the direct sum decomposition

$$\wedge^2 V \simeq \left(\wedge^2 W\right) \oplus (W \otimes L)$$

and its dual

$$\wedge^2 V^\vee \simeq \left(\wedge^2 W^\vee\right) \oplus (W^\vee \otimes L^\vee).$$

We may regard ω as $\omega + 0 \in \wedge^2 V^\vee$ and define

$$A(\omega) := \{\lambda \cdot \omega + \alpha \in \left(\wedge^2 W^\vee\right) \oplus (W^\vee \otimes L^\vee) \mid \lambda \in \mathbb{C},\ \alpha \in W^\vee \otimes L^\vee\} \subseteq \wedge^2 V^\vee.$$

Then $A(\omega)^\perp \subseteq \wedge^2 V$ consists of points that are killed by ω as well as all linear forms

doi:10.6342/NTU202500514

$\alpha \in W^\vee \otimes L^\vee$. Therefore, we have the non-transverse intersection

$$\mathsf{IGr}(2, W) = \mathsf{Gr}(2, V) \cap \mathbb{P}(A(\omega)^\perp) \subseteq \mathbb{P}(\wedge^2 V).$$

Note that $\mathsf{IGr}(2, W)$ is the zero locus of a general section of $\mathscr{O}_{\mathsf{Gr}(2,W)}(1)$ in $\mathsf{Gr}(2, W)$, and $\mathsf{Gr}(2, W)$ is also the zero locus of a section of $\mathscr{U}_V^\vee$, the dual of the tautological subbundle, in $\mathsf{Gr}(2, V)$ (we will give the detail in the proof of Corollary 4.4.2), so the intersection is an intersection as a scheme.

We put it into a Cartesian diagram

$$\begin{tikzcd}
\mathsf{IGr}(2, W) \arrow[r, hook] \arrow[d, hook] & \mathsf{Gr}(2, V) \arrow[d, hook] \\
\mathbb{P}(A(\omega)^\perp) \arrow[r, hook] & \mathbb{P}(\wedge^2 V)
\end{tikzcd} \tag{4.3}$$

As we mentioned earlier, $\mathsf{IGr}(2, W)$ is a quadric hypersurface inside $\mathbb{P}^4$, and $\mathsf{IGr}(2, W)$ is also the quadric 3-fold obtained by the hyperplane section of $\mathsf{Gr}(2, W)$.

On the other side, for such ω, the intersection $\mathbf{P} := \mathsf{Gr}(2, V^\vee) \cap \mathbb{P}(A(\omega)) \subseteq \mathbb{P}(\wedge^2 V^\vee)$ is not transverse, either. Here, we can think of $\mathsf{Gr}(2, V^\vee)$ as $\mathsf{Pf}(2, V^\vee)$, the space of degenerate 2-forms. In this way, we see that for $[\lambda\omega + \alpha] \in \mathbf{P}$, λ must be zero since ω is non-degenerate. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}(W) \arrow[r, hook] \arrow[d, hook] & \mathsf{Gr}(2, V^\vee) \arrow[d, hook] \\
\mathbb{P}(A(\omega)) \arrow[r, hook] & \mathbb{P}(\wedge^2 V^\vee)
\end{tikzcd}$$

Actually, $\mathbf{P} = \mathbb{P}(W^\vee \otimes L^\vee)$. Indeed, let $\{e_1, \ldots, e_4\}$ be a basis of W and $\{e_5\}$ be a basis of L. Then we can write $\alpha \in \mathbf{P}$ as $\alpha = \gamma \wedge e_5^\vee$ where $\gamma \in W^\vee$ and $e_5^\vee$ is the dual basis of e_5. Therefore, $[\alpha] \in \mathbb{P}(W^\vee \otimes L^\vee)$.

doi:10.6342/NTU202500514

Conversely, for $[\alpha] \in \mathbb{P}(W^\vee \otimes L^\vee) \simeq \mathbb{P}(\mathrm{Hom}(L, W^\vee))$, the plane $\alpha(L) \oplus L^\vee$ in $V^\vee$ gives an element in $\mathsf{Gr}(2, V^\vee)$. Thus, $\mathbf{P} = \mathbb{P}(W^\vee \otimes L^\vee) \simeq \mathbb{P}(\mathrm{Hom}(W, L^\vee)) \simeq \mathbb{P}(W)$.

Remark 4.3.1. In [MM83, page 117], Mori and Mukai prove that for Fano 3-fold with $b_2 = 2$ (which, in this case, is the same as the Picard rank) of type $(E_1 - E_1)$, there are 6 deformation types. Our case is exactly type 2) in the list.

4.4 Calabi-Yau pair of Homogeneous Roof of Type C_2

4.4.1 Quadric 3-folds and spinor bundles

Before we start, we review some basic properties of quadric 3-folds and spinor bundles. Let Q^3 be a smooth quadric 3-fold, meaning a smooth variety Q that admits a closed embedding $\iota \colon Q^3 \hookrightarrow \mathbb{P}^4$, such that Q^3 is a quadric hypersurface. We denote $\mathscr{O}_{Q^3}(1) := \iota^* \mathscr{O}_{\mathbb{P}^4}(1)$.

Suppose the quadric 3-fold Q^3 is a linear section of a quadric 4-fold $\mathsf{Gr}(2, 4) \simeq Q^4$. On Q^3, recall that the **spinor bundle** is the pull-back of the tautological subbundle on $\mathsf{Gr}(2, 4)$. It is also the pullback of the dual of the tautological quotient bundle on $\mathsf{Gr}(2, 4)$ [Ott88, Theorem 1.4 (i)]. The following is classical and we will use it in the proof of Corollary 4.4.2.

Proposition 4.4.1. There exists an exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}_{Q^3}^{\oplus 4} \to \mathscr{S}^\vee \to 0, \tag{4.4}$$

and $\mathscr{S}^\vee \simeq \mathscr{S}(1)$, where $\mathscr{S}(1) := \mathscr{S} \otimes \mathscr{O}_{Q^3}(1)$.

doi:10.6342/NTU202500514

Proof. The exact sequence (4.4) follows from the tautological exact sequence (4.1) on Grassmannian $\mathsf{Gr}(2,4)$ and the fact that the spinor bundle is the pullback of the dual of the tautological quotient bundle.

For $\mathscr{S}^\vee$, since it is of rank 2 and it is, by definition, the pullback of tautological subbundle on $\mathsf{Gr}(2,4)$, we have

$$\mathscr{S} \simeq \mathscr{S}^\vee \otimes \wedge^2 \mathscr{S} \simeq \mathscr{S}^\vee \otimes \mathscr{O}_{Q^3}(-1).$$

$\square$

4.4.2 Degeneration

Recall the setting in Section 4.3.2. Let

$$\mathsf{IFl}(1,2;W) := \{L \subseteq P \subseteq W \mid \dim L = 1,\ \dim P = 2,\ P \text{ is isotropic}\}$$

be the partial isotropic flag variety, $\mathsf{IGr}(2,W)$ be the isotropic Grassmannian of W. Consider the following diagram

$$\begin{tikzcd} & \mathsf{IFl}(1,2;W) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}(W) & & \mathsf{IGr}(2,W) \end{tikzcd} \tag{4.5}$$

Here, p_1 and p_2 are the canonical projections.

In fact, these three varieties can be described as homogeneous varieties. Let G be a simple simply connected algebraic group of type C_2. The Dynkin diagram of G is $\bullet\!\Leftarrow\!\bullet$. Using the crossed Dynkin diagram, $\mathsf{IFl}(1,2;W)$ corresponds to $\times\!\Leftarrow\!\times$, $\mathbb{P}(W)$ corresponds to $\times\!\Leftarrow\!\bullet$, and $\mathsf{IGr}(2,W)$ corresponds to $\bullet\!\Leftarrow\!\times$. The diagram (4.5) is sometimes called the ho-

doi:10.6342/NTU202500514

mogeneous roof of type C_2 (see [Kan22, Section 5]). We will use the fact that $\mathsf{IFl}(1,2;W)$ is a projective bundle over both $\mathbb{P}(W)$ and $\mathsf{IGr}(2,W)$.

Both $\mathbb{P}(W)$ and $\mathsf{IGr}(2,W)$ are Fano varieties of Picard rank 1. The ample generator for $\mathbb{P}(W)$ is $\mathscr{O}_{\mathbb{P}(W)}(1)$ and the ample generator for $\mathsf{IGr}(2,W)$ is $\mathscr{O}_{\mathsf{IGr}(2,W)}(1) := \det \mathscr{S}^\vee$, where $\mathscr{S}$ is the spinor bundle on $\mathsf{IGr}(2,W)$. By Borel-Weil Theorem [Ott95, Theorem 10.6], the ample line bundles $\mathscr{O}_{\mathbb{P}(W)}(1)$ and $\mathscr{O}_{\mathsf{IGr}(2,W)}(1)$ are very ample.

Consider the ample line bundle $\mathscr{L} := p_1^*\mathscr{O}_{\mathbb{P}(W)}(1) \otimes p_2^*\mathscr{O}_{\mathsf{IGr}(2,W)}(1)$, which is also very ample. Take $s \in H^0(\mathsf{IFl}(1,2;W), \mathscr{L})$ be a general section and let

$$C_1' := Z(p_{1*}s) \subseteq \mathbb{P}(W)$$
$$C_2' := Z(p_{2*}s) \subseteq \mathsf{IGr}(2,W).$$

(We use the notation C_i' to align with the notation introduced in Section **??**.) Since $\mathscr{L}$ is a very ample line bundle, it is globally generated. By [Muk92, Theorem 1.8], $p_{i*}\mathscr{L}$ are also globally generated. By generality of s, C_i' are smooth schemes [Muk92, Theorem 1.10].

With Theorem 3.2.1, we can prove the following.

Corollary 4.4.2. The Shinder-Zhang's elliptic curves degenerate to the elliptic curve $C_2' = Z(p_{2*}s)$.

Proof of Corollary 4.4.2. Recall the diagram (4.3). $\mathbb{P}(A(\omega)^\perp)$ is the zero locus of a section σ of $\mathscr{O}(1)^{\oplus 5}$ in $\mathbb{P}(\wedge^2 V)$. The non-transverse intersection $\mathsf{IGr}(2,W) = Z(\sigma|_{\mathsf{Gr}(2,V)})$ is smooth. Also, recall that the Shinder-Zhang's elliptic curve is the smooth zero locus of a general section of $\mathscr{O}_{\mathsf{Gr}(2,V)}(1)^{\oplus 5}$ in the Grassmannian $\mathsf{Gr}(2,V)$ (Section 4.2).

Let $C = \mathbb{A}^1$ be the affine line and $\pi_1 \colon \mathsf{Gr}(2,V) \times C \to \mathsf{Gr}(2,V)$ be the first pro-

doi:10.6342/NTU202500514

jection. We choose another general section $\tau \in H^0(\mathbb{P}(\wedge^2 V), \mathscr{O}(1)^{\oplus 5})$ such that $\mathcal{M} \subseteq \mathsf{Gr}(2,V) \times C$ is defined to be the zero locus of the section

$$s = \sigma + t\tau \in H^0(\mathsf{Gr}(2,V) \times C, \pi_1^* \mathscr{O}_{\mathsf{Gr}(2,V)}(1)^{\oplus 5})$$

and for $t \neq 0$, the fiber $\mathcal{M}_t$ over $t \in C$ is the Shinder-Zhang's elliptic curve. The flat limit of $\mathcal{M} \setminus \mathcal{M}_0$ is, by Theorem 3.2.1, the zero locus $Z(\tau') \subseteq \mathsf{IGr}(2,W)$ of $\tau' \in H^0(\mathsf{IGr}(2,W), \mathscr{E}_W)$, where τ' is the image under the composition

$$H^0(\mathbb{P}(\wedge^2), \mathscr{O}(1)^{\oplus 5}) \to H^0(\mathsf{IGr}(2,W), \mathscr{O}(1)^{\oplus 5}|_{\mathsf{IGr}(2,W)}) \to H^0(\mathsf{IGr}(2,W), \mathscr{E}_{\mathsf{IGr}(2,W)}).$$

Since the flat limit can be described as the zero locus of the section τ' of the excess bundle $\mathscr{E}_{\mathsf{IGr}(2,W)}$, our goal is to show that $\mathscr{E}_{\mathsf{IGr}(2,W)}$ is isomorphic to the bundle $p_{2*}\mathscr{L}$. Then the desired result follows.

First we show that the excess normal bundle $\mathscr{E}_{\mathsf{IGr}(2,W)}$ is $\mathscr{S}^\vee(1)$, where $\mathscr{S}$ is the spinor bundle of the quadric 3-fold $\mathsf{IGr}(2,W)$. As $\mathbb{P}(A(\omega)^\perp) \simeq \mathbb{P}^4$ is the zero locus of a section in $H^0(\mathbb{P}(\wedge^2 V), \mathscr{O}(1)^{\oplus 5})$, the normal bundle

$$\mathscr{N}_{\mathbb{P}(A(\omega)^\perp)/\mathbb{P}(\wedge^2 V)} \simeq \mathscr{O}(1)^{\oplus 5}|_{\mathbb{P}(A(\omega)^\perp)} \simeq \mathscr{O}_{\mathbb{P}(A(\omega)^\perp)}(1)^{\oplus 5}.$$

By the definition of excess normal bundle (Definition 3.1.1),

$$\mathscr{E}_{\mathsf{IGr}(2,W)} = \frac{\mathscr{N}_{\mathbb{P}(A(\omega)^\perp)/\mathbb{P}(\wedge^2 V)}|_{\mathsf{IGr}(2,W)}}{\mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,V)}} \simeq \frac{\mathscr{O}_{\mathsf{IGr}(2,W)}(1)^{\oplus 5}}{\mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,V)}}.$$

Consider the embedding

$$\mathsf{IGr}(2,W) \hookrightarrow \mathsf{Gr}(2,W) \hookrightarrow \mathsf{Gr}(2,V).$$

doi:10.6342/NTU202500514

Since these three varieties are all smooth, we have the normal bundle exact sequence

$$0 \to \mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,W)} \to \mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,V)} \to \mathscr{N}_{\mathsf{Gr}(2,W)/\mathsf{Gr}(2,V)}|_{\mathsf{IGr}(2,W)} \to 0. \tag{4.6}$$

To compute the normal bundle $\mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,W)}$, we notice that $\mathsf{IGr}(2,W) \subseteq \mathsf{Gr}(2,W)$ is the zero locus of a global section of $\mathscr{O}_{\mathsf{Gr}(2,W)}(1)$ given by the sympletic form ω as described in Section 4.3.1. Thus, $\mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,W)} \simeq \mathscr{O}_{\mathsf{IGr}(2,W)}(1)$.

On the other hand, the Grassmannian $\mathsf{Gr}(2,W)$ is the zero locus of a section of $H^0(\mathsf{Gr}(2,V), \mathscr{U}_V^\vee)$, where $\mathscr{U}_V^\vee$ is the dual of tautological subbundle of $\mathsf{Gr}(2,V)$. Indeed, consider the canonical exact sequence $0 \to W \to V \to L \to 0$. Then any 2-plane P lies in W if and only if the projection $V \to L \simeq \mathbb{C}$ restricts to P is 0. The projection gives a section of $\mathscr{U}_V^\vee$ via the canonical inclusion $\mathscr{U} \to V \otimes \mathscr{O}_{\mathsf{Gr}(2,V)}$. Thus, $\mathscr{N}_{\mathsf{Gr}(2,W)/\mathsf{Gr}(2,V)} \simeq \mathscr{U}_V^\vee|_{\mathsf{Gr}(2,W)} \simeq \mathscr{U}_W^\vee$, where $\mathscr{U}_W^\vee$ is the dual of tautological subbundle of $\mathsf{Gr}(2,W)$. Note that if we view $\mathsf{IGr}(2,W)$ as a quadric 3-fold, then $\mathscr{U}_W^\vee|_{\mathsf{IGr}(2,W)} \simeq \mathscr{S}^\vee$, where $\mathscr{S}$ is the spinor bundle on $\mathsf{IGr}(2,W)$.

Therefore, by (4.6),

$$\mathscr{E}_{\mathsf{IGr}(2,W)} = \frac{\mathscr{O}_{\mathsf{IGr}(2,W)}(1)^{\oplus 5}}{\mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,V)}} \simeq \frac{\mathscr{O}_{\mathsf{IGr}(2,W)}(1)^{\oplus 5} / \mathscr{O}_{\mathsf{IGr}(2,W)}(1)}{\mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,V)} / \mathscr{O}_{\mathsf{IGr}(2,W)}(1)} \simeq \frac{\mathscr{O}_{\mathsf{IGr}(2,W)}(1)^{\oplus 4}}{\mathscr{S}^\vee}. \tag{4.7}$$

Notice that in the first isomorphism, we use the fact that $\mathscr{O}_{\mathsf{Gr}(2,V)}(1)$ restricts to $\mathsf{Gr}(2,W)$ is exactly $\mathscr{O}_{\mathsf{Gr}(2,W)}(1)$ given by the Plücker embedding. This is because $\mathbb{P}(\wedge^2 W) \hookrightarrow \mathbb{P}(\wedge^2 V)$ is a linear embedding.

doi:10.6342/NTU202500514

The inclusion

$$\mathscr{S}^\vee \simeq \mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,V)} \,/\, \mathscr{O}_{\mathsf{IGr}(2,W)}(1) \hookrightarrow W \otimes \mathscr{O}_{\mathsf{IGr}(2,W)}(1) \simeq \mathscr{O}_{\mathsf{IGr}(2,W)}(1)^{\oplus 4}$$

inducing (4.7) is obtained from the natural inclusion of the normal bundle

$$\mathscr{N}_{\mathsf{IGr}(2,W)/\mathsf{Gr}(2,V)} \hookrightarrow \mathscr{N}_{\mathbb{P}(A(\omega)^\perp)/\mathbb{P}(\wedge^2 V)}|_{\mathsf{IGr}(2,W)} \simeq V \otimes \mathscr{O}_{\mathsf{IGr}(2,W)}(1) \simeq \mathscr{O}_{\mathsf{IGr}(2,W)}(1)^{\oplus 5}$$

modding out $\mathscr{O}(1)$.

On the other hand, we have the spinor bundle exact sequence (4.4)

$$0 \to \mathscr{S} \to \mathscr{O}^{\oplus 4}_{\mathsf{IGr}(2,W)} \to \mathscr{S}^\vee \to 0$$

on the quadric 3-fold $\mathsf{IGr}(2,W)$. It is the restriction of the tautological exact sequence on $\mathsf{Gr}(2,W)$ to $\mathsf{IGr}(2,W)$. Tensoring by $\mathscr{O}(1)$ and using the isomorphism $\mathscr{S}^\vee \simeq \mathscr{S}(1)$, we have

$$0 \to \mathscr{S}^\vee \to \mathscr{O}_{\mathsf{IGr}(2,W)}(1)^{\oplus 4} \to \mathscr{S}^\vee(1) \to 0.$$

These two inclusions coincide. Therefore, combining (4.7), we conclude that

$$\mathscr{E}_{\mathsf{IGr}(2,W)} \simeq \mathscr{S}^\vee(1).$$

Next, we show that on $\mathsf{IGr}(2,W)$, we have $p_{2*}\mathscr{L} \simeq \mathscr{S}^\vee(1)$. By the projection formula, $p_{2*}\mathscr{L} = p_{2*}(p_1^*\mathscr{O}(1) \otimes p_2^*\mathscr{O}(1)) \simeq p_{2*}p_1^*\mathscr{O}(1) \otimes \mathscr{O}(1)$. By [Ott88, (2)], $p_{2*}p_1^*\mathscr{O}(1) \simeq \mathscr{S}^\vee$. Therefore,

$$p_{2*}p_1^*\mathscr{O}(1) \otimes \mathscr{O}(1) \simeq \mathscr{S}^\vee(1) \simeq \mathscr{E}_{\mathsf{IGr}(2,W)}.$$

doi:10.6342/NTU202500514

Thus, the flat limit of $\mathcal{M} \setminus \mathcal{M}_0$ is C_2', which is the zero locus of $p_{2*}\mathscr{L} \simeq \mathscr{E}_{\mathsf{IGr}(2,W)}$. $\square$

Remark 4.4.3. We expect that the $\mathbb{P}^3$ side also holds, but we are not able to prove it yet.

Remark 4.4.4. In [IIM19, Proposition 5.1], they show that the Calabi-Yau 3-fold in G_2-Grassmannian X defined by the zero locus of the bundle $\mathscr{S}^\vee \otimes \wedge^2 \mathscr{Q}$ is the degeneration of Calabi-Yau 3-folds obtained by linear section in $\mathsf{Gr}(2,7)$. Apart from this, in [KK16, Proposition 7.1], they show that the family of linear sections of Pfaffian degenerates to the Calabi-Yai 3-fold in the other G_2-Grassmannian, which is a quadric 5-fold.

Remark 4.4.5. Recently in [Ram24, Proposition 3.3], M. Rampazzo proved that all Calabi-Yau pairs associated to a homogeneous roof are L-equivalent.

doi:10.6342/NTU202500514

References

[BC09] Lev Borisov and Andrei Căldăraru. “The Pfaffian-Grassmannian derived equivalence”. In: J. Algebraic Geom. 18.2 (2009), pp. 201–222. ISSN: 1056-3911,1534-7486. DOI: 10.1090/S1056-3911-08-00496-7. URL: https://doi.org/10.1090/S1056-3911-08-00496-7.

[Bor18] Lev A. Borisov. “The class of the affine line is a zero divisor in the Grothendieck ring”. In: J. Algebraic Geom. 27.2 (2018), pp. 203–209. ISSN: 1056-3911,1534-7486. DOI: 10.1090/jag/701. URL: https://doi.org/10.1090/jag/701.

[Bor66] Armand Borel. “Linear algebraic groups”. In: Algebraic Groups and Discontinuous Subgroups (I
Amer. Math. Soc., Providence, RI, 1966, pp. 3–19.

[Efi18] Alexander I. Efimov. “Some remarks on L-equivalence of algebraic varieties”. In: Selecta Math. (N.S.) 24.4 (2018), pp. 3753–3762. ISSN: 1022-1824,1420-9020. DOI: 10.1007/s00029-017-0374-y. URL: https://doi.org/10.1007/s00029-017-0374-y.

[EH16] David Eisenbud and Joe Harris. 3264 and all that—a second course in algebraic geometry. Cambridge University Press, Cambridge, 2016, pp. xiv+616. ISBN: 978-1-

doi:10.6342/NTU202500514

107-60272-4; 978-1-107-01708-5. DOI: 10.1017/CBO9781139062046. URL: https://doi.org/10.1017/CBO9781139062046.

[Ful98] William Fulton. Intersection theory. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470. ISBN: 3-540-62046-X; 0-387-98549-2. DOI: 10.1007/978-1-4612-1700-8. URL: https://doi.org/10.1007/978-1-4612-1700-8.

[Har77] Robin Hartshorne. Algebraic geometry. Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[IIM19] Daisuke Inoue, Atsushi Ito, and Makoto Miura. "Complete intersection Calabi-Yau manifolds with respect to homogeneous vector bundles on Grassmannians". In: Math. Z. 292.1-2 (2019), pp. 677–703. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-018-2163-5. URL: https://doi.org/10.1007/s00209-018-2163-5.

[Ito+19] Atsushi Ito et al. "The class of the affine line is a zero divisor in the Grothendieck ring: via G_2-Grassmannians". In: J. Algebraic Geom. 28.2 (2019), pp. 245–250. ISSN: 1056-3911,1534-7486. DOI: 10.1090/jag/731. URL: https://doi.org/10.1090/jag/731.

[Kan22] Akihiro Kanemitsu. "Mukai pairs and simple K-equivalence". In: Math. Z. 302.4 (2022), pp. 2037–2057. ISSN: 0025-5874,1432-1823. DOI: 10.1007/

doi:10.6342/NTU202500514

s00209-022-03126-0. URL: https://doi.org/10.1007/s00209-022-03126-0.

[KK16] Grzegorz Kapustka and Michał Kapustka. "Calabi-Yau threefolds in $\mathbb{P}^6$". In: Ann. Mat. Pura Appl. (4) 195.2 (2016), pp. 529–556. ISSN: 0373-3114,1618-1891. DOI: 10.1007/s10231-015-0476-0. URL: https://doi.org/10.1007/s10231-015-0476-0.

[KS18] Alexander Kuznetsov and Evgeny Shinder. "Grothendieck ring of varieties, D- and L-equivalence, and families of quadrics". In: Selecta Math. (N.S.) 24.4 (2018), pp. 3475–3500. ISSN: 1022-1824,1420-9020. DOI: 10.1007/s00029-017-0344-4. URL: https://doi.org/10.1007/s00029-017-0344-4.

[Kuz06] A. G. Kuznetsov. "Hyperplane sections and derived categories". In: Izv. Ross. Akad. Nauk Ser. M 70.3 (2006), pp. 23–128. ISSN: 1607-0046,2587-5906. DOI: 10.1070/IM2006v070n03ABEH002318. URL: https://doi.org/10.1070/IM2006v070n03ABEH002318.

[Kuz18] Alexander Kuznetsov. "Derived equivalence of Ito-Miura-Okawa-Ueda Calabi-Yau 3-folds". In: J. Math. Soc. Japan 70.3 (2018), pp. 1007–1013. ISSN: 0025-5645,1881-1167. DOI: 10.2969/jmsj/76827682. URL: https://doi.org/10.2969/jmsj/76827682.

[LS24] Hsueh-Yung Lin and Evgeny Shinder. "Motivic invariants of birational maps". In: Ann. of Math. (2) 199.1 (2024), pp. 445–478. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2024.199.1.6. URL: https://doi.org/10.4007/annals.2024.199.1.6.

[Mar16] Nicolas Martin. “The class of the affine line is a zero divisor in the Grothendieck ring: an improvement”. In: C. R. Math. Acad. Sci. Paris 354.9 (2016), pp. 936–939. ISSN: 1631-073X,1778-3569. DOI: 10.1016/j.crma.2016.05.016. URL: https://doi.org/10.1016/j.crma.2016.05.016.

[MM83] Shigefumi Mori and Shigeru Mukai. “On Fano 3-folds with $B_2 \geq 2$”. In: Algebraic varieties and analytic varieties (Tokyo, 1981). Vol. 1. Adv. Stud. Pure Math. North-Holland, Amsterdam, 1983, pp. 101–129. ISBN: 0-444-86612-4. DOI: 10.2969/aspm/00110101. URL: https://doi.org/10.2969/aspm/00110101.

[Muk92] Shigeru Mukai. “Polarized $K3$ surfaces of genus 18 and 20”. In: Complex projective geometry (T Vol. 179. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1992, pp. 264–276. ISBN: 0-521-43352-5. DOI: 10.1017/CBO9780511662652.019. URL: https://doi.org/10.1017/CBO9780511662652.019.

[Ott88] Giorgio Ottaviani. “Spinor bundles on quadrics”. In: Trans. Amer. Math. Soc. 307.1 (1988), pp. 301–316. ISSN: 0002-9947,1088-6850. DOI: 10.2307/2000764. URL: https://doi.org/10.2307/2000764.

[Ott95] Giorgio Ottaviani. “Rational homogeneous varieties”. In: Lecture notes for the summer school ir (1995).

[Ram21] Marco Rampazzo. Calabi-Yau fibrations, simple K-equivalence and mutations. 2021. arXiv: 2006.06330 [math.AG]. URL: https://arxiv.org/abs/2006.06330.

[Ram24] Marco Rampazzo. “New counterexamples to the birational Torelli theorem for Calabi-Yau manifolds”. In: Proc. Amer. Math. Soc. 152.6 (2024), pp. 2317–

doi:10.6342/NTU202500514

2331. ISSN: 0002-9939,1088-6826. DOI: 10.1090/proc/16745. URL: https://doi.org/10.1090/proc/16745.

[SZ20] Evgeny Shinder and Ziyu Zhang. "L-equivalence for degree five elliptic curves, elliptic fibrations and K3 surfaces". In: Bull. Lond. Math. Soc. 52.2 (2020), pp. 395–409. ISSN: 0024-6093,1469-2120. DOI: 10.1112/blms.12339. URL: https://doi.org/10.1112/blms.12339.

[Vak] Ravi Vakil. The Rising Sea: Foundations of Algebraic Geometry. URL: https://math.stanford.edu/~vakil/216blog/FOAGsep0824public.pdf.

doi:10.6342/NTU202500514