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Master's Thesis

公私立大學學費差異之影響

The Impact of Tuition Gaps Between Public and Private Colleges

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國立臺灣大學碩士學位論文

口試委員會審定書

公私立大學學費差異之影響

The Impact of Tuition Gaps Between Public and Private Colleges

本論文係林咏壎君（R12323010）在國立臺灣大學經濟學研究所完成之碩士學位論文，於民國 114 年 7 月 10 日承下列考試委員審查通過及口試及格，特此證明

口試委員：________________________

（指導教授）

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摘要

本文探討在存在借貸限制的情況下，公私立大學學費差異與財務補助政策如何影響學生與學校之間的配對結果與個體福利。研究動機源自台灣高等教育的制度背景：公立大學通常提供較高品質且收費較低的教育，而政府亦開始推動補助私立大學學費的政策，以期減輕其學生的經濟負擔。本文透過納入學費，延伸 Fernández and Galí (1999) 的競爭式配對模型（tournament-based matching model），用以精確地呈現台灣的大學體系並加以分析。

我們使用一組微分方程系統進行數值模擬，比較兩種政策情境：(1) 僅私立大學收取學費；(2) 透過提高公立學費與降低私立學費，使所有學校學費逐步趨於一致，並維持整體學費收入不變。此設計可用以隔離並分析縮小學費差距所產生的再分配效果。

模擬結果顯示，縮小公私立學費差異可提升整體社會福利，尤其有利於能力高但財力不足的學生。本研究指出，若能精確調整學費結構，將有助於提升階層化教育體系中的公平性與效率。

關鍵字：學費差距、學生與學校配對、借貸限制、高等教育、教育補助政策

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Abstract

This paper examines how tuition gaps between public and private universities, along with financial aid policies, affect student-school matching and individual welfare under borrowing constraints. Motivated by Taiwan's higher education context—where public universities offer higher quality at lower cost, and where the government has recently begun promoting tuition subsidies for private institutions—we extend the Fernández and Galí (1999) matching model by integrating tuition, to more accurately represent the Taiwanese university system.

We simulate two policy scenarios using a system of differential equations: (1) only private universities charge tuition; and (2) tuition is gradually equalized across institutions by increasing public college tuition and decreasing private college tuition, with total tuition revenue held constant. This design allows us to isolate the redistributive effects of narrowing the tuition gap.

The results show that reducing tuition disparities improves overall welfare, especially

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for high-ability, low-wealth students. Our findings highlight the importance of carefully calibrated tuition structures in enhancing both access and efficiency in stratified education systems.

Keywords: Tuition gap；Student-school matching；Borrowing constraints；Higher education；Tuition subsidy policy

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Chapter 1 Introduction

To reduce educational inequality and financial barriers in higher education, the Ministry of Education (MOE) in Taiwan launched a tuition subsidy program targeting students in private universities. The policy aims to narrow the tuition gap between public and private institutions and promote choice based on student ability rather than family income.

Specifically, the program provides a direct tuition reduction of NT$35,000 for students attending private universities per year. In addition, economically disadvantaged students at both public and private institutions receive supplementary financial aid ranging from NT$15,000 to NT$20,000. The initiative is further supported by three complementary measures: (1) full tuition waivers for senior high school and vocational school students, (2) improvements to the student loan application and repayment system, and (3) targeted financial assistance to low-income households.

These policies collectively aim to reduce educational cost disparities and support students in pursuing academic paths aligned with their talents and interests, regardless of socioeconomic background.

According to the Regulations Governing Tuition and Miscellaneous Fee Collection for Institutions of Higher Education in Taiwan, universities and colleges are granted the authority to propose their own tuition levels, which must then be approved by the MOE.

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Newly established institutions may initially set tuition fees independently, subject to MOE review and approval. Any subsequent fee adjustments must undergo the same approval process. Consequently, while fee levels vary across institutions, the variation within each institutional category (i.e., among public or among private institutions) tends to be relatively small.

The substantial difference in tuition between public and private institutions primarily stems from their distinct financial structures. According to data from the MOE, tuition and miscellaneous fees account for only about 20% of annual revenue at public institutions, which rely heavily on government subsidies—approximately a 70:30 ratio between public funding and self-generated income. In contrast, tuition revenues at private institutions often exceed 50% of their total income, reflecting a greater dependence on student contributions.

Moreover, when excluding highly selective programs such as medicine, public universities in Taiwan generally exhibit higher admission scores than their private counterparts. This trend is illustrated in Figures 1.1 and 1.2, which are based on data from two official sources: (1)*University Admission Committee(大學考試入學分發委員會)*, which published the minimum admission scores and number of admitted students for each program in the 2024 (113th academic year) university placement process; and (2) *the Joint Committee of Technological and Vocational College Admission (JCTV, 技專校院招生委員會聯合會)*, which provided the statistical distribution of admission scores for four-year technological universities and two-year junior colleges during the same academic year. These figures respectively depict the average admission thresholds for general universities and technological institutions, thereby highlighting the correlation between institutional type and entrance selectivity.

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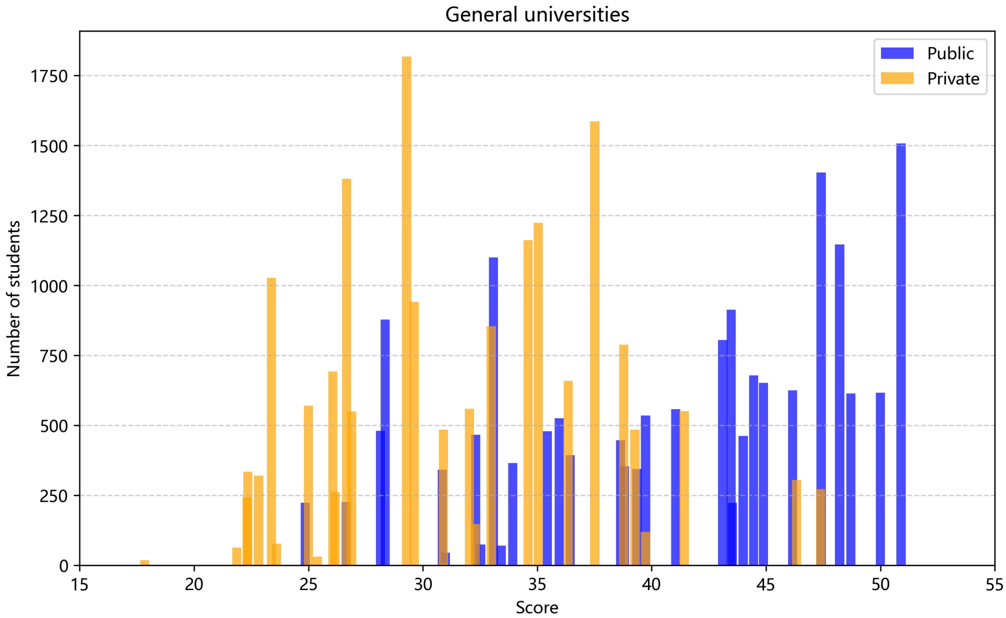

Figure 1.1: Admission Score and Enrollment: Public vs. Private General Universities

Source: University Admission Committee (大學考試入學分發委員會), 2024

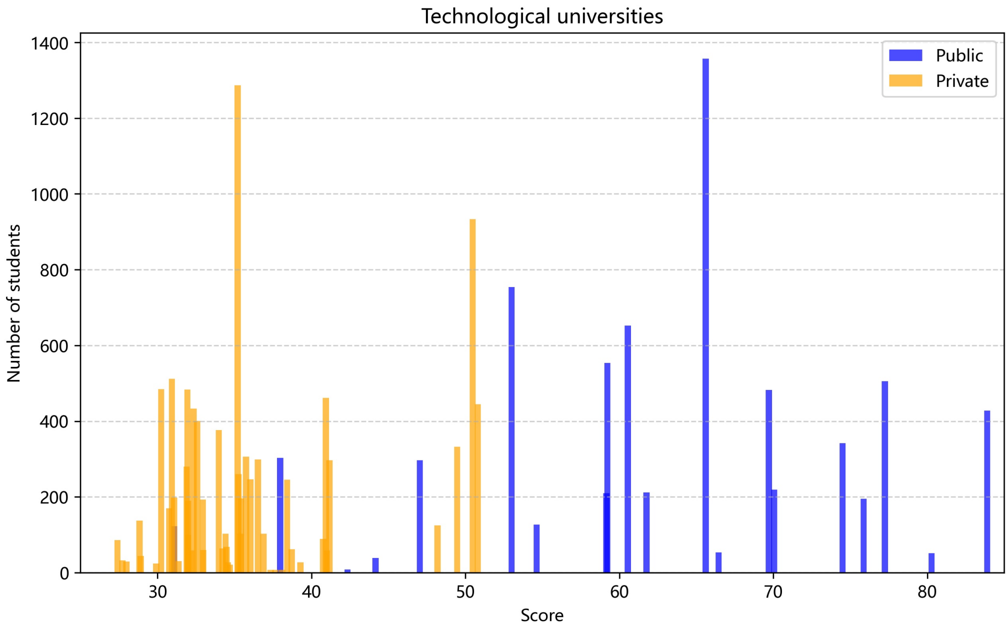

Figure 1.2: Admission Score and Enrollment: Public vs. Private Universities of Technology

Source: the Joint Committee of Technological and Vocational College Admis sion (JCTV, 技專校院招生委員會聯合會), 2024

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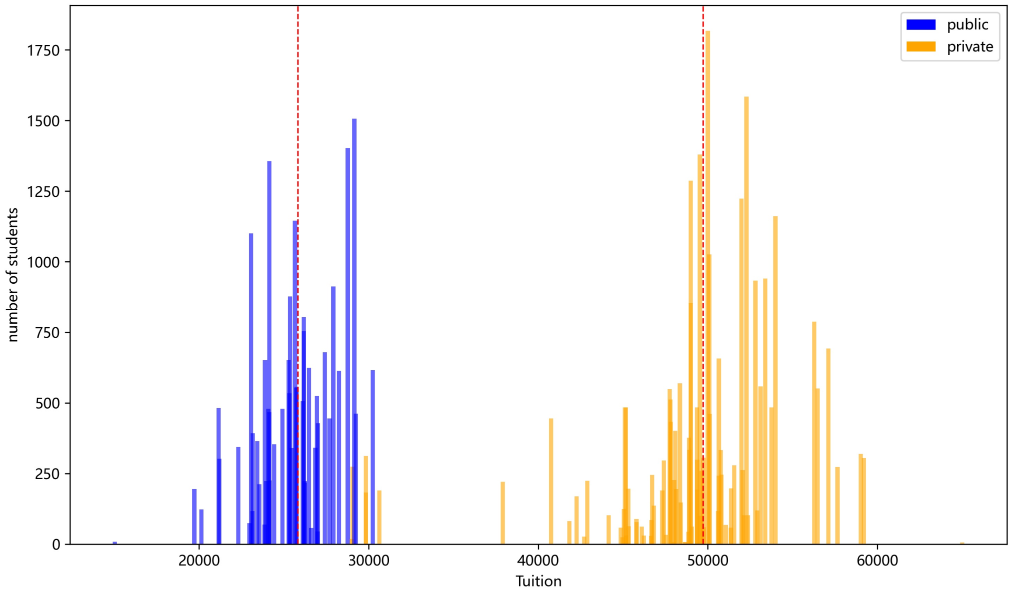

Figure 1.3: Public vs. Private University Tuition and Enrollment

Source: University Affairs Information Disclosure Platform (大專校院校務資訊公開平臺), 2024

Figure 1.3 presents tuition data drawn from the University Affairs Information Disclosure Platform (大專校院校務資訊公開平臺). The results indicate a clear tuition gap: on average, public institutions charge NT$25,842 per semester, while private institutions charge NT$49,735—nearly NT$25,000 —highlighting a significant structural disparity in the financial burden borne by students.

To explore the policy implications of tuition subsidies in a stratified education system, I adapt the matching model of Fernandez and Gali (1999) to reflect institutional features of Taiwan's higher education sector. In Taiwan, public universities are generally perceived to be of higher quality than private ones, yet public institutions charge significantly lower tuition fees. Despite this inverse relationship between quality and price, the government provides tuition subsidies only to students attending private universities.

This raises an important policy question: Does subsidizing access to lower-quality but higher-cost private universities improve equity and efficiency—or does it misallocate

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talent by incentivizing high-ability, low-wealth students away from better-quality public institutions?

To explore these questions, we develop a model in which students differ in ability and wealth, and compete for school placement through costly signaling under borrowing constraints. We simulate two policy scenarios using a system of differential equations: (1) only private universities charge tuition; and (2) tuition is gradually equalized across institutions by increasing public university tuition and decreasing private university tuition, with total tuition revenue held constant. This design allows us to isolate the redistributive impact of narrowing the tuition gap.

Results indicate that narrowing the tuition gap—either through subsidies for private colleges or modest increases in public tuition—improves overall welfare and enhances access for financially constrained students. Importantly, the benefits are not limited to private school students; even those attending public institutions are better off, as reduced competition for public seats leads to more efficient student-school matching. Charging high tuition at lower-ranked institutions is particularly inefficient, as it distorts enrollment incentives and intensifies pressure on public universities. By narrowing the gap, the system not only expands access and improves sorting by ability and wealth, but also increases overall enrollment. As total education expenditures decline, financially constrained students are better able to afford higher-quality schools. These findings offer important insights for designing tuition policies that promote both equity and efficiency in stratified education systems.

The remainder of this paper proceeds as follows: Section 2 presents the model; Section 3 analyzes theoretical results under perfect capital markets and borrowing constraints;

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Section 4 examines the role of wealth constraints; and Section 5 concludes.

1.1 Literature review

Prior work has examined how competition between public and private schools, combined with peer-group effects, shapes educational outcomes and social stratification. Epple and Romano (1998) develop a general equilibrium model with heterogeneous students (by income and ability) and peer effects, showing that private schools engage in income- and ability-based tuition discrimination, and that equilibrium leads to stratification across schools. While voucher policies may increase sorting and improve welfare for high-ability students, public schools' inability to internalize peer externalities results in overall inefficiency.

Cremer and Maldonado (2013) further explore mixed oligopoly in education, where public and private schools coexist but differ in objectives. Their two-stage model shows that while a single public school can restore efficiency without peer effects, once peer-group dynamics are introduced, no configuration achieves efficient outcomes. Importantly, the presence of public provision alone is insufficient to ensure welfare gains, highlighting the need for targeted policy instruments to correct market failures in education.

In a complementary direction, Fernandez and Gali (1999) shift attention from institutional structures to the allocation mechanisms themselves—comparing markets and tournaments under borrowing constraints. Their theoretical framework shows that the way students are assigned to educational resources fundamentally affects equity and efficiency when capital markets are imperfect. Specifically, even though tournaments involve socially wasteful signaling expenditures, they can outperform market allocations by en-

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abling high-ability but low-wealth individuals to access better resources.

Extending this line of inquiry, Fernández and Rogerson (1996) develop a dynamic intergenerational model that incorporates marital sorting, fertility behavior, educational investment, and borrowing constraints. Their calibrated simulations show that increases in assortative matching amplify long-run income inequality, particularly when educational attainment depends on family income and when the returns to education are sensitive to the relative supply of skilled labor. Together, these studies underscore that educational inequality is shaped not only by institutional competition or public-private mix, but also by the deeper structure of who gets matched to what opportunity under financial constraints.

Hopkins and Kornienko (2010) analyze how different types of inequality affect welfare in tournament-based settings. While Fernandez and Gali (1999) focus on borrowing constraints, Hopkins and Kornienko (2010) distinguish between inequality in endowments (e.g., ability, wealth) and rewards (e.g., school quality). They show that reducing reward inequality improves welfare by limiting excessive competition, whereas reducing endowment inequality can worsen outcomes by intensifying positional rivalry. Their results offer useful insights for evaluating redistribution in rank-based education systems.

Abdulkadiroğlu and Sönmez (2003) reformulate school choice as a mechanism design problem, focusing on strategy-proofness and efficiency in student assignment. They compare the Gale–Shapley student-optimal stable mechanism and the top trading cycles mechanism—both direct and strategy-proof—highlighting the trade-off between eliminating justified envy and achieving Pareto efficiency. While their framework assumes fixed priorities and capacities, their emphasis on assignment mechanisms as determinants of access and welfare complements the matching model of Fernandez and Gali (1999).

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Chapter 2 Model

We extend the Fernandez and Gali (1999) matching model by integrating tuition, to more accurately represent the Taiwanese university system. The economy is composed of a continuum of individuals, each identified by their endowed ability a and initial wealth w. For simplicity, we assume that students are uniformly distributed over the unit square $[0, 1]^2$, implying that ability and wealth are independently assigned across individuals. we consider the case of schools with varying levels of quality. The quality of each school is represented by an index s, which is assumed to follow a uniform distribution over $[0, 1]$.

A student with ability a assigned to a school s produces an output level $X(a, s)$, where $X : I^2 \to \mathbb{R}_+$ is interpreted as a production function. We assume that X is twice continuously differentiable and bounded, with $X_a > 0$ and $X_s > 0$, meaning that output increases in both ability and school quality. Furthermore, we assume $X_{as} > 0$. This indicates that the interaction between student ability and school quality enhances output through a complementary relationship in the production process.

To simplify exposition, we impose two additional assumptions: (i) $X(0, 0) = 0$, meaning that an student with the lowest ability attending the lowest-tier school obtains zero output; and (ii) $X_s(a, s) < 1$ for all $(a, s) \in I^2$.

Students make decisions based on the allocation mechanism and the availability of

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capital markets. Each student chooses an expenditure level—potentially borrowing if credit is available—and is then assigned to a school. After completing education, the student earns income, repays any debt, and consumes. The student's objective is to maximize utility from consumption, which is given by total income $X(a, s)$ minus educational expenditures and the additional tuition cost associated with attending either a public or private university. We assume the utility of not attending school is zero.

We define the allocation as a mapping $S : I^2 \to [0, 1] \cup \{\emptyset\}$ that matches a school of quality $S(a, w)$ to each student, where students are distinguished by their ability level a and initial wealth endowment w. Since some students may not go to any school in equilibrium, we denote $S(a, w) = \emptyset$ for this case.

Let $\bar{s} \in (0, 1)$. Schools with quality levels in the interval $[0, \bar{s}]$ are classified as private, while those in $(\bar{s}, 1]$ are classified as public. The student cost function is defined as

$$C(s) = c \cdot \mathbf{1}\{s \leq \bar{s}\},$$

indicating that a student pays an additional fee c if assigned to a private school (i.e., when $s \leq \bar{s}$).

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Chapter 3 The Tournament

In this paper, we view exams as a tournament that allocates each student to a school with a quality ranking corresponding to the student's position in the competition. The signaling mechanism is defined by a function $V : I \times \mathbb{R}_+ \to \mathbb{R}$, where $V(a, e)$ denotes the signal (or score) generated by a student with ability a and effort level e.

We assume that $V_a > 0$ and $V_e > 0$ for all a, and that $V(a, 0) = 0$, implying that a strictly positive expenditure is required to generate a positive signal. It is frequently convenient to analyze the corresponding cost function $e(v, a)$, implicitly defined by $V(a, e(v, a)) = v$. That is, for a given signal level v, $e(v, a)$ denotes the minimum expenditure needed by a student with ability a to generate that signal.

Using the implicit function theorem to $V(a, e(v, a)) = v$, we obtain the following partial derivatives:

$$e_a = -\frac{V_a}{V_e} < 0, \quad e_v = \frac{1}{V_e}. \tag{3.1}$$

We define the cumulative distribution function of signals in the economy as a mapping $F : \mathbb{R}_+ \to [0, 1]$, *i.e.*

$$F(v) = \int_0^1 \int_0^1 \mathbf{1}\left[v - V\left(a, E(a, w)\right)\right] dw\, da, \quad \forall v \in \mathbb{R}_+,$$

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where $E(a,w)$ represents the resources used by student (a,w). With F, we can get that the expenditure required by each student to achieve his equilibrium signal is $e(F^{-1}(s),a)$ since

$$F(V(a,e(F^{-1}(s),a))) = F(F^{-1}(s)) = s.$$

3.1 Equilibrium in complete economy

An equilibrium under tournament-based competition and perfect capital markets consists of a feasible allocation S and a signal distribution F satisfying, $\forall(a,w) \in I^2$, $\forall s \in I$

$$S(a,w) = F(V(a,e(F^{-1}(S(a,w),a))) \tag{3.2}$$

$$X(a,S(a,w)) - e(F^{-1}(s),a) - C(S(a,w)) \geq X(a,s) - e(F^{-1}(s),a) - C(s) \tag{3.3}$$

$$F(v) = \int_0^1 \int_0^1 \mathbf{1}\{v - V(a,e(F^{-1}(S(a,w)),a))\}dwda \tag{3.4}$$

$$X(a,S(a,w)) - e(F^{-1}(S(a,w),a)) - C(S(a,w)) \geq 0 \tag{3.5}$$

Condition (3.2) states the allocation rule about match between students and schools with their rank in the signaling distribution. Condition (3.3) says that all the students maximize their utility. Condition (3.4) says the signaling distribution makes the expenditure required by each student to acheive his equilibrium. Condition (3.5) is the incentive compatible constraint.

By some adequate change in Fernandez and Gali (1999), we can get the similar allo-

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cation rule (See Appendix A),

$$S(a,w) = \begin{cases} a \text{ if } X(a,a) - c \geq 0 \\ \emptyset \text{ otherwise} \end{cases} \tag{3.6}$$

For the case $S(a^*) = \bar{s}$ (whuch implies $a = \bar{s}$), we require the following incentive compatibility condition:

$$\begin{aligned} U(a^*, w^*) &= X(a^*, S(a^*, w^*)) - e(F^{-1}(S(a^*, w^*)), a^*) - C(S(a^*, w^*)) \\ &\geq X(a^* + \epsilon, S(a^* + \epsilon, w^*)) - e(F^{-1}(S(a^* + \epsilon, w^*)), a^* + \epsilon) \end{aligned} \tag{3.7}$$

By continuity of X and e,

$$\lim_{\epsilon \to 0^+} e(F^{-1}(S(a^* + \epsilon, w^*)), a^*) - e(F^{-1}(S(a^*, w^*)), a^*) = c \tag{3.8}$$

Since $X_a > 0$, there exists $\underline{a}$ such that $X(a, S(a)) - c = X(a,a) - c \geq 0$ for $a \geq \underline{a}$. Let $e(\underline{a}) = 0$,

$$e(a) = \int_{\underline{a}}^{a} X_s(z,z) - V_a(z, e(z))/V_e(z, e(z))\, dz + \textbf{const} \tag{3.9}$$

Then by (3.8), $\forall a \in [\underline{a}, 1]$,

$$e(F^{-1}(a), a) = \int_{\underline{a}}^{a} X_s(z,z) - V_a(z, e(z))/V_e(z, e(z))\, dz + c \cdot \mathbf{1}_{\{a > \bar{s}\}} \tag{3.10}$$

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Bring (3.10) back to (3.9), we get

$$
\begin{aligned}
U(a,w) &= X(a,S(a,w)) - e(F^{-1}(a),a) - C(S(a,w)) \\
&= X(a,a) - \int_{\underline{a}}^{a} X_s(z,z) - V_a(z,e(z))/V_e(z,e(z))\,dz - c
\end{aligned} \tag{3.11}
$$

3.1.1 Practical example

Consider $X(a,s) = 2as + s$, $V(a,e) = a + e$, then $e(v,a) = v - a$.

In the general case, which means there's no difference between private and public school, we can get

$$e(F^{-1}(a),a) = \int_0^a X_s(z,z) - V_a(z,e(z))/V_e(z,e(z))dz = a^2 \tag{3.12}$$

$$\begin{cases} F(v) = \int_0^1 \mathbf{1}\{v - a^2 - a\}\,da = \frac{-1+\sqrt{1+4v}}{2} \\ F^{-1}(s) = s + s^2 \end{cases} \tag{3.13}$$

$$V(a,E(a,w)) = a^2 + a \tag{3.14}$$

$$U(a,w) = X(a,S(a)) - e(F^{-1}(a),a) = 2a^2 + a - a^2 = a^2 + a \tag{3.15}$$

Now we consider the difference of private and public school. First, we can find $\underline{s}$, satisfying $X(\underline{s},\underline{s}) = c$, so $\underline{s} = \frac{-1+\sqrt{1+8c}}{4}$. In the jump case, we can get

$$e(F^{-1}(a),a) = a^2 - \underline{s}^2 - C(a) + C(0) = E(a,w)$$

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$$F(v) = \int_0^{s^*} \mathbf{1}\{v - [a^2 - \underline{s}^2 + a]]\}\, da + \int_{s^*}^{1} \mathbf{1}\{v - [a^2 - \underline{s}^2 + a + c]]\}\, da$$
$$= \begin{cases} \frac{-1+\sqrt{1+4(v+\underline{s}^2-c)}}{2} & \text{if } \bar{s} \le \frac{-1+\sqrt{1+4(v+\underline{s}^2-c)}}{2} \\ \bar{s} & \text{if } \bar{s} \in (\frac{-1+\sqrt{1+4(v+\underline{s}^2-c)}}{2}, \frac{-1+\sqrt{1+4(v+\underline{s}^2)}}{2}] \\ \frac{-1+\sqrt{1+4(v+\underline{s}^2)}}{2} & \text{if } \bar{s} > \frac{-1+\sqrt{1+4(v+\underline{s}^2)}}{2} \end{cases}$$

and

$$V(a, E(a,w)) = a^2 - \underline{s}^2 + a - C(a) + C(0) = a^2 - \underline{s}^2 + a + c \cdot \mathbf{1}\{a \ge s^*\}$$

$$U(a,w) = X(a, S(a)) - e(F^{-1}(a), a) = a^2 - \underline{s}^2 + a - c$$

For the student with $a < \underline{s} = \frac{-1+\sqrt{1+8c}}{4}$ will have negative utility, who will choose not to enter any school. Figure 3.1 compares the two scenarios: the blue line represents the differentiated tuition case, and the orange line depicts the baseline case.

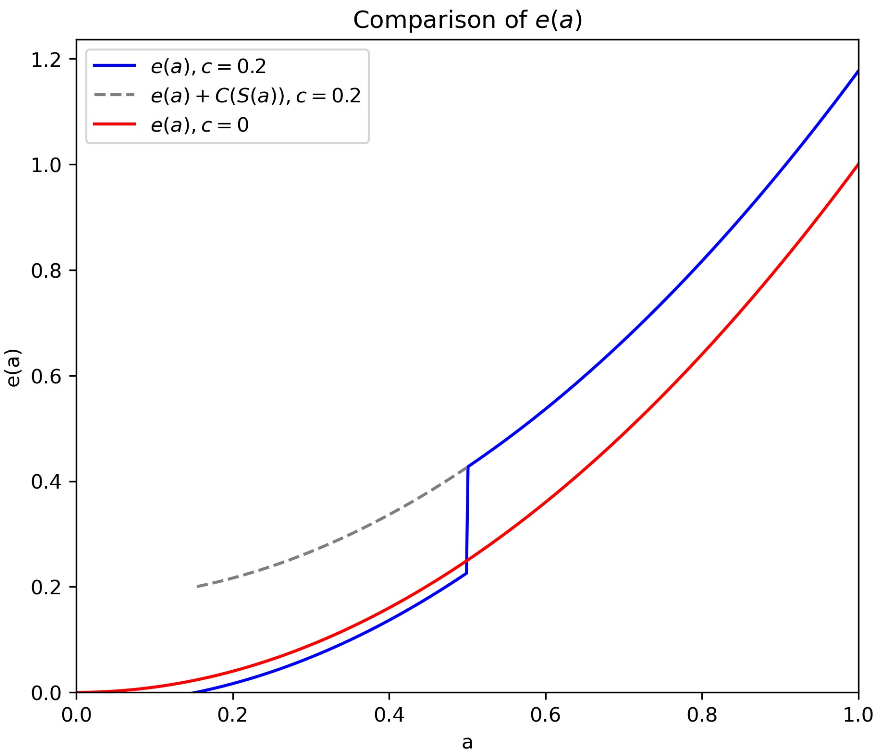

Figure 3.1: Expenditure $e(a)$ under perfect capital market

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3.2 With borrowing constraint

Under borrowing constraints, an equilibrium characterized by a feasible allocation S, signal distribution F, expenditure e and fee $C(s)$, should satisfy requirements as under perfect captital markets except the maximize utility condition (3.17) should consider the borrowing constraint:

$$S(a,w) = F(V(a, e(F^{-1}(S(a,w)), a))) \tag{3.16}$$

$$X(a, S(a,w)) - e(F^{-1}(S(a,w)), a) - C(S(a,w)) \geq X(a,s) - e(F^{-1}(s), a) - C(s)$$
$$\forall (a,w) \in I \text{ with } e(F^{-1}(S(a,w), a)) + C(S(a,w)) \leq w \tag{3.17}$$

$$F(v) = \int_0^1 \int_0^1 \mathbf{1}\{v - V(a, e(F^{-1}(S(a,w)), a))\} dw da \tag{3.18}$$

$$X(a, S(a,w)) - e(F^{-1}(S(a,w)), a) - C(S(a,w)) \geq 0 \tag{3.19}$$

The necessary condition for maximizing utility is:

$$X_s(a, S(a,w)) \geq \frac{\mathrm{d}}{\mathrm{d}s}(e(F^{-1}(S(a,w)), a) + C(S(a,w))) \tag{3.20}$$

where the equation hold for the unconstrained student.

By the similar process in Fernandez and Gali (1999), we can get:

Now, denoted $\underline{a}(s)$: the lowest ability level assigned to school s in equilibrium, or can be seen as the inverse function of $S(a,1) \equiv \mathbf{S}(a)$. following proposition characterizes

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the allocation rule: $S(a, w) = s$ if and only if $(a, w) \in R(s) \cup T(s)$, where

$$R(s) := \{a = \underline{a}(s),\ w \geq e(F^{-1}(s), a) + C(s)\}$$

$$T(s) := \{a > \underline{a}(s),\ w = e(F^{-1}(s), a) + C(s)\}$$

Property 3.2.1. $S(a, w) = s \iff (a, w) \in R(s) \cup T(s)$.

Proof. See appendix. $\square$

By Property 3.1, we know that school s will have unconstrained student with $a = \underline{a}(s)$ (says $R(s)$), and constrained student with $a > \underline{a}(s)$ (says $T(s)$). By market clearance condition, the amount of school rank higher than s, must equal the student whose $a \geq \underline{a}(s)$ and with wealth $w \geq e(F^{-1}(s), \underline{a}(s)) + C(s)$. Hence, $1 - s = \int_{\underline{a}(s)}^{1} [1 - e(F^{-1}(s), a) - C(s)] \mathrm{d}a$, or can be written as

$$\underline{a}(s) = s - \int_{\underline{a}(s)}^{1} [e(F^{-1}(s), a) + C(s)] \mathrm{d}a \tag{3.21}$$

Denoted $\mathbf{e}(a) = e(F^{-1}(\mathbf{S}(a)), a)$, and note that $\mathbf{S}(\underline{a}(s)) = s$, we have

$\mathbf{e}(\underline{a}(s)) = e(F^{-1}(s), \underline{a}(s))$. By First order condition of (3.20),

$$X_s(\underline{a}(s), s) = (\mathbf{e}(\underline{a}(s)) + C(s))' = (e(F^{-1}(s), \underline{a}(s)) + C(s))'$$

If $s \neq \bar{s}$, we can get

$$X_s(\underline{a}(s), s) = e_v(F^{-1}(s), \underline{a}(s))(F^{-1})'(s) > 0 \tag{3.22}$$

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and if at somewhere differentiable s, we have

$$X_s(\underline{a}(s), s) - C'(s) = e_v(F^{-1}(s), \underline{a}(s))F^{-1}(s) \tag{3.23}$$

By differentiating (3.21), we get

$$\begin{aligned}
\underline{a}'(s) &= 1 + \underline{a}'(s)(\mathbf{e}(\underline{a}(s)) + C(s)) - \int_{\underline{a}(s)}^{1} [e(F^{-1}(s), a) + C(s)]' \mathrm{d}a \\
&= 1 + \underline{a}'(s)(\mathbf{e}(\underline{a}(s)) + C(s)) - \int_{\underline{a}(s)}^{1} X_s(\underline{a}(s), s) \frac{e_v(F^{-1}(s), a)(F^{-1})'(s)}{e_v(F^{-1}(s), \underline{a}(s))(F^{-1})'(s)} \mathrm{d}a \\
&= \frac{1 - X_s(\underline{a}(s), s) \cdot \int_{\underline{a}(s)}^{1} \frac{e_v(F^{-1}(s), a)}{e_v(F^{-1}(s), \underline{a}(s))} da}{1 - \mathbf{e}(\underline{a}(s)) - C(s)}
\end{aligned} \tag{3.24}$$

The first equation is due to Leibniz integral rule, and the second equation is due to (3.22). Since $e_{va} \leq 0$, $\int_{\underline{a}(s)}^{1} \frac{e_v(F^{-1}(s), a)}{e_v(F^{-1}(s), \underline{a}(s))} da \leq 1$. Together with $X_s < 1$, we can conclude that $\underline{a}'(s) > 0$.

By the allocation rule (3.2), we have $F^{-1}(s) = V(\underline{a}(s), \mathbf{e}(\underline{a}(s)))$. Then differentiate it by s, we get $(F^{-1})' = V_a \cdot \underline{a}' + V_e \cdot \mathbf{e}' \cdot \underline{a}' = \underline{a}'(V_a + V_e \cdot \mathbf{e}')$. Since $V_e = \frac{1}{e_v}$ (the result from (3.1)), then $\mathbf{e}' = \frac{e_v (F^{-1})'}{\underline{a}'} - \frac{V_a}{V_e} = \frac{X_s}{\underline{a}'} - \frac{V_a}{V_e}$. Thus, by (3.24),

$$\begin{aligned}
\mathbf{e}'(\underline{a}(s)) &= \frac{X_s(\underline{a}(s), s)}{\underline{a}'(s)} - \frac{V_a(\underline{a}(s), \mathbf{e}(\underline{a}(s)))}{V_e(\underline{a}(s), \mathbf{e}(\underline{a}(s)))} \\
&= \frac{1 - \mathbf{e}(\underline{a}(s)) - C(s)}{\frac{1}{X_s(\underline{a}(s), s)} - \int_{\underline{a}(s)}^{1} \frac{e_v(F^{-1}(s), z)}{e_v(F^{-1}(s), \underline{a}(s))} \mathrm{d}z} - \frac{V_a(\underline{a}(s), \mathbf{e}(\underline{a}(s)))}{V_e(\underline{a}(s), \mathbf{e}(\underline{a}(s)))}
\end{aligned} \tag{3.25}$$

or can be written as

$$\mathbf{e}'(a) = \frac{1 - \mathbf{e}(a) - C(\mathbf{S}(a))}{\frac{1}{X_s(a, \mathbf{S}(a))} - \int_{a}^{1} \frac{e_v(F^{-1}(\mathbf{S}(a)), z)}{e_v(F^{-1}(\mathbf{S}(a)), a)} \mathrm{d}z} - \frac{V_a(a, \mathbf{e}(a))}{V_e(a, \mathbf{e}(a))} \tag{3.26}$$

At the same time, since $\mathbf{S}$ is increasing, there exists $\tilde{a}, \tilde{s} = \mathbf{S}(\tilde{a})$ such that $X(\tilde{a}, \tilde{s}) =$

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$C(\tilde{s})$ and $\mathbf{e}(\tilde{a}) = 0$, which makes an unconstrained student with ability $\tilde{a}$ indifferent to go to school or not. Thus, the ability in $[0, \tilde{a})$ would not go to school (which makes them get negative utility), and $[0, \tilde{s})$ wouldn't have any students (since monotonicity).

By solving (3.24) and (3.25) simultaneously with $X(\tilde{a}, \tilde{s}) = C(\tilde{s})$ and $\mathbf{e}(\tilde{a}) = 0$, and the initial condition $\underline{a}(1) = 1$ (or $\mathbf{S}(1) = 1$), we can get $\underline{a}(s)$ and $\mathbf{e}(a)$.

The next two properties show that:

i If both public and private schools exist (i.e. $\bar{s} > 0$), then the equilibrium always occurs at the lowest-quality school that the student is willing to attend, denoted by $\tilde{s}$, which is necessarily a private school.

ii For the same reason as in the unconstrained case, we have $\mathbf{e}(\bar{a}^+) = \mathbf{e}(\bar{a}) + c$. Moreover, note that $\mathbf{e}(a) + C(\mathbf{S}(a))$ is continuous in a.

Property 3.2.2. *If $\bar{s} > 0$, then $\tilde{s} \leq \bar{s}$.*

Proof. See Appendix. □

Property 3.2.3. *Suppose $\bar{s} > 0$, and consider $\underline{a}(\bar{s}) = \bar{a}$, the type of the unconstrained student who attends school $\bar{s}$ (the highest-ranked private school). Let $\bar{a}^+ = \underline{a}(\bar{s}^+)$. Then,*

$$\mathbf{e}(\bar{a}^+) = \mathbf{e}(\bar{a}) + c.$$

Proof. See Appendix. □

Since $\mathbf{e}(a) + C(s) \geq c \; \forall a \in [\tilde{a}, 1]$, student with wealth less than c would not go to school in wealth constraint case. Since $e_a \leq 0$, $e(F^{-1}(\tilde{s}), a) = 0$ for $a \geq \tilde{a}$.

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After solving the unconstrained case, note that the constrained student who goes to school $\bar{s}$, will not choose $s < \bar{s}$ since X is increasing and their wealth $w = \mathbf{e}(a') + C(\mathbf{S}(a'))$ for some a', because of the continuity of the function $\mathbf{e}(a) + C(\mathbf{S}(a))$, will not make him better off.

3.2.1 Numerical Method and Practical Example

As the differential equations in this model do not admit a closed-form solution, we employ numerical methods to simulate the model's behavior. The following subsections describe the implementation procedure using Python.

1. Fix a constant c and choose suitable functional forms for $X(a, s)$ and $V(a, e)$ such that the following conditions are satisfied:

$$X_a > 0, \quad 0 < X_s < 1 - c, \quad X_{as} > 0, \quad V_a > 0, \quad V_e > 0, \quad V(a, 0) = 0.$$

2. Based on equations (3.24) and (3.25), derive the system of differential equations jointly characterizing $\underline{a}(s)$ and $\mathbf{e}(a)$.

3. Impose the initial condition $\underline{a}(1) = 1$, which reflects the assumption that the best (unconstrained) student attends the best school.

4. (a) Implement an iterative procedure (or manually adjust step by step), where for each trial a value of the initial condition $\mathbf{e}(1) = init\_e$ is selected. Using the initial conditions together with the system of differential equations derived in step 2, solve for a discrete numerical solution

$$\{s_i, \ \underline{a}(s_i), \ \mathbf{e}(\underline{a}(s_i))\}_{i=0}^{n} \quad \text{where } s_n = 1,$$

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by applying the `ode45` numerical solver.

(b) Set a tolerance level ϵ, and determine whether there exists a triple $(\tilde{s},\, \underline{a}(\tilde{s}),\, \mathbf{e}(\underline{a}(\tilde{s})))$ such that

$$|X(\underline{a}(\tilde{s}),\tilde{s})-c|<\epsilon, \quad |\mathbf{e}(\underline{a}(\tilde{s}))-0|<\epsilon.$$

(c) Since $\underline{a}'(s)>0$, the function $X(\underline{a}(s),s)$ is strictly increasing in s. Therefore, if a solution exists, it is unique. Iteratively refine the initial guess $init\_e$, reduce the tolerance ϵ, and examine the stability and convergence of the solution.

5. Record the resulting value $\tilde{s}$ and the corresponding $init\_e$. Plot the outcomes accordingly.

Let $X(a,s)=(1-\delta_1)as+\frac{\delta_1}{2}(a+s)$, then $0<X_s(a,s)=(1-\delta_1)a+\frac{\delta_1}{2}<1-c$ if $\delta_1>2c$. Let $V(a,e)=(a+\delta_2)e$, and hence $e(v,a)=\frac{v}{(a+\delta_2)}$, $e_v(v,a)=\frac{1}{a+\delta_2}$.

Then we have

$$\begin{aligned}\underline{a}'(s)&=\frac{1-X_s(\underline{a}(s),s)\cdot\int_{\underline{a}(s)}^1\frac{e_v(F^{-1}(s),a)}{e_v(F^{-1}(s),\underline{a}(s))}da}{1-\mathbf{e}(\underline{a}(s))-C(s)}\\&=\frac{1-[(1-\delta_1)\underline{a}(s)+\frac{\delta_1}{2}](\underline{a}(s)+\delta_2)\cdot\ln(\frac{1+\delta_2}{\underline{a}(s)+\delta_2})}{1-\mathbf{e}(\underline{a}(s))-C(s)}\end{aligned}$$

with $\underline{a}(1)=1$,and

$$\begin{aligned}\mathbf{e}'(a)&=\frac{1-\mathbf{e}(a)-C(\mathbf{S}(a))}{\frac{1}{X_s(a,\mathbf{S}(a))}-\int_a^1\frac{e_v(F^{-1}(\mathbf{S}(a)),z)}{e_v(F^{-1}(\mathbf{S}(a)),a)}\mathrm{d}z}-\frac{V_a(a,\mathbf{e}(a))}{V_e(a,\mathbf{e}(a))}\\&=\frac{1-\mathbf{e}(a)-C(\mathbf{S}(a))}{\frac{1}{X_s(a,\mathbf{S}(a))}-(a+\delta_2)\cdot\ln(\frac{1+\delta_2}{a+\delta_2})}-\frac{\mathbf{e}(a)}{a+\delta_2}\end{aligned}$$

where $X(\underline{a}(\tilde{s}),\tilde{s})=c,\ \mathbf{e}(\underline{a}(\tilde{s}))+C(\tilde{s})=c.$

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Chapter 4 Discussion

In this section, we examine two policy directions for narrowing the tuition gap between private and public universities under the assumption of borrowing constraints. The first scenario reduces tuition charged by private institutions. The second direction includes two designs: first, introducing a modest tuition fee at public universities while reducing private tuition; second, applying a uniform tuition across all institutions. Both designs are adjusted such that total tuition revenue remains constant across cases. This framework allows us to evaluate how different methods of narrowing tuition gaps affect student-school matching outcomes and individual welfare.

The analysis extends the numerical method presented in the previous section, using the same functional forms. We set the parameters as $\delta_1 = \delta_2 = 0.25,\ c = 0.1$ and $\bar{s} = 0.5$.

4.1 Scenario 1: Only Private Universities Charge Tuition

Given a fixed proportion of public and private universities ($\bar{s} = 0.5$), we examine the effects of increasing the tuition gap between the two sectors. The results are illustrated in Figure 4.1 and Figure 4.2.

When the additional tuition for private schools is high ($c = 0.1$), students with ability

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$a \in [0, 0.2044)$ opt not to enroll in any school, and those with wealth $w \leq 0.1$ are similarly excluded. Consequently, schools with quality $s \in [0, 0.2665)$ are left without any students.

Reducing the tuition to $c = 0.05$ leads to improved access: students with ability $a \in [0, 0.1237)$ and wealth $w \leq 0.05$ still do not enroll, but the range of unattended schools shrinks to $s \in [0, 0.1725)$.

When the additional tuition is fully eliminated ($c = 0$), students of all ability levels choose to enroll, and all schools are populated. In this case, the threshold function $\underline{a}(s)$—defined as the minimum student ability required to attend school s—is described by the red line in the corresponding figure, and satisfies $\underline{a}(s) < s$ for all s.

This change in the tuition structure leads to several notable effects. Now we focus on $c = 0.1$ case versus $c = 0$. First, among unconstrained students with ability levels $a \in [0.2044, 1]$, total educational expenditure increases. For lower-ability students attending private schools, the rise in tuition induces a decrease in effort, as the marginal increase in total cost is less than 0.1. In contrast, students attending public institutions increase their learning effort in response to the wider tuition gap. Around the boundary $\bar{s} = 0.5$, a noticeable kink appears in the slope of total expenditure: at $c = 0.1$, the threshold $\underline{a}(0.5) = 0.4322$, and students attending private schools exhibit a steeper cost-effort slope than those attending public schools. This discontinuity in slope becomes smoother as the tuition gap narrows.

Second, the function $\underline{a}(s)$ shifts downward with the increase in tuition gap, indicating that for any given school s, the minimum admitted student ability becomes lower. This can be interpreted as a crowding-out effect: students who were previously able to afford enrollment are pushed out due to rising tuition, and are replaced by less capable but

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wealthier students who compensate for their lower ability with increased effort.

4.2 Scenario 2: Equalizing Tuition Across Institutions

We consider schools in the interval $[0, 0.5]$ as private and those in $(0.5, 1]$ as public. The tuition fee function is given by:

$$C(s) = 0.1 \times \mathbf{1}_{\{s \leq 0.5\}}.$$

Under this setting, the lowest ability and the corresponding school quality at the lower bound of enrollment are $(\underline{a}, \underline{s}) = (0.2045, 0.2665)$. The total tuition paid by students in private schools is calculated as $(0.5 - 0.2045) \times 0.1 = 0.023347$. Figure 4.3 illustrates the utility distribution of students over the domain $[a, w] \in [0, 1]^2$.

We next consider a modified tuition structure in which private school fees are set at $c_{\text{pri}} = 0.05$, and public schools also charge a fee denoted by c_{pub}. The corresponding tuition fee function is given by:

$$C(s) = 0.05 * \mathbf{1}_{\{s \leq 0.5\}} + c_{pub} * \mathbf{1}_{\{s > 0.5\}}$$

To ensure the same total tuition revenue, we slightly modify the original code to construct an iterative procedure over c_{pub}, adjusting its value until the total tuition collected remains constant. Through this process, we obtain $c_{\text{pub}} = 0.0127$, and the corresponding threshold values are $(\underline{a}, \underline{s}) = (0.1217, 0.1603)$.

Using the same coding procedure, we generate Figure 4.4, which illustrates the utility distribution of students over the domain $[a, w] \in [0, 1]^2$.

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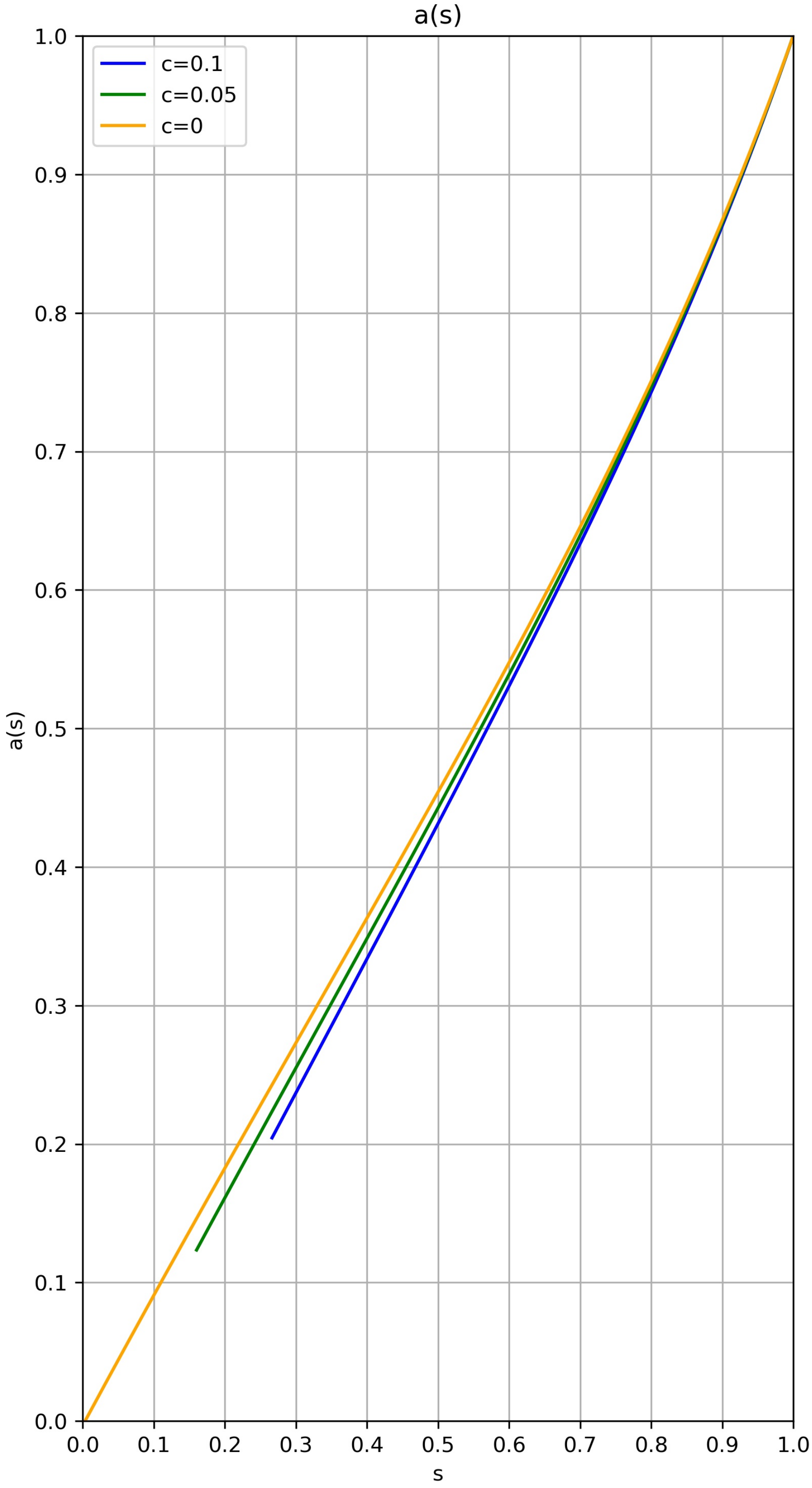

Figure 4.1: Threshold ability $\underline{a}(s)$ under different private tuition levels c

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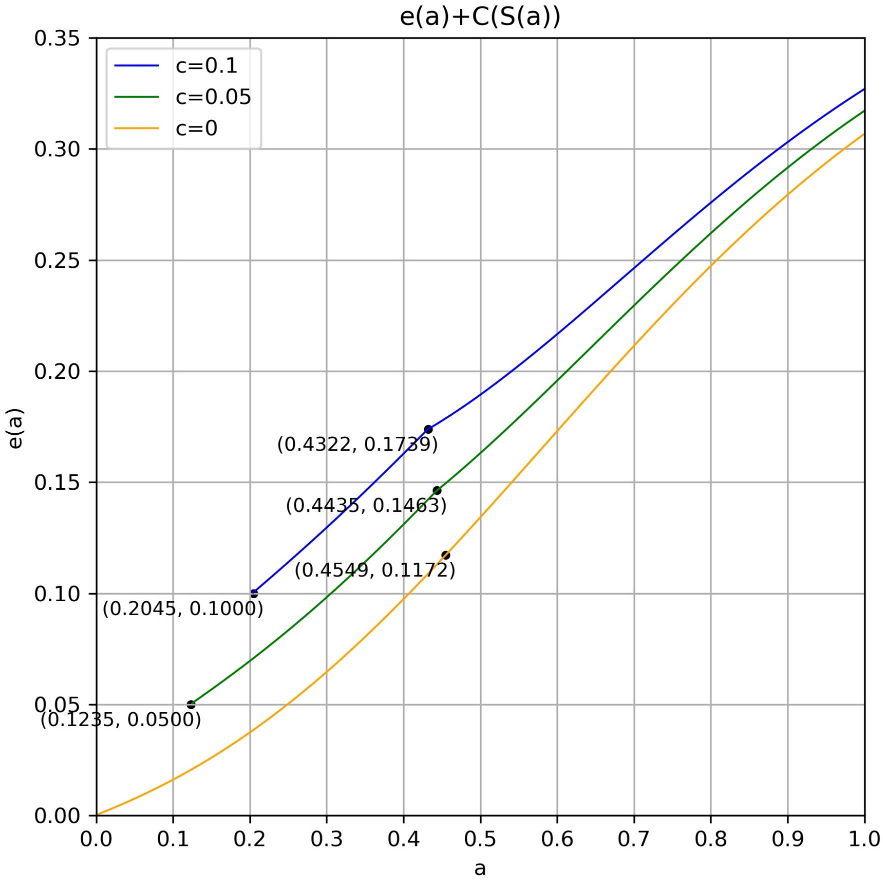

Figure 4.2: Total educational cost $e(a) + C(s)$ under different private tuition levels c

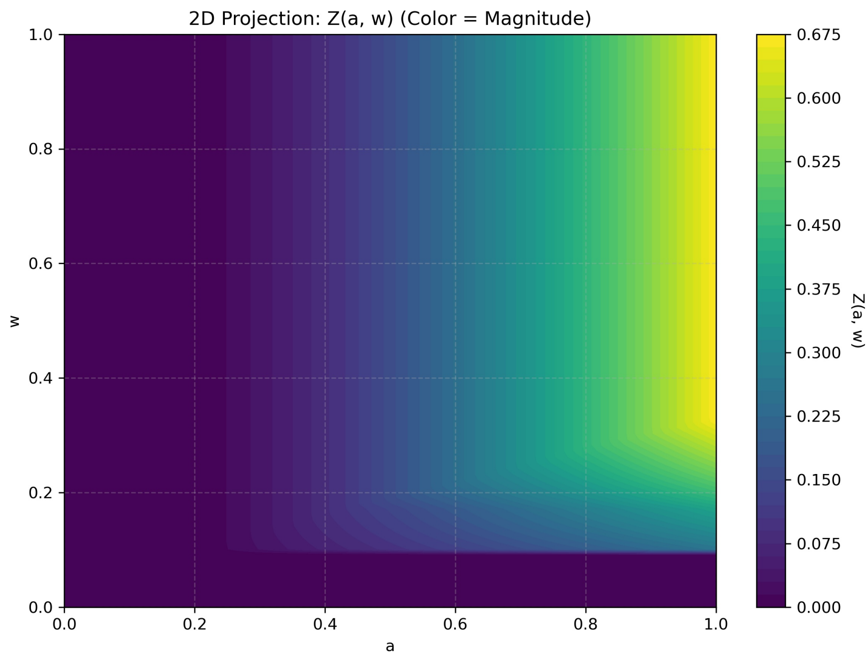

Figure 4.3: Utility $U(a, w)$ over $(a, w) \in [0, 1]^2$ under $C(s) = 0.1 \cdot \mathbf{1}_{\{s \leq 0.5\}}$

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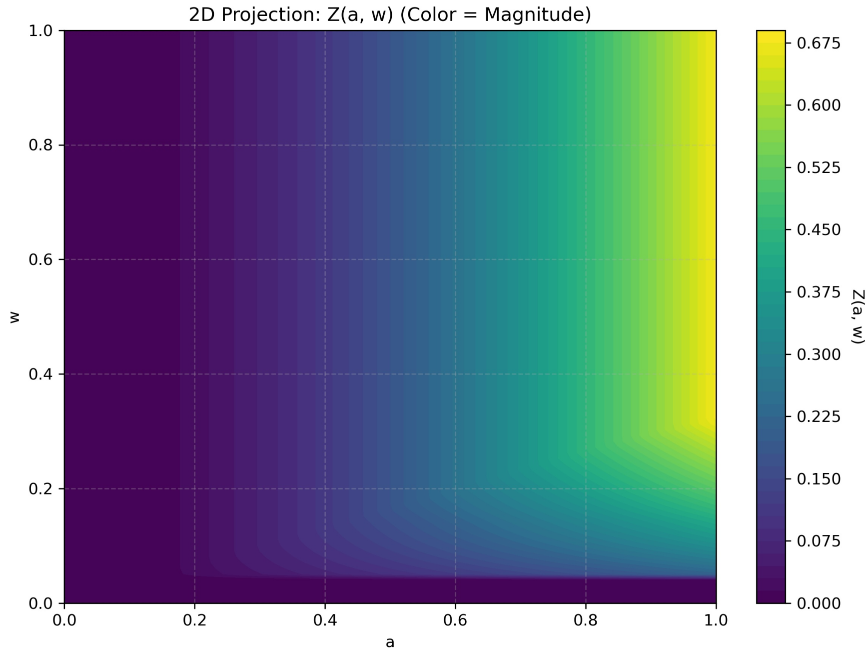

Figure 4.4: Utility $U(a, w)$ over $(a, w) \in [0, 1]^2$ under $C(s) = 0.05 * \mathbf{1}_{\{s \leq 0.5\}} + c_{pub} * \mathbf{1}_{\{s > 0.5\}}$

Furthermore, we consider the case in which both public and private schools charge the same tuition, meaning that there is no gap between the two tuition levels. We denote this common fee by c_{same}, which is chosen such that the total tuition revenue remains the same as in the previous scenarios. Specifically, the tuition fee function becomes:

$$C(s) = c_{\text{same}}.$$

Following a similar computational procedure, iterating c_{same} until the total fee to be the same, we obtain $c_{\text{same}} = 0.0257$ and $(\underline{a}, \underline{s}) = (0.0707, 0.0960)$. See Figure 4.5 for the total cost structure. The shaded regions in the figure represent the areas where fees are charged.

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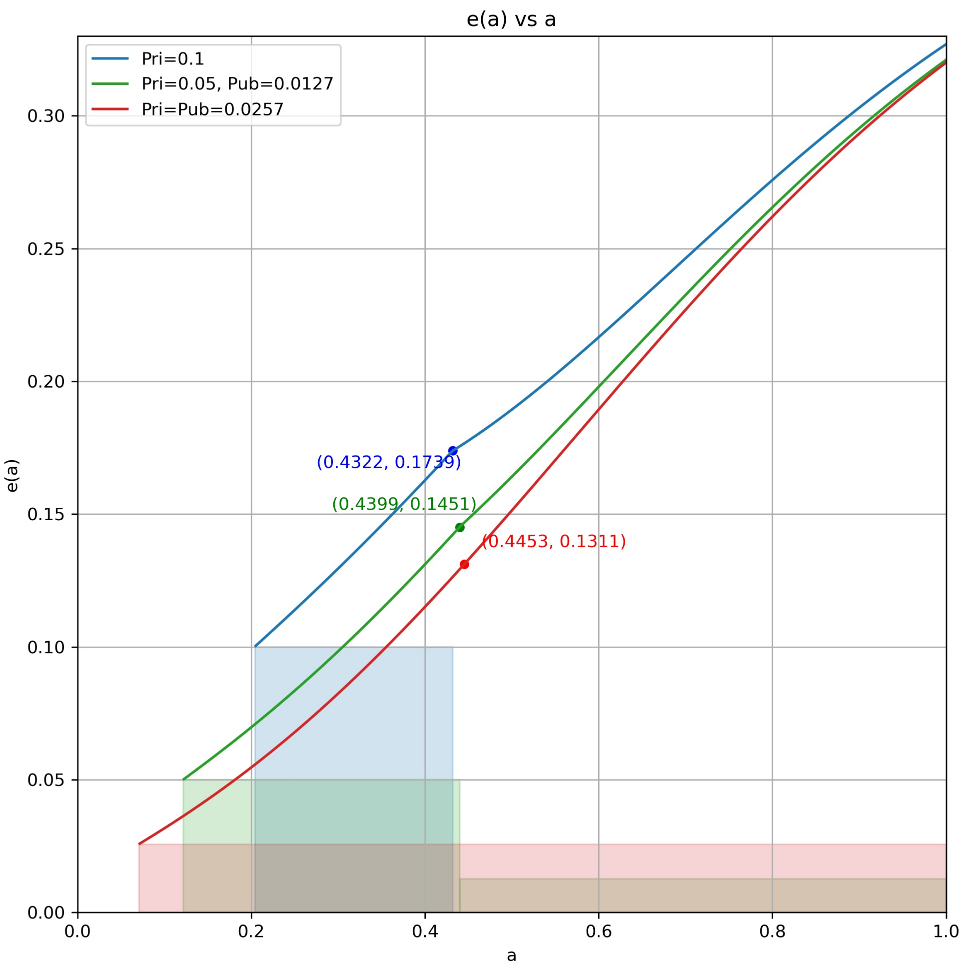

Figure 4.5: Total educational cost $e(a) + C(S(a))$ under different combinations of public and private tuition levels

4.2.1 Welfare Comparison: Who Benefits from Tuition Reform?

Next, we compute the utility difference between the high private fee regime and the alternative regime featuring lower private fees equal to public fee. The resulting changes in individual utility are presented in Figure 4.7.

Overall, **all students experience an increase in utility** under the latter regime. This improvement is primarily driven by reduced competition for public school seats, particularly among high-ability students. Under the high private fee regime, the large cost differential between private and public schools (e.g., a gap of 0.1) strongly incentivized high-ability students to choose private schools. The introduction of lower private fees equal

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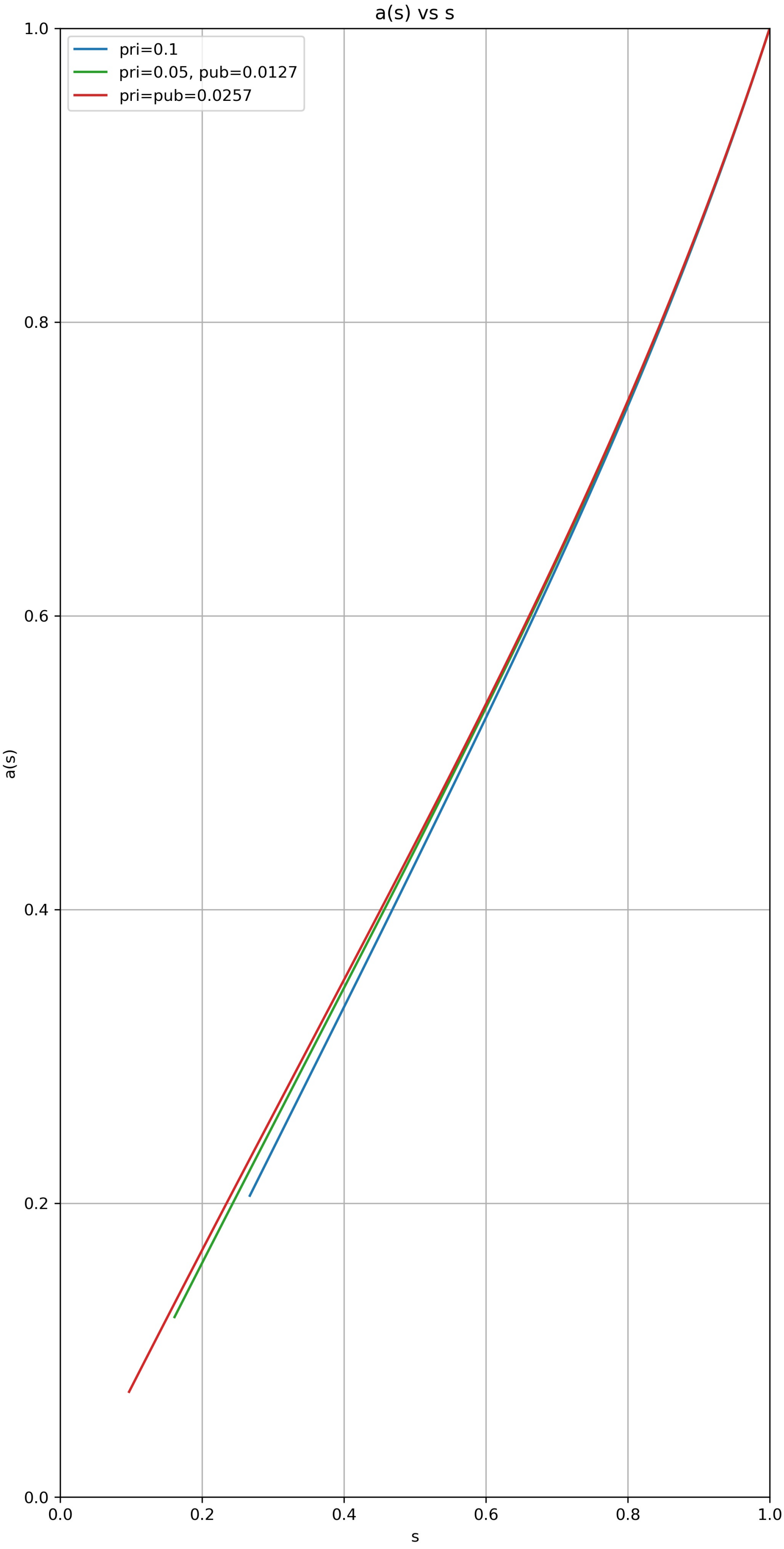

Figure 4.6: Threshold ability $\underline{a}(s)$ under different combinations of public and private tuition levels

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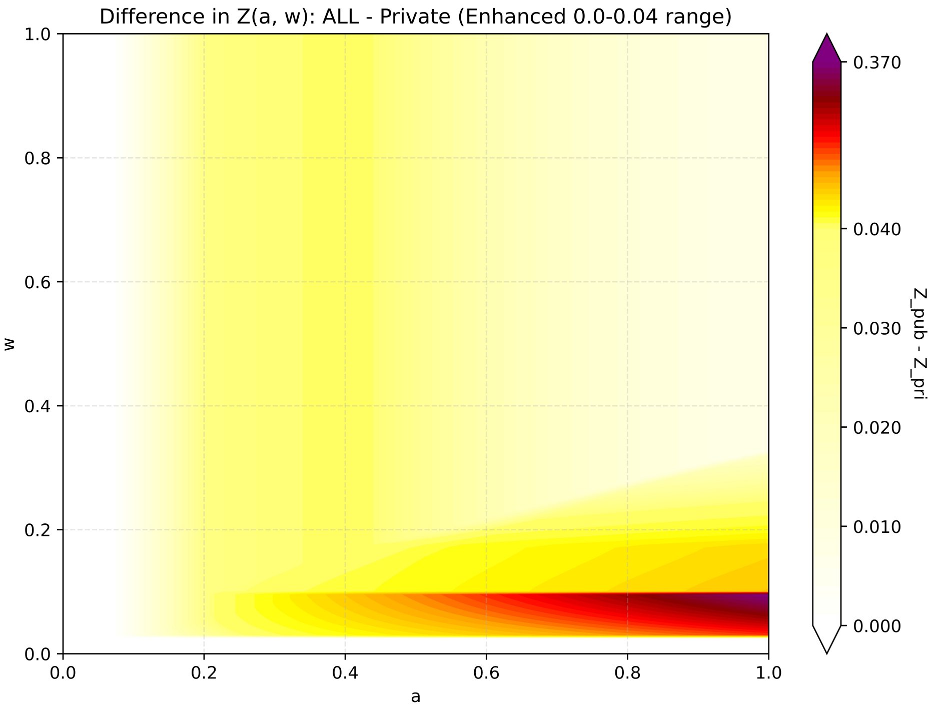

Figure 4.7: Utility difference $U_{\text{uniform}}(a, w) - U_{\text{private}}(a, w)$ across $(a, w) \in [0, 1]^2$

to public fee (where $c_{\text{pri}} - c_{\text{pub}} = 0.05 - 0.0127$) narrows this gap, thereby reducing the pressure to 0 on public school access.

When the tuition gap was large, high-ability, low-wealth ($w \in [0.0257, 0.1]$) students were unable to attend any school. Narrowing the gap allows them to re-enter the system and benefit significantly. Additionally, students with slightly higher wealth also experience notable welfare improvements, as the easing of financial constraints allows them to access higher-quality school options. Interestingly, the increase in utility is more substantial for students enrolled in private schools compared to those in public schools, suggesting that the new regime not only improves access but also enhances sorting efficiency across school types.

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Chapter 5 Conclusion

Based on our previous discussion, we found that a Pareto improvement for the entire student population can be achieved either by reducing private school tuition under the current fee structure or by redirecting this reduction to increase public school student expenditure.

The primary reason is that the main cause of expenditure distortion is the tuition gap between public and private institutions. Whether through reducing private school tuition or increasing public school fees, narrowing this gap diminishes resource allocation distortions. This approach prevents excessive competition and ensures talented students are not barred from appropriate educational institutions due to financial constraints.

Notably, implementing a policy that both reduces private school tuition and introduces fees at public institutions does not reduce public school students' utility—since the tuition they would pay remains less than the loss incurred from excessive competition under the distorted resource allocation.

Furthermore, lowering private school tuition enables more students to attend these institutions with relatively lower expenditure than before, while non-financially constrained students of equal ability can access comparatively better schools.

Therefore, policies aimed at narrowing the tuition gap between public and private

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institutions warrant encouragement, particularly when they facilitate more efficient allocation for all students while maintaining the same overall tuition revenue.

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Appendix A — Details about the tournament

A.1 The perfect capital case

by FOC, we have that feasible allocation $S(a, w)$ can be supported by the thresholds$F^{-1}(s)$ if $\forall(a, w)$, s satisfies

$$X_s - e_v \cdot (F^{-1})'(s) - C'(s) = 0 \tag{A.1}$$

and SOC

$$X_{ss} - e_{vv} \cdot (F^{-1})'^2 - e_v \cdot (F^{-1})''(s) - c''(s) < 0 \tag{A.2}$$

Denote $G = X_s(a, S(a, w)) - e_v(F^{-1}(S(a, w)), a)(F^{-1})'(S(a, w)) - c'(S(a, w))$ and $f(s) = F^{-1}(s)$ for simplicity.

By Implicit function theorem on $s(a) \neq s^*$,

$$\frac{\delta s}{\delta w} = -\frac{\delta G}{\delta w} / \frac{\delta G}{\delta s} \tag{A.3}$$

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with

$$\frac{\delta G}{\delta w} = X_{ss}\frac{\delta s}{\delta w} - e_{vv}(f')^2\frac{\delta s}{\delta w} - f''e_v\frac{\delta s}{\delta w} - c'(S)\frac{\delta s}{\delta w}, \tag{A.4}$$

$$\frac{\delta G}{\delta s} = X_{ss} - e_{vv}(f')^2 - f''e_v - c'(S) \tag{A.5}$$

Thus we can get $S(a,w) = S(a,w') \equiv S(a)$.

To get $S'(a)$, consider $\frac{d}{da}X_s$:

$$\frac{d}{da}X_s = X_{as} + X_{ss}\cdot S' = [e_{vv}\cdot(F^{-1})'^2 + e_v\cdot(F^{-1})'' + c'']S' + e_{va}\cdot(F^{-1})' \tag{A.6}$$

since $e_{va} \geq 0$ and $X_{as} \geq 0$, by SOC,

$$S'(a) = -\frac{X_{as} - e_{va}\cdot(F^{-1})'}{X_{ss} - e_{vv}\cdot(F^{-1})'^2 - e_v\cdot(F^{-1})'' - c''} > 0 \tag{A.7}$$

In order to satisfy feasibility, for any $s' \in [0,1]$, by (A.7),

$$s' = \Phi(1,s') = \int_0^1\int_0^1 \mathbf{1}_{\{s'-S(z)\}}dzdw \Rightarrow S(a) = a \tag{A.8}$$

For the case $s(a^*) \neq s^*$, note that there's no a such that $s(a) = s^*$ for (A.8). If $s(s^*) > s^*$, $a = s^* + \epsilon$ will deviate, and similar as the case $s(s^*) < s^*$, hence we can conclude that $S(a) = a$ for $a \in [0,1]$.

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A.1.1 The Jump point

For the case $S(a^*) = s^*$ (whuch means $a = s^*$), we need

$$\begin{aligned} U(a^*, w^*) &= X(a^*, S(a^*, w^*)) - e(F^{-1}(S(a^*, w^*)), a^*) - C(S(a^*, w^*)) \\ &\geq X(a^* + \epsilon, S(a^* + \epsilon, w^*)) - e(F^{-1}(S(a^* + \epsilon, w^*)), a^* + \epsilon) \end{aligned} \tag{A.9}$$

By continuity of X and e,

$$\lim_{\epsilon \to 0^+} e(F^{-1}(S(a^* + \epsilon, w^*)), a^*) - e(F^{-1}(S(a^*, w^*)), a^*) \geq c \tag{A.10}$$

To maximize all students' utility, (A.10) will be equal:

$$e(F^{-1}(S(a^{*+}, w^*)), a^*) = e(F^{-1}(S(a^*, w^*)), a^*) + c.$$

Let $s^* \in (0, 1)$ ，rank $[0, s^*]$ be 私立 and rank $(s^*, 1]$ be 公立.

$$U(a, w) = X(a, S(a, w)) - e(F^{-1}(S(a, w)), a) - C(S(a, w)) \tag{A.11}$$

where $C(S(a, w)) = c \cdot \mathbf{1}_{\{S(a,w) \leq s^*\}}$.

Denote $e(a) = e(F^{-1}(a), a)$, then

$$e'(a) = e_v(F^{-1}(a), a))(F^{-1})'(a) + e_a = X_s(a, a) - V_a/V_e \tag{A.12}$$

Since $X_a > 0$, there exists $\underline{a}$ such that$X(a, S(a)) - c = X(a, a) - c \geq 0$ for $a \geq \underline{a}$. Let $e(\underline{a}) = 0$,

$$e(a) = \int_{\underline{a}}^{a} X_s(z, z) - V_a(z, e(z))/V_e(z, e(z)) + \mathbf{d}z + \mathbf{const} \tag{A.13}$$

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Then by (A.10), $\forall a \in [\underline{a}, 1]$,

$$e(F^{-1}(a), a) = \int_{\underline{a}}^{a} X_s(z,z) - V_a(z, e(z))/V_e(z, e(z))\mathbf{d}z + c \cdot \mathbf{1}_{\{a > s^*\}} \tag{A.14}$$

Bring (A.14) back to (A.11), we get

$$U(a, w) = X(a, a) - \int_{\underline{a}}^{a} X_s(z,z) - V_a(z, e(z))/V_e(z, e(z))\mathbf{d}z - c \tag{A.15}$$

A.2 The borrowing constraint case

The first lemma says that students attending school $\underline{s}$ have no expenditures, and the second lemma says that **in equilibrium**, no student spends more than 1. Thus, the FOC will hold for some student.

Lemma 1. $e(F^{-1}(\underline{s}), a) = 0 \; \forall a \in [0, 1]$.

Proof. If $\underline{s} = 0$, by (3.18), we can get $F(0) = 0$. Then

$$V(a, e(F^{-1}(0), a)) = F^{-1}(0) = 0 \implies e(F^{-1}(0), a) = 0 \text{ since } V_e > 0 \text{ and } V(a, 0) = 0$$

If $\underline{s}$>0, suppose $e(F^{-1}(\underline{s}), a') = k > 0$ for some $a' > 0$. Since there alway exists $s' = \underline{s} - \epsilon$ and $C(s') = C(\underline{s})$, and this school have no student, which means entering this school will not have extra expenditure other than the fee $C(s)$. Then, by continuity, the student will prefer s' than $\underline{s}$, which is a contradiction. □

Lemma 2. *if* $X_s < 1 - c$*, then* $e(F^{-1}(S(a, w)), a) + C(S(a, w)) \leq 1 \; \forall s \in [\underline{s}, 1]$

Proof. Take (a, w) with $S(a, w) = s$ and $e(F^{-1}(s), a)) + C(s) > 1$ for some s. First, $s = \underline{s}$ holds by Lemma 1.

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For $s > \underline{s}$, By Revealed preference, $X(a,s) - e(F^{-1}(S(a,w),a)) - C(s) > X(a,\underline{s}) - C(\underline{s})$. But we have

$$\begin{aligned} X(a,s) - e(F^{-1}(s),a) - C(s) &= X(a,\underline{s}) + \int_{\underline{s}}^{s} X_s(a,z)\mathrm{d}z - e(F^{-1}(s),a) - C(s) \\ &\leq X(a,\underline{s}) + \int_{\underline{s}}^{s} X_s(a,z)\mathrm{d}z - 1 \\ &< X(a,\underline{s}) + (1-c)\cdot(s-\underline{s}) - 1 \leq X(a,\underline{s}) - C(\underline{s}) \rightarrow\leftarrow \end{aligned}$$

□

Let $Q(s)$ represent the set of students who can afford to go to school s,denoted as $Q(s) = \{(a,w) \in I^2, a \geq a_s, w \geq e(F^{-1}(s),a)) + C(s)\}$, where a_s stands for the lowest ability that can afford to go to school s by spending less than 1, that is, $e(F^{-1}(s),a_s) = 1$ (since $e_a \leq 0$). Note that $(a,w) \in Q(1)$ are affordable to go to any school in $[0,1]$.

Lemma 3. $\forall (a,w),(a',w') \in Q(1)$, *(i)* $a > a' \Rightarrow S(a,w) > S(a',w')$
(ii) $S(a,w) = S(a,w')$.

Proof. (i) Suppose $a > a'$, $s = S(a,w) \leq S(a',w') = s'$. By revealed preference,

$$X(a,s) - e(F^{-1}(s),a)) - C(s) \geq X(a,s') - e(F^{-1}(s'),a)) - C(s')$$
$$X(a',s') - e(F^{-1}(s'),a')) - C(s') \geq X(a',s) - e(F^{-1}(s),a')) - C(s)$$

$$\begin{aligned} &\Rightarrow 0 > (X(a',s') - X(a',s)) - (X(a,s') - X(a,s)) \\ &\quad \geq (e(F^{-1}(s)),a) - e(F^{-1}(s)),a') - (e(F^{-1}(s'),a) - e(F^{-1}(s'),a')) > 0 \rightarrow\leftarrow \end{aligned}$$

by the fact of single crossing property and $e_{va} < 0$,

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If $s = s'$, since in $Q(1)$, the student are all unconstrained, $X_s(a, s) = \frac{d}{ds}[e(F^{-1}(s), a) + C(s)] = X_s(a', s) \Rightarrow a = a' \rightarrow\leftarrow$ by the fact that $X_{as} > 0$.

(ii) if $s = S(a, w) > S(a, w') = s'$: for a, s, s' are indifferent for person in $Q(1)$, hence we have the FOC holds in both cases. By (i), if $a'' > a$, a'' prefers s than $[s', s)$; if $a'' < a$, a'' prefers s' than $(s', s]$. Hence, in $Q(1)$, only student with ability=a would go to school $[s', s]$.

Notice that it is only zero measure. Thus, we can know that $[s', s]$ is filled by $\{(a, w) : w \in [e(F^{-1}(s'), a)) + C(s'), e(F^{-1}(s), a)) + C(s)] \text{ for some } a \in I\}$. By Market clearance,

$$
\begin{aligned}
s - s' &\leq \int_0^1 (e(F^{-1}(s), z)) + C(s)) - (e(F^{-1}(s'), z)) + C(s'))\mathrm{d}z \\
&= \int_0^1 \int_{s'}^{s} (e(F^{-1}(s), z)) + C(s))' ds dz \\
&\leq \int_{s'}^{s} X_s(1, s) ds < s - s' \rightarrow\leftarrow
\end{aligned}
$$

The last two inequalities are due to $X_s(z, s) \geq (e(F^{-1}(s), z)) + C(s))'$ and $X_{as} > 0$ and $X_s < 1 - c < 1$.

□

Now, we want to say that $\forall a \in [0, 1]$, if the student with wealth$= 1$ choose to go to school, then he is effectively unconstrained.

Consider $(a, w) = (a_1, 1)$, the student of the lowest ability who can afford to go to school $s = 1$(says $e(F^{-1}(1), a_1) + C(1) = 1$). According to the previous two lemmas, the student would go to $S(a_1, 1) = s_1 < 1$ and spend less than 1.

Then we can set $Q(s_1)$, note that every student in $Q(s_1)$ are unconstrained by the

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single crossing property (since in $Q(s_1)$ with $a < a_1$ cannot attend $s > s_1$ even if he can afford it). Then we can find a_2: the lowest ability student who can afford to s_1(says $e(F^{-1}(s_1), a_2) + C(s_1) = 1$). Thus, we can also find $s_2 = S(a_2, 1)$.

Thus, by iteration, we can get a sequence $Q(s_1), Q(s_2), \cdots$ such that any student $(a, 1) \in Q(s_j)$ is effectively unconstrained for all j, and will converge to a_k, s_k with $S(a_k, 1) = s_k$. Note that a_k is the point to determine whether to go to school(if $a < a_k, S(a, 1) = \emptyset$). Otherwise, denote $s' = s_k$, a_s as the lowest ability student who can afford to s, by implicit function theorem, we will have

$$0 = \lim_{s \to s'} \frac{\mathrm{d}a_s}{\mathrm{d}s} = -\frac{\frac{\mathrm{d}}{\mathrm{d}s}[e(F^{-1}(s'), a_{s'}) + C(s')]}{e_a(F^{-1}(s'), a_{s'})} = -\frac{X_s(a_{s'}, s')}{e_a(F^{-1}(s'), a_{s'})} > 0 \ \rightarrow\leftarrow$$

Hence, we can conclude that $\forall a \in [a_k, 1]$, $(a, 1)$ are effectively unconstrained and the FOC will hold:

$$X_s(a, S(a, 1)) = (e(F^{-1}(S(a, 1)), a) + C(S(a, 1)))' \tag{A.16}$$

Define $\mathbf{S}(a) = S(a, 1)$, and note that any student (a, w) with $w > e(F^{-1}(\mathbf{S}(a)), a)$ will be able to afford the school chosen by the student with the same ability and highest wealth, which is the direct result of **Lemma 3**.(ii).

Lemma 4. *for $a \in [a_k, 1]$, (i) $\mathbf{S}$ is strictly increasing, (ii) $\mathbf{S}(1) = 1$, (iii) $\mathbf{S}$ is continuous, (iv) $\mathbf{S}$ is (a.e.) differentiable.*

Proof. (i) By Lemma 3.

(ii) By the strict monotonicity of **S** and market clearing.

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(iii) If $s_L = \lim_{z\to a_-} \mathbf{S}(z) < \lim_{z\to a_+} \mathbf{S}(z) = s_H$, monotonicity of $\mathbf{S}$ implies that no unconstrained student will attend schools in the interval (s_L, s_H). we can know that $[s_L, s_H]$ is filled by $\{(a,w) : w \in [e(F^{-1}(s_L), a)) + C(s_L), e(F^{-1}(s_H), a)) + C(s_H)]$ for some $a \in [a_k, 1]\}$. By Market clearance,

$$
\begin{aligned}
s_H - s_L &\leq \int_{a_k}^{1} (e(F^{-1}(s_H), z)) + C(s_H)) - (e(F^{-1}(s_L), z)) + C(s_L))dz \\
&= \int_{a_k}^{1}\int_{s_L}^{s_H} (e(F^{-1}(s), z) + C(s))'\mathrm{d}s\ \mathrm{d}z \leq \int_{s_L}^{s_H} X_s(1,s)ds < s_H - s_L \rightarrow\leftarrow
\end{aligned}
$$

Thus, $\lim_{z\to a_-} \mathbf{S}(z) = \lim_{z\to a_+} \mathbf{S}(z)$ and $\mathbf{S}$ is continuous.

(iv) Follows from (i) and the boundedness of the range of $\mathbf{S}$. □

□

A.2.1 Proof of properties

Proof of property 3.1

Proof. By Lemma 3, all students in $R(s)$ will go to school s. Then we fix s.

First, $w \in [0, e(F^{-1}(s), \underline{a}(s)) + C(s))$will go to other school.

Second, (a, w), with $w \in (e(F^{-1}(s), \underline{a}(s)) + C(s), 1]$ and $a > \underline{a}(s)$ won't go to s since they will choose $s' > s$.

Third, similarly, (a, w) with $w \in [e(F^{-1}(s), \underline{a}(s)) + C(s), 1]$ and $a < \underline{a}(s)$ won't go to s since they will choose $s' < s$.

Finally, students in $T(s) = \{(a,w) : w = e(F^{-1}(s), \underline{a}(s)) + C(s),\ a > \underline{a}(s)\}$ can't afford $s' > s$, and prefer s to any $s \in [s_k, s)$, so they go to s. □

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Proof of property 3.2

Proof. Suppose $\tilde{s} > \bar{s}$, then by Lemma 1, $\mathbf{e}(\tilde{a}) = 0$.

Also we have $X(\tilde{a}, \tilde{s}) - \mathbf{e}(\tilde{a}) - C(\tilde{s}) = X(\tilde{a}, \tilde{s}) = 0$ to make unconstrained student indifferent to go to school or not. However, $X(\tilde{a}, \tilde{s}) = 0$ implies that $\tilde{a} = \tilde{s} = 0 < \bar{s}$, which is a contradiction. $\square$

Proof of property 3.3

Proof. First, we have $C(\bar{s}^+) = 0$ and $C(\bar{s}) = c$. If $\mathbf{e}(\bar{a}^+) < \mathbf{e}(\bar{a}) + c$, then

$$X(\bar{a}, \bar{s}) - \mathbf{e}(\bar{a}) - C(\bar{s}) < X(\bar{a}, \bar{s}) - \mathbf{e}(\bar{a}^+) - C(\bar{s}^+) = X(\bar{a}, \bar{s}^+) - \mathbf{e}(\underline{a}(\bar{s}^+)) - C(\bar{s}^+)$$

If $\mathbf{e}(\bar{a}^+) > \mathbf{e}(\bar{a}) + c$, then

$$X(\bar{a}^+, \bar{s}^+) - \mathbf{e}(\bar{a}^+) - C(\bar{s}^+) < X(\bar{a}^+, \bar{s}^+) - \mathbf{e}(\bar{a}) - C(\bar{s}) = X(\bar{a}^+, \bar{s}) - \mathbf{e}(\underline{a}(\bar{s})) - C(\bar{s})$$

by continuity of X, violating IC constraint. $\square$

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