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# 一維非厄密特量子系統之不可約化參數

# Indecomposability Parameters in Non-Hermitian Quantum Systems in One Dimension

黃福祥

Fu-Hsiang Huang

指導教授: 謝長澤 博士

Advisor: Chang-Tse Hsieh Ph.D.

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## MASTER'S THESIS ACCEPTANCE CERTIFICATE
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一維非厄密特量子系統之不可約化參數

（論文中文題目）(Chinese title of Master's thesis)

Indecomposability Parameters in Non-Hermitian Quantum Systems in One Dimension

（論文英文題目）(English title of Master's thesis)

本論文係 黃福祥 (姓名) R10222098 （學號）在國立臺灣大學物理學系(系/所/學位學程)完成之碩士學位論文，於民國 114 年 1 月 22 日承下列考試委員審查通過及口試及格，特此證明。

The undersigned, appointed by the Department / Institute of Physics on 22 (date) 1 (month) 2025 (year) have examined a Master's thesis entitled above presented by Fu-Hsiang Huang (name) R10222098 (student ID) candidate and hereby certify that it is worthy of acceptance.

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位得以追隨菩薩的腳步，前往西方極樂世界。



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# 摘要

共形不變性通常出現在臨界系統中，此時關聯長度發散，這種情況在無能隙的厄米量子系統（熱力學極限下）中尤為常見。然而，當厄米性被破壞時，情況會變得更加微妙。在本工作中，我們提出了一個僅包含質量項與二階導數項、而不含一階導數（動能）項的 $(1+1)$ 維非厄米自由費米子量子場論。有趣的是，該系統具有線性色散的無能隙譜。我們通過構造一組滿足 Virasoro 代數且中心荷 $c=-2$ 的無限多算符，證明該模型在低能下具有共形不變性，這反映了系統的非么正性。此外，該模型的希爾伯特空間呈現不可分解的 Jordan-cell 結構，表明它是一個對數共形場論。我們同時給出此非厄米場論的晶格實現，並從中識別出一個表徵系統不可分解性的普適量。結果顯示，我們模型的中心荷與不可分解參數與辛費米子理論完全一致，暗示兩者之間存在密切聯繫。

**關鍵字：**非厄密特量子力學、對數共形場論、不可約化參數、維拉宿代數、辛費米子、量子場論



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# Abstract

Conformal invariance typically emerges in critical systems where the correlation length diverges, as is generally the case for Hermitian quantum systems without a spectral gap in the thermodynamic limit. However, the situation becomes more subtle when Hermiticity is broken. In this work, we propose a $(1+1)$-dimensional non-Hermitian free-fermion quantum field theory containing only mass and second-derivative terms, without a first-derivative (kinetic) term. Interestingly, this system possesses a gapless spectrum with linear dispersion. By constructing an infinite set of operators that satisfy the Virasoro algebra, we demonstrate that this model is conformally invariant at low energy, with a resulting central charge of $-2$, reflecting the system's non-unitarity. Furthermore, the Hilbert space of this model exhibits indecomposable Jordan-cell structures, indicating that the theory is a logarithmic conformal field theory. We also give a lattice realization of our proposed non-Hermitian field theory, from which we identify a universal quantity characterizing the system's indecomposability. It turns out that the central charge and in-


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decomposability parameter of our model are identical to those of the symplectic fermion theory, suggesting a close connection between the two.

**Keywords:** Non-Hermitian Quantum mechanics, Logarithmic conformal field theory, Indecomposability Parameters, Virasoro algebra, Symplectic fermion, Quantum field theory



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# Contents





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doi:10.6342/NTU202504331

# List of Figures





doi:10.6342/NTU202504331

doi:10.6342/NTU202504331

# Chapter 1 Introduction

## 1.1 Background and motivation

Critical quantum many-body systems often exhibit emergent scale invariance in their low-energy, long-wavelength limits. When this scale invariance enlarges to the full conformal group, the effective description is a conformal field theory (CFT), whose powerful algebraic constraints—most notably the Virasoro algebra—control spectra, correlation functions, and universal finite-size effects. In Hermitian one-dimensional gapless systems, this expectation is by now standard: linear low-energy dispersion and a traceless, symmetric stress tensor typically signal conformality.

However, non-Hermitian quantum mechanics (NHQM) has brought an enriched landscape in which these familiar implications must be reconsidered. Non-Hermitian Hamiltonians naturally arise in effective descriptions of open systems, gain-loss settings in photonics and cold atoms, non-equilibrium steady states, and in certain stochastic or disordered models. Mathematically, they feature biorthogonal left-right eigenbases and can display exceptional points where the Hamiltonian becomes non-diagonalizable. At such points the Hilbert space forms indecomposable modules under the symmetry algebra: the representation is reducible but cannot be decomposed as a direct sum of irreducibles because Jordan blocks "glue" states together.


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Exactly at this interface—gapless physics with non-unitarity—logarithmic CFTs (LCFTs) enter. LCFTs retain conformal symmetry, but a non-diagonalizable action of $L_0$ produces logarithms in correlators and extends representation theory from highest-weight modules to indecomposable staggered modules [1, 2]. A key quantitative probe is the indecomposability parameter that measures the overlap between a primary field and its logarithmic partner inside a Jordan cell. Determining this parameter provides representation-level information that complements standard CFT data such as central charge and scaling dimensions.

These considerations raise a basic conceptual question that motivates this work: **When Hermiticity is broken, does gaplessness—especially with linear dispersion—still guarantee conformal symmetry, and if so, what algebraic structure and universal data characterize the resulting non-unitary critical theory?**

This thesis addresses the question in a concrete, analytically tractable setting. We construct and analyze a (1+1)-dimensional non-Hermitian free-fermion field theory whose Hamiltonian omits the usual first-derivative (kinetic) term and retains only a non-Hermitian mass term together with a second-derivative term. Despite this unusual structure, the continuum dispersion is linear and gapless, suggesting an emergent CFT. We show that this expectation is correct but with a crucial twist: the emergent theory is logarithmic with central charge $c = -2$, and its Hilbert space exhibits Jordan-cell indecomposability. We further give a lattice realization and identify a universal indecomposability parameter that matches that of the well-known symplectic fermion LCFT [3, 4].

Two technical pillars underlie our analysis. First, on the field-theory side we explicitly construct Virasoro generators and verify their algebra—including the central term—


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thereby establishing conformality in the infrared despite non-Hermiticity. Second, on the lattice we adapt the Koo–Saleur construction [5], which realizes scaling Virasoro modes as Fourier transforms of local Hamiltonian and momentum densities. Together, these approaches let us bridge continuum and lattice viewpoints, compute the central charge $c$, and extract the indecomposability parameter directly in a microscopic framework.

A broader methodological point is that, in non-Hermitian settings, familiar "CFT diagnostics" such as logarithmic entanglement scaling can persist even when the algebraic structure responsible for them is obscured at the lattice level. By building the Virasoro structure explicitly and quantifying indecomposability, we provide a field-theoretic and representation-theoretic confirmation of conformality that goes beyond entropic evidence, in a model simple enough to serve as a testbed for future non-Hermitian critical phenomena.

## 1.2 Outline

In Chapter 2, we introduce the foundational concepts of non-Hermitian quantum mechanics. Following the demonstration in [6], we show that if a system exhibits $\mathcal{PT}$ symmetry, its eigenvalues can remain real despite the Hamiltonian being non-Hermitian. In other words, the Hermiticity condition in conventional quantum theory can be relaxed while still maintaining real physical observables. We also introduce the mathematical framework of biorthogonal Hilbert space, which is commonly used to address non-Hermitian systems and will serve as a key tool in subsequent chapters.

Chapter 3 provides an overview of logarithmic conformal field theory and explores the connection between non-diagonalizable Hamiltonians and logarithmic behavior. We de-


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rive the correlation function between a primary field and its logarithmic partner, leading to the definition of the indecomposability parameter. This parameter plays a role analogous to that of the central charge and is expected to aid in classifying models within LCFTs. In the latter part of the chapter, we follow the procedure outlined in [7] to present two illustrative examples in conformal field theory, calculating their respective indecomposability parameters.

In Chapter 4, we turn our attention to a special model characterized by a Hamiltonian containing only a mass term and a second-derivative term. By constructing the Virasoro generators and computing the commutators between them, we verify that the system preserves conformal symmetry and exhibits a central charge of $-2$ in the continuum limit. Furthermore, we provide a lattice realization of the non-Hermitian field theory. Applying the methods from [8], we indirectly demonstrate that this massive model exhibits properties characteristic of symplectic fermions and determine its indecomposability parameter to be $-1$.


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# Chapter 2 Non-Hermitian quantum mechanics

Since its proposal in the 1990s, Non-Hermitian Quantum Mechanics (NHQM) has gained significant attention among physicists, particularly in the study of non-equilibrium phenomena, open systems, and critical systems in condensed matter physics [9], as well as in atomic, molecular, and optical physics. On the other hand, in mathematical physics, several studies suggest that the Hilbert space in NHQM exhibits behavior analogous to spacetime in general relativity, specifically that it is curved [9–11]. In this chapter, we present the work of Bender and Boettcher, which discovered that if a system possesses $\mathcal{PT}$ symmetry, its Hamiltonian can still have real eigenvalues, even if it is non-Hermitian.

## 2.1 Real Spectrum and $\mathcal{PT}$ symmetry

In order to ensure that physical measurements yield real values, we typically require the measurement operator to be Hermitian (or, in matrix representation, a self-adjoint matrix). This condition possesses several powerful properties, such as the operator having real eigenvalues corresponding to physical measurements. Moreover, the eigenvectors are complete and orthogonal to each other. However, Recently, several articles have sug-


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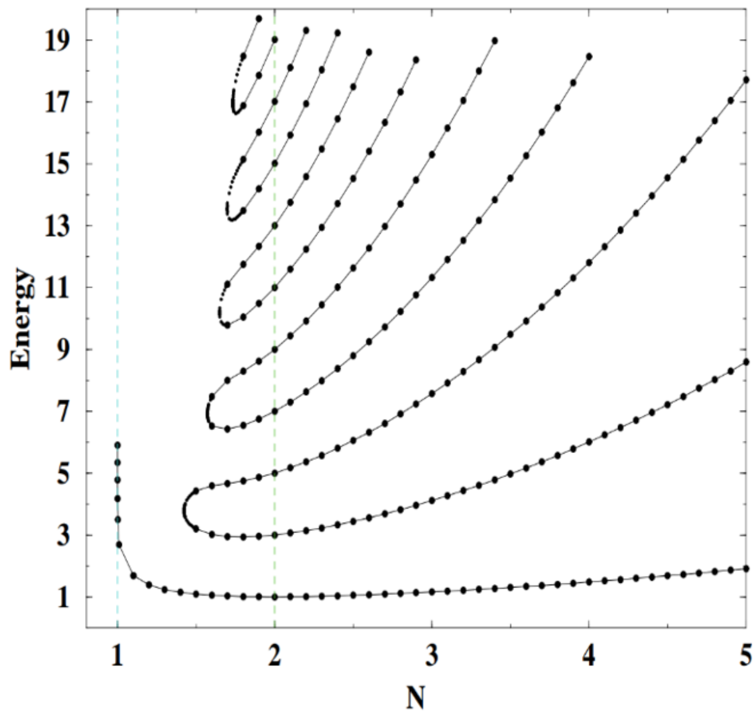

Figure 2.1: Spectrum of the Hamiltonian $H = p^2 - (ix)^N$.

gested that if a system displays $\mathcal{PT}$ symmetry, the eigenvalues tend to be real [6]. $\mathcal{PT}$ symmetry implies that the system remains invariant under both parity and time reverse. In the domain of Hamiltonian mechanics, both position $(x)$ and momentum $(p)$ serve as the fundamental quantities of the system. Thus, any observable can be expressed in terms of $x$ and $p$. When the system exhibits a parity reverse, both $x$ and $p$ will be inverted to $-x$ and $-p$, respectively. Similarly, for a time transformation, $x$ and $p$ will transform into $x$ and $-p$, respectively. However, due to a fundamental principle of quantum mechanics, defined as $[x, p] = i\hbar$, the imaginary number $i$ should also change in sign.

Here, we exhibit an example in [6]. Consider an elementary Hamiltonian

$$H = p^2 - (ix)^N, N \in \mathbb{R}, \tag{2.1}$$

the energy spectrum of this Hamiltonian, as illustrated in Fig.[2.1], can be determined through the semi-classical method. When $N \geq 2$, the system is under $\mathcal{PT}$ symmetry,


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which means that there are no complex eigenvalues present in the spectrum. When $N = 2$ the Hamiltonian depict a quantum harmonic oscillator for which the energy is $E_n = (n + 1/2)\omega$. However, when $N$ falls below 2, the $\mathcal{PT}$ symmetry is *spontaneously broken*, leading to the emergence of complex eigenvalues in the spectrum, while the number of real eigenvalues becomes finite. In addition, we refer to the breakpoint of ($N = 2$), which divides the topological phase ($N > 2$) from the $\mathcal{PT}$ broken phase ($N < 2$) as a *critical point*. The system undergoes a *phase transition* from one site to another. For the range $1 < N \leq 1.422$, only one real eigenvalue is present, whose value tends toward infinity as $N$ approaches 1. When $N$ falls below 1, no real eigenvalue appears in the spectrum.

## 2.2 Biorthogonal bases

In conventional quantum mechanics, observables are represented by operators. The standard formulation requires these operators to be Hermitian, ensuring that the outcomes of physical measurements are real. Furthermore, Hermitian operators guarantee the orthogonality and completeness of their eigenbases—two essential features for constructing a well-defined Hilbert space. However, in non-Hermitian quantum mechanics, these powerful tools are generally not available. To explore this, we introduce a generic complex Hamiltonian $\hat{K}$, which can be decomposed into two Hermitian operators $\hat{H}$ and $\hat{\Gamma}$, as follows:

$$\hat{K} = \hat{H} + i\hat{\Gamma},$$

with eigenstates $\{|\phi_n\rangle\}$ and corresponding eigenvalus $\{k_n\}$ satisfying

$$\hat{K}|\phi_n\rangle = k_n|\phi_n\rangle. \tag{2.2}$$


doi:10.6342/NTU202504331

This implies the relation

$$\langle \phi_n | \hat{K}^\dagger = \langle \phi_n | k_n^*. \tag{2.3}$$

To recover a notion of orthogonality in this non-Hermitian framework, we introduce the eigenstates $\{|\psi_n\rangle\}$ of the adjoint Hamiltonian $\hat{K}^\dagger$:

$$\hat{K}^\dagger |\psi_n\rangle = \kappa_n |\psi_n\rangle, \quad \langle \psi_n | \hat{K} = \langle \psi_n | \kappa_n^*. \tag{2.4}$$

In general, the eigenstates of $\hat{K}$ are not orthogonal:

$$\langle \phi_n | \phi_m \rangle = 2i \frac{\langle \phi_n | \hat{\Gamma} | \phi_m \rangle}{k_m - k_n^*} = 2 \frac{\langle \phi_m | \hat{H} | \phi_n \rangle}{k_m + k_n^*}. \tag{2.5}$$

In Hermitian cases, $\{\langle \psi_n |\}$ is just adjoint of $\{|\phi_n\rangle\}$. However, in general, $\{|\phi_n\rangle\}$ and $\{|\psi_n\rangle\}$ are mutually independent. Hence, all calculation in the Hilbert space require both eigenbases. We call $\{|\phi_n\rangle\}$ are the right-eigenbases of $\hat{K}$, and the corresponding eigenvalues are $k_n$, while $\{\langle \psi_n |\}$ are the left-eigenbases, the corresponding eigenvalues are $\kappa_n^*$. Despite the lack of Hermiticity, the eigenstates can still be linearly independent and form a complete basis.[1] To demonstrate this, following the argument in [12], suppose the set $|\phi_n\rangle$ is linearly dependent. Then there exists a non-zero set of coefficients $c_n$ such that

$$\sum_n c_n |\phi_n\rangle = 0. \tag{2.6}$$

Multiplying from the left by $\langle \psi_m |$ yields, for each $m$, each $m$,

$$c_m \langle \psi_m | \phi_m \rangle = 0, \tag{2.7}$$

[1] In this section, we assume the Hamiltonian is diagonalizable.


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where we have used the orthogonality condition

$$\langle\psi_m|\phi_n\rangle = \delta_{mn}\langle\psi_m|\phi_m\rangle. \tag{2.8}$$

Furthermore, from Eq.(2.2) and Eq.(2.4) , we find:

$$\langle\psi_m|\hat{K}|\phi_n\rangle = k_m^*\langle\psi_m|\phi_n\rangle = \kappa_n\langle\psi_m|\phi_n\rangle. \tag{2.9}$$

This leads to the conclusion that if $\langle\psi_m|\phi_n\rangle \neq 0$, then $k_m^* = \kappa_n$. Conversely, if $k_m^* \neq \kappa_n$, then $\langle\psi_m|\phi_n\rangle = 0$. It is not possible for all $\langle\psi_m|\phi_n\rangle$ to vanish, must exist at least one $k_m$ such that $k_m^* = \kappa_n$. Assuming the eigenstates are non-degenerate, there exists a unique $m$ such that this holds. Without losing generality, we relabel the states such that $k_n^* = \kappa_n$ for all $n$. Therefore, we obtain $\langle\psi_n|\phi_n\rangle \neq 0$, while $\langle\psi_m|\phi_n\rangle = 0$ for $m \neq n$, establishing the biorthogonality relation in Eq.(2.8). It follows that all $c_m = 0$, and hence, the eigenstates $|\phi_n\rangle$ are linearly independent. So far, we have shown that the two sets of eigenstates, $|\phi_n\rangle$ and $|\psi_n\rangle$, together form a biorthogonal Hilbert space. With these biorthogonal bases, many calculations from conventional quantum mechanics can be generalized to the non-Hermitian framework. For example, the completeness relation in the biorthogonal Hilbert space is expressed as:

$$\sum_n \frac{|\phi_n\rangle\langle\psi_n|}{\langle\psi_n|\phi_n\rangle} = 1. \tag{2.10}$$

Furthermore, the projection operator onto the $n$-th eigenstate is defined as:

$$\hat{\Pi}_n = \frac{|\phi_n\rangle\langle\psi_n|}{\langle\psi_n|\phi_n\rangle}, \tag{2.11}$$

which satisfies

$$(1-\hat{\Pi}_n)|\phi_n\rangle = 0, \quad (1-\hat{\Pi}_n^\dagger)|\psi_n\rangle = 0.$$


doi:10.6342/NTU202504331

The non-Hermitian Hamiltonian $\hat{K}$ can then be expressed in terms of these projection operators as:

$$\hat{K} = \sum_n k_n \hat{\Pi}_n, \tag{2.12}$$

where $k_n$ are the eigenvalues of $\hat{K}$, satisfying Eq.(2.2). Similarly, its adjoint Hamiltonian is given by:

$$\hat{K}^\dagger = \sum_n \kappa_n^* \hat{\Pi}_n^\dagger, \tag{2.13}$$

where $\kappa_n$ are the eigenvalues of $\hat{K}^\dagger$, satisfying Eq.(2.4).

For two arbitrary states $|V\rangle$ and $\langle W|$, their inner product can be written as:

$$\langle W|V\rangle = \sum_n \frac{\langle W|\phi_n\rangle\langle\psi_n|V\rangle}{\langle\psi_n|\phi_n\rangle}. \tag{2.14}$$

These mathematical tools will serve as the foundation for solving non-Hermitian quantum problems in the following sections.


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# Chapter 3 The indecomposibility parameter and the logarithmic conformal field theory

## 3.1 Logarithmic conformal field theory

In Logarithmic Conformal Field Theories, the logarithmic term appears in the correlation function [1, 7]. In this chapter, we verify that it is related to the zeroth Virasoro generator $L_0$ which is non-diagonalizable. In conformal field theory, the operator $L_0 + \bar{L}_0$ represents the Hamiltonian. When the Hamiltonian cannot be diagonalized, the entire theory is under non-Hermitian regime. This also leads to an indecomposable operator product expansion (OPE).

Given a local observable $\Phi$, which is expanded in scaling operators $\phi_i$, each operator has conformal dimension $h_i$, its two point function in $\mathbb{R}^d$ has the form

$$\langle \Phi(r)\Phi(0)\rangle \sim \sum_{ij} \frac{a_{ij}}{r^{h_i+h_j}}. \tag{3.1}$$


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Conformal invariance implies that $a_{ij} = 0$ when $h_i \neq h_j$. The logarithmic behavior appears in the singularity that the parameter $n$ in conformal dimensions $h(n)$ and amplitudes $a(n)$ tends to a certain number $n_c$, such that a pair of conformal dimensions $h$ and $\tilde{h}$ collide, namely, $h - \tilde{h} \to 0$. Meanwhile, the corresponding amplitudes $a \sim -\tilde{a} \to \infty$, and the $a(h - \tilde{h})$ converges to the finite value which is proportional to $r^{-2h} \log r$. Note that the number $n$ can be a parameter in some model, for example, $n$ in $O(n)$ model or $Q$ in the $Q$-state Potts model [1].

Suppose that

$$\langle \phi(r)\phi(0) \rangle = \frac{A(n)}{n - n_c} r^{-2h(n)}, \tag{3.2}$$

$$\langle \tilde{\phi}(r)\tilde{\phi}(0) \rangle = -\frac{\tilde{A}(n)}{n - n_c} r^{-2\tilde{h}(n)}, \tag{3.3}$$

$$\langle \phi(r)\tilde{\phi}(0) \rangle = 0. \tag{3.4}$$

where $A(n)$ and $\tilde{A}(n)$ tend to same value while $h(n)$ and $\tilde{h}(n)$ are not same when $n \to n_c$. The last equation is given by conformal invariance when $h(n) \neq \tilde{h}(n)$. Then we artificially define

$$D \equiv \phi - \tilde{\phi}, \quad C \equiv (h(n) - \tilde{h}(n))\phi, \tag{3.5}$$

then, as $n \to n_c$

$$\lim_{n \to n_c} \langle D(r)D(0) \rangle = \lim_{n \to n_c} -\frac{A(n_c)}{r^{4h(n)}} \frac{r^{2h(n)} - r^{2\tilde{h}(n)}}{n - n_c} = -\frac{2\alpha(\ln r)}{r^{2h(n_c)}}, \tag{3.6}$$

$$\lim_{n \to n_c} \langle C(r)D(0) \rangle = \lim_{n \to n_c} \frac{A(n_c)}{r^{2h(n)}} \frac{h(n) - \tilde{h}(n)}{n - n_c} = \frac{\alpha}{r^{2h(n_c)}}, \tag{3.7}$$

$$\lim_{n \to n_c} \langle C(r)C(0) \rangle = \lim_{n \to n_c} \frac{A(n)}{n - n_c} \frac{(h - \tilde{h})^2}{n - n_c} = 0, \tag{3.8}$$


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where $\alpha = (h'(n_c) - \tilde{h}'(n_c))A(n_c)$. The logarithmic operator is a generalization of the primary operator. Primary states, created by primary operators, are eigenstates of the Virasoro generator $L_0$ and are annihilated by $L_n$ for all $n > 0$ [2].

If $L_n$ are primary operators that satisfy $L_0|\phi\rangle = h|\phi\rangle$ and $L_0|\tilde{\phi}\rangle = \tilde{h}|\tilde{\phi}\rangle$ for $n \neq n_c$, so that

$$
\begin{aligned}
L_0|C\rangle &= h|C\rangle, \\
L_0|D\rangle &= |C\rangle + \tilde{h}(n)|D\rangle, \\
&\to |C\rangle + h|D\rangle.
\end{aligned}
\tag{3.9}
$$

Clearly, $|C\rangle$ is a primary state with conformal dimension $h$. However,, when $L_0$ acts on $|D\rangle$, the result contains both $|D\rangle$ and $|C\rangle$. In this case, $|D\rangle$ is referred to as the *logarithmic partner* of $|C\rangle$, and it is a *generalized eigenstate* of $L_0$. This Jordan block structure underlies the presence of logarithmic behavior in correlation functions, a hallmark of logarithmic conformal field theory (LCFT). On the other hand, $L_0$ and $(C, D)$ can be expressed in matrix form:

$$
L_0 \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} h & 0 \\ 1 & h \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}.
\tag{3.10}
$$

We see when the system $(n)$ tends to a singularity or critical point $(n_c)$, $(C, D)$ spans the Jordan cell of $L_0$. As for a higher rank Jordan cell, it arises from the collision of more than two operators. It seems like we artificially suppose that amplitude $a(n)$ and $\tilde{a}(n)$ diverge in the same way; however, we shall argue that this indeed occurs naturally when the theory possesses an internal symmetry [1].

Consider a pair of logarithmic operators $\{\phi(z), \psi(z)\}$. In the matrix representation, $\phi$ serves as the eigenvector in the Jordan block matrix, while $\psi$ functions as the correspond-


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$$
\begin{tikzcd}
 & \psi \arrow[dl, "\beta^{-1}A^{-}"'] \arrow[dd, "L_0 - I"] \arrow[dr] & \\
\xi \arrow[dr, "A^{+}"'] & & \rho \arrow[dl] \\
 & \phi &
\end{tikzcd}
$$

Figure 3.1: The staggered Virasoro module, the picture shows the conformal transform within $\phi, \psi, \xi$ and $\rho$.

ing generalized eigenvector. Therefore, both fields carry the same conformal dimension, denoted as $h$, and satisfy the equation $\phi(z) = H\psi(z)$, where $H$ is the Hamiltonian. The form of the correlation function is determined by invariance under global conformal transformations[1] [7]:

$$
\begin{aligned}
\langle \phi(z)\phi(0) \rangle &= 0, \\
\langle \phi(z)\psi(0) \rangle &= \frac{\beta}{z^{2h}}, \\
\langle \psi(z)\psi(0) \rangle &= \frac{\theta - 2\beta \ln z}{z^{2h}},
\end{aligned}
\tag{3.11}
$$

where $\beta$ and $\theta$ are two parameters. $\theta$ can be canceled by a transformation $\psi \to \psi - \frac{\theta}{2\beta}$, while $\beta$ characterizes the structure of the Jordan cell. In a staggered Virasoro module, these fields are organized in a diamond-shaped structure, as illustrated in Fig.3.1. Each arrow signifies the action of the conformal generators on the fields. In the upper left segment of Fig.3.1, the Virasoro generator $A^{-}$ decreases the conformal dimension of $\psi(z)$. Therefore, if the conformal dimension of $A^{+}$ is depicted as a positive number $n$, then the conformal dimension of $\xi(z)$ is $h_\xi = h_\psi - n$, and that of $\phi(z)$ is $h_\phi = h_\xi + n = h_\psi$. As such, the conformal dimensions of different fields adhere to the condition $h_\xi \leq h_\phi = h_\psi \leq h_\rho$.

[1] Note that only the model with c = 0 has the following relation.


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Meanwhile, the field $\phi(z)$ is the descendant of $\xi(z)$:

$$\phi(z) = A^{+}\xi(z),$$
$$A^{+} = L_{-n} + \alpha_1 L_{-n+1}L_{-1} + ... + \alpha_{(P(n)-1)}L_{-1}^{n}, \tag{3.12}$$

where $n = h - h_\xi$, $L_n$ is corresponding $n$-th order Virasoro generator and $P(n)$ is the number of partitions of the integer $n$. The coefficients $\alpha_i$ are uniquely determined by the null-state condition, that is $L_n\phi = 0$. If $\psi(z)$ and $\phi(z)$ satisfy these relations, we say that $\psi(z)$ is the logarithmic partner of $\phi(z)$. Furthermore, we define the parameter $\beta$ as the overlap of the generalized and conventional eigenvector, indicated by [7]

$$\beta = \langle\psi|\phi\rangle = \frac{\langle\psi|A^{+}|\xi\rangle^2}{\langle\psi|\phi\rangle}. \tag{3.13}$$

This parameter is also known as the *indecomposability parameter* or *logarithmic coupling*. In order to ensure the uniqueness of $\beta$, we constrain $\xi(z)$ is normalized ($\langle\xi|\xi\rangle = 1$). Then, it becomes straightforward to express the relationship between $\psi(z)$ and $\xi(z)$ :

$$A^{-}\psi(z) = \beta\xi(z). \tag{3.14}$$

On the other hand, the order of the Virasoro generator $A^{+}$, depends on where the Jordan cell is located. Specifically, it pertains to the exceptional point at which level. For instance, if the Jordan is situated at the system's third excited state, then $n = 3$. In this case $A_3 = L_{-1}^3 - 8L_{-2}L_{-1} + 12L_{-3}$.[7]

Figuring out the indecomposability parameter can be an arduous task. For instance, in lattice models, the conventional eigenvector, denoted as $\phi$, in the Jordan block always yields a scalar product[2] of zero, even on the lower sites. As a result, the normalized

[2]Lattice scalar product [7]: When performing an inner product operation, the left vector should be transformed into a Hermitian conjugate before it is combined with the right vector. In this situation, the


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vector $\xi$ must be introduced, which allows for the uniqueness of $\phi$ to be established as $\phi(z) = A^{+}\xi(z)$. On the other hand, prior to calculating the indecomposability parameter, we need to clarify what the off-diagonal element within the Jordan block is. For instance, in a non-diagonalizable Hamiltonian, the Jordan block could be defined as follows:

$$J_1 = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix} \quad \text{or} \quad J_2 = \begin{pmatrix} h & a \\ 0 & h \end{pmatrix}.$$

The relation between two corresponding generalized eigenvector $\psi_1$ and $\psi_2$ is

$$\psi_2 = a\psi_1. \tag{3.15}$$

## 3.2 Indecomposability Parameters

We use two examples to show how the logarithm terms appear in operators products. First, introducing the Kac formula:

$$\begin{aligned} c &= 1 - \frac{6}{x(x+1)}, \\ h_{r,s} &= \frac{[(x+1)r - xs]^2 - 1}{4x(x+1)}. \end{aligned} \tag{3.16}$$

It connects the two important parameters in this theory, central charge $c$ and conformal dimension $h_{r,s}$, $r$, $s$ characterize the operator. The two parameters parameterize by another number $x$, which characterize the system.

norm of the vector would always be positive. However, should we remove the complex conjugate from the left vector, the norm may then become zero or even negative. This instance is referred to as a scalar product.


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### 3.2.1 $c \to 0$ catastrophe

As we have seen, logarithmic terms can appear in two-point correlation functions. Such terms may also arise in operator product expansions when singularities are present. These phenomena typically occur in conformal field theories (CFTs) with vanishing central charge, $c = 0$. Theories with zero central charge play a significant role in the study of condensed matter systems, particularly those involving quenched disorder. Notable examples include the $O(n)$ model in the limit $n \to 0$, which describes self-avoiding walks, and $Q$-state Potts model as $Q \to 1$, which corresponds to percolation [1, 2]. Consider the percolation, with $r = 1, s = 5, h_{1,5} = 2$, which is the chargeless system $(x = 2)$. The conformal invariance fixes the form of the OPE [1, 2, 7, 13].

$$\Phi_h(z)\Phi_h(0) = \frac{\alpha_\Phi}{z^{2h}}[1 + \frac{2h}{c}z^2 T(0) + ...], \tag{3.17}$$

where $T(z) = L_{-2}I$ is the stress tensor of the system. Clearly, the expression will diverse as $c \to 0$. We can normalize the field $\Phi_{1,5}$ such that the OPE reads

$$\Phi_h(z)\Phi_h(0) \sim \frac{\alpha_\Phi}{z^{2h}}[1 + \frac{2h}{c}z^2 T(0) + z^{h_{1,5}}\Phi_{1,5}(0) + ...]. \tag{3.18}$$

Then we define a new field $t(z)$ as

$$\Phi_{1,5}(z) = \frac{2h\langle T \mid T\rangle}{c\beta(\epsilon)}t(z) - \frac{2h}{c}T(z), \tag{3.19}$$

where

$$\beta(\epsilon) = -\frac{\langle T \mid T\rangle}{h_{1,5} - 2} \tag{3.20}$$


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and

$$\langle T \mid T \rangle = \frac{c}{2}. \tag{3.21}$$

It is useful to rewrite the central charge in the form

$$c = 2\beta(2 - h_{1,5}). \tag{3.22}$$

Putting these back to Eq.(3.18), we obtain

$$\begin{aligned}
\Phi_h(z)\Phi_h(0) &\sim \frac{\alpha_\Phi}{z^{2h}}[1 + \frac{2h}{c}z^2 T(0) + z^{h_{1,5}}\left(\frac{2h\langle T \mid T\rangle}{c\beta(\epsilon)}t(0) - \frac{2h}{c}T(0)\right) + ...], \\
&\sim \frac{\alpha_\Phi}{z^{2h}}[1 + \frac{2h}{c}(z^2 - z^{h_{1,5}})T(0) + \frac{h}{\beta(\epsilon)}z^{h_{1,5}}t(0) + ...], \\
&\sim \frac{\alpha_\Phi}{z^{2h}}[1 + \frac{h(z^2 - z^{h_{1,5}})}{\beta(2 - h_{1,5})}T(0) + \frac{h}{\beta(\epsilon)}z^{h_{1,5}}t(0) + ...].
\end{aligned} \tag{3.23}$$

To remove the ill-defined term in the OPE, we make the central charge infinitesimally close to 0, namely $x \to 2 + \epsilon$. As a result, the conformal dimension $h_{1,5}$ will be 2 in this limit, leading to the OPE

$$\Phi_h(z)\Phi_h(0) \sim \frac{\alpha_\Phi}{z^{2h}}[1 + \frac{h}{\beta}z^2\left(T(0)\ln(z) + t(0)\right) + ...]. \tag{3.24}$$

Note that the appearance of a logarithmic term in Eq.(3.24). Similarly, if we consider the chargeless case in the dilute polymers, in which $r = 3, s = 1, h_{1,5} = 2$, the similar result can be obtained. By the definition above, indecomposability parameters in each system are found to be respectively,

$$\beta_{percolation} = -\lim_{\epsilon\to 0}\frac{c/2}{h_{1,5} - 2} = -\frac{5}{8}, \tag{3.25}$$

$$\beta_{polymer} = -\lim_{\epsilon\to 0}\frac{c/2}{h_{3,1} - 2} = \frac{5}{6}. \tag{3.26}$$


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In this case, the field $t(z)$ becomes the logarithmic partner of $T(z)$. Therefore, the two logarithmic operators satisfy the relations mentioned above:

$$\begin{aligned}\langle T(z)\phi(0)\rangle &= 0,\\ \langle T(z)t(0)\rangle &= \frac{\beta}{z^4},\\ \langle t(z)t(0)\rangle &= \frac{\theta - 2\beta lnz}{z^4}.\end{aligned} \tag{3.27}$$

where $\theta$ is a constant.

### 3.2.2 Symplectic fermions

Let's generalize our view to the unusual case where the theory has $c = -2$ ($x = 1$). We focus on the theory of symplectic fermions, which describes the scaling limit of dense polymers on the lattice [3, 4]. Notably, there is no interaction leading to the Jordan cell structure, and this can be understood in terms of free fermions. Furthermore, the indecomposability parameters can be computed using the free fermion representation [7]. In this theory, as the central charge approaches $-2$, we focus on the level 2 case[3], which has $A^+ = L_{-1}^2 - 2L_{-2}$. The system is characterized by $r = 1$, $s = 1 + 2j$, with $j \in \mathbb{N}$. The first excited state $\xi$ has a conformal dimension approaching 1, while $\phi$ and $\psi$ are both second excited states, each with a conformal dimension of 3. According to the

[3] We just arbitrarily choose a level, it doesn't matter if you choose another level.


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arrangement in Fig.[3.1], we can list the relation of each state and parameter.

$$
\begin{aligned}
L_0\phi &= 3\phi \\
L_0\psi &= 3\psi + \phi \\
\phi &= A^+\xi \\
A^-\psi &= \beta\xi \\
A^+ &= L_{-1}^2 - 2L_{-2}
\end{aligned}
\tag{3.28}
$$

Note that the conformal dimensions and the central charges are parameterized by an infinitesimal number $\epsilon$. In this generic (non-logarithmic) CFT, The OPE of a general operator $\Phi_h$ with itself reads [7]

$$
\Phi_h(z)\Phi_h(0) \sim \frac{\alpha_\Phi}{z^{2h-h_\xi}}[\xi(0) + \frac{1}{2}z\partial\xi(0) + \alpha_{(-2)}z^2L_{-2}\xi(0) + \alpha_{(-1,-1)}z^2L_{-1}^2\xi(0) + ...]. \tag{3.29}
$$

Similarly, the coefficients $\alpha_{(-2)}$ and $\alpha_{(-1,-1)}$ are given by conformal invariance and diverge as $\epsilon \to 0$. Note that $\alpha$ and $h$ are both parameterized by $\epsilon$. That is,

$$
\alpha_{(-2)} = \frac{4h}{27\epsilon} + \frac{1+2h}{27} + \mathcal{O}(\epsilon), \tag{3.30}
$$

$$
\alpha_{(-1,-1)} = -\frac{2h}{27\epsilon} + \frac{4+2h}{27} + \mathcal{O}(\epsilon), \tag{3.31}
$$

$$
h = h_{1,5} = 1 + \frac{3}{2}\epsilon + \mathcal{O}(\epsilon^2). \tag{3.32}
$$


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And we know the field $\phi = (L_{-1}^2 - 2L_{-2})\xi$, then the OPE become

$$
\begin{aligned}
&\Phi_h(z)\Phi_h(0) \\
&\sim \frac{\alpha_\Phi}{z^{2h-h_\xi}}\left[\xi(0) + \frac{1}{2}z\partial\xi(0) + \alpha_{(-2)}z^2L_{-2}\xi(0) + \alpha_{(-1,-1)}z^2(\phi(0) + 2L_{-2}\xi(0)) + \ldots\right] \\
&\sim \frac{\alpha_\Phi}{z^{2h-h_\xi}}\left[\xi(0) + \frac{1}{2}z\partial\xi(0) + \alpha_{(-1,-1)}z^2\phi(0) + \left(2\alpha_{(-1,-1)} + \alpha_{(-2)}\right)z^2L_{-2}\xi(0) + \ldots\right].
\end{aligned}
\tag{3.33}
$$

We define the new field $\psi$ which conformal dimension is $h_\psi = h_{1,7} = 3 + 3\epsilon + \mathcal{O}(\epsilon^2)$.

Putting it into 3.2.2, which becomes

$$
\begin{aligned}
&\Phi_h(z)\Phi_h(0) \\
&\sim \frac{\alpha_\Phi}{z^{2h-1}}\left[z^{h_\xi-1}\xi(0) + \frac{z^{h_\xi}}{2}\partial\xi(0) + \alpha_{(-1,-1)}z^{h_\xi+1}\phi(0) + \alpha_{reg}z^{h_\xi+1}L_{-2}\xi(0) + z^{h_\psi-1}\Phi_{1,7}(0) + \ldots\right],
\end{aligned}
\tag{3.34}
$$

where $\alpha_{reg} = 2\alpha_{(-1,-1)} + \alpha_{(-2)} = \dfrac{9+4h}{27} + \mathcal{O}(\epsilon)$. Meanwhile, we define $\psi(z)$ as

$$
\Phi_{1,7}(z) = \frac{\alpha_{(-1,-1)}\langle\phi \mid \phi\rangle}{\beta(\epsilon)}\psi(z) - \alpha_{(-1,-1)}\phi(z),
\tag{3.35}
$$

where $\beta(\epsilon) = -\frac{\langle\phi|\phi\rangle}{h_\psi - h_\xi - 2}$. We can change $\Phi_{1,7}(z)$ into this form:

$$
\Phi_{1,7}(z) = \alpha_{(-1,-1)}(2 - h_\psi + h_\xi)\psi(z) - \alpha_{(-1,-1)}\phi(z),
\tag{3.36}
$$

Finally,

$$
\begin{aligned}
\Phi_h(z)\Phi_h(0) \sim \frac{\alpha_\Phi}{z^{2h-1}}[&z^{h_\xi-1}\xi(0) + \frac{z^{h_\xi}}{2}\partial\xi(0) + \alpha^{(-1,-,1)}z^{h_\xi+1}\phi(0) + \alpha_{reg}z^{h_\xi+1}L_{-2}\xi(0) \\
&+\alpha^{(-1,-,1)}z^{h_\phi-1}(2 - h_\phi + h_\xi)\psi(0) - \alpha^{(-1,-,1)}z^{h_\psi+1}\phi(0) + \ldots] \\
= \frac{\alpha_\Phi}{z^{2h-1}}[&\xi(0) + \frac{z}{2}\partial\xi(0) + \frac{9+4h}{27}z^2L_{-2}\xi(0) + \frac{h}{9}z^2\psi(0) + \frac{h}{9}z^2\phi(0)lnz + \ldots].
\end{aligned}
\tag{3.37}
$$



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Note that when $\epsilon \to 0$, the two terms in OPE become:

$$\begin{aligned} \alpha^{(-1,-,1)}(z^{h_\xi+1} - z^{h_\psi+1}) &= \left(-\frac{2h}{27\epsilon} + \frac{4+2h}{27} + \mathcal{O}(\epsilon)\right)(2 - h_\psi + h_\xi) \\ &= \left(-\frac{2h}{27\epsilon} + \frac{4+2h}{27} + \mathcal{O}(\epsilon)\right)\left(-\frac{3}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) \\ &= \frac{h}{9} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3.38}$$

It is straightforward to obtain $\beta$ from the definition given above:

$$\beta_{1,7} = \lim_{\epsilon\to 0} \beta(\epsilon) = -\frac{\langle\phi|\phi\rangle}{h_\phi - h_\xi - 2}. \tag{3.39}$$

The numerator, namely the inner product $\langle\phi|\phi\rangle$, can be expressed as

$$\langle\phi|\phi\rangle = \langle A^+\xi|A^+\xi\rangle, \tag{3.40}$$

where $A^+ = L_{-1}^2 - 2L_{-2}$. Therefore, we have

$$\langle\phi|\phi\rangle = \langle(L_{-1}^2 - 2L_{-2})\xi|(L_{-1}^2 - 2L_{-2})\xi\rangle. \tag{3.41}$$

Expanding this expression and using the adjoint properties of the Virasoro generators, we obtain:

$$\langle\phi|\phi\rangle = \langle\xi|L_1^2 L_{-1}^2|\xi\rangle - 2\langle\xi|L_1^2 L_{-2}|\xi\rangle - 2\langle\xi|L_2 L_{-1}^2|\xi\rangle + 4\langle\xi|L_2 L_{-2}|\xi\rangle. \tag{3.42}$$

By commuting the Virasoro generators appropriately and applying the Virasoro algebra relations, we simplify the expression. Since $\xi$ is a normalized primary field, it satisfies $L_n|\xi\rangle = 0$ for all $n > 0$, $L_0|\xi\rangle = h_\xi|\xi\rangle$ and $\langle\xi|\xi\rangle = 1$. Using these properties, the inner


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product reduces to:

$$\langle\phi|\phi\rangle = 4h_\xi(2h_\xi+1) - 12h_\xi + \left(4h_\xi - \frac{c}{2}\right). \tag{3.43}$$

Expanding the conformal weight $h_\xi$ and the central charge $c$ in terms of $\epsilon$, we obtain:

$$\langle\phi|\phi\rangle = 27\epsilon + \mathcal{O}(\epsilon^2). \tag{3.44}$$

Therefore, the indecomposability parameter can be obtained as

$$\beta = -\frac{\langle\phi|\phi\rangle}{h_\psi - h_\xi - 2} = -18, \tag{3.45}$$

Next, we turn our attention to the ground states. In this case, we take

$$A = L_{-1},\ h_\phi = h_\psi = h_{1,5} = 1 + \frac{3}{2}\epsilon + \mathcal{O}(\epsilon^2),\ h_\xi = h_{1,3} = \epsilon + \mathcal{O}(\epsilon^2). \tag{3.46}$$

Following the same procedure as above, we find that the indecomposability parameter for the ground state sector is

$$\beta = \beta_{1,5} = -1. \tag{3.47}$$


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# Chapter 4 Non-Hermitian model of free fermions and conformal invariance

## 4.1 Non-Hermitian free-fermion field theory and the Virasoro operators

We consider a (1+1)d non-Hermitian free-fermion field theory, which includes both a non-Hermitian mass term and a second-order derivative term but without kinetic term. Note that, since we are working within non-Hermitian quantum field theory, the quantized field operator should be expressed in the biorthogonal formalism. Here, $+$ and $-$ indicate the right- and left-moving chiral fermions. The Hamiltonian is given by:

$$H = \frac{1}{a}\Delta \int dx\, \psi_-^\dagger \psi_+ - a\Delta \int dx\, \left(\partial_x \psi_-^\dagger \partial_x \psi_+ - \partial_x \psi_+^\dagger \partial_x \psi_-\right), \tag{4.1}$$

while the field momentum is written as:

$$P = \int dx\, i \left(\psi_-^\dagger \partial_x \psi_- - \partial_x \psi_+^\dagger \psi_+\right). \tag{4.2}$$


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where $a$ is the lattice constant, and $\Delta$ is a parameter characterizing the system.

We perform a Fourier transform to express the system in momentum space. The fermionic operators in real and momentum space are related as:

$$\psi_{+/-}(x) = \sum_k e^{-ikx}\tilde{\psi}_{+/-}(k), \quad k = \frac{2\pi m}{L} \tag{4.3}$$

Substituting into the Hamiltonian, we obtain:

$$H = \frac{\Delta}{a}\sum_k \tilde{\psi}_-^\dagger(k)\tilde{\psi}_+(k) - a\Delta\sum_k k^2\left(\tilde{\psi}_-^\dagger(k)\tilde{\psi}_+(k) - \tilde{\psi}_+^\dagger(k)\tilde{\psi}_-(k)\right), \tag{4.4}$$

and the field momentum:

$$P = \sum_k k\left(\tilde{\psi}_-^\dagger(k)\tilde{\psi}_-(k) - \tilde{\psi}_+^\dagger(k)\tilde{\psi}_+(k)\right) \tag{4.5}$$

This can be expressed in matrix form as:

$$H = \sum_k \begin{pmatrix} \tilde{\psi}_-^\dagger(k) & \tilde{\psi}_+^\dagger(k) \end{pmatrix} \begin{pmatrix} 0 & \Delta(\frac{1}{a} - ak^2) \\ a\Delta k^2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_-(k) \\ \tilde{\psi}_+(k) \end{pmatrix}. \tag{4.6}$$

We define the Bloch Hamiltonian as:

$$\mathcal{H}(k) = \begin{pmatrix} 0 & \Delta(\frac{1}{a} - ak^2) \\ a\Delta k^2 & 0 \end{pmatrix}$$

When $k \neq 0$, we can diagonalize $\mathcal{H}(k)$ by changing to a new basis:

$$\mathcal{H}_D(k) = S^{-1}(k)\mathcal{H}(k)S(k) = \begin{pmatrix} k\Delta\sqrt{1-a^2k^2} & 0 \\ 0 & -k\Delta\sqrt{1-a^2k^2} \end{pmatrix}, \tag{4.7}$$


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where the eigenbases are given by

$$S(k) = \begin{pmatrix} \sqrt{1-a^2k^2} & -\sqrt{1-a^2k^2} \\ ak & ak \end{pmatrix}, \quad S^{-1}(k) = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{1-a^2k^2}} & \frac{1}{ak} \\ \frac{-1}{\sqrt{1-a^2k^2}} & \frac{1}{ak} \end{pmatrix}. \tag{4.8}$$

Note that $S(k)$ is not invertible when $k = 0$, where H(k) forms a non-diagonalizable Jordan block. The Hamiltonian is then expressed as

$$H = \left(\frac{L}{2\pi v_F}\right)\sum_{k\neq 0}\begin{pmatrix} B^{L\dagger}(k) & C^{L\dagger}(k)\end{pmatrix}\begin{pmatrix} k\Delta & 0 \\ 0 & -k\Delta \end{pmatrix}\begin{pmatrix} B^R(k) \\ C^R(k)\end{pmatrix} + \left(\frac{L}{2\pi a}\right)\tilde{\psi}_-^{L\dagger}(0)\tilde{\psi}_+^R(0) \tag{4.9}$$

in terms of the biorthogonal eigenmode operators

$$B^{L\dagger}(k) = ak\tilde{\psi}_+^\dagger(k) + \sqrt{1-a^2k^2}\tilde{\psi}_-^\dagger(k), \quad C^{L\dagger}(k) = ak\tilde{\psi}_+^\dagger(k) - \sqrt{1-a^2k^2}\tilde{\psi}_-^\dagger(k),$$
$$B^R(k) = \frac{1}{2ak}\tilde{\psi}_+(k) + \frac{1}{2\sqrt{1-a^2k^2}}\tilde{\psi}_-(k), \quad C^R(k) = \frac{1}{2ak}\tilde{\psi}_+(k) - \frac{1}{2\sqrt{1-a^2k^2}}\tilde{\psi}_-(k),$$

where the superscripts $R$ and $L$ denote the right and left bases, respectively.

Generally, if a system is massive, it does not exhibit conformal symmetry because it is not scale-invariant. However, in this case, although the Hamiltonian includes a mass term, it is evident from Eq. (4.7) that in the limit $a \to 0$, the dispersion relation becomes linear, which implies that the system is gapless. Moreover, the coefficient of the linear term corresponds to the Fermi velocity, given by $v_F = \Delta$. The Koo-Saleur formula provides a relation between higher-order Virasoro generators and the Hamiltonian and momentum densities. These are defined as [5, 14]:

$$H_n = \frac{L}{2\pi v_F}\int dx\, e^{iq_n x}\mathcal{H}(x), \quad P_n = \frac{L}{2\pi}\int dx\, e^{iq_n x}\mathcal{P}(x), \tag{4.10}$$

where $q_n = \frac{2\pi n}{L}$, $v_F$ is the Fermi velocity, and $\mathcal{H}(x)$, $\mathcal{P}(x)$ are the Hamiltonian and field


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momentum density respectively. Transforming to momentum space, the generators become

$$H_n = \left(\frac{L}{2\pi v_F}\right)\left\{\frac{\Delta}{a}\sum_k \tilde{\psi}_-^\dagger\tilde{\psi}_+(k)\right.$$
$$\left. + a\Delta\sum_k k(k-q_n)\left(\tilde{\psi}_+^\dagger(k-q_n)\tilde{\psi}_-(k) - \tilde{\psi}_-^\dagger(k-q_n)\tilde{\psi}_+(k)\right)\right\}, \tag{4.11}$$
$$P_n = \left(\frac{L}{2\pi}\right)\sum_k k\tilde{\psi}_-^\dagger(k-q_n)\tilde{\psi}_-(k) - (k-q_n)\tilde{\psi}_+^\dagger(k-q_n)\tilde{\psi}_+(k).$$

In matrix form, the expression for $H_n$ becomes:

$$H_n = \left(\frac{L}{2\pi v_F}\right)\sum_k \begin{pmatrix}\tilde{\psi}_-^\dagger(k-q_n) & \tilde{\psi}_+^\dagger(k-q_n)\end{pmatrix}\begin{pmatrix}0 & (\frac{1}{a}-ak(k-q_n))\Delta \\ ak(k-q_n)\Delta & 0\end{pmatrix}\begin{pmatrix}\tilde{\psi}_-(k) \\ \tilde{\psi}_+(k)\end{pmatrix} \tag{4.12}$$

Using the change-of-basis matrix $S(k)$ in Eq. (4.8), we find that $H_n$ can be brought into a diagonal form (in the limit $a/L \to 0^+$):

$$\begin{aligned}H_n = &\left(\frac{L}{2\pi v_F}\right)\sum_{k\neq 0, k\neq q_n}\begin{pmatrix}B^{L\dagger}(k-q_n) & C^{L\dagger}(k-q_n)\end{pmatrix}S^{-1}(k-q_n)\\ &\begin{pmatrix}0 & (\frac{1}{a}-ak(k-q_n))\Delta \\ ak(k-q_n)\Delta & 0\end{pmatrix}S(k)\begin{pmatrix}B^R(k) \\ C^R(k)\end{pmatrix}\\ &+\left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}_-^\dagger(0)\tilde{\psi}_+(q_n) + \tilde{\psi}_-^\dagger(-q_n)\tilde{\psi}_+(0)\right).\\ = &\left(\frac{L}{2\pi v_F}\right)\sum_{k\neq 0, k\neq q_n}\begin{pmatrix}B^{L\dagger}(k-q_n) & C^{L\dagger}(k-q_n)\end{pmatrix}\begin{pmatrix}k\Delta & 0 \\ 0 & -k\Delta\end{pmatrix}\begin{pmatrix}B^R(k) \\ C^R(k)\end{pmatrix}\\ &+\left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}_-^\dagger(0)\tilde{\psi}_+(q_n) + \tilde{\psi}_-^\dagger(-q_n)\tilde{\psi}_+(0)\right).\end{aligned} \tag{4.13}$$


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Similarly, the momentum generator $P_n$ in the new basis is written as

$$P_n = \frac{L}{2\pi} \sum_{k\neq 0, k\neq q_n} k\left(B^{L\dagger}(k-q_n)B^R(k) + C^{L\dagger}(k-q_n)C^R(k)\right) + q_n\left(\tilde{\psi}^\dagger_+(-q_n)\tilde{\psi}_+(0) + \tilde{\psi}^\dagger_-(0)\tilde{\psi}_-(q_n)\right). \tag{4.14}$$

Due to $k = \frac{2\pi m}{L}$, the summation index in Eq.(4.13) and Eq.(4.14) can be changed from $k$ to $m$. Accordingly, the expressions for the generators $H_n$ and $P_n$ become:

$$H_n = \sum_{m\neq 0, m\neq n} m\left(B^{L\dagger}_{m-n}B^R_m - C^{L\dagger}_{m-n}C^R_m\right) + \left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{+,n} + \tilde{\psi}^\dagger_{-,-n}\tilde{\psi}_{+,0}\right), \tag{4.15}$$

and

$$P_n = \sum_{m\neq 0, m\neq n} m\left(B^{L\dagger}_{m-n}B^R_m + C^{L\dagger}_{m-n}C^R_m\right) + n\left(\tilde{\psi}^\dagger_{+,-n}\tilde{\psi}_{+,0} + \tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{-,n}\right), \tag{4.16}$$

As for physical Hamiltonian and momentum, read:

$$H = \sum_{m\neq 0} m\left(B^{L\dagger}_m B^R_m - C^{L\dagger}_m C^R_m\right) + \left(\frac{L}{2\pi a}\right)\tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{+,0}, \tag{4.17}$$

$$P = \sum_{m\neq 0} m\left(B^{L\dagger}_m B^R_m + C^{L\dagger}_m C^R_m\right), \tag{4.18}$$

respectively. It is a little different with the zero mode generators, from Koo-Saleur formula:

$$H_n = \frac{L}{2\pi v_F}\int dx\, e^{iq_n x}\mathcal{H}(x).$$

Therefore,

$$H_0 = \frac{L}{2\pi v_F}\int dx\,\mathcal{H}(x) = \frac{L}{2\pi v_F}H,$$

where $H$ is the physical Hamiltonian.


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## 4.2 Conformal symmetry from the Virasoro algebra

To determine whether our model exhibits conformal symmetry, one effective approach is to verify whether the generators constructed in the previous section satisfy the Virasoro algebra. The Virasoro algebra is given by

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\
[L_n, \bar{L}_m] &= 0
\end{aligned}
\tag{4.19}
$$

In this section, we focus on the $n+m \neq 0$ cases, and Eq.(4.19) is equivalent to:

$$
\begin{aligned}
[H_n, H_m] &= (n-m)P_{n+m}, \\
[H_n, P_m] &= (n-m)H_{n+m}, \\
[P_n, P_m] &= (n-m)P_{n+m}.
\end{aligned}
\tag{4.20}
$$

In real space, the $(1+1)$-dimensional fermion commutation relations are given by

$$
\left\{\psi^\dagger_\alpha(x), \psi_\beta(x')\right\} = \delta_{\alpha,\beta}\delta(x-x'), \quad \left\{\psi^\dagger_\alpha(x), \psi^\dagger_\beta(x')\right\} = \left\{\psi_\alpha(x), \psi_\beta(x')\right\} = 0 \tag{4.21}
$$

where $\alpha, \beta \in \{+,-\}$ and $\gamma, \delta \in \{R, L\}$. These relations translate into the momentum space as

$$
\left\{\psi^\dagger_\alpha(k), \psi_\beta(k')\right\} = \delta_{\alpha,\beta}\delta_{k,k'}, \quad \left\{\psi^\dagger_\alpha(k), \psi^\dagger_\beta(k')\right\} = \left\{\psi_\alpha(k), \psi_\beta(k')\right\} = 0 \tag{4.22}
$$


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Therefore, using Eq.(4.21), we deduce the commutation relations in the new basis states:

$$\begin{aligned} \left\{B_L^\dagger(k), B_R(k')\right\} = \delta_{k,k'}, \quad \left\{C_L^\dagger(k), C_R(k')\right\} = \delta_{k,k'}, \\ \left\{B_L^\dagger(k), C_R(k')\right\} = \left\{C_L^\dagger(k), B_R(k')\right\} = 0. \end{aligned} \tag{4.23}$$

With the commutation relations in the new basis, we can now analyze the structure and algebra of the higher-order Virasoro generators. For simplicity, in the following calculations, we denote $q_n = \frac{2\pi n}{L}$ as $q$, and $q_m = \frac{2\pi m}{L}$ as $q'$. Furthermore, we separate generators into the diagonal and zero-mode parts:

$$H_n = H_n^{(d)} + H_n^{(0)}, \quad P_n = P_n^{(d)} + P_n^{(0)} \tag{4.24}$$

**1. Commutator $[H_n, H_m]$**

$$\begin{aligned} [H_n, H_m] &= [H_n^{(d)} + H_n^{(0)}, H_m^{(d)} + H_m^{(0)}] \\ &= (n-m)\left(\frac{L}{2\pi}\right) \sum_{k\neq 0, k\neq q+q'} k\left(B^{L\dagger}(k-(q+q'))B^R(k) + C^{L\dagger}(k-(q+q'))C^R(k)\right) \\ &+ (n-m)\left(\tilde{\psi}_+^\dagger(-(q+q'))\tilde{\psi}_+(0) + \tilde{\psi}_-^\dagger(0)\tilde{\psi}_-(q+q')\right), \end{aligned}$$

the commutator is same with

$$\begin{aligned} (n-m)P_{n+m} &= (n-m)\left(\frac{L}{2\pi}\right) \sum_{k\neq 0, k\neq q+q'} k\left(B^{L\dagger}(k-(q+q'))B^R(k) + C^{L\dagger}(k-(q+q'))C^R(k)\right) \\ &+ (n-m)\left(\tilde{\psi}_+^\dagger(-(q+q'))\tilde{\psi}_+(0) + \tilde{\psi}_-^\dagger(0)\tilde{\psi}_-(q+q')\right) \\ &= (n-m)(P_{n+m}^{(d)} + P_{n+m}^{(0)}). \end{aligned}$$

Thus, we have verified the first identity in Eq.(4.20):

$$[H_n, H_m] = (n-m)P_{n+m}.$$


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**2 .Commutator** $[H_n, P_m]$

$$
\begin{aligned}
[H_n, P_m] &= [H_n^{(d)} + H_n^{(0)}, P_m^{(d)} + P_m^{(0)}] \\
&= (n-m)\left(\frac{L}{2\pi}\right) \sum_{k\neq 0, k\neq q+q'} k\left(B^{L\dagger}(k-(q+q'))B^R(k) - C^{L\dagger}(k-(q+q'))C^R(k)\right) \\
&+ (n-m)\left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}_-^\dagger(0)\tilde{\psi}_+(q+q') + \tilde{\psi}_-^\dagger(-(q+q'))\tilde{\psi}_+(0)\right).
\end{aligned}
$$

Meanwhile, from Eq.(4.13), we find:

$$
\begin{aligned}
(n-m)H_{n+m} &= (n-m)\left(\frac{L}{2\pi}\right) \sum_{k\neq 0, k\neq q+q'} k\left(B^{L\dagger}(k-(q+q'))B^R(k) - C^{L\dagger}(k-(q+q'))C^R(k)\right) \\
&+ (n-m)\left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}_-^\dagger(0)\tilde{\psi}_+(q+q') + \tilde{\psi}_-^\dagger(-(q+q'))\tilde{\psi}_+(0)\right) \\
&= (n-m)(H_{n+m}^{(d)} + H_{n+m}^{(0)}).
\end{aligned}
$$

Therefore, the second identity in Eq.(4.20) is verified:

$$
[H_n, P_m] = (n-m)H_{n+m}.
$$

**3. Commutator** $[P_n, P_m]$

$$
\begin{aligned}
[P_n, P_m] &= [P_n^{(d)} + P_n^{(0)}, P_m^{(d)} + P_m^{(0)}] \\
&= (n-m)\left(\frac{L}{2\pi}\right) \sum_{k\neq 0, k\neq q+q'} k\left(B^{L\dagger}(k-(q+q'))B^R(k) + C^{L\dagger}(k-(q+q'))C^R(k)\right) \\
&+ (n-m)\left(\tilde{\psi}_+^\dagger(-(q+q'))\tilde{\psi}_+(0) + \tilde{\psi}_-^\dagger(0)\tilde{\psi}_-(q+q')\right).
\end{aligned}
$$


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Clearly, this matches the definition:

$$(n-m)P_{n+m} = (n-m)\left(\frac{L}{2\pi}\right)\sum_{k\neq 0, k\neq q+q'} k\left(B^{L\dagger}(k-(q+q'))B^{R}(k) + C^{L\dagger}(k-(q+q'))C^{R}(k)\right)$$
$$+ (n-m)\left(\tilde{\psi}_{+}^{\dagger}(-(q+q'))\tilde{\psi}_{+}(0) + \tilde{\psi}_{-}^{\dagger}(0)\tilde{\psi}_{-}(q+q')\right)$$
$$= (n-m)(P_{n+m}^{(d)} + P_{n+m}^{(0)}).$$

Thus, we have verified the final identity in Eq.(4.20):

$$[P_n, P_m] = (n-m)P_{n+m}.$$

Therefore, we have shown that when the lattice constant $a$ becomes infinitesmial, the non-Hermitian model in Eq.(4.1) indeed obeys conformal symmetry.

## 4.3 Central charge

In the previous section, we proved that the second-derivative model given in Eq.(4.1) satisfies the Virasoro algebra in the continuum limit for the case where the sum of the generator indices is nonzero. We now aim to determine the central charge of the model through the Virasoro algebra. To do so, we consider the commutator of two Virasoro generators, $[L_n, L_{-n}]$, where $n$ is a natural number. The algebra implies:

$$[L_n, L_{-n}] = 2nL_0 + \frac{c}{12}n(n^2-1), \tag{4.25}$$

where $c$ is the central charge. The high-order generators $H_n$ and $P_n$ are given by Eq. (4.13) and Eq.(4.14), respectively. Following the method described in [15], we write the


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generators into normal-order form. For $n > 0$, the Hamiltonian generator becomes:

$$\begin{aligned} H_n = &\sum_{m>n} m\left(B^{L\dagger}_{m-n}B^R_m + C^R_m C^{L\dagger}_{m-n}\right) + \sum_{0<m<n} m\left(D^{L\dagger}_{n-m}B^R_m + C^R_m F^{L\dagger}_{n-m}\right) \\ &+ \sum_{m>0} m\left(D^R_m D^{L\dagger}_{m+n} + F^{L\dagger}_{m+n}F^R_m\right) + \left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{+,n} + \tilde{\psi}^\dagger_{-,-n}\tilde{\psi}_{+,0}\right), \end{aligned} \tag{4.26}$$

where we define the $m < 0$ mode operators as: $D^{L\dagger}_m = B^{L\dagger}_{-m}$, $D^R_m = B^R_{-m}$, $F^{L\dagger}_m = C^{L\dagger}_{-m}$, $F^R_m = C^R_{-m}$. Similarly, the corresponding negative generator is:

$$\begin{aligned} H_{-n} = &\sum_{m>0} m\left(B^{L\dagger}_{m+n}B^R_m + C^R_m C^{L\dagger}_{m+n}\right) + \sum_{0<m<n} m\left(D^R_m B^{L\dagger}_{n-m} + C^{L\dagger}_{n-m}F^R_m\right) \\ &+ \sum_{m>n} m\left(D^R_m D^{L\dagger}_{m-n} + F^{L\dagger}_{m-n}F^R_m\right) + \left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{+,-n} + \tilde{\psi}^\dagger_{-,n}\tilde{\psi}_{+,0}\right), \end{aligned} \tag{4.27}$$

Using the same approach, the momentum generators (for $n > 0$) become:

$$\begin{aligned} P_n = &\sum_{m>n} m\left(B^{L\dagger}_{m-n}B^R_m + C^{L\dagger}_{m-n}C^R_m\right) + \sum_{0<m<n} m\left(D^{L\dagger}_{n-m}B^R_m + F^{L\dagger}_{n-m}C^R_m\right) \\ &+ \sum_{m>0} m\left(D^R_m D^{L\dagger}_{m+n} + F^R_m F^{L\dagger}_{m+n}\right) + n\left(\tilde{\psi}^\dagger_{+,-n}\tilde{\psi}_{+,0} + \tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{-,n}\right), \end{aligned} \tag{4.28}$$

$$\begin{aligned} P_{-n} = &\sum_{m>0} m\left(B^{L\dagger}_{m+n}B^R_m + C^{L\dagger}_{m+n}C^R_m\right) + \sum_{0<m<n} m\left(D^R_m B^{L\dagger}_{n-m} + F^R_m C^{L\dagger}_{n-m}\right) \\ &+ \sum_{m>n} m\left(D^R_m D^{L\dagger}_{m-n} + F^R_m F^{L\dagger}_{m-n}\right) - n\left(\tilde{\psi}^\dagger_{+,n}\tilde{\psi}_{+,0} + \tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{-,-n}\right), \end{aligned} \tag{4.29}$$

The zero-order Virasoro generator follows from Eq.(4.17) and Eq.(4.18):

$$L_0 = \sum_{m>0} m\left(B^{L\dagger}_m B^R_m + D^R_m D^{L\dagger}_m\right) + \left(\frac{L}{4\pi a}\right)\left(\tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{+,-n} + \tilde{\psi}^\dagger_{-,n}\tilde{\psi}_{+,0}\right). \tag{4.30}$$

With these expressions, we are now equipped to identify the Virasoro generators and compute their commutators in order to extract the central charge of the theory. The Virasoro


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generator can be obtain by linear combination of Hamiltonian and momentum generator:

$$L_n = \frac{1}{2}\left(H_n + P_n\right). \tag{4.31}$$

Therefore,

$$\begin{aligned} L_n = &\sum_{m>n} m\left(B^{L\dagger}_{m-n} B^R_m\right) + \sum_{0<m<n} m\left(D^{L\dagger}_{n-m} B^R_m\right) + \sum_{m>0} m\left(D^R_m D^{L\dagger}_{m+n}\right) \\ &+ \frac{1}{2}\left(\left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{+,n} + \tilde{\psi}^\dagger_{-,-n}\tilde{\psi}_{+,0}\right) + n\left(\tilde{\psi}^\dagger_{+,-n}\tilde{\psi}_{+,0} + \tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{-,n}\right)\right), \end{aligned} \tag{4.32}$$

$$\begin{aligned} L_{-n} = &\sum_{m>0} m\left(B^{L\dagger}_{m+n} B^R_m\right) + \sum_{0<m<n} m\left(D^R_m B^{L\dagger}_{n-m}\right) + \sum_{m>n} m\left(D^R_m D^{L\dagger}_{m-n}\right) \\ &+ \frac{1}{2}\left(\left(\frac{L}{2\pi a}\right)\left(\tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{+,-n} + \tilde{\psi}^\dagger_{-,n}\tilde{\psi}_{+,0}\right) - n\left(\tilde{\psi}^\dagger_{+,n}\tilde{\psi}_{+,0} + \tilde{\psi}^\dagger_{-,0}\tilde{\psi}_{-,-n}\right)\right), \end{aligned} \tag{4.33}$$

where $n > 0$. Hence, the commutator of positive and negative mode generators is

$$[L_n, L_{-n}] = \sum_{m>0} 2nm\left(B^{L\dagger}_m B^R_m + D^R_m D^{L\dagger}_m\right) - \frac{1}{6}n(n^2-1). \tag{4.34}$$

From 4.30, we know

$$[L_n, L_{-n}] = 2nL_0 - \frac{1}{6}n(n^2-1). \tag{4.35}$$

Therefore, if the Virasoro algebra hold in this case, Eq.(4.35) should same as Eq.(4.25). Hence, the central charge of the model is $-2$. As for the anti-holomorphic part, the system also satisfies the Virasoro algebra:

$$\left[\bar{L}_n, \bar{L}_m\right] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{4.36}$$

where both the holomorphic and anti-holomorphic parts share the same central charge $-2$.



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## 4.4 Lattice realization

From the previous section, we know that the non-Hermitian model Eq. (4.1), which includes only a mass term and a second derivative term, surprisingly satisfies the Virasoro algebra. In this section, we shift our focus to a lattice model that serves as a finite-system regularization of the continuum field theory in Eq. (4.1). Specifically, we consider a one-dimensional chain of free fermions described by the following non-Hermitian Hamiltonian density:

$$e_j = i(-1)^j \left[c_j c_{j+1}^\dagger - c_j^\dagger c_{j+1} + i(c_j^\dagger c_j - c_{j+1}^\dagger c_{j+1})\right], \quad 1 \le j \le L. \tag{4.37}$$

Here, $L$ is the total number of lattice sites, and $N$ is half that number. $c_j^{(\dagger)}$ are fermionic operators in real space satisfying

$$\{c_i, c_j\} = \left\{c_i^\dagger, c_j^\dagger\right\} = 0, \quad \left\{c_i, c_j^\dagger\right\} = \delta_{ij}. \tag{4.38}$$

We now decompose the lattice fermion operators into right- and left-moving modes:

$$c_j = e^{ikx}\psi_+(x) + e^{-ikx}\psi_-(x), \quad c_{j+1} = e^{ik(x+a)}\psi_+(x+a) + e^{-ik(x+a)}\psi_-(x+a). \tag{4.39}$$

where $a$ is the lattice constant, and $\psi_{+/-}(x)$ are the continuous fermionic operator [1]. We focus on the region near $k_F = \pm\frac{\pi}{2a}$ where the dispersion relation becomes linear. After detailed calculation, the Hamiltonian reduces to Eq.(4.1) in the continuum limit. This confirms that the lattice model also exhibits conformal symmetry in that limit. We transform

[1] The indices $+$ and $-$ on the fermionic operators refer to the right- and left-moving modes.


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these operators into momentum space using the Fourier transform. Setting:

$$c_j = \frac{1}{\sqrt{L}} \sum_{p_m} e^{ijp_m} \theta_{p_m}, \quad c_j^\dagger = \frac{1}{\sqrt{L}} \sum_{p_m} e^{-ijp_m} \theta_{p_m}^\dagger. \tag{4.40}$$

Substituting into Eq.(4.37), we get

$$e_j = \frac{(-1)^j}{L} \sum_{p_m p'_m} \left( -ie^{-i(j+1)p_m} e^{ijp'_m} - ie^{-ijp_m} e^{i(j+1)p'_m} - e^{-ijp_m} e^{ijp'_m} + e^{-i(j+1)p_m} e^{i(j+1)p'_m} \right) \theta_{p_m}^\dagger \theta_{p'_m}. \tag{4.41}$$

Hence, the Hamiltonian with periodic boundary consition is

$$H = -\sum_{j=1}^{L} e_j = 2 \sum_{p_m} (1 + \sin p_m) \theta_{p_m}^\dagger \theta_{\pi + p_m}. \tag{4.42}$$

We redefine the momentum $p$ by subtracting $\frac{\pi}{2}$, i.e., $p \to p - \pi/2$, and convert $p_m$ to the index $m$:

$$H = 2 \sum_m (1 - \cos p_m) \theta_{p_m - \frac{\pi}{2}}^\dagger \theta_{p_m + \frac{\pi}{2}}, \quad p_m = \frac{m\pi}{N}.$$

In the second half of the sum, where $m = N + 1 \sim 2N$, we shift the range to $m = 1 \sim N$ by converting $p \to p + \pi$:

$$\begin{aligned}
&= \sum_{m=1}^{N} 2(1 - \cos p_m) \theta_{p_m - \frac{\pi}{2}}^\dagger \theta_{p_m + \frac{\pi}{2}} + \sum_{m=N+1}^{2N} 2(1 - \cos p_m) \theta_{p_m - \frac{\pi}{2}}^\dagger \theta_{p_m + \frac{\pi}{2}} \\
&= 2 \sum_{m=1}^{N} (1 - \cos p_m) \theta_{p_m - \frac{\pi}{2}}^\dagger \theta_{p_m + \frac{\pi}{2}} + (1 + \cos p_m) \theta_{p_m + \frac{\pi}{2}}^\dagger \theta_{p_m - \frac{\pi}{2}}.
\end{aligned}$$

Finally, we write the Hamiltonian in matrix form:

$$H = 2 \sum_{m=1}^{N} \begin{pmatrix} \theta_{p-\pi/2}^\dagger & \theta_{p+\pi/2}^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 - \cos p \\ 1 + \cos p & 0 \end{pmatrix} \begin{pmatrix} \theta_{p-\pi/2} \\ \theta_{p+\pi/2} \end{pmatrix}, \tag{4.43}$$


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where we define the Bloch Hamiltonian:

$$\mathcal{H} \equiv \begin{pmatrix} 0 & 1-\cos p \\ 1+\cos p & 0 \end{pmatrix}.$$

When $m = N$ ($p = \pi$), $\mathcal{H}$ becomes non-diagonalizable, and this term must be treated separately. We apply a biorthogonal basis expansion with eigenvalues $E^{\pm}$:

$$\mathcal{H}\begin{Bmatrix} v_R^+ \\ v_R^- \end{Bmatrix} = \begin{matrix} E^+ \\ E^- \end{matrix}\begin{Bmatrix} v_R^+ \\ v_R^- \end{Bmatrix};\ \mathcal{H}^\dagger\begin{Bmatrix} v_L^+ \\ v_L^- \end{Bmatrix} = \begin{matrix} E^+ \\ E^- \end{matrix}\begin{Bmatrix} v_L^+ \\ v_L^- \end{Bmatrix}, \tag{4.44}$$

We construct the biorthogonal transformation matrices:

$$U_R = \{v_R^+, v_R^-\} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tan\frac{p}{2} & \tan\frac{p}{2} \\ 1 & -1 \end{pmatrix}, \quad U_L = \{v_L^+, v_L^-\} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cot\frac{p}{2} & \cot\frac{p}{2} \\ 1 & -1 \end{pmatrix}. \tag{4.45}$$

These matrices satisfy

$$U_L^\dagger \mathcal{H} U_R = \mathcal{H}_D, \quad \text{and} \quad U_L^\dagger U_R = \mathbb{1}. \tag{4.46}$$

where $\mathcal{H}_D = \begin{pmatrix} \sin p & 0 \\ 0 & -\sin p \end{pmatrix}$, and $\mathbb{1}$ is the identity matrix. Therefore, the full Hamiltonian is

$$\begin{aligned} H &= 2\sum_{m=1}^{N-1}\begin{pmatrix} \theta^\dagger_{p-\pi/2} & \theta^\dagger_{p+\pi/2} \end{pmatrix}\begin{pmatrix} 0 & 1-\cos p \\ 1+\cos p & 0 \end{pmatrix}\begin{pmatrix} \theta_{p-\pi/2} \\ \theta_{p+\pi/2} \end{pmatrix} + (m = N \text{ term}) \\ &= \sum_{m=1}^{N-1} 2\begin{pmatrix} \chi^\dagger_{R,p} & \eta^\dagger_{R,p} \end{pmatrix}\begin{pmatrix} \sin p & 0 \\ 0 & -\sin p \end{pmatrix}\begin{pmatrix} \chi_{L,p} \\ \eta_{L,p} \end{pmatrix} + (m = N \text{ term}) \\ &= \sum_{m=1}^{N-1} 2\sin p\left(\chi^\dagger_{R,p}\chi_{L,p} - \eta^\dagger_{R,p}\eta_{L,p}\right) + (m = N \text{ term}). \end{aligned} \tag{4.47}$$


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When $m = N$, $p = \pi$, $\mathcal{H}$ becomes

$$\mathcal{H}_{non-diag} = \begin{pmatrix} 0 & 2 \\ 0 & 0` \end{pmatrix}, \tag{4.48}$$

$$H_{non-diag} = 4\theta^{\dagger}_{\pi/2}\theta_{3\pi/2} = 4\chi^{\dagger}_{0}\eta_{0}.$$

Therefore, the Hamiltonian can be expressed by $\chi_p$ and $\eta_p$.

$$H = \sum_{m=1}^{N-1} 2\sin p \left(\chi^{\dagger}_{R,p}\chi_{L,p} - \eta^{\dagger}_{R,p}\eta_{L,p}\right) + 4\chi^{\dagger}_{0}\eta_{0}, \tag{4.49}$$

where

$$\begin{aligned}
&\chi^{\dagger}_{R,p} = \frac{1}{\sqrt{2}}\left(\sqrt{\tan\frac{p}{2}}\theta^{\dagger}_{p-\frac{\pi}{2}} + \sqrt{\cot\frac{p}{2}}\theta^{\dagger}_{p+\frac{\pi}{2}}\right), \chi_{L,p} = \frac{1}{\sqrt{2}}\left(\sqrt{\cot\frac{p}{2}}\theta_{p-\frac{\pi}{2}} + \sqrt{\tan\frac{p}{2}}\theta_{p+\frac{\pi}{2}}\right), \\
&\eta^{\dagger}_{R,p} = \frac{1}{\sqrt{2}}\left(\sqrt{\tan\frac{p}{2}}\theta^{\dagger}_{p-\frac{\pi}{2}} - \sqrt{\cot\frac{p}{2}}\theta^{\dagger}_{p+\frac{\pi}{2}}\right), \eta_{L,p} = \frac{1}{\sqrt{2}}\left(\sqrt{\cot\frac{p}{2}}\theta_{p-\frac{\pi}{2}} - \sqrt{\tan\frac{p}{2}}\theta_{p+\frac{\pi}{2}}\right), \\
&\quad \chi^{\dagger}_{0} = \theta^{\dagger}_{\frac{\pi}{2}}, \quad \chi_{0} = \theta_{\frac{\pi}{2}}, \quad \eta^{\dagger}_{0} = \theta^{\dagger}_{\frac{3\pi}{2}}, \quad \eta_{0} = \theta_{\frac{3\pi}{2}}.
\end{aligned} \tag{4.50}$$

Thus, $\chi_p$ and $\eta_p$ satisfy anti-commutation relations,

$$\left\{\chi_{L,p}, \chi^{\dagger}_{R,p}\right\} = \left\{\eta_{L,p}, \eta^{\dagger}_{R,p}\right\} = \delta_{p,q}. \tag{4.51}$$

On the other hand, the annihilation modes are

$$\chi_{L,p}\Omega = \eta^{\dagger}_{R,p}\Omega = \chi^{\dagger}_{0}\Omega = \eta_{0}\Omega = 0.$$

We will clarify the relationship between the creation and annihilation operators and the corresponding states in more detail in the following section. Following the discussion and


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the method in [8], we rewrite Eq.(4.49) in the normal-ordered form as

$$H = 2\sum_{m=1}^{N-1} \sin p \left(\chi^{\dagger}_{R,p}\chi_{L,p} - \eta_{L,p}\eta^{\dagger}_{R,p}\right) + 4\chi^{\dagger}_0\eta_0 - 2\sum_{m=1}^{N-1} \sin p,$$

where the last term refers to the ground state energy [8]. Next, we linearize the dispersion relation near $p = 0$ and $p = \pi$ in the limit $N \to \infty$. In this limit, we introduce the left-moving modes $\bar{\chi}^{(\dagger)}_p = \chi^{(\dagger)}_{\pi-p}$ and $\bar{\eta}^{(\dagger)}_p = \eta^{(\dagger)}_{\pi-p}$. Then the Hamiltonian can be written as

$$H = \frac{4\pi}{N}\sum_{m>0} m\left(\chi^{\dagger}_{R,p}\chi_{L,p} + \bar{\chi}^{\dagger}_{R,p}\bar{\chi}_{L,p} + \eta_{L,p}\eta^{\dagger}_{R,p} + \bar{\eta}_{L,p}\bar{\eta}^{\dagger}_{R,p}\right) + 4\chi^{\dagger}_0\eta_0 + \langle \mathrm{vac}\,|H|\,\mathrm{vac}\rangle\,,$$

where the ground state energy

$$\langle \mathrm{vac}\,|H|\,\mathrm{vac}\rangle = -2\sum_{m=1}^{N-1}\sin p = -2\cot\frac{\pi}{N} = -\frac{2N}{\pi} + \frac{2\pi}{3N} + \mathcal{O}(1/N).$$

The Hamiltonian in this system can be decomposed into diagonal and non-diagonal components. In the framework of CFT, the Hamiltonian is expressed in terms of the Virasoro generators. Thus, it can be written as:

$$H = H^{(d)} + H^{(n)} = -\frac{2N}{\pi} + \frac{4\pi}{N}\left((L_0 + \bar{L}_0)^{(d)} + (L_0 + \bar{L}_0)^{(n)} - \frac{c}{12}\right) + \mathcal{O}(1/N).$$

Here, the diagonal parts of the Hamiltonian in the continuum limit is

$$(L_0 + \bar{L}_0)^{(d)} = \sum_{m>0} m\left(\chi^{\dagger}_{R,p}\chi_{L,p} + \bar{\chi}^{\dagger}_{R,p}\bar{\chi}_{L,p} + \eta_{L,p}\eta^{\dagger}_{R,p} + \bar{\eta}_{L,p}\bar{\eta}^{\dagger}_{R,p}\right),$$

while the non-diagonal parts is

$$(L_0 + \bar{L}_0)^{(n)} = \frac{N}{\pi}\chi^{\dagger}_0\eta_0.$$


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Afterwards, we introduce another notation $\psi$ to replace the $\chi_p$ and $\eta_p$, the relations are [8]

$$\begin{aligned}\psi^1_m = \sqrt{m}\chi_{L,p},\ \psi^2_m = \sqrt{m}\bar{\eta}^\dagger_{R,p},\ \bar{\psi}^1_m = \sqrt{m}\bar{\chi}_{L,p},\ \bar{\psi}^2_m = -\sqrt{m}\bar{\eta}^\dagger_{R,p},\ \psi^2_0 = \bar{\psi}^2_0 = \sqrt{\frac{L}{\pi}}\chi^\dagger_0,\\ \psi^1_{-m} = -\sqrt{m}\bar{\eta}_{L,p},\ \psi^2_{-m} = \sqrt{m}\chi^\dagger_{R,p},\ \bar{\psi}^1_{-m} = \sqrt{m}\eta_{L,p},\ \bar{\psi}^2_{-m} = \sqrt{m}\bar{\chi}^\dagger_{R,p},\ \psi^1_0 = \bar{\psi}^1_0 = \sqrt{\frac{L}{\pi}}\eta_0.\end{aligned} \tag{4.52}$$

From the Eq.(4.50), we know the anti-commutation relation

$$\left\{\psi^\alpha_m, \psi^\beta_{m'}\right\} = mJ^{\alpha\beta}\delta_{m+m',0}, \qquad \alpha,\beta \in \{1,2\},$$

where $J^{\alpha\beta}$ is the $2\times 2$ symplectic matrix. Therefore, for the continuum limit,

$$L_0 + \bar{L}_0 - \frac{c}{12} = \sum_{m>0}\left(\psi^2_{-m}\psi^1_m - \psi^1_{-m}\psi^2_m + \bar{\psi}^2_{-m}\bar{\psi}^1_m - \bar{\psi}^1_{-m}\bar{\psi}^2_m\right) + 2\psi^2_0\psi^1_0 - \frac{c}{12}. \tag{4.53}$$

It is related to the Hamiltonian, which characterizes the system

$$\frac{L}{2\pi}\left(H + \frac{2N}{\pi}\right) \mapsto L_0 + \bar{L}_0 - \frac{c}{12}. \tag{4.54}$$

## 4.5 Indecomposability parameter

In this section, we follow the method in [8] and figure out the indecomposability parameter of the lattice model in the continuum limit. The Koo-Saleur formula indicates that the high-order Virasoro generators built on the lattice system are [16]:

$$\begin{aligned}H_n = L_n + \bar{L}_{-n} &= -\frac{L}{2\pi v_F}\sum_{j=1}^{L} e^{ijq_n}(e_j - e_\infty) + \frac{c}{12}\delta_{n.0},\\ P_n = L_n - \bar{L}_{-n} &= -\frac{iL}{2\pi v_F^2}\sum_{j=1}^{L} e^{ijq_n}[e_j, e_{j+1}],\end{aligned} \tag{4.55}$$


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where $v_F$ is Fermi velocity. Here, our focus is on the lowest $\beta$ (more preciesly, $\beta_{1,5}$), meaning that the conformal operator $A$ in Eqs.(3.13) is equal to $L_{-1}$. Therefore, we need to compute $H_{-1}$ and $P_{-1}$ respectively, and obtain $L_{-1}$ by linear combining $H_{-1}$ and $P_{-1}$. First of all, we put the local Hamiltonian into the formula, therefore,

$$
\begin{aligned}
&-\sum_{j=1}^{L} e^{-iqj} e_j \\
&= -\sum_{m=1}^{N} \left[\left(e^{i(p+q)} + e^{-ip} - 1 - e^{iq}\right) \theta^{\dagger}_{p-\frac{\pi}{2}} \theta_{p+q+\frac{\pi}{2}} - \left(e^{i(p+q)} + e^{-ip} + 1 + e^{iq}\right) \theta^{\dagger}_{p+\frac{\pi}{2}} \theta_{p-q+\frac{\pi}{2}}\right].
\end{aligned}
\tag{4.56}
$$

Suppose exist $m = \alpha$ in the series, such that

$$
p_\alpha + q + \frac{\pi}{2} = \frac{3\pi}{2} \text{ or } \frac{\pi}{2}, \quad p_\alpha + q - \frac{\pi}{2} = \frac{3\pi}{2} \text{ or } \frac{\pi}{2}.
$$

Therefore, we take $p_\alpha + q = \pi,\ \alpha + n = N,\ n \in \mathbb{N}$

$$
\begin{aligned}
&-\sum_{j=1}^{L} e^{-iqj} e_j \\
&= -\sum_{m=1}^{\alpha-1} \left[(e^{i(p+q)} + e^{-ip} - 1 - e^{iq})\theta^{\dagger}_{p-\frac{\pi}{2}} \theta_{p+q+\frac{\pi}{2}} - (e^{i(p+q)} + e^{-ip} + 1 + e^{iq})\theta^{\dagger}_{p+\frac{\pi}{2}} \theta_{p+q-\frac{\pi}{2}}\right] \\
&- (e^{i\pi} + e^{i(q-\pi)} - 1 - e^{iq})\theta^{\dagger}_{\frac{\pi}{2}-q} \theta_{\frac{3\pi}{2}} \\
&- \sum_{m=\alpha+1}^{N-1} \left[(e^{i(p+q)} + e^{-ip} - 1 - e^{iq})\theta^{\dagger}_{p-\frac{\pi}{2}} \theta_{p+q+\frac{\pi}{2}} - (e^{i(p+q)} + e^{-ip} + 1 + e^{iq})\theta^{\dagger}_{p+\frac{\pi}{2}} \theta_{p+q-\frac{\pi}{2}}\right] \\
&- \left(e^{i(\pi+q)} + e^{-i\pi} - 1 - e^{iq}\right) \theta^{\dagger}_{\frac{\pi}{2}} \theta_{q+\frac{3\pi}{2}} + \left(e^{i(\pi+q)} + e^{-i\pi} + 1 + e^{iq}\right) \theta^{\dagger}_{\frac{3\pi}{2}} \theta_{q+\frac{\pi}{2}} \\
&= -\sum_{m\neq\alpha} \left[\left(e^{i(p+q)} + e^{-ip} - 1 - e^{iq}\right) \theta^{\dagger}_{p-\frac{\pi}{2}} \theta_{p+q+\frac{\pi}{2}} - \left(e^{i(p+q)} + e^{-ip} + 1 + e^{iq}\right) \theta^{\dagger}_{p+\frac{\pi}{2}} \theta_{p+q-\frac{\pi}{2}}\right] \\
&+ 2\left(1 + e^{iq}\right) \theta^{\dagger}_{\frac{\pi}{2}-q} \theta_{\frac{3\pi}{2}} + 2\left(1 + e^{iq}\right) \theta^{\dagger}_{\frac{\pi}{2}} \theta_{q-\frac{\pi}{2}}.
\end{aligned}
\tag{4.57}
$$



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Put the Eq.(4.50) into result, we obtain

$$-\sum_{j=1}^{L} e^{-iqj} e_j = -\sum_{m\neq\alpha} [...] + 2e^{i\frac{q}{2}}\sqrt{\sin q}\left[\chi_0^\dagger(\chi_q+\eta_q)+(\chi_{\pi-q}^\dagger+\eta_{\pi-q}^\dagger)\eta_0\right]. \quad (4.58)$$

Hence,

$$H_{-1} = -\frac{L}{2\pi v_F}\left\{\sum_{m\neq\alpha} [...] + 2e^{i\frac{q}{2}}\sqrt{\sin q}\left[\chi_0^\dagger(\chi_q+\eta_q)+(\chi_{\pi-q}^\dagger+\eta_{\pi-q}^\dagger)\eta_0\right]\right\}. \quad (4.59)$$

Next, we try to calculate the $P_{-1}$. First, we start with the commutator:

$$\begin{aligned}[e_j, e_{j+1}] &= -\left[ic_j^\dagger c_{j+1} + ic_{j+1}^\dagger c_j + c_j^\dagger c_j - c_{j+1}^\dagger c_{j+1},\, ic_{j+1}^\dagger c_{j+2} + ic_{j+2}^\dagger c_{j+1} + c_{j+1}^\dagger c_{j+1} - c_{j+2}^\dagger c_{j+2}\right] \\ &= c_j^\dagger c_{j+2} - c_{j+2}^\dagger c_j - ic_j^\dagger c_{j+1} + ic_{j+1}^\dagger c_j + ic_{j+1}^\dagger c_{j+2} - ic_{j+2}^\dagger c_{j+1}.\end{aligned} \quad (4.60)$$

Then, turning the commutator into momentum space,

$$\begin{aligned}[e_j, e_{j+1}] &= \frac{1}{L}\sum_{pp'} e^{i(p'-p)j}\left(e^{i2p'} - e^{-i2p} - ie^{ip'} + ie^{-ip} - ie^{-ip}e^{2iR_2} + ie^{-i2p}e^{ip'}\right)\theta_p^\dagger\theta_{p'} \\ &= \frac{1}{L}\sum_{pp'} e^{i(p'-p)j}\left(i - e^{-ip}\right)\left(e^{-ip} - e^{ip'}\right)\left(1 + ie^{ip'}\right)\theta_p^\dagger\theta_{p'}.\end{aligned} \quad (4.61)$$


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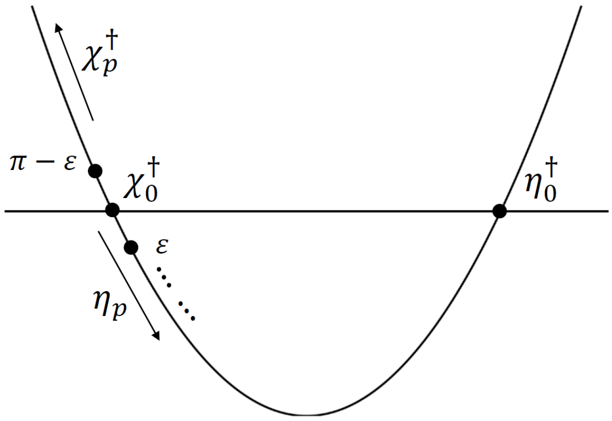


Figure 4.1: The spectrum of the many-body state and the roles of $\chi_0^\dagger$ and $\eta_p$ in this system.

And we sum over the commutator.

$$
\begin{aligned}
\sum_{j=1}^{L} e^{-iqj}[e_j, e_{j+1}] &= \frac{1}{L}\sum_{pp'j} e^{i(p'-p-q)j}\left(i-e^{-ip}\right)\left(e^{-ip}-e^{ip'}\right)\left(1+ie^{ip'}\right)\theta_p^\dagger\theta_{p'} \\
&= \sum_p \left[i-e^{-i\left(p+\frac{\pi}{2}\right)}\right]\left[e^{-i\left(p+\frac{\pi}{2}\right)}-e^{i\left(p+\frac{\pi}{2}+q\right)}\right]\left[1+ie^{i\left(p+q+\frac{\pi}{2}\right)}\right]\theta^\dagger_{p+\frac{\pi}{2}}\theta_{p+q+\frac{\pi}{2}} \\
&= \sum_{m=1}^{2N}\left(1+e^{-ip}\right)\left(e^{-ip}+e^{i(p+q)}\right)\left(1-e^{i(p+q)}\right)\theta^\dagger_{p+\frac{\pi}{2}}\theta_{p+q+\frac{\pi}{2}} \\
&= \sum_{m=1}^{N}\left(1+e^{-ip}\right)\left(e^{-ip}+e^{i(p+q)}\right)\left(1-e^{i(p+q)}\right)\theta^\dagger_{p+\frac{\pi}{2}}\theta_{p+q+\frac{\pi}{2}} \\
&+\left(1-e^{-ip}\right)\left(-e^{-ip}e^{i(p+q)}\right)\left(1+e^{i(p+q)}\right)\theta^\dagger_{p-\frac{\pi}{2}}\theta_{p+q-\frac{\pi}{2}} \\
&= \sum_{m\neq\alpha}[...] - 4ie^{iq}\sin q\left(\theta^\dagger_{\frac{\pi}{2}}\theta_{q+\frac{\pi}{2}}-\theta^\dagger_{\frac{3\pi}{2}-q}\theta_{\frac{3\pi}{2}}\right) \\
&= \sum_{m\neq\alpha}[...] - 4ie^{iq}\cos\left(\frac{q}{2}\right)\sqrt{\sin q}\left[\chi_0^\dagger\left(\chi_{L,q}-\eta_{L,q}\right)-\left(\chi^\dagger_{R,\pi-q}-\eta^\dagger_{R,\pi-q}\right)\eta_0\right]
\end{aligned}
\tag{4.62}
$$

Therefore,

$$
P_{-1} = -\frac{iL}{2\pi v_F^2}\left\{\sum_{m\neq\alpha}[...] - 4ie^{iq}\cos\left(\frac{q}{2}\right)\sqrt{\sin q}\left[\chi_0^\dagger\left(\chi_{L,q}-\eta_{L,q}\right)-\left(\chi^\dagger_{R,\pi-q}-\eta^\dagger_{R,\pi-q}\right)\eta_0\right]\right\}. \tag{4.63}
$$

Consider the many-body state spectrum as shown in Fig.4.1. The horizontal line represents zero energy; states below the line are filled with particles. There are two zero energy


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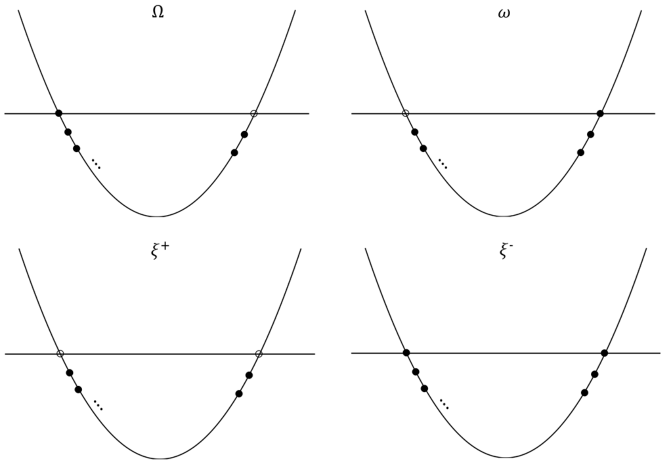

Figure 4.2: The spectrum of four ground states

states in the spectrum. The operators $\chi_0^\dagger$ and $\eta_0^\dagger$ correspond to filling a fermion in the left and right zero energy states, respectively. When the annihilation operator $\eta_p$ acts on the many-body state, it removes particles from the states below the zero energy level. Conversely, the operator $\chi_p^\dagger$ creates particles in the states above the zero energy level. Now we can define the four many-body ground states as follows:

1. $\xi^-$: $N-1$ particles fill up the Dirac sea, leaving two zero-energy states empty.

2. $\xi^+$: Similar to $\xi^-$ , but with the two zero-energy states filled.

3. $\Omega$: The left zero-energy state is filled, while the right one is empty.

4. $\omega$: The right zero-energy state is filled, while the left one is empty.

The spectrum of the four ground are shown as Fig.[4.2], meanwhile, these ground states can be arranged like Fig.[4.3]. Meanwhile, $\phi$ and $\psi$ are first excited state, which satisfy [8]

$$\phi = \chi^\dagger_{\pi-\epsilon}\Omega = \eta_\epsilon\Omega, \quad \psi = -\epsilon\chi^\dagger_{\pi-\epsilon}\omega = \epsilon\eta_\epsilon\omega. \tag{4.64}$$

The spectra of $\phi$ and $\psi$ are shown in Fig.[4.4]. Although the upper arrangement of the spectra differs from the lower one, both carry the same energy because we assume the


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$$
\begin{tikzcd}
 & \omega \arrow[dl, "\chi_0^\dagger"'] \arrow[dr, "-\eta_0"] \arrow[dd, "H"'] & \\
\xi^+ \arrow[dr, "\eta_0"'] & & \xi^- \arrow[dl, "\chi_0^\dagger"] \\
 & \Omega &
\end{tikzcd}
$$

Figure 4.3: The relation between four ground states.

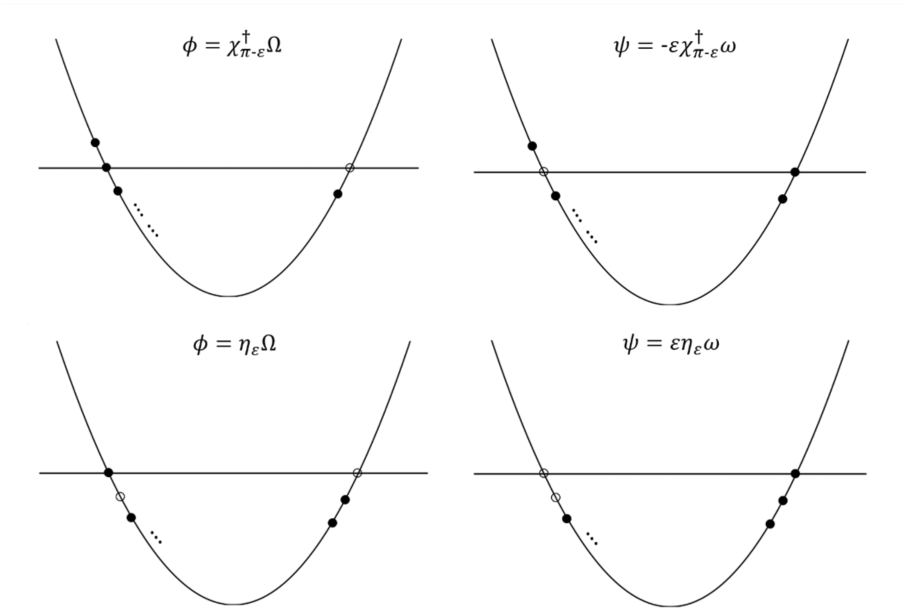


Figure 4.4: The spectrum of two first excited states.



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particles are very close together (i.e., the momentum step $\epsilon \to 0$). Finally, according to the analysis of $\chi_p$ and $\eta_p$ acting on different many-body states, we know that only the terms with $\chi_0$ and $\eta_0$ will not vanish when $H_{-1}$ and $P_{-1}$ act on $\xi$. Therefore,

$$
\begin{aligned}
&\big(L_{-1}+\bar{L}_1\big)\big|\,\xi\rangle = H_{-1}|\xi\rangle \\
&= -\frac{L}{\pi v_F}\left(e^{i\frac{q}{2}}\sqrt{\sin q}\left[\chi_0^\dagger(\chi_{L,q}+\eta_{L,q})+(\chi^\dagger_{R,\pi-q}+\eta^\dagger_{R,\pi-q})\eta_0\right]\right)|\xi\rangle \\
&= -\frac{1}{q}e^{i\frac{q}{2}}\sqrt{\sin q}|\phi\rangle, \\
&\big(L_{-1}-\bar{L}_1\big)\big|\,\xi\rangle = P_{-1}|\xi\rangle \\
&= -\frac{L}{\pi v_F^2}\left(2e^{iq}\cos\left(\frac{q}{2}\right)\sqrt{\sin q}\left[\chi_0^\dagger\left(\chi_{L,q}-\eta_{L,q}\right)-\left(\chi^\dagger_{R,\pi-q}-\eta^\dagger_{R,\pi-q}\right)\eta_0\right]\right)|\xi\rangle \\
&= -\frac{1}{q}e^{iq}\cos\frac{q}{2}\sqrt{\sin q}|\phi\rangle.
\end{aligned}
\tag{4.65}
$$

Finally, we obtain

$$
\langle\psi|L_{-1}|\xi\rangle = -\frac{1}{q}e^{iq}\cos\frac{q}{2}\sqrt{\sin q}\langle\psi|\phi\rangle. \tag{4.66}
$$

Hence, the indecomposability parameter of the Jordan cell associated with the first excited states (the "level-1" indecomposability parameter) is

$$
\beta_{1,5} = \lim_{q\to 0}\frac{|\langle\psi|L_{-1}|\xi\rangle|^2}{\langle\psi|\phi\rangle} = -1. \tag{4.67}
$$

Note that we use the assumption from [16], which is $\langle\psi|\phi\rangle = -q$.

In a similar manner, one can derive explicit lattice expressions for the indecomposability parameters associated with other excited states. As an illustrative case, for the states that, in the continuum limit, correspond to the primary fields and their logarithmic partners with chiral conformal dimension $h_{1,7}$, we obtain

$$
\beta_{1,7} = \lim_{q\to 0}[-18+O(q^2)] = -18. \tag{4.68}
$$


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The lattice computations of the indecomposability parameters at the first two levels, $\beta_{1,5}$ and $\beta_{1,7}$, quantitatively match the CFT benchmarks of the symplectic fermion theory with $c = -2$: from the field‑theory side one has $\beta_{1,5} = -1$ and $\beta_{1,7} = -18$, while our lattice evaluations yield the same values. This agreement, together with the verified Virasoro structure and $c = -2$, strongly indicates that the infrared fixed point of our non-Hermitian model lies in the symplectic fermion LCFT universality class.


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# Chapter 5 Summary

In this thesis we set out to explore a very simple question with surprisingly rich consequences: what happens to conformal symmetry when Hermiticity is broken? In ordinary one-dimensional quantum systems we are used to the idea that if a system is gapless and has linear dispersion, then the long-distance theory is a conformal field theory. But in the non-Hermitian setting, this connection is no longer automatic. The possibility of non-diagonalizable Hamiltonians and the appearance of exceptional points bring new structures into play, and one is naturally led to the territory of logarithmic conformal field theories.

The model we studied was deliberately minimal: a free fermion in one spatial dimension with a non-Hermitian mass term and a second-derivative term, but without the standard first-derivative kinetic term. Despite its unusual appearance, the continuum spectrum turned out to be gapless with linear dispersion. This raised the tantalizing possibility that the system might hide a conformal structure. By explicitly constructing Virasoro generators, both in the continuum theory and on the lattice using the Koo–Saleur approach, we showed that the algebra does indeed close and that the central charge is $c = -2$. This already placed the theory within the family of logarithmic CFTs, which are known to be non-unitary and to possess Jordan-cell structures in their Hilbert spaces.


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An important part of the story was the role of indecomposability. In a Hermitian CFT, the Hilbert space typically splits neatly into irreducible representations. Here, by contrast, we found indecomposable modules where $L_0$ acts non-diagonalizably. This was made quantitative by computing the indecomposability parameter on the lattice, which measures the overlap between a primary state and its logarithmic partner. The remarkable result was that our model produced the same value as in the symplectic fermion theory, a classic example of a $c = -2$ logarithmic CFT. This showed that our non-Hermitian free fermion and the symplectic fermion belong to the same universality class, despite their very different starting points.

Stepping back, the lessons of this work are broader than the details of the model. The main message is that gaplessness alone does not guarantee standard conformality once Hermiticity is given up. Instead, new universality classes can emerge, governed by logarithmic CFTs with indecomposable structures. Entanglement entropy scaling, which often suggests conformality, needs to be supplemented by direct algebraic checks of the Virasoro symmetry and by computation of invariants such as the indecomposability parameter. When these are in place, one can be confident that a non-Hermitian model is genuinely conformal, even if its Hilbert space looks very different from the familiar Hermitian case.

Looking forward, there are many directions that build naturally on what we have done here. Adding interactions or disorder could reveal new non-unitary universality classes. Studying higher-dimensional or more symmetric models might expose other kinds of logarithmic structures. And since non-Hermitian Hamiltonians often arise as effective models of open or driven systems, there may be experimental routes to realizing such logarithmic CFTs in photonic or cold-atom setups. From a computational point of view, the ability to measure indecomposability on the lattice opens the door to numerical approaches, such as


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tensor-network simulations adapted to non-Hermitian systems.

To conclude, the thesis has shown that even a simple non-Hermitian free-fermion model can lead to a rich and surprising conformal structure. What began as an unusual Hamiltonian with an odd combination of terms turned out to realize a well-known logarithmic CFT with $c = -2$. By combining analytic field theory, algebraic analysis, and lattice computations, we were able to demonstrate this connection in detail. The results highlight how conformality can survive in non-Hermitian critical systems, but in a form that is indecomposable and non-unitary. This adds another piece to the puzzle of how universality works beyond Hermitian quantum mechanics, and points the way to further explorations at the boundary between condensed matter, quantum field theory, and mathematical physics.


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