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多物種的羅特卡-弗爾特拉擴散競爭方程組之非單調行波解

Non-monotone traveling wave solutions for the n-species Lotka–Volterra competitive system with diffusion

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口試委員會審定書

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本論文係吳政軒君（R11221019）在國立臺灣大學數學系完成之碩士學位論文，於民國 113 年 6 月 25 日承下列考試委員審查通過及口試及格，特此證明

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所　　長：

摘要

這篇論文主要研究多物種的羅特卡-弗爾特拉擴散競爭系統 (Lotka-Volterra competitive system with diffusion)。我們透過研究行波解以了解該系統，並成功證明連接 $\vec{O} := (0, 0, \cdots, 0)$ 和 $\vec{e_1} := (1, 0, \cdots, 0)$ 兩個平衡態的非單調解的存在性。關於這方面的研究在過去的文獻中相當稀少。然而，這類非單調解在生態學中具有重要意義，它可以啟發我們發現一些特殊現象。我們主要的研究方法為利用 Schauder 不動點定理，以及合適的上下解來證明解的存在性，並通過縮小區間的方法來描述 $z \to \infty$ 時的漸近行為。另外，透過證明不存在速度小於某個特定值 s^* 的解，我們找出該系統行波解的最小速度。

關鍵字： 羅特卡-弗爾特拉競爭、上下解

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Abstract

This focuses on the n-species Lotka-Volterra competitive system with diffusion. Understanding traveling wave solutions is essential for gaining insights into this dynamical system. We successfully show the existence of non-monotonic pulse-front traveling wave solutions that connect two equilibriums $\vec{O} := (0, \cdots, 0)$ and $\vec{e_1} := (1, 0, \cdots, 0)$. These solutions are significant in ecology and can inspire the exploration of other intriguing phenomena within the Lotka-Volterra system. To prove the existence of traveling wave solutions, we rely on the application of the Schauder fixed-point theorem and appropriate upper-lower solutions. A key breakthrough in our work is the construction of these suitable upper-lower solutions for the competition system. Additionally, the concept of shrinking rectangles is employed to deduce the asymptotic behavior when $z \to \infty$. Furthermore, by proving the non-existence of traveling wave solutions at speeds below a critical threshold s^*, we identify the minimum speed of the traveling wave solutions for this model.

Keywords: Lotka−Volterra competitive system, upper-lower-solution

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1 Introduction

Competition systems are crucial for describing interspecific and intraspecific competition through mathematical equations and simulations. They offer a quantitative insight into species interactions within ecosystems. By comprehending competition systems, biologists can more accurately predict and explain phenomena such as species richness, species distribution, and evolutionary dynamics within ecological systems. Many studies on competition systems, such as those referenced in [1], [7], and [11], have been conducted in recent years.

In this paper, we consider the n-species competition with diffusion below:

$$\begin{cases} (u_1)_t = d_1(u_1)_{xx} + r_1u_1(1 - \sum\limits_{j=1}^{n} c_{1j}u_j),\ x \in \mathbb{R}, t > 0, \\ (u_2)_t = d_2(u_2)_{xx} + r_2u_2(1 - \sum\limits_{j=1}^{n} c_{2j}u_j),\ x \in \mathbb{R}, t > 0, \\ \quad \vdots \\ (u_n)_t = d_n(u_n)_{xx} + r_nu_n(1 - \sum\limits_{j=1}^{n} c_{nj}u_j),\ x \in \mathbb{R}, t > 0, \end{cases} \tag{1.1}$$

with $u = (u_1, \cdots, u_n)$. In this system, the diffusion rates d_i, the intrinsic growth rates r_i, and the competition coefficients c_{ij} are considered to be positive for all $i, j = 1, \cdots, n$. We assumed $c_{ii} = 1\,\forall\, i = 1, \cdots, n$ and $0 < \sum\limits_{j=2}^{n} c_{1j} < 1 < c_{i1}\,\forall\, i > 1$.

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To know more about this system, we study the existence of traveling wave solutions of (1.1). We denote $z := x - st$ and let $u(x,t) := (u_i(x,t))_{i=1}^n = (\phi_i(z))_{i=1}^n =: \phi(z)$ which connects $\vec{O} = (0, \cdots, 0)$ and $\vec{e_1} = (1, 0, \cdots, 0)$. That is, $\phi(-\infty) := \lim_{z \to -\infty} \phi(z) = \vec{O}$ and $\phi(\infty) := \lim_{z \to \infty} \phi(z) = \vec{e_1}$. Then, we have the ODE system below:

$$\begin{cases} d_1\phi_1'' + s\phi_1' + r_1\phi_1(1 - \sum_{j=1}^{n} c_{1j}\phi_j) = 0, \ z \in \mathbb{R}, \\ d_2\phi_2'' + s\phi_2' + r_2\phi_2(1 - \sum_{j=1}^{n} c_{2j}\phi_j) = 0, \ z \in \mathbb{R}, \\ \quad \vdots \\ d_n\phi_n'' + s\phi_n' + r_n\phi_n(1 - \sum_{j=1}^{n} c_{nj}\phi_j) = 0, \ z \in \mathbb{R}, \end{cases} \tag{1.2}$$

with $\phi \in [0,1]$. The conditions about c_{ij} above cause equation (1.2) to be unstable at $\vec{O}$ but stable at $\vec{e_1}$. Note that the wave speed s must be determined when dealing with this system.

Set $s^* := \min_{i=1,\cdots,n}\{-2\sqrt{d_i r_i}\}$. Then s^* is the minimal wave speed in this research. That is, there exists a solution ϕ to be the solution of the system (1.2) only if $s \leq s^*$. Moreover, we use Schauder's fixed-point theorem in conjunction with a suitable upper-lower solution to prove the existence of ϕ. This method has been used on many research like [9], [12] and [14].

In this paper, we write $u := (u_1, \cdots, u_n) > v := (v_1, \cdots, v_n) > c$ if $u_i > v_i > c \,\forall\, i$, and $<, \geq, \leq$ use the similar definition. We use the notation $\sum_i$ to sum all $i = 1, \cdots, n$ and $\sum_{j \neq i}$ to sum $j = 1$ to $j = n$ except $j = i$.

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2 General Theory

In this chapter, we will define upper-lower solutions $(\overline{\phi}, \underline{\phi})$. First, we define two function spaces X and X_0 below:

$$X := \{\phi = (\phi_1, \cdots, \phi_n) | \phi : \mathbb{R} \to \mathbb{R}^n \text{ is continuous}\},$$

$$X_0 := \{0 \leq \phi \leq 1 : \phi \in X\}.$$

Next, we define $F = (F_1, \cdots, F_n)$ for some constant $\lambda > r_i \sum_{j=1}^{n} c_{ij} \, \forall \, i = 1, \cdots, n$:

$$F_i(y) := \lambda y_i + r_i y_i (1 - \sum_{j=1}^{n} c_{ij} y_j) \, \forall \, i = 1, \cdots, n.$$

Therefore, F_i is increasing only with respect to y_i, but decreasing with respect to $y_j \, \forall \, j \neq i$, for $0 \leq y \leq 1$ and any $i = 1, \cdots, n$. We can rewrite (1.2) as

$$\begin{cases} d_1 \phi_1'' + s\phi_1' - \lambda \phi_1 + F_1(\phi) = 0, \; z \in \mathbb{R}, \\ d_2 \phi_2'' + s\phi_2' - \lambda \phi_2 + F_2(\phi) = 0, \; z \in \mathbb{R}, \\ \quad \vdots \\ d_n \phi_n'' + s\phi_n' - \lambda \phi_n + F_n(\phi) = 0, \; z \in \mathbb{R}. \end{cases} \tag{2.1}$$

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Consider the operator $P = (P_1, \cdots, P_n) : X \to X_0$ below:

$$P_i(\phi)[z] := \frac{1}{d_i(v_i^+ - v_i^-)} \left(\int_{-\infty}^{z} v_i^- e^{v_i^-(z-t)} + \int_{z}^{\infty} v_i^+ e^{v_i^+(z-t)} \right) F_i(\phi)[t] dt \, \forall\, i, \quad (2.2)$$

where $v_i^\pm$ is different real roots of $d_i x^2 + sx - \lambda = 0$. Clearly, (2.2) satisfies the equation $d_i P_i'' + sP_i' - \lambda P_i + F_i(\phi) = 0 \,\forall\, i$. Therefore, if ϕ is a fixed point of P, then ϕ is a solution of (2.1).

Definition 1. $(\overline{\phi}, \underline{\phi}) = ((\overline{\phi_1}, \cdots, \overline{\phi_n}), (\underline{\phi_1}, \cdots, \underline{\phi_n}))$ is called a pair of upper-lower solution of (1.2) if $\overline{\phi}, \underline{\phi} \in X_0$ and their first and second derivative are both bounded such that the following inequalities hold:

$$\begin{aligned} U_i(z) &:= d_i \overline{\phi_i}'' + s\overline{\phi_i}' + r_i \overline{\phi_i}(1 - \overline{\phi_i} - \sum_{j \neq i} c_{ij} \underline{\phi_j}) \leq 0, \\ L_i(z) &:= d_i \underline{\phi_i}'' + s\underline{\phi_i}' + r_i \underline{\phi_i}(1 - \underline{\phi_i} - \sum_{j \neq i} c_{ij} \overline{\phi_j}) \geq 0, \end{aligned} \quad (2.3)$$

for all $i = 1, \cdots, n$ and $z \notin E$, where E is a finite set in $\mathbb{R}$.

Lemma 1. Define the set $[g, h] := \{f \text{ is continuous} : g \leq f \leq h\}$ for some functions g, h. Given $s < 0$ and $(\overline{\phi}, \underline{\phi})$ is an upper-lower solution of (1.2) such that:

$$\begin{aligned} &(1)\ \overline{\phi} \geq \underline{\phi} \,\forall\, z \in \mathbb{R}. \\ &(2)\ \overline{\phi}'(z^-) \geq \overline{\phi}'(z^+) \text{ and } \underline{\phi}'(z^-) \leq \underline{\phi}'(z^+) \,\forall\, z \in E. \end{aligned} \quad (2.4)$$

Then there exists a fixed point $\phi = (\phi_1, \cdots, \phi_n) \in [\underline{\phi}, \overline{\phi}]$ of P; therefore, ϕ is a solution of (1.2).

Proof. We can check that P maps $\Phi := \{\phi \in [\underline{\phi}, \overline{\phi}]\}$ to itself, and it is completely continuous with respect to some complete weighted normed space on the nonempty bounded convex closed set Φ. By applying the Schauder fixed-point theorem, we can conclude that

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there exists a fixed point ϕ for the operator P. Therefore, ϕ is a solution of (1.2).

More detailed information can be located in the Lemma 2.3 of the reference [4] when considering $k = 1$ as $\alpha \in (0, \min_{i=1,\cdots,n}\{-c_{i1}\})$.

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3 Upper-lower Solution

In this chapter, we follow the construction in [15] and give a pair of upper-lower solution $(\overline{\phi}, \underline{\phi})$ of (1.2) for $s \le s^*$, then the solution ϕ exists by Lemma 1. First, we set some variables for any $i = 1, \cdots, n$ below:

$$f_i(x) := d_i x^2 + sx + r_i,$$

$$\alpha_i^{\pm} := \frac{-s \pm \sqrt{s^2 - 4d_i r_i}}{2d_i} \text{ are two real roots of } f_i.$$

We write α_i instead of $\alpha_i^{\pm}$ if $\alpha_i^+ = \alpha_i^-$.

For $s \le s^*$, set

$$h_i(z) := \begin{cases} e^{\alpha_i^- z} - q_i e^{\mu_i \alpha_i^- z} & \text{if } \alpha_i^+ > \alpha_i^- \\ (-\alpha_i e z - q_i(-z)^{1/2}) e^{\alpha_i z} & \text{if } \alpha_i^+ = \alpha_i^- \end{cases},$$

$$\gamma_i := \begin{cases} \frac{-\ln q_i}{(\mu_i - 1)\alpha_i^-} & \text{if } \alpha_i^+ > \alpha_i^- \\ -(\frac{q_i}{\alpha_i e})^2 & \text{if } \alpha_i^+ = \alpha_i^- \end{cases} \text{ is the unique zero of } h_i,$$

$$z_M := \arg\max(h_1) < \gamma_1,$$

for some constants $q_i, \mu_i > 1$ with the inequality $\alpha_i^- \mu_i < \alpha_i^- + \alpha_j^- \ \forall\, i, j$. We do not require $i \ne j$, so the inequality says that $\mu_i < \frac{\alpha_i^- + \alpha_j^-}{\alpha_i^-} = 2$ when $i = j$.

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Define $\overline{\phi}, \underline{\phi}$ below:

$$\overline{\phi_i} := \begin{cases} \begin{cases} 1 & \text{if } z \geq -\alpha_i^{-1} \\ -\alpha_i e z \cdot e^{\alpha_i z} & \text{if } z < -\alpha_i^{-1} \end{cases} & \text{when } \alpha_i^{\pm} = \alpha_i \\ \begin{cases} 1 & \text{if } z \geq 0 \\ e^{\alpha_i^- z} & \text{if } z < 0 \end{cases} & \text{when } \alpha_i^+ > \alpha_i^- \end{cases} \quad \forall\, i \geq 1, \tag{3.1}$$

$$\underline{\phi_1} := \begin{cases} \delta & \text{if } z \geq z_1 \\ h_i(z) & \text{if } z < z_1 \end{cases} \text{ and } \underline{\phi_i} := \begin{cases} 0 & \text{if } z \geq \gamma_i \\ h_i(z) & \text{if } z < \gamma_i \end{cases} \quad \forall\, i > 1,$$

where $z_1 \in (z_M, \gamma_1)$ with $h_1'(z_1) < 0$ such that $0 < \delta = h_1(z_1) < 1 - \sum_{j \neq 1} c_{1j}$.

Theorem 1. Given $s \leq s^*$ and suitable parameters $\mu_1, \cdots, \mu_n$, there exists $q_1, \cdots, q_n$ such that (3.1) is a pair of upper-lower solution of (1.2).

Proof. We need to check (2.4) and (2.3) hold. All of them are easy by direct computation except $L_i \geq 0$ when $z < \gamma_i$ for any i. Note that $-\alpha_i^{-1} e z \geq 1 \,\forall\, z \leq -\alpha_i^{-1}$, so we estimate that $\overline{\phi_j} \leq -\alpha_i e z \cdot e^{\alpha_i z}$ for any $j \neq i$. We divide it to four cases and using the fact that $\sup_{z \leq 0} (-z)^{\lambda} e^{\eta z} = (\frac{\lambda}{\eta e})^{\lambda}$ for positive λ, η in case 2 and 4.

Case 1: $\alpha_i^+ > \alpha_i^-$ and $z \geq -\alpha_j^{-1}$ for some $j \neq i$.

$$\begin{aligned} L_i &= d_i h_i'' + s h_i' + r_i h_i (1 - h_i - \sum_{j \neq i} c_{ij} \overline{\phi_j}) \\ &= -q_i e^{\mu_i \alpha_i^- z} f_i(\mu_i \alpha_i^-) - r_i h_i (h_i + \sum_{j \neq i} c_{ij} \overline{\phi_j}) \\ &\geq -q_i e^{\mu_i \alpha_i^- z} f_i(\mu_i \alpha_i^-) - r_i e^{\alpha_i^- z} (\sum_{j \neq i} c_{ij} + e^{\alpha_i^- z}) \\ &\geq 0 \quad \text{if } q_i \geq \frac{r_i e^{\alpha_i^- z}}{-e^{\mu_i \alpha_i^- z} f_i(\mu_i \alpha_i^-)} (\sum_{j \neq i} c_{ij} + e^{\alpha_i^- z}) \,\forall\, z \in [\min_{j \neq i}(-\alpha_j^{-1}), 0]. \end{aligned}$$

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Case 2: $\alpha_i^+ > \alpha_i^-$ and $z < -\alpha_j^{-1}\, \forall\, j \neq i$.

$$
\begin{aligned}
L_i &= -q_i e^{\mu_i\alpha_i^- z} f_i(\mu_i\alpha_i^-) - r_i h_i(h_i + \sum_{j\neq i} c_{ij}\overline{\phi_j}) \\
&\geq -q_i e^{\mu_i\alpha_i^- z} f_i(\mu_i\alpha_i^-) - r_i e^{\alpha_i^- z}(e^{\alpha_i^- z} - \sum_{j\neq i} c_{ij}\alpha_j e z \cdot e^{\alpha_j z}) \\
&= e^{\mu_i\alpha_i^- z}\left(-q_i f_i(\mu_i\alpha_i^-) - r_i e^{(2-\mu_i)\alpha_i^- z} - r_i \sum_{j\neq i} c_{ij}\alpha_j e(-z)\cdot e^{(\alpha_i^- + \alpha_j - \mu_i\alpha_i^-)z}\right) \\
&\geq e^{\mu_i\alpha_i^- z}\left(-q_i f_i(\mu_i\alpha_i^-) - r_i - r_i \sum_{j\neq i} \frac{c_{ij}\alpha_j}{\alpha_i^- + \alpha_j - \mu_i\alpha_i^-}\right) \\
&\geq 0 \quad \text{if } q_i \geq \frac{r_i}{-f_i(\mu_i\alpha_i)}\left(1 + \sum_{j\neq i}\frac{c_{ij}\alpha_j}{\alpha_i^- + \alpha_j - \mu_i\alpha_i^-}\right).
\end{aligned}
$$

Case 3: $\alpha_i^\pm = \alpha_i$ and $z \geq -\alpha_j^{-1}$ for some $j \neq i$.

$$
\begin{aligned}
L_i &\geq d_i\underline{\phi_i}'' + s\underline{\phi_i}' + r\underline{\phi_i} - r_i\underline{\phi_i}(\sum_{j\neq i} c_{ij} + \underline{\phi_i}) \\
&\geq q_i\cdot\frac{1}{4}d_i(-z)^{-3/2}e^{\alpha_i z} + r_i\alpha_i e z\cdot e^{\alpha_i z}(\sum_{j\neq i} c_{ij} - \alpha_i e z\cdot e^{\alpha_i z}) \\
&\geq 0 \quad \text{if } q_i \geq \frac{4r_i\alpha_i e}{d_i}(-z)^{5/2}(\sum_{j\neq i} c_{ij} - \alpha_i e z\cdot e^{\alpha_i z})\,\forall\, z \in [\min_{j\neq i}(-\alpha_j^{-1}), 0].
\end{aligned}
$$

Case 4: $\alpha_i^\pm = \alpha_i$ and $z < -\alpha_j^{-1}\, \forall\, j \neq i$.

$$
\begin{aligned}
L_i &\geq q_i\cdot\frac{1}{4}d_i(-z)^{-3/2}e^{\alpha_i z} - r_i\alpha_i e z\cdot e^{\alpha_i z}(\alpha_i e z\cdot e^{\alpha_1 z} + \sum_{j\neq i} c_{ij}\alpha_j e z\cdot e^{\alpha_j z}) \\
&= (-z)^{-3/2}e^{\alpha_i z}\left(\frac{1}{4}q_i d_i - (-z)^{7/2} r_i\alpha_i e^2 \sum_j c_{ij}\alpha_j e^{\alpha_j z}\right) \\
&\geq (-z)^{-3/2}e^{\alpha_i z}\left(\frac{1}{4}q_i d_i - r_i\alpha_i e^2 \sum_j (\frac{7}{2\alpha_j e})^{7/2}\right) \\
&\geq 0 \quad \text{if } q_i \geq \frac{4r_i\alpha_i e^2}{d_i}\sum_j(\frac{7}{2\alpha_j e})^{7/2}.
\end{aligned}
$$

By these four cases, we finish the proof that (3.1) is a pair of upper-lower solution of (1.2) when $q_1, \cdots, q_n$ are all big enough. Therefore, there exists a solution ϕ of (1.2).

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We give an example of upper-lower solution here. Given $n = 3$ and coefficients $d_i = i, r_i = 2^{3-i}$ and $c_{ij} = i/j$:

$$\begin{cases} \phi_1'' + s\phi_1' + 4\phi_1(1 - \phi_1 - \frac{1}{2}\phi_2 - \frac{1}{3}\phi_3) = 0, \\ 2\phi_2'' + s\phi_2' + 2\phi_2(1 - 2\phi_1 - \phi_2 - \frac{2}{3}\phi_3) = 0, \\ 3\phi_3'' + s\phi_3' + \phi_3(1 - 3\phi_1 - \frac{3}{2}\phi_2 - \phi_3) = 0. \end{cases}$$

In this system, $s^* = -4$. When $s = s^* = -4$, then $\alpha_1 = 2, \alpha_2 = 1$ and $\alpha_3^{\pm} = 1, 1/3$. We can choose $\mu_i = 1.1\,\forall\, i,\ q_1 = 30000, q_2 = 3500, q_3 = 400$ and $\delta = \exp(-10^9)$ by the above estimate.

We show the upper-lower solutions of ϕ_1, ϕ_2 and ϕ_3 below by Geogebra:

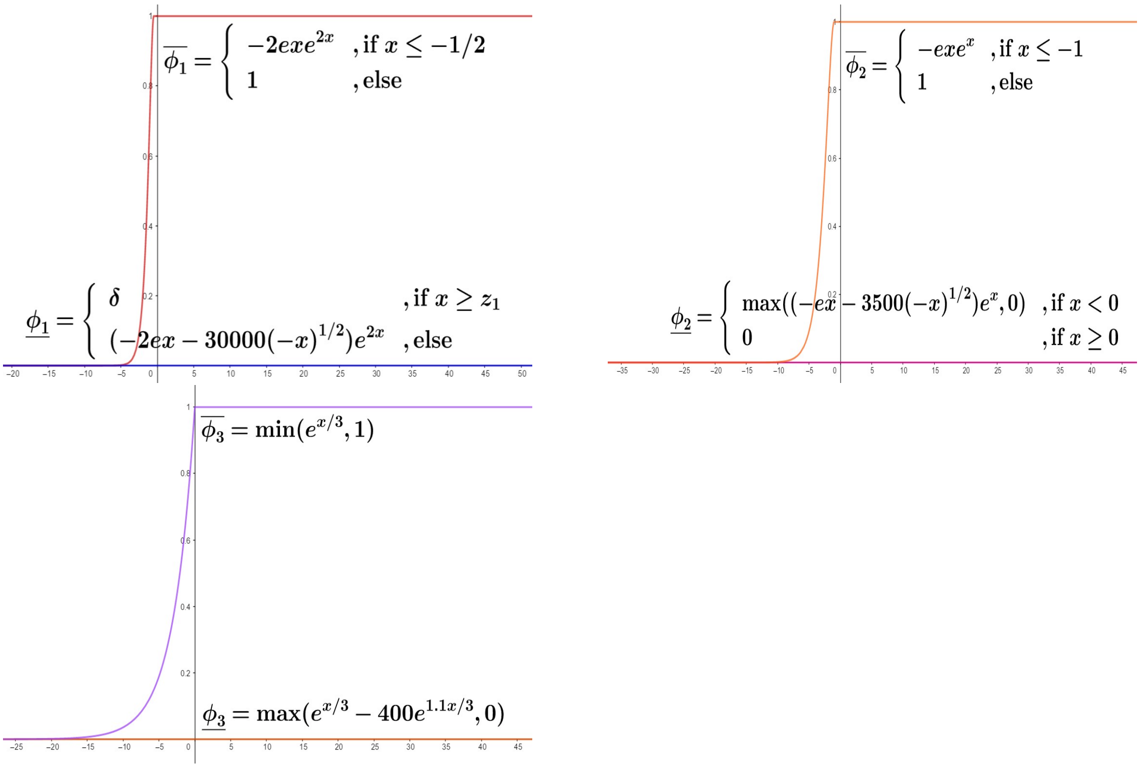

Figure 3.1: The graph of upper-lower solutions of ϕ_1, ϕ_2 and ϕ_3

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4 Asymptotic Behavior

In this chapter, we will show the asymptotic behavior of the solution ϕ by shrinking rectangles, which way is similar to [3] and [6].

Theorem 2. Given $s \leq s^*$. The solution ϕ of (1.2) with $\underline{\phi} \leq \phi \leq \overline{\phi}$ connects $\vec{O}$ and $\vec{e_1}$, where $(\overline{\phi}, \underline{\phi})$ is given at (3.1).

Proof.

Define $\phi^- = \liminf_{z\to\infty} \phi(z)$ and $\phi^+ = \limsup_{z\to\infty} \phi(z)$ for the solution of (1.2). Let $m_1 = (1-\theta)(\delta-\varepsilon^2)+\theta$, $M_1 = (1-\theta)(1+\varepsilon^2)+\theta$ and $m_i = 0$, $M_i = (1-\theta)(1+\varepsilon^2)\,\forall\, i > 1$ for some $\sqrt{\delta} > \varepsilon > 0$. Then $m \nearrow \vec{e_1}$ and $M \searrow \vec{e_1}$ and $\theta \to 1^-$ with $m(0) \leq \phi^- \leq \phi^+ < M(0)$.

Define $\theta_0 := \sup\{0 \leq \theta < 1 : m(\theta) \leq \phi^- \leq \phi^+ < M(\theta)\}$. Then either $M_i(\theta_0) = \phi^+$ for some i or $m_1(\theta_0) = \phi_1^-$ must hold. We will show that each of them leads the contradiction. Since the way to get the contradiction from these cases is similar, we only deal with the case $m_1(\theta_0) = \phi_1^-$.

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First, we set functions ρ_1 and σ_i of θ for $\theta < 1$ and $i = 1, \cdots, n$ below:

$$
\begin{aligned}
\rho_1 &:= 1 - m_1 - \sum_{j>1} c_{1j} M_j = -\delta(1-\theta) + \sum_{j>1} (1 - c_{1j}) M_j > 0 \\
\sigma_i &:= 1 - M_i - \sum_{j \neq i} c_{ij} m_j = 1 - c_{i1} m_1 + m_i - M_i \\
&= \begin{cases} -(1-\theta)\varepsilon^2 & \text{if } i = 1 \\ 1 - c_{i1}\big((1-\theta)(\delta - \varepsilon^2) + \theta\big) - (1-\theta)(1+\varepsilon^2) & \text{if } i > 1 \end{cases} \\
&< 0 \quad \text{since } c_{i1} \geq 1
\end{aligned}
$$

If ϕ_1 is monotone for large z, then $\phi_1(\infty) = m_1(\theta_0)$ and

$$
\begin{aligned}
\int_0^n r_1 \phi_1 (1 - \sum_j c_{1j} \phi_j) dz = \int_0^n -d_1 \phi_1'' - s\phi_1' dz \\
= -d_1(\phi_1'(n) - \phi_1'(0)) - s(\phi_1(n) - \phi_1(0))
\end{aligned}
$$

Since $\liminf\limits_{z\to\infty} \phi_1(1 - \sum\limits_j c_{1j}\phi_j) \geq m_1(\theta_0)\rho_1 > 0$, the left hand side goes to ∞ when $n \to \infty$ but the right hand side is uniform bounded, which is a contradiction.

For the other case that ϕ is oscillatory for large z, we choose an increasing sequence of local minimum points $\{z_n\}$. Then $\phi_1''(z_n) \geq 0, \phi_1'(z_n) = 0 \, \forall \, n$, and $\phi_1(z_n) \to m_1(\theta_0)$. Then $0 = \liminf\limits_{n\to\infty} \big(d_1\phi_1'' + s\phi_1' + r_1\phi_1(1 - \sum\limits_j c_{1j}\phi_j)\big)(z_n) \geq r_1 m_1(\theta_0)\rho_1 > 0, (\rightarrow\leftarrow)$.

Therefore $\theta_0 = 1$ and $\vec{e_1} = m(1) \leq \phi^- \leq \phi^+ \leq M(1) = \vec{e_1}$, which implies that $\phi^- = \phi^+ = \vec{e_1}$. That is, $\phi(\infty)$ exists and equal to $\vec{e_1}$.

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5 Minimal Speed

In this chapter, we will show that there is no solution of (1.2) when $s > s^*$ by contradiction, which implies that s^* is the minimal speed in this system.

Theorem 3. There is no solution of (1.2) for all $s > s^*$.

Proof.

We divide it into two cases: either $s \in (s^*, 0)$ or $s \geq 0$. When the first case occurs, one of the linear terms in (1.2) oscillates on the left side. Then by Hartman-Grobman theorem [8], this solution is also oscillatory on the left side and does not converge to 0.

When the another case happens, since $\phi(-\infty) = \vec{O}$, then $1 - \sum\limits_j c_{1j}\phi_j > 0 \,\forall\, z < -N$. Integrate one part of the first equation in (1.2) and get

$$
\begin{aligned}
0 &< r_1 \int_{-\infty}^{x} \phi_1(1 - \sum_j c_{1j}\phi_j)dz \qquad \forall\, x < -N \\
&= \int_{-\infty}^{x} -d_1\phi_1'' - s\phi_1' dz \leq -d_1\phi_1'(z)
\end{aligned}
$$

Therefore $0 < \int_{-\infty}^{-N} -d_1\phi_1'(z)dz = -d_1\phi_1(-N) \leq 0, \quad (\rightarrow\leftarrow)$.

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