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量子計量學中適界的可達性

Achievability of Tight Bound in Quantum Metrology

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本論文係 江丞澤 (姓名) R11942055 (學號) 在國立臺灣大學電信工程學研究所完成之碩士學位論文，於民國 113 年 6 月 24 日承下列考試委員審查通過及口試及格，特此證明。

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摘要

在量子系統中高精度地估計參數是許多領域的重要工作，從工程、物理到生物學皆是如此。Holevo-Cramer-Rao界限（HCRB）是量子計量學中的一個重要的適界（tight bound）。一般來說，它在漸近（asymptotical）情況下是可達到的（achievable），但在有限樣本（finite copy）——特別是單樣本（single shot）——的情況下並不總是可達到的。在這本論文中，我們刻畫了估計量（estimator）和最佳測量在單樣本下飽和HCRB的幾個必要/充分條件。特別是，我們提供了兩個關於HCRB可達性的刻畫。其中之一（**定理4.1**）作為物理圖像和達到HCRB的充分條件。我們提供了一個條件（**定理4.1**中的條件5），它作為該定理的刻畫。在某些條件下，這種刻畫可以簡化為其他量子Cramer-Rao界限的可達性必要條件（見頁碼30的討論）。然而，總的來說，該定理並不是HCRB可達性的必要條件。另一個刻畫（**定理5.1**）提供了HCRB可達性的數學刻畫，這是必要且充分的條件。通過這種刻畫（**定理5.1**），其他量子Cramer-Rao界限的可達性可以很容易地確立（**推論5.2**和**推論5.3**）。

關鍵詞： 量子計量學，Holevo-Cramer-Rao界限，可達性。



Abstract

Estimating parameter in quantum system with high precision is an important work in many field, ranging from engineering, physics to biology. The Holevo Cramer-Rao bound (HCRB) is an important tight bound in quantum metrology. In general, it is asymptotic achievable, while it is not always achievable with finite copy, in particular, single shot level. In this work, we characterize several necessary/sufficient conditions for the estimator and the optimal measurement to saturate the HCRB in single copy level. In particular, we develop 2 characterizations for the achievability of the HCRB. First one of them (**Theorem 4.1**) is served as a physical picture and sufficient condition for saturating the HCRB. We provide a condition (condition 5 in **Theorem 4.1**) which is served as a characterization of the theorem. With some condition, this characterization can be reduced to a necessary condition for the achievability of the other quantum Cramer-Rao bound (see discussion on page 30). In general, however, the theorem is not a necessary condition for the achievability of the HCRB. Second one of them (**Theorem 5.1**) gives a mathematical characterization for the saturation of the HCRB, it is necessary and sufficient condition. With this characterization (**Theorem 5.1**), the achievability of other quantum Cramer-Rao bound can be easily established (**Corollary 5.2** and **Corollary 5.3**).

Keywords: Quantum metrology, Holevo-Cramer-Rao bound, Achievability.





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Introduction to the Thesis

Quantum metrology is a field that focuses on estimating parameters in physical systems with high precision [DGG20; Liu+19a; Liu+22; MD16; SK20; TB16]. It has a wide range of applications, including gravitational wave detection [Adh14; DBS13], quantum imaging [Alb+20], phase estimation [BD16; Pez+17; YZF14], channel estimation [AD22; EMD11; Gór+20; LJW15; ZJ21], etc.

In quantum metrology, people establish various quantum versions of the Cramér-Rao bound (CRB) and design many protocols to saturate them, including the symmetric logarithmic derivative CRB (SLD CRB) [BC94; Hel67; Hel69], right logarithmic derivative CRB (RLD CRB) [Hol11; YL73], Holevo-Cramér-Rao bound (HCRB) [AFD19; Hay05; Hol11; Suz16; Suz19], and Nagaoka-Hayashi bound (NHB) [Con+21; Con+23; Nag05]. The SLD CRB has a wide range of applications in quantum metrology. In the single parameter case, it is always saturable [Liu+19b; Par09]. In phase estimation and unitary parameter estimation [Hum+13; Pez+17; YZF14], the SLD CRB is also saturable in the asymptotic regime. However, it is not always achievable for general state/channel estimation problems, even in the asymptotic regime. The RLD CRB is particularly useful for Gaussian state estimation and estimation in the \mathcal{D} -invariant model [Suz16; Suz19; YL73], but it is not necessarily achievable in both finite and asymptotic regimes. The HCRB is probably the most important bound in general quantum metrology. It is not always achievable with a finite number of copies, but it is always asymptotically achievable for *any* quantum statistical model with some mild regularity conditions [HM08; KG09; YCH19; YFG13]. Despite the HCRB being asymptotically achievable, it requires collective measurement in a large number of copies, which is extremely hard to implement in practical usage [Con+21; Con+23; YHT+20]. Recently, the NHB has been generalized from two parameters to multiple parameters [Con+21; Nag05]. It is a tighter bound than the HCRB in the finite copy regime and asymptotically coincides with the HCRB.

In practical scenarios, we do not have infinite copies to attain tight bounds. How to and when we can saturate the various bounds in finite copies, particularly a single copy, is significantly important. It is known that measurement incompatibility prevents us from attaining the quantum CRB [CCY22a; RJD16; Yan+19; Nur24]. Several necessary and sufficient conditions for achieving the SLD CRB are characterized in [Yan+19; Nur24]. However, regarding conditions on the optimal measurement to saturate the RLD, NHB, and HCRB in finite copies, to the author's best knowledge, there is no such characterization for those bounds.

In this work, we provide several necessary and sufficient conditions for the achievability of the HCRB, characterizing the existence of an estimator and optimal measurement. In particular, we give two characterizations for the achievability of the HCRB. The first characterization (**Theorem 4.1**) gives a sufficient condition for the achievability of the HCRB, and we provide a condition (Condition 5 in **Theorem 4.1**) that serves as a

characterization for the theorem. Although this condition is neither a necessary nor sufficient condition for the achievability of the HCRB, it can be reduced to a necessary condition for the saturation of the SLD CRB. Hence, it serves as an information geometrical characterization for quantum statistical models related to the HCRB [CCY22a; Suz19]. The second characterization (**Theorem 5.1**) gives a mathematical characterization for the saturation of the HCRB. It is a necessary and sufficient condition, revealing the relationship between the optimal estimator/measurement and a special operator (defined in eq. (3.2)).

This thesis is organized as follows. In the first part, we will give an introduction to quantum metrology. **Chapter 1** presents elements of classical estimation theory and quantum measurement. To provide the motivation for studying the HCRB, we introduce various quantum Fisher information and the HCRB in **Section 2.2** and **Section 2.3** of **Chapter 2**. In **Section 2.4** of **Chapter 2**, we demonstrate several, but not exhaustive, challenges in the saturation of the HCRB, and state the main problem that we want to solve. In the second part, we will demonstrate the main results of this thesis. **Chapter 3** sets up some preliminary ingredients for proving the main results. **Chapter 4** presents our first main result (**Theorem 4.1**); **Chapter 5** demonstrates our second main result (**Theorem 5.1**). This thesis concludes with a summary and outlook in **Chapter 6**.



Part I

Introduction to Quantum Metrology



Chapter 1

Mathematical Preliminary

The goal of quantum metrology is to estimate the unknown parameter in a quantum system. To get information about the state, one needs to measure, obtain a probability distribution, and perform statistics based on the distribution. In **Section 1.1**, we will introduce measurement in quantum systems. In **Section 1.2**, we will introduce how to estimate an unknown parameter for a parametrized probability distribution.

1.1 Element of Quantum Theory

A quantum system can be described by a density operator in a Hilbert space.

Definition 1.1 (State Postulate). Let \mathcal{H} be a (finite dimensional)¹ Hilbert space. A linear operator ρ on \mathcal{H} is called a density operator if it satisfies the following three conditions:

$$\rho = \rho^\dagger, \quad \text{Tr}\rho = 1, \quad \rho \geq 0$$

the above inequality is in matrix inequality sense, that is, ρ is positive semi-definite.

To extract information from a quantum system, one needs to perform a measurement on the quantum system.

Definition 1.2 (Quantum Measurement). Let \mathcal{H} be a Hilbert space. We call a collection of positive semi-definite hermitian operators on \mathcal{H} : $\{M_\omega\}_{\omega \in \Omega}$ a positive operator-valued measure (POVM) if and only if the operators satisfy that

$$\sum_{\omega} M_\omega = \text{id}_{\mathcal{H}}$$

where $\text{id}_{\mathcal{H}}$ is the identity operator on \mathcal{H} . Given a quantum state ρ , the probability of getting outcome ω is given by

$$p(\omega) = \text{Tr}\rho M_\omega$$

For rigorous mathematical treatment of measurement, see [Hol11], [Hay05].

Example 1.3. Let the quantum state ρ be given by $\rho = \frac{1}{2} (\text{id}_2 + \sum_j \theta^j \sigma_j)$, where id_2 is the 2-dimensional identity operator, $\{\sigma_j\}_j$ is the Pauli operators.

¹Though out this thesis, we only consider finite dimensional quantum system.



- Choosing the measurement:

$$\mathcal{M} \equiv \left\{ \frac{1}{2} |y_+\rangle \langle y_+|, \frac{1}{2} |y_-\rangle \langle y_-|, \frac{1}{2} |z_+\rangle \langle z_+|, \frac{1}{2} |z_-\rangle \langle z_-| \right\}$$

where $|y_\pm\rangle$ (resp. $|z_\pm\rangle$) are the eigenvectors of σ_y (resp. σ_z) with eigenvalue ± 1 . By the measurement postulate, the outcome probability is given by

$$p(y_\pm) = \frac{1}{4} (1 \pm \theta^y), \quad p(z_\pm) = \frac{1}{4} (1 \pm \theta^z)$$

- Choosing the measurement:

$$\mathcal{M} \equiv \left\{ \frac{1}{2} |z_+\rangle \langle z_+|, \frac{1}{2} |z_-\rangle \langle z_-| \right\}$$

The outcome probability is given by

$$p(z_\pm) = \frac{1}{2} (1 \pm \theta^z)$$

From the definition of measurement and the example above, we see that the outcome alphabet and distribution will depend on the measurement chosen by us.

Definition 1.4. Let $\{\Pi_\omega\}_{\omega \in \Omega}$ be a POVM on \mathcal{H} . It is called projection valued-measure (PVM) if and only if, for all ω , Π_ω is a projection operator, and they satisfy that

$$\Pi_\omega \Pi_{\omega'} = \delta_{\omega\omega'} \Pi_\omega, \quad \forall \omega, \omega'$$

That is, they are not only projection operators but are also orthogonal to each other.

In physics, a PVM is sometimes called a perfect measurement or noiseless measurement; in contrast, a POVM is sometimes called a noisy measurement. From the point of view of an information theorist, a POVM can be realized as a randomized decision, while a PVM can be realized as a deterministic decision. Here we introduce a theorem that states that every noisy measurement (POVM) can be realized as a perfect measurement on an extended space (sometimes such a space is called the environment).

Theorem 1.5 (Naimark Extension Theorem, [Hay16; Hol11; Wil11]). Given a POVM $\{M_\omega\}_{\omega \in \Omega}$ on \mathcal{H} . There exists a Hilbert space $\hat{\mathcal{H}}$, pure state σ on $\hat{\mathcal{H}}$ and a PVM $\{\Pi_\omega\}_{\omega \in \Omega}$ on $\mathcal{H} \otimes \hat{\mathcal{H}}$ such that

$$\text{Tr} \rho M_\omega = \text{Tr} (\rho \otimes \sigma) \Pi_\omega, \quad \forall \omega \in \Omega$$

1.2 Element of Parameter Estimation

In general estimation scheme, we consider an alphabet $\Omega = \{\omega_1, \dots, \omega_k\}$ to be our output, and a family of distribution p_θ indexed by parameter $\theta \in \Theta \subset \mathbb{R}^P$, where P is number of parameter. The goal of estimation is that estimate the parameter θ from an observing data.

Example 1.6. Given a coin, whose outcome probability is given by

$$p(\text{H}) = \theta, \quad p(\text{T}) = 1 - \theta$$



Suppose that we do identical experiment N times, and get n head, $N - n$ tail. How to guess the probability θ ?

Since $p(\mathbf{H}) = \theta$, the likelihood of this scenario is given by:

$$L = \theta^n (1 - \theta)^{N-n}$$

we guess the parameter by maximize the possibility

$$\begin{aligned} \partial_\theta L = 0 &= n\theta^{n-1} (1 - \theta)^{N-n} - (N - 1)\theta^n (1 - \theta)^{N-n-1} \\ \Rightarrow \partial_\theta L = 0 &\Leftrightarrow \theta = \frac{n}{N} \end{aligned}$$

One can take the second derivative test to show that $\theta = \frac{n}{N}$ indeed maximize the probability.

Example 1.7. Consider the i.i.d. tensor product state $\rho^{\otimes n}$ with $\rho = \frac{1}{2} (\text{id}_2 + \sum_j \theta^j \sigma_j)$. Choosing the measurement be identical, independent, $\Pi_{z_\pm} \equiv |z_\pm\rangle \langle z_\pm|$ on each system. Suppose that we get the output string: $\omega^n = \omega_1 \omega_2 \dots \omega_n$, where $\omega_j = z_+$ or z_- . Given each subsystem, the outcome probability is given by

$$p(z_\pm) = \frac{1}{2} (1 \pm \theta^z)$$

The likelihood of the string ω^n is

$$L(\theta | \omega^n) = p^{\otimes n}(\omega^n) = \prod_{j=1}^n p(\omega_j) = \left(\frac{1}{2} (1 + \theta^z)\right)^{n_+} \left(\frac{1}{2} (1 - \theta^z)\right)^{n_-}$$

where n_+ (resp. n_-) is the number of $+$ (resp. $-$) in the outcome string. We estimate the parameter θ^z by maximize the likelihood, that is, solving the equation:

$$\partial_\theta L(\theta | \omega^n) = 0 \Leftrightarrow \theta = \frac{n_+ - n_-}{n}$$

The purpose of an estimator is to estimate the true value of a parameter θ . In the following we define a concept which gives a criterion on accuracy of an estimator.

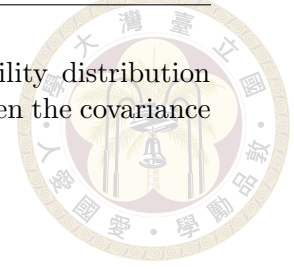
Definition 1.8 (Locally Unbiased Estimator). An estimator $\hat{\theta}$ is called locally unbiased if and only if it satisfies that

$$\begin{aligned} \mathbb{E} \left[\hat{\theta}_j \right] \Big|_{\theta=\theta_0} &= \sum_{\omega} \hat{\theta}_j(\omega) p_\theta(\omega) \Big|_{\theta=\theta_0} = \theta_{0,j}, \quad \forall j \\ \partial_i \mathbb{E} \left[\hat{\theta}_j \right] \Big|_{\theta=\theta_0} &= \sum_{\omega} \hat{\theta}_j(\omega) \partial_i p_\theta(\omega) \Big|_{\theta=\theta_0} = \delta_{ij}, \quad \forall i, j \end{aligned}$$

where $\partial_j \equiv \frac{\partial}{\partial \theta_j}$.

The above definition says that an estimator is locally unbiased if and only if at point θ_0 , $\mathbb{E} \left[\hat{\theta}_j \right]$ equal to $\theta_{0,j}$ up to first order. Note that the estimators in **Example 1.6** and **Example 1.7** are all locally unbiased. For comparison between unbiased and locally unbiased, please see **Remark** on page 8.

In the following, we give a theorem that quantify the quality on precision of an estimator.



Theorem 1.9 (Cramer-Rao Bound). Let p_θ be a family of probability distribution indexed by $\theta \in \Theta \subset \mathbb{R}^P$. Let $\hat{\theta}$ be a locally unbiased estimator at θ_0 , then the covariance matrix Σ would satisfy the Cramer-Rao inequality:

$$\Sigma \geq J^{-1}$$

where the component of the matrix Σ and J is given by

$$\begin{aligned} \Sigma_{ij} &= \mathbb{E} \left[\left(\hat{\theta}_i - \theta_{0,i} \right) \left(\hat{\theta}_j - \theta_{0,j} \right) \right] \Big|_{\theta=\theta_0} \\ &= \sum_{\omega} \left(\hat{\theta}_i(\omega) - \theta_{0,i} \right) \left(\hat{\theta}_j(\omega) - \theta_{0,j} \right) p_\theta(\omega) \Big|_{\theta=\theta_0} \\ J_{ij} &= \mathbb{E} [(\partial_i \log p_\theta) (\partial_j \log p_\theta)] \Big|_{\theta=\theta_0} = \sum_{\omega} (\partial_i \log p_\theta(\omega)) (\partial_j \log p_\theta(\omega)) \Big|_{\theta=\theta_0} \\ &= \sum_{\omega} \frac{(\partial_i p_\theta(\omega)) (\partial_j p_\theta(\omega))}{p_\theta(\omega)} \Big|_{\theta=\theta_0} \end{aligned}$$

The matrix J is called the Fisher information matrix.

The meaning of the above theorem is that the Fisher information serves as the best attainable bound for covariance. It can be viewed as an uncertainty relation for a random variable.



Chapter 2

Element of Quantum Metrology

2.1 Problem Formulation in Quantum Metrology

In quantum metrology, people consider the problem where a parameter is encoded on a quantum state. In general, the parameter encoded on a state can characterize the quantum system. Here are some examples.

Example 2.1. The Gibbs state can be written in the following form:

$$\rho_\beta = \frac{e^{-\beta H}}{Z}, \quad Z = \text{Tr} e^{-\beta H}$$

where β is the temperature (more precisely, the inverse of the temperature) parameter, and H is a positive semi-definite hermitian operator. Usually, a Gibbs state characterizes a system at thermal equilibrium. If one can estimate the temperature with high precision, one can characterize the system at thermal equilibrium.

Example 2.2. Consider the qubit state:

$$\rho_\theta = \frac{1}{2} \left(\text{id}_2 + \sum_j \theta^j \sigma_j \right)$$

where $\theta = (\theta^x, \theta^y, \theta^z)$ is the Bloch vector, and σ_j are the Pauli operators. If one can estimate the Bloch vector, one can access the information of the quantum state.

Example 2.3. Consider a unitary operator U_θ with some phases θ encoded on the operator (maybe single- or multi- parameter). Let ρ_0 be a state, then

$$\rho_\theta = U_\theta \rho_0 U_\theta^\dagger$$

the information of θ is encoded on the state ρ_θ . If one can estimate θ , one can characterize the unitary channel.

Example 2.4. Consider a randomized Pauli channel with

$$\rho_{\mathbf{p}} = \sum_{j=0}^3 p_j \sigma_j \rho_0 \sigma_j$$

where ρ_0 is a state, $\sigma_0 = \text{id}_2$, $\sum_j p_j = 1$. The probability (randomized) parameter is encoded on the state $\rho_{\mathbf{p}}$. If one can estimate \mathbf{p} , one can characterize the randomized Pauli channel.

The goal of quantum metrology is to estimate the parameter with high precision. In general, we consider a quantum statistical family $\{\rho_\theta\}_\theta$ indexed by parameter $\theta \in \Theta \subset \mathbb{R}^P$. To get information in the quantum state, we perform a measurement on the state. After applying a measurement on ρ_θ , we obtain a probability distribution: $p_{\theta;M}(\omega) \equiv \text{Tr}\rho_\theta M_\omega$, and get a Fisher information matrix:

$$(J_M)_{ij} = \mathbb{E} [(\partial_i \log p_{\theta;M}) (\partial_j \log p_{\theta;M})] = \sum_{\omega} \frac{(\text{Tr}\partial_i \rho_\theta M_\omega) (\text{Tr}\partial_j \rho_\theta M_\omega)}{\text{Tr}\rho_\theta M_\omega}$$

where $\partial_j (\cdot) \equiv \frac{\partial}{\partial \theta_j} (\cdot)$. Trace the inverse of the information matrix with a weight matrix $W > 0$ (a real symmetric positive definite matrix), then we obtain a scalar quantity: $\text{tr}W (J_M)^{-1}$. We see that for different measurement $\{M_\omega\}$, we obtain different Fisher information matrices. As introduced in **Theorem 1.9**, Fisher information represents the best attainable lower bound on the covariance of an estimator. To get the best precision bound, we optimize over all possible POVM, and thus we define the following quantity.

Definition (Most Informative Bound). Let $W \in \mathcal{L}(\mathbb{R}^P) \equiv \mathbb{R}^{P \times P}$ be a real, positive definite, symmetric matrix,

$$C^{\text{MI}} [W] \equiv \min_{M \text{ is a POVM}} \text{tr}W (J_M)^{-1}$$

We use Tr to denote trace (physical operation) on quantum system, tr to denote trace on matrix space.

For the above bound, one has the following characterization.

Theorem (Exercise 6.44, [Hay16]).

$$C^{\text{MI}} [W] = \min \left\{ \text{tr}W \Sigma \mid \hat{\theta} \text{ is locally unbiased, } M \text{ is POVM} \right\}$$

where

$$\Sigma_{ij} = \sum_{\omega} (\hat{\theta}_i - \theta_{0,i}) (\hat{\theta}_j - \theta_{0,j}) \text{Tr}\rho_\theta M_\omega \quad (2.1)$$

If we denote $X_j = \sum_{\omega} (\hat{\theta}_j(\omega) - \theta_{0,j}) M_\omega$, then the above local unbiasedness condition can be rewritten as

$$\text{Tr}\rho X_j = 0, \quad \text{Tr}\partial_i \rho X_j = \delta_{ij}$$

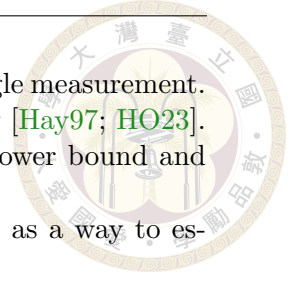
From the above, we define the local unbiasedness for set of hermitian operators.

Definition. Let X_j be a set of hermitian operators, it is called locally unbiased if and only if it satisfies

$$\text{Tr}\rho X_j = 0, \quad \text{Tr}\partial_i \rho X_j = \delta_{ij}$$

we denote the set of all locally unbiasedness hermitian operator by \mathcal{X} , and when a set of hermitian operator X_j is locally unbiased, we denote that $\vec{X} \in \mathcal{X}$.

Remark. In contrast to classical statistics, which usually focuses on “unbiased” estimation, quantum estimation adapts the concept of “locally unbiased” estimation. This adaptation arises because, in general, the optimal measurement depends on the parameter. Different measurements yield different outcomes, resulting in different statistics. An estimator that performs optimally with respect to one measurement at a true value may not perform optimally with respect to another true value.



The most informative bound is the best attainable bound from a single measurement. However, optimizing over all possible POVMs is extremely challenging [Hay97; HO23]. A common approach to minimizing a quantity is first to establish a lower bound and then attempt to achieve equality.

In the next section, we introduce the quantum Fisher information as a way to establish lower bounds on the most informative bound.

2.2 Quantum Fisher Information

We start with the single parameter case, and introduce the concept of symmetric logarithmic derivative (SLD) operator.

Definition. The operator L_j^S (for each j) satisfy the equation:

$$\partial_j \rho = \frac{1}{2} \{ \rho, L_j^S \}$$

is called the SLD operator, where on the above, $\{A, B\} = AB + BA$. The matrix:

$$(J^S)_{ij} \equiv \frac{1}{2} \text{Tr} \rho \{ L_i^S, L_j^S \}$$

is called SLD Fisher information matrix. The quantity:

$$C^S [W] \equiv \text{tr} W (J^S)^{-1}$$

is called SLD CRB. In particular, when parameter is 1-dimensional, that is, single parameter, the SLD operator is the operator which satisfy the equation:

$$\partial_\theta \rho = \frac{1}{2} \{ \rho, L_\theta^S \}$$

and the SLD Fisher information is given by

$$\frac{1}{2} \text{Tr} \rho \{ L_\theta^S, L_\theta^S \} = \text{Tr} \rho (L_\theta^S)^2$$

For single parameter, given a measurement, our goal is to maximize the classical Fisher information over all possible measurement, that is,

$$\max_{M \text{ is a POVM}} J_M = \max_{M \text{ is a POVM}} \sum_{\omega} \frac{(\text{Tr} \partial_\theta \rho M_\omega)^2}{\text{Tr} \rho M_\omega}$$

It is hard to directly optimize over all possible POVM. Hence, one common way is that set an upper bound to the quantity, and try to saturate it. Given arbitrary a POVM $\{M_\omega\}$, we have that

$$\begin{aligned} J_M &= \sum_{\omega} \frac{(\text{Tr} \partial_\theta \rho M_\omega)^2}{\text{Tr} \rho M_\omega} \\ &= \sum_{\omega} \frac{(\text{Tr} \frac{1}{2} \{ \rho, L_\theta^S \} M_\omega)^2}{\text{Tr} \rho M_\omega} \\ &= \sum_{\omega} \frac{(\text{Re} \text{Tr} \rho M_\omega L_\theta^S)^2}{\text{Tr} \rho M_\omega} \end{aligned}$$



$$\begin{aligned}
 &\stackrel{(a)}{\leq} \sum_{\omega} \frac{|\text{Tr} \rho M_{\omega} L_{\theta}^S|^2}{\text{Tr} \rho M_{\omega}} = \sum_{\omega} \frac{|\text{Tr} (\sqrt{M_{\omega}} \sqrt{\rho})^{\dagger} (\sqrt{M_{\omega}} L_{\theta}^S \sqrt{\rho})|^2}{\text{Tr} \rho M_{\omega}} \\
 &\stackrel{(b)}{\leq} \sum_{\omega} \frac{1}{\text{Tr} \rho M_{\omega}} \text{Tr} (\sqrt{M_{\omega}} \sqrt{\rho})^{\dagger} \sqrt{M_{\omega}} \sqrt{\rho} \text{Tr} (\sqrt{M_{\omega}} L_{\theta}^S \sqrt{\rho})^{\dagger} (\sqrt{M_{\omega}} L_{\theta}^S \sqrt{\rho}) \\
 &= \sum_{\omega} \text{Tr} \sqrt{\rho} L_{\theta} M_{\omega} L_{\theta} \sqrt{\rho} = \text{Tr} \rho (L_{\theta}^S)^2
 \end{aligned}$$

where in (a), we use the fact that for any complex number z , $|\text{Re}z|^2 \leq |z|^2$; in (b), we use the Cauchy-Schwartz inequality: $|\text{Tr} A^{\dagger} B|^2 \leq \text{Tr} A^{\dagger} A \text{Tr} B^{\dagger} B$ applied on $A = \sqrt{M_{\omega}} \sqrt{\rho}$, $B = \sqrt{M_{\omega}} L_{\theta}^S \sqrt{\rho}$. The above derivation holds for any POVM M , and hence we have that

$$\max_{M \text{ is POVM}} J_M \leq \text{Tr} \rho (L_{\theta}^S)^2$$

Equalities are saturated when

$$\begin{aligned}
 \text{Im} \text{Tr} \rho M_{\omega} L_{\theta} &= 0 \\
 \frac{\sqrt{M_{\omega}} \sqrt{\rho}}{\text{Tr} M_{\omega} \rho} &= \frac{\sqrt{M_{\omega}} L_{\theta} \sqrt{\rho}}{\text{Tr} \rho M_{\omega} L_{\theta}}
 \end{aligned}$$

Now, choosing $M_{\omega} = \Pi_{\omega} = |\ell_{\omega}\rangle \langle \ell_{\omega}|$ be eigenvector of L_{θ}^S , we see that the above inequality can all be saturated. This shows that for single parameter, we have

$$\max_{M \text{ is POVM}} J_M = \text{Tr} \rho (L_{\theta}^S)^2, \quad \partial_{\theta} \rho = \frac{1}{2} \{\rho, L_{\theta}^S\}$$

As for multiparameter case, one has that $J_M \leq J^S$, or $C^{\text{MI}} \geq C^S$ (For standard proof, please see [ATD20; Liu+19a; SK20; Wat13]. For different approach, please see [Yan+19]. For proof by numerical approach, please see [HO23]). In general, the equality can not be saturated. To saturate the equality, the measurement should satisfy the certain conditions, actually, one has the following theorem.

Theorem 2.5 (Theorem 1, Theorem 2 [Yan+19]). Given a set of measurement operators: M_{ω} with $\sum_{\omega} M_{\omega} = \text{id}_{\mathcal{H}}$, and a quantum statistical family ρ_{θ} , then, SLD CRB is saturable if and only if following condition holds:

- If measurement operator is regular, that is, $\text{Tr} \rho M_{\omega} \neq 0$, then

$$M_{\omega} L_j^S \sqrt{\rho} = \xi_j^{\omega} M_{\omega} \sqrt{\rho}, \quad \xi_j^{\omega} = \frac{\text{Tr} \rho M_{\omega} L_j^S}{\text{Tr} \rho M_{\omega}}, \quad \forall j, \omega$$

- If measurement operator is non-regular, that is, $\text{Tr} \rho M_{\omega} = 0$, then

$$M_{\omega} L_j^S \sqrt{\rho} = \eta_{ij}^{\omega} M_{\omega} L_j^S \sqrt{\rho}, \quad \eta_{ij}^{\omega} = \frac{\text{Tr} \rho L_j^S M_{\omega} L_i^S}{\text{Tr} \rho L_j^S M_{\omega} L_j^S}, \quad \forall \omega, i, j$$

As a necessary condition for the achievability of the SLD CRB, one has the following characterization.

Theorem 2.6 (Theorem 3, the partial commutativity condition, [Yan+19]). Suppose that there exists a measurement M_{ω} such that SLD CRB is saturable, that is, $J_M = J^S$, then, the following condition must holds

$$\sqrt{\rho} [L_i^S, L_j^S] \sqrt{\rho} = 0$$

In general, the identity mentioned above does not hold; therefore, such a measurement typically does not exist. Recently, a new characterization has been found for the saturability of the SLD CRB [Nur24].

The SLD CRB has many applications in unitary/phase estimation problems [Hum+13; Pez+17; YZF14] and in resource theory of asymmetry [Bud23; CH23; Mar22; ST23].

In classical estimation theory, unbiased and efficient estimator does not always exist; however, maximum likelihood estimator is asymptotically unbiased and efficient. Whereas for the SLD CRB, even in asymptotic regime, it is not always achievable (meaning, there does not necessarily exist a measurement such that $J_M \rightarrow J^S$ asymptotically). For the asymptotic achievability, one has the following characterization:

Theorem ([RJD16]). The SLD CRB is asymptotically achievable if and only if

$$\text{Tr} \rho [L_i^S, L_j^S] = 0$$

Besides using the SLD Fisher information matrix to establish a lower bound, one can also use the right logarithmic derivative (RLD) Fisher information to establish a lower bound. Let's first give the definition of the RLD CRB.

Definition. The operator L_j^R satisfy the equation:

$$\partial_j \rho = \rho L_j^R$$

is called the RLD operator. The matrix:

$$(J^R)_{ij} \equiv \text{Tr} (L_i^R)^\dagger \rho L_j^R$$

is called RLD Fisher information matrix. The quantity:

$$C^R [W] \equiv \text{tr} W \text{Re} (J^R)^{-1} + \left\| \sqrt{W} \text{Im} (J^R)^{-1} \sqrt{W} \right\|_1$$

is called the RLD CRB, where for a matrix A , $\|A\|_1 \equiv \text{tr} |A|$, and $|A| = \sqrt{A^\dagger A}$.

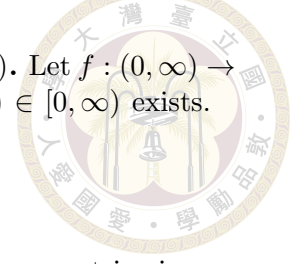
The reason that we use $\text{tr} W \text{Re} (J^R)^{-1} + \left\| \sqrt{W} \text{Im} (J^R)^{-1} \sqrt{W} \right\|_1$ as the RLD CRB rather than $\text{tr} W (J^R)^{-1}$ will be explained in the next section, as we introduce the Holevo Cramer-Rao bound. The RLD CRB is a lower bound to C^{MI} , and it is particularly useful in Gaussian state estimation and \mathcal{D} -invariant model [Hol11; YL73]. For its asymptotic achievability, one has the following characterization:

Theorem (Corollary B.2, [YFG13]; Theorem 2.5, [Suz16]). The RLD CRB is asymptotically achievable if and only if the quantum statistical model is \mathcal{D} -invariant.

In classical estimation theory, we have $L_j \equiv \partial_j \log p$, or, one has that $\partial_j p = p L_j$. The SLD/RLD operator satisfy the relation:

$$\begin{aligned} \partial_j \rho &= \frac{1}{2} \{ \rho, L_j^S \} \\ \partial_j \rho &= \rho L_j^R \end{aligned}$$

We see that the SLD/RLD are particularly two non-commutative generalization of classical Fisher information. Besides SLD/RLD, is there any other non-commutative generalization of Fisher information? The answer is yes. In general, quantum Fisher information can be defined as follow.



Definition 2.7 (Quantum f -Fisher information, Definition A.3 [Sag22]). Let $f : (0, \infty) \rightarrow (0, \infty)$ be operator monotone and suppose that $f(0) := \lim_{x \rightarrow +0} f(x) \in [0, \infty)$ exists. Let ρ be a positive-definite (normalized) state, and

$$\mathcal{K}_\rho \equiv f(\text{Ad}_\rho) \mathcal{R}_\rho$$

then, quantum f -Fisher information is defined as a matrix whose component is given by

$$J_{ij}^f \equiv \langle \partial_i \rho, \mathcal{K}_\rho^{-1}(\partial_j \rho) \rangle_{\text{HS}} = \text{Tr} \partial_i \rho \mathcal{K}_\rho^{-1}(\partial_j \rho)$$

where $\text{Ad}_\rho, \mathcal{R}_\rho$ are both super operators defined as

$$\text{Ad}_\rho(X) = \rho X \rho^{-1}, \quad \mathcal{R}_\rho(X) = X \rho$$

In particular, if $f(x) = \frac{1}{2}(x+1)$, it corresponds to the SLD Fisher information; if $f(x) = 1$, it corresponds to the RLD Fisher information. The quantum f -Fisher information is the unique monotone metric on matrix space, for discussion on f -Fisher information, please see [Hay16; Pet96; PG11; Sag22], for one of its application, see [Gao+23].

2.3 Holevo Cramer-Rao Bound

As we have seen in **Section 2.2**, quantum Fisher information-based bounds (in particular, the SLD/RLD CRB) are not necessarily asymptotically achievable. Is there any bound that is asymptotically achievable, just like the maximum likelihood estimator in classical estimation? The answer is yes; such a bound is called the Holevo-Cramer-Rao bound (HCRB). It is asymptotically achievable and tighter than the SLD/RLD CRB. In this section, we will introduce the HCRB.

The Holevo bound is based on the following inequality (**Theorem 2.8**) and the matrix optimization (**Lemma 2.9**).

Theorem 2.8 (Lemma 3, [ATD20]; [Nag05]; Lemma 1, [HM08]; Equation 6.6.55 [Hol11]). Given a measurement $\{M_\omega\}$ and locally unbiased estimator $\hat{\theta}(\omega)$, then,

$$\Sigma \geq Z \tag{2.2}$$

where Σ is covariance matrix as defined in eq. (2.1), and Z is a matrix defined by

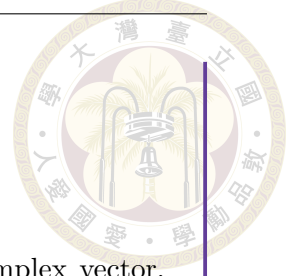
$$Z_{ij} = \text{Tr} X_i \rho X_j$$

and $X_j = \sum_\omega (\hat{\theta}_j(\omega) - \theta_{0,j}) M_\omega$.

The proof of the theorem can be found in the various reference as cited above, we include the proof for reader's convenience.

Proof for theorem 2.8. Let $f : \Omega \rightarrow \mathbb{C}$ be a complex-valued function, $\{M_\omega\}$ be a POVM on \mathcal{H} , and $F \equiv \sum_\omega f(\omega) M_\omega$. We have the following inequality:

$$\sum_\omega (f(\omega) - F) M_\omega (f(\omega) - F)^\dagger \geq 0$$



The above matrix inequality can be modified as

$$\sum_{\omega} |f(\omega)|^2 M_{\omega} \geq FF^{\dagger}$$

Let $f(\omega) = c^{\dagger} (\hat{\theta}(\omega) - \theta_0)$, where $c \in \mathbb{C}^P$ is a P -dimensional complex vector, and c^{\dagger} is the conjugate transpose of the complex vector. This will imply that $F = \sum_{\ell} \bar{c}_{\ell} X_{\ell}$, where \bar{c}_{ℓ} means the complex conjugate of the complex number c_{ℓ} . By the above matrix inequality, we have that

$$c^{\dagger} \Sigma c = \sum_{\omega} \left| c^{\dagger} (\hat{\theta}(\omega) - \theta_0) \right|^2 M_{\omega} \geq \sum_{ij} \bar{c}_i c_j X_i X_j$$

Trace both side of the matrix inequality with $\text{Tr}\rho$, we then have that for any $c \in \mathbb{C}^P$,

$$c^{\dagger} \Sigma c \geq c^{\dagger} Z c$$

Hence, we have the matrix inequality: $\Sigma \geq Z$. □

Lemma 2.9 (Lemma 6.6.1, [Hol11]). Let R be a complex hermitian matrix and W be a positive definite matrix, then

$$\min_{V \geq \pm R} \{\text{Tr} WV\} = \text{TrAbs}\{WR\} = \left\| \sqrt{WR} \sqrt{W} \right\|_1$$

The minimum is achieved for $V = W^{-1} \text{Abs}\{WR\}$, where for positive operator A and hermitian operator B , $\text{Abs}\{AB\}$ is defined as

$$\text{Abs}\{AB\} \equiv \sqrt{A} \left| \sqrt{AB} \sqrt{A} \right| \sqrt{A}^{-1} \Rightarrow \text{TrAbs}\{AB\} = \left\| \sqrt{AB} \sqrt{A} \right\|_1$$

and for a matrix X , $|X| = \sqrt{X^{\dagger} X}$, $\|X\|_1 = \text{Tr}|X|$.

Proof. Original proof is due to [Hol11], we include proof for reader's convenience. Consider the matrix inequality: $V \geq \pm R$. For any positive definite W , we have that

$$\sqrt{W} V \sqrt{W} \geq \pm \sqrt{W} R \sqrt{W}$$

Let $(\lambda_j, |e_j\rangle)$ be eigenpair (pair of eigenvalue and eigenvector) of $\sqrt{W} R \sqrt{W}$. For each j , we have that

$$\langle e_j, \sqrt{W} V \sqrt{W} e_j \rangle \geq \pm \lambda_j \Rightarrow \langle e_j, \sqrt{W} V \sqrt{W} e_j \rangle \geq |\lambda_j|$$

Summing over all j , we arrive that

$$\text{Tr} WV = \sum_j \langle e_j, \sqrt{W} V \sqrt{W} e_j \rangle \geq \sum_j |\lambda_j| = \left\| \sqrt{W} R \sqrt{W} \right\|_1$$

The above holds for any $V \geq \pm R$. We have that

$$\min_{V \geq \pm R} \text{Tr} WV \geq \left\| \sqrt{W} R \sqrt{W} \right\|_1$$



However, taking $V = W^{-1/2} \left| \sqrt{W} R \sqrt{W} \right| W^{-1/2}$, we have that

$$\text{Tr} WV = \text{Tr} \left| \sqrt{W} R \sqrt{W} \right| = \left\| \sqrt{W} R \sqrt{W} \right\|_1$$

Therefore, we conclude that

$$\min_{V \geq \pm R} \text{Tr} WV = \left\| \sqrt{W} R \sqrt{W} \right\|_1$$

□

Now, back to the matrix inequality (2.2). The meaning of this is that the classical covariance is greater than the quantum covariance, or the quantum covariance can serve as a lower bound to the classical covariance. But notice that the matrix Z is possibly complex. Saturating the matrix inequality would directly require Z to be a real matrix, which is a stringent condition for a general quantum statistical model. To remedy this defect, we combine **Theorem 2.8** and **Lemma 2.9**, resulting in the following theorem.

Theorem 2.10 (Theorem 3, [ATD20]; [Nag05]; Equation 19, [HM08]). Let $\{M_\omega\}$ be a measurement, $\hat{\theta}(\omega)$ be a locally unbiased estimator, Σ , X_j , Z are defined as in **Theorem 2.8**, and W be a real positive definite matrix, then

$$\text{tr} W \Sigma \geq \text{tr} W \text{Re} Z + \left\| \sqrt{W} \text{Im} Z \sqrt{W} \right\|_1 \quad (2.3)$$

Before the proof of the theorem, we first establish a lemma.

Lemma 2.11 (Lemma 4, [ATD20]). Let A be a hermitian matrix with $A \geq 0$, then, $\text{Re} A \geq \pm i \text{Im} A$.

Proof of Lemma 2.11. Since $A \geq 0$, we have that

$$\begin{aligned} \forall v, (\mathbb{R} \ni) \langle v, Av \rangle \geq 0 &\Rightarrow \forall v, \overline{\langle v, Av \rangle} \geq 0 \Rightarrow \forall v, \langle v, \bar{A}v \rangle \geq 0 \\ &\Rightarrow \bar{A} \geq 0 \end{aligned}$$

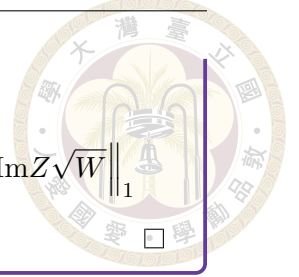
where \bar{A} is entry-wise complex conjugate of A . Since $A \geq 0 \Rightarrow \bar{A} \geq 0$, this implies that

$$\begin{aligned} A \geq 0 &\Rightarrow \text{Re} A + i \text{Im} A \geq 0 \Rightarrow \text{Re} A \geq -i \text{Im} A \\ \bar{A} \geq 0 &\Rightarrow \text{Re} A - i \text{Im} A \geq 0 \Rightarrow \text{Re} A \geq i \text{Im} A \\ &\Rightarrow \text{Re} A \geq \pm \text{Im} A \end{aligned}$$

□

Proof of Theorem 2.10. The proof of the theorem can be found in the various references as cited above. Here, the author includes the proof for reader's convenience. By **Theorem 2.8** and **Lemma 2.11**, we have the matrix inequality:

$$\Sigma \geq Z \Rightarrow \Sigma - \text{Re} Z \geq \pm i \text{Im} Z$$



By **Lemma 2.9**, we directly arrive that

$$\mathrm{tr}W\Sigma - \mathrm{tr}W\mathrm{Re}Z \geq \left\| \sqrt{W}\mathrm{Im}Z\sqrt{W} \right\|_1 \Rightarrow \mathrm{tr}W\Sigma \geq \mathrm{tr}W\mathrm{Re}Z + \left\| \sqrt{W}\mathrm{Im}Z\sqrt{W} \right\|_1$$

For the clarity of the following discussion, let W be the identity matrix of suitable size. The quantity on the right-hand side of equation (2.3) represents the minimum among all real matrices V that satisfy the inequality $V \geq Z$. Specifically, this minimum is given by $V = \mathrm{Re}Z + |\mathrm{Im}Z|$. Since the matrix $\mathrm{Re}Z + |\mathrm{Im}Z|$ is real, it is possible to saturate the inequality without imposing further restrictions on the quantum system, such as requiring Z to be a real matrix. Moreover, by optimizing the lower bound (2.3) through the minimization of the modified quantum covariance ($\mathrm{Re}Z + |\mathrm{Im}Z|$) using the optimal locally unbiased estimator, we obtain the celebrated HCRB.

Definition (Holevo-Cramer-Rao Bound, [Nag05], [Hol11]). The quantity:

$$C^H[W] = \min_{\vec{X} \in \mathcal{X}} \left\{ \mathrm{tr}W\mathrm{Re}Z + \left\| \sqrt{W}\mathrm{Im}Z\sqrt{W} \right\|_1 \right\}$$

is called the Holevo-Cramer-Rao bound, and we have the inequality:

$$C^{\mathrm{MI}} \geq C^H$$

The HCRB is in general not achievable in finite copy; however, it is always achievable in the asymptotic regime with some mild regularity conditions. Roughly speaking, the HCRB is achievable in Gaussian state estimation schemes. Quantum statistical models with i.i.d. extensions are locally asymptotically normal, and hence the HCRB is asymptotically achievable. For detailed information on convergence and different approaches, see the various references [GK06; KG09; YFG13; YCH19] on quantum local asymptotic normality.

Furthermore, the HCRB is a tighter bound than the SLD CRB and RLD CRB, and we state the following theorem.

Theorem ([Hol11]).

$$C^H \geq \max \{C^S, C^R\}$$

Proof. Here we mention the proof for the theorem. For $C^H \geq C^S$, see for example, corollary 3, [ATD20]; section C., [RJD16]; [Nag05]; [HM08]. For $C^H \geq C^R$, see for example, [HM08], [Nag05]. \square

Theorem ([RJD16]). $C^H = C^S$ if and only if $\mathrm{Tr}\rho [L_i^S, L_j^S] = 0$, and if $C^H = C^S$, the optimizer of the HCRB is

$$L^{S,j} \equiv \sum_{\ell} \left((J^S)^{-1} \right)_{\ell j} L_{\ell}^S$$

Theorem (Theorem 2.5, [Suz16]). $C^H = C^R$ if and only if the model is \mathcal{D} -invariant, and if model is \mathcal{D} -invariant, the optimizer of the HCRB is

$$L^{R,j} \equiv \sum_{\ell} \left((J^R)^{-1} \right)_{\ell j} L_{\ell}^R$$

Recently, the HCRB is cast as a convex optimization problem [AFD19; Con+21; HO23], and hence it can be computed efficiently.

Here, we summarize the various definitions and theorems.

Definition 2.12 (Most Informative Bound). Let $W \in \mathcal{L}(\mathbb{R}^P) \equiv \mathbb{R}^{P \times P}$ be a real, positive definite, symmetric matrix,

$$C^{\text{MI}}[W] \equiv \min_{M \text{ is a POVM}} \text{tr} W (J_M)^{-1}$$

We use Tr to denote trace (physical operation) on quantum system, tr to denote trace on matrix space.

Definition 2.13. Let X_j be a set of hermitian operators, it is called locally unbiased if and only if it satisfies

$$\text{Tr} \rho X_j = 0, \quad \text{Tr} \partial_i \rho X_j = \delta_{ij}$$

we denote the set of all locally unbiasedness hermitian operator by \mathcal{X} , and when a set of hermitian operator X_j is locally unbiased, we denote that $\vec{X} \in \mathcal{X}$.

Theorem 2.14 (Exercise 6.44, [Hay16]).

$$C^{\text{MI}}[W] = \min \left\{ \text{tr} W \Sigma \mid \hat{\theta} \text{ is locally unbiased, } M \text{ is POVM} \right\}$$

where

$$\Sigma_{ij} = \sum_{\omega} \left(\hat{\theta}_i - \theta_{0,i} \right) \left(\hat{\theta}_j - \theta_{0,j} \right) \text{Tr} \rho_{\theta} M_{\omega}$$

Definition 2.15. The operator L_j^{S} satisfy the equation:

$$\partial_j \rho = \frac{1}{2} \{ \rho, L_j^{\text{S}} \}$$

is called SLD operator, where on the above, $\{A, B\} = AB + BA$. The matrix:

$$(J^{\text{S}})_{ij} \equiv \frac{1}{2} \text{Tr} \rho \{ L_i^{\text{S}}, L_j^{\text{S}} \}$$

is called SLD Fisher information matrix. The quantity:

$$C^{\text{S}}[W] \equiv \text{tr} W (J^{\text{S}})^{-1}$$

is called SLD CRB. In particular, when parameter is 1-dimensional, that is, single parameter, SLD operator is operator which satisfy the equation:

$$\partial_{\theta} \rho = \frac{1}{2} \{ \rho, L_{\theta}^{\text{S}} \}$$

and the SLD Fisher information is given by

$$\frac{1}{2} \text{Tr} \rho \{ L_{\theta}^{\text{S}}, L_{\theta}^{\text{S}} \} = \text{Tr} \rho (L_{\theta}^{\text{S}})^2$$

Definition 2.16. The operator L_j^{R} satisfy the equation:

$$\partial_j \rho = \rho L_j^{\text{R}}$$



is called RLD operator. The matrix:

$$(J^{\text{R}})_{ij} \equiv \text{Tr} (L_i^{\text{R}})^\dagger \rho L_j^{\text{R}}$$

is called RLD Fisher information matrix. The quantity:

$$C^{\text{R}} [W] \equiv \text{tr} W \text{Re} (J^{\text{R}})^{-1} + \left\| \sqrt{W} \text{Im} (J^{\text{R}})^{-1} \sqrt{W} \right\|_1$$

is called RLD CRB.

Theorem 2.17 ([Hol11]).

$$C^{\text{H}} \geq \max \{C^{\text{S}}, C^{\text{R}}\}$$

Theorem 2.18 ([RJD16]). $C^{\text{H}} = C^{\text{S}}$ if and only if $\text{Tr} \rho [L_i^{\text{S}}, L_j^{\text{S}}] = 0$, and if $C^{\text{H}} = C^{\text{S}}$, then the optimizer of the HCRB is

$$L^{\text{S},j} \equiv \sum_{\ell} \left((J^{\text{S}})^{-1} \right)_{\ell j} L_{\ell}^{\text{S}}$$

Theorem 2.19 (Theorem 2.5, [Suz16]). $C^{\text{H}} = C^{\text{R}}$ if and only if the model is \mathcal{D} -invariant, and if model is \mathcal{D} -invariant, then the optimizer of the HCRB is

$$L^{\text{R},j} \equiv \sum_{\ell} \left((J^{\text{R}})^{-1} \right)_{\ell j} L_{\ell}^{\text{R}}$$

Definition 2.20 (Holevo-Cramer-Rao Bound, [Nag05], [Hol11]). The quantity:

$$C^{\text{H}}[W] = \min_{\vec{X} \in \mathcal{X}} \left\{ \text{tr} W \text{Re} Z + \left\| \sqrt{W} \text{Im} Z \sqrt{W} \right\|_1 \right\}$$

is called the Holevo-Cramer-Rao bound, and we have the inequality:

$$C^{\text{MI}} \geq C^{\text{H}}$$

2.4 Statement of the Problem

The HCRB is the ultimate precision that we can achieve (in the asymptotic regime). In practical schemes, however, we do not have infinite copies of the state. To saturate the HCRB in a large number of copies, one needs to perform collective measurements on a large tensor product of the state, which is extremely challenging in experimental settings [Con+21]. Saturating the HCRB in finite copies is particularly important and highly non-trivial work. To the best of the author's knowledge, the achievability of the HCRB is only characterized for pure state and Gaussian state estimation, where the HCRB is achievable and optimal measurements can be constructed (see, for example, [Mat02; YFG13]). For the general finite-dimensional mixed state estimation problem, no general characterization is given. Characterizing the achievability of the HCRB is non-trivial due to its inherent complexity. The following are some reasons.

1. The HCRB comes from a matrix optimization (**Lemma 2.9**), and it is not like the usual Fisher information bound, which is established based on the Cauchy-Schwartz inequality. This means that saturation of the HCRB cannot directly apply methods like those in [Yan+19], which fully utilize the Cauchy-Schwartz inequality to establish and characterize conditions on the measurement that saturate the SLD CRB.

2. Endowing a matrix space with a metric allows tools from geometry to be applied to the matrix space. The connection between geometry and physics becomes apparent. Geometrical objects like torsion and curvature can help in studying properties of a quantum statistical model. For example, if one assigns the SLD metric to a quantum statistical model, the achievability of the SLD CRB for pure state and faithful state models is fully characterized [AN00; Hay05; NF23]. Another example is the RLD CRB, which is highly related to \mathcal{D} -invariant models and characterizes the achievability of the tight bound in Gaussian state models [Hol11; Suz16; Suz19; YL73]. To the best of the author's knowledge, no study has been conducted on information geometry by endowing the HCRB as a metric on matrix space. This is possibly because the HCRB, in general, is not a monotone metric on matrix space, which prevents us from applying techniques like those in [Gao+23], where full use was made of operator Jensen's inequality and the integral representation of operator convex functions to derive relations between conditions on recoverability and the monotonic metric.

The contribution of this work is to provide some partial answers to the following problem:

When will there exists a POVM such that $C^{\text{MI}} = C^{\text{H}}$?



Part II

Achievability of Tight Bound in Quantum Metrology



Chapter 3

Some Preliminary Results for The Characterizations

In this chapter, we introduce some preliminary for our main result.

Definition 3.1. Let $V^H = \text{Re}Z^H + V_I^H$, where $Z^H = \text{Tr}\rho X_j^H X_i^H$, X_j^H is optimizer of C^H , that is,

$$X_j^H = \underset{\tilde{X} \in \mathcal{X}}{\text{argmin}} C^H[W] \quad (3.1)$$

and

$$V_I^H = W^{-\frac{1}{2}} \left| \sqrt{W} \text{Im} Z^H \sqrt{W} \right| W^{-\frac{1}{2}}$$

Theorem 3.2 (Achievability in Extended Space). The following two conditions are equivalent.

1. For any $\mathbb{R}^{P \times P} \ni W > 0$, there exists a pair of estimator, measurement $\left\{ \left(\hat{\theta}_i(\omega), M_\omega \right) \right\}_{\omega=1}^{|\Omega|}$ (where Ω is output alphabet and $|\Omega|$ is its cardinality) such that the covariance matrix of the estimator and the quantum covariance matrix satisfy the following relation:

$$\Sigma = V^H$$

where V^H is defined in **Definition 3.1**. That is, there exists a locally unbiased estimator, POVM such that $C^{\text{MI}}[W] = C^H[W]$.

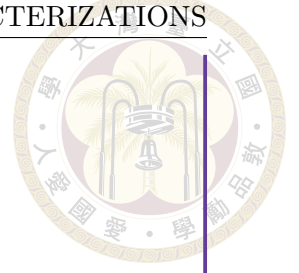
2. There exists a Hilbert space $\hat{\mathcal{H}}$, a pure state σ on it, a set of observable \tilde{X}_i on $\mathcal{L}(\mathcal{H} \otimes \hat{\mathcal{H}})$ with

$$\begin{aligned} [\tilde{X}_i, \tilde{X}_j] &= 0 \\ \text{Tr} \partial_i(\rho \otimes \sigma) \tilde{X}_j &= \delta_{ij}, \quad \text{Tr}(\rho \otimes \sigma) \tilde{X}_j \tilde{X}_i = V^H \end{aligned}$$

Proof. (1 \Rightarrow 2) Suppose that there exists locally unbiased estimator and POVM such that

$$C^{\text{MI}}[W] = C^H[W] \Leftrightarrow \Sigma = V^H$$

by Naimark theorem, there exists a Hilbert space $\hat{\mathcal{H}}$, pure state σ on it and a



PVM Π_ω on $\mathcal{H} \otimes \hat{\mathcal{H}}$ such that

$$\text{Tr} \rho M_\omega = \text{Tr} (\rho \otimes \sigma) \Pi_\omega$$

defining the operator by

$$\tilde{X}_j \equiv \sum_{\omega} \left(\hat{\theta}_j(\omega) - \theta_{0,j} \right) \Pi_\omega \equiv \sum_{\omega} \tilde{x}_{j,\omega} \Pi_\omega$$

where $\tilde{x}_{j,\omega} \equiv \hat{\theta}_j(\omega) - \theta_{0,j}$. It is easy to see that \tilde{X}_j commute, and we have that

$$\begin{aligned} \text{Tr} (\rho \otimes \sigma) \tilde{X}_j &= \sum_{\omega} \text{Tr} (\rho \otimes \sigma) \tilde{x}_{j,\omega} \Pi_\omega \\ &= \sum_{\omega} \tilde{x}_{j,\omega} \text{Tr} \rho M_\omega \\ &= \sum_{\omega} \left(\hat{\theta}_j(\omega) - \theta_{0,j} \right) \text{Tr} \rho M_\omega = 0 \end{aligned}$$

$$\begin{aligned} \text{Tr} \partial_i (\rho \otimes \sigma) \tilde{X}_j &= \sum_{\omega} \partial_i \text{Tr} (\rho \otimes \sigma) \tilde{x}_{j,\omega} \Pi_\omega \\ &= \partial_i \text{Tr} \rho \left(\sum_{\omega} \tilde{x}_{j,\omega} M_\omega \right) \\ &= \partial_i \text{Tr} \rho X_j = \delta_{ij} \end{aligned}$$

Next,

$$\begin{aligned} \text{Tr} (\rho \otimes \sigma) \tilde{X}_j \tilde{X}_i &= \text{Tr} (\rho \otimes \sigma) \left(\sum_{\omega, m} \tilde{x}_{j,\omega} \tilde{x}_{i,\omega'} \Pi_\omega \Pi_{\omega'} \right) \\ &= \sum_{\omega, \omega'} \text{Tr} (\rho \otimes \sigma) \tilde{x}_{j,\omega} \tilde{x}_{i,\omega'} \delta_{\omega\omega'} \Pi_\omega \\ &= \sum_{\omega} \tilde{x}_{i,\omega} \tilde{x}_{j,\omega} \text{Tr} (\rho \otimes \sigma) \Pi_\omega \\ &= \sum_{\omega} \tilde{x}_{i,\omega} \tilde{x}_{j,\omega} \text{Tr} \rho M_\omega = \Sigma_{ij} \end{aligned}$$

(2 \Rightarrow 1) Suppose that there exists a Hilbert space $\hat{\mathcal{H}}$, a pure state σ on it, and a set of observable \tilde{X}_j such that

$$\begin{aligned} [\tilde{X}_i, \tilde{X}_j] &= 0 \\ \text{Tr} (\rho \otimes \sigma) \tilde{X}_j &= 0, \quad \text{Tr} \partial_i (\rho \otimes \sigma) \tilde{X}_j = \delta_{ij} \\ \text{Tr} (\rho \otimes \sigma) \tilde{X}_j \tilde{X}_i &= V^H \end{aligned}$$

Let spectral decomposition of \tilde{X}_i be

$$\tilde{X}_j \equiv \sum_{\omega} \tilde{x}_{j,\omega} \Pi_\omega = \sum_{\omega} \tilde{x}_{j,\omega} |e_\omega\rangle \langle e_\omega|$$



defining the estimator and measurement by

$$\begin{aligned}\hat{\theta}_j(\omega) &= \theta_{0,j} + \text{Tr} \tilde{X}_j \Pi_\omega = \theta_{0,j} + \tilde{x}_{j,\omega} \\ M_\omega &\equiv \langle \varphi | \Pi_\omega | \varphi \rangle, \quad X_i = \sum_\omega \tilde{x}_{i,\omega}(\omega) M_\omega\end{aligned}$$

where with abuse of notation, we denote that $\langle \varphi | \Pi_\omega | \varphi \rangle \equiv (\text{id}_{\mathcal{H}} \otimes \langle \varphi |) \Pi_\omega (\text{id}_{\mathcal{H}} \otimes | \varphi \rangle)$, and $\sigma = | \varphi \rangle \langle \varphi |$. Then, we have

$$\begin{aligned}\sum_\omega \hat{\theta}_j(\omega) \text{Tr} \rho M_\omega &= \sum_\omega (\theta_{0,j} + \tilde{x}_{j,\omega}) \text{Tr} (\rho \otimes \sigma) \Pi_\omega \\ &= \text{Tr} (\rho \otimes \sigma) \left(\theta_{0,j} + \sum_\omega \tilde{x}_{j,\omega} \Pi_\omega \right) \\ &= \theta_{0,j} + \text{Tr} (\rho \otimes \sigma) \tilde{X}_j = \theta_{0,j}\end{aligned}$$

$$\begin{aligned}\text{Tr} \partial_i \rho X_j &= \sum_\omega \partial_i \text{Tr} \rho \tilde{x}_{j,\omega} M_\omega \\ &= \sum_\omega \partial_i \text{Tr} (\rho \otimes \sigma) \tilde{x}_{j,\omega} \Pi_\omega \\ &= \text{Tr} \partial_i (\rho \otimes \sigma) \tilde{X}_j = \delta_{ij}\end{aligned}$$

and

$$\begin{aligned}\Sigma_{ij} &= \sum_\omega \tilde{x}_{i,\omega} \tilde{x}_{j,\omega} \text{Tr} \rho M_\omega \\ &= \sum_\omega \tilde{x}_{i,\omega} \tilde{x}_{j,\omega} \text{Tr} (\rho \otimes \sigma) \Pi_\omega \\ &= \sum_{\omega, \omega'} \text{Tr} (\rho \otimes \sigma) \tilde{x}_{i,\omega} \tilde{x}_{j,\omega} \delta_{\omega\omega'} \Pi_\omega \\ &= \text{Tr} (\rho \otimes \sigma) \left(\sum_{\omega, \omega'} \tilde{x}_{j,\omega} \tilde{x}_{i,\omega'} \Pi_\omega \Pi_{\omega'} \right) = \text{Tr} (\rho \otimes \sigma) \tilde{X}_j \tilde{X}_i\end{aligned}$$

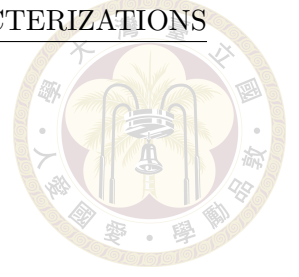
□

Condition 1 is just the statement of achievability; condition 2 tells us that if such estimator and measurement exists, then it is nothing but measuring a set of commutative observables on a larger space (extended by Naimark extension theorem). The extended space together with the special operator (defined in eq. (3.2)), would help us discuss the achievability of the HCRB.

Lemma 3.3 (Lemma B.9, in supplement material of [YFG13]). Given $P \times P$ positive semidefinite hermitian matrix J . Then there exists a Hilbert space $\hat{\mathcal{H}}$, pure state σ on it, and set of operators B_1, \dots, B_P such that

$$\text{Tr} \sigma B_j = 0, \quad \text{Tr} \sigma B_j B_i = J_{ij}$$

We now introduce a very important operator. Observing that $V_1^H - i \text{Im} Z^H$ is a positive semi-definite matrix, by **Lemma 3.3**, there exists Hilbert space $\hat{\mathcal{H}}$, pure state



σ on it, and observables B_j such that

$$\text{Tr}\sigma B_j = 0, \quad \forall j, \quad \text{Tr}\sigma B_j B_i = (V_I^H - i\text{Im}Z^H)_{ij}$$

Proposition 3.4. Let $\hat{\mathcal{H}}, \sigma, B_j$ satisfy that

$$\text{Tr}\sigma B_j = 0, \quad \text{Tr}\sigma B_j B_i = V_I^H = W^{-\frac{1}{2}} \left| \sqrt{W} \text{Im}Z^H \sqrt{W} \right| W^{-\frac{1}{2}}$$

then the operator $\bar{X}_j \in \mathcal{L}(\mathcal{H} \otimes \hat{\mathcal{H}})$ (defined as follow) is locally unbiased estimator and its quantum covariance matrix is just the optimum matrix of the HCRB, here

$$\bar{X}_j \equiv X_j^H \otimes \text{id}_{\hat{\mathcal{H}}} + \text{id}_{\mathcal{H}} \otimes B_j \quad (3.2)$$

and X_j^H is the optimizer of C^H , as eq. (3.1)

Proof. For the unbiasedness:

$$\begin{aligned} \text{Tr}(\rho \otimes \sigma) \bar{X}_j &= \text{Tr}(\rho \otimes \sigma) (X_j^H \otimes \text{id}_{\hat{\mathcal{H}}} + \text{id}_{\mathcal{H}} \otimes B_j) \\ &= (\text{Tr}\rho X_j^H) (\text{Tr}\sigma) + (\text{Tr}\rho) (\text{Tr}\sigma B_j) \\ &= 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{Tr}\partial_i(\rho \otimes \sigma) \bar{X}_j &= \partial_i \text{Tr}((\rho X_j^H) \otimes \sigma + \rho \otimes (\sigma B_j)) \\ &= (\text{Tr}\partial_i \rho X_j^H) (\text{Tr}\sigma) + (\text{Tr}\partial_i \rho) (\text{Tr}\sigma B_j) \\ &= \delta_{ij} \end{aligned}$$

for quantum covariance,

$$\begin{aligned} \text{Tr}(\rho \otimes \sigma) \bar{X}_j \bar{X}_i &= \text{Tr}(\rho \otimes \sigma) (X_j^H X_i^H \otimes \text{id}_{\hat{\mathcal{H}}} + X_j^H \otimes B_i + X_i^H \otimes B_j + \text{id}_{\mathcal{H}} \otimes B_j B_i) \\ &= \text{Tr}\rho X_j^H X_i^H + (\text{Tr}\rho X_j^H) (\text{Tr}\sigma B_i) + (\text{Tr}\rho X_i^H) (\text{Tr}\sigma B_j) + \text{Tr}\sigma B_j B_i \\ &= (\text{Re}Z^H + i\text{Im}Z^H)_{ij} + (V_I^H - i\text{Im}Z^H)_{ij} \\ &= (\text{Re}Z^H + V_I^H)_{ij} = (V^H)_{ij} \end{aligned}$$

□

By studying the properties of the operators \bar{X}_j , one can characterize the quantum statistical model.

Lemma 3.5. Let \mathcal{V} be a finite dimensional, *complex* inner product space, $\{|v_j\rangle\}_{j=1}^P \subset \mathcal{V}$, $P \leq M \leq \dim \mathcal{V}$ and $\langle \cdot, \cdot \rangle$ be inner product on it. Suppose that $\langle v_i, v_j \rangle \in \mathbb{R}$ for any i, j , then, there exists a set of orthonormal vectors $\{|e_k\rangle\}_{k=1}^M$ with $\text{span}_{\mathbb{R}}\{|v_j\rangle\} \subset \text{span}_{\mathbb{R}}\{|e_k\rangle\}$ such that

$$\langle e_k, v_j \rangle \in \mathbb{R}, \quad \forall j, k$$

that is, in basis $|e_k\rangle, |v_j\rangle$ is real vector (vectors with real component).



Proof. Performing Gram Schmidt process then we get the result. Let

$$\begin{aligned} |u_1\rangle &= |v_1\rangle \\ |u_2\rangle &= |v_2\rangle - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_1\rangle} |u_1\rangle \\ |u_3\rangle &= |v_3\rangle - \frac{\langle u_2, v_3\rangle}{\langle u_2, u_2\rangle} |u_2\rangle - \frac{\langle u_1, v_3\rangle}{\langle u_1, u_1\rangle} |u_1\rangle \\ &\vdots \end{aligned}$$

we see that the coefficient $\langle u_i, v_j\rangle / \langle u_i, u_i\rangle$ are all real, and $\{|u_j\rangle\}_{j=1}^P$ forms a set of orthogonal vectors, $\text{span}_{\mathbb{R}}\{|u_j\rangle\} = \text{span}_{\mathbb{R}}\{|v_j\rangle\}$, and

$$|v_j\rangle = \sum_{\ell} u_{j,\ell} |u_{\ell}\rangle$$

the coefficient are all real. Let $|e_k\rangle = \frac{|u_k\rangle}{\sqrt{\langle u_k, u_k\rangle}}$, and extend the set of vectors to orthonormal set $\{|e_k\rangle\}_{k=1}^M$, then we get the result. \square

Lemma 3.6. Let \mathcal{V} be a finite dimensional, *real* inner product space, and $\langle \cdot, \cdot \rangle$ be inner product on \mathcal{V} . Suppose that $\{|u_j\rangle\}_{j=1}^P$, $\{|v_j\rangle\}_{j=1}^P$ ($P \leq \dim \mathcal{V}$) be two sets of vectors such that

$$\langle u_i, u_j \rangle = \langle v_i, v_j \rangle$$

for all i, j . That is $\{|u_j\rangle\}_{j=1}^P$, $\{|v_j\rangle\}_{j=1}^P$ admits same Gram matrix. Let $\{|e_j\rangle\}_{j=1}^{\dim \mathcal{V}}$ be a orthonormal basis with $|u_j\rangle = \sum_{\ell} u_{j,\ell} |e_{\ell}\rangle$, then, there exists a orthonormal basis $|f_{\ell}\rangle$ such that under this basis, $|v_j\rangle = \sum_{\ell} v_{j,\ell} |f_{\ell}\rangle$ we have $u_{j,\ell} = v_{j,\ell}$.

Proof. First extending vectors $\{|u_j\rangle\}_{j=1}^P$ to a basis $\{|u'_j\rangle\}_{j=1}^{\dim \mathcal{V}}$ with

$$\begin{aligned} |u'_j\rangle &= |u_j\rangle, \quad \forall 1 \leq j \leq P \\ \langle u'_j, u'_k \rangle &= 0, \quad \forall 1 \leq j \leq P, P+1 \leq k \leq \dim \mathcal{V} \\ \langle u'_j, u'_k \rangle &= \delta_{jk}, \quad \forall P+1 \leq j, k \leq \dim \mathcal{V} \end{aligned}$$

that is, $|u'_j\rangle$ for $j = P+1, \dots, \dim \mathcal{V}$ is orthogonal to $|u'_j\rangle$ for $j = 1, \dots, P$ and further, for $j = P+1, \dots, \dim \mathcal{V}$, $|u'_j\rangle$ is orthonormal. Do the similar things for $|v_j\rangle$, and extending them to $|v'_j\rangle$. We see that $\{|u'_j\rangle\}_{j=1}^{\dim \mathcal{V}}$, $\{|v'_j\rangle\}_{j=1}^{\dim \mathcal{V}}$ are both basis of \mathcal{V} , further, they admits same gram matrix with

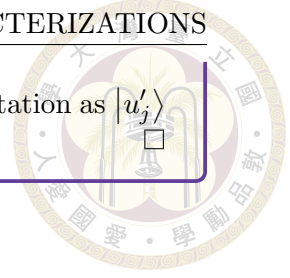
$$\langle u'_i, u'_j \rangle = \langle v'_i, v'_j \rangle = \begin{pmatrix} g_{ij} & 0 \\ 0 & \delta_{ij} \end{pmatrix}, \quad g_{ij} = \langle u_i, u_j \rangle = \langle v_i, v_j \rangle$$

Since $|u'_j\rangle$, $|v'_j\rangle$ are both basis and admits same Gram matrix, there exists a unitary operator Q such that $Q |u'_j\rangle = |v'_j\rangle$. Defining that $|f_{\ell}\rangle \equiv Q |e_{\ell}\rangle$, we see that

$$|v'_j\rangle = Q |u'_j\rangle = Q \sum_{\ell} [u'_j]_{\ell}^e |e_{\ell}\rangle = \sum_{\ell} [u'_j]_{\ell}^e |f_{\ell}\rangle = \sum_{\ell} [v'_j]_{\ell}^f |f_{\ell}\rangle$$

CHAPTER 3. SOME PRELIMINARY RESULTS FOR THE
CHARACTERIZATIONS

this means that, under basis $|f_\ell\rangle, |v'_j\rangle$ has same component representation as $|u'_2\rangle$ in basis $|e_\ell\rangle$. \square





Chapter 4

1-st Characterization

Condition 1 first appeared in the supplement material of the paper [YFG13], as a proof of the achievability of the HCRB for the pure state case. The main contribution of **Theorem 4.1** is that the author generalizes this condition to the mixed state case, develops two equivalent conditions (condition 2 and 3), and provides a necessary condition (condition 5) which is served as a characterization for the theorem.

Theorem 4.1 (1-st Characterization). The relations for following conditions are:

$$1 \Leftrightarrow 2 \Rightarrow 3 \Rightarrow 1 \Rightarrow 4 \Leftrightarrow 5$$

in particular 1, 2, 3 are (equivalent and) sufficient for achivability of HCRB, that is, conditions in **Theorem 3.2**.

1. There exists Hilbert space $\hat{\mathcal{H}}$, pure state σ on it, a set of commutative observables \tilde{X}_j such that $\tilde{X}_j(\rho \otimes \sigma) = \bar{X}_j(\rho \otimes \sigma)$, where \bar{X}_j is defined as in **Proposition 3.4**. Equivalently, there exists hermitian operators K_j with $K_j\rho = \rho K_j = 0$ such that

$$\tilde{X}_j = \bar{X}_j + K_j$$

are all commute.

2. There exists real number $\tilde{x}_{j,\omega}$ basis $|e_\omega\rangle$ such that it satisfy the equation:

$$(\rho \otimes \sigma) \bar{X}_j |e_\omega\rangle = \tilde{x}_{j,\omega} (\rho \otimes \sigma) |e_\omega\rangle$$

3. There exists basis $|e_\omega\rangle$ such that

$$\tilde{x}_{j,\omega} = \frac{\langle e_\omega, \bar{X}_j \psi_m \rangle}{\langle e_\omega, \psi_m \rangle}$$

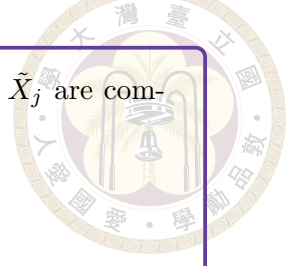
and $\tilde{x}_{j,\omega}$ is real and independent of m , where for each m , $|\psi_m\rangle$ is eigenvector of $\rho \otimes \sigma$.

4. There exists Hilbert space $\hat{\mathcal{H}}$, pure state σ on it such that

$$(\rho \otimes \sigma)^{\frac{1}{2}} [\bar{X}_i, \bar{X}_j] (\rho \otimes \sigma)^{\frac{1}{2}} = 0$$

5. The optimizer of C^H satisfy the equation:

$$\sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho} = 2i (\text{Im} Z^H)_{ji} \rho = (\text{Tr} \rho [X_i^H, X_j^H]) \rho \quad (4.1)$$



Proof. **1** \Rightarrow **2**. Suppose that there exists $\tilde{X}_j = \bar{X}_j + K_j$ such that \tilde{X}_j are commutative, $K_j(\rho \otimes \sigma) = (\rho \otimes \sigma)K_j = 0$, then, set

$$\tilde{X}_j = \sum_{\omega} \tilde{x}_{j,\omega} |e_{\omega}\rangle \langle e_{\omega}|$$

then, we see that $(\rho \otimes \sigma)\tilde{X}_j = (\rho \otimes \sigma)\bar{X}_j$ and

$$(\rho \otimes \sigma)\tilde{X}_j |e_{\omega}\rangle = \rho \bar{X}_j |e_{\omega}\rangle = \tilde{x}_{j,\omega} (\rho \otimes \sigma) |e_{\omega}\rangle$$

2 \Rightarrow **1**. Suppose that there exists real numbers $\tilde{x}_{j,\omega}$ and basis $|e_{\omega}\rangle$ such that

$$(\rho \otimes \sigma)\bar{X}_j |e_{\omega}\rangle = \tilde{x}_{j,\omega} (\rho \otimes \sigma) |e_{\omega}\rangle$$

defining that $\tilde{X}_j = \sum_{\omega} \tilde{x}_{j,\omega} |e_{\omega}\rangle \langle e_{\omega}|$, we see that \tilde{X}_j are commute and

$$\begin{aligned} (\rho \otimes \sigma)\bar{X}_j |e_{\omega}\rangle &= \tilde{x}_{j,\omega} (\rho \otimes \sigma) |e_{\omega}\rangle \\ \Rightarrow (\rho \otimes \sigma)\bar{X}_j |e_{\omega}\rangle \langle e_{\omega}| &= \tilde{x}_{j,\omega} (\rho \otimes \sigma) |e_{\omega}\rangle \langle e_{\omega}| \\ \Rightarrow \sum_{\omega} (\rho \otimes \sigma)\bar{X}_j |e_{\omega}\rangle \langle e_{\omega}| &= \sum_{\omega} \tilde{x}_{j,\omega} (\rho \otimes \sigma) |e_{\omega}\rangle \langle e_{\omega}| \\ \Rightarrow (\rho \otimes \sigma)\bar{X}_j &= (\rho \otimes \sigma)\tilde{X}_j \end{aligned}$$

where we have used the fact that $\sum_{\omega} |e_{\omega}\rangle \langle e_{\omega}| = \text{id}_{\mathcal{H} \otimes \mathcal{H}}$.

2 \Rightarrow **3**. Suppose there exists a basis $|e_{\omega}\rangle$ and real numbers $\tilde{x}_{j,\omega}$ such that $\rho \bar{X}_j |e_{\omega}\rangle = \tilde{x}_{j,\omega} \rho |e_{\omega}\rangle$, inner product both side by $|\psi_m\rangle$, eigenvector of $\rho \otimes \sigma$, we get that

$$\lambda_m \langle \psi_m, \bar{X}_j e_{\omega} \rangle = \tilde{x}_{j,\omega} \lambda_m \langle \psi_m, e_{\omega} \rangle$$

and for the above equation holds for any m , we then get that

$$\tilde{x}_{j,\omega} = \frac{\langle e_{\omega}, \bar{X}_j \psi_m \rangle}{\langle e_{\omega}, \psi_m \rangle}$$

$\tilde{x}_{j,\omega}$ is independent of m .

3 \Rightarrow **1**. Suppose that there exists basis $|e_{\omega}\rangle$ such that the value

$$\tilde{x}_{j,\omega} = \frac{\langle e_{\omega}, \bar{X}_j \psi_m \rangle}{\langle e_{\omega}, \psi_m \rangle}$$

is real and independent of m , defining that

$$\tilde{X}_j \equiv \sum_{\omega} \tilde{x}_{j,\omega} |e_{\omega}\rangle \langle e_{\omega}|$$

we then have that

$$\begin{aligned} \tilde{X}_j \rho &= \sum_{\omega m} \frac{\langle e_{\omega}, \bar{X}_j \psi_m \rangle}{\langle e_{\omega}, \psi_m \rangle} |e_{\omega}\rangle \langle e_{\omega}| \lambda_m |\psi_m\rangle \langle \psi_m| \\ &= \sum_{\omega m} \langle e_{\omega}, \bar{X}_j \psi_m \rangle \lambda_m |e_{\omega}\rangle \langle \psi_m| \end{aligned}$$



$$\begin{aligned}
&= \sum_{\omega m} |e_\omega\rangle \langle e_\omega, \bar{X}_j \lambda_m \psi_m\rangle \langle \psi_m| \\
&= \bar{X}_j \rho
\end{aligned}$$

and \bar{X}_j are all commutative.

1 \Rightarrow 4. Suppose that \tilde{X}_j are commuting observables and $\tilde{X}_j (\rho \otimes \sigma) = \bar{X}_j (\rho \otimes \sigma)$, then we have that

$$\begin{aligned}
&(\rho \otimes \sigma)^{1/2} [\bar{X}_i, \bar{X}_j] (\rho \otimes \sigma)^{1/2} \\
&= (\rho \otimes \sigma)^{1/2} \bar{X}_i \bar{X}_j (\rho \otimes \sigma)^{1/2} - (\rho \otimes \sigma)^{1/2} \bar{X}_j \bar{X}_i (\rho \otimes \sigma)^{1/2} \\
&= \left(\bar{X}_i (\rho \otimes \sigma)^{1/2} \right)^\dagger \bar{X}_j (\rho \otimes \sigma)^{1/2} - \left(\bar{X}_j (\rho \otimes \sigma)^{1/2} \right)^\dagger \bar{X}_i (\rho \otimes \sigma)^{1/2} \\
&= \left(\tilde{X}_i (\rho \otimes \sigma)^{1/2} \right)^\dagger \tilde{X}_j (\rho \otimes \sigma)^{1/2} - \left(\tilde{X}_j (\rho \otimes \sigma)^{1/2} \right)^\dagger \tilde{X}_i (\rho \otimes \sigma)^{1/2} \\
&= (\rho \otimes \sigma)^{1/2} \underbrace{[\tilde{X}_i, \tilde{X}_j]}_{=0} (\rho \otimes \sigma)^{1/2} = 0
\end{aligned}$$

4 \Rightarrow 5. By the fact that $(\rho \otimes \sigma)^{\frac{1}{2}} [\bar{X}_i, \bar{X}_j] (\rho \otimes \sigma)^{\frac{1}{2}} = 0$, we see that

$$\begin{aligned}
&[\bar{X}_i, \bar{X}_j] = [X_i^H, X_j^H] \otimes \text{id}_{\hat{\mathcal{H}}} + \text{id}_{\mathcal{H}} \otimes [B_i, B_j] \\
&\Rightarrow (\sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho}) \otimes \sigma + \rho \otimes (\sqrt{\sigma} [B_i, B_j] \sqrt{\sigma}) = 0
\end{aligned}$$

By the fact that $\sigma = |\varphi\rangle \langle \varphi|$, the above can be reduced to that

$$\begin{aligned}
&(\sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho}) \otimes \sigma + \rho \otimes (\langle \varphi | [B_i, B_j] | \varphi \rangle \sigma) = 0 \\
&\Rightarrow \left(\sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho} + \left(2i \text{Im} \hat{Z} \right)_{ji} \rho \right) \otimes \sigma = 0
\end{aligned}$$

By that $\hat{Z}_{ij} = \text{Tr} \sigma B_j B_i \Rightarrow 2i \text{Im} \hat{Z}_{ji} = \text{Tr} \sigma [B_i, B_j]$ and $\text{Im} \hat{Z} = -\text{Im} Z^H$, we have that

$$\sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho} = 2i (\text{Im} Z^H)_{ji} \rho = (\text{Tr} \rho [X_i^H, X_j^H]) \rho$$

5 \Rightarrow 4. Suppose that $\sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho} = \left(\text{Tr} \rho [X_i^H, X_j^H] \right) \rho$, by **Lemma 3.3**, there exists a Hilbert space $\hat{\mathcal{H}}$, pure state σ on it and set of observables such that $\text{Tr} \sigma B_j = 0$, $\text{Tr} \sigma B_j B_i = V_1^H - i \text{Im} Z^H$. Now, we have that $\bar{X}_j = X_j^H \otimes \text{id}_{\hat{\mathcal{H}}} + \text{id}_{\mathcal{H}} \otimes B_j$, and

$$\begin{aligned}
(\rho \otimes \sigma)^{1/2} [\bar{X}_i, \bar{X}_j] (\rho \otimes \sigma)^{1/2} &= \underbrace{\left(\sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho} + \left(2i \text{Im} \hat{Z} \right)_{ji} \rho \right)}_{=0, \text{ by } \sqrt{\rho} [X_i^H, X_j^H] \sqrt{\rho} = (\text{Tr} \rho [X_i^H, X_j^H]) \rho} \otimes \sigma = 0
\end{aligned}$$

Sufficiency for **Theorem 3.2**. Since we have that $\tilde{X}_j (\rho \otimes \sigma)^{\frac{1}{2}} = \bar{X}_j (\rho \otimes \sigma)^{\frac{1}{2}}$, we see that

$$\text{Tr} \tilde{X}_j \rho = \text{Tr} \bar{X}_j \rho = 0$$

$$\partial_i \text{Tr} (\rho \otimes \sigma) \tilde{X}_j = \text{Tr} \partial_i (\rho \otimes \sigma) \tilde{X}_j$$



$$\begin{aligned}
 &= 2\text{ReTr} \left(\partial_i (\rho \otimes \sigma)^{1/2} \right) \left(\tilde{X}_j (\rho \otimes \sigma)^{1/2} \right) \\
 &= 2\text{ReTr} \left(\partial_i (\rho \otimes \sigma)^{1/2} \right) \left(\bar{X}_j (\rho \otimes \sigma) \right) \\
 &= \text{Tr} \partial_i (\rho \otimes \sigma) \bar{X}_j = \delta_{ij}
 \end{aligned}$$

and

$$\begin{aligned}
 \text{Tr} (\rho \otimes \sigma) \tilde{X}_j \tilde{X}_i &= \text{Tr} \left(\tilde{X}_j (\rho \otimes \sigma)^{\frac{1}{2}} \right)^\dagger \tilde{X}_i (\rho \otimes \sigma)^{\frac{1}{2}} \\
 &= \text{Tr} \left(\bar{X}_j (\rho \otimes \sigma)^{\frac{1}{2}} \right)^\dagger \bar{X}_i (\rho \otimes \sigma)^{\frac{1}{2}} \\
 &= \text{Tr} (\rho \otimes \sigma) \bar{X}_j \bar{X}_i = V^H
 \end{aligned}$$

□

Observables \bar{X}_j are really special operators; they are locally unbiased with an optimal covariance matrix, but they are not necessarily commuting. The equation $\tilde{X}_j \rho = \bar{X}_j \rho$ means that from the viewpoint of the state, these two operators are equivalent. Condition 1 tells us that if one can find commutative observables equivalent to \bar{X}_j (from the perspective of ρ), then the HCRB is saturable. An estimation scheme consists of one measurement and statistics based on the measurement (the estimator). Condition 2 tells us that the construction of the optimal measurement is just solving an eigenvector equation on the support of ρ . Condition 3 tells us how to construct the estimator based on the measurement. Condition 4 means that if 1 is satisfied, then it is necessary for \bar{X}_j to commute on the support of the state. Condition 5 serves as an equivalent commutativity condition in the original space. Such a condition is independent of the extended space. That is, to check the commutativity condition, one can directly check in the original space and need not always extend the space to a larger space.

Condition 5 serves as a partial commutativity condition for the optimizer of the HCRB. Note that when the model is asymptotically classical, that is, $\text{Tr} \rho [L_i^S, L_j^S] = 0$ (see **Theorem 2.18**), the partial commutativity condition for the HCRB is reduced to the partial commutativity condition for the SLD operators, which is expressed as

$$\sqrt{\rho} [L_i^S, L_j^S] \sqrt{\rho} = 0. \quad (4.2)$$

Note that the above condition is necessary for the saturation of the SLDCRB. From our approach, Condition 1 is sufficient but not necessary for achieving the HCRB. Hence, Condition 5, which is necessary for Condition 1, is neither necessary nor sufficient for achieving the HCRB. However, it can be reduced to a necessary condition for achieving the SLD CRB.

In Gaussian state estimation, the canonical observables R_j satisfy the canonical commutation relation:

$$\frac{i}{2} [R_i, R_j] = S_{ij} \text{id}_{\mathcal{H}} \quad (4.3)$$

where S_{ij} is element of real skew-symmetric matrix. Multiplying both side of the equation by $\sqrt{\rho}$ from left and right, we have

$$\frac{i}{2} \sqrt{\rho} [R_i, R_j] \sqrt{\rho} = S_{ij} \rho$$



Therefore, partial commutativity condition 5 can be viewed as finite dimensional analogue of eq. (4.3). Note that it is impossible for any two finite dimensional hermitian operators to satisfy eq. (4.3), since for any two finite dimensional hermitian operator A, B , we have

$$\text{Tr}[A, B] = 0, \quad \text{Tr} \text{id}_{\mathcal{H}} = \dim \mathcal{H}$$

which implies that $[A, B] \not\propto \text{id}_{\mathcal{H}}$.

Proposition 4.2 (Theorem B.10, supplement material of [YFG13]). For any pure state parametric family, the HCRB is achievable.

Proof. For proof, see Theorem B.10 of [YFG13]. Essentially, it is because that the pure state model always satisfy condition 1 in **Theorem 4.1**. \square





Chapter 5

2-nd Characterization

Theorem 5.1 (Characterization of Achievability of HCRB). The following condition is necessary and sufficient for **Theorem 3.2**.

- There exists a Hilbert space $\hat{\mathcal{H}}$, a pure state σ on it, a PVM $|e_\omega\rangle \in \mathcal{H} \otimes \hat{\mathcal{H}}$ and orthonormal (w.r.t. Hilbert Schmidt inner product on $\mathcal{L}(\mathcal{H} \otimes \hat{\mathcal{H}})$) linear operators $\mathbf{b}_\omega \in \mathcal{L}(\mathcal{H} \otimes \hat{\mathcal{H}})$ such that

$$\begin{aligned} \text{span}_{\mathbb{R}} \{ \bar{X}_j \sqrt{\rho \otimes \sigma} \} &\subset \text{span}_{\mathbb{R}} \{ \mathbf{b}_\omega \} \\ \sqrt{\langle e_\omega, (\rho \otimes \sigma) e_\omega \rangle} &= \langle \mathbf{b}_\omega, \sqrt{\rho \otimes \sigma} \rangle_S \\ \partial_i \sqrt{\langle e_\omega, (\rho \otimes \sigma) e_\omega \rangle} &= \langle \mathbf{b}_\omega, \partial_i \sqrt{\rho \otimes \sigma} \rangle_S \end{aligned}$$

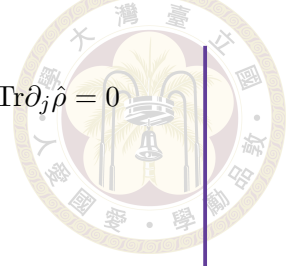
where \bar{X}_j is defined as in eq. (3.2), $\partial_i \sqrt{\rho \otimes \sigma}$ is derivative of operator $\sqrt{\rho \otimes \sigma}$ with respect to parameter θ_i , $\langle A, B \rangle_S \equiv \text{Re} \langle A, B \rangle_{\text{HS}}$.

Proof. In the following, we denote $\rho \otimes \sigma$ by $\hat{\rho}$, $\partial_i (\hat{\rho}^{1/2}) \equiv \hat{\rho}_{;i}^{1/2}$. (Condition 2 in **Theorem 3.2** \Rightarrow **Theorem 5.1**) First, observing that the following two Gram matrix are equal under inner product $\langle A, B \rangle_S = \text{Re} \langle A, B \rangle_{\text{HS}}$

$$\begin{aligned} &\begin{matrix} \tilde{X}_j \hat{\rho}^{1/2} & \hat{\rho}_{;j}^{1/2} & \hat{\rho}^{1/2} \\ \tilde{X}_i \hat{\rho}^{1/2} & \begin{pmatrix} V_{ij}^H & \frac{1}{2} \delta_{ij} & 0 \\ \frac{1}{2} \delta_{ij} & \langle \hat{\rho}_{;i}^{1/2}, \hat{\rho}_{;j}^{1/2} \rangle_S & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \hat{\rho}_{;i}^{1/2} & & \\ \hat{\rho}^{1/2} & & \end{matrix} \\ &\begin{matrix} \bar{X}_j \hat{\rho}^{1/2} & \hat{\rho}_{;j}^{1/2} & \hat{\rho}^{1/2} \\ \bar{X}_i \hat{\rho}^{1/2} & \begin{pmatrix} V_{ij}^H & \frac{1}{2} \delta_{ij} & 0 \\ \frac{1}{2} \delta_{ij} & \langle \hat{\rho}_{;i}^{1/2}, \hat{\rho}_{;j}^{1/2} \rangle_S & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \hat{\rho}_{;i}^{1/2} & & \\ \hat{\rho}^{1/2} & & \end{matrix} \end{aligned}$$

where we have used the facts that

$$\begin{aligned} \delta_{ij} &= \text{Tr} \partial_i \hat{\rho} \tilde{X}_j = \left(\text{Tr} \hat{\rho}_{;i}^{1/2} \tilde{X}_j \hat{\rho}^{1/2} + \hat{\rho}^{1/2} \tilde{X}_j \hat{\rho}_{;i}^{1/2} \right) \\ &= 2 \text{Re} \langle \hat{\rho}_{;i}^{1/2}, \tilde{X}_j \hat{\rho}^{1/2} \rangle_{\text{HS}} = 2 \langle \hat{\rho}_{;i}^{1/2}, \tilde{X}_j \hat{\rho}^{1/2} \rangle_S \end{aligned}$$



$$\langle \hat{\rho}^{1/2}, \hat{\rho}_{;j}^{1/2} \rangle_S = \text{Re} \langle \hat{\rho}^{1/2}, \hat{\rho}_{;j}^{1/2} \rangle_{\text{HS}} = \text{Tr} \left(\hat{\rho}^{1/2} \hat{\rho}_{;j}^{1/2} + \hat{\rho}_{;j}^{1/2} \hat{\rho}^{1/2} \right) = \text{Tr} \partial_j \hat{\rho} = 0$$

Defining $\mathbf{b}'_\omega = \frac{1}{\sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle}} |e_\omega\rangle \langle e_\omega| \hat{\rho}^{1/2}$, we have

$$\begin{aligned} \text{Tr} \mathbf{b}'_\omega{}^\dagger \mathbf{b}'_\omega &= \text{Tr} \hat{\rho}^{1/2} (|e_{\omega'}\rangle \langle e_{\omega'}|) (|e_\omega\rangle \langle e_\omega|) \hat{\rho}^{1/2} \\ &= \frac{\delta_{\omega'\omega}}{\sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle} \sqrt{\langle e_{\omega'}, \hat{\rho} e_{\omega'} \rangle}} \langle e_{\omega'}, \hat{\rho} e_\omega \rangle \\ &= \delta_{\omega'\omega} \end{aligned}$$

which means

$$\langle \mathbf{b}'_{\omega'}, \mathbf{b}'_\omega \rangle_S = \langle \mathbf{b}'_{\omega'}, \mathbf{b}'_\omega \rangle_{\text{HS}} = \delta_{\omega'\omega}$$

Further,

$$\begin{aligned} \langle \mathbf{b}'_\omega, \hat{\rho}^{1/2} \rangle_S &= \langle \mathbf{b}'_\omega, \hat{\rho}^{1/2} \rangle_{\text{HS}} = \sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle} \\ \langle \mathbf{b}'_\omega, \hat{\rho}_{;i}^{1/2} \rangle_S &= \frac{1}{\sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle}} \langle |e_\omega\rangle \langle e_\omega| \hat{\rho}^{1/2}, \hat{\rho}_{;i}^{1/2} \rangle_S \\ &= \frac{1}{2\sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle}} \langle e_\omega, \left(\hat{\rho}_{;i}^{1/2} \hat{\rho}^{1/2} + \hat{\rho}^{1/2} \hat{\rho}_{;i}^{1/2} \right) e_\omega \rangle \\ &= \frac{1}{2\sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle}} \langle e_\omega, \partial_i \hat{\rho} e_\omega \rangle = \partial_i \sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle} \end{aligned}$$

by **Lemma 3.5** and **Lemma 3.6**, there exists orthonormal (with respect to Hilbert-Schmidt inner product) linear operators \mathbf{b}_ω such that

$$\begin{aligned} \langle \mathbf{b}'_\omega, \hat{\rho}^{1/2} \rangle_S &= \langle \mathbf{b}_\omega, \hat{\rho}^{1/2} \rangle_S = \sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle} \\ \langle \mathbf{b}'_\omega, \hat{\rho}_{;i}^{1/2} \rangle_S &= \langle \mathbf{b}_\omega, \hat{\rho}_{;i}^{1/2} \rangle_S = \partial_i \sqrt{\langle e_\omega, \hat{\rho} e_\omega \rangle} \end{aligned}$$

(**Theorem 5.1** \Rightarrow Condition 2 in **Theorem 3.2**) Assume **Theorem 5.1**, define that

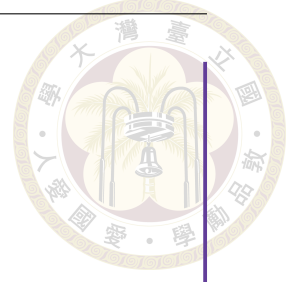
$$\tilde{x}_{j,\omega} = \frac{\langle \mathbf{b}_\omega, \bar{X}_j \hat{\rho}^{1/2} \rangle_S}{\langle \mathbf{b}_\omega, \hat{\rho}^{1/2} \rangle_S}$$

and let $\tilde{X}_j \equiv \sum_\omega \tilde{x}_{j,\omega} |e_\omega\rangle \langle e_\omega|$. It is obvious that \tilde{X}_j are all commute, it is suffices to show that \tilde{X}_j admits optimum covariance matrix and is locally unbiased, that is, \tilde{X}_j satisfy the following:

$$\begin{aligned} \text{Tr} \tilde{X}_j \hat{\rho} \tilde{X}_j &= V_{ij}^{\text{H}} \\ \text{Tr} \hat{\rho} \tilde{X}_j &= 0, \quad \text{Tr} \partial_i \hat{\rho} \tilde{X}_j = \delta_{ij} \end{aligned}$$

for optimum covariance matrix, we have that

$$\begin{aligned} \text{Tr} \tilde{X}_i \hat{\rho} \tilde{X}_j &= \sum_\omega \tilde{x}_{i,\omega} \tilde{x}_{j,\omega} \langle e_\omega, \hat{\rho} e_\omega \rangle \\ &= \sum_\omega \frac{\langle \mathbf{b}_\omega, \bar{X}_i \hat{\rho}^{1/2} \rangle_S \langle \mathbf{b}_\omega, \bar{X}_j \hat{\rho}^{1/2} \rangle_S}{\langle \mathbf{b}_\omega, \hat{\rho}^{1/2} \rangle_S^2} \langle \mathbf{b}_\omega, \hat{\rho}^{1/2} \rangle_S^2 \end{aligned}$$



$$\begin{aligned}
&= \sum_{\omega} \langle \bar{X}_i \hat{\rho}^{1/2}, \mathbf{b}_{\omega} \rangle_S \langle \mathbf{b}_{\omega}, \bar{X}_j \hat{\rho}^{1/2} \rangle_S \\
&= \langle \bar{X}_i \hat{\rho}^{1/2}, \bar{X}_j \hat{\rho}^{1/2} \rangle_S = V_{ij}^H
\end{aligned}$$

for locally unbiasedness,

$$\begin{aligned}
\text{Tr} \hat{\rho} \tilde{X}_j &= \sum_{\omega} \tilde{x}_{j,\omega} \langle e_{\omega}, \hat{\rho} e_{\omega} \rangle \\
&= \sum_{\omega} \frac{\langle \mathbf{b}_{\omega}, \bar{X}_j \hat{\rho}^{1/2} \rangle_S}{\langle \mathbf{b}_{\omega}, \hat{\rho}^{1/2} \rangle_S} \langle \mathbf{b}_{\omega}, \hat{\rho}^{1/2} \rangle_S^2 \\
&= \sum_{\omega} \langle \hat{\rho}^{1/2}, \mathbf{b}_{\omega} \rangle_S \langle \mathbf{b}_{\omega}, \bar{X}_j \hat{\rho}^{1/2} \rangle_S \\
&= \langle \hat{\rho}^{1/2}, \bar{X}_j \hat{\rho}^{1/2} \rangle_S = \text{Tr} \hat{\rho} \bar{X}_j = 0
\end{aligned}$$

$$\begin{aligned}
\text{Tr} \partial_i \hat{\rho} \tilde{X}_j &= \sum_{\omega} \tilde{x}_{j,\omega} \underbrace{\partial_i \langle e_{\omega}, \hat{\rho} e_{\omega} \rangle}_{\langle e_{\omega}, \partial_i \hat{\rho} e_{\omega} \rangle} \\
&= \sum_{\omega} \frac{\langle \mathbf{b}_{\omega}, \bar{X}_j \hat{\rho}^{1/2} \rangle_S}{\langle \mathbf{b}_{\omega}, \hat{\rho}^{1/2} \rangle_S} 2 \sqrt{\langle e_{\omega}, \hat{\rho} e_{\omega} \rangle} \partial_i \sqrt{\langle e_{\omega}, \hat{\rho} e_{\omega} \rangle} \\
&= 2 \sum_{\omega} \langle \bar{X}_j \hat{\rho}^{1/2}, \mathbf{b}_{\omega} \rangle_S \langle \mathbf{b}_{\omega}, \hat{\rho}_{;i}^{1/2} \rangle_S \\
&= 2 \langle \bar{X}_j \hat{\rho}^{1/2}, \hat{\rho}_{;i}^{1/2} \rangle_S = \delta_{ij}
\end{aligned}$$

□

Equivalence between Condition 2 in **Theorem 3.2** and **Theorem 5.1** states that, optimal estimator exists if and only if it can be constructed by component of $\bar{X}_j (\rho \otimes \sigma)^{1/2}$ (under some basis \mathbf{b}_{ω}); the probability given by optimal measurement $|e_{\omega}\rangle$ and the probability given by the basis \mathbf{b}_{ω} is equal up to first order (at true value θ_0), that is:

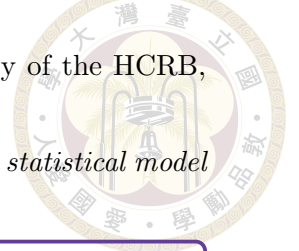
$$\langle \mathbf{b}_{\omega}, (\rho \otimes \sigma)^{1/2} \rangle_S = \sqrt{\langle e_{\omega}, (\rho \otimes \sigma) e_{\omega} \rangle}, \quad \langle \mathbf{b}_{\omega}, \partial_i (\rho \otimes \sigma)^{1/2} \rangle_S = \partial_i \sqrt{\langle e_{\omega}, (\rho \otimes \sigma) e_{\omega} \rangle}$$

From the construction of the estimator, we see that given *any* orthonormal linear operator \mathbf{b}_{ω} (with respect to inner product $\langle \cdot, \cdot \rangle_S$) with $\text{span}_{\mathbb{R}} \{\mathbf{b}_{\omega}\} \supset \text{span}_{\mathbb{R}} \{\bar{X}_j \hat{\rho}^{1/2}, \hat{\rho}^{1/2}\}$, the estimator:

$$\hat{\theta}_j(\omega) - \theta_{0,j} \equiv \tilde{x}_{j,\omega} = \frac{\langle \mathbf{b}_{\omega}, \bar{X}_j (\rho \otimes \sigma)^{1/2} \rangle_S}{\langle \mathbf{b}_{\omega}, (\rho \otimes \sigma)^{1/2} \rangle_S}$$

with the probability distribution: $p_{\omega} \equiv \langle \mathbf{b}_{\omega}, (\rho \otimes \sigma)^{1/2} \rangle_S^2$, are always locally unbiased and admits optimal covariance. There not always, however, exists a PVM $|e_{\omega}\rangle$ with

$$\begin{aligned}
\langle e_{\omega}, (\rho \otimes \sigma) e_{\omega} \rangle &= \langle \mathbf{b}_{\omega}, (\rho \otimes \sigma)^{1/2} \rangle_S^2 \\
\partial_i \sqrt{\langle e_{\omega}, (\rho \otimes \sigma) e_{\omega} \rangle} &= \langle \mathbf{b}_{\omega}, \partial_i (\rho \otimes \sigma)^{1/2} \rangle_S
\end{aligned}$$



Theorem 5.1 is not only the characterization for the achievability of the HCRB, but also applicable to the achievability of other bounds.

Corollary 5.2. *The SLD CRB is achievable if and only if the quantum statistical model satisfies **Theorem 5.1** and **Theorem 2.18**.*

Proof. By **Theorem 2.18**, we have $C^H = C^S$. When **Theorem 5.1** is satisfied, C^H is achievable. Hence, C^S is achievable. □

Similarly, we have the following corollary.

Corollary 5.3. *The RLD CRB is achievable if and only if the quantum statistical model satisfies **Theorem 5.1** and **Theorem 2.19**.*

In the following we demonstrate some examples which the HCRB is achievable.

Example 5.4 (Pure state model). Let $\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ be a pure state model, $\hat{\mathcal{H}}, \sigma \equiv |\varphi\rangle\langle\varphi|$, B_j and \bar{X}_j be defined as in **Proposition 3.4**. Defining that $|l_j\rangle = \bar{X}_j|\psi\rangle|\varphi\rangle$, we have

$$\text{Tr}\bar{X}_i(\rho \otimes \sigma)\bar{X}_j = \langle\psi|\langle\varphi|\bar{X}_i\bar{X}_j|\psi\rangle|\varphi\rangle = V_{ij}^H$$

is real matrix, there exists an orthonormal basis $|e_\omega\rangle \in \mathcal{H} \otimes \hat{\mathcal{H}}$ such that $\langle e_\omega, l_j\rangle$ are all real. Let

$$\mathbf{b}_\omega \equiv |e_\omega\rangle\langle\psi \otimes \varphi|$$

It is straightforward to verify that $\mathbf{b}_\omega, |e_\omega\rangle$ satisfy **Theorem 5.1**.

Example 5.5. Consider the model which satisfies **Theorem 4.1**. Let $|e_\omega\rangle$ be eigenbasis of observables \tilde{X}_j , and $\mathbf{b}_\omega \equiv |e_\omega\rangle\langle e_\omega|(\rho \otimes \sigma)^{1/2}$, it is straightforward to show that $\mathbf{b}_\omega, |e_\omega\rangle$ satisfy **Theorem 5.1**.



Chapter 6

Summary and Outlook

6.1 Summary

In this thesis, we demonstrate two characterizations for the achievability of the HCRB: **Theorem 4.1** and **Theorem 5.1**. They characterize several conditions for the existence of the estimator and the optimal measurement that saturate the HCRB at the single-shot level.

Theorem 4.1 is a sufficient but not necessary condition for the achievability of the HCRB. However, the operator characterization (eq. (4.1)) admits significant connections to other physical theories. In particular, with **Theorem 2.18**, it can be reduced to a necessary condition for the achievability of the SLD CRB. On the other hand, the form of eq. (4.1) is also similar to the CCR algebra in Gaussian states, and hence, it serves as a finite-dimensional analogue of the CCR algebra. Therefore, it can be viewed as a clue to establish the physical meaning of the achievability of the HCRB.

Theorem 5.1 is both necessary and sufficient for the achievability of the HCRB. It states that the optimal (locally unbiased) estimator can be constructed via components of the operator $\bar{X}_j (\rho \otimes \sigma)^{1/2}$ (under some basis \mathbf{b}_ω), and the probability generated by the optimal measurement ($\sqrt{p} = \sqrt{e_\omega, (\rho \otimes \sigma) e_\omega}$) is equal to the probability generated by the basis \mathbf{b}_ω ($\langle \mathbf{b}_\omega, (\rho \otimes \sigma) \rangle_{\mathcal{S}}$) up to the first order. The theorem not only characterizes the achievability of the HCRB but also establishes the achievability of other quantum Cramer-Rao bounds (**Corollary 5.2** and **Corollary 5.3**).

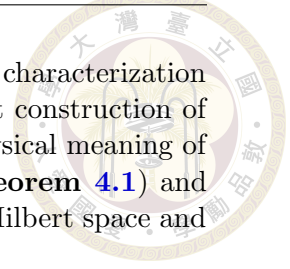
6.2 Outlook

Our work potentially admits many applications, including quantum channel estimation problems and quantum information geometry. Our characterizations (**Theorem 4.1** and **Theorem 5.1**), however, each admit their drawbacks.

Theorem 4.1 is not a necessary condition for the achievability of the HCRB. In particular, eq. (4.1) is neither necessary nor sufficient for the achievability of the HCRB. A future objective would be to modify **Theorem 4.1** to a necessary condition while preserving the physical characterization (eq. (4.1)). The author believes that this condition potentially serves as an information geometrical characterization for quantum statistical models, just like the role of the partial commutativity condition (**Theorem 2.6**) for the SLD CRB, and the \mathcal{D} -invariant model for the RLD CRB.

Theorem 5.1 is both a necessary and sufficient condition for the achievability of the HCRB. However, the characterization is a mathematical one and does not give

a physical picture of the achievability of the HCRB. Moreover, the characterization depends on an extended Hilbert space, and we do not give an explicit construction of the optimal measurement. A future objective would be to find the physical meaning of **Theorem 5.1** (like the operator characterization in eq. (4.1) in **Theorem 4.1**) and characterize **Theorem 5.1** so that it can dispense with the extended Hilbert space and provide an explicit construction of the optimal measurement.





Bibliography

- [Abi+23] Paolo Abiuso et al. “Characterizing (non-)Markovianity through Fisher information”. In: *SciPost Physics* 15.1 (July 2023). ISSN: 2542-4653. DOI: [10.21468/scipostphys.15.1.014](https://doi.org/10.21468/scipostphys.15.1.014). URL: <http://dx.doi.org/10.21468/SciPostPhys.15.1.014>.
- [AD22] Francesco Albarelli and Rafał Demkowicz-Dobrzański. “Probe Incompatibility in Multiparameter Noisy Quantum Metrology”. In: *Phys. Rev. X* 12 (1 Mar. 2022), p. 011039. DOI: [10.1103/PhysRevX.12.011039](https://doi.org/10.1103/PhysRevX.12.011039). URL: <https://link.aps.org/doi/10.1103/PhysRevX.12.011039>.
- [Adh14] Rana X. Adhikari. “Gravitational radiation detection with laser interferometry”. In: *Rev. Mod. Phys.* 86 (1 Feb. 2014), pp. 121–151. DOI: [10.1103/RevModPhys.86.121](https://doi.org/10.1103/RevModPhys.86.121). URL: <https://link.aps.org/doi/10.1103/RevModPhys.86.121>.
- [AFD19] Francesco Albarelli, Jamie F. Friel, and Animesh Datta. “Evaluating the Holevo Cramér-Rao Bound for Multiparameter Quantum Metrology”. In: *Phys. Rev. Lett.* 123 (20 Nov. 2019), p. 200503. DOI: [10.1103/PhysRevLett.123.200503](https://doi.org/10.1103/PhysRevLett.123.200503). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.123.200503>.
- [Alb+20] F. Albarelli et al. “A perspective on multiparameter quantum metrology: From theoretical tools to applications in quantum imaging”. In: *Physics Letters A* 384.12 (2020), p. 126311. ISSN: 0375-9601. DOI: <https://doi.org/10.1016/j.physleta.2020.126311>. URL: <https://www.sciencedirect.com/science/article/pii/S0375960120301109>.
- [AN00] Shun-ichi Amari and Hiroshi Nagaoka. *Methods of Information Geometry*. Trans. by Daishi Harada. Vol. 191. Translations of Mathematical Monographs. Translated by Daishi Harada. Providence, RI: American Mathematical Society, 2000. ISBN: 978-0-8218-4302-4. DOI: [10.1090/mmono/191](https://doi.org/10.1090/mmono/191).
- [Arr+14] G. Arrad et al. “Increasing Sensing Resolution with Error Correction”. In: *Phys. Rev. Lett.* 112 (15 Apr. 2014), p. 150801. DOI: [10.1103/PhysRevLett.112.150801](https://doi.org/10.1103/PhysRevLett.112.150801). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.112.150801>.
- [ATD20] Francesco Albarelli, Mankei Tsang, and Animesh Datta. *Upper bounds on the Holevo Cramér-Rao bound for multiparameter quantum parametric and semiparametric estimation*. 2020. arXiv: [1911.11036](https://arxiv.org/abs/1911.11036) [quant-ph].
- [BC94] Samuel L. Braunstein and Carlton M. Caves. “Statistical distance and the geometry of quantum states”. In: *Phys. Rev. Lett.* 72 (22 May 1994), pp. 3439–3443. DOI: [10.1103/PhysRevLett.72.3439](https://doi.org/10.1103/PhysRevLett.72.3439). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.72.3439>.

- [BD16] Tillmann Baumgratz and Animesh Datta. “Quantum Enhanced Estimation of a Multidimensional Field”. In: *Phys. Rev. Lett.* 116 (3 Jan. 2016), p. 030801. DOI: [10.1103/PhysRevLett.116.030801](https://doi.org/10.1103/PhysRevLett.116.030801). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.116.030801>.
- [Bud23] Agung Budiyo. “Operational interpretation and estimation of quantum trace-norm asymmetry based on weak-value measurement and some bounds”. In: *Phys. Rev. A* 108 (1 July 2023), p. 012431. DOI: [10.1103/PhysRevA.108.012431](https://doi.org/10.1103/PhysRevA.108.012431). URL: <https://link.aps.org/doi/10.1103/PhysRevA.108.012431>.
- [CCY22a] Hongzhen Chen, Yu Chen, and Haidong Yuan. “Incompatibility measures in multiparameter quantum estimation under hierarchical quantum measurements”. In: *Phys. Rev. A* 105 (6 June 2022), p. 062442. DOI: [10.1103/PhysRevA.105.062442](https://doi.org/10.1103/PhysRevA.105.062442). URL: <https://link.aps.org/doi/10.1103/PhysRevA.105.062442>.
- [CCY22b] Hongzhen Chen, Yu Chen, and Haidong Yuan. “Information Geometry under Hierarchical Quantum Measurement”. In: *Phys. Rev. Lett.* 128 (25 June 2022), p. 250502. DOI: [10.1103/PhysRevLett.128.250502](https://doi.org/10.1103/PhysRevLett.128.250502). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.128.250502>.
- [CH23] Zishi Chen and Xueyuan Hu. “Resource theory of dephasing estimation in multiqubit systems”. In: *Phys. Rev. A* 108 (3 Sept. 2023), p. 032415. DOI: [10.1103/PhysRevA.108.032415](https://doi.org/10.1103/PhysRevA.108.032415). URL: <https://link.aps.org/doi/10.1103/PhysRevA.108.032415>.
- [Con+21] Lorcan O Conlon et al. “Efficient computation of the Nagaoka–Hayashi bound for multiparameter estimation with separable measurements”. In: *npj Quantum Information* 7.1 (2021), p. 110. DOI: [10.1038/s41534-021-00414-1](https://doi.org/10.1038/s41534-021-00414-1). URL: <https://doi.org/10.1038/s41534-021-00414-1>.
- [Con+23] Lorcan Conlon et al. “The Gap Persistence Theorem Between Nagaoka–Hayashi Bound and Holevo Bound for Quantum Multiparameter Estimation”. In: *IEICE Technical Report; IEICE Tech. Rep.* 123.14 (2023), pp. 55–60.
- [DBS13] Rafał Demkowicz-Dobrzański, Konrad Banaszek, and Roman Schnabel. “Fundamental quantum interferometry bound for the squeezed-light-enhanced gravitational wave detector GEO 600”. In: *Phys. Rev. A* 88 (4 Oct. 2013), p. 041802. DOI: [10.1103/PhysRevA.88.041802](https://doi.org/10.1103/PhysRevA.88.041802). URL: <https://link.aps.org/doi/10.1103/PhysRevA.88.041802>.
- [DGG20] Rafał Demkowicz-Dobrzański, Wojciech Górecki, and Mădălin Guță. “Multiparameter estimation beyond quantum Fisher information”. In: *Journal of Physics A: Mathematical and Theoretical* 53.36 (Aug. 2020), p. 363001. DOI: [10.1088/1751-8121/ab8ef3](https://doi.org/10.1088/1751-8121/ab8ef3). URL: <https://dx.doi.org/10.1088/1751-8121/ab8ef3>.
- [Dor+09] U. Dorner et al. “Optimal Quantum Phase Estimation”. In: *Phys. Rev. Lett.* 102 (4 Jan. 2009), p. 040403. DOI: [10.1103/PhysRevLett.102.040403](https://doi.org/10.1103/PhysRevLett.102.040403). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.102.040403>.

- [Dür+14] W. Dür et al. “Improved Quantum Metrology Using Quantum Error Correction”. In: *Phys. Rev. Lett.* 112 (8 Feb. 2014), p. 080801. DOI: [10.1103/PhysRevLett.112.080801](https://doi.org/10.1103/PhysRevLett.112.080801). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.112.080801>.
- [EMD11] BM Escher, Ruynet Lima de Matos Filho, and Luiz Davidovich. “General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology”. In: *Nature Physics* 7.5 (May 2011), pp. 406–411. DOI: [10.1038/nphys1958](https://doi.org/10.1038/nphys1958). URL: <https://doi.org/10.1038/nphys1958>.
- [Gao+23] Li Gao et al. *Sufficient statistic and recoverability via Quantum Fisher Information metrics*. 2023. arXiv: [2302.02341](https://arxiv.org/abs/2302.02341) [quant-ph].
- [GK06] Mădălin Guță and Jonas Kahn. “Local asymptotic normality for qubit states”. In: *Phys. Rev. A* 73 (5 May 2006), p. 052108. DOI: [10.1103/PhysRevA.73.052108](https://doi.org/10.1103/PhysRevA.73.052108). URL: <https://link.aps.org/doi/10.1103/PhysRevA.73.052108>.
- [Gór+20] Wojciech Górecki et al. “Optimal probes and error-correction schemes in multi-parameter quantum metrology”. In: *Quantum* 4 (July 2020), p. 288. ISSN: 2521-327X. DOI: [10.22331/q-2020-07-02-288](https://doi.org/10.22331/q-2020-07-02-288). URL: <https://doi.org/10.22331/q-2020-07-02-288>.
- [Hay02] Masahito Hayashi. “Two quantum analogues of Fisher information from a large deviation viewpoint of quantum estimation”. In: *Journal of Physics A: Mathematical and General* 35.36 (Aug. 2002), p. 7689. DOI: [10.1088/0305-4470/35/36/302](https://doi.org/10.1088/0305-4470/35/36/302). URL: <https://dx.doi.org/10.1088/0305-4470/35/36/302>.
- [Hay05] Masahito Hayashi. *Asymptotic theory of quantum statistical inference: selected papers*. World Scientific, 2005.
- [Hay16] Masahito Hayashi. *Quantum information theory*. Springer Berlin, Heidelberg, 2016. DOI: [10.1007/978-3-662-49725-8](https://doi.org/10.1007/978-3-662-49725-8). URL: <https://doi.org/10.1007/978-3-662-49725-8>.
- [Hay97] Masahito Hayashi. “A Linear Programming Approach to Attainable Cramér-Rao Type Bounds”. In: *Quantum Communication, Computing, and Measurement*. Ed. by O. Hirota, A. S. Holevo, and C. M. Caves. Boston, MA: Springer US, 1997, pp. 99–108. ISBN: 978-1-4615-5923-8. DOI: [10.1007/978-1-4615-5923-8_11](https://doi.org/10.1007/978-1-4615-5923-8_11). URL: https://doi.org/10.1007/978-1-4615-5923-8_11.
- [Hel67] C.W. Helstrom. “Minimum mean-squared error of estimates in quantum statistics”. In: *Physics Letters A* 25.2 (1967), pp. 101–102. ISSN: 0375-9601. DOI: [https://doi.org/10.1016/0375-9601\(67\)90366-0](https://doi.org/10.1016/0375-9601(67)90366-0). URL: <https://www.sciencedirect.com/science/article/pii/0375960167903660>.
- [Hel69] Carl W. Helstrom. “Quantum Detection and Estimation Theory”. In: *Journal of Statistical Physics* 1.2 (June 1, 1969), pp. 231–252. ISSN: 1572-9613. DOI: [10.1007/BF01007479](https://doi.org/10.1007/BF01007479). URL: <https://doi.org/10.1007/BF01007479>.

- [HM08] Masahito Hayashi and Keiji Matsumoto. “Asymptotic performance of optimal state estimation in qubit system”. In: *Journal of Mathematical Physics* 49.10 (Oct. 2008), p. 102101. ISSN: 0022-2488. DOI: [10.1063/1.2988130](https://doi.org/10.1063/1.2988130). eprint: https://pubs.aip.org/aip/jmp/article-pdf/doi/10.1063/1.2988130/15610995/102101_1_online.pdf. URL: <https://doi.org/10.1063/1.2988130>.
- [HO23] Masahito Hayashi and Yingkai Ouyang. “Tight Cramér-Rao type bounds for multiparameter quantum metrology through conic programming”. In: *Quantum* 7 (Aug. 2023), p. 1094. ISSN: 2521-327X. DOI: [10.22331/q-2023-08-29-1094](https://doi.org/10.22331/q-2023-08-29-1094). URL: <https://doi.org/10.22331/q-2023-08-29-1094>.
- [Hol11] Alexander S Holevo. *Probabilistic and statistical aspects of quantum theory*. Vol. 1. Edizioni della Normale Pisa, 2011. DOI: [10.1007/978-88-7642-378-9](https://doi.org/10.1007/978-88-7642-378-9). URL: <https://doi.org/10.1007/978-88-7642-378-9>.
- [Hum+13] Peter C. Humphreys et al. “Quantum Enhanced Multiple Phase Estimation”. In: *Phys. Rev. Lett.* 111 (7 Aug. 2013), p. 070403. DOI: [10.1103/PhysRevLett.111.070403](https://link.aps.org/doi/10.1103/PhysRevLett.111.070403). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.111.070403>.
- [JAG19] H. Rangani Jahromi, M. Amini, and M. Ghanaatian. “Multiparameter estimation, lower bound on quantum Fisher information, and non-Markovianity witnesses of noisy two-qubit systems”. In: *Quantum Information Processing* 18.11 (2019), p. 338. DOI: [10.1007/s11128-019-2446-8](https://doi.org/10.1007/s11128-019-2446-8). URL: <https://doi.org/10.1007/s11128-019-2446-8>.
- [Kes+14] E. M. Kessler et al. “Quantum Error Correction for Metrology”. In: *Phys. Rev. Lett.* 112 (15 Apr. 2014), p. 150802. DOI: [10.1103/PhysRevLett.112.150802](https://link.aps.org/doi/10.1103/PhysRevLett.112.150802). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.112.150802>.
- [KG09] Jonas Kahn and Mădălin Guță. “Local asymptotic normality for finite dimensional quantum systems”. In: *Communications in Mathematical Physics* 289.2 (July 2009), pp. 597–652. DOI: [10.1007/s00220-009-0787-3](https://doi.org/10.1007/s00220-009-0787-3). URL: <https://doi.org/10.1007/s00220-009-0787-3>.
- [Liu+19a] Jing Liu et al. “Quantum Fisher information matrix and multiparameter estimation”. In: *Journal of Physics A: Mathematical and Theoretical* 53.2 (Dec. 2019), p. 023001. DOI: [10.1088/1751-8121/ab5d4d](https://dx.doi.org/10.1088/1751-8121/ab5d4d). URL: <https://dx.doi.org/10.1088/1751-8121/ab5d4d>.
- [Liu+19b] Jing Liu et al. “Quantum Fisher information matrix and multiparameter estimation”. In: *Journal of Physics A: Mathematical and Theoretical* 53.2 (Dec. 2019), p. 023001. DOI: [10.1088/1751-8121/ab5d4d](https://dx.doi.org/10.1088/1751-8121/ab5d4d). URL: <https://dx.doi.org/10.1088/1751-8121/ab5d4d>.
- [Liu+22] Jing Liu et al. “Optimal Scheme for Quantum Metrology”. In: *Advanced Quantum Technologies* 5.1 (2022), p. 2100080. DOI: <https://doi.org/10.1002/qute.202100080>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/qute.202100080>.
- [LJW15] Jing Liu, Xiao-Xing Jing, and Xiaoguang Wang. “Quantum metrology with unitary parametrization processes”. In: *Scientific reports* 5.1 (Feb. 2015), p. 8565. DOI: [10.1038/srep08565](https://doi.org/10.1038/srep08565). URL: <https://doi.org/10.1038/srep08565>.

- [Mar22] Iman Marvian. “Operational Interpretation of Quantum Fisher Information in Quantum Thermodynamics”. In: *Phys. Rev. Lett.* 129 (19 Oct. 2022), p. 190502. DOI: [10.1103/PhysRevLett.129.190502](https://doi.org/10.1103/PhysRevLett.129.190502). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.129.190502>.
- [Mat02] K Matsumoto. “A new approach to the Cramér-Rao-type bound of the pure-state model”. In: *Journal of Physics A: Mathematical and General* 35.13 (Mar. 2002), p. 3111. DOI: [10.1088/0305-4470/35/13/307](https://doi.org/10.1088/0305-4470/35/13/307). URL: <https://dx.doi.org/10.1088/0305-4470/35/13/307>.
- [MD16] Tillmann Baumgratz Magdalena Szczykulska and Animesh Datta. “Multi-parameter quantum metrology”. In: *Advances in Physics: X* 1.4 (2016), pp. 621–639. DOI: [10.1080/23746149.2016.1230476](https://doi.org/10.1080/23746149.2016.1230476). URL: <https://doi.org/10.1080/23746149.2016.1230476>.
- [Nag05] H. Nagaoka. “A new approach to Cramér-Rao bounds for quantum state estimation”. In: *Asymptotic Theory of Quantum Statistical Inference: Selected Papers*. Ed. by M. Hayashi. IEICE Technical Report, 89, 228, IT 89-42, 9-14, (1989). Singapore: World Scientific, 2005, pp. 100–112.
- [NF23] Hiroshi Nagaoka and Akio Fujiwara. *Autoparallelity of Quantum Statistical Manifolds in The Light of Quantum Estimation Theory*. 2023. arXiv: [2307.03431](https://arxiv.org/abs/2307.03431) [quant-ph].
- [Nur24] Hendra I. Nurdin. “Saturability of the Quantum Cramér-Rao Bound in Multiparameter Quantum Estimation at the Single-Copy Level”. In: *IEEE Control Systems Letters* 8 (2024), pp. 376–381. DOI: [10.1109/LCSYS.2024.3382330](https://doi.org/10.1109/LCSYS.2024.3382330).
- [Par09] Matteo G. A. Paris. “Quantum Estimation for quantum technology”. In: *International Journal of Quantum Information* 07.suppl01 (2009), pp. 125–137. DOI: [10.1142/S0219749909004839](https://doi.org/10.1142/S0219749909004839). eprint: <https://doi.org/10.1142/S0219749909004839>. URL: <https://doi.org/10.1142/S0219749909004839>.
- [Pet96] Dénes Petz. “Monotone metrics on matrix spaces”. In: *Linear Algebra and its Applications* 244 (1996), pp. 81–96. ISSN: 0024-3795. DOI: [https://doi.org/10.1016/0024-3795\(94\)00211-8](https://doi.org/10.1016/0024-3795(94)00211-8). URL: <https://www.sciencedirect.com/science/article/pii/0024379594002118>.
- [Pez+17] Luca Pezzè et al. “Optimal Measurements for Simultaneous Quantum Estimation of Multiple Phases”. In: *Phys. Rev. Lett.* 119 (13 Sept. 2017), p. 130504. DOI: [10.1103/PhysRevLett.119.130504](https://doi.org/10.1103/PhysRevLett.119.130504). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.119.130504>.
- [PG11] D. PETZ and C. GHINEA. “Introduction to quantum Fisher information”. In: *Quantum Probability and Related Topics*. WORLD SCIENTIFIC, Jan. 2011. DOI: [10.1142/9789814338745_0015](https://doi.org/10.1142/9789814338745_0015). URL: http://dx.doi.org/10.1142/9789814338745_0015.
- [RJD16] Sammy Ragy, Marcin Jarzyna, and Rafał Demkowicz-Dobrzański. “Compatibility in multiparameter quantum metrology”. In: *Phys. Rev. A* 94 (5 Nov. 2016), p. 052108. DOI: [10.1103/PhysRevA.94.052108](https://doi.org/10.1103/PhysRevA.94.052108). URL: <https://link.aps.org/doi/10.1103/PhysRevA.94.052108>.

- [Sag22] Takahiro Sagawa. *Entropy, Divergence, and Majorization in Classical and Quantum Thermodynamics*. Vol. 16. Springer Singapore, Mar. 2022. DOI: 10.1007/978-981-16-6644-5. URL: <https://doi.org/10.1007/978-981-16-6644-5>.
- [Sek+17] Pavel Sekatski et al. “Quantum metrology with full and fast quantum control”. In: *Quantum* 1 (Sept. 2017), p. 27. ISSN: 2521-327X. DOI: 10.22331/q-2017-09-06-27. URL: <https://doi.org/10.22331/q-2017-09-06-27>.
- [SK20] Jasminder S. Sidhu and Pieter Kok. “Geometric perspective on quantum parameter estimation”. In: *AVS Quantum Science* 2.1 (Feb. 2020), p. 014701. ISSN: 2639-0213. DOI: 10.1116/1.5119961. URL: <https://doi.org/10.1116/1.5119961>.
- [SLH15] Hongting Song, Shunlong Luo, and Yan Hong. “Quantum non-Markovianity based on the Fisher-information matrix”. In: *Phys. Rev. A* 91 (4 Apr. 2015), p. 042110. DOI: 10.1103/PhysRevA.91.042110. URL: <https://link.aps.org/doi/10.1103/PhysRevA.91.042110>.
- [ST23] Tomohiro Shitara and Hiroyasu Tajima. *The i.i.d. State Convertibility in the Resource Theory of Asymmetry for Finite Groups and Lie groups*. 2023. arXiv: 2312.15758 [quant-ph].
- [STM11] Takanori Sugiyama, Peter S. Turner, and Mio Mura0. “Error probability analysis in quantum tomography: A tool for evaluating experiments”. In: *Phys. Rev. A* 83 (1 Jan. 2011), p. 012105. DOI: 10.1103/PhysRevA.83.012105. URL: <https://link.aps.org/doi/10.1103/PhysRevA.83.012105>.
- [Suz16] Jun Suzuki. “Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem”. In: *Journal of Mathematical Physics* 57.4 (Apr. 2016), p. 042201. ISSN: 0022-2488. DOI: 10.1063/1.4945086. eprint: https://pubs.aip.org/aip/jmp/article-pdf/doi/10.1063/1.4945086/14750827/042201_1_online.pdf. URL: <https://doi.org/10.1063/1.4945086>.
- [Suz19] Jun Suzuki. “Information Geometrical Characterization of Quantum Statistical Models in Quantum Estimation Theory”. In: *Entropy* 21.7 (2019). ISSN: 1099-4300. DOI: 10.3390/e21070703. URL: <https://www.mdpi.com/1099-4300/21/7/703>.
- [TAD20] Mankei Tsang, Francesco Albarelli, and Animesh Datta. “Quantum Semiparametric Estimation”. In: *Phys. Rev. X* 10 (3 July 2020), p. 031023. DOI: 10.1103/PhysRevX.10.031023. URL: <https://link.aps.org/doi/10.1103/PhysRevX.10.031023>.
- [TB16] Michael A. Taylor and Warwick P. Bowen. “Quantum metrology and its application in biology”. In: *Physics Reports* 615 (2016), pp. 1–59. ISSN: 0370-1573. DOI: <https://doi.org/10.1016/j.physrep.2015.12.002>. URL: <https://www.sciencedirect.com/science/article/pii/S0370157315005001>.
- [Wat13] Yu Watanabe. *Formulation of uncertainty relation between error and disturbance in quantum measurement by using quantum estimation theory*. Springer Tokyo, 2013. DOI: 10.1007/978-4-431-54493-7. URL: <https://doi.org/10.1007/978-4-431-54493-7>.

- [Wil11] Mark M Wilde. “From classical to quantum Shannon theory”. In: *arXiv preprint arXiv:1106.1445* (2011). DOI: [10.48550/arXiv.1106.1445](https://doi.org/10.48550/arXiv.1106.1445). URL: <https://doi.org/10.48550/arXiv.1106.1445>.
- [Yan+19] Jing Yang et al. “Optimal measurements for quantum multiparameter estimation with general states”. In: *Phys. Rev. A* 100 (3 Sept. 2019), p. 032104. DOI: [10.1103/PhysRevA.100.032104](https://doi.org/10.1103/PhysRevA.100.032104). URL: <https://link.aps.org/doi/10.1103/PhysRevA.100.032104>.
- [YCH19] Yuxiang Yang, Giulio Chiribella, and Masahito Hayashi. “Attaining the ultimate precision limit in quantum state estimation”. In: *Communications in Mathematical Physics* 368 (May 2019), pp. 223–293. DOI: [10.1007/s00220-019-03433-4](https://doi.org/10.1007/s00220-019-03433-4). URL: <https://doi.org/10.1007/s00220-019-03433-4>.
- [YFG13] Koichi Yamagata, Akio Fujiwara, and Richard D. Gill. “Quantum local asymptotic normality based on a new quantum likelihood ratio”. In: *The Annals of Statistics* 41.4 (2013), pp. 2197–2217. DOI: [10.1214/13-AOS1147](https://doi.org/10.1214/13-AOS1147). URL: <https://doi.org/10.1214/13-AOS1147>.
- [YHT+20] Y. Yuan, Z. Hou, J. F. Tang, et al. “Direct estimation of quantum coherence by collective measurements”. In: *npj Quantum Information* 6.46 (2020). DOI: [10.1038/s41534-020-0280-6](https://doi.org/10.1038/s41534-020-0280-6). URL: <https://doi.org/10.1038/s41534-020-0280-6>.
- [YL73] H. Yuen and M. Lax. “Multiple-parameter quantum estimation and measurement of nonselfadjoint observables”. In: *IEEE Transactions on Information Theory* 19.6 (1973), pp. 740–750. DOI: [10.1109/TIT.1973.1055103](https://doi.org/10.1109/TIT.1973.1055103).
- [YZF14] Jie-Dong Yue, Yu-Ran Zhang, and Heng Fan. “Quantum-enhanced metrology for multiple phase estimation with noise”. In: *Scientific reports* 4.1 (2014), p. 5933. DOI: [10.1038/srep05933](https://doi.org/10.1038/srep05933). URL: <https://doi.org/10.1038/srep05933>.
- [ZJ21] Sisi Zhou and Liang Jiang. “Asymptotic Theory of Quantum Channel Estimation”. In: *PRX Quantum* 2 (1 Mar. 2021), p. 010343. DOI: [10.1103/PRXQuantum.2.010343](https://doi.org/10.1103/PRXQuantum.2.010343). URL: <https://link.aps.org/doi/10.1103/PRXQuantum.2.010343>.