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質環上（弱）喬登型式的可加函數

Additive maps of (weak) Jordan types of prime rings

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# 國立臺灣大學博士學位論文 口試委員會審定書

## 質環上 (弱) 喬登型式的可加函數 Additive maps of (weak) Jordan types of prime rings

本論文係林政輝君 (D05221002) 在國立臺灣大學數學系完成之  
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## 中文摘要



在本篇論文中，我們將研究兩種質環上（弱）喬登型式的可加函數。在此我們令  $R$  為一個質環，且  $C$  為其廣義中心子，而  $Q_{ml}(R)$  及  $Q_{ms}(R)$  則分別代表  $R$  的左邊及雙邊極大商環。

首先我們研究  $R$  上頭的喬登  $\tau$  導算之結構，其中  $\tau$  為  $R$  的反自同構。如果一個可加函數  $\delta: R \rightarrow Q_{ms}(R)$  滿足對於所有  $x \in R$  都有  $\delta(x^2) = \delta(x)x^\tau + x\delta(x)$ ，則我們稱其為喬登  $\tau$  導算。另外，我們稱  $x \mapsto ax^\tau - xa$  型式的函數為 X-內喬登  $\tau$  導算，其中  $a$  是  $Q_{ms}(R)$  中的元素。在此我們證明了，當  $\tau$  為第二型時，喬登  $\tau$  導算的結構可被完全決定，這推廣了李秋坤教授及筆者在 2015 年的結果。定理敘述如下：

- 令  $R$  為一個非交換的質環且  $\tau$  為其上之反自同構。如果  $\tau$  是第二型，則所有在  $R$  上頭的喬登  $\tau$  導算皆為 X-內喬登  $\tau$  導算。

當  $\tau$  為第一型時，我們還有得到下面的結果：

- 令  $R$  為一個質 GPI 環且  $\text{char} R \neq 2$ ，並假設  $\tau$  為其上第一型的反自同構。如果  $\text{deg } \tau^2 \neq 2$ ，則所有在  $R$  上頭的喬登  $\tau$  導算皆為 X-內喬登  $\tau$  導算。

接下來我們研究  $R$  上頭的弱喬登導算之結構。如果一個可加函數  $\delta: R \rightarrow Q_{ml}(R)$  滿足所有  $x \in R$  都有  $\delta(x^2) - \delta(x)x - x\delta(x) \in C$ ，則我們稱其為弱喬登導算。在此我們完整給出了弱喬登導算的結構，其中  $\dim_C RC > 4$  的情況如下：

- 令  $R$  為一個質環且  $\dim_C RC > 4$ ，並假設  $\delta: R \rightarrow Q_{ml}(R)$  為一個弱喬登導算。

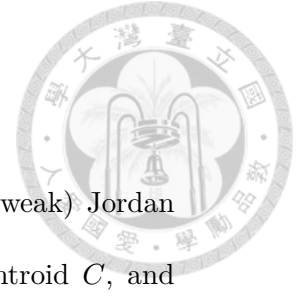
- (i) 如果  $\text{char } R \neq 2$ ，則  $\delta$  是一個導算。



(ii) 如果  $\text{char } R = 2$ ，則存在一個導算  $d: R \rightarrow Q_{ml}(R)$  和一個可加函數  $\nu: R \rightarrow C$  使得  $\delta = d + \nu$ 。

另外， $\dim_C RC = 4$  的情況也有決定出弱喬登導算的結構，但由於敘述較複雜，請讀者觀看內文的 Theorem 4.6。作為此結構定理的應用，我們推廣了 Brešar 在 1993 年的定理，其內容為關於可加函數  $\delta: R \rightarrow RC + C$  滿足對於所有  $x \in R$  都有  $[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$  的結構。

**關鍵字：**質環；泛函恆等式；GPI 環；喬登  $\tau$  導算；弱喬登導算；左邊（雙邊）極大商環。



## Abstract

In the dissertation, we study two kinds of additive maps of (weak) Jordan types on prime rings. Let  $R$  be a prime ring with extended centroid  $C$ , and let  $Q_{ml}(R)$  (resp.  $Q_{ms}(R)$ ) denote the maximal left (resp. symmetric) ring of quotients of  $R$ .

Firstly, we investigate the structure of Jordan  $\tau$ -derivations of  $R$ , where  $\tau$  is an anti-automorphism of  $R$ . An additive map  $\delta: R \rightarrow Q_{ms}(R)$  is called a Jordan  $\tau$ -derivation if  $\delta(x^2) = \delta(x)x^\tau + x\delta(x)$  for all  $x \in R$ . A Jordan  $\tau$ -derivation  $\delta$  of  $R$  is called X-inner if there exists  $a \in Q_{ms}(R)$  such that  $\delta(x) = ax^\tau - xa$  for all  $x \in R$ . We completely determine Jordan  $\tau$ -derivations of  $R$  when  $\tau$  is of the second kind, which generalizes Lee and the author's result in 2015 as follows.

- *Let  $R$  be a noncommutative prime ring with an anti-automorphism  $\tau$ . If  $\tau$  is of the second kind, then any Jordan  $\tau$ -derivation of  $R$  is X-inner.*

We also get the following characterization when  $\tau$  is of the first kind.

- *Let  $R$  be a prime GPI-ring,  $\text{char}R \neq 2$ , and let  $\tau$  be an anti-automorphism of  $R$ , which is of the first kind. If  $\deg \tau^2 \neq 2$ , then any Jordan  $\tau$ -derivation of  $R$  is X-inner.*

Secondly, we study the structure of weak Jordan derivations of  $R$ . An additive map  $\delta: R \rightarrow Q_{ml}(R)$  is called a weak Jordan derivation if  $\delta(x^2) - \delta(x)x - x\delta(x) \in C$  for all  $x \in R$ . Here we give a complete characterization of weak Jordan



derivations of  $R$ . Precisely, we prove the following.

- Let  $R$  be a prime ring with  $\dim_C RC > 4$ , and let  $\delta: R \rightarrow Q_{ml}(R)$  be a weak

Jordan derivation.

(i) If  $\text{char } R \neq 2$ , then  $\delta$  is a derivation.

(ii) If  $\text{char } R = 2$ , then  $\delta = d + \nu$ , where  $d: R \rightarrow Q_{ml}(R)$  is a derivation and  $\nu: R \rightarrow C$  is an additive map.

We also give a complete characterization for the case that  $\dim_C RC = 4$  (see Theorem 4.6). The characterization can be applied to generalize Brešar's theorem in 1993 concerning additive maps  $\delta: R \rightarrow RC + C$  satisfying  $[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$  for all  $x \in R$ .

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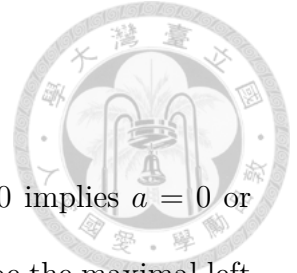
*Key words and phrases:* Prime ring; functional identity; GPI-ring; Jordan  $\tau$ -derivation; weak Jordan derivation; maximal left (symmetric) ring of quotients.

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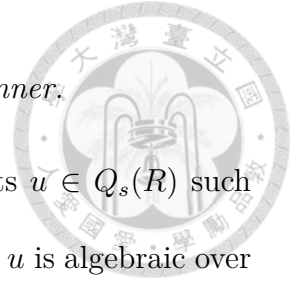


# 1 Introduction

Throughout,  $R$  denotes a prime ring, that is, for  $a, b \in R$ ,  $aRb = 0$  implies  $a = 0$  or  $b = 0$ , and let  $Z(R)$  be the center of  $R$ . Let  $Q_{ml}(R)$  (resp.  $Q_{ms}(R)$ ) be the maximal left (resp. symmetric) ring of quotients of  $R$ , and let  $Q_s(R)$  be the Martindale symmetric ring of quotients of  $R$ . The center of  $Q_{ml}(R)$ , denoted by  $C$  is called the extended centroid of  $R$ . In this case,  $C$  is always a field. It is well-known that  $Q_s(R) \subseteq Q_{ms}(R) \subseteq Q_{ml}(R)$  and  $Z(Q_{ms}(R)) = Z(Q_s(R)) = C$ . We refer the reader to [2] for details.

Let  $\tau$  be an anti-automorphism of  $R$ . An additive map  $\delta: R \rightarrow Q_{ms}(R)$  is called a Jordan  $\tau$ -derivation if  $\delta(x^2) = \delta(x)x^\tau + x\delta(x)$  for all  $x \in R$ . A Jordan  $\tau$ -derivation  $\delta$  of  $R$  is called X-inner if there exists  $a \in Q_{ms}(R)$  such that  $\delta(x) = ax^\tau - xa$  for all  $x \in R$ . In 2015, Lee and the author [20] proved the following: If  $R$  is either a non GPI-ring or a PI-ring, then every Jordan  $\tau$ -derivation of  $R$  is X-inner except when both  $\text{char}R = 2$  and  $\dim_C RC = 4$ . Therefore, it keeps unknown when  $R$  is a GPI-ring but is not a PI-ring. In order to solve the unknown case, in Chapter 2 we develop some results concerning certain functional identities with an anti-automorphism  $\tau$ . In Chapter 3, we use these results to give a complete characterization of Jordan  $\tau$ -derivations of  $R$  when  $\tau$  is of the second kind.

**Theorem 3.12.** *Let  $R$  be a noncommutative prime ring with an anti-automorphism  $\tau$ .*



If  $\tau$  is of the second kind, then any Jordan  $\tau$ -derivation of  $R$  is  $X$ -inner.

By an  $X$ -inner automorphism  $\sigma$  of  $R$ , we mean that there exists  $u \in Q_s(R)$  such that  $\sigma(x) = uxu^{-1}$  for all  $x \in R$ . In this case, we define  $\deg \sigma = m$  if  $u$  is algebraic over  $C$  with minimal degree  $m$ . We define  $\deg \sigma = \infty$  otherwise. By Kharchenko's theorem,  $\tau^2$  is an  $X$ -inner automorphism when  $R$  is a prime GPI-ring and  $\tau$  is of the first kind. We get the following result for the first kind case.

**Theorem 3.14.** *Let  $R$  be a prime GPI-ring with  $\text{char} R \neq 2$  and with  $\tau$  an anti-automorphism of the first kind. If  $\deg \tau^2 \neq 2$ , then any Jordan  $\tau$ -derivation of  $R$  is  $X$ -inner.*

In 1993, Brešar [5] proved that an additive map  $\delta: R \rightarrow RC + C$  satisfying  $\delta(x^2) - x\delta(x) - \delta(x)x \in C$  for all  $x \in R$  is a derivation if  $\text{char}(R) \neq 2, 3$  and  $\dim_C RC > 4$ . Several years later, Brešar et al. characterized weak Lie derivations (i.e., additive maps  $\delta: R \rightarrow Q_{ml}(R)$  satisfy  $\delta([x, y]) - [\delta(x), y] - [x, \delta(y)] \in C$  for all  $x, y \in R$ ) when  $\dim_C RC \geq 16$  (see the book [8]). We use their fashion to define a “weak Jordan derivation” to be an additive map  $\delta: R \rightarrow Q_{ml}(R)$  satisfying  $\delta(x^2) - \delta(x)x - x\delta(x) \in C$  for all  $x \in R$ . In Chapter 4, we completely determine its structure as follows.

**Theorem 4.5.** *Let  $R$  be a prime ring with  $\dim_C RC > 4$ , and let  $\delta: R \rightarrow Q_{ml}(R)$  be a weak Jordan derivation.*



(i) If  $\text{char } R \neq 2$ , then  $\delta$  is a derivation.

(ii) If  $\text{char } R = 2$ , then  $\delta = d + \nu$ , where  $d: R \rightarrow Q_{ml}(R)$  is a derivation and  $\nu: R \rightarrow C$  is an additive map.

**Theorem 4.6.** Let  $R$  be a prime ring with  $\dim_C RC = 4$ , and let  $\delta: R \rightarrow RC$  be a weak Jordan derivation.

(i) If  $\text{char } R \neq 2$ , then there exists a field extension  $F$  of  $C$  such that  $RC$  can be embedded into  $M_2(F)$  and

$$\delta(x) = d(x) + [a, x] + L(x) + \zeta(x)$$

for all  $x \in R$ , where  $d: R \rightarrow RC$  is a derivation,  $a \in M_2(F)$  and  $L, \zeta: M_2(F) \rightarrow M_2(F)$  are  $F$ -linear maps. Moreover, there exist  $\beta_i \in F$ ,  $1 \leq i \leq 6$ , such that

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \left( \beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for all  $x = (x_{ij}) \in M_2(F)$ . In this case,  $\delta$  is a derivation if and only if all  $\beta_i = 0$ .



(ii) If  $\text{char } R = 2$ , then there exists a field extension  $F$  of  $C$  such that  $RC$  can be embedded into  $M_2(F)$  and

$$\delta(x) = d(x) + \nu(x) + [a, x] + L(x) + \zeta(x)$$

for all  $x \in R$ , where  $d: R \rightarrow RC$  is a derivation,  $\nu: R \rightarrow C$  is an additive map,  $a \in M_2(F)$ , and  $L, \zeta: M_2(F) \rightarrow M_2(F)$  are  $F$ -linear maps. Moreover, there exist  $\beta_i \in F$ ,  $1 \leq i \leq 6$ , such that

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \left( \beta_1(x_{11} + x_{22}) + \beta_2 x_{12} + \beta_3 x_{21} \right) I_2$$

for all  $x = (x_{ij}) \in M_2(F)$ .

Conversely, an additive map  $\delta: R \rightarrow Q_{ml}(R)$  satisfying (i) or (ii) is a weak Jordan derivation.

Moreover, we use the above two theorems to generalize Brešar's theorem [5, Theorem 4].

**Theorem 4.9.** Let  $R$  be a prime ring with  $\text{char } R \neq 2$ , and let  $\delta: R \rightarrow RC + C$  be an



*additive map satisfying*

$$[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$$

*for all  $x \in R$ .*

(i) *If  $\dim_C RC > 4$ , then*

$$\delta(x) = \gamma x + d(x) + \mu(x)$$

*for all  $x \in R$ , where  $\gamma \in C$ ,  $d: R \rightarrow RC + C$  is a derivation, and  $\mu: R \rightarrow C$  is an additive map.*

(ii) *If  $\dim_C RC = 4$ , then*

$$\delta(x) = d(x) + \mu(x) + [a, x] + L(x) + \zeta(x)$$

*for all  $x \in R$ , where  $d: R \rightarrow RC$  is a derivation,  $\mu: R \rightarrow C$  is an additive map,  $a \in M_2(F)$ , and  $L, \zeta: M_2(F) \rightarrow M_2(F)$  are as in Theorem 4.6 (i).*

Note that all results mentioned above have been published as journal papers (see [22] and [23]). The method of characterizing weak Jordan derivations developed by the author was also applied to studying weak Jordan  $*$ -derivations by Siddeeqe et al. (see [32]).



## 2 Preliminary

### 2.1 GPI-rings, PI-rings, and functional identities

Let  $R$  be a prime ring with extended centroid  $C$ . We first introduce prime GPI-rings and PI-rings and their structure theorems. To be precise, set  $Q := Q_{ml}(R)$ . Let  $X$  be an infinite set and  $C\langle X \rangle$  be the free  $C$ -algebra on  $X$ . Define  $Q_C\langle X \rangle$  to be the free product of  $Q$  and  $C\langle X \rangle$ . Elements of  $Q_C\langle X \rangle$  are called generalized polynomials. A generalized polynomial is said to be nontrivial if it is nonzero in  $Q_C\langle X \rangle$ . Let  $U$  be an additive subgroup of  $R$ . By a generalized polynomial identity (GPI) on  $U$  we mean an element  $\phi = \phi(x_1, \dots, x_n)$  in  $Q_C\langle X \rangle$  such that  $\phi(r_1, \dots, r_n) = 0$  for all  $r_1, \dots, r_n \in U$ . In this case, we say that  $U$  satisfies a GPI  $\phi$ . We say that  $R$  is a GPI-ring if  $R$  satisfies a nontrivial GPI. The following is a famous structure theorem due to Martindale (see [24, Theorem 3] or [2, Theorem 6.1.6]).

**Theorem 2.1.** *Let  $R$  be a prime ring with extended centroid  $C$ . Then  $R$  is a GPI-ring if and only if its central closure  $RC$  contains a nonzero idempotent  $e$  such that  $eRC$  is a minimal right ideal of  $RC$  and  $eRCe$  is a finite-dimensional division algebra over  $C$ .*

We view  $C\langle X \rangle$  as a  $C$ -subalgebra of  $Q_C\langle X \rangle$ . We say  $R$  is a PI-ring if it satisfies a nontrivial element of  $C\langle X \rangle$  whose coefficients are in  $\{1, -1\}$ . The following describes the structure of prime PI-rings (see [25], [26], and [9]).



**Theorem 2.2.** *Let  $R$  be a prime PI-ring with extended centroid  $C$ . Then*

- (a)  $RC$  is a finite-dimensional central simple algebra over  $C$ .
- (b)  $Z(R)$  intersects every nonzero ideal of  $R$  nontrivially.
- (c)  $C$  is the quotient field of  $Z(R)$ .

Next we introduce some useful results concerning functional identities of prime rings. These play a key role in characterizing weak Jordan derivations and Jordan  $\tau$ -derivations.

Before stating them, we fix some notations. Let  $m$  be a positive integer,  $I, J \subseteq \{1, 2, \dots, m\}$ , and  $a, b$  be non-negative integers. Let  $E_{iu}, F_{jv}: R^{m-1} \rightarrow Q_{ml}(R)$ ,  $i \in I$ ,  $j \in J$ ,  $0 \leq u \leq a$ ,  $0 \leq v \leq b$ . Fix  $t \in Q_{ml}(R)$  and  $V = \sum_{i=0}^{\infty} Ct^i$ . We say that  $\deg t = n$  if  $t$  is algebraic of minimal degree  $n$  over  $C$ . Moreover,  $\deg t = \infty$  if  $t$  is not algebraic over  $C$ . For any maps  $f: R^{r-1} \rightarrow Q_{ml}(R)$  and  $g: R^{r-2} \rightarrow Q_{ml}(R)$  we write

$$f^i(\bar{x}_r) = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r)$$

and

$$g^{ij}(\bar{x}_r) = g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_r),$$

where  $\bar{x}_r = (x_1, \dots, x_r) \in R^r$ . We need the following important theorems due to Beidar and Martindale [3].



**Theorem 2.3.** ([3, Theorem 2.4]) *Suppose that  $\deg t > a + |I|$  and*

$$\sum_{i \in I} \sum_{u=0}^a E_{iu}^i x_i t^u \in V$$

*for all  $x_1, x_2, \dots, x_m \in R$ . Then each  $E_{iu} = 0$ .*

**Theorem 2.4.** ([3, Theorem 2.5]) *Suppose that  $\deg t > a + |I| - 1$  and*

$$\sum_{i \in I} \sum_{u=0}^a E_{iu}^i x_i t^u = 0$$

*for all  $x_1, x_2, \dots, x_m \in R$ . Then each  $E_{iu} = 0$ .*

**Theorem 2.5.** ([3, Corollary 2.11]) *Let  $E_i, F_j: R^{m-1} \rightarrow Q_{ml}(R)$ ,  $i \in I$ ,  $j \in J$  such that*

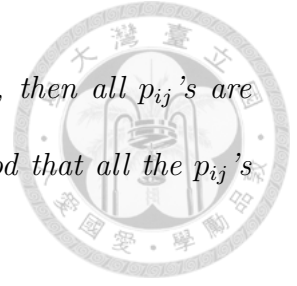
$$\sum_{i \in I} E_i^i(x_1, \dots, x_m) x_i + \sum_{j \in J} x_j F_j^j(x_1, \dots, x_m) \in C$$

*for all  $x_1, x_2, \dots, x_m \in R$ . Suppose that  $\dim_C RC > (\max\{|I|, |J|\})^2$ . Then there exist unique maps  $p_{ij}: R^{m-2} \rightarrow Q_{ml}(R)$  and  $\lambda_k: R^{m-1} \rightarrow C$  such that*

$$E_i^i = \sum_{\substack{j \in J \\ j \neq i}} x_j p_{ij}^{ij} + \lambda_i^i,$$

$$F_j^j = - \sum_{\substack{i \in I \\ i \neq j}} p_{ij}^{ij} x_i - \lambda_j^j,$$





where  $\lambda_k = 0$  if  $k \notin I \cap J$ . If  $E_i$ 's and  $F_j$ 's are  $(m - 1)$ -additive, then all  $p_{ij}$ 's are  $(m - 2)$ -additive and all  $\lambda_k$ 's are  $(m - 1)$ -additive. (It is understood that all the  $p_{ij}$ 's are equal to 0 if  $m = 1$ .)

## 2.2 Functional identities with an anti-automorphism

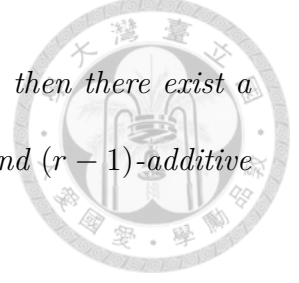
Let  $R$  be a prime ring. It is well-known that an automorphism (resp. anti-automorphism) of  $R$  can be uniquely extended to an automorphism (resp. anti-automorphism) of  $Q_s(R)$  (see [2, Proposition 2.5.3] for the automorphism case and [2, Proposition 2.5.4] for the anti-automorphism case). An automorphism (or anti-automorphism)  $g$  is said to be of the first kind if  $\beta^g = \beta$  for all  $\beta \in C$ . Otherwise,  $g$  is said to be of the second kind.

To study Jordan  $\tau$ -derivations in the next chapter, we have to develop some results concerning functional identities with an anti-automorphism. In [21], Lee dealt with functional identities on prime rings with an automorphism. Now, we will follow his viewpoint to get useful results. Our purpose in the section is to prove the following theorem.

**Theorem 2.6.** *Let  $R$  be a prime ring with an anti-automorphism  $\tau$  of the second kind.*

*Suppose that  $E_{it}, F_{\ell 1}: R^{r-1} \rightarrow Q_{ml}(R)$  are  $(r - 1)$ -additive maps such that*

$$\sum_{i=1}^r E_{i1}^i(\bar{x}_r)x_i + \sum_{i=1}^r E_{i2}^i(\bar{x}_r)x_i^\tau + \sum_{\ell=1}^r x_\ell F_{\ell 1}^\ell(\bar{x}_r) \in C \quad (1)$$



for  $\bar{x}_r \in R^r$ , where  $1 \leq i, \ell \leq r$  and  $t = 1, 2$ . If  $R$  is not a PI-ring, then there exist a nonzero ideal  $I$  of  $R$ ,  $(r - 2)$ -additive maps  $p_{it\ell 1}: I^{r-2} \rightarrow Q_{ml}(R)$ , and  $(r - 1)$ -additive maps  $\lambda_{i1}: I^{r-1} \rightarrow C$  such that

$$E_{i1}^i(\bar{x}_r) = \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i1\ell 1}^{i\ell}(\bar{x}_r) + \lambda_{i1}^i(\bar{x}_r)$$

$$E_{i2}^i(\bar{x}_r) = \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i2\ell 1}^{i\ell}(\bar{x}_r)$$

and

$$F_{\ell 1}^\ell(\bar{x}_r) = - \sum_{\substack{1 \leq i \leq r \\ i \neq \ell}} p_{i1\ell 1}^{i\ell}(\bar{x}_r) x_i - \sum_{\substack{1 \leq i \leq r \\ i \neq \ell}} p_{i2\ell 1}^{i\ell}(\bar{x}_r) x_i^\tau - \lambda_{\ell 1}^\ell(\bar{x}_r)$$

for all  $\bar{x}_r \in I^r$ , where  $1 \leq i, \ell \leq r$  and  $t = 1, 2$ .

**Corollary 2.7.** Let  $R$  be a prime ring with an anti-automorphism  $\tau$  of the second kind.

Suppose that  $E_i, F_\ell: R^{r-1} \rightarrow Q_{ml}(R)$  are  $(r - 1)$ -additive maps such that

$$\sum_{i=1}^r E_i^i(\bar{x}_r) x_i^\tau + \sum_{\ell=1}^r x_\ell F_\ell^\ell(\bar{x}_r) \in C$$

for  $\bar{x}_r \in R^r$ , where  $1 \leq i, \ell \leq r$ . If  $R$  is not a PI-ring, then there exist a nonzero ideal



$I$  of  $R$  and  $(r - 2)$ -additive maps  $p_{i\ell}: I^{r-2} \rightarrow Q_{ml}(R)$  such that

$$E_i^i(\bar{x}_r) = \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i\ell}^{i\ell}(\bar{x}_r)$$

and

$$F_\ell^\ell(\bar{x}_r) = - \sum_{\substack{1 \leq i \leq r \\ i \neq \ell}} p_{i\ell}^{i\ell}(\bar{x}_r) x_i^\tau$$

for all  $\bar{x}_r \in I^r$ , where  $1 \leq i, \ell \leq r$ .

*Proof.* By Theorem 2.6, there exist a nonzero ideal  $I$  of  $R$ ,  $(r - 2)$ -additive maps  $p_{i\ell 1}: I^{r-2} \rightarrow Q_{ml}(R)$ , and  $(r - 1)$ -additive maps  $\lambda_{i1}: I^{r-1} \rightarrow C$  such that

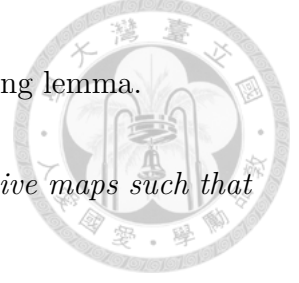
$$0 = \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i\ell 1}^{i\ell}(\bar{x}_r) + \lambda_{i1}^i(\bar{x}_r),$$

$$E_i^i(\bar{x}_r) = \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i2\ell 1}^{i\ell}(\bar{x}_r),$$

and

$$F_\ell^\ell(\bar{x}_r) = - \sum_{\substack{1 \leq i \leq r \\ i \neq \ell}} p_{i1\ell 1}^{i\ell}(\bar{x}_r) x_i - \sum_{\substack{1 \leq i \leq r \\ i \neq \ell}} p_{i2\ell 1}^{i\ell}(\bar{x}_r) x_i^\tau - \lambda_{\ell 1}^\ell(\bar{x}_r)$$

for all  $\bar{x}_r \in I^r$ , where  $1 \leq i, \ell \leq r$  and  $t = 1, 2$ . In view of Theorem 2.3,  $p_{i1\ell 1} = 0$  and  $\lambda_{i1} = 0$  for  $1 \leq i, \ell \leq r$ . The proof is complete by putting  $p_{i\ell} = p_{i2\ell 1}$ .  $\square$



To begin with the proof of Theorem 2.6, we first give the following lemma.

**Lemma 2.8.** *Suppose that  $E_i, F_\ell: R^{r-1} \rightarrow Q_{ml}(R)$  are  $(r-1)$ -additive maps such that*

$$\sum_{i=1}^r E_i^i(\bar{x}_r)x_i + \sum_{\ell=1}^r F_\ell^\ell(\bar{x}_r)x_\ell^\tau \in C \quad (2)$$

for  $\bar{x}_r \in R^r$ , where  $1 \leq i, \ell \leq r$ . If  $R$  is not a PI-ring, then there exists a nonzero ideal  $I$  of  $R$  such that  $E_i^i = F_\ell^\ell = 0$  on  $I^r$  for  $1 \leq i, \ell \leq r$ .

Before proving it, we define the following notation (see [21]). For a map  $f: R^{r-1} \rightarrow Q_{ml}(R)$  and  $t \neq i$ , we write

$$f^i(\bar{x}_r; \{y\}_t) = f(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_r)$$

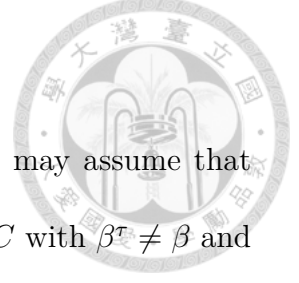
where  $z_t = y$  and  $z_j = x_j$  for  $j \neq t$ , i.e., we replace  $x_t$  by  $y$  in  $f^i(\bar{x}_r)$ .

*Proof of Lemma 2.8.* Let  $A := \{1, 2, \dots, r\}$  and

$$L := \{\ell \in A \mid \text{there exists a nonzero ideal } J \text{ of } R \text{ such that } F_\ell^\ell = 0 \text{ on } J^r\}.$$

We proceed the proof by induction on  $r - |L|$ .

Suppose first that  $r - |L| = 0$ , i.e.,  $L = A$ . Then there exists a nonzero ideal  $J$  such that  $F_\ell^\ell = 0$  on  $J^r$  for all  $\ell = 1, \dots, r$ . Thus  $\sum_{i=1}^r E_i^i(\bar{x}_r)x_i \in C$  for all  $\bar{x}_r \in J^r$ . By



Theorem 2.3,  $E_i^i = 0$  on  $J^r$  for all  $i = 1, \dots, r$ , as asserted.

Suppose next that  $r - |L| \geq 1$ . Without loss of generality, we may assume that  $r \notin L$ . Then, for any nonzero ideal  $U$  of  $R$ ,  $F_r^r \neq 0$  on  $U^r$ . Fix  $\beta \in C$  with  $\beta^\tau \neq \beta$  and choose a nonzero ideal  $K$  of  $R$  such that  $\beta K \subseteq R$ . Then, by (2), we have

$$\begin{aligned} & \sum_{i=1}^{r-1} (E_i^i(\bar{x}_r; \{\beta x_r\}_r) - \beta E_i^i(\bar{x}_r)) x_i \\ & + \sum_{\ell=1}^{r-1} (F_\ell^\ell(\bar{x}_r; \{\beta x_r\}_r) - \beta F_\ell^\ell(\bar{x}_r)) x_\ell^\tau + (\beta^\tau - \beta) F_r^r(\bar{x}_r) x_r^\tau \in C \end{aligned} \quad (3)$$

for all  $\bar{x}_r \in K^r$ . Let  $K_1 = K \cap K^\tau$ . Then  $K_1$  is an ideal of  $R$  such that  $K_1^{\tau^{-1}} \subseteq K$  and, by (3), we have

$$\sum_{i=1}^{r-1} \tilde{E}_i^i(\bar{x}_r) x_i + F_r^r(\bar{x}_r) x_r + \sum_{\ell=1}^{r-1} \tilde{F}_\ell^\ell(\bar{x}_r) x_\ell^\tau \in C \quad (4)$$

for all  $\bar{x}_r \in K_1^r$ , where

$$\tilde{E}_i^i(\bar{x}_r) = (\beta^\tau - \beta)^{-1} (E_i^i(\bar{x}_r; \{\beta x_r^{\tau^{-1}}\}_r) - \beta E_i^i(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r))$$

and

$$\tilde{F}_\ell^\ell(\bar{x}_r) = (\beta^\tau - \beta)^{-1} (F_\ell^\ell(\bar{x}_r; \{\beta x_r^{\tau^{-1}}\}_r) - \beta F_\ell^\ell(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r)).$$



Set

$L_1 := \{\ell \mid 1 \leq \ell \leq r-1, \text{ there exists a nonzero ideal } J \text{ of } R \text{ such that } \tilde{F}_\ell^\ell = 0 \text{ on } J^r\}$ .

Let  $\ell \in L$ . Then  $1 \leq \ell \leq r-1$  and there is a nonzero ideal  $N$  of  $R$  such that  $F_\ell^\ell = 0$  on  $N^r$ . By the definition of  $\tilde{F}_\ell^\ell$ , there exists a nonzero ideal  $M$  of  $R$  contained in  $N$  such that  $\tilde{F}_\ell^\ell = 0$  on  $M^r$  and so  $\ell \in L_1$ . Thus  $|L| \leq |L_1|$  and  $r - |L| \geq r - |L_1| > (r-1) - |L_1|$ . By applying the induction hypothesis on (4), we have  $F_\ell^\ell = 0$  on  $W^r$  for some nonzero ideal  $W$  of  $R$ , a contradiction.  $\square$

*Proof of Theorem 2.6.* Let  $A := \{1, 2, \dots, r\}$  and

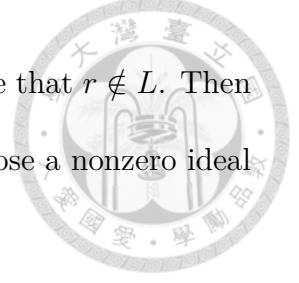
$$L := \{i \in A \mid \text{there exists a nonzero ideal } J \text{ of } R \text{ such that } E_{i2}^i = 0 \text{ on } J^r\}.$$

We proceed the proof by induction on  $r - |L|$ .

Assume first that  $r - |L| = 0$ , i.e.,  $L = A$ . Then  $E_{i2}^i = 0$  on  $U^r$  for some nonzero ideal  $U$  of  $R$  and so (1) becomes

$$\sum_{i=1}^r E_{i1}^i(\bar{x}_r)x_i + \sum_{\ell=1}^r x_\ell F_{\ell 1}^\ell(\bar{x}_r) \in C$$

for  $\bar{x}_r \in U^r$ . Hence the result follows from Theorem 2.5.



Assume next that  $r - |L| \geq 1$ . Without loss of generality, assume that  $r \notin L$ . Then  $E_{r2}^r \neq 0$  on any nonzero ideal of  $R$ . Let  $\beta \in C$  with  $\beta^\tau \neq \beta$  and choose a nonzero ideal  $J$  of  $R$  such that  $\beta J \subseteq R$ . Then, by (1), we have

$$\begin{aligned} & \sum_{i=1}^{r-1} (E_{i1}^i(\bar{x}_r; \{\beta x_r\}_r) - \beta E_{i1}^i(\bar{x}_r)) x_i \\ & + \sum_{i=1}^{r-1} (E_{i2}^i(\bar{x}_r; \{\beta x_r\}_r) - \beta E_{i2}^i(\bar{x}_r)) x_i^\tau + E_{r2}^r(\bar{x}_r)(\beta^\tau - \beta) x_r^\tau \\ & + \sum_{\ell=1}^{r-1} x_\ell (F_{\ell 1}^\ell(\bar{x}_r; \{\beta x_r\}_r) - \beta F_{\ell 1}^\ell(\bar{x}_r)) \in C \end{aligned}$$

for all  $\bar{x}_r \in J^r$ . Let

$$\tilde{E}_{i1}^i(\bar{x}_r) = (\beta^\tau - \beta)^{-1} (E_{i1}^i(\bar{x}_r; \{\beta x_r\}_r) - \beta E_{i1}^i(\bar{x}_r)),$$

$$\tilde{E}_{i2}^i(\bar{x}_r) = (\beta^\tau - \beta)^{-1} (E_{i2}^i(\bar{x}_r; \{\beta x_r\}_r) - \beta E_{i2}^i(\bar{x}_r)),$$

and

$$\tilde{F}_{\ell 1}^\ell(\bar{x}_r) = (\beta^\tau - \beta)^{-1} (F_{\ell 1}^\ell(\bar{x}_r; \{\beta x_r\}_r) - \beta F_{\ell 1}^\ell(\bar{x}_r)).$$

Then

$$\sum_{i=1}^{r-1} \tilde{E}_{i1}^i(\bar{x}_r) x_i + \sum_{i=1}^{r-1} \tilde{E}_{i2}^i(\bar{x}_r) x_i^\tau + E_{r2}^r(\bar{x}_r) x_r^\tau + \sum_{\ell=1}^{r-1} x_\ell \tilde{F}_{\ell 1}^\ell(\bar{x}_r) \in C \quad (5)$$



for all  $\bar{x}_r \in J^r$ . Choose a nonzero ideal  $J_1$  of  $R$  contained in  $J$  so that  $J_1^{r-1} \subseteq J$ . By

(5), we have

$$\begin{aligned} & \sum_{i=1}^{r-1} \tilde{E}_{i1}^i(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r) x_i + \sum_{i=1}^{r-1} \tilde{E}_{i2}^i(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r) x_i^\tau \\ & + E_{r2}^r(\bar{x}_r) x_r + \sum_{\ell=1}^{r-1} x_\ell \tilde{F}_{\ell 1}^\ell(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r) \in C \end{aligned} \quad (6)$$

for all  $\bar{x}_r \in J_1^r$ . Set  $G_{i2}^i(\bar{x}_r) := \tilde{E}_{i2}^i(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r)$  and

$L_1 := \{i \mid 1 \leq i \leq r-1, \text{ there exists a nonzero ideal } J \text{ of } R \text{ such that } G_{i2}^i = 0 \text{ on } J^r\}$ .

Let  $i \in L$  and  $i \neq r$ . Then there exists a nonzero ideal  $N$  of  $R$  such that  $E_{i2}^i = 0$  on  $N^r$ . From the definition of  $G_{i2}^i$ , there is a nonzero ideal  $M$  of  $R$  contained in  $N$  such that  $G_{i2}^i = 0$  on  $M^r$ , and so  $i \in L_1$ . Thus

$$r - |L| \geq r - |L_1| > (r-1) - |L_1|.$$

By the induction hypothesis, there exist a nonzero ideal  $J_2$  of  $R$  contained in  $J_1$  and  $(r-2)$ -additive maps  $p_{r2\ell 1}: J_2^{r-2} \rightarrow Q_{ml}(R)$  such that

$$E_{r2}^r(\bar{x}_r) = \sum_{\ell=1}^{r-1} x_\ell p_{r2\ell 1}^{r\ell}(\bar{x}_r)$$





for all  $\bar{x}_r \in J_2^r$ . Substituting it into (1), we have

$$\sum_{i=1}^r E_{i1}^i(\bar{x}_r)x_i + \sum_{i=1}^{r-1} E_{i2}^i(\bar{x}_r)x_i^\tau + \sum_{\ell=1}^{r-1} x_\ell \left( F_{\ell 1}^\ell(\bar{x}_r) + p_{r2\ell 1}^{\tau\ell}(\bar{x}_r)x_r^\tau \right) + x_r F_{r1}^r(\bar{x}_r) \in C \quad (7)$$

for all  $\bar{x}_r \in J_2^r$ . By the induction hypothesis, there is a nonzero ideal  $I$  of  $R$  contained in  $J_2$  and  $(r-2)$ -additive maps  $p_{i2\ell 1}: I^{r-2} \rightarrow Q_{ml}(R)$  such that

$$E_{i2}^i(\bar{x}_r) = \sum_{\substack{\ell=1 \\ \ell \neq i}}^r x_\ell p_{i2\ell 1}^{i\ell}(\bar{x}_r)$$

for all  $\bar{x}_r \in I^r$  and  $1 \leq i \leq r-1$ . Thus (7) becomes

$$\sum_{i=1}^r E_{i1}^i(\bar{x}_r)x_i + \sum_{\ell=1}^r x_\ell \left( F_{\ell 1}^\ell(\bar{x}_r) + \sum_{\substack{i=1 \\ i \neq \ell}}^r p_{i2\ell 1}^{i\ell}(\bar{x}_r)x_i^\tau \right) \in C$$

for all  $\bar{x}_r \in I^r$ . According to Theorem 2.5, there exist  $(r-2)$ -additive maps  $p_{i1\ell 1}: I^{r-2} \rightarrow Q_{ml}(R)$  and  $(r-1)$ -additive maps  $\lambda_{i1}: I^{r-1} \rightarrow C$  such that

$$E_{i1}^i(\bar{x}_r) = \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i1\ell 1}^{i\ell}(\bar{x}_r) + \lambda_{i1}^i(\bar{x}_r)$$

and

$$F_{\ell 1}^{\ell}(\bar{x}_r) + \sum_{\substack{i=1 \\ i \neq \ell}}^r p_{i 2 \ell 1}^{i \ell}(\bar{x}_r) x_i^{\tau} = - \sum_{\substack{1 \leq i \leq r \\ i \neq \ell}} p_{i 1 \ell 1}^{i \ell}(\bar{x}_r) x_i - \lambda_{\ell 1}^{\ell}(\bar{x}_r)$$

for all  $\bar{x}_r \in I^r$ , where  $1 \leq i, \ell \leq r$ , as asserted. □



### 3 Jordan $\tau$ -derivations of prime GPI-rings



#### 3.1 Motivation

Let  $R$  be a prime ring with an anti-automorphism  $\tau$ . An additive map  $\delta: R \rightarrow Q_{ms}(R)$  is called a Jordan  $\tau$ -derivation of  $R$  if  $\delta(x^2) = x\delta(x) + \delta(x)x^\tau$  for all  $x \in R$ . A Jordan  $\tau$ -derivation  $\delta$  of  $R$  is said to be inner (resp. X-inner) if there exists  $a \in R$  (resp.  $a \in Q_{ms}(R)$ ) such that  $\delta(x) = ax^\tau - xa$  for  $x \in R$ . Note that if  $\delta: R \rightarrow Q_{ms}(R)$  is a Jordan  $\tau$ -derivation, then

$$\delta(xy + yx) = \delta(x)y^\tau + y\delta(x) + \delta(y)x^\tau + x\delta(y) \quad (8)$$

for all  $x, y \in R$ .

Let  $A$  be a ring. Suppose that  $*$ :  $A \rightarrow A$  is an involution of  $A$ , i.e.,  $*$  is an anti-automorphism of  $A$  such that  $(x^*)^* = x$  for all  $x \in A$ . The problem of the representability of quadratic forms by bilinear forms is connected with the structure of Jordan  $*$ -derivations (see [29] and [30]). In 1989, Brešar and Vukman proved the following (see [4, Theorem 1]).

**Theorem 3.1.** *Let  $A$  be a unital ring with involution  $*$ . Suppose that  $A$  contains  $\frac{1}{2}$  and an invertible skew-hermitian element  $\mu$  (i.e.,  $\mu^* = -\mu$ ) which lies in  $Z(A)$ . Then*



every Jordan  $*$ -derivation from  $A$  into itself is inner.

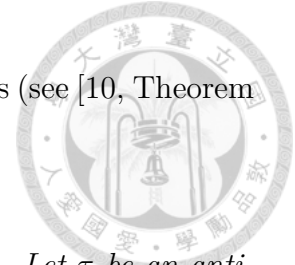
In particular, every Jordan  $*$ -derivation of a unital complex  $*$ -algebra is inner. Indeed, given any complex  $*$ -algebra  $A$ , we always assume  $(\beta x)^* = \bar{\beta}x^*$  for all  $x \in A$  and  $\beta \in \mathbb{C}$ . Therefore we can always find an invertible skew-hermitian element in  $Z(A)$ .

Let  $H$  be a real (resp. complex) Hilbert space with  $\dim_{\mathbb{R}}H > 1$  (resp.  $\dim_{\mathbb{C}}H > 1$ ). Let  $\mathcal{B}(H)$  stand for the algebra of all bounded linear operators on the Hilbert space  $H$  and let  $\mathcal{A}$  be a standard operator algebra on  $H$ , i.e.,  $\mathcal{A}$  is a subalgebra of  $\mathcal{B}(H)$  containing the subalgebra of all bounded finite rank operators (see [31]). Then  $\mathcal{B}(H)$  can be endowed with a canonical involution, say  $*$ . It is known that  $\mathcal{A}$  is a prime algebra with nonzero socle. Moreover,  $Q_{ms}(\mathcal{A}) = Q_{ms}(\mathcal{B}(H)) = \mathcal{B}(H)$  (see [10, Theorem 1.3]). In 1990, Šemrl proved the following (see [28, Theorem 2.3]).

**Theorem 3.2.** *Let  $H$  be a real Hilbert space with  $\dim_{\mathbb{R}}H > 1$ . Let  $D: \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  be a Jordan  $*$ -derivation. Then there exists a unique  $T \in \mathcal{B}(H)$  such that  $D(S) = ST - TS^*$  for all  $S \in \mathcal{B}(H)$ , i.e.,  $D$  is inner.*

In 1994, Šemrl showed the following (see [31]).

**Theorem 3.3.** *Let  $H$  be a complex Hilbert space with  $\dim_{\mathbb{C}}H > 1$  and let  $\mathcal{A}$  be a standard operator algebra on  $H$ . Suppose that  $J: \mathcal{A} \rightarrow \mathcal{B}(H)$  is a Jordan  $*$ -derivation. Then there exists a unique  $T \in \mathcal{B}(H)$  such that  $J(A) = AT - TA^*$  for all  $A \in \mathcal{A}$ , i.e.,  $J$  is  $X$ -inner.*



In 2013, Chuang et al. extended Šemrl's theorems above as follows (see [10, Theorem 1.2]).

**Theorem 3.4.** *Let  $R$  be a prime ring, which is not a division ring. Let  $\tau$  be an anti-automorphism of  $R$  and let  $\delta: R \rightarrow Q_s(R)$  be a Jordan  $\tau$ -derivation. If  $\text{char}R \neq 2$  and the socle of  $R$  is nonzero, then  $\delta$  is  $X$ -inner.*

Moreover, Lee et al. completely determined the structure of Jordan  $*$ -derivations of prime rings (see [11], [17], and [18]). To be precise, we state its conclusion.

**Theorem 3.5.** ([18, Theorem 1.2]) *Let  $R$  be a prime ring with involution  $*$ , which is not commutative. Then any Jordan  $*$ -derivation of  $R$  is  $X$ -inner except when  $\text{char}R = 2$  and  $\dim_C RC = 4$ .*

They also gave an example of non  $X$ -inner Jordan  $*$ -derivations when  $\text{char}R = 2$  and  $\dim_C RC = 4$  (see [18, Example 3.2]). For the general anti-automorphism case, Lee and the author [20] proved the following results.

**Theorem 3.6.** ([20, Theorem 2.1]) *Let  $R$  be a prime ring with an anti-automorphism  $\tau$ . Suppose that  $R$  is not a GPI-ring. Then any Jordan  $\tau$ -derivation  $\delta: R \rightarrow Q_{ms}(R)$  is  $X$ -inner.*

**Theorem 3.7.** ([20, Theorem 2.8]) *Let  $D$  be a division ring, which is not commutative, with center  $C$  and let  $\tau$  be an anti-automorphism of  $D$ . Then any Jordan  $\tau$ -derivation*



$\delta: D \rightarrow D$  is *X-inner* except when  $\text{char}D = 2$  and  $\dim_C D = 4$ .

**Theorem 3.8.** ([20, Theorem 2.9]) *Let  $R$  be a prime PI-ring, which is not commutative, and let  $\tau$  be an anti-automorphism of  $R$ . Then any Jordan  $\tau$ -derivation  $\delta: R \rightarrow RC$  is X-inner except when  $\text{char}R = 2$  and  $\dim_C RC = 4$ .*

By the above theorems, any Jordan  $\tau$ -derivation of  $R$  is X-inner if either  $R$  is not a GPI-ring or  $R$  is a PI-ring except when  $\text{char}R = 2$  and  $\dim_C RC = 4$ . In order to completely characterize Jordan  $\tau$ -derivations of  $R$ , they raised the following question.

**Question A.** Let  $R$  be a prime GPI-ring, which is not commutative, with an anti-automorphism  $\tau$ . Suppose that neither  $R$  is a PI-ring nor  $R$  is a division ring. Is any Jordan  $\tau$ -derivation of  $R$  X-inner?

We remark that, by Theorem 2.1, if  $R$  is both a prime GPI-ring and a division ring, then it is a PI-ring and Question A is solved by Theorem 3.8 in this case. Hence Question A is reduced to the case that  $R$  is a prime GPI-ring but is not a PI-ring.

## 3.2 Results

Let  $R$  be a prime ring with an anti-automorphism  $\tau$ . Recall that  $\tau$  can be uniquely extended to an anti-automorphism of  $Q_s(R)$  and that  $\tau$  is said to be of the first kind if  $\beta^\tau = \beta$  for all  $\beta \in C$ . Otherwise,  $\tau$  is said to be of the second kind. In order to deal with the second kind case, we need the following.



**Lemma 3.9.** ([20, Lemma 2.7]) *Suppose that  $\beta^\tau \neq \beta$  for some  $\beta \in Z(R)$ . Then any Jordan  $\tau$ -derivation  $\delta: R \rightarrow Q_{ms}(R)$  is X-inner. In fact,  $\delta(x) = ax^\tau - xa$  for all  $x \in R$ , where  $a = (\beta^\tau - \beta)^{-1}\delta(\beta)$ .*

By applying Lemma 3.9, we get the following result.

**Theorem 3.10.** *Let  $R$  be a noncommutative prime PI-ring with an anti-automorphism  $\tau$ . If  $\tau$  is of the second kind, then any Jordan  $\tau$ -derivation  $\delta: R \rightarrow Q_{ms}(R)$  is X-inner.*

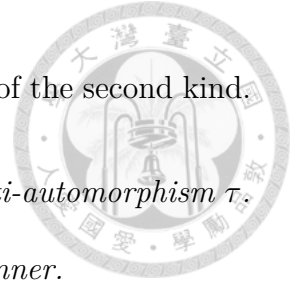
*Proof.* Since  $R$  is a prime PI-ring, it follows from Theorem 2.2 that  $Z(R) \neq 0$  and  $C$  is the quotient field of  $Z(R)$ . Thus there exists  $\beta \in Z(R)$  such that  $\beta^\tau \neq \beta$  because  $\tau$  is of the second kind. By Lemma 3.9,  $\delta$  is X-inner, as desired.  $\square$

We remark that the lemma below holds for an arbitrary ring  $R$  (see [17, Lemma 2.3]).

**Lemma 3.11.** *Let  $B: R \times R \rightarrow A$  be a bi-additive map and let  $f, g: R \rightarrow A$  be additive maps, where  $A$  is an additive group. Suppose that  $B(x, y) = f(xy) + g(yx)$  for all  $x, y \in R$ . Then*

$$B(xw, yz) - B(x, wyz) = B(zxw, y) - B(zx, wy)$$

*for all  $w, x, y, z \in R$ .*



Now we can give an affirmative answer to Question A when  $\tau$  is of the second kind.

**Theorem 3.12.** *Let  $R$  be a noncommutative prime ring with an anti-automorphism  $\tau$ .*

*If  $\tau$  is of the second kind, then any Jordan  $\tau$ -derivation of  $R$  is  $X$ -inner.*

*Proof.* According to Theorem 3.10, we can assume that  $R$  is not a PI-ring. Let  $\delta: R \rightarrow Q_{ms}(R)$  be a Jordan  $\tau$ -derivation. Define the bi-additive map  $B: R \times R \rightarrow Q_{ms}(R)$  by  $B(x, y) = \delta(xy + yx)$  for  $x, y \in R$ . It follows from Lemma 3.11 that

$$B(xw, yz) - B(x, wyz) = B(zxw, y) - B(zx, wy)$$

for all  $x, y, z, w \in R$ . So, by (8), we have

$$\begin{aligned} & \left( \delta(yz)w^\tau - \delta(wyz) \right) x^\tau + \left( \delta(xw)z^\tau - \delta(zxw) \right) y^\tau \\ & + \left( \delta(wy)x^\tau - \delta(y)w^\tau x^\tau \right) z^\tau + \left( \delta(zx)y^\tau - \delta(x)z^\tau y^\tau \right) w^\tau \\ & + x \left( w\delta(yz) - \delta(wyz) \right) + y \left( z\delta(xw) - \delta(zxw) \right) \\ & + z \left( x\delta(wy) - xw\delta(y) \right) + w \left( y\delta(zx) - yz\delta(x) \right) = 0 \end{aligned}$$

for all  $x, y, z, w \in R$ . According to Corollary 2.7, there exist a nonzero ideal  $I_1$  of  $R$





and bi-additive maps  $r_{13}, r_{23}, r_{43}: I_1^2 \rightarrow Q_{ml}(R)$  such that

$$x\left(\delta(wy) - w\delta(y)\right) = -r_{13}(y, w)x^\tau - r_{23}(x, w)y^\tau - r_{43}(x, y)w^\tau$$

for all  $x, y, w \in I_1$ . Again, there are additive maps  $p, q: I_2 \rightarrow Q_{ml}(R)$  so that

$$\delta(wy) - w\delta(y) = -p(y)w^\tau - q(w)y^\tau$$

for all  $y, w \in I_2$ , where  $I_2$  is a nonzero ideal of  $R$  contained in  $I_1$ . Let  $x, y, t \in I_2$ . We have

$$\delta(xy) = x\delta(y) - p(y)x^\tau - q(x)y^\tau \tag{9}$$

Replacing  $x$  by  $tx$  in (9), we obtain

$$\delta(txy) = tx\delta(y) - p(y)x^\tau t^\tau - q(tx)y^\tau.$$

Left-multiplying (9) by  $t$ , we get  $t\delta(xy) = tx\delta(y) - tp(y)x^\tau - tq(x)y^\tau$ . Thus,

$$\delta(txy) - t\delta(xy) = \left(tq(x) - q(tx)\right)y^\tau - p(y)x^\tau t^\tau + tp(y)x^\tau.$$



Replacing  $x, y$  by  $t, xy$  respectively in (9), we have

$$\delta(txy) - t\delta(xy) = -p(xy)t^\tau - q(t)y^\tau x^\tau.$$

Comparing the two equalities above, we see that

$$\left(tq(x) - q(tx)\right)y^\tau + \left(p(xy) - p(y)x^\tau\right)t^\tau + \left(tp(y) + q(t)y^\tau\right)x^\tau = 0$$

for all  $x, y, t \in I_2$ . By Lemma 2.8,

$$tq(x) = q(tx), \quad p(xy) = p(y)x^\tau, \quad \text{and} \quad tp(y) = -q(t)y^\tau$$

for all  $x, y, t \in I_3$ , where  $I_3$  is a nonzero ideal of  $R$  contained in  $I_2$ . According to [16, Lemma 2.1], there is  $a \in Q_{ml}(R)$  such that  $q(x) = xa$  for  $x \in I_3$ . So

$$tp(y) = -q(t)y^\tau = -tay^\tau$$

for  $t, y \in I_3$ , i.e.,  $I_3(p(y) + ay^\tau) = 0$  for all  $y \in I_3$ . Thus,  $p(y) = -ay^\tau$  and it follows from (9) that

$$\delta(xy) - x\delta(y) = ay^\tau x^\tau - xay^\tau$$



for  $x, y \in I_3$ . Let  $\tilde{\delta}: I_3 \rightarrow Q_{ml}(R)$  be defined by  $\tilde{\delta}(x) = ax^\tau - xa$  for all  $x \in I_3$ . Then  $\tilde{\delta}(xy) = ay^\tau x^\tau - xya$  and

$$(\tilde{\delta} - \delta)(xy) = xay^\tau - xya - x\delta(y) = x(\tilde{\delta} - \delta)(y),$$

for  $x, y \in I_3$ . So there exists  $c \in Q_{ml}(R)$  such that  $(\tilde{\delta} - \delta)(x) = xc$  for all  $x \in I_3$  (see [16, Lemma 2.1]). Define  $J := \tilde{\delta} - \delta$ , a Jordan  $\tau$ -derivation of  $I_3$ . Thus,  $x^2c = J(x^2) = xJ(x) + J(x)x^\tau = x^2c + xcx^\tau$  for all  $x \in I_3$ ; that is,  $xcx^\tau = 0$  for all  $x \in I_3$ . By [10, Lemma 2.2],  $c = 0$  follows, i.e.,  $\delta = \tilde{\delta}$  on  $I_3$ . Therefore,  $\delta(x) = ax^\tau - xa$  for  $x \in I_3$ . By [20, Lemma 2.6],  $a \in Q_{ms}(R)$ . Finally, we will show that  $\delta$  is X-inner. Let  $x \in I_3$  and  $y \in R$ . Then

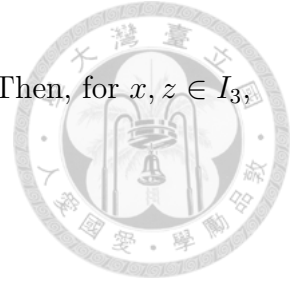
$$\delta(xy + yx) = ay^\tau x^\tau + ax^\tau y^\tau - xya - yxa$$

and

$$\begin{aligned} \delta(xy + yx) &= \delta(x)y^\tau + y\delta(x) + \delta(y)x^\tau + x\delta(y) \\ &= ax^\tau y^\tau - xay^\tau + yax^\tau - yxa + \delta(y)x^\tau + x\delta(y). \end{aligned}$$

Comparing these equations, we have

$$\left(\delta(y) - (ay^\tau - ya)\right)x^\tau + x\left(\delta(y) - (ay^\tau - ya)\right) = 0$$



for all  $x \in I_3$  and  $y \in R$ . Fix  $y \in R$  and set  $q := \delta(y) - (ay^\tau - ya)$ . Then, for  $x, z \in I_3$ ,

$$xzq = -q(xz)^\tau = -qz^\tau x^\tau = zqx^\tau = -zxq$$

and so  $(xz + zx)q = 0$  for all  $x, z \in I_3$ . This implies  $(xz + zx)q = 0$  for all  $x, z \in Q_{ml}(R)$ .

If  $\text{char} R \neq 2$ , let  $z = 1$  and so  $2xq = 0$  for all  $x \in Q_{ml}(R)$  implying  $q = 0$ . If  $\text{char} R = 2$ , then  $[Q_{ml}(R), Q_{ml}(R)]q = 0$  forcing  $q = 0$ . Hence  $\delta(y) = ay^\tau - ya$  for all  $y \in R$ , as desired.  $\square$

We next consider the case that  $\tau$  is of the first kind. By an X-inner automorphism we mean an automorphism of the form  $x \mapsto uxu^{-1}$  for all  $x \in R$ , where  $u \in Q_s(R)$ . Kharchenko proved that, given an automorphism  $\sigma$  of a prime GPI-ring, if  $\sigma$  is of the first kind, then it is X-inner (see [14, Proof of Proposition 2]). By Kharchenko's theorem,  $\tau^2$  is X-inner when  $R$  is a prime GPI-ring and  $\tau$  is of the first kind. The complexity of the question we will study depends on that of  $\tau^2$ .

**Definition 3.13.** *Let  $R$  be a prime GPI-ring with an automorphism  $\sigma$  of the first kind. Then there exists  $u \in Q_s(R)$  such that  $x^\sigma = uxu^{-1}$  for all  $x \in R$ . We say that  $\deg \sigma = m$  if  $u$  is algebraic of minimal degree  $m$  over  $C$ . Moreover,  $\deg \sigma = \infty$  if  $u$  is not algebraic over  $C$ .*

Clearly,  $\deg \sigma$  is independent of the element  $u$  we choose. Also, if  $\deg \tau^2 = 1$ , then



$\tau$  is an involution and Question A has been solved by Theorem 3.5. The following is the second main theorem and we will prove it in the next section.

**Theorem 3.14.** *Let  $R$  be a prime GPI-ring with  $\text{char}R \neq 2$  and with  $\tau$  an anti-automorphism of the first kind. If  $\deg \tau^2 \neq 2$ , then any Jordan  $\tau$ -derivation of  $R$  is  $X$ -inner.*

We remark that the case of  $\deg \tau^2 = 2$  keeps unknown.

### 3.3 The first kind case

The goal of this section is to prove Theorem 3.14. Let  $R$  be a prime GPI-ring with an anti-automorphism  $\tau$  of the first kind. Let  $u \in Q_s(R)$  be fixed such that  $x^{\tau^2} = uxu^{-1}$  for all  $x \in R$ .

**Lemma 3.15.**  $u^\tau u = uu^\tau \in C$ .

*Proof.* Let  $x \in R$ . Then

$$u^\tau ux = u^\tau x^{\tau^2} u = (x^\tau u)^\tau u = (u(x^\tau)^{\tau^{-2}})^\tau u = xu^\tau u.$$

Hence  $u^\tau u = uu^\tau \in C$ . □

Now, by Lemma 3.15, we fix  $\beta := u^\tau u = uu^\tau \in C$  and so  $u^\tau = \beta u^{-1}$ . Since  $R$  is a prime GPI-ring,  $RC$  is a primitive ring with nonzero socle and so  $Q_{ms}(RC) = Q_s(RC)$ .



In view of Theorem 3.4, the aim of this section is to extend  $\delta$  to a Jordan  $\tau$ -derivation of  $RC$  when  $\text{char}R \neq 2$ . The following result plays a key role.

**Lemma 3.16.** *Let  $f: R \rightarrow Q_{ml}(R)$  be an additive map such that*

$$xf(y) + f(y)x^\tau = yf(x) + f(x)y^\tau \quad (10)$$

for all  $x, y \in R$ . If  $\deg \tau^2 > 2$ , then  $f = 0$ .

Note that this result is better than [22, Lemma 3.2].

*Proof.* Choose a nonzero ideal such that  $uI_1 \subseteq R$ . Then, by replacing  $x$  with  $ux$  in (10),

$$uxf(y)u + \beta f(y)x^\tau = yf(ux)u + f(ux)y^\tau u$$

for  $x \in I_1$  and  $y \in R$ . Also, by (10),

$$uxf(y)u + uf(y)x^\tau u = uyf(x)u + uf(x)y^\tau u.$$

Comparing the two equations, we have

$$\beta f(y)x^\tau - uf(y)x^\tau u = yf(ux)u - uyf(x)u + (f(ux) - uf(x))y^\tau u \quad (11)$$



for  $x \in I_1$  and  $y \in R$ . Replacing  $y$  by  $uy$  in (11),

$$\beta f(uy)x^\tau - uf(uy)x^\tau u = uyf(ux)u - u^2yf(x)u + \beta(f(ux) - uf(x))y^\tau$$

for  $x, y \in I_1$ . Also, by (11),

$$u\beta f(y)x^\tau - u^2f(y)x^\tau u = uyf(ux)u - u^2yf(x)u + u(f(ux) - uf(x))y^\tau u.$$

Comparing the two equations, we have

$$\begin{aligned} & \beta(f(uy) - uf(y))x^\tau - u(f(uy) - uf(y))x^\tau u \\ &= \beta(f(ux) - uf(x))y^\tau - u(f(ux) - uf(x))y^\tau u \end{aligned}$$

for all  $x, y \in I_1$ . By Theorem 2.4,  $f(uy) = uf(y)$  for all  $y \in I_1$ . So (11) becomes

$$\beta f(y)x^\tau - uf(y)x^\tau u = yuf(x)u - yf(x)u \tag{12}$$

for all  $x, y \in I_1$ . Choose a nonzero ideal  $I_2$  of  $R$  contained in  $I_1$  such that  $uI_2 \subseteq I_1$ .

Replacing  $x$  by  $ux$  in (12), we have

$$\beta^2 f(y)x^\tau - \beta uf(y)x^\tau u = yu^2 f(x)u^2 - yuf(x)u^2$$



for all  $x, y \in I_2$ . On the other hand, (12) implies

$$\beta^2 f(y)x^\tau - \beta u f(y)x^\tau u = \beta \left( y u f(x) u - u y f(x) u \right)$$

for all  $x, y \in I_2$ . Comparing the two equations, we have

$$y \left( \beta u f(x) - u^2 f(x) u \right) + u y \left( u f(x) u - \beta f(x) \right) = 0$$

for all  $x, y \in I_2$ . Similarly,  $u f(x) u = \beta f(x)$  for all  $x \in I_2$ . Thus (12) becomes

$$\beta f(y)x^\tau - u f(y)x^\tau u = \beta y f(x) - u y f(x) u \tag{13}$$

for all  $x, y \in I_2$ . Choose a nonzero ideal  $I_3$  of  $R$  contained in  $I_2$  such that  $I_3 u \subseteq I_2$ .

Replacing  $y$  by  $yu$  in (13),

$$\begin{aligned} & \beta f(yu)x^\tau - u f(yu)x^\tau u \\ &= \beta y u f(x) - u y u f(x) u \\ &= \beta^2 y f(x) u^{-1} - \beta u y f(x) \\ &= \beta \left( \beta y f(x) - u y f(x) u \right) u^{-1} \\ &= \beta^2 f(y)x^\tau u^{-1} - \beta u f(y)x^\tau \end{aligned}$$





for all  $x, y \in I_3$ . So we get

$$\beta^2 f(y)x^\tau - (\beta u f(y) + \beta f(yu))x^\tau u + u f(yu)x^\tau u^2 = 0$$

for all  $x, y \in I_3$ . By Theorem 2.4,  $f(y) = 0$  for all  $y \in I_3$ . Thus it follows from (10) that

$$y f(x) + f(x)y^\tau = x f(y) + f(y)x^\tau = 0$$

for all  $x \in R$  and  $y \in I_3$ . By applying the same argument as the last part of the proof of Theorem 3.12, we have  $f = 0$ , as desired.  $\square$

**Remark 3.17.** In Lemma 3.16, the case for  $\deg \tau^2 = 1$  had been solved by Beidar and Martindale [3]. However, the solution of (10) is still unknown when  $\deg \tau^2 = 2$ .

**Lemma 3.18.** *Suppose  $\deg \tau^2 > 2$ . Let  $\alpha \in C$  and  $I$  be a nonzero ideal of  $R$  such that  $\alpha I \subseteq R$ . Then  $\delta(\alpha x) = \alpha \delta(x)$  for all  $x \in I$ .*

*Proof.* Let  $x, y \in I$ . Then

$$\delta((\alpha x)y + y(\alpha x)) = \delta(\alpha x)y^\tau + y\delta(\alpha x) + \alpha\delta(y)x^\tau + \alpha x\delta(y)$$



and

$$\delta(x(\alpha y) + (\alpha y)x) = \alpha\delta(x)y^\tau + \alpha y\delta(x) + \delta(\alpha y)x^\tau + x\delta(\alpha y).$$

Comparing the two equations, we have

$$\left(\delta(\alpha y) - \alpha\delta(y)\right)x^\tau + x\left(\delta(\alpha y) - \alpha\delta(y)\right) = \left(\delta(\alpha x) - \alpha\delta(x)\right)y^\tau + y\left(\delta(\alpha x) - \alpha\delta(x)\right).$$

By Lemma 3.16,  $\delta(\alpha x) = \alpha\delta(x)$  for all  $x \in I$ . □

**Lemma 3.19.** *Suppose  $\deg \tau^2 > 2$ . Then every Jordan  $\tau$ -derivation  $\delta$  of  $R$  can be extended to a Jordan  $\tau$ -derivation of  $RC$ .*

*Proof.* Choose a subset  $\{w_i\}_{i \in \Phi}$  of  $R$  which is a basis of  $RC$  over  $C$ , where  $\Phi$  is a nonempty well-ordered set. Then any element of  $RC$  can be written as the form  $\sum_{i \in \Phi} \alpha_i w_i$ , where  $\alpha_i = 0$  for all but finitely many  $i \in \Phi$ . Recall that  $Q_{ms}(R) = Q_{ms}(RC)$ . Define  $\tilde{\delta}: RC \rightarrow Q_{ms}(RC)$  by

$$\tilde{\delta}\left(\sum_{i \in \Phi} \alpha_i w_i\right) = \sum_{i \in \Phi} \alpha_i \delta(w_i).$$

Then it is clearly a well-defined additive map since  $\{w_i\}_{i \in \Phi}$  forms a basis of  $RC$  over  $C$ . We claim that  $\tilde{\delta}|_R = \delta$ . Let  $x \in R$ . Write  $x = \sum_{i \in \Phi} \alpha_i w_i$ . By Lemma 3.18, there is



a nonzero ideal  $I$  of  $R$  such that  $\alpha_i I \subseteq R$  and  $\delta(\alpha_i y) = \alpha_i \delta(y)$  for all  $i \in \Phi$  and  $y \in I$ .

Let  $y \in I$ . Then

$$\begin{aligned}\delta(xy + yx) &= \sum_{i \in \Phi} \delta((\alpha_i y)w_i + w_i(\alpha_i y)) \\ &= \sum_{i \in \Phi} \delta(\alpha_i y)w_i^\tau + \delta(w_i)\alpha_i y^\tau + w_i\delta(\alpha_i y) + \alpha_i y\delta(w_i) \\ &= \delta(y)x^\tau + x\delta(y) + \sum_{i \in \Phi} (\alpha_i \delta(w_i)y^\tau + \alpha_i y\delta(w_i)).\end{aligned}$$

Comparing this with (8), we have

$$\left(\delta(x) - \sum_{i \in \Phi} \alpha_i \delta(w_i)\right)y^\tau + y\left(\delta(x) - \sum_{i \in \Phi} \alpha_i \delta(w_i)\right) = 0$$

for all  $y \in I$ . By applying the same argument as the last part of the proof of Theorem 3.12, we have  $\delta(x) = \sum_{i \in \Phi} \alpha_i \delta(w_i) = \tilde{\delta}(x)$  and so the claim holds.

Finally we show that  $\tilde{\delta}$  is a Jordan  $\tau$ -derivation. Let  $x = \sum_{i \in \Phi} \alpha_i w_i \in RC$ . For



each  $i, j \in \Phi$ , write  $w_i w_j = \sum_{k \in \Phi} \gamma_k^{ij} w_k \in R$ , where  $\gamma_k^{ij} \in C$ . Then

$$\begin{aligned}
\tilde{\delta}(x^2) &= \tilde{\delta}\left(\sum_{i,j} \alpha_i \alpha_j w_i w_j\right) \\
&= \tilde{\delta}\left(\sum_{i,j} \alpha_i \alpha_j \sum_{k \in \Phi} \gamma_k^{ij} w_k\right) \\
&= \sum_{i,j} \alpha_i \alpha_j \sum_{k \in \Phi} \gamma_k^{ij} \delta(w_k) \\
&= \sum_{i,j} \alpha_i \alpha_j \tilde{\delta}(w_i w_j) \\
&= \sum_{i,j} \alpha_i \alpha_j \delta(w_i w_j) \\
&= \sum_i \alpha_i^2 \delta(w_i^2) + \sum_{i < j} \alpha_i \alpha_j \delta(w_i w_j + w_j w_i) \\
&= \sum_i \alpha_i^2 \left( w_i \delta(w_i) + \delta(w_i) w_i^\tau \right) \\
&\quad + \sum_{i < j} \alpha_i \alpha_j \left( \delta(w_i) w_j^\tau + w_j \delta(w_i) + \delta(w_j) w_i^\tau + w_i \delta(w_j) \right) \\
&= x \tilde{\delta}(x) + \tilde{\delta}(x) x^\tau.
\end{aligned}$$

Hence the proof of Lemma 3.19 is complete.  $\square$

Applying Lemma 3.19, we are now ready to prove Theorem 3.14.

*Proof of Theorem 3.14.* In view of Theorem 3.8, we can assume that  $R$  is not a PI-ring. By Theorem 3.5, we also assume that  $\deg \tau^2 > 2$ . Recall that if  $R$  is a prime GPI-ring and  $RC$  is a division ring, then  $R$  is a PI-ring. So  $RC$  is not a division ring. Let  $\delta: R \rightarrow Q_{ms}(R)$  be a Jordan  $\tau$ -derivation of  $R$ . Since the socle of  $RC$  is nonzero,

$Q_{ms}(RC) = Q_s(RC)$ . By Lemma 3.19,  $\delta$  can be extended to a Jordan  $\tau$ -derivation  $\tilde{\delta}: RC \rightarrow Q_s(RC)$  of  $RC$ . According to Theorem 3.4, there exists  $a \in Q_s(RC)$  such that  $\tilde{\delta}(x) = ax^\tau - xa$  for  $x \in RC$ . In particular,  $\delta(x) = ax^\tau - xa$  for all  $x \in R$ . As a consequence of [2, Proposition 2.1.10],  $Q_{ms}(R) = Q_{ms}(RC) = Q_s(RC)$  and so  $a \in Q_{ms}(R)$ . Hence  $\delta$  is X-inner, as desired.  $\square$



## 4 Weak Jordan derivations

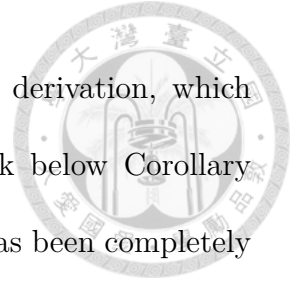
### 4.1 Motivation

Let  $R$  be a prime ring with extended centroid  $C$ . An additive map  $d: R \rightarrow Q_{ml}(R)$  is called a derivation if  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$  and is called a Jordan derivation (resp. Lie derivation) if  $d(x^2) = d(x)x + xd(x)$  for all  $x \in R$  (resp.  $d([x, y]) = [d(x), y] + [x, d(y)]$  for all  $x, y \in R$ ), where  $[a, b] := ab - ba$  for  $a, b \in Q_{ml}(R)$ . Clearly, any derivation is a Jordan derivation but a Jordan derivation is not in general a derivation. In 1957, Herstein proved the following theorem.

**Theorem 4.1.** ([12, Theorem 3.1]) *Let  $R$  be a prime ring with  $\text{char } R \neq 2$ . Then every Jordan derivation of  $R$  is a derivation.*

We remark that though a Jordan derivation defined in [12] maps  $R$  to itself, the same proof is also valid for our definition. In 2014, Lee and the author completely described Jordan derivations of a prime ring  $R$  with  $\text{char } R = 2$  as follows.

**Theorem 4.2.** ([19, Theorem 2.2]) *Let  $R$  be a prime ring with  $\text{char } R = 2$ . An additive map  $\delta: R \rightarrow Q_{ml}(R)$  is a Jordan derivation if and only if there exist a derivation  $d: R \rightarrow Q_{ml}(R)$  and an additive map  $\mu: R \rightarrow C$  such that  $\delta = d + \mu$  and  $\mu(x^2) = 0$  for all  $x \in R$ .*



They also gave an example to show the existence of a Jordan derivation, which is not a derivation, on  $R$  when  $\text{char } R = 2$  (see [19, The remark below Corollary 2.4]). Therefore the structure of Jordan derivations of prime rings has been completely determined.

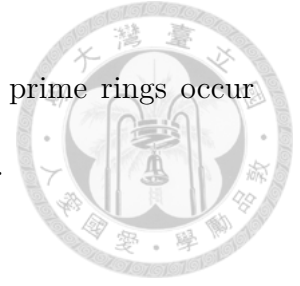
As a generalization of Jordan derivations (resp. Lie derivations), an additive map  $\delta: R \rightarrow Q_{ml}(R)$  is called a weak Jordan derivation (resp. weak Lie derivation) if  $\delta(x^2) - x\delta(x) - \delta(x)x \in C$  for all  $x \in R$  (resp.  $\delta([x, y]) - [\delta(x), y] - [x, \delta(y)] \in C$  for all  $x, y \in R$ ). In order to characterize additive maps  $d: R \rightarrow R$  satisfying  $[d(x^2) - xd(x) - d(x)x, x] = 0$  for all  $x \in R$  (see [5, Theorem 4]), Brešar proved the following result.

**Theorem 4.3.** ([5, pp. 541–542]) *Let  $R$  be a prime ring of characteristic different from 2 and 3. Suppose that  $\delta: R \rightarrow RC + C$  is a weak Jordan derivation. If  $\dim_C RC > 4$ , then  $\delta$  is a derivation.*

Also, in [8, Remark 6.8], Brešar et al. studied weak Lie derivations. In particular, they characterized weak Lie derivations of  $R$  when  $\dim_C RC > 9$  as follows.

**Theorem 4.4.** *Let  $R$  be a prime ring with  $\dim_C RC > 9$ . If  $\delta: R \rightarrow Q_{ml}(R)$  is a weak Lie derivation, then there exist a derivation  $d: R \rightarrow Q_{ml}(R)$  and an additive map  $\nu: R \rightarrow C$  such that  $\delta = d + \nu$ .*

We remark that every weak Jordan derivation of  $R$  is a weak Lie derivation when  $\text{char } R = 2$ . The aim of the chapter is to determine the structure of weak Jordan



derivations of  $R$ . On the other hand, weak Jordan derivations of prime rings occur canonically when one studies Jordan derivations of semiprime rings.

## 4.2 Results

Let  $R$  be a prime ring with extended centroid  $C$ . We give a complete characterization of weak Jordan derivations of prime rings. The following is the first result.

**Theorem 4.5.** *Let  $R$  be a prime ring with  $\dim_C RC > 4$ , and let  $\delta: R \rightarrow Q_{ml}(R)$  be a weak Jordan derivation.*

- (i) *If  $\text{char } R \neq 2$ , then  $\delta$  is a derivation.*
- (ii) *If  $\text{char } R = 2$ , then  $\delta = d + \nu$ , where  $d: R \rightarrow Q_{ml}(R)$  is a derivation and  $\nu: R \rightarrow C$  is an additive map.*

To state the second result, we will fix some notations. Suppose  $\dim_C RC = n^2 < \infty$ . It follows from Theorem 2.2 that  $RC$  is a finite-dimensional central simple algebra over  $C$  and  $C$  is the quotient field of  $Z(R)$ . In this case, we have  $Q_{ml}(R) = RC$ . By the well-known Wedderburn-Artin theorem,  $RC \cong M_m(\Delta)$  for some division ring  $\Delta$  and  $m \leq n$ . Let  $F$  be a separable maximal subfield of  $\Delta$ . Then  $RC \otimes_C F \cong M_n(F)$  and we regard  $RC$  as a  $C$ -subalgebra of  $M_n(F)$ . Denote  $I_n$  for the identity matrix in  $M_n(F)$ . Notice that, if  $RC$  is not a division ring and  $\dim_C RC = 4$ , then  $F = C$  and  $RC \cong M_2(C)$ . Also, if  $RC = \Delta$  is a division ring with  $\dim_C RC = 4$ , then  $F$  is a Galois extension over





C. We are now ready to state the second result.

**Theorem 4.6.** *Let  $R$  be a prime ring with  $\dim_C RC = 4$ , and let  $\delta: R \rightarrow Q_{ml}(R)$  be a weak Jordan derivation.*

(i) *If  $\text{char } R \neq 2$ , then*

$$\delta(x) = d(x) + [a, x] + L(x) + \zeta(x)$$

for all  $x \in R$ , where  $d: R \rightarrow RC$  is a derivation,  $a \in M_2(F)$  and  $L, \zeta: M_2(F) \rightarrow M_2(F)$  are  $F$ -linear maps. Moreover, there exist  $\beta_i \in F$ ,  $1 \leq i \leq 6$ , such that

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \left( \beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for all  $x = (x_{ij}) \in M_2(F)$ . In this case,  $\delta$  is a derivation if and only if all  $\beta_i = 0$ .

(ii) *If  $\text{char } R = 2$ , then*

$$\delta(x) = d(x) + \nu(x) + [a, x] + L(x) + \zeta(x)$$



for all  $x \in R$ , where  $d: R \rightarrow RC$  is a derivation,  $\nu: R \rightarrow C$  is an additive map,  $a \in M_2(F)$ , and  $L, \zeta: M_2(F) \rightarrow M_2(F)$  are  $F$ -linear maps. Moreover, there exist  $\beta_i \in F$ ,  $1 \leq i \leq 6$ , such that

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \left( \beta_1(x_{11} + x_{22}) + \beta_2 x_{12} + \beta_3 x_{21} \right) I_2$$

for all  $x = (x_{ij}) \in M_2(F)$ .

Conversely, an additive map  $\delta: R \rightarrow Q_{ml}(R)$  satisfying (i) or (ii) is a weak Jordan derivation.

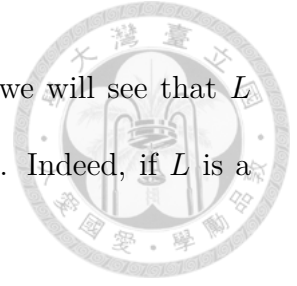
**Remark 4.7.** The additive maps  $L$  and  $\zeta$  in Theorem 4.6 are weak Jordan derivations.

In fact, they satisfy

$$L(x^2) - xL(x) - L(x)x = -\left( \beta_4 x_{21}^2 + \beta_5 x_{12}^2 + \beta_6 x_{12}x_{21} \right) I_2$$

and

$$\zeta(x^2) - x\zeta(x) - \zeta(x)x = -\left( \beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21} \right) (x_{11} - x_{22}) I_2$$



for all  $x = (x_{ij}) \in M_2(F)$ . By substituting some appropriate  $x_{ij}$ , we will see that  $L$  (resp.  $\zeta$ ) is a Jordan derivation if and only if  $L = 0$  (resp.  $\zeta = 0$ ). Indeed, if  $L$  is a Jordan derivation, then

$$\beta_4x^2 + \beta_5y^2 + \beta_6xy = 0$$

for all  $x, y \in F$ . Substituting  $(x, y)$  by  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , we get  $\beta_4 = \beta_5 = \beta_6 = 0$ .

Suppose next that  $\zeta$  is a Jordan derivation. Then

$$\left(\beta_1(x - y) + \beta_2z + \beta_3w\right)(x - y) = 0$$

for all  $x, y, z, w \in F$ . Substituting  $(x, y, z, w)$  by  $(1, 0, 0, 0)$ ,  $(1, 0, 1, 0)$ , and  $(1, 0, 0, 1)$ , we get  $\beta_1 = \beta_2 = \beta_3 = 0$ , as asserted.

A map  $q: R \rightarrow RC + C$  is called a trace of a biadditive map if there is a biadditive map  $B: R \times R \rightarrow RC + C$  such that  $q(x) = B(x, x)$  for all  $x \in R$ . Brešar and Šemrl characterized commuting traces of biadditive maps when  $\text{char } R \neq 2$  (see [5, Theorem 1] and [7, Theorem 3.1]). The following is the conclusion.

**Theorem 4.8.** *Let  $R$  be a prime ring with  $\text{char } R \neq 2$  and  $q: R \rightarrow RC + C$  be a trace of a biadditive map such that  $[q(x), x] = 0$  for all  $x \in R$ . Then there exist  $\lambda \in C$ , an*



additive map  $\mu: R \rightarrow C$ , and a trace of a biadditive map  $\nu: R \rightarrow C$  such that

$$q(x) = \lambda x^2 + \mu(x)x + \nu(x)$$

for all  $x \in R$ . Moreover, in case  $\dim_C RC = 4$  one may take  $\lambda = 0$ .

As an application of Theorem 4.5 and Theorem 4.6, the following generalizes Brešar's theorem [5, Theorem 4].

**Theorem 4.9.** *Let  $R$  be a prime ring with  $\text{char } R \neq 2$ , and let  $\delta: R \rightarrow RC + C$  be an additive map satisfying*

$$[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$$

for all  $x \in R$ .

(i) *If  $\dim_C RC > 4$ , then*

$$\delta(x) = \gamma x + d(x) + \mu(x)$$

for all  $x \in R$ , where  $\gamma \in C$ ,  $d: R \rightarrow RC + C$  is a derivation, and  $\mu: R \rightarrow C$  is an additive map.



(ii) If  $\dim_C RC = 4$ , then

$$\delta(x) = d(x) + \mu(x) + [a, x] + L(x) + \zeta(x)$$

for all  $x \in R$ , where  $d: R \rightarrow RC$  is a derivation,  $\mu: R \rightarrow C$  is an additive map,  $a \in M_2(F)$ , and  $L, \zeta: M_2(F) \rightarrow M_2(F)$  are as in Theorem 4.6 (i).

*Proof.* We will follow the proof of [5, Theorem 4] and apply Theorem 4.5 and Theorem 4.6. Let  $q: R \rightarrow RC + C$  be the map defined by  $q(x) = \delta(x^2) - x\delta(x) - \delta(x)x$  for  $x \in R$ . Then  $q$  is a trace of a biadditive map. According to Theorem 4.8, there exist  $\lambda \in C$ , an additive map  $\tilde{\mu}: R \rightarrow C$ , and a trace of a biadditive map  $\tilde{\nu}: R \rightarrow C$  such that

$$\delta(x^2) - x\delta(x) - \delta(x)x = \lambda x^2 + \tilde{\mu}(x)x + \tilde{\nu}(x)$$

for all  $x \in R$ . Note that we may take  $\lambda = 0$  if  $\dim_C RC = 4$ . Define an additive map  $D: R \rightarrow RC + C$  by

$$D(x) = \delta(x) + \lambda x + \frac{1}{2}\tilde{\mu}(x), \quad x \in R.$$



We claim that  $D$  is a weak Jordan derivation. Indeed, let  $x \in R$ . Then

$$\begin{aligned} D(x^2) &= \delta(x^2) + \lambda x^2 + \frac{1}{2}\tilde{\mu}(x^2) \\ &= x\delta(x) + \delta(x)x + 2\lambda x^2 + \tilde{\mu}(x)x + \tilde{\nu}(x) + \frac{1}{2}\tilde{\mu}(x^2) \end{aligned}$$

and  $xD(x) + D(x)x = x\delta(x) + \delta(x)x + 2\lambda x^2 + \tilde{\mu}(x)x$ . Thus

$$D(x^2) - xD(x) - D(x)x = \tilde{\nu}(x) + \frac{1}{2}\tilde{\mu}(x^2) \in C$$

and hence  $D$  is a weak Jordan derivation.

If  $\dim_C RC > 4$ , then it follows from Theorem 4.5 (i) that  $D$  is a derivation, and hence

$$\delta(x) = \gamma x + d(x) + \mu(x),$$

where  $\gamma = -\lambda$ ,  $d = D$ , and  $\mu(x) = -\frac{1}{2}\tilde{\mu}(x)$ .

Suppose that  $\dim_C RC = 4$ . Then  $RC + C = RC$  and, by Theorem 4.6 (i),  $D$  is of the form

$$D(x) = d(x) + [a, x] + L(x) + \zeta(x),$$

where  $d: R \rightarrow RC$  is a derivation,  $a \in M_2(F)$ , and  $L, \zeta: M_2(F) \rightarrow M_2(F)$  are as in



Theorem 4.6 (i). Hence

$$\delta(x) = d(x) + \mu(x) + [a, x] + L(x) + \zeta(x),$$

where  $\mu(x) = -\frac{1}{2}\tilde{\mu}(x)$ . □

### 4.3 Proofs of Theorem 4.5 and 4.6

Let  $R$  be a prime ring and let  $\delta: R \rightarrow Q_{ml}(R)$  be a weak Jordan derivation. Thus,

$$\mu(x) := \delta(x^2) - x\delta(x) - \delta(x)x \in C \tag{14}$$

for all  $x \in R$ . Linearizing it, we get

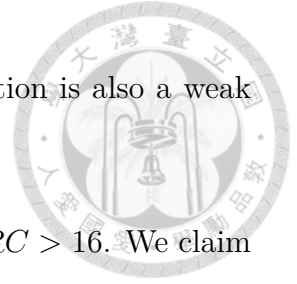
$$\lambda(x, y) := \delta(xy + yx) - x\delta(y) - y\delta(x) - \delta(x)y - \delta(y)x \in C \tag{15}$$

for all  $x, y \in R$ .

**Proposition 4.10.** *Let  $\delta: R \rightarrow Q_{ml}(R)$  be a weak Jordan derivation.*

(i) *If  $\text{char } R \neq 2$  and  $\dim_C RC > 16$ , then  $\delta$  is a derivation.*

(ii) *If  $\text{char } R = 2$  and  $\dim_C RC > 9$ , then  $\delta = d + \nu$ , where  $d: R \rightarrow Q_{ml}(R)$  is a derivation and  $\nu: R \rightarrow C$  is an additive map.*



*Proof.* We first prove (ii). Since  $\text{char } R = 2$ , a weak Jordan derivation is also a weak Lie derivation by (15). Therefore (ii) follows from Theorem 4.4.

We turn to the proof of (i). Suppose that  $\text{char } R \neq 2$  and  $\dim_C RC > 16$ . We claim that there exist additive maps  $p, q: R \rightarrow Q_{ml}(R)$  and a bi-additive map  $\phi: R^2 \rightarrow C$  such that

$$\delta(xy) - \delta(x)y = xp(y) + yq(x) + \phi(x, y) \quad (16)$$

for all  $x, y \in R$ . Let  $B(x, y) := \delta(xy + yx)$ . By Lemma 3.11,

$$B(xw, yz) - B(x, wyz) = B(zxw, y) - B(zx, wy)$$

for all  $x, y, z, w \in R$ . It follows from (15) that

$$\begin{aligned} & (\delta(wy)z - \delta(wyz))x + (\delta(zx)w - \delta(zxw))y \\ & + (\delta(xw)y - \delta(x)wy)z + (\delta(yz)x - \delta(y)zx)w \\ & + x(w\delta(yz) - \delta(wyz)) + y(z\delta(xw) - \delta(zxw)) \\ & + z(x\delta(wy) - xw\delta(y)) + w(y\delta(zx) - yz\delta(x)) \in C \end{aligned}$$

for all  $x, y, z, w \in R$ . According to Theorem 2.5, there are bi-additive maps  $\tilde{p}, \tilde{q}, \tilde{s}: R^2 \rightarrow$





$Q_{ml}(R)$  and a 3-additive map  $\tilde{r}: R^3 \rightarrow C$  such that

$$(\delta(xw) - \delta(x)w)y = x\tilde{p}(y, w) + y\tilde{q}(x, w) + w\tilde{s}(x, y) + \tilde{r}(x, y, w)$$

for all  $x, y, w \in R$ . By Theorem 2.5 again, there exist additive maps  $p, q: R \rightarrow Q_{ml}(R)$

and a bi-additive map  $\phi: R^2 \rightarrow C$  such that

$$\delta(xw) - \delta(x)w = xp(w) + wq(x) + \phi(x, w)$$

for all  $x, w \in R$ . Hence (16) holds.

Next, let  $x, y, z \in R$ . By (16), we have

$$\delta(x(yz)) - \delta(x)yz = xp(yz) + yzq(x) + \phi(x, yz),$$

$$\delta((xy)z) - \delta(xy)z = xyp(z) + zq(xy) + \phi(xy, z).$$

Thus,

$$(\delta(xy) - \delta(x)y)z - x(p(yz) - yp(z)) - yzq(x) + zq(xy) \in C.$$

So, by Theorem 2.5, there is an additive map  $d: R \rightarrow Q_{ml}(R)$  such that

$$p(yz) - yp(z) = d(y)z.$$



By (17) again,

$$p(xyz) - xp(yz) = d(x)yz,$$

$$p(xyz) - xyp(z) = d(xy)z.$$

Thus,

$$-xd(y)z = x(yp(z) - p(yz)) = (d(x)y - d(xy))z$$

and so

$$(d(xy) - d(x)y - xd(y))z = 0.$$

This proves that  $d$  is a derivation. So, by (17),

$$(p - d)(yz) = y(p - d)(z).$$

By [16, Lemma 2.1], there exists  $a \in Q_{ml}(R)$  such that  $d(x) = p(x) - xa$  for all  $x \in R$ .



So

$$d(xy) - d(x)y = xd(y) = xp(y) - xya.$$

Compare it with (16) and let  $\tilde{\delta} = \delta - d$ , and then we have

$$\tilde{\delta}(xy) - \tilde{\delta}(x)y - xya - yq(x) \in C. \quad (18)$$

It follows from (18) that

$$\tilde{\delta}(xyz) - \tilde{\delta}(x)yz - xyz a - yzq(x) \in C,$$

$$\tilde{\delta}(xyz) - \tilde{\delta}(xy)z - xyz a - zq(xy) \in C.$$

Thus,

$$(\tilde{\delta}(xy) - \tilde{\delta}(x)y)z - yzq(x) + zq(xy) \in C.$$

By Theorem 2.5, there is an additive map  $A: R \rightarrow Q_{ml}(R)$  such that

$$zq(x) = A(x)z$$

and so  $q(x) \in C$  for all  $x \in R$ . Let  $Q = \tilde{\delta} + q$ . Then, by (18), we get

$$Q(xy) - Q(x)y - xya \in C.$$



Let  $x, y, z \in R$ . Then, by (19),

$$Q(xyz) - Q(x)yz - xyz a \in C,$$

$$Q(xyz) - Q(xy)z - xyz a \in C.$$

So  $(Q(xy) - Q(x)y)z \in C$ . Since  $R$  is not commutative, we see that  $Q(xy) - Q(x)y = 0$  for all  $x, y \in R$ . It follows from this and (19) that  $a = 0$ ,  $p = d$  and so  $p$  is a derivation.

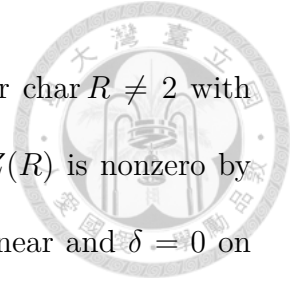
Furthermore, since  $Q = \delta - d + q$ , we have

$$-xQ(x) = Q(x^2) - Q(x)x - xQ(x) = \mu(x) + q(x^2) - 2q(x)x$$

and so

$$x(Q(x) - 2q(x)) \in C.$$

Thus,  $Q(x) = 2q(x) \in C$  for all  $x \in R$ , and hence  $\delta = d + \nu$ , where  $d: R \rightarrow Q_{ml}(R)$  is a derivation and  $\nu = q: R \rightarrow C$  is an additive map. Since  $\text{char } R \neq 2$ ,  $\nu(x)x \in C$  for all  $x \in R$  and thus  $\nu = 0$ , as desired. □



According to Proposition 4.10, we may assume that  $R$  is either  $\text{char } R \neq 2$  with  $\dim_C RC \leq 16$  or  $\text{char } R = 2$  with  $\dim_C RC \leq 9$ . In either case,  $Z(R)$  is nonzero by Theorem 2.2. We next claim that  $\delta$  can be assumed to be  $Z(R)$ -linear and  $\delta = 0$  on  $Z(R)$ .

**Lemma 4.11.** *Assume that  $\dim_C RC < \infty$ . Let  $\delta: R \rightarrow RC$  be a weak Jordan derivation.*

(i) *If  $\text{char } R \neq 2$ , then there is a derivation  $d: R \rightarrow RC$  such that  $\delta - d$  is  $Z(R)$ -linear and  $\delta = d$  on  $Z(R)$ .*

(ii) *If  $\text{char } R = 2$ , then there exist a derivation  $d: R \rightarrow RC$  and an additive map  $\nu: R \rightarrow C$  such that  $\delta + d + \nu$  is  $Z(R)$ -linear and  $\delta + d + \nu = 0$  on  $Z(R)$ .*

*Proof.* (i) Assume that  $\text{char } R \neq 2$ . Let  $\beta, \gamma \in Z(R)$  and  $x \in R$ . Then, by (15),

$$2\delta((\beta\gamma)x) = 2\beta\gamma\delta(x) + \delta(\beta\gamma)x + x\delta(\beta\gamma) + \lambda(\beta\gamma, x),$$

and

$$\begin{aligned} 2\delta(\beta(\gamma x)) &= 2\beta\delta(\gamma x) + \delta(\beta)\gamma x + \gamma x\delta(\beta) + \lambda(\beta, \gamma x) \\ &= \beta(2\gamma\delta(x) + \delta(\gamma)x + x\delta(\gamma) + \lambda(\gamma, x)) + \delta(\beta)\gamma x + \gamma x\delta(\beta) + \lambda(\beta, \gamma x) \\ &= 2\beta\gamma\delta(x) + (\beta\delta(\gamma) + \delta(\beta)\gamma)x + x(\beta\delta(\gamma) + \delta(\beta)\gamma) + \beta\lambda(\gamma, x) + \lambda(\beta, \gamma x). \end{aligned}$$



Comparing the two equations, we have

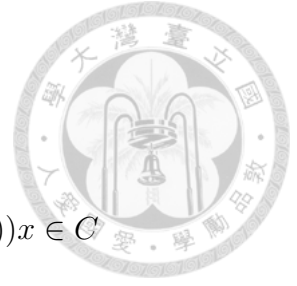
$$(\delta(\beta\gamma) - \beta\delta(\gamma) - \delta(\beta)\gamma)x + x(\delta(\beta\gamma) - \beta\delta(\gamma) - \delta(\beta)\gamma) \in C.$$

Thus,  $\delta(\beta\gamma) = \beta\delta(\gamma) + \delta(\beta)\gamma$  and so  $\delta|_{Z(R)}: Z(R) \rightarrow RC$  is a derivation. Next we claim that  $\delta(Z(R)) \subseteq C$ . Let  $0 \neq \beta \in Z(R)$  and  $x \in R$ . Then

$$\begin{aligned} 2\delta(\beta^2 x^2) &= 2\beta^2\delta(x^2) + x^2\delta(\beta^2) + \delta(\beta^2)x^2 + \lambda(\beta^2, x^2) \\ &= 2\beta^2(x\delta(x) + \delta(x)x + \mu(x)) + 2\beta(x^2\delta(\beta) + \delta(\beta)x^2) + \lambda(\beta^2, x^2), \end{aligned}$$

and

$$\begin{aligned} 2\delta((\beta x)^2) &= 2\beta x\delta(\beta x) + 2\delta(\beta x)\beta x + 2\mu(\beta x) \\ &= \beta x(2\beta\delta(x) + x\delta(\beta) + \delta(\beta)x + \lambda(\beta, x)) \\ &\quad + (2\beta\delta(x) + x\delta(\beta) + \delta(\beta)x + \lambda(\beta, x))\beta x + 2\mu(\beta x) \\ &= 2\beta^2(x\delta(x) + \delta(x)x) + \beta(x^2\delta(\beta^2) + \delta(\beta^2)x^2) \\ &\quad + 2\beta x\delta(\beta)x + 2\beta\lambda(\beta, x)x + 2\mu(\beta x). \end{aligned}$$



Comparing the two equations, we get

$$x\beta(x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x)) + \beta(x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x))x \in C$$

and so

$$x(x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x)) + (x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x))x \in C$$

for all  $x \in R$ . This means

$$[x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x), x^2] = 0$$

for all  $x \in R$ . Note that the proof of [6, Theorem 1] is also valid when we replace  $f: R \rightarrow R$  by  $f: R \rightarrow RC$ . Thus, there exist  $\alpha \in C$  and an additive map  $\xi: R \rightarrow C$  such that  $x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x) = \alpha x + \xi(x)$ . So  $x(\delta(\beta) - \alpha) - \delta(\beta)x \in C$  and hence  $\delta(\beta) \in C$ . This proves the claim. Since  $C$  is the quotient field of  $Z(R)$ ,  $\delta|_{Z(R)}$  can be extended to a derivation from  $C$  to  $C$ . According to [13, Theorem 6] (also see [27, Theorem 4.1] and [1]), it can also be extended to a derivation  $d: RC \rightarrow RC$ . Thus,  $(\delta - d)(Z(R)) = 0$ . Let  $J := \delta - d$ ,  $0 \neq \beta \in Z(R)$ , and  $x \in R$ . Then by (15) we have

$$J(\beta x) = \beta J(x) + \frac{1}{2}\lambda(\beta, x).$$



Thus,

$$2J(\beta^2 x^2) = 2\beta^2 J(x^2) + \lambda(\beta^2, x^2),$$

and

$$\begin{aligned} 2J((\beta x)^2) &= 2(\beta x J(\beta x) + J(\beta x)\beta x + \mu(\beta x)) \\ &= 2\beta x \left( \beta J(x) + \frac{1}{2}\lambda(\beta, x) \right) + 2 \left( \beta J(x) + \frac{1}{2}\lambda(\beta, x) \right) \beta x + 2\mu(\beta x) \\ &= 2\beta^2 (xJ(x) + J(x)x) + 2\beta\lambda(\beta, x)x + 2\mu(\beta x). \end{aligned}$$

Comparing the two equations, we have  $2\beta\lambda(\beta, x)x \in C$  and so  $\lambda(\beta, x) = 0$ . Therefore

(i) has been proved.

(ii) Assume that  $\text{char } R = 2$ . Let  $0 \neq \beta \in Z(R)$ . Then, by (15), we have  $[\delta(\beta), x] \in C$  for all  $x \in R$ . Thus,  $\delta(Z(R)) \subseteq C$ . Also, by letting  $y = \beta x$  in (15), we get  $[\delta(\beta x) + \beta\delta(x), x] \in C$  for all  $x \in R$ . According to [15, Theorem 2], there exist  $d: Z(R) \rightarrow C$  and  $\psi: Z(R) \times R \rightarrow C$  such that

$$\delta(\beta x) = \beta\delta(x) + d(\beta)x + \psi(\beta, x) \tag{20}$$

for all  $x \in R$  and  $\beta \in Z(R)$ . We will show that  $d: Z(R) \rightarrow C$  is a derivation. Let





$\beta, \gamma \in Z(R)$  and  $x \in R$ . Then

$$\delta((\beta\gamma)x) + \beta\gamma\delta(x) + d(\beta\gamma)x \in C,$$

$$\delta(\beta(\gamma x)) + \beta\delta(\gamma x) + d(\beta)\gamma x \in C,$$

$$\beta\delta(\gamma x) + \beta\gamma\delta(x) + \beta d(\gamma)x \in C.$$

Considering the sum of the above three equations, we have

$$(d(\beta\gamma) + \beta d(\gamma) + d(\beta)\gamma)x \in C$$

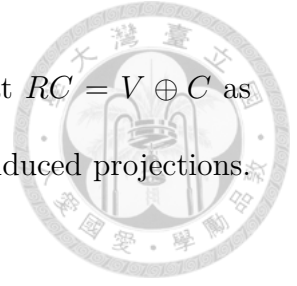
for  $\beta, \gamma \in Z(R)$  and  $x \in R$ . This implies that  $d(\beta\gamma) = \beta d(\gamma) + d(\beta)\gamma$  for all  $\beta, \gamma \in Z(R)$ , that is,  $d$  is a derivation. Since  $C$  is the quotient field of  $Z(R)$ ,  $d$  can be uniquely extended to a derivation from  $C$  to itself. By [13, Theorem 6], it can be extended to a derivation from  $RC$  to itself, say the same  $d$ . Let  $J := \delta + d$ . Then, by (20), we have

$$J(\beta x) + \beta J(x) \in C,$$

$$J(Z(R)) \subseteq Z(R),$$

and

$$J(x^2) + [x, J(x)] \in C,$$



for  $\beta \in Z(R)$  and  $x \in R$ . Choose a  $C$ -subspace  $V$  of  $RC$  such that  $RC = V \oplus C$  as vector spaces over  $C$ . Let  $\pi_1: RC \rightarrow V$  and  $\pi_2: RC \rightarrow C$  be the induced projections.

Let  $\tilde{J} := \pi_1 \circ J$  and  $\nu := \pi_2 \circ J$ . Then  $\tilde{J} = \delta + d + \nu$  and

$$\tilde{J}(\beta x) = \beta \tilde{J}(x),$$

$$\tilde{J}(Z(R)) = 0,$$

and

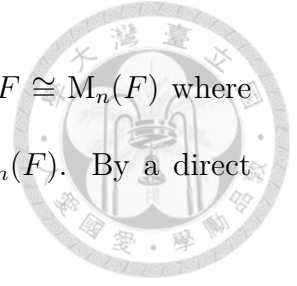
$$\tilde{J}(x^2) + [x, \tilde{J}(x)] \in C,$$

for  $\beta \in Z(R)$  and  $x \in R$ , as desired. Hence (ii) is proved. □

Now we assume that  $\delta$  is  $Z(R)$ -linear and  $\delta(Z(R)) = 0$ . Since  $C$  is the quotient field of  $Z(R)$ , the map  $\delta$  can be uniquely extended to a map from  $RC$  into itself (denoted by  $\delta$  also) satisfying

$$\left\{ \begin{array}{l} \delta(x^2) - x\delta(x) - \delta(x)x \in C \\ \delta(cx) = c\delta(x) \\ \delta(C) = 0 \end{array} \right.$$

for all  $x \in RC$  and  $c \in C$ . Since  $\delta: RC \rightarrow RC$  is  $C$ -linear, by the universal property of tensor product, there is a unique additive map  $\bar{\delta}$  from  $RC \otimes_C F$  into itself which



maps  $x \otimes \beta$  to  $\delta(x) \otimes \beta$  for all  $x \in RC$  and  $\beta \in F$ . Since  $RC \otimes_C F \cong M_n(F)$  where  $n = \sqrt{\dim_C RC} > 1$ , we can view  $\bar{\delta}$  as a map from  $M_n(F)$  to  $M_n(F)$ . By a direct calculation, we have

$$\begin{cases} \bar{\delta}(x^2) - x\bar{\delta}(x) - \bar{\delta}(x)x \in F \\ \bar{\delta}(cx) = c\bar{\delta}(x) \\ \bar{\delta}(F) = 0 \end{cases}$$

for all  $x \in M_n(F)$  and  $c \in F$ . Now we will determine the map  $\bar{\delta}$ . Let  $e_{ij}$ 's,  $1 \leq i, j \leq n$ , be the standard matrix units of  $M_n(F)$ . It suffices to determine  $\bar{\delta}(e_{ij})$  for all  $i, j$  because  $\bar{\delta}$  is  $F$ -linear. For convenience, we set

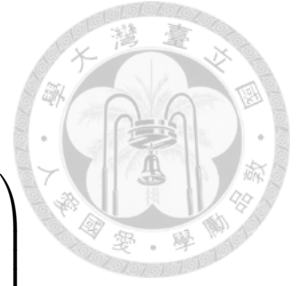
$$\bar{\delta}(e_{ij}) = (\alpha_{kl}^{ij}) := \sum_{k,l=1}^n \alpha_{kl}^{ij} e_{kl}.$$

**Lemma 4.12.** *Let  $F$  be a field with  $\text{char } F \neq 2$  and  $\bar{\delta}: M_n(F) \rightarrow M_n(F)$  be an  $F$ -linear weak Jordan derivation such that  $\bar{\delta}(F) = 0$ . Then  $\bar{\delta}$  is an inner derivation.*

*Proof.* We will compute  $\alpha_{kl}^{ij}$  by the following steps.

**Step 1.** Calculate (14) for  $x = e_{ii}$ ,  $i = 1, \dots, 4$ . For  $x = e_{11}$ ,

$$\bar{\delta}(e_{11}^2) - e_{11}\bar{\delta}(e_{11}) - \bar{\delta}(e_{11})e_{11} = \begin{pmatrix} -\alpha_{11}^{11} & 0 & 0 & 0 \\ 0 & \alpha_{22}^{11} & \alpha_{23}^{11} & \alpha_{24}^{11} \\ 0 & \alpha_{32}^{11} & \alpha_{33}^{11} & \alpha_{34}^{11} \\ 0 & \alpha_{42}^{11} & \alpha_{43}^{11} & \alpha_{44}^{11} \end{pmatrix} \in F$$



implies

$$\alpha_{22}^{11} = \alpha_{33}^{11} = \alpha_{44}^{11} = -\alpha_{11}^{11}$$

and

$$\alpha_{23}^{11} = \alpha_{24}^{11} = \alpha_{32}^{11} = \alpha_{34}^{11} = \alpha_{42}^{11} = \alpha_{43}^{11} = 0.$$

So

$$\bar{\delta}(e_{11}) = \begin{pmatrix} \alpha_{11}^{11} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{21}^{11} & -\alpha_{11}^{11} & 0 & 0 \\ \alpha_{31}^{11} & 0 & -\alpha_{11}^{11} & 0 \\ \alpha_{41}^{11} & 0 & 0 & -\alpha_{11}^{11} \end{pmatrix}.$$

Similarly, we have

$$\bar{\delta}(e_{22}) = \begin{pmatrix} \alpha_{11}^{22} & \alpha_{12}^{22} & 0 & 0 \\ \alpha_{21}^{22} & -\alpha_{11}^{22} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{32}^{22} & \alpha_{11}^{11} & 0 \\ 0 & \alpha_{42}^{22} & 0 & \alpha_{11}^{22} \end{pmatrix},$$

$$\bar{\delta}(e_{33}) = \begin{pmatrix} \alpha_{11}^{33} & 0 & \alpha_{13}^{33} & 0 \\ 0 & \alpha_{11}^{33} & \alpha_{23}^{33} & 0 \\ \alpha_{31}^{33} & \alpha_{32}^{33} & -\alpha_{11}^{33} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{43}^{33} & \alpha_{11}^{33} \end{pmatrix},$$

and

$$\bar{\delta}(e_{44}) = \begin{pmatrix} \alpha_{11}^{44} & 0 & 0 & \alpha_{14}^{44} \\ 0 & \alpha_{11}^{44} & 0 & \alpha_{24}^{44} \\ 0 & 0 & \alpha_{11}^{44} & \alpha_{34}^{44} \\ \alpha_{41}^{44} & \alpha_{42}^{44} & \alpha_{43}^{44} & -\alpha_{11}^{44} \end{pmatrix}.$$





**Step 2.** Since  $\bar{\delta}(I_4) = 0$ , we have

$$\begin{aligned} \alpha_{11}^{11} &= \alpha_{11}^{22} = \alpha_{11}^{33} = \alpha_{11}^{44} = 0, \\ \alpha_{41}^{44} &= -\alpha_{41}^{11}, \quad \alpha_{42}^{44} = -\alpha_{42}^{22}, \quad \alpha_{43}^{44} = -\alpha_{43}^{33}, \\ \alpha_{14}^{44} &= -\alpha_{14}^{11}, \quad \alpha_{24}^{44} = -\alpha_{24}^{22}, \quad \alpha_{34}^{44} = -\alpha_{34}^{33}, \\ \alpha_{21}^{22} &= -\alpha_{21}^{11}, \quad \alpha_{12}^{22} = -\alpha_{12}^{11}, \\ \alpha_{31}^{33} &= -\alpha_{31}^{11}, \quad \alpha_{13}^{33} = -\alpha_{13}^{11}, \\ \alpha_{23}^{33} &= -\alpha_{23}^{22}, \quad \alpha_{32}^{33} = -\alpha_{32}^{22}. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{\delta}(e_{11}) &= \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{21}^{11} & 0 & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 & 0 \\ \alpha_{41}^{11} & 0 & 0 & 0 \end{pmatrix}, \\ \bar{\delta}(e_{22}) &= \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 & 0 \\ -\alpha_{21}^{11} & 0 & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{32}^{22} & 0 & 0 \\ 0 & \alpha_{42}^{22} & 0 & 0 \end{pmatrix}, \end{aligned}$$



$$\bar{\delta}(e_{33}) = \begin{pmatrix} 0 & 0 & -\alpha_{13}^{11} & 0 \\ 0 & 0 & -\alpha_{23}^{22} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & 0 & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{43}^{33} & 0 \end{pmatrix},$$

and

$$\bar{\delta}(e_{44}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{14}^{11} \\ 0 & 0 & 0 & -\alpha_{24}^{22} \\ 0 & 0 & 0 & -\alpha_{34}^{33} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & 0 \end{pmatrix}.$$

**Step 3.** For each  $x = e_{ij}$ ,  $i \neq j$ , calculate (14), and then calculate (15) for  $y = e_{kk}$ ,

$k = 1, 2, 3$ . For example, by (14), we have

$$e_{12}\bar{\delta}(e_{12}) + \bar{\delta}(e_{12})e_{12} = \begin{pmatrix} \alpha_{21}^{12} & \alpha_{22}^{12} & \alpha_{23}^{12} & \alpha_{24}^{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{11}^{12} & 0 & 0 \\ 0 & \alpha_{21}^{12} & 0 & 0 \\ 0 & \alpha_{31}^{12} & 0 & 0 \\ 0 & \alpha_{41}^{12} & 0 & 0 \end{pmatrix} \in F.$$

Thus,

$$\alpha_{21}^{12} = \alpha_{23}^{12} = \alpha_{24}^{12} = \alpha_{31}^{12} = \alpha_{41}^{12} = 0,$$

$$\alpha_{22}^{12} = -\alpha_{11}^{12},$$

and so

$$\bar{\delta}(e_{12}) = \begin{pmatrix} \alpha_{11}^{12} & \alpha_{12}^{12} & \alpha_{13}^{12} & \alpha_{14}^{12} \\ 0 & -\alpha_{11}^{12} & 0 & 0 \\ 0 & \alpha_{32}^{12} & \alpha_{33}^{12} & \alpha_{34}^{12} \\ 0 & \alpha_{42}^{12} & \alpha_{43}^{12} & \alpha_{44}^{12} \end{pmatrix}.$$







Let  $x = e_{12}$  and  $y = e_{33}$ . By (15), we get

$$\begin{aligned}
 & e_{12}\bar{\delta}(e_{33}) + \bar{\delta}(e_{33})e_{12} + e_{33}\bar{\delta}(e_{12}) + \bar{\delta}(e_{12})e_{33} \\
 &= \begin{pmatrix} 0 & 0 & -\alpha_{23}^{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\alpha_{31}^{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_{32}^{12} & \alpha_{33}^{12} & \alpha_{34}^{12} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \alpha_{13}^{12} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33}^{12} & 0 \\ 0 & 0 & \alpha_{43}^{12} & 0 \end{pmatrix} \in F,
 \end{aligned}$$

and so

$$\alpha_{13}^{12} = \alpha_{23}^{22},$$

$$\alpha_{32}^{12} = \alpha_{31}^{11},$$

$$\alpha_{33}^{12} = \alpha_{34}^{12} = \alpha_{43}^{12} = 0.$$

These give

$$\bar{\delta}(e_{12}) = \begin{pmatrix} \alpha_{11}^{12} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{14}^{12} \\ 0 & -\alpha_{11}^{12} & 0 & 0 \\ 0 & \alpha_{31}^{11} & 0 & 0 \\ 0 & \alpha_{42}^{12} & 0 & \alpha_{44}^{12} \end{pmatrix}.$$



Let  $x = e_{12}$  and  $y = e_{22}$ , and compute (15). Then

$$\alpha_{11}^{12} = -\alpha_{21}^{11},$$

$$\alpha_{44}^{12} = 0,$$

$$\alpha_{14}^{12} = \alpha_{24}^{22}.$$

These give

$$\bar{\delta}(e_{12}) = \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{21}^{11} & 0 & 0 \\ 0 & \alpha_{31}^{11} & 0 & 0 \\ 0 & \alpha_{42}^{12} & 0 & 0 \end{pmatrix}.$$

Again, let  $x = e_{12}$  and  $y = e_{11}$ , and compute (15). Then  $\alpha_{42}^{12} = \alpha_{41}^{11}$  and so



$$\bar{\delta}(e_{12}) = \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{21}^{11} & 0 & 0 \\ 0 & \alpha_{31}^{11} & 0 & 0 \\ 0 & \alpha_{41}^{11} & 0 & 0 \end{pmatrix}.$$

Similarly, we have

$$\bar{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 & 0 & 0 \\ \alpha_{21}^{21} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{32}^{22} & 0 & 0 & 0 \\ \alpha_{42}^{22} & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\delta}(e_{13}) = \begin{pmatrix} -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{13}^{13} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{21}^{11} & 0 \\ 0 & 0 & \alpha_{31}^{11} & 0 \\ 0 & 0 & \alpha_{41}^{11} & 0 \end{pmatrix}, \bar{\delta}(e_{31}) = \begin{pmatrix} -\alpha_{13}^{11} & 0 & 0 & 0 \\ -\alpha_{23}^{22} & 0 & 0 & 0 \\ \alpha_{31}^{31} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{43}^{33} & 0 & 0 & 0 \end{pmatrix},$$



$$\begin{aligned}
 \bar{\delta}(e_{14}) &= \begin{pmatrix} -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & \alpha_{14}^{14} \\ 0 & 0 & 0 & \alpha_{21}^{11} \\ 0 & 0 & 0 & \alpha_{31}^{11} \\ 0 & 0 & 0 & \alpha_{41}^{11} \end{pmatrix}, \bar{\delta}(e_{41}) = \begin{pmatrix} -\alpha_{14}^{11} & 0 & 0 & 0 \\ -\alpha_{24}^{22} & 0 & 0 & 0 \\ -\alpha_{34}^{33} & 0 & 0 & 0 \\ \alpha_{41}^{41} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \end{pmatrix}, \\
 \bar{\delta}(e_{23}) &= \begin{pmatrix} 0 & 0 & -\alpha_{12}^{11} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{23}^{23} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{32}^{22} & 0 \\ 0 & 0 & \alpha_{42}^{22} & 0 \end{pmatrix}, \bar{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{32}^{32} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix}, \\
 \bar{\delta}(e_{24}) &= \begin{pmatrix} 0 & 0 & 0 & -\alpha_{12}^{11} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & \alpha_{24}^{24} \\ 0 & 0 & 0 & \alpha_{32}^{22} \\ 0 & 0 & 0 & \alpha_{42}^{22} \end{pmatrix}, \bar{\delta}(e_{42}) = \begin{pmatrix} 0 & -\alpha_{14}^{11} & 0 & 0 \\ 0 & -\alpha_{24}^{22} & 0 & 0 \\ 0 & -\alpha_{34}^{33} & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{42}^{42} & \alpha_{23}^{22} & \alpha_{24}^{22} \end{pmatrix}, \\
 \bar{\delta}(e_{34}) &= \begin{pmatrix} 0 & 0 & 0 & -\alpha_{13}^{11} \\ 0 & 0 & 0 & -\alpha_{23}^{22} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & \alpha_{34}^{34} \\ 0 & 0 & 0 & \alpha_{43}^{33} \end{pmatrix}, \bar{\delta}(e_{43}) = \begin{pmatrix} 0 & 0 & -\alpha_{14}^{11} & 0 \\ 0 & 0 & -\alpha_{24}^{22} & 0 \\ 0 & 0 & -\alpha_{34}^{33} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{43}^{43} & \alpha_{34}^{33} \end{pmatrix}.
 \end{aligned}$$

**Step 4.** Calculate (15) for  $x = e_{ij}$  and  $y = e_{ji}$ ,  $i < j$ . For example, if  $x = e_{12}$  and



$y = e_{21}$ , then

$$\begin{aligned}
 & \bar{\delta}(e_{11} + e_{22}) - e_{12}\bar{\delta}(e_{21}) - \bar{\delta}(e_{21})e_{12} - e_{21}\bar{\delta}(e_{12}) - \bar{\delta}(e_{12})e_{21} \\
 = & \begin{pmatrix} 0 & 0 & \alpha_{13}^{11} & \alpha_{14}^{11} \\ 0 & 0 & \alpha_{23}^{22} & \alpha_{24}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & 0 & 0 \\ \alpha_{41}^{11} & \alpha_{42}^{22} & 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_{21}^{21} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 & 0 \\ 0 & \alpha_{21}^{21} & 0 & 0 \\ 0 & \alpha_{32}^{22} & 0 & 0 \\ 0 & \alpha_{42}^{22} & 0 & 0 \end{pmatrix} \\
 - & \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_{12}^{12} & 0 & 0 & 0 \\ \alpha_{21}^{11} & 0 & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 & 0 \\ \alpha_{41}^{11} & 0 & 0 & 0 \end{pmatrix} \in F
 \end{aligned}$$

giving  $\alpha_{21}^{21} = -\alpha_{12}^{12}$ . Hence, we get a new

$$\bar{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 & 0 & 0 \\ -\alpha_{12}^{12} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{32}^{22} & 0 & 0 & 0 \\ \alpha_{42}^{22} & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, we have new



$$\bar{\delta}(e_{31}) = \begin{pmatrix} -\alpha_{13}^{11} & 0 & 0 & 0 \\ -\alpha_{23}^{22} & 0 & 0 & 0 \\ -\alpha_{13}^{13} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{43}^{33} & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\delta}(e_{41}) = \begin{pmatrix} -\alpha_{14}^{11} & 0 & 0 & 0 \\ -\alpha_{24}^{22} & 0 & 0 & 0 \\ -\alpha_{34}^{33} & 0 & 0 & 0 \\ -\alpha_{14}^{14} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \end{pmatrix},$$

$$\bar{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{21}^{11} & -\alpha_{23}^{23} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix},$$

$$\bar{\delta}(e_{42}) = \begin{pmatrix} 0 & -\alpha_{14}^{11} & 0 & 0 \\ 0 & -\alpha_{24}^{22} & 0 & 0 \\ 0 & -\alpha_{34}^{33} & 0 & 0 \\ -\alpha_{21}^{11} & -\alpha_{24}^{24} & \alpha_{23}^{22} & \alpha_{24}^{22} \end{pmatrix},$$

and

$$\bar{\delta}(e_{43}) = \begin{pmatrix} 0 & 0 & -\alpha_{14}^{11} & 0 \\ 0 & 0 & -\alpha_{24}^{22} & 0 \\ 0 & 0 & -\alpha_{34}^{33} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & -\alpha_{34}^{34} & \alpha_{34}^{33} \end{pmatrix}.$$



**Step 5.** Calculate (15) for  $(x, y) \in \{(e_{12}, e_{31}), (e_{12}, e_{41}), (e_{13}, e_{41})\}$ . For example, if

$(x, y) = (e_{12}, e_{31})$ , then (15) gives

$$\begin{aligned} & \bar{\delta}(e_{32}) - e_{12}\bar{\delta}(e_{31}) - \bar{\delta}(e_{31})e_{12} - e_{31}\bar{\delta}(e_{12}) - \bar{\delta}(e_{12})e_{31} \\ &= \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{21}^{11} & -\alpha_{23}^{23} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix} - \begin{pmatrix} -\alpha_{23}^{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ 0 & -\alpha_{13}^{13} & 0 & 0 \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix} \\ & - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_{23}^{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in F. \end{aligned}$$

Thus, we have  $\alpha_{23}^{23} = -\alpha_{12}^{12} + \alpha_{13}^{13}$ . Similarly,  $\alpha_{24}^{24} = -\alpha_{12}^{12} + \alpha_{14}^{14}$  and  $\alpha_{34}^{34} = -\alpha_{13}^{13} + \alpha_{14}^{14}$ .



**Step 6.** As a result, we get

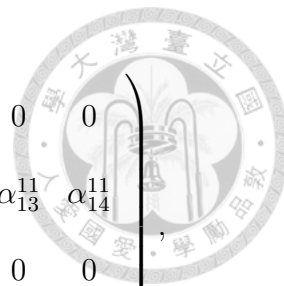
$$\bar{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{21}^{11} & 0 & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 & 0 \\ \alpha_{41}^{11} & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\delta}(e_{22}) = \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 & 0 \\ -\alpha_{21}^{11} & 0 & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{32}^{22} & 0 & 0 \\ 0 & \alpha_{42}^{22} & 0 & 0 \end{pmatrix},$$


$$\bar{\delta}(e_{33}) = \begin{pmatrix} 0 & 0 & -\alpha_{13}^{11} & 0 \\ 0 & 0 & -\alpha_{23}^{22} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & 0 & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{43}^{33} & 0 \end{pmatrix},$$

$$\bar{\delta}(e_{44}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{14}^{11} \\ 0 & 0 & 0 & -\alpha_{24}^{22} \\ 0 & 0 & 0 & -\alpha_{34}^{33} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & 0 \end{pmatrix},$$





$$\begin{aligned}
 \bar{\delta}(e_{12}) &= \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{21}^{11} & 0 & 0 \\ 0 & \alpha_{31}^{11} & 0 & 0 \\ 0 & \alpha_{41}^{11} & 0 & 0 \end{pmatrix}, \quad \bar{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 & 0 & 0 \\ -\alpha_{12}^{12} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{32}^{22} & 0 & 0 & 0 \\ \alpha_{42}^{22} & 0 & 0 & 0 \end{pmatrix}, \\
 \bar{\delta}(e_{13}) &= \begin{pmatrix} -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{13}^{13} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{21}^{11} & 0 \\ 0 & 0 & \alpha_{31}^{11} & 0 \\ 0 & 0 & \alpha_{41}^{11} & 0 \end{pmatrix}, \quad \bar{\delta}(e_{31}) = \begin{pmatrix} -\alpha_{13}^{11} & 0 & 0 & 0 \\ -\alpha_{23}^{22} & 0 & 0 & 0 \\ -\alpha_{13}^{13} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{43}^{33} & 0 & 0 & 0 \end{pmatrix}, \\
 \bar{\delta}(e_{14}) &= \begin{pmatrix} -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & \alpha_{14}^{14} \\ 0 & 0 & 0 & \alpha_{21}^{11} \\ 0 & 0 & 0 & \alpha_{31}^{11} \\ 0 & 0 & 0 & \alpha_{41}^{11} \end{pmatrix}, \quad \bar{\delta}(e_{41}) = \begin{pmatrix} -\alpha_{14}^{11} & 0 & 0 & 0 \\ -\alpha_{24}^{22} & 0 & 0 & 0 \\ -\alpha_{34}^{33} & 0 & 0 & 0 \\ -\alpha_{14}^{14} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \end{pmatrix}, \\
 \bar{\delta}(e_{23}) &= \begin{pmatrix} 0 & 0 & -\alpha_{12}^{11} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & -\alpha_{12}^{12} + \alpha_{13}^{13} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{32}^{22} & 0 \\ 0 & 0 & \alpha_{42}^{22} & 0 \end{pmatrix}, \quad \bar{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} - \alpha_{13}^{13} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix},
 \end{aligned}$$



$$\bar{\delta}(e_{24}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{12}^{11} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & -\alpha_{12}^{12} + \alpha_{14}^{14} \\ 0 & 0 & 0 & \alpha_{32}^{22} \\ 0 & 0 & 0 & \alpha_{42}^{22} \end{pmatrix}, \quad \bar{\delta}(e_{42}) = \begin{pmatrix} 0 & -\alpha_{14}^{11} & 0 & 0 \\ 0 & -\alpha_{24}^{22} & 0 & 0 \\ 0 & -\alpha_{34}^{33} & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} - \alpha_{14}^{14} & \alpha_{23}^{22} & \alpha_{24}^{22} \end{pmatrix},$$

$$\bar{\delta}(e_{34}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{13}^{11} \\ 0 & 0 & 0 & -\alpha_{23}^{22} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & -\alpha_{13}^{13} + \alpha_{14}^{14} \\ 0 & 0 & 0 & \alpha_{43}^{33} \end{pmatrix}, \quad \bar{\delta}(e_{43}) = \begin{pmatrix} 0 & 0 & -\alpha_{14}^{11} & 0 \\ 0 & 0 & -\alpha_{24}^{22} & 0 \\ 0 & 0 & -\alpha_{34}^{33} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{13}^{13} - \alpha_{14}^{14} & \alpha_{34}^{33} \end{pmatrix}.$$

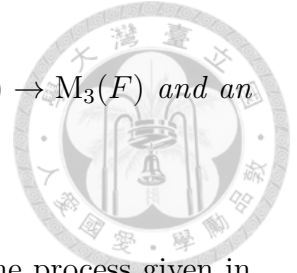
Now, let

$$a = \begin{pmatrix} 0 & -\alpha_{12}^{11} & -\alpha_{13}^{11} & -\alpha_{14}^{11} \\ \alpha_{21}^{11} & -\alpha_{12}^{12} & -\alpha_{23}^{22} & -\alpha_{24}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & -\alpha_{13}^{13} & -\alpha_{34}^{33} \\ \alpha_{41}^{11} & \alpha_{42}^{22} & \alpha_{43}^{33} & -\alpha_{14}^{14} \end{pmatrix} \in M_4(F).$$

Therefore, by a direct computation, we have  $\bar{\delta}(x) = [a, x]$ , for all  $x \in M_4(F)$ , as desired.  $\square$

**Lemma 4.13.** *Let  $F$  be a field and  $\bar{\delta}: M_3(F) \rightarrow M_3(F)$  an  $F$ -linear weak Jordan derivation such that  $\bar{\delta}(F) = 0$ .*

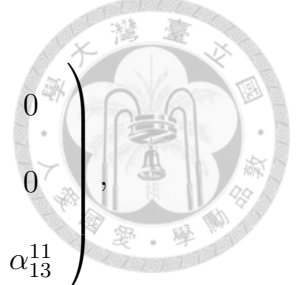
(i) *If  $\text{char } F \neq 2$ , then  $\bar{\delta}$  is an inner derivation.*



(ii) If  $\text{char } F = 2$ , then there exist an inner derivation  $\bar{d}: M_3(F) \rightarrow M_3(F)$  and an  $F$ -linear map  $\bar{v}: M_3(F) \rightarrow F$  such that  $\bar{\delta} = \bar{d} + \bar{v}$ .

*Proof.* For (i), assume that  $\text{char } F \neq 2$ . Then, by following the same process given in the proof of Lemma 4.12, we have

$$\begin{aligned} \bar{\delta}(e_{11}) &= \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{21}^{11} & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 \end{pmatrix}, \\ \bar{\delta}(e_{22}) &= \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 \\ -\alpha_{21}^{11} & 0 & \alpha_{23}^{22} \\ 0 & \alpha_{32}^{22} & 0 \end{pmatrix}, \\ \bar{\delta}(e_{33}) &= \begin{pmatrix} 0 & 0 & -\alpha_{13}^{11} \\ 0 & 0 & -\alpha_{23}^{22} \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & 0 \end{pmatrix}, \\ \bar{\delta}(e_{12}) &= \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} \\ 0 & \alpha_{21}^{11} & 0 \\ 0 & \alpha_{31}^{11} & 0 \end{pmatrix}, \quad \bar{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 & 0 \\ -\alpha_{12}^{12} & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{32}^{22} & 0 & 0 \end{pmatrix}, \end{aligned}$$



$$\bar{\delta}(e_{13}) = \begin{pmatrix} -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{13}^{13} \\ 0 & 0 & \alpha_{21}^{11} \\ 0 & 0 & \alpha_{31}^{11} \end{pmatrix}, \quad \bar{\delta}(e_{31}) = \begin{pmatrix} -\alpha_{13}^{11} & 0 & 0 \\ -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{13}^{13} & \alpha_{12}^{11} & \alpha_{13}^{11} \end{pmatrix},$$

$$\bar{\delta}(e_{23}) = \begin{pmatrix} 0 & 0 & -\alpha_{12}^{11} \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & -\alpha_{12}^{12} + \alpha_{13}^{13} \\ 0 & 0 & \alpha_{32}^{22} \end{pmatrix}, \quad \bar{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 \\ 0 & -\alpha_{23}^{22} & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} - \alpha_{13}^{13} & \alpha_{23}^{22} \end{pmatrix}.$$

Let

$$a = \begin{pmatrix} 0 & -\alpha_{12}^{11} & -\alpha_{13}^{11} \\ \alpha_{21}^{11} & -\alpha_{12}^{12} & -\alpha_{23}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & -\alpha_{13}^{13} \end{pmatrix} \in M_3(F).$$

Then  $\bar{\delta}(x) = [a, x]$ , for all  $x \in M_3(F)$ , as desired.

For (ii), assume that  $\text{char } F = 2$ . Then, by following the same process given in the proof of Lemma 4.12, we have

$$\bar{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{21}^{11} & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 \end{pmatrix} + \alpha_{11}^{11} I_3,$$



$$\bar{\delta}(e_{22}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & 0 \\ \alpha_{21}^{11} & 0 & \alpha_{23}^{22} \\ 0 & \alpha_{32}^{22} & 0 \end{pmatrix} + \alpha_{11}^{22} I_3,$$

$$\bar{\delta}(e_{33}) = \begin{pmatrix} 0 & 0 & \alpha_{13}^{11} \\ 0 & 0 & \alpha_{23}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & 0 \end{pmatrix} + (\alpha_{11}^{11} + \alpha_{11}^{22}) I_3,$$

$$\bar{\delta}(e_{12}) = \begin{pmatrix} \alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} \\ 0 & \alpha_{21}^{11} & 0 \\ 0 & \alpha_{31}^{11} & 0 \end{pmatrix} + (\alpha_{11}^{12} + \alpha_{21}^{11}) I_3,$$

$$\bar{\delta}(e_{21}) = \begin{pmatrix} \alpha_{12}^{11} & 0 & 0 \\ \alpha_{12}^{12} & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{32}^{22} & 0 & 0 \end{pmatrix} + (\alpha_{11}^{21} + \alpha_{12}^{11}) I_3,$$



$$\bar{\delta}(e_{13}) = \begin{pmatrix} \alpha_{31}^{11} & \alpha_{32}^{22} & \alpha_{13}^{13} \\ 0 & 0 & \alpha_{21}^{11} \\ 0 & 0 & \alpha_{31}^{11} \end{pmatrix} + (\alpha_{11}^{13} + \alpha_{31}^{11})I_3,$$

$$\bar{\delta}(e_{31}) = \begin{pmatrix} \alpha_{13}^{11} & 0 & 0 \\ \alpha_{23}^{22} & 0 & 0 \\ \alpha_{13}^{13} & \alpha_{12}^{11} & \alpha_{13}^{11} \end{pmatrix} + (\alpha_{11}^{31} + \alpha_{13}^{11})I_3.$$

$$\bar{\delta}(e_{23}) = \begin{pmatrix} 0 & 0 & \alpha_{12}^{11} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & \alpha_{12}^{12} + \alpha_{13}^{13} \\ 0 & 0 & \alpha_{32}^{22} \end{pmatrix} + (\alpha_{22}^{23} + \alpha_{32}^{22})I_3,$$

$$\bar{\delta}(e_{32}) = \begin{pmatrix} 0 & \alpha_{13}^{11} & 0 \\ 0 & \alpha_{23}^{22} & 0 \\ \alpha_{21}^{11} & \alpha_{12}^{12} + \alpha_{13}^{13} & \alpha_{23}^{22} \end{pmatrix} + (\alpha_{22}^{32} + \alpha_{23}^{22})I_3.$$



Let

$$a = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & \alpha_{13}^{13} \end{pmatrix} \in M_3(F).$$

Then  $\bar{\delta} = \bar{d} + \bar{\nu}$ , where  $\bar{d}(x) = [a, x]$  and

$$\begin{aligned} \bar{\nu}(x) = & (\alpha_{11}^{11}x_{11} + \alpha_{11}^{22}x_{22} + (\alpha_{11}^{11} + \alpha_{11}^{22})x_{33} + (\alpha_{11}^{12} + \alpha_{21}^{11})x_{12} + (\alpha_{11}^{21} + \alpha_{12}^{11})x_{21} \\ & + (\alpha_{11}^{13} + \alpha_{31}^{11})x_{13} + (\alpha_{11}^{31} + \alpha_{13}^{11})x_{31} + (\alpha_{22}^{23} + \alpha_{32}^{22})x_{23} + (\alpha_{22}^{32} + \alpha_{23}^{22})x_{32})I_3, \end{aligned}$$

for all  $x = (x_{ij}) \in M_3(F)$ , as asserted. □

**Lemma 4.14.** *Let  $F$  be a field and  $\bar{\delta}: M_2(F) \rightarrow M_2(F)$  be an  $F$ -linear weak Jordan derivation such that  $\bar{\delta}(F) = 0$ . Then  $\bar{\delta} = \bar{d} + L + \zeta$  for some inner derivation  $\bar{d}: M_2(F) \rightarrow M_2(F)$  and  $F$ -linear maps  $L, \zeta: M_2(F) \rightarrow M_2(F)$ . The  $F$ -linear maps  $L$  and  $\zeta$  are of the forms*

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = (\beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



for all  $x = (x_{ij}) \in M_2(F)$ , where

$$\begin{aligned}\beta_1 &= \alpha_{11}^{11}, \quad \beta_2 = \alpha_{11}^{12} + \alpha_{21}^{11}, \quad \beta_3 = \alpha_{11}^{21} + \alpha_{12}^{11}, \\ \beta_4 &= \alpha_{12}^{21}, \quad \beta_5 = \alpha_{21}^{12}, \quad \beta_6 = \alpha_{12}^{12} + \alpha_{21}^{21}.\end{aligned}$$

Moreover,  $\bar{\delta}$  is a derivation if and only if  $\beta_i = 0$  for all  $i = 1, \dots, 6$ .

*Proof.* We follow the same process step by step as given in the proof of Lemma 4.12.

**Step 1.** Calculate (14) for  $x = e_{ii}$ ,  $i = 1, 2$ . For  $x = e_{11}$ ,

$$\bar{\delta}(e_{11}^2) - e_{11}\bar{\delta}(e_{11}) - \bar{\delta}(e_{11})e_{11} = \begin{pmatrix} -\alpha_{11}^{11} & 0 \\ 0 & \alpha_{22}^{11} \end{pmatrix} \in F$$

implies  $-\alpha_{11}^{11} = \alpha_{22}^{11}$ . Similarly, for  $x = e_{22}$ , we have  $-\alpha_{11}^{22} = \alpha_{22}^{22}$ . Thus

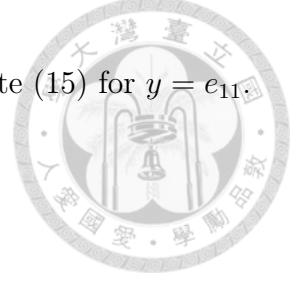
$$\bar{\delta}(e_{11}) = \begin{pmatrix} \alpha_{11}^{11} & \alpha_{12}^{11} \\ \alpha_{21}^{11} & -\alpha_{11}^{11} \end{pmatrix}, \quad \bar{\delta}(e_{22}) = \begin{pmatrix} \alpha_{11}^{22} & \alpha_{12}^{22} \\ \alpha_{21}^{22} & -\alpha_{11}^{22} \end{pmatrix}.$$

**Step 2.** Since  $\bar{\delta}(I_2) = 0$ , we have  $\alpha_{11}^{11} + \alpha_{11}^{22} = 0$ ,  $\alpha_{12}^{11} + \alpha_{12}^{22} = 0$ , and  $\alpha_{21}^{11} + \alpha_{21}^{22} = 0$ .

From these, we have

$$\bar{\delta}(e_{22}) = \begin{pmatrix} -\alpha_{11}^{11} & -\alpha_{12}^{11} \\ -\alpha_{21}^{11} & \alpha_{11}^{11} \end{pmatrix}.$$





**Step 3.** For each  $x = e_{ij}$ ,  $i \neq j$ , calculate (14), and then calculate (15) for  $y = e_{11}$ .

For  $x = e_{12}$ ,

$$e_{12}\bar{\delta}(e_{12}) + \bar{\delta}(e_{12})e_{12} = \begin{pmatrix} \alpha_{21}^{12} & \alpha_{22}^{12} + \alpha_{11}^{12} \\ 0 & \alpha_{21}^{12} \end{pmatrix} \in F$$

implies  $\alpha_{22}^{12} = -\alpha_{11}^{12}$ . Thus

$$\bar{\delta}(e_{12}) = \begin{pmatrix} \alpha_{11}^{12} & \alpha_{12}^{12} \\ \alpha_{21}^{12} & -\alpha_{11}^{12} \end{pmatrix}.$$

Similarly, for  $x = e_{21}$ , we have  $\alpha_{22}^{21} = -\alpha_{11}^{21}$  and so

$$\bar{\delta}(e_{21}) = \begin{pmatrix} \alpha_{11}^{21} & \alpha_{12}^{21} \\ \alpha_{21}^{21} & -\alpha_{11}^{21} \end{pmatrix}.$$

Note that

$$\bar{\delta}(e_{12}) - e_{12}\bar{\delta}(e_{11}) - \bar{\delta}(e_{11})e_{12} - e_{11}\bar{\delta}(e_{12}) - \bar{\delta}(e_{12})e_{11} = -(\alpha_{11}^{12} + \alpha_{21}^{11})I_2$$

and

$$\bar{\delta}(e_{21}) - e_{21}\bar{\delta}(e_{11}) - \bar{\delta}(e_{11})e_{21} - e_{11}\bar{\delta}(e_{21}) - \bar{\delta}(e_{21})e_{11} = -(\alpha_{11}^{21} + \alpha_{12}^{11})I_2.$$



**Step 4.** Calculate (15) for  $x = e_{12}$  and  $y = e_{21}$ . We have

$$e_{12}\bar{\delta}(e_{21}) + \bar{\delta}(e_{21})e_{12} + e_{21}\bar{\delta}(e_{12}) + \bar{\delta}(e_{12})e_{21} = (\alpha_{12}^{12} + \alpha_{21}^{21})I_2.$$

**Step 5.** In  $M_2(F)$  case, this step does not exist.

**Step 6.** As a result, we get

$$\bar{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} \\ \alpha_{21}^{11} & 0 \end{pmatrix} + \alpha_{11}^{11} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\bar{\delta}(e_{22}) = \begin{pmatrix} 0 & -\alpha_{12}^{11} \\ -\alpha_{21}^{11} & 0 \end{pmatrix} + (-\alpha_{11}^{11}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\bar{\delta}(e_{12}) = \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} \\ 0 & \alpha_{21}^{11} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_{21}^{12} & 0 \end{pmatrix} + (\alpha_{11}^{12} + \alpha_{21}^{11}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\bar{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 \\ -\alpha_{12}^{12} & \alpha_{12}^{11} \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{12}^{21} \\ \alpha_{12}^{12} + \alpha_{21}^{21} & 0 \end{pmatrix} + (\alpha_{11}^{21} + \alpha_{12}^{11}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$



Let

$$a = \begin{pmatrix} 0 & -\alpha_{12}^{11} \\ \alpha_{21}^{11} & -\alpha_{12}^{12} \end{pmatrix} \in M_2(F)$$

and  $\bar{d}(x) = [a, x]$  for  $x \in M_2(F)$ . Then  $\bar{\delta} = \bar{d} + L + \zeta$ , where  $L$  and  $\zeta$  are of the forms as in the description of this Lemma.

Finally we show the last statement. If  $\beta_i = 0$  for all  $i = 1, \dots, 6$ , then  $\bar{\delta} = \bar{d}$  is a derivation. Conversely, assume that  $\bar{\delta}$  is a derivation. In particular, it is a Jordan derivation. Note that  $\bar{\delta}$  is a Jordan derivation if and only if

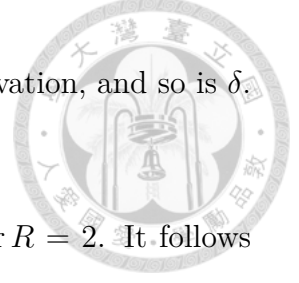
$$\bar{\delta}(x^2) - x\bar{\delta}(x) - \bar{\delta}(x)x = 0$$

and

$$\bar{\delta}(xy + yx) - x\bar{\delta}(y) - y\bar{\delta}(x) - \bar{\delta}(x)y - \bar{\delta}(y)x = 0$$

for all  $x, y \in \{e_{ij}\}_{i,j=1,2}$ . From the above computations, we see that  $\beta_i = 0$  for all  $i = 1, \dots, 6$ . Hence the last statement holds.  $\square$

*Proof of Theorem 4.5.* Recall that, by Proposition 4.10, we can assume that  $\dim_C RC \leq 16$  if  $\text{char } R \neq 2$ , and  $\dim_C RC \leq 9$  if  $\text{char } R = 2$ . Also, it follows from Lemma 4.11 that we can assume that  $\delta$  is  $Z(R)$ -linear and  $\delta(Z(R)) = 0$ , and let  $\bar{\delta}: M_n(F) \rightarrow M_n(F)$  be an extension of  $\delta$  as above, where  $n = \sqrt{\dim_C RC} > 2$ . For (i), suppose that  $\dim_C RC \leq 16$



and  $\text{char } R \neq 2$ . Then, by Lemma 4.12 and Lemma 4.13,  $\bar{\delta}$  is a derivation, and so is  $\delta$ .

Hence (i) holds.

Now we turn to prove (ii). Assume that  $\dim_C RC = 9$  and  $\text{char } R = 2$ . It follows from Lemma 4.13 that there exist an  $a \in RC \otimes_C F$  and an  $F$ -linear map  $\bar{\nu}: RC \otimes_C F \rightarrow F$  such that  $\bar{\delta}(x) = [a, x] + \bar{\nu}(x)$  for all  $x \in RC \otimes_C F$ . If  $F = C$ , then  $a \in RC$  and  $\bar{\nu}$  maps  $R$  into  $C$ , as asserted. Suppose that  $\dim_C F = 3$ . Let  $\{1, w_2, w_3\}$  be a basis of  $F$  over  $C$ . Then

$$a = a_1 \otimes 1 + a_2 \otimes w_2 + a_3 \otimes w_3,$$

for some  $a_1, a_2, a_3 \in RC$ . Thus,


$$[a, x] = [a_1, x] \otimes 1 + [a_2, x] \otimes w_2 + [a_3, x] \otimes w_3$$

for all  $x \in R$ . Also, we write

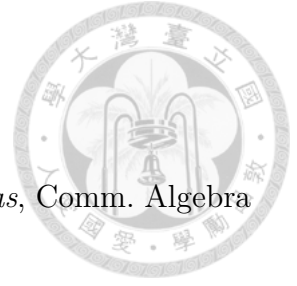
$$\bar{\nu}(x) = \nu_1(x) \otimes 1 + \nu_2(x) \otimes w_2 + \nu_3(x) \otimes w_3$$

for all  $x \in R$ , where  $\nu_i: RC \rightarrow C$ . Thus, we have

$$\delta(x) \otimes 1 = ([a_1, x] + \nu_1(x)) \otimes 1 + ([a_2, x] + \nu_2(x)) \otimes w_2 + ([a_3, x] + \nu_3(x)) \otimes w_3$$

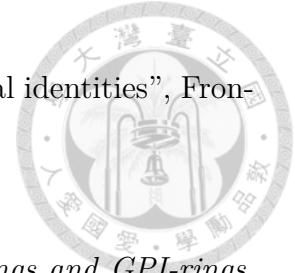
for all  $x \in R$ . This means  $\delta(x) = [a_1, x] + \nu(x)$  for  $x \in R$ , where  $\nu = \nu_1|_R : R \rightarrow C$ , as desired. □

*Proof of Theorem 4.6.* By Lemma 4.11, we can assume that  $\delta$  is  $Z(R)$ -linear and  $\delta(Z(R)) = 0$ , and let  $\bar{\delta}: M_2(F) \rightarrow M_2(F)$  be an extension of  $\delta$  as above. Hence Theorem 4.6 follows directly from Lemma 4.14. □



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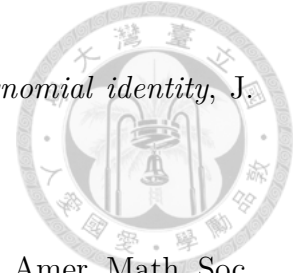


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