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質環上（弱）喬登型式的可加函數

Additive maps of (weak) Jordan types of prime rings

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國立臺灣大學博士學位論文
口試委員會審定書

質環上（弱）喬登型式的可加函數
Additive maps of (weak) Jordan types of prime rings

本論文係林政輝君（D05221002）在國立臺灣大學數學系完成之博士學位論文，於民國 109 年 07 月 17 日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

在本篇論文中，我們將研究兩種質環上（弱）喬登型式的可加函數。在此我們令 R 為一個質環，且 C 為其廣義中心子，而 $Q_{ml}(R)$ 及 $Q_{ms}(R)$ 則分別代表 R 的左邊及雙邊極大商環。

首先我們研究 R 上頭的喬登 τ 導算之結構，其中 τ 為 R 的反自同構。如果一個可加函數 $\delta\colon R \to Q_{ms}(R)$ 滿足對於所有 $x \in R$ 都有 $\delta(x^2) = \delta(x)x^\tau + x\delta(x)$，則我們稱其為喬登 τ 導算。另外，我們稱 $x \mapsto ax^\tau - xa$ 型式的函數為 X-內喬登 τ 導算，其中 a 是 $Q_{ms}(R)$ 中的元素。在此我們證明了，當 τ 為第二型時，喬登 τ 導算的結構可被完全決定，這推廣了李秋坤教授及筆者在 2015 年的結果。定理敘述如下：

- 令 R 為一個非交換的質環且 τ 為其上之反自同構。如果 τ 是第二型，則所有在 R 上頭的喬登 τ 導算皆為 X-內喬登 τ 導算。

當 τ 為第一型時，我們還有得到下面的結果：

- 令 R 為一個質 GPI 環且 $\mathrm{char}R \neq 2$，並假設 τ 為其上第一型的反自同構。如果 $\deg \tau^2 \neq 2$，則所有在 R 上頭的喬登 τ 導算皆為 X-內喬登 τ 導算。

接下來我們研究 R 上頭的弱喬登導算之結構。如果一個可加函數 $\delta\colon R \to Q_{ml}(R)$ 滿足所有 $x \in R$ 都有 $\delta(x^2) - \delta(x)x - x\delta(x) \in C$，則我們稱其為弱喬登導算。在此我們完整給出了弱喬登導算的結構，其中 $\dim_C RC > 4$ 的情況如下：

- 令 R 為一個質環且 $\dim_C RC > 4$，並假設 $\delta\colon R \to Q_{ml}(R)$ 為一個弱喬登導算。

  (i) 如果 $\mathrm{char}\, R \neq 2$，則 δ 是一個導算。

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(ii) 如果 $\operatorname{char} R = 2$，則存在一個導算 $d\colon R \to Q_{ml}(R)$ 和一個可加函數 $\nu\colon R \to C$ 使得 $\delta = d + \nu$。

另外，$\dim_C RC = 4$ 的情況也有決定出弱喬登導算的結構，但由於敘述較複雜，請讀者觀看內文的 Theorem 4.6。作為此結構定理的應用，我們推廣了 Brešar 在 1993 年的定理，其內容為關於可加函數 $\delta\colon R \to RC + C$ 滿足對於所有 $x \in R$ 都有 $[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$ 的結構。

關鍵字：質環；泛函恆等式；GPI 環；喬登 τ 導算；弱喬登導算；左邊（雙邊）極大商環。

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Abstract

In the dissertation, we study two kinds of additive maps of (weak) Jordan types on prime rings. Let R be a prime ring with extended centroid C, and let $Q_{ml}(R)$ (resp. $Q_{ms}(R)$) denote the maximal left (resp. symmetric) ring of quotients of R.

Firstly, we investigate the structure of Jordan τ-derivations of R, where τ is an anti-automorphism of R. An additive map $\delta\colon R \to Q_{ms}(R)$ is called a Jordan τ-derivation if $\delta(x^2) = \delta(x)x^\tau + x\delta(x)$ for all $x \in R$. A Jordan τ-derivation δ of R is called X-inner if there exists $a \in Q_{ms}(R)$ such that $\delta(x) = ax^\tau - xa$ for all $x \in R$. We completely determine Jordan τ-derivations of R when τ is of the second kind, which generalizes Lee and the author's result in 2015 as follows.

- *Let R be a noncommutative prime ring with an anti-automorphism τ. If τ is of the second kind, then any Jordan τ-derivation of R is X-inner.*

We also get the following characterization when τ is of the first kind.

- *Let R be a prime GPI-ring,* $\mathrm{char} R \neq 2$*, and let τ be an anti-automorphism of R, which is of the first kind. If* $\deg \tau^2 \neq 2$*, then any Jordan τ-derivation of R is X-inner.*

Secondly, we study the structure of weak Jordan derivations of R. An additive map $\delta\colon R \to Q_{ml}(R)$ is called a weak Jordan derivation if $\delta(x^2) - \delta(x)x - x\delta(x) \in C$ for all $x \in R$. Here we give a complete characterization of weak Jordan

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derivations of R. Precisely, we prove the following.

- *Let R be a prime ring with $\dim_C RC > 4$, and let $\delta\colon R \to Q_{ml}(R)$ be a weak Jordan derivation.*

  (i) *If* $\operatorname{char} R \neq 2$, *then δ is a derivation.*

  (ii) *If* $\operatorname{char} R = 2$, *then $\delta = d + \nu$, where $d\colon R \to Q_{ml}(R)$ is a derivation and $\nu\colon R \to C$ is an additive map.*

We also give a complete characterization for the case that $\dim_C RC = 4$ (see Theorem 4.6). The characterization can be applied to generalize Brešar's theorem in 1993 concerning additive maps $\delta\colon R \to RC + C$ satisfying $[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$ for all $x \in R$.

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Key words and phrases: Prime ring; functional identity; GPI-ring; Jordan τ-derivation; weak Jordan derivation; maximal left (symmetric) ring of quotients.

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1 Introduction

Throughout, R denotes a prime ring, that is, for $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$, and let $Z(R)$ be the center of R. Let $Q_{ml}(R)$ (resp. $Q_{ms}(R)$) be the maximal left (resp. symmetric) ring of quotients of R, and let $Q_s(R)$ be the Martindale symmetric ring of quotients of R. The center of $Q_{ml}(R)$, denoted by C is called the extended centroid of R. In this case, C is always a field. It is well-known that $Q_s(R) \subseteq Q_{ms}(R) \subseteq Q_{ml}(R)$ and $Z(Q_{ms}(R)) = Z(Q_s(R)) = C$. We refer the reader to [2] for details.

Let τ be an anti-automorphism of R. An additive map $\delta\colon R \to Q_{ms}(R)$ is called a Jordan τ-derivation if $\delta(x^2) = \delta(x)x^\tau + x\delta(x)$ for all $x \in R$. A Jordan τ-derivation δ of R is called X-inner if there exists $a \in Q_{ms}(R)$ such that $\delta(x) = ax^\tau - xa$ for all $x \in R$. In 2015, Lee and the author [20] proved the following: If R is either a non GPI-ring or a PI-ring, then every Jordan τ-derivation of R is X-inner except when both $\mathrm{char} R = 2$ and $\dim_C RC = 4$. Therefore, it keeps unknown when R is a GPI-ring but is not a PI-ring. In order to solve the unknown case, in Chapter 2 we develop some results concerning certain functional identities with an anti-automorphism τ. In Chapter 3, we use these results to give a complete characterization of Jordan τ-derivations of R when τ is of the second kind.

Theorem 3.12. *Let R be a noncommutative prime ring with an anti-automorphism τ.*

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If τ is of the second kind, then any Jordan τ-derivation of R is X-inner.

By an X-inner automorphism σ of R, we mean that there exists $u \in Q_s(R)$ such that $\sigma(x) = uxu^{-1}$ for all $x \in R$. In this case, we define $\deg \sigma = m$ if u is algebraic over C with minimal degree m. We define $\deg \sigma = \infty$ otherwise. By Kharchenko's theorem, τ^2 is an X-inner automorphism when R is a prime GPI-ring and τ is of the first kind. We get the following result for the first kind case.

Theorem 3.14. *Let R be a prime GPI-ring with* $\mathrm{char} R \neq 2$ *and with τ an anti-automorphism of the first kind. If* $\deg \tau^2 \neq 2$*, then any Jordan τ-derivation of R is X-inner.*

In 1993, Brešar [5] proved that an additive map $\delta\colon R \to RC + C$ satisfying $\delta(x^2) - x\delta(x) - \delta(x)x \in C$ for all $x \in R$ is a derivation if $\mathrm{char}\,(R) \neq 2, 3$ and $\dim_C RC > 4$. Several years later, Brešar et al. characterized weak Lie derivations (i.e., additive maps $\delta\colon R \to Q_{ml}(R)$ satisfy $\delta([x, y]) - [\delta(x), y] - [x, \delta(y)] \in C$ for all $x, y \in R$) when $\dim_C RC \geq 16$ (see the book [8]). We use their fashion to define a "weak Jordan derivation" to be an additive map $\delta\colon R \to Q_{ml}(R)$ satisfying $\delta(x^2) - \delta(x)x - x\delta(x) \in C$ for all $x \in R$. In Chapter 4, we completely determine its structure as follows.

Theorem 4.5. *Let R be a prime ring with* $\dim_C RC > 4$*, and let $\delta\colon R \to Q_{ml}(R)$ be a weak Jordan derivation.*

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(i) *If* $\operatorname{char} R \neq 2$, *then* δ *is a derivation.*

(ii) *If* $\operatorname{char} R = 2$, *then* $\delta = d + \nu$, *where* $d\colon R \to Q_{ml}(R)$ *is a derivation and* $\nu\colon R \to C$ *is an additive map.*

Theorem 4.6. *Let* R *be a prime ring with* $\dim_C RC = 4$, *and let* $\delta\colon R \to RC$ *be a weak Jordan derivation.*

(i) *If* $\operatorname{char} R \neq 2$, *then there exists a field extension* F *of* C *such that* RC *can be embedded into* $\mathrm{M}_2(F)$ *and*

$$\delta(x) = d(x) + [a, x] + L(x) + \zeta(x)$$

for all $x \in R$, *where* $d\colon R \to RC$ *is a derivation,* $a \in \mathrm{M}_2(F)$ *and* $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ *are* F*-linear maps. Moreover, there exist* $\beta_i \in F$, $1 \leq i \leq 6$, *such that*

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \Big(\beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21}\Big) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for all $x = (x_{ij}) \in \mathrm{M}_2(F)$. *In this case,* δ *is a derivation if and only if all* $\beta_i = 0$.

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(ii) *If* $\operatorname{char} R = 2$*, then there exists a field extension* F *of* C *such that* RC *can be embedded into* $\mathrm{M}_2(F)$ *and*

$$\delta(x) = d(x) + \nu(x) + [a, x] + L(x) + \zeta(x)$$

for all $x \in R$*, where* $d\colon R \to RC$ *is a derivation,* $\nu\colon R \to C$ *is an additive map,* $a \in \mathrm{M}_2(F)$*, and* $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ *are* F*-linear maps. Moreover, there exist* $\beta_i \in F$, $1 \le i \le 6$*, such that*

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \Big(\beta_1(x_{11} + x_{22}) + \beta_2 x_{12} + \beta_3 x_{21}\Big) I_2$$

for all $x = (x_{ij}) \in \mathrm{M}_2(F)$*.*

Conversely, an additive map $\delta\colon R \to Q_{ml}(R)$ *satisfying* (i) *or* (ii) *is a weak Jordan derivation.*

Moreover, we use the above two theorems to generalize Brešar's theorem [5, Theorem 4].

Theorem 4.9. *Let* R *be a prime ring with* $\operatorname{char} R \neq 2$*, and let* $\delta\colon R \to RC + C$ *be an*

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additive map satisfying

$$[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$$

for all $x \in R$.

(i) *If* $\dim_C RC > 4$, *then*

$$\delta(x) = \gamma x + d(x) + \mu(x)$$

for all $x \in R$, *where* $\gamma \in C$, $d\colon R \to RC + C$ *is a derivation, and* $\mu\colon R \to C$ *is an additive map.*

(ii) *If* $\dim_C RC = 4$, *then*

$$\delta(x) = d(x) + \mu(x) + [a, x] + L(x) + \zeta(x)$$

for all $x \in R$, *where* $d\colon R \to RC$ *is a derivation,* $\mu\colon R \to C$ *is an additive map,* $a \in \mathrm{M}_2(F)$, *and* $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ *are as in Theorem 4.6* (i).

Note that all results mentioned above have been published as journal papers (see [22] and [23]). The method of characterizing weak Jordan derivations developed by the author was also applied to studying weak Jordan $*$-derivations by Siddeeque et al. (see [32]).

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2 Preliminary

2.1 GPI-rings, PI-rings, and functional identities

Let R be a prime ring with extended centroid C. We first introduce prime GPI-rings and PI-rings and their structure theorems. To be precise, set $Q := Q_{ml}(R)$. Let X be an infinite set and $C\langle X\rangle$ be the free C-algebra on X. Define $Q_C\langle X\rangle$ to be the free product of Q and $C\langle X\rangle$. Elements of $Q_C\langle X\rangle$ are called generalized polynomials. A generalized polynomial is said to be nontrivial if it is nonzero in $Q_C\langle X\rangle$. Let U be an additive subgroup of R. By a generalized polynomial identity (GPI) on U we mean an element $\phi = \phi(x_1, \ldots, x_n)$ in $Q_C\langle X\rangle$ such that $\phi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U$. In this case, we say that U satisfies a GPI ϕ. We say that R is a GPI-ring if R satisfies a nontrivial GPI. The following is a famous structure theorem due to Martindale (see [24, Theorem 3] or [2, Theorem 6.1.6]).

Theorem 2.1. *Let R be a prime ring with extended centroid C. Then R is a GPI-ring if and only if its central closure RC contains a nonzero idempotent e such that eRC is a minimal right ideal of RC and $eRCe$ is a finite-dimensional division algebra over C.*

We view $C\langle X\rangle$ as a C-subalgebra of $Q_C\langle X\rangle$. We say R is a PI-ring if it satisfies a nontrivial element of $C\langle X\rangle$ whose coefficients are in $\{1, -1\}$. The following describes the structure of prime PI-rings (see [25], [26], and [9]).

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Theorem 2.2. *Let R be a prime PI-ring with extended centroid C. Then*

(a) *RC is a finite-dimensional central simple algebra over C.*

(b) *$Z(R)$ intersects every nonzero ideal of R nontrivially.*

(c) *C is the quotient field of $Z(R)$.*

Next we introduce some useful results concerning functional identities of prime rings. These play a key role in characterizing weak Jordan derivations and Jordan τ-derivations.

Before stating them, we fix some notations. Let m be a positive integer, $I, J \subseteq \{1, 2, \ldots, m\}$, and a, b be non-negative integers. Let $E_{iu}, F_{jv} \colon R^{m-1} \to Q_{ml}(R)$, $i \in I$, $j \in J$, $0 \leq u \leq a$, $0 \leq v \leq b$. Fix $t \in Q_{ml}(R)$ and $V = \sum_{i=0}^{\infty} Ct^i$. We say that $\deg t = n$ if t is algebraic of minimal degree n over C. Moreover, $\deg t = \infty$ if t is not algebraic over C. For any maps $f \colon R^{r-1} \to Q_{ml}(R)$ and $g \colon R^{r-2} \to Q_{ml}(R)$ we write

$$f^i(\bar{x}_r) = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r)$$

and

$$g^{ij}(\bar{x}_r) = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r),$$

where $\bar{x}_r = (x_1, \ldots, x_r) \in R^r$. We need the following important theorems due to Beidar and Martindale [3].

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Theorem 2.3. ([3, Theorem 2.4]) *Suppose that* $\deg t > a + |I|$ *and*

$$\sum_{i \in I} \sum_{u=0}^{a} E_{iu}^{i} x_i t^u \in V$$

for all $x_1, x_2, \ldots, x_m \in R$. *Then each* $E_{iu} = 0$.

Theorem 2.4. ([3, Theorem 2.5]) *Suppose that* $\deg t > a + |I| - 1$ *and*

$$\sum_{i \in I} \sum_{u=0}^{a} E_{iu}^{i} x_i t^u = 0$$

for all $x_1, x_2, \ldots, x_m \in R$. *Then each* $E_{iu} = 0$.

Theorem 2.5. ([3, Corollary 2.11]) *Let* $E_i, F_j \colon R^{m-1} \to Q_{ml}(R)$, $i \in I$, $j \in J$ *such that*

$$\sum_{i \in I} E_i^i(x_1, \ldots, x_m) x_i + \sum_{j \in J} x_j F_j^j(x_1, \ldots, x_m) \in C$$

for all $x_1, x_2, \ldots, x_m \in R$. *Suppose that* $\dim_C RC > \big(\max\{|I|, |J|\}\big)^2$. *Then there exist unique maps* $p_{ij} \colon R^{m-2} \to Q_{ml}(R)$ *and* $\lambda_k \colon R^{m-1} \to C$ *such that*

$$E_i^i = \sum_{\substack{j \in J \\ j \neq i}} x_j p_{ij}^{ij} + \lambda_i^i,$$

$$F_j^j = -\sum_{\substack{i \in I \\ i \neq j}} p_{ij}^{ij} x_i - \lambda_j^j,$$

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where $\lambda_k = 0$ *if* $k \notin I \cap J$. *If* E_i*'s and* F_j*'s are* $(m-1)$*-additive, then all* p_{ij}*'s are* $(m-2)$*-additive and all* λ_k*'s are* $(m-1)$*-additive. (It is understood that all the* p_{ij}*'s are equal to* 0 *if* $m = 1$*.)*

2.2 Functional identities with an anti-automorphism

Let R be a prime ring. It is well-known that an automorphism (resp. anti-automorphism) of R can be uniquely extended to an automorphism (resp. anti-automorphism) of $Q_s(R)$ (see [2, Proposition 2.5.3] for the automorphism case and [2, Proposition 2.5.4] for the anti-automorphism case). An automorphism (or anti-automorphism) g is said to be of the first kind if $\beta^g = \beta$ for all $\beta \in C$. Otherwise, g is said to be of the second kind.

To study Jordan τ-derivations in the next chapter, we have to develop some results concerning functional identities with an anti-automorphism. In [21], Lee dealt with functional identities on prime rings with an automorphism. Now, we will follow his viewpoint to get useful results. Our purpose in the section is to prove the following theorem.

Theorem 2.6. *Let* R *be a prime ring with an anti-automorphism* τ *of the second kind. Suppose that* $E_{it}, F_{\ell 1}\colon R^{r-1} \to Q_{ml}(R)$ *are* $(r-1)$*-additive maps such that*

$$\sum_{i=1}^{r} E_{i1}^{i}(\bar{x}_r)x_i + \sum_{i=1}^{r} E_{i2}^{i}(\bar{x}_r)x_i^{\tau} + \sum_{\ell=1}^{r} x_\ell F_{\ell 1}^{\ell}(\bar{x}_r) \in C \qquad (1)$$

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for $\bar{x}_r \in R^r$, where $1 \le i, \ell \le r$ and $t = 1, 2$. If R is not a PI-ring, then there exist a nonzero ideal I of R, $(r-2)$-additive maps $p_{it\ell 1}\colon I^{r-2} \to Q_{ml}(R)$, and $(r-1)$-additive maps $\lambda_{i1}\colon I^{r-1} \to C$ such that

$$E^i_{i1}(\bar{x}_r) = \sum_{\substack{1\le \ell\le r\\ \ell\ne i}} x_\ell p^{i\ell}_{i1\ell 1}(\bar{x}_r) + \lambda^i_{i1}(\bar{x}_r)$$

$$E^i_{i2}(\bar{x}_r) = \sum_{\substack{1\le \ell\le r\\ \ell\ne i}} x_\ell p^{i\ell}_{i2\ell 1}(\bar{x}_r)$$

and

$$F^\ell_{\ell 1}(\bar{x}_r) = -\sum_{\substack{1\le i\le r\\ i\ne \ell}} p^{i\ell}_{i1\ell 1}(\bar{x}_r)x_i - \sum_{\substack{1\le i\le r\\ i\ne \ell}} p^{i\ell}_{i2\ell 1}(\bar{x}_r)x_i^\tau - \lambda^\ell_{\ell 1}(\bar{x}_r)$$

for all $\bar{x}_r \in I^r$, where $1 \le i, \ell \le r$ and $t = 1, 2$.

Corollary 2.7. *Let R be a prime ring with an anti-automorphism τ of the second kind. Suppose that $E_i, F_\ell\colon R^{r-1} \to Q_{ml}(R)$ are $(r-1)$-additive maps such that*

$$\sum_{i=1}^{r} E^i_i(\bar{x}_r)x_i^\tau + \sum_{\ell=1}^{r} x_\ell F^\ell_\ell(\bar{x}_r) \in C$$

for $\bar{x}_r \in R^r$, where $1 \le i, \ell \le r$. If R is not a PI-ring, then there exist a nonzero ideal

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I of R and $(r-2)$-additive maps $p_{i\ell}\colon I^{r-2} \to Q_{ml}(R)$ such that

$$E_i^i(\bar{x}_r) = \sum_{\substack{1\leq \ell\leq r \\ \ell\neq i}} x_\ell p_{i\ell}^{i\ell}(\bar{x}_r)$$

and

$$F_\ell^\ell(\bar{x}_r) = -\sum_{\substack{1\leq i\leq r \\ i\neq \ell}} p_{i\ell}^{i\ell}(\bar{x}_r) x_i^\tau$$

for all $\bar{x}_r \in I^r$, where $1 \leq i, \ell \leq r$.

Proof. By Theorem 2.6, there exist a nonzero ideal I of R, $(r-2)$-additive maps $p_{it\ell 1}\colon I^{r-2} \to Q_{ml}(R)$, and $(r-1)$-additive maps $\lambda_{i1}\colon I^{r-1} \to C$ such that

$$0 = \sum_{\substack{1\leq \ell\leq r \\ \ell\neq i}} x_\ell p_{i1\ell 1}^{i\ell}(\bar{x}_r) + \lambda_{i1}^i(\bar{x}_r),$$

$$E_i^i(\bar{x}_r) = \sum_{\substack{1\leq \ell\leq r \\ \ell\neq i}} x_\ell p_{i2\ell 1}^{i\ell}(\bar{x}_r),$$

and

$$F_\ell^\ell(\bar{x}_r) = -\sum_{\substack{1\leq i\leq r \\ i\neq \ell}} p_{i1\ell 1}^{i\ell}(\bar{x}_r) x_i - \sum_{\substack{1\leq i\leq r \\ i\neq \ell}} p_{i2\ell 1}^{i\ell}(\bar{x}_r) x_i^\tau - \lambda_{\ell 1}^\ell(\bar{x}_r)$$

for all $\bar{x}_r \in I^r$, where $1 \leq i, \ell \leq r$ and $t = 1, 2$. In view of Theorem 2.3, $p_{i1\ell 1} = 0$ and $\lambda_{i1} = 0$ for $1 \leq i, \ell \leq r$. The proof is complete by putting $p_{i\ell} = p_{i2\ell 1}$. □

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To begin with the proof of Theorem 2.6, we first give the following lemma.

Lemma 2.8. *Suppose that $E_i, F_\ell \colon R^{r-1} \to Q_{ml}(R)$ are $(r-1)$-additive maps such that*

$$\sum_{i=1}^{r} E_i^i(\bar{x}_r)x_i + \sum_{\ell=1}^{r} F_\ell^\ell(\bar{x}_r)x_\ell^\tau \in C \tag{2}$$

for $\bar{x}_r \in R^r$, where $1 \le i, \ell \le r$. If R is not a PI*-ring, then there exists a nonzero ideal I of R such that $E_i^i = F_\ell^\ell = 0$ on I^r for $1 \le i, \ell \le r$.*

Before proving it, we define the following notation (see [21]). For a map $f \colon R^{r-1} \to Q_{ml}(R)$ and $t \ne i$, we write

$$f^i(\bar{x}_r; \{y\}_t) = f(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_r)$$

where $z_t = y$ and $z_j = x_j$ for $j \ne t$, i.e., we replace x_t by y in $f^i(\bar{x}_r)$.

Proof of Lemma 2.8. Let $A := \{1, 2, \ldots, r\}$ and

$$L := \{\ell \in A \mid \text{there exists a nonzero ideal } J \text{ of } R \text{ such that } F_\ell^\ell = 0 \text{ on } J^r\}.$$

We proceed the proof by induction on $r - |L|$.

Suppose first that $r - |L| = 0$, i.e., $L = A$. Then there exists a nonzero ideal J such that $F_\ell^\ell = 0$ on J^r for all $\ell = 1, \ldots, r$. Thus $\sum_{i=1}^{r} E_i^i(\bar{x}_r)x_i \in C$ for all $\bar{x}_r \in J^r$. By

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Theorem 2.3, $E_i^i = 0$ on J^r for all $i = 1, \ldots, r$, as asserted.

Suppose next that $r - |L| \geq 1$. Without loss of generality, we may assume that $r \notin L$. Then, for any nonzero ideal U of R, $F_r^r \neq 0$ on U^r. Fix $\beta \in C$ with $\beta^\tau \neq \beta$ and choose a nonzero ideal K of R such that $\beta K \subseteq R$. Then, by (2), we have

$$\sum_{i=1}^{r-1} \big(E_i^i(\bar{x}_r; \{\beta x_r\}_r) - \beta E_i^i(\bar{x}_r)\big) x_i \qquad (3)$$
$$+ \sum_{\ell=1}^{r-1} \big(F_\ell^\ell(\bar{x}_r; \{\beta x_r\}_r) - \beta F_\ell^\ell(\bar{x}_r)\big) x_\ell^\tau + (\beta^\tau - \beta) F_r^r(\bar{x}_r) x_r^\tau \in C$$

for all $\bar{x}_r \in K^r$. Let $K_1 = K \cap K^\tau$. Then K_1 is an ideal of R such that $K_1^{\tau^{-1}} \subseteq K$ and, by (3), we have

$$\sum_{i=1}^{r-1} \widetilde{E}_i^i(\bar{x}_r) x_i + F_r^r(\bar{x}_r) x_r + \sum_{\ell=1}^{r-1} \widetilde{F}_\ell^\ell(\bar{x}_r) x_\ell^\tau \in C \qquad (4)$$

for all $\bar{x}_r \in K_1^r$, where

$$\widetilde{E}_i^i(\bar{x}_r) = (\beta^\tau - \beta)^{-1} \big(E_i^i(\bar{x}_r; \{\beta x_r^{\tau^{-1}}\}_r) - \beta E_i^i(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r)\big)$$

and

$$\widetilde{F}_\ell^\ell(\bar{x}_r) = (\beta^\tau - \beta)^{-1} \big(F_\ell^\ell(\bar{x}_r; \{\beta x_r^{\tau^{-1}}\}_r) - \beta F_\ell^\ell(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r)\big).$$

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Set

$$L_1 := \{\ell \mid 1 \leq \ell \leq r-1, \text{ there exists a nonzero ideal } J \text{ of } R \text{ such that } \widetilde{F}_\ell^\ell = 0 \text{ on } J^r\}.$$

Let $\ell \in L$. Then $1 \leq \ell \leq r-1$ and there is a nonzero ideal N of R such that $F_\ell^\ell = 0$ on N^r. By the definition of $\widetilde{F}_\ell^\ell$, there exists a nonzero ideal M of R contained in N such that $\widetilde{F}_\ell^\ell = 0$ on M^r and so $\ell \in L_1$. Thus $|L| \leq |L_1|$ and $r-|L| \geq r-|L_1| > (r-1)-|L_1|$. By applying the induction hypothesis on (4), we have $F_\ell^\ell = 0$ on W^r for some nonzero ideal W of R, a contradiction. □

Proof of Theorem 2.6. Let $A := \{1, 2, \ldots, r\}$ and

$$L := \{i \in A \mid \text{there exists a nonzero ideal } J \text{ of } R \text{ such that } E_{i2}^i = 0 \text{ on } J^r\}.$$

We proceed the proof by induction on $r - |L|$.

Assume first that $r - |L| = 0$, i.e., $L = A$. Then $E_{i2}^i = 0$ on U^r for some nonzero ideal U of R and so (1) becomes

$$\sum_{i=1}^{r} E_{i1}^i(\bar{x}_r)x_i + \sum_{\ell=1}^{r} x_\ell F_{\ell 1}^\ell(\bar{x}_r) \in C$$

for $\bar{x}_r \in U^r$. Hence the result follows from Theorem 2.5.

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Assume next that $r - |L| \geq 1$. Without loss of generality, assume that $r \notin L$. Then $E^r_{r2} \neq 0$ on any nonzero ideal of R. Let $\beta \in C$ with $\beta^\tau \neq \beta$ and choose a nonzero ideal J of R such that $\beta J \subseteq R$. Then, by (1), we have

$$\sum_{i=1}^{r-1} \big(E^i_{i1}(\bar{x}_r; \{\beta x_r\}_r) - \beta E^i_{i1}(\bar{x}_r)\big) x_i + \sum_{i=1}^{r-1} \big(E^i_{i2}(\bar{x}_r; \{\beta x_r\}_r) - \beta E^i_{i2}(\bar{x}_r)\big) x_i^\tau + E^r_{r2}(\bar{x}_r)(\beta^\tau - \beta) x_r^\tau + \sum_{\ell=1}^{r-1} x_\ell \big(F^\ell_{\ell1}(\bar{x}_r; \{\beta x_r\}_r) - \beta F^\ell_{\ell1}(\bar{x}_r)\big) \in C$$

for all $\bar{x}_r \in J^r$. Let

$$\widetilde{E}^i_{i1}(\bar{x}_r) = (\beta^\tau - \beta)^{-1} \big(E^i_{i1}(\bar{x}_r; \{\beta x_r\}_r) - \beta E^i_{i1}(\bar{x}_r)\big),$$

$$\widetilde{E}^i_{i2}(\bar{x}_r) = (\beta^\tau - \beta)^{-1} \big(E^i_{i2}(\bar{x}_r; \{\beta x_r\}_r) - \beta E^i_{i2}(\bar{x}_r)\big),$$

and

$$\widetilde{F}^\ell_{\ell1}(\bar{x}_r) = (\beta^\tau - \beta)^{-1} \big(F^\ell_{\ell1}(\bar{x}_r; \{\beta x_r\}_r) - \beta F^\ell_{\ell1}(\bar{x}_r)\big).$$

Then

$$\sum_{i=1}^{r-1} \widetilde{E}^i_{i1}(\bar{x}_r) x_i + \sum_{i=1}^{r-1} \widetilde{E}^i_{i2}(\bar{x}_r) x_i^\tau + E^r_{r2}(\bar{x}_r) x_r^\tau + \sum_{\ell=1}^{r-1} x_\ell \widetilde{F}^\ell_{\ell1}(\bar{x}_r) \in C \qquad (5)$$

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for all $\bar{x}_r \in J^r$. Choose a nonzero ideal J_1 of R contained in J so that $J_1^{\tau^{-1}} \subseteq J$. By (5), we have

$$\sum_{i=1}^{r-1} \widetilde{E}^i_{i1}(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r)x_i + \sum_{i=1}^{r-1} \widetilde{E}^i_{i2}(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r)x_i^{\tau} \tag{6}$$
$$+E^r_{r2}(\bar{x}_r)x_r + \sum_{\ell=1}^{r-1} x_\ell \widetilde{F}^\ell_{\ell 1}(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r) \in C$$

for all $\bar{x}_r \in J_1^r$. Set $G^i_{i2}(\bar{x}_r) := \widetilde{E}^i_{i2}(\bar{x}_r; \{x_r^{\tau^{-1}}\}_r)$ and

$$L_1 := \{i \mid 1 \leq i \leq r-1, \text{ there exists a nonzero ideal } J \text{ of } R \text{ such that } G^i_{i2} = 0 \text{ on } J^r\}.$$

Let $i \in L$ and $i \neq r$. Then there exists a nonzero ideal N of R such that $E^i_{i2} = 0$ on N^r. From the definition of G^i_{i2}, there is a nonzero ideal M of R contained in N such that $G^i_{i2} = 0$ on M^r, and so $i \in L_1$. Thus

$$r - |L| \geq r - |L_1| > (r-1) - |L_1|.$$

By the induction hypothesis, there exist a nonzero ideal J_2 of R contained in J_1 and $(r-2)$-additive maps $p_{r2\ell 1} \colon J_2^{r-2} \to Q_{ml}(R)$ such that

$$E^r_{r2}(\bar{x}_r) = \sum_{\ell=1}^{r-1} x_\ell p^{r\ell}_{r2\ell 1}(\bar{x}_r)$$

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for all $\bar{x}_r \in J_2^r$. Substituting it into (1), we have

$$\sum_{i=1}^{r} E_{i1}^i(\bar{x}_r)x_i + \sum_{i=1}^{r-1} E_{i2}^i(\bar{x}_r)x_i^\tau + \sum_{\ell=1}^{r-1} x_\ell\Big(F_{\ell 1}^\ell(\bar{x}_r) + p_{r2\ell 1}^{r\ell}(\bar{x}_r)x_r^\tau\Big) + x_r F_{r1}^r(\bar{x}_r) \in C \quad (7)$$

for all $\bar{x}_r \in J_2^r$. By the induction hypothesis, there is a nonzero ideal I of R contained in J_2 and $(r-2)$-additive maps $p_{i2\ell 1}\colon I^{r-2} \to Q_{ml}(R)$ such that

$$E_{i2}^i(\bar{x}_r) = \sum_{\substack{\ell=1 \\ \ell\neq i}}^{r} x_\ell p_{i2\ell 1}^{i\ell}(\bar{x}_r)$$

for all $\bar{x}_r \in I^r$ and $1 \leq i \leq r-1$. Thus (7) becomes

$$\sum_{i=1}^{r} E_{i1}^i(\bar{x}_r)x_i + \sum_{\ell=1}^{r} x_\ell\Big(F_{\ell 1}^\ell(\bar{x}_r) + \sum_{\substack{i=1 \\ i\neq \ell}}^{r} p_{i2\ell 1}^{i\ell}(\bar{x}_r)x_i^\tau\Big) \in C$$

for all $\bar{x}_r \in I^r$. According to Theorem 2.5, there exist $(r-2)$-additive maps $p_{i1\ell 1}\colon I^{r-2} \to Q_{ml}(R)$ and $(r-1)$-additive maps $\lambda_{i1}\colon I^{r-1} \to C$ such that

$$E_{i1}^i(\bar{x}_r) = \sum_{\substack{1\leq \ell\leq r \\ \ell\neq i}} x_\ell p_{i1\ell 1}^{i\ell}(\bar{x}_r) + \lambda_{i1}^i(\bar{x}_r)$$

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and

$$F^{\ell}_{\ell 1}(\bar{x}_r) + \sum_{\substack{i=1 \\ i\neq\ell}}^{r} p^{i\ell}_{i2\ell 1}(\bar{x}_r)x^{\tau}_i = -\sum_{\substack{1\leq i\leq r \\ i\neq\ell}} p^{i\ell}_{i1\ell 1}(\bar{x}_r)x_i - \lambda^{\ell}_{\ell 1}(\bar{x}_r)$$

for all $\bar{x}_r \in I^r$, where $1 \leq i, \ell \leq r$, as asserted. □

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3 Jordan τ-derivations of prime GPI-rings

3.1 Motivation

Let R be a prime ring with an anti-automorphism τ. An additive map $\delta\colon R \to Q_{ms}(R)$ is called a Jordan τ-derivation of R if $\delta(x^2) = x\delta(x) + \delta(x)x^\tau$ for all $x \in R$. A Jordan τ-derivation δ of R is said to be inner (resp. X-inner) if there exists $a \in R$ (resp. $a \in Q_{ms}(R)$) such that $\delta(x) = ax^\tau - xa$ for $x \in R$. Note that if $\delta\colon R \to Q_{ms}(R)$ is a Jordan τ-derivation, then

$$\delta(xy + yx) = \delta(x)y^\tau + y\delta(x) + \delta(y)x^\tau + x\delta(y) \tag{8}$$

for all $x, y \in R$.

Let A be a ring. Suppose that $*\colon A \to A$ is an involution of A, i.e., $*$ is an anti-automorphism of A such that $(x^*)^* = x$ for all $x \in A$. The problem of the representability of quadratic forms by bilinear forms is connected with the structure of Jordan $*$-derivations (see [29] and [30]). In 1989, Brešar and Vukman proved the following (see [4, Theorem 1]).

Theorem 3.1. *Let A be a unital ring with involution $*$. Suppose that A contains $\frac{1}{2}$ and an invertible skew-hermitian element μ (i.e., $\mu^* = -\mu$) which lies in $Z(A)$. Then*

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every Jordan $*$-derivation from A into itself is inner.

In particular, every Jordan $*$-derivation of a unital complex $*$-algebra is inner. Indeed, given any complex $*$-algebra A, we always assume $(\beta x)^* = \bar{\beta}x^*$ for all $x \in A$ and $\beta \in \mathbb{C}$. Therefore we can always find an invertible skew-hermitian element in $Z(A)$.

Let H be a real (resp. complex) Hilbert space with $\dim_{\mathbb{R}} H > 1$ (resp. $\dim_{\mathbb{C}} H > 1$). Let $\mathcal{B}(H)$ stand for the algebra of all bounded linear operators on the Hilbert space H and let $\mathcal{A}$ be a standard operator algebra on H, i.e., $\mathcal{A}$ is a subalgebra of $\mathcal{B}(H)$ containing the subalgebra of all bounded finite rank operators (see [31]). Then $\mathcal{B}(H)$ can be endowed with a canonical involution, say $*$. It is known that $\mathcal{A}$ is a prime algebra with nonzero socle. Moreover, $Q_{ms}(\mathcal{A}) = Q_{ms}(\mathcal{B}(H)) = \mathcal{B}(H)$ (see [10, Theorem 1.3]). In 1990, Šemrl proved the following (see [28, Theorem 2.3]).

Theorem 3.2. *Let H be a real Hilbert space with $\dim_{\mathbb{R}} H > 1$. Let $D\colon \mathcal{B}(H) \to \mathcal{B}(H)$ be a Jordan $*$-derivation. Then there exists a unique $T \in \mathcal{B}(H)$ such that $D(S) = ST - TS^*$ for all $S \in \mathcal{B}(H)$, i.e., D is inner.*

In 1994, Šemrl showed the following (see [31]).

Theorem 3.3. *Let H be a complex Hilbert space with $\dim_{\mathbb{C}} H > 1$ and let $\mathcal{A}$ be a standard operator algebra on H. Suppose that $J\colon \mathcal{A} \to \mathcal{B}(H)$ is a Jordan $*$-derivation. Then there exists a unique $T \in \mathcal{B}(H)$ such that $J(A) = AT - TA^*$ for all $A \in \mathcal{A}$, i.e., J is X-inner.*

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In 2013, Chuang et al. extended Šemrl's theorems above as follows (see [10, Theorem 1.2]).

Theorem 3.4. *Let R be a prime ring, which is not a division ring. Let τ be an anti-automorphism of R and let $\delta\colon R \to Q_s(R)$ be a Jordan τ-derivation. If char$R \neq 2$ and the socle of R is nonzero, then δ is X-inner.*

Moreover, Lee et al. completely determined the structure of Jordan $*$-derivations of prime rings (see [11], [17], and [18]). To be precise, we state its conclusion.

Theorem 3.5. ([18, Theorem 1.2]) *Let R be a prime ring with involution $*$, which is not commutative. Then any Jordan $*$-derivation of R is X-inner except when* $\mathrm{char}R = 2$ *and* $\dim_C RC = 4$.

They also gave an example of non X-inner Jordan $*$-derivations when $\mathrm{char}R = 2$ and $\dim_C RC = 4$ (see [18, Example 3.2]). For the general anti-automorphism case, Lee and the author [20] proved the following results.

Theorem 3.6. ([20, Theorem 2.1]) *Let R be a prime ring with an anti-automorphism τ. Suppose that R is not a* GPI-*ring. Then any Jordan τ-derivation $\delta\colon R \to Q_{ms}(R)$ is X-inner.*

Theorem 3.7. ([20, Theorem 2.8]) *Let D be a division ring, which is not commutative, with center C and let τ be an anti-automorphism of D. Then any Jordan τ-derivation*

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$\delta\colon D \to D$ is X-inner except when $\mathrm{char}D = 2$ *and* $\dim_C D = 4$.

Theorem 3.8. ([20, Theorem 2.9]) *Let R be a prime* PI-*ring, which is not commutative, and let τ be an anti-automorphism of R. Then any Jordan τ-derivation $\delta\colon R \to RC$ is X-inner except when* $\mathrm{char}R = 2$ *and* $\dim_C RC = 4$.

By the above theorems, any Jordan τ-derivation of R is X-inner if either R is not a GPI-ring or R is a PI-ring except when $\mathrm{char}R = 2$ and $\dim_C RC = 4$. In order to completely characterize Jordan τ-derivations of R, they raised the following question.

Question A. Let R be a prime GPI-ring, which is not commutative, with an anti-automorphism τ. Suppose that neither R is a PI-ring nor R is a division ring. Is any Jordan τ-derivation of R X-inner?

We remark that, by Theorem 2.1, if R is both a prime GPI-ring and a division ring, then it is a PI-ring and Question A is solved by Theorem 3.8 in this case. Hence Question A is reduced to the case that R is a prime GPI-ring but is not a PI-ring.

3.2 Results

Let R be a prime ring with an anti-automorphism τ. Recall that τ can be uniquely extended to an anti-automorphism of $Q_s(R)$ and that τ is said to be of the first kind if $\beta^\tau = \beta$ for all $\beta \in C$. Otherwise, τ is said to be of the second kind. In order to deal with the second kind case, we need the following.

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Lemma 3.9. ([20, Lemma 2.7]) *Suppose that $\beta^\tau \neq \beta$ for some $\beta \in Z(R)$. Then any Jordan τ-derivation $\delta\colon R \to Q_{ms}(R)$ is X-inner. In fact, $\delta(x) = ax^\tau - xa$ for all $x \in R$, where $a = (\beta^\tau - \beta)^{-1}\delta(\beta)$.*

By applying Lemma 3.9, we get the following result.

Theorem 3.10. *Let R be a noncommutative prime PI-ring with an anti-automorphism τ. If τ is of the second kind, then any Jordan τ-derivation $\delta\colon R \to Q_{ms}(R)$ is X-inner.*

Proof. Since R is a prime PI-ring, it follows from Theorem 2.2 that $Z(R) \neq 0$ and C is the quotient field of $Z(R)$. Thus there exists $\beta \in Z(R)$ such that $\beta^\tau \neq \beta$ because τ is of the second kind. By Lemma 3.9, δ is X-inner, as desired. □

We remark that the lemma below holds for an arbitrary ring R (see [17, Lemma 2.3]).

Lemma 3.11. *Let $B\colon R \times R \to A$ be a bi-additive map and let $f, g\colon R \to A$ be additive maps, where A is an additive group. Suppose that $B(x, y) = f(xy) + g(yx)$ for all $x, y \in R$. Then*

$$B(xw, yz) - B(x, wyz) = B(zxw, y) - B(zx, wy)$$

for all $w, x, y, z \in R$.

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Now we can give an affirmative answer to Question A when τ is of the second kind.

Theorem 3.12. *Let R be a noncommutative prime ring with an anti-automorphism τ. If τ is of the second kind, then any Jordan τ-derivation of R is X-inner.*

Proof. According to Theorem 3.10, we can assume that R is not a PI-ring. Let $\delta\colon R \to Q_{ms}(R)$ be a Jordan τ-derivation. Define the bi-additive map $B\colon R \times R \to Q_{ms}(R)$ by $B(x, y) = \delta(xy + yx)$ for $x, y \in R$. It follows from Lemma 3.11 that

$$B(xw, yz) - B(x, wyz) = B(zxw, y) - B(zx, wy)$$

for all $x, y, z, w \in R$. So, by (8), we have

$$\begin{aligned}\Big(\delta(yz)w^\tau - \delta(wyz)\Big)x^\tau + \Big(\delta(xw)z^\tau - \delta(zxw)\Big)y^\tau \\ +\Big(\delta(wy)x^\tau - \delta(y)w^\tau x^\tau\Big)z^\tau + \Big(\delta(zx)y^\tau - \delta(x)z^\tau y^\tau\Big)w^\tau \\ +x\Big(w\delta(yz) - \delta(wyz)\Big) + y\Big(z\delta(xw) - \delta(zxw)\Big) \\ +z\Big(x\delta(wy) - xw\delta(y)\Big) + w\Big(y\delta(zx) - yz\delta(x)\Big) = 0\end{aligned}$$

for all $x, y, z, w \in R$. According to Corollary 2.7, there exist a nonzero ideal I_1 of R

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and bi-additive maps $r_{13}, r_{23}, r_{43} \colon I_1^2 \to Q_{ml}(R)$ such that

$$x\Big(\delta(wy) - w\delta(y)\Big) = -r_{13}(y,w)x^\tau - r_{23}(x,w)y^\tau - r_{43}(x,y)w^\tau$$

for all $x, y, w \in I_1$. Again, there are additive maps $p, q \colon I_2 \to Q_{ml}(R)$ so that

$$\delta(wy) - w\delta(y) = -p(y)w^\tau - q(w)y^\tau$$

for all $y, w \in I_2$, where I_2 is a nonzero ideal of R contained in I_1. Let $x, y, t \in I_2$. We have

$$\delta(xy) = x\delta(y) - p(y)x^\tau - q(x)y^\tau \tag{9}$$

Replacing x by tx in (9), we obtain

$$\delta(txy) = tx\delta(y) - p(y)x^\tau t^\tau - q(tx)y^\tau.$$

Left-multiplying (9) by t, we get $t\delta(xy) = tx\delta(y) - tp(y)x^\tau - tq(x)y^\tau$. Thus,

$$\delta(txy) - t\delta(xy) = \Big(tq(x) - q(tx)\Big)y^\tau - p(y)x^\tau t^\tau + tp(y)x^\tau.$$

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Replacing x, y by t, xy respectively in (9), we have

$$\delta(txy) - t\delta(xy) = -p(xy)t^\tau - q(t)y^\tau x^\tau.$$

Comparing the two equalities above, we see that

$$\Big(tq(x) - q(tx)\Big)y^\tau + \Big(p(xy) - p(y)x^\tau\Big)t^\tau + \Big(tp(y) + q(t)y^\tau\Big)x^\tau = 0$$

for all $x, y, t \in I_2$. By Lemma 2.8,

$$tq(x) = q(tx),\ p(xy) = p(y)x^\tau,\ \text{and } tp(y) = -q(t)y^\tau$$

for all $x, y, t \in I_3$, where I_3 is a nonzero ideal of R contained in I_2. According to [16, Lemma 2.1], there is $a \in Q_{ml}(R)$ such that $q(x) = xa$ for $x \in I_3$. So

$$tp(y) = -q(t)y^\tau = -tay^\tau$$

for $t, y \in I_3$, i.e., $I_3\big(p(y) + ay^\tau\big) = 0$ for all $y \in I_3$. Thus, $p(y) = -ay^\tau$ and it follows from (9) that

$$\delta(xy) - x\delta(y) = ay^\tau x^\tau - xay^\tau$$

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for $x, y \in I_3$. Let $\tilde{\delta}\colon I_3 \to Q_{ml}(R)$ be defined by $\tilde{\delta}(x) = ax^\tau - xa$ for all $x \in I_3$. Then $\tilde{\delta}(xy) = ay^\tau x^\tau - xya$ and

$$(\tilde{\delta} - \delta)(xy) = xay^\tau - xya - x\delta(y) = x(\tilde{\delta} - \delta)(y),$$

for $x, y \in I_3$. So there exists $c \in Q_{ml}(R)$ such that $(\tilde{\delta} - \delta)(x) = xc$ for all $x \in I_3$ (see [16, Lemma 2.1]). Define $J := \tilde{\delta} - \delta$, a Jordan τ-derivation of I_3. Thus, $x^2c = J(x^2) = xJ(x) + J(x)x^\tau = x^2c + xcx^\tau$ for all $x \in I_3$; that is, $xcx^\tau = 0$ for all $x \in I_3$. By [10, Lemma 2.2], $c = 0$ follows, i.e., $\delta = \tilde{\delta}$ on I_3. Therefore, $\delta(x) = ax^\tau - xa$ for $x \in I_3$. By [20, Lemma 2.6], $a \in Q_{ms}(R)$. Finally, we will show that δ is X-inner. Let $x \in I_3$ and $y \in R$. Then

$$\delta(xy + yx) = ay^\tau x^\tau + ax^\tau y^\tau - xya - yxa$$

and

$$\begin{aligned} \delta(xy + yx) &= \delta(x)y^\tau + y\delta(x) + \delta(y)x^\tau + x\delta(y) \\ &= ax^\tau y^\tau - xay^\tau + yax^\tau - yxa + \delta(y)x^\tau + x\delta(y). \end{aligned}$$

Comparing these equations, we have

$$\Big(\delta(y) - (ay^\tau - ya)\Big)x^\tau + x\Big(\delta(y) - (ay^\tau - ya)\Big) = 0$$

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for all $x \in I_3$ and $y \in R$. Fix $y \in R$ and set $q := \delta(y) - (ay^\tau - ya)$. Then, for $x, z \in I_3$,

$$xzq = -q(xz)^\tau = -qz^\tau x^\tau = zqx^\tau = -zxq$$

and so $(xz+zx)q = 0$ for all $x, z \in I_3$. This implies $(xz+zx)q = 0$ for all $x, z \in Q_{ml}(R)$. If $\text{char}R \neq 2$, let $z = 1$ and so $2xq = 0$ for all $x \in Q_{ml}(R)$ implying $q = 0$. If $\text{char}R = 2$, then $[Q_{ml}(R), Q_{ml}(R)]q = 0$ forcing $q = 0$. Hence $\delta(y) = ay^\tau - ya$ for all $y \in R$, as desired. □

We next consider the case that τ is of the first kind. By an X-inner automorphism we mean an automorphism of the form $x \mapsto uxu^{-1}$ for all $x \in R$, where $u \in Q_s(R)$. Kharchenko proved that, given an automorphism σ of a prime GPI-ring, if σ is of the first kind, then it is X-inner (see [14, Proof of Proposition 2]). By Kharchenko's theorem, τ^2 is X-inner when R is a prime GPI-ring and τ is of the first kind. The complexity of the question we will study depends on that of τ^2.

Definition 3.13. *Let R be a prime GPI-ring with an automorphism σ of the first kind. Then there exists $u \in Q_s(R)$ such that $x^\sigma = uxu^{-1}$ for all $x \in R$. We say that* $\deg\sigma = m$ *if u is algebraic of minimal degree m over C. Moreover,* $\deg\sigma = \infty$ *if u is not algebraic over C.*

Clearly, $\deg\sigma$ is independent of the element u we choose. Also, if $\deg\tau^2 = 1$, then

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τ is an involution and Question A has been solved by Theorem 3.5. The following is the second main theorem and we will prove it in the next section.

Theorem 3.14. *Let R be a prime GPI-ring with* $\text{char}R \neq 2$ *and with τ an anti-automorphism of the first kind. If* $\deg \tau^2 \neq 2$*, then any Jordan τ-derivation of R is X-inner.*

We remark that the case of $\deg \tau^2 = 2$ keeps unknown.

3.3 The first kind case

The goal of this section is to prove Theorem 3.14. Let R be a prime GPI-ring with an anti-automorphism τ of the first kind. Let $u \in Q_s(R)$ be fixed such that $x^{\tau^2} = uxu^{-1}$ for all $x \in R$.

Lemma 3.15. $u^\tau u = uu^\tau \in C$.

Proof. Let $x \in R$. Then

$$u^\tau ux = u^\tau x^{\tau^2} u = (x^\tau u)^\tau u = (u(x^\tau)^{\tau^{-2}})^\tau u = xu^\tau u.$$

Hence $u^\tau u = uu^\tau \in C$. □

Now, by Lemma 3.15, we fix $\beta := u^\tau u = uu^\tau \in C$ and so $u^\tau = \beta u^{-1}$. Since R is a prime GPI-ring, RC is a primitive ring with nonzero socle and so $Q_{ms}(RC) = Q_s(RC)$.

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In view of Theorem 3.4, the aim of this section is to extend δ to a Jordan τ-derivation of RC when $\mathrm{char}R \neq 2$. The following result plays a key role.

Lemma 3.16. *Let $f\colon R \to Q_{ml}(R)$ be an additive map such that*

$$xf(y) + f(y)x^{\tau} = yf(x) + f(x)y^{\tau} \tag{10}$$

for all $x, y \in R$. If $\deg \tau^2 > 2$, then $f = 0$.

Note that this result is better than [22, Lemma 3.2].

Proof. Choose a nonzero ideal such that $uI_1 \subseteq R$. Then, by replacing x with ux in (10),

$$uxf(y)u + \beta f(y)x^{\tau} = yf(ux)u + f(ux)y^{\tau}u$$

for $x \in I_1$ and $y \in R$. Also, by (10),

$$uxf(y)u + uf(y)x^{\tau}u = uyf(x)u + uf(x)y^{\tau}u.$$

Comparing the two equations, we have

$$\beta f(y)x^{\tau} - uf(y)x^{\tau}u = yf(ux)u - uyf(x)u + \Big(f(ux) - uf(x)\Big)y^{\tau}u \tag{11}$$

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for $x \in I_1$ and $y \in R$. Replacing y by uy in (11),

$$\beta f(uy)x^\tau - uf(uy)x^\tau u = uyf(ux)u - u^2yf(x)u + \beta\Big(f(ux) - uf(x)\Big)y^\tau$$

for $x, y \in I_1$. Also, by (11),

$$u\beta f(y)x^\tau - u^2 f(y)x^\tau u = uyf(ux)u - u^2yf(x)u + u\Big(f(ux) - uf(x)\Big)y^\tau u.$$

Comparing the two equations, we have

$$\begin{aligned} & \beta\Big(f(uy) - uf(y)\Big)x^\tau - u\Big(f(uy) - uf(y)\Big)x^\tau u \\ = \ & \beta\Big(f(ux) - uf(x)\Big)y^\tau - u\Big(f(ux) - uf(x)\Big)y^\tau u \end{aligned}$$

for all $x, y \in I_1$. By Theorem 2.4, $f(uy) = uf(y)$ for all $y \in I_1$. So (11) becomes

$$\beta f(y)x^\tau - uf(y)x^\tau u = yuf(x)u - uyf(x)u \tag{12}$$

for all $x, y \in I_1$. Choose a nonzero ideal I_2 of R contained in I_1 such that $uI_2 \subseteq I_1$. Replacing x by ux in (12), we have

$$\beta^2 f(y)x^\tau - \beta uf(y)x^\tau u = yu^2 f(x)u^2 - uyuf(x)u^2$$

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for all $x, y \in I_2$. On the other hand, (12) implies

$$\beta^2 f(y)x^\tau - \beta uf(y)x^\tau u = \beta\Big(yuf(x)u - uyf(x)u\Big)$$

for all $x, y \in I_2$. Comparing the two equations, we have

$$y\Big(\beta uf(x) - u^2 f(x)u\Big) + uy\Big(uf(x)u - \beta f(x)\Big) = 0$$

for all $x, y \in I_2$. Similarly, $uf(x)u = \beta f(x)$ for all $x \in I_2$. Thus (12) becomes

$$\beta f(y)x^\tau - uf(y)x^\tau u = \beta yf(x) - uyf(x)u \tag{13}$$

for all $x, y \in I_2$. Choose a nonzero ideal I_3 of R contained in I_2 such that $I_3 u \subseteq I_2$. Replacing y by yu in (13),

$$\begin{aligned} & \beta f(yu)x^\tau - uf(yu)x^\tau u \\ = \;& \beta yuf(x) - uyuf(x)u \\ = \;& \beta^2 yf(x)u^{-1} - \beta uyf(x) \\ = \;& \beta\Big(\beta yf(x) - uyf(x)u\Big)u^{-1} \\ = \;& \beta^2 f(y)x^\tau u^{-1} - \beta uf(y)x^\tau \end{aligned}$$

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for all $x, y \in I_3$. So we get

$$\beta^2 f(y)x^\tau - \Big(\beta u f(y) + \beta f(yu)\Big)x^\tau u + u f(yu)x^\tau u^2 = 0$$

for all $x, y \in I_3$. By Theorem 2.4, $f(y) = 0$ for all $y \in I_3$. Thus it follows from (10) that

$$yf(x) + f(x)y^\tau = xf(y) + f(y)x^\tau = 0$$

for all $x \in R$ and $y \in I_3$. By applying the same argument as the last part of the proof of Theorem 3.12, we have $f = 0$, as desired. □

Remark 3.17. In Lemma 3.16, the case for $\deg \tau^2 = 1$ had been solved by Beidar and Martindale [3]. However, the solution of (10) is still unknown when $\deg \tau^2 = 2$.

Lemma 3.18. *Suppose* $\deg \tau^2 > 2$. *Let* $\alpha \in C$ *and* I *be a nonzero ideal of* R *such that* $\alpha I \subseteq R$. *Then* $\delta(\alpha x) = \alpha\delta(x)$ *for all* $x \in I$.

Proof. Let $x, y \in I$. Then

$$\delta((\alpha x)y + y(\alpha x)) \quad = \quad \delta(\alpha x)y^\tau + y\delta(\alpha x) + \alpha\delta(y)x^\tau + \alpha x\delta(y)$$

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and

$$\delta(x(\alpha y) + (\alpha y)x) \quad = \quad \alpha\delta(x)y^\tau + \alpha y\delta(x) + \delta(\alpha y)x^\tau + x\delta(\alpha y).$$

Comparing the two equations, we have

$$\Big(\delta(\alpha y) - \alpha\delta(y)\Big)x^\tau + x\Big(\delta(\alpha y) - \alpha\delta(y)\Big) = \Big(\delta(\alpha x) - \alpha\delta(x)\Big)y^\tau + y\Big(\delta(\alpha x) - \alpha\delta(x)\Big).$$

By Lemma 3.16, $\delta(\alpha x) = \alpha\delta(x)$ for all $x \in I$. □

Lemma 3.19. *Suppose* $\deg \tau^2 > 2$*. Then every Jordan* τ*-derivation* δ *of* R *can be extended to a Jordan* τ*-derivation of* RC*.*

Proof. Choose a subset $\{w_i\}_{i\in\Phi}$ of R which is a basis of RC over C, where Φ is a nonempty well-ordered set. Then any element of RC can be written as the form $\sum_{i\in\Phi} \alpha_i w_i$, where $\alpha_i = 0$ for all but finitely many $i \in \Phi$. Recall that $Q_{ms}(R) = Q_{ms}(RC)$. Define $\widetilde{\delta}\colon RC \to Q_{ms}(RC)$ by

$$\widetilde{\delta}\Big(\sum_{i\in\Phi} \alpha_i w_i\Big) = \sum_{i\in\Phi} \alpha_i \delta(w_i).$$

Then it is clearly a well-defined additive map since $\{w_i\}_{i\in\Phi}$ forms a basis of RC over C. We claim that $\widetilde{\delta}|_R = \delta$. Let $x \in R$. Write $x = \sum_{i\in\Phi} \alpha_i w_i$. By Lemma 3.18, there is

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a nonzero ideal I of R such that $\alpha_i I \subseteq R$ and $\delta(\alpha_i y) = \alpha_i \delta(y)$ for all $i \in \Phi$ and $y \in I$. Let $y \in I$. Then

$$\begin{aligned}
\delta(xy + yx) &= \sum_{i\in\Phi} \delta\big((\alpha_i y)w_i + w_i(\alpha_i y)\big) \\
&= \sum_{i\in\Phi} \delta(\alpha_i y)w_i^{\tau} + \delta(w_i)\alpha_i y^{\tau} + w_i\delta(\alpha_i y) + \alpha_i y\delta(w_i) \\
&= \delta(y)x^{\tau} + x\delta(y) + \sum_{i\in\Phi} \big(\alpha_i\delta(w_i)y^{\tau} + \alpha_i y\delta(w_i)\big).
\end{aligned}$$

Comparing this with (8), we have

$$\Big(\delta(x) - \sum_{i\in\Phi} \alpha_i\delta(w_i)\Big)y^{\tau} + y\Big(\delta(x) - \sum_{i\in\Phi} \alpha_i\delta(w_i)\Big) = 0$$

for all $y \in I$. By applying the same argument as the last part of the proof of Theorem 3.12, we have $\delta(x) = \sum_{i\in\Phi} \alpha_i\delta(w_i) = \widetilde{\delta}(x)$ and so the claim holds.

Finally we show that $\widetilde{\delta}$ is a Jordan τ-derivation. Let $x = \sum_{i\in\Phi} \alpha_i w_i \in RC$. For

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each $i, j \in \Phi$, write $w_i w_j = \sum_{k\in\Phi} \gamma_k^{ij} w_k \in R$, where $\gamma_k^{ij} \in C$. Then

$$
\begin{aligned}
\widetilde{\delta}(x^2) &= \widetilde{\delta}\Big(\sum_{i,j} \alpha_i\alpha_j w_i w_j\Big) \\
&= \widetilde{\delta}\Big(\sum_{i,j} \alpha_i\alpha_j \sum_{k\in\Phi} \gamma_k^{ij} w_k\Big) \\
&= \sum_{i,j} \alpha_i\alpha_j \sum_{k\in\Phi} \gamma_k^{ij} \delta(w_k) \\
&= \sum_{i,j} \alpha_i\alpha_j \widetilde{\delta}(w_i w_j) \\
&= \sum_{i,j} \alpha_i\alpha_j \delta(w_i w_j) \\
&= \sum_{i} \alpha_i^2 \delta(w_i^2) + \sum_{i<j} \alpha_i\alpha_j \delta(w_i w_j + w_j w_i) \\
&= \sum_{i} \alpha_i^2 \Big(w_i\delta(w_i) + \delta(w_i) w_i^\tau\Big) \\
&\quad + \sum_{i<j} \alpha_i\alpha_j \Big(\delta(w_i) w_j^\tau + w_j \delta(w_i) + \delta(w_j) w_i^\tau + w_i \delta(w_j)\Big) \\
&= x\widetilde{\delta}(x) + \widetilde{\delta}(x) x^\tau.
\end{aligned}
$$

Hence the proof of Lemma 3.19 is complete. □

Applying Lemma 3.19, we are now ready to prove Theorem 3.14.

Proof of Theorem 3.14. In view of Theorem 3.8, we can assume that R is not a PI-ring. By Theorem 3.5, we also assume that $\deg \tau^2 > 2$. Recall that if R is a prime GPI-ring and RC is a division ring, then R is a PI-ring. So RC is not a division ring. Let $\delta\colon R \to Q_{ms}(R)$ be a Jordan τ-derivation of R. Since the socle of RC is nonzero,

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$Q_{ms}(RC) = Q_s(RC)$. By Lemma 3.19, δ can be extended to a Jordan τ-derivation $\widetilde{\delta}\colon RC \to Q_s(RC)$ of RC. According to Theorem 3.4, there exists $a \in Q_s(RC)$ such that $\widetilde{\delta}(x) = ax^\tau - xa$ for $x \in RC$. In particular, $\delta(x) = ax^\tau - xa$ for all $x \in R$. As a consequence of [2, Proposition 2.1.10], $Q_{ms}(R) = Q_{ms}(RC) = Q_s(RC)$ and so $a \in Q_{ms}(R)$. Hence δ is X-inner, as desired. □

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4 Weak Jordan derivations

4.1 Motivation

Let R be a prime ring with extended centroid C. An additive map $d\colon R \to Q_{ml}(R)$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ and is called a Jordan derivation (resp. Lie derivation) if $d(x^2) = d(x)x + xd(x)$ for all $x \in R$ (resp. $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in R$), where $[a, b] := ab - ba$ for $a, b \in Q_{ml}(R)$. Clearly, any derivation is a Jordan derivation but a Jordan derivation is not in general a derivation. In 1957, Herstein proved the following theorem.

Theorem 4.1. ([12, Theorem 3.1]) *Let R be a prime ring with* $\operatorname{char} R \neq 2$. *Then every Jordan derivation of R is a derivation.*

We remark that though a Jordan derivation defined in [12] maps R to itself, the same proof is also valid for our definition. In 2014, Lee and the author completely described Jordan derivations of a prime ring R with $\operatorname{char} R = 2$ as follows.

Theorem 4.2. ([19, Theorem 2.2]) *Let R be a prime ring with* $\operatorname{char} R = 2$. *An additive map $\delta\colon R \to Q_{ml}(R)$ is a Jordan derivation if and only if there exist a derivation $d\colon R \to Q_{ml}(R)$ and an additive map $\mu\colon R \to C$ such that $\delta = d + \mu$ and $\mu(x^2) = 0$ for all $x \in R$.*

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They also gave an example to show the existence of a Jordan derivation, which is not a derivation, on R when $\operatorname{char} R = 2$ (see [19, The remark below Corollary 2.4]). Therefore the structure of Jordan derivations of prime rings has been completely determined.

As a generalization of Jordan derivations (resp. Lie derivations), an additive map $\delta\colon R \to Q_{ml}(R)$ is called a weak Jordan derivation (resp. weak Lie derivation) if $\delta(x^2) - x\delta(x) - \delta(x)x \in C$ for all $x \in R$ (resp. $\delta([x, y]) - [\delta(x), y] - [x, \delta(y)] \in C$ for all $x, y \in R$). In order to characterize additive maps $d\colon R \to R$ satisfying $[d(x^2) - xd(x) - d(x)x, x] = 0$ for all $x \in R$ (see [5, Theorem 4]), Brešar proved the following result.

Theorem 4.3. ([5, pp. 541–542]) *Let R be a prime ring of characteristic different from 2 and 3. Suppose that $\delta\colon R \to RC + C$ is a weak Jordan derivation. If $\dim_C RC > 4$, then δ is a derivation.*

Also, in [8, Remark 6.8], Brešar et al. studied weak Lie derivations. In particular, they characterized weak Lie derivations of R when $\dim_C RC > 9$ as follows.

Theorem 4.4. *Let R be a prime ring with $\dim_C RC > 9$. If $\delta\colon R \to Q_{ml}(R)$ is a weak Lie derivation, then there exist a derivation $d\colon R \to Q_{ml}(R)$ and an additive map $\nu\colon R \to C$ such that $\delta = d + \nu$.*

We remark that every weak Jordan derivation of R is a weak Lie derivation when $\operatorname{char} R = 2$. The aim of the chapter is to determine the structure of weak Jordan

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derivations of R. On the other hand, weak Jordan derivations of prime rings occur canonically when one studies Jordan derivations of semiprime rings.

4.2 Results

Let R be a prime ring with extended centroid C. We give a complete characterization of weak Jordan derivations of prime rings. The following is the first result.

Theorem 4.5. *Let R be a prime ring with $\dim_C RC > 4$, and let $\delta\colon R \to Q_{ml}(R)$ be a weak Jordan derivation.*

(i) *If $\operatorname{char} R \neq 2$, then δ is a derivation.*

(ii) *If $\operatorname{char} R = 2$, then $\delta = d + \nu$, where $d\colon R \to Q_{ml}(R)$ is a derivation and $\nu\colon R \to C$ is an additive map.*

To state the second result, we will fix some notations. Suppose $\dim_C RC = n^2 < \infty$. It follows from Theorem 2.2 that RC is a finite-dimensional central simple algebra over C and C is the quotient field of $Z(R)$. In this case, we have $Q_{ml}(R) = RC$. By the well-known Wedderburn-Artin theorem, $RC \cong \mathrm{M}_m(\Delta)$ for some division ring Δ and $m \leq n$. Let F be a separable maximal subfield of Δ. Then $RC \otimes_C F \cong \mathrm{M}_n(F)$ and we regard RC as a C-subalgebra of $\mathrm{M}_n(F)$. Denote I_n for the identity matrix in $\mathrm{M}_n(F)$. Notice that, if RC is not a division ring and $\dim_C RC = 4$, then $F = C$ and $RC \cong \mathrm{M}_2(C)$. Also, if $RC = \Delta$ is a division ring with $\dim_C RC = 4$, then F is a Galois extension over

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C. We are now ready to state the second result.

Theorem 4.6. *Let R be a prime ring with $\dim_C RC = 4$, and let $\delta\colon R \to Q_{ml}(R)$ be a weak Jordan derivation.*

(i) *If* $\operatorname{char} R \neq 2$, *then*

$$\delta(x) = d(x) + [a, x] + L(x) + \zeta(x)$$

for all $x \in R$, where $d\colon R \to RC$ is a derivation, $a \in \mathrm{M}_2(F)$ and $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ are F-linear maps. Moreover, there exist $\beta_i \in F$, $1 \leq i \leq 6$, such that

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \Big(\beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21}\Big) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for all $x = (x_{ij}) \in \mathrm{M}_2(F)$. In this case, δ is a derivation if and only if all $\beta_i = 0$.

(ii) *If* $\operatorname{char} R = 2$, *then*

$$\delta(x) = d(x) + \nu(x) + [a, x] + L(x) + \zeta(x)$$

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for all $x \in R$, *where* $d\colon R \to RC$ *is a derivation,* $\nu\colon R \to C$ *is an additive map,* $a \in \mathrm{M}_2(F)$, *and* $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ *are* F*-linear maps. Moreover, there exist* $\beta_i \in F$, $1 \le i \le 6$, *such that*

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = \Big(\beta_1(x_{11} + x_{22}) + \beta_2 x_{12} + \beta_3 x_{21}\Big) I_2$$

for all $x = (x_{ij}) \in \mathrm{M}_2(F)$.

Conversely, an additive map $\delta\colon R \to Q_{ml}(R)$ *satisfying* (i) *or* (ii) *is a weak Jordan derivation.*

Remark 4.7. The additive maps L and ζ in Theorem 4.6 are weak Jordan derivations. In fact, they satisfy

$$L(x^2) - xL(x) - L(x)x = -\Big(\beta_4 x_{21}^2 + \beta_5 x_{12}^2 + \beta_6 x_{12} x_{21}\Big) I_2$$

and

$$\zeta(x^2) - x\zeta(x) - \zeta(x)x = -\Big(\beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21}\Big)\Big(x_{11} - x_{22}\Big) I_2$$

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for all $x = (x_{ij}) \in \mathrm{M}_2(F)$. By substituting some appropriate x_{ij}, we will see that L (resp. ζ) is a Jordan derivation if and only if $L = 0$ (resp. $\zeta = 0$). Indeed, if L is a Jordan derivation, then

$$\beta_4 x^2 + \beta_5 y^2 + \beta_6 xy = 0$$

for all $x, y \in F$. Substituting (x, y) by $(1, 0)$, $(0, 1)$, and $(1, 1)$, we get $\beta_4 = \beta_5 = \beta_6 = 0$. Suppose next that ζ is a Jordan derivation. Then

$$\Big(\beta_1(x - y) + \beta_2 z + \beta_3 w\Big)\big(x - y\big) = 0$$

for all $x, y, z, w \in F$. Substituting (x, y, z, w) by $(1, 0, 0, 0)$, $(1, 0, 1, 0)$, and $(1, 0, 0, 1)$, we get $\beta_1 = \beta_2 = \beta_3 = 0$, as asserted.

A map $q \colon R \to RC + C$ is called a trace of a biadditive map if there is a biadditive map $B \colon R \times R \to RC + C$ such that $q(x) = B(x, x)$ for all $x \in R$. Brešar and Šemrl characterized commuting traces of biadditive maps when $\operatorname{char} R \neq 2$ (see [5, Theorem 1] and [7, Theorem 3.1]). The following is the conclusion.

Theorem 4.8. *Let R be a prime ring with* $\operatorname{char} R \neq 2$ *and $q \colon R \to RC + C$ be a trace of a biadditive map such that $[q(x), x] = 0$ for all $x \in R$. Then there exist $\lambda \in C$, an*

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additive map $\mu\colon R \to C$*, and a trace of a biadditive map* $\nu\colon R \to C$ *such that*

$$q(x) = \lambda x^2 + \mu(x)x + \nu(x)$$

for all $x \in R$*. Moreover, in case* $\dim_C RC = 4$ *one may take* $\lambda = 0$*.*

As an application of Theorem 4.5 and Theorem 4.6, the following generalizes Brešar's theorem [5, Theorem 4].

Theorem 4.9. *Let* R *be a prime ring with* $\operatorname{char} R \neq 2$*, and let* $\delta\colon R \to RC + C$ *be an additive map satisfying*

$$[\delta(x^2) - x\delta(x) - \delta(x)x, x] = 0$$

for all $x \in R$*.*

(i) *If* $\dim_C RC > 4$*, then*

$$\delta(x) = \gamma x + d(x) + \mu(x)$$

for all $x \in R$*, where* $\gamma \in C$*,* $d\colon R \to RC + C$ *is a derivation, and* $\mu\colon R \to C$ *is an additive map.*

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(ii) *If* $\dim_C RC = 4$, *then*

$$\delta(x) = d(x) + \mu(x) + [a, x] + L(x) + \zeta(x)$$

for all $x \in R$, *where* $d\colon R \to RC$ *is a derivation,* $\mu\colon R \to C$ *is an additive map,* $a \in \mathrm{M}_2(F)$, *and* $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ *are as in Theorem 4.6* (i).

Proof. We will follow the proof of [5, Theorem 4] and apply Theorem 4.5 and Theorem 4.6. Let $q\colon R \to RC + C$ be the map defined by $q(x) = \delta(x^2) - x\delta(x) - \delta(x)x$ for $x \in R$. Then q is a trace of a biadditive map. According to Theorem 4.8, there exist $\lambda \in C$, an additive map $\tilde{\mu}\colon R \to C$, and a trace of a biadditive map $\tilde{\nu}\colon R \to C$ such that

$$\delta(x^2) - x\delta(x) - \delta(x)x = \lambda x^2 + \tilde{\mu}(x)x + \tilde{\nu}(x)$$

for all $x \in R$. Note that we may take $\lambda = 0$ if $\dim_C RC = 4$. Define an additive map $D\colon R \to RC + C$ by

$$D(x) = \delta(x) + \lambda x + \frac{1}{2}\tilde{\mu}(x), \quad x \in R.$$

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We claim that D is a weak Jordan derivation. Indeed, let $x \in R$. Then

$$\begin{aligned} D(x^2) &= \delta(x^2) + \lambda x^2 + \frac{1}{2}\tilde{\mu}(x^2) \\ &= x\delta(x) + \delta(x)x + 2\lambda x^2 + \tilde{\mu}(x)x + \tilde{\nu}(x) + \frac{1}{2}\tilde{\mu}(x^2) \end{aligned}$$

and $xD(x) + D(x)x = x\delta(x) + \delta(x)x + 2\lambda x^2 + \tilde{\mu}(x)x$. Thus

$$D(x^2) - xD(x) - D(x)x = \tilde{\nu}(x) + \frac{1}{2}\tilde{\mu}(x^2) \in C$$

and hence D is a weak Jordan derivation.

If $\dim_C RC > 4$, then it follows from Theorem 4.5 (i) that D is a derivation, and hence

$$\delta(x) = \gamma x + d(x) + \mu(x),$$

where $\gamma = -\lambda$, $d = D$, and $\mu(x) = -\frac{1}{2}\tilde{\mu}(x)$.

Suppose that $\dim_C RC = 4$. Then $RC + C = RC$ and, by Theorem 4.6 (i), D is of the form

$$D(x) = d(x) + [a, x] + L(x) + \zeta(x),$$

where $d\colon R \to RC$ is a derivation, $a \in \mathrm{M}_2(F)$, and $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ are as in

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Theorem 4.6 (i). Hence

$$\delta(x) = d(x) + \mu(x) + [a, x] + L(x) + \zeta(x),$$

where $\mu(x) = -\frac{1}{2}\tilde{\mu}(x)$. □

4.3 Proofs of Theorem 4.5 and 4.6

Let R be a prime ring and let $\delta\colon R \to Q_{ml}(R)$ be a weak Jordan derivation. Thus,

$$\mu(x) := \delta(x^2) - x\delta(x) - \delta(x)x \in C \tag{14}$$

for all $x \in R$. Linearizing it, we get

$$\lambda(x, y) := \delta(xy + yx) - x\delta(y) - y\delta(x) - \delta(x)y - \delta(y)x \in C \tag{15}$$

for all $x, y \in R$.

Proposition 4.10. *Let $\delta\colon R \to Q_{ml}(R)$ be a weak Jordan derivation.*

(i) *If $\operatorname{char} R \neq 2$ and $\dim_C RC > 16$, then δ is a derivation.*

(ii) *If $\operatorname{char} R = 2$ and $\dim_C RC > 9$, then $\delta = d + \nu$, where $d\colon R \to Q_{ml}(R)$ is a derivation and $\nu\colon R \to C$ is an additive map.*

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Proof. We first prove (ii). Since char $R = 2$, a weak Jordan derivation is also a weak Lie derivation by (15). Therefore (ii) follows from Theorem 4.4.

We turn to the proof of (i). Suppose that char $R \neq 2$ and $\dim_C RC > 16$. We claim that there exist additive maps $p, q\colon R \to Q_{ml}(R)$ and a bi-additive map $\phi\colon R^2 \to C$ such that

$$\delta(xy) - \delta(x)y = xp(y) + yq(x) + \phi(x, y) \tag{16}$$

for all $x, y \in R$. Let $B(x, y) := \delta(xy + yx)$. By Lemma 3.11,

$$B(xw, yz) - B(x, wyz) = B(zxw, y) - B(zx, wy)$$

for all $x, y, z, w \in R$. It follows from (15) that

$$\begin{aligned}
&(\delta(wy)z - \delta(wyz))x + (\delta(zx)w - \delta(zxw))y \\
&+(\delta(xw)y - \delta(x)wy)z + (\delta(yz)x - \delta(y)zx)w \\
&+x(w\delta(yz) - \delta(wyz)) + y(z\delta(xw) - \delta(zxw)) \\
&+z(x\delta(wy) - xw\delta(y)) + w(y\delta(zx) - yz\delta(x)) \in C
\end{aligned}$$

for all $x, y, z, w \in R$. According to Theorem 2.5, there are bi-additive maps $\tilde{p}, \tilde{q}, \tilde{s}\colon R^2 \to$

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$Q_{ml}(R)$ and a 3-additive map $\tilde{r}\colon R^3 \to C$ such that

$$(\delta(xw) - \delta(x)w)y = x\tilde{p}(y, w) + y\tilde{q}(x, w) + w\tilde{s}(x, y) + \tilde{r}(x, y, w)$$

for all $x, y, w \in R$. By Theorem 2.5 again, there exist additive maps $p, q\colon R \to Q_{ml}(R)$ and a bi-additive map $\phi\colon R^2 \to C$ such that

$$\delta(xw) - \delta(x)w = xp(w) + wq(x) + \phi(x, w)$$

for all $x, w \in R$. Hence (16) holds.

Next, let $x, y, z \in R$. By (16), we have

$$\delta\big(x(yz)\big) - \delta(x)yz = xp(yz) + yzq(x) + \phi(x, yz),$$
$$\delta\big((xy)z\big) - \delta(xy)z = xyp(z) + zq(xy) + \phi(xy, z).$$

Thus,

$$\big(\delta(xy) - \delta(x)y\big)z - x\big(p(yz) - yp(z)\big) - yzq(x) + zq(xy) \in C.$$

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So, by Theorem 2.5, there is an additive map $d\colon R \to Q_{ml}(R)$ such that

$$p(yz) - yp(z) = d(y)z. \tag{17}$$

By (17) again,

$$p(xyz) - xp(yz) = d(x)yz,$$

$$p(xyz) - xyp(z) = d(xy)z.$$

Thus,

$$-xd(y)z = x(yp(z) - p(yz)) = (d(x)y - d(xy))z$$

and so

$$(d(xy) - d(x)y - xd(y))z = 0.$$

This proves that d is a derivation. So, by (17),

$$(p - d)(yz) = y(p - d)(z).$$

By [16, Lemma 2.1], there exists $a \in Q_{ml}(R)$ such that $d(x) = p(x) - xa$ for all $x \in R$.

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So

$$d(xy) - d(x)y = xd(y) = xp(y) - xya.$$

Compare it with (16) and let $\widetilde{\delta} = \delta - d$, and then we have

$$\widetilde{\delta}(xy) - \widetilde{\delta}(x)y - xya - yq(x) \in C. \tag{18}$$

It follows from (18) that

$$\widetilde{\delta}(xyz) - \widetilde{\delta}(x)yz - xyza - yzq(x) \in C,$$

$$\widetilde{\delta}(xyz) - \widetilde{\delta}(xy)z - xyza - zq(xy) \in C.$$

Thus,

$$(\widetilde{\delta}(xy) - \widetilde{\delta}(x)y)z - yzq(x) + zq(xy) \in C.$$

By Theorem 2.5, there is an additive map $A\colon R \to Q_{ml}(R)$ such that

$$zq(x) = A(x)z$$

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and so $q(x) \in C$ for all $x \in R$. Let $Q = \widetilde{\delta} + q$. Then, by (18), we get

$$Q(xy) - Q(x)y - xya \in C. \tag{19}$$

Let $x, y, z \in R$. Then, by (19),

$$Q(xyz) - Q(x)yz - xyza \in C,$$

$$Q(xyz) - Q(xy)z - xyza \in C.$$

So $(Q(xy) - Q(x)y)z \in C$. Since R is not commutative, we see that $Q(xy) - Q(x)y = 0$ for all $x, y \in R$. It follows from this and (19) that $a = 0$, $p = d$ and so p is a derivation. Furthermore, since $Q = \delta - d + q$, we have

$$-xQ(x) = Q(x^2) - Q(x)x - xQ(x) = \mu(x) + q(x^2) - 2q(x)x$$

and so

$$x(Q(x) - 2q(x)) \in C.$$

Thus, $Q(x) = 2q(x) \in C$ for all $x \in R$, and hence $\delta = d + \nu$, where $d\colon R \to Q_{ml}(R)$ is a derivation and $\nu = q\colon R \to C$ is an additive map. Since $\operatorname{char} R \neq 2$, $\nu(x)x \in C$ for all $x \in R$ and thus $\nu = 0$, as desired. $\square$

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According to Proposition 4.10, we may assume that R is either $\operatorname{char} R \neq 2$ with $\dim_C RC \leq 16$ or $\operatorname{char} R = 2$ with $\dim_C RC \leq 9$. In either case, $Z(R)$ is nonzero by Theorem 2.2. We next claim that δ can be assumed to be $Z(R)$-linear and $\delta = 0$ on $Z(R)$.

Lemma 4.11. *Assume that* $\dim_C RC < \infty$. *Let* $\delta\colon R \to RC$ *be a weak Jordan derivation.*

(i) *If* $\operatorname{char} R \neq 2$, *then there is a derivation* $d\colon R \to RC$ *such that* $\delta - d$ *is* $Z(R)$*-linear and* $\delta = d$ *on* $Z(R)$.

(ii) *If* $\operatorname{char} R = 2$, *then there exist a derivation* $d\colon R \to RC$ *and an additive map* $\nu\colon R \to C$ *such that* $\delta + d + \nu$ *is* $Z(R)$*-linear and* $\delta + d + \nu = 0$ *on* $Z(R)$.

Proof. (i) Assume that $\operatorname{char} R \neq 2$. Let $\beta, \gamma \in Z(R)$ and $x \in R$. Then, by (15),

$$2\delta((\beta\gamma)x) = 2\beta\gamma\delta(x) + \delta(\beta\gamma)x + x\delta(\beta\gamma) + \lambda(\beta\gamma, x),$$

and

$$\begin{aligned} 2\delta(\beta(\gamma x)) &= 2\beta\delta(\gamma x) + \delta(\beta)\gamma x + \gamma x\delta(\beta) + \lambda(\beta, \gamma x) \\ &= \beta(2\gamma\delta(x) + \delta(\gamma)x + x\delta(\gamma) + \lambda(\gamma, x)) + \delta(\beta)\gamma x + \gamma x\delta(\beta) + \lambda(\beta, \gamma x) \\ &= 2\beta\gamma\delta(x) + (\beta\delta(\gamma) + \delta(\beta)\gamma)x + x(\beta\delta(\gamma) + \delta(\beta)\gamma) + \beta\lambda(\gamma, x) + \lambda(\beta, \gamma x). \end{aligned}$$

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Comparing the two equations, we have

$$(\delta(\beta\gamma) - \beta\delta(\gamma) - \delta(\beta)\gamma)x + x(\delta(\beta\gamma) - \beta\delta(\gamma) - \delta(\beta)\gamma) \in C.$$

Thus, $\delta(\beta\gamma) = \beta\delta(\gamma) + \delta(\beta)\gamma$ and so $\delta|_{Z(R)}\colon Z(R) \to RC$ is a derivation. Next we claim that $\delta(Z(R)) \subseteq C$. Let $0 \neq \beta \in Z(R)$ and $x \in R$. Then

$$\begin{aligned} 2\delta(\beta^2 x^2) &= 2\beta^2\delta(x^2) + x^2\delta(\beta^2) + \delta(\beta^2)x^2 + \lambda(\beta^2, x^2) \\ &= 2\beta^2(x\delta(x) + \delta(x)x + \mu(x)) + 2\beta(x^2\delta(\beta) + \delta(\beta)x^2) + \lambda(\beta^2, x^2), \end{aligned}$$

and

$$\begin{aligned} 2\delta((\beta x)^2) &= 2\beta x\delta(\beta x) + 2\delta(\beta x)\beta x + 2\mu(\beta x) \\ &= \beta x(2\beta\delta(x) + x\delta(\beta) + \delta(\beta)x + \lambda(\beta, x)) \\ &\quad +(2\beta\delta(x) + x\delta(\beta) + \delta(\beta)x + \lambda(\beta, x))\beta x + 2\mu(\beta x) \\ &= 2\beta^2(x\delta(x) + \delta(x)x) + \beta(x^2\delta(\beta^2) + \delta(\beta^2)x^2) \\ &\quad +2\beta x\delta(\beta)x + 2\beta\lambda(\beta, x)x + 2\mu(\beta x). \end{aligned}$$

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Comparing the two equations, we get

$$x\beta(x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x)) + \beta(x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x))x \in C$$

and so

$$x(x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x)) + (x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x))x \in C$$

for all $x \in R$. This means

$$[x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x), x^2] = 0$$

for all $x \in R$. Note that the proof of [6, Theorem 1] is also valid when we replace $f\colon R \to R$ by $f\colon R \to RC$. Thus, there exist $\alpha \in C$ and an additive map $\xi\colon R \to C$ such that $x\delta(\beta) - \delta(\beta)x - \lambda(\beta, x) = \alpha x + \xi(x)$. So $x(\delta(\beta) - \alpha) - \delta(\beta)x \in C$ and hence $\delta(\beta) \in C$. This proves the claim. Since C is the quotient field of $Z(R)$, $\delta|_{Z(R)}$ can be extended to a derivation from C to C. According to [13, Theorem 6] (also see [27, Theorem 4.1] and [1]), it can also be extended to a derivation $d\colon RC \to RC$. Thus, $(\delta - d)(Z(R)) = 0$. Let $J := \delta - d$, $0 \neq \beta \in Z(R)$, and $x \in R$. Then by (15) we have

$$J(\beta x) = \beta J(x) + \frac{1}{2}\lambda(\beta, x).$$

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Thus,

$$2J(\beta^2x^2) = 2\beta^2J(x^2) + \lambda(\beta^2, x^2),$$

and

$$\begin{aligned} 2J((\beta x)^2) &= 2(\beta xJ(\beta x) + J(\beta x)\beta x + \mu(\beta x)) \\ &= 2\beta x\Big(\beta J(x) + \frac{1}{2}\lambda(\beta, x)\Big) + 2\Big(\beta J(x) + \frac{1}{2}\lambda(\beta, x)\Big)\beta x + 2\mu(\beta x)) \\ &= 2\beta^2(xJ(x) + J(x)x) + 2\beta\lambda(\beta, x)x + 2\mu(\beta x). \end{aligned}$$

Comparing the two equations, we have $2\beta\lambda(\beta, x)x \in C$ and so $\lambda(\beta, x) = 0$. Therefore (i) has been proved.

(ii) Assume that $\operatorname{char} R = 2$. Let $0 \neq \beta \in Z(R)$. Then, by (15), we have $[\delta(\beta), x] \in C$ for all $x \in R$. Thus, $\delta(Z(R)) \subseteq C$. Also, by letting $y = \beta x$ in (15), we get $[\delta(\beta x) + \beta\delta(x), x] \in C$ for all $x \in R$. According to [15, Theorem 2], there exist $d\colon Z(R) \to C$ and $\psi\colon Z(R) \times R \to C$ such that

$$\delta(\beta x) = \beta\delta(x) + d(\beta)x + \psi(\beta, x) \tag{20}$$

for all $x \in R$ and $\beta \in Z(R)$. We will show that $d\colon Z(R) \to C$ is a derivation. Let

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$\beta, \gamma \in Z(R)$ and $x \in R$. Then

$$\delta((\beta\gamma)x) + \beta\gamma\delta(x) + d(\beta\gamma)x \in C,$$
$$\delta(\beta(\gamma x)) + \beta\delta(\gamma x) + d(\beta)\gamma x \in C,$$
$$\beta\delta(\gamma x) + \beta\gamma\delta(x) + \beta d(\gamma)x \in C.$$

Considering the sum of the above three equations, we have

$$(d(\beta\gamma) + \beta d(\gamma) + d(\beta)\gamma)x \in C$$

for $\beta, \gamma \in Z(R)$ and $x \in R$. This implies that $d(\beta\gamma) = \beta d(\gamma) + d(\beta)\gamma$ for all $\beta, \gamma \in Z(R)$, that is, d is a derivation. Since C is the quotient field of $Z(R)$, d can be uniquely extended to a derivation from C to itself. By [13, Theorem 6], it can be extended to a derivation from RC to itself, say the same d. Let $J := \delta + d$. Then, by (20), we have

$$J(\beta x) + \beta J(x) \in C,$$
$$J(Z(R)) \subseteq Z(R),$$

and

$$J(x^2) + [x, J(x)] \in C,$$

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for $\beta \in Z(R)$ and $x \in R$. Choose a C-subspace V of RC such that $RC = V \oplus C$ as vector spaces over C. Let $\pi_1 \colon RC \to V$ and $\pi_2 \colon RC \to C$ be the induced projections. Let $\widetilde{J} := \pi_1 \circ J$ and $\nu := \pi_2 \circ J$. Then $\widetilde{J} = \delta + d + \nu$ and

$$\widetilde{J}(\beta x) = \beta \widetilde{J}(x),$$
$$\widetilde{J}(Z(R)) = 0,$$

and

$$\widetilde{J}(x^2) + [x, \widetilde{J}(x)] \in C,$$

for $\beta \in Z(R)$ and $x \in R$, as desired. Hence (ii) is proved. $\square$

Now we assume that δ is $Z(R)$-linear and $\delta(Z(R)) = 0$. Since C is the quotient field of $Z(R)$, the map δ can be uniquely extended to a map from RC into itself (denoted by δ also) satisfying

$$\begin{cases} \delta(x^2) - x\delta(x) - \delta(x)x \in C \\ \delta(cx) = c\delta(x) \\ \delta(C) = 0 \end{cases}$$

for all $x \in RC$ and $c \in C$. Since $\delta \colon RC \to RC$ is C-linear, by the universal property of tensor product, there is a unique additive map $\overline{\delta}$ from $RC \otimes_C F$ into itself which

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maps $x \otimes \beta$ to $\delta(x) \otimes \beta$ for all $x \in RC$ and $\beta \in F$. Since $RC \otimes_C F \cong \mathrm{M}_n(F)$ where $n = \sqrt{\dim_C RC} > 1$, we can view $\overline{\delta}$ as a map from $\mathrm{M}_n(F)$ to $\mathrm{M}_n(F)$. By a direct calculation, we have

$$\begin{cases} \overline{\delta}(x^2) - x\overline{\delta}(x) - \overline{\delta}(x)x \in F \\ \overline{\delta}(cx) = c\overline{\delta}(x) \\ \overline{\delta}(F) = 0 \end{cases}$$

for all $x \in \mathrm{M}_n(F)$ and $c \in F$. Now we will determine the map $\overline{\delta}$. Let e_{ij}'s, $1 \leq i, j \leq n$, be the standard matrix units of $\mathrm{M}_n(F)$. It suffices to determine $\overline{\delta}(e_{ij})$ for all i, j because $\overline{\delta}$ is F-linear. For convenience, we set

$$\overline{\delta}(e_{ij}) = (\alpha_{kl}^{ij}) := \sum_{k,l=1}^{n} \alpha_{kl}^{ij} e_{kl}.$$

Lemma 4.12. *Let F be a field with* $\operatorname{char} F \neq 2$ *and $\overline{\delta}\colon \mathrm{M}_4(F) \to \mathrm{M}_4(F)$ be an F-linear weak Jordan derivation such that $\overline{\delta}(F) = 0$. Then $\overline{\delta}$ is an inner derivation.*

Proof. We will compute α_{kl}^{ij} by the following steps.

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Step 1. Calculate (14) for $x = e_{ii}$, $i = 1, ..., 4$. For $x = e_{11}$,

$$\overline{\delta}(e_{11}^2) - e_{11}\overline{\delta}(e_{11}) - \overline{\delta}(e_{11})e_{11} = \begin{pmatrix} -\alpha_{11}^{11} & 0 & 0 & 0 \\ 0 & \alpha_{22}^{11} & \alpha_{23}^{11} & \alpha_{24}^{11} \\ 0 & \alpha_{32}^{11} & \alpha_{33}^{11} & \alpha_{34}^{11} \\ 0 & \alpha_{42}^{11} & \alpha_{43}^{11} & \alpha_{44}^{11} \end{pmatrix} \in F$$

implies

$$\alpha_{22}^{11} = \alpha_{33}^{11} = \alpha_{44}^{11} = -\alpha_{11}^{11}$$

and

$$\alpha_{23}^{11} = \alpha_{24}^{11} = \alpha_{32}^{11} = \alpha_{34}^{11} = \alpha_{42}^{11} = \alpha_{43}^{11} = 0.$$

So

$$\overline{\delta}(e_{11}) = \begin{pmatrix} \alpha_{11}^{11} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{21}^{11} & -\alpha_{11}^{11} & 0 & 0 \\ \alpha_{31}^{11} & 0 & -\alpha_{11}^{11} & 0 \\ \alpha_{41}^{11} & 0 & 0 & -\alpha_{11}^{11} \end{pmatrix}.$$

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Similarly, we have

$$\overline{\delta}(e_{22}) = \begin{pmatrix} \alpha_{11}^{22} & \alpha_{12}^{22} & 0 & 0 \\ \alpha_{21}^{22} & -\alpha_{11}^{22} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{32}^{22} & \alpha_{11}^{11} & 0 \\ 0 & \alpha_{42}^{22} & 0 & \alpha_{11}^{22} \end{pmatrix},$$

$$\overline{\delta}(e_{33}) = \begin{pmatrix} \alpha_{11}^{33} & 0 & \alpha_{13}^{33} & 0 \\ 0 & \alpha_{11}^{33} & \alpha_{23}^{33} & 0 \\ \alpha_{31}^{33} & \alpha_{32}^{33} & -\alpha_{11}^{33} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{43}^{33} & \alpha_{11}^{33} \end{pmatrix},$$

and

$$\overline{\delta}(e_{44}) = \begin{pmatrix} \alpha_{11}^{44} & 0 & 0 & \alpha_{14}^{44} \\ 0 & \alpha_{11}^{44} & 0 & \alpha_{24}^{44} \\ 0 & 0 & \alpha_{11}^{44} & \alpha_{34}^{44} \\ \alpha_{41}^{44} & \alpha_{42}^{44} & \alpha_{43}^{44} & -\alpha_{11}^{44} \end{pmatrix}.$$

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Step 2. Since $\overline{\delta}(I_4) = 0$, we have

$$\alpha_{11}^{11} = \alpha_{11}^{22} = \alpha_{11}^{33} = \alpha_{11}^{44} = 0,$$

$$\alpha_{41}^{44} = -\alpha_{41}^{11},\ \alpha_{42}^{44} = -\alpha_{42}^{22},\ \alpha_{43}^{44} = -\alpha_{43}^{33},$$

$$\alpha_{14}^{44} = -\alpha_{14}^{11},\ \alpha_{24}^{44} = -\alpha_{24}^{22},\ \alpha_{34}^{44} = -\alpha_{34}^{33},$$

$$\alpha_{21}^{22} = -\alpha_{21}^{11},\ \alpha_{12}^{22} = -\alpha_{12}^{11},$$

$$\alpha_{31}^{33} = -\alpha_{31}^{11},\ \alpha_{13}^{33} = -\alpha_{13}^{11},$$

$$\alpha_{23}^{33} = -\alpha_{23}^{22},\ \alpha_{32}^{33} = -\alpha_{32}^{22}.$$

Thus,

$$\overline{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{21}^{11} & 0 & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 & 0 \\ \alpha_{41}^{11} & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{22}) = \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 & 0 \\ -\alpha_{21}^{11} & 0 & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{32}^{22} & 0 & 0 \\ 0 & \alpha_{42}^{22} & 0 & 0 \end{pmatrix},$$

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$$\overline{\delta}(e_{33}) = \begin{pmatrix} 0 & 0 & -\alpha_{13}^{11} & 0 \\ 0 & 0 & -\alpha_{23}^{22} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & 0 & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{43}^{33} & 0 \end{pmatrix},$$

and

$$\overline{\delta}(e_{44}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{14}^{11} \\ 0 & 0 & 0 & -\alpha_{24}^{22} \\ 0 & 0 & 0 & -\alpha_{34}^{33} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & 0 \end{pmatrix}.$$

Step 3. For each $x = e_{ij}$, $i \neq j$, calculate (14), and then calculate (15) for $y = e_{kk}$, $k = 1, 2, 3$. For example, by (14), we have

$$e_{12}\overline{\delta}(e_{12}) + \overline{\delta}(e_{12})e_{12} = \begin{pmatrix} \alpha_{21}^{12} & \alpha_{22}^{12} & \alpha_{23}^{12} & \alpha_{24}^{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{11}^{12} & 0 & 0 \\ 0 & \alpha_{21}^{12} & 0 & 0 \\ 0 & \alpha_{31}^{12} & 0 & 0 \\ 0 & \alpha_{41}^{12} & 0 & 0 \end{pmatrix} \in F.$$

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Thus,

$$\alpha_{21}^{12} = \alpha_{23}^{12} = \alpha_{24}^{12} = \alpha_{31}^{12} = \alpha_{41}^{12} = 0,$$

$$\alpha_{22}^{12} = -\alpha_{11}^{12},$$

and so

$$\overline{\delta}(e_{12}) = \begin{pmatrix} \alpha_{11}^{12} & \alpha_{12}^{12} & \alpha_{13}^{12} & \alpha_{14}^{12} \\ 0 & -\alpha_{11}^{12} & 0 & 0 \\ 0 & \alpha_{32}^{12} & \alpha_{33}^{12} & \alpha_{34}^{12} \\ 0 & \alpha_{42}^{12} & \alpha_{43}^{12} & \alpha_{44}^{12} \end{pmatrix}.$$

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Let $x = e_{12}$ and $y = e_{33}$. By (15), we get

$$e_{12}\overline{\delta}(e_{33}) + \overline{\delta}(e_{33})e_{12} + e_{33}\overline{\delta}(e_{12}) + \overline{\delta}(e_{12})e_{33}$$

$$= \begin{pmatrix} 0 & 0 & -\alpha_{23}^{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\alpha_{31}^{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_{32}^{12} & \alpha_{33}^{12} & \alpha_{34}^{12} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \alpha_{13}^{12} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33}^{12} & 0 \\ 0 & 0 & \alpha_{43}^{12} & 0 \end{pmatrix} \in F,$$

and so

$$\alpha_{13}^{12} = \alpha_{23}^{22},$$

$$\alpha_{32}^{12} = \alpha_{31}^{11},$$

$$\alpha_{33}^{12} = \alpha_{34}^{12} = \alpha_{43}^{12} = 0.$$

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These give

$$\overline{\delta}(e_{12}) = \begin{pmatrix} \alpha_{11}^{12} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{14}^{12} \\ 0 & -\alpha_{11}^{12} & 0 & 0 \\ 0 & \alpha_{31}^{11} & 0 & 0 \\ 0 & \alpha_{42}^{12} & 0 & \alpha_{44}^{12} \end{pmatrix}.$$

Let $x = e_{12}$ and $y = e_{22}$, and compute (15). Then

$$\alpha_{11}^{12} = -\alpha_{21}^{11},$$

$$\alpha_{44}^{12} = 0,$$

$$\alpha_{14}^{12} = \alpha_{24}^{22}.$$

These give

$$\overline{\delta}(e_{12}) = \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{21}^{11} & 0 & 0 \\ 0 & \alpha_{31}^{11} & 0 & 0 \\ 0 & \alpha_{42}^{12} & 0 & 0 \end{pmatrix}.$$

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Again, let $x = e_{12}$ and $y = e_{11}$, and compute (15). Then $\alpha_{42}^{12} = \alpha_{41}^{11}$ and so

$$\overline{\delta}(e_{12}) = \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{21}^{11} & 0 & 0 \\ 0 & \alpha_{31}^{11} & 0 & 0 \\ 0 & \alpha_{41}^{11} & 0 & 0 \end{pmatrix}.$$

Similarly, we have

$$\overline{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 & 0 & 0 \\ \alpha_{21}^{21} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{32}^{22} & 0 & 0 & 0 \\ \alpha_{42}^{22} & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{13}) = \begin{pmatrix} -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{13}^{13} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{21}^{11} & 0 \\ 0 & 0 & \alpha_{31}^{11} & 0 \\ 0 & 0 & \alpha_{41}^{11} & 0 \end{pmatrix}, \overline{\delta}(e_{31}) = \begin{pmatrix} -\alpha_{13}^{11} & 0 & 0 & 0 \\ -\alpha_{23}^{22} & 0 & 0 & 0 \\ \alpha_{31}^{31} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{43}^{33} & 0 & 0 & 0 \end{pmatrix},$$

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$$\overline{\delta}(e_{14}) = \begin{pmatrix} -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & \alpha_{14}^{14} \\ 0 & 0 & 0 & \alpha_{21}^{11} \\ 0 & 0 & 0 & \alpha_{31}^{11} \\ 0 & 0 & 0 & \alpha_{41}^{11} \end{pmatrix}, \overline{\delta}(e_{41}) = \begin{pmatrix} -\alpha_{14}^{11} & 0 & 0 & 0 \\ -\alpha_{24}^{22} & 0 & 0 & 0 \\ -\alpha_{34}^{33} & 0 & 0 & 0 \\ \alpha_{41}^{41} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \end{pmatrix},$$

$$\overline{\delta}(e_{23}) = \begin{pmatrix} 0 & 0 & -\alpha_{12}^{11} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{23}^{23} & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{32}^{22} & 0 \\ 0 & 0 & \alpha_{42}^{22} & 0 \end{pmatrix}, \overline{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{32}^{32} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{24}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{12}^{11} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & \alpha_{24}^{24} \\ 0 & 0 & 0 & \alpha_{32}^{22} \\ 0 & 0 & 0 & \alpha_{42}^{22} \end{pmatrix}, \overline{\delta}(e_{42}) = \begin{pmatrix} 0 & -\alpha_{14}^{11} & 0 & 0 \\ 0 & -\alpha_{24}^{22} & 0 & 0 \\ 0 & -\alpha_{34}^{33} & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{42}^{42} & \alpha_{23}^{22} & \alpha_{24}^{22} \end{pmatrix},$$

$$\overline{\delta}(e_{34}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{13}^{11} \\ 0 & 0 & 0 & -\alpha_{23}^{22} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & \alpha_{34}^{34} \\ 0 & 0 & 0 & \alpha_{43}^{33} \end{pmatrix}, \overline{\delta}(e_{43}) = \begin{pmatrix} 0 & 0 & -\alpha_{14}^{11} & 0 \\ 0 & 0 & -\alpha_{24}^{22} & 0 \\ 0 & 0 & -\alpha_{34}^{33} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{43}^{43} & \alpha_{34}^{33} \end{pmatrix}.$$

Step 4. Calculate (15) for $x = e_{ij}$ and $y = e_{ji}$, $i < j$. For example, if $x = e_{12}$ and

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$y = e_{21}$, then

$$\overline{\delta}(e_{11}+e_{22}) - e_{12}\overline{\delta}(e_{21}) - \overline{\delta}(e_{21})e_{12} - e_{21}\overline{\delta}(e_{12}) - \overline{\delta}(e_{12})e_{21}$$

$$= \begin{pmatrix} 0 & 0 & \alpha_{13}^{11} & \alpha_{14}^{11} \\ 0 & 0 & \alpha_{23}^{22} & \alpha_{24}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & 0 & 0 \\ \alpha_{41}^{11} & \alpha_{42}^{22} & 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_{21}^{21} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 & 0 \\ 0 & \alpha_{21}^{21} & 0 & 0 \\ 0 & \alpha_{32}^{22} & 0 & 0 \\ 0 & \alpha_{42}^{22} & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_{12}^{12} & 0 & 0 & 0 \\ \alpha_{21}^{11} & 0 & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 & 0 \\ \alpha_{41}^{11} & 0 & 0 & 0 \end{pmatrix} \in F$$

giving $\alpha_{21}^{21} = -\alpha_{12}^{12}$. Hence, we get a new

$$\overline{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 & 0 & 0 \\ -\alpha_{12}^{12} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{32}^{22} & 0 & 0 & 0 \\ \alpha_{42}^{22} & 0 & 0 & 0 \end{pmatrix}.$$

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Similarly, we have new

$$\overline{\delta}(e_{31}) = \begin{pmatrix} -\alpha_{13}^{11} & 0 & 0 & 0 \\ -\alpha_{23}^{22} & 0 & 0 & 0 \\ -\alpha_{13}^{13} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{43}^{33} & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{41}) = \begin{pmatrix} -\alpha_{14}^{11} & 0 & 0 & 0 \\ -\alpha_{24}^{22} & 0 & 0 & 0 \\ -\alpha_{34}^{33} & 0 & 0 & 0 \\ -\alpha_{14}^{14} & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \end{pmatrix},$$

$$\overline{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{21}^{11} & -\alpha_{23}^{23} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{42}) = \begin{pmatrix} 0 & -\alpha_{14}^{11} & 0 & 0 \\ 0 & -\alpha_{24}^{22} & 0 & 0 \\ 0 & -\alpha_{34}^{33} & 0 & 0 \\ -\alpha_{21}^{11} & -\alpha_{24}^{24} & \alpha_{23}^{22} & \alpha_{24}^{22} \end{pmatrix},$$

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and

$$\overline{\delta}(e_{43}) = \begin{pmatrix} 0 & 0 & -\alpha_{14}^{11} & 0 \\ 0 & 0 & -\alpha_{24}^{22} & 0 \\ 0 & 0 & -\alpha_{34}^{33} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & -\alpha_{34}^{34} & \alpha_{34}^{33} \end{pmatrix}.$$

Step 5. Calculate (15) for $(x, y) \in \{(e_{12}, e_{31}), (e_{12}, e_{41}), (e_{13}, e_{41})\}$. For example, if $(x, y) = (e_{12}, e_{31})$, then (15) gives

$$\overline{\delta}(e_{32}) - e_{12}\overline{\delta}(e_{31}) - \overline{\delta}(e_{31})e_{12} - e_{31}\overline{\delta}(e_{12}) - \overline{\delta}(e_{12})e_{31}$$

$$= \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{21}^{11} & -\alpha_{23}^{23} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix} - \begin{pmatrix} -\alpha_{23}^{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 & 0 \\ 0 & -\alpha_{23}^{22} & 0 & 0 \\ 0 & -\alpha_{13}^{13} & 0 & 0 \\ 0 & \alpha_{43}^{33} & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_{23}^{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in F.$$

Thus, we have $\alpha_{23}^{23} = -\alpha_{12}^{12} + \alpha_{13}^{13}$. Similarly, $\alpha_{24}^{24} = -\alpha_{12}^{12} + \alpha_{14}^{14}$ and $\alpha_{34}^{34} = -\alpha_{13}^{13} + \alpha_{14}^{14}$.

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Step 6. As a result, we get

$$\overline{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} & \alpha_{14}^{11} \\ \alpha_{21}^{11} & 0 & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 & 0 \\ \alpha_{41}^{11} & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{22}) = \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 & 0 \\ -\alpha_{21}^{11} & 0 & \alpha_{23}^{22} & \alpha_{24}^{22} \\ 0 & \alpha_{32}^{22} & 0 & 0 \\ 0 & \alpha_{42}^{22} & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{33}) = \begin{pmatrix} 0 & 0 & -\alpha_{13}^{11} & 0 \\ 0 & 0 & -\alpha_{23}^{22} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & 0 & \alpha_{34}^{33} \\ 0 & 0 & \alpha_{43}^{33} & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{44}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{14}^{11} \\ 0 & 0 & 0 & -\alpha_{24}^{22} \\ 0 & 0 & 0 & -\alpha_{34}^{33} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & 0 \end{pmatrix},$$

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$$\overline{\delta}(e_{12}) = \begin{pmatrix} -\alpha^{11}_{21} & \alpha^{12}_{12} & \alpha^{22}_{23} & \alpha^{22}_{24} \\ 0 & \alpha^{11}_{21} & 0 & 0 \\ 0 & \alpha^{11}_{31} & 0 & 0 \\ 0 & \alpha^{11}_{41} & 0 & 0 \end{pmatrix}, \ \overline{\delta}(e_{21}) = \begin{pmatrix} -\alpha^{11}_{12} & 0 & 0 & 0 \\ -\alpha^{12}_{12} & \alpha^{11}_{12} & \alpha^{11}_{13} & \alpha^{11}_{14} \\ \alpha^{22}_{32} & 0 & 0 & 0 \\ \alpha^{22}_{42} & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{13}) = \begin{pmatrix} -\alpha^{11}_{31} & -\alpha^{22}_{32} & \alpha^{13}_{13} & \alpha^{33}_{34} \\ 0 & 0 & \alpha^{11}_{21} & 0 \\ 0 & 0 & \alpha^{11}_{31} & 0 \\ 0 & 0 & \alpha^{11}_{41} & 0 \end{pmatrix}, \ \overline{\delta}(e_{31}) = \begin{pmatrix} -\alpha^{11}_{13} & 0 & 0 & 0 \\ -\alpha^{22}_{23} & 0 & 0 & 0 \\ -\alpha^{13}_{13} & \alpha^{11}_{12} & \alpha^{11}_{13} & \alpha^{11}_{14} \\ \alpha^{33}_{43} & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{14}) = \begin{pmatrix} -\alpha^{11}_{41} & -\alpha^{22}_{42} & -\alpha^{33}_{43} & \alpha^{14}_{14} \\ 0 & 0 & 0 & \alpha^{11}_{21} \\ 0 & 0 & 0 & \alpha^{11}_{31} \\ 0 & 0 & 0 & \alpha^{11}_{41} \end{pmatrix}, \ \overline{\delta}(e_{41}) = \begin{pmatrix} -\alpha^{11}_{14} & 0 & 0 & 0 \\ -\alpha^{22}_{24} & 0 & 0 & 0 \\ -\alpha^{33}_{34} & 0 & 0 & 0 \\ -\alpha^{14}_{14} & \alpha^{11}_{12} & \alpha^{11}_{13} & \alpha^{11}_{14} \end{pmatrix},$$

$$\overline{\delta}(e_{23}) = \begin{pmatrix} 0 & 0 & -\alpha^{11}_{12} & 0 \\ -\alpha^{11}_{31} & -\alpha^{22}_{32} & -\alpha^{12}_{12} + \alpha^{13}_{13} & \alpha^{33}_{34} \\ 0 & 0 & \alpha^{22}_{32} & 0 \\ 0 & 0 & \alpha^{22}_{42} & 0 \end{pmatrix}, \ \overline{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha^{11}_{13} & 0 & 0 \\ 0 & -\alpha^{22}_{23} & 0 & 0 \\ -\alpha^{11}_{21} & \alpha^{12}_{12} - \alpha^{13}_{13} & \alpha^{22}_{23} & \alpha^{22}_{24} \\ 0 & \alpha^{33}_{43} & 0 & 0 \end{pmatrix},$$

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$$\overline{\delta}(e_{24}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{12}^{11} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & -\alpha_{12}^{12}+\alpha_{14}^{14} \\ 0 & 0 & 0 & \alpha_{32}^{22} \\ 0 & 0 & 0 & \alpha_{42}^{22} \end{pmatrix}, \ \overline{\delta}(e_{42}) = \begin{pmatrix} 0 & -\alpha_{14}^{11} & 0 & 0 \\ 0 & -\alpha_{24}^{22} & 0 & 0 \\ 0 & -\alpha_{34}^{33} & 0 & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12}-\alpha_{14}^{14} & \alpha_{23}^{22} & \alpha_{24}^{22} \end{pmatrix},$$

$$\overline{\delta}(e_{34}) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{13}^{11} \\ 0 & 0 & 0 & -\alpha_{23}^{22} \\ -\alpha_{41}^{11} & -\alpha_{42}^{22} & -\alpha_{43}^{33} & -\alpha_{13}^{13}+\alpha_{14}^{14} \\ 0 & 0 & 0 & \alpha_{43}^{33} \end{pmatrix}, \ \overline{\delta}(e_{43}) = \begin{pmatrix} 0 & 0 & -\alpha_{14}^{11} & 0 \\ 0 & 0 & -\alpha_{24}^{22} & 0 \\ 0 & 0 & -\alpha_{34}^{33} & 0 \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{13}^{13}-\alpha_{14}^{14} & \alpha_{34}^{33} \end{pmatrix}.$$

Now, let

$$a = \begin{pmatrix} 0 & -\alpha_{12}^{11} & -\alpha_{13}^{11} & -\alpha_{14}^{11} \\ \alpha_{21}^{11} & -\alpha_{12}^{12} & -\alpha_{23}^{22} & -\alpha_{24}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & -\alpha_{13}^{13} & -\alpha_{34}^{33} \\ \alpha_{41}^{11} & \alpha_{42}^{22} & \alpha_{43}^{33} & -\alpha_{14}^{14} \end{pmatrix} \in \mathrm{M}_4(F).$$

Therefore, by a direct computation, we have $\overline{\delta}(x) = [a, x]$, for all $x \in \mathrm{M}_4(F)$, as desired. □

Lemma 4.13. *Let F be a field and $\overline{\delta}\colon \mathrm{M}_3(F) \to \mathrm{M}_3(F)$ an F-linear weak Jordan derivation such that $\overline{\delta}(F) = 0$.*

(i) *If* $\operatorname{char} F \neq 2$, *then $\overline{\delta}$ is an inner derivation.*

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(ii) *If* char $F = 2$*, then there exist an inner derivation* $\overline{d}\colon \mathrm{M}_3(F) \to \mathrm{M}_3(F)$ *and an* F*-linear map* $\overline{\nu}\colon \mathrm{M}_3(F) \to F$ *such that* $\overline{\delta} = \overline{d} + \overline{\nu}$*.*

Proof. For (i), assume that char $F \neq 2$. Then, by following the same process given in the proof of Lemma 4.12, we have

$$\overline{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{21}^{11} & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{22}) = \begin{pmatrix} 0 & -\alpha_{12}^{11} & 0 \\ -\alpha_{21}^{11} & 0 & \alpha_{23}^{22} \\ 0 & \alpha_{32}^{22} & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{33}) = \begin{pmatrix} 0 & 0 & -\alpha_{13}^{11} \\ 0 & 0 & -\alpha_{23}^{22} \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & 0 \end{pmatrix},$$

$$\overline{\delta}(e_{12}) = \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} \\ 0 & \alpha_{21}^{11} & 0 \\ 0 & \alpha_{31}^{11} & 0 \end{pmatrix}, \ \overline{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 & 0 \\ -\alpha_{12}^{12} & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{32}^{22} & 0 & 0 \end{pmatrix},$$

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$$\overline{\delta}(e_{13}) = \begin{pmatrix} -\alpha_{31}^{11} & -\alpha_{32}^{22} & \alpha_{13}^{13} \\ 0 & 0 & \alpha_{21}^{11} \\ 0 & 0 & \alpha_{31}^{11} \end{pmatrix}, \ \overline{\delta}(e_{31}) = \begin{pmatrix} -\alpha_{13}^{11} & 0 & 0 \\ -\alpha_{23}^{22} & 0 & 0 \\ -\alpha_{13}^{13} & \alpha_{12}^{11} & \alpha_{13}^{11} \end{pmatrix},$$

$$\overline{\delta}(e_{23}) = \begin{pmatrix} 0 & 0 & -\alpha_{12}^{11} \\ -\alpha_{31}^{11} & -\alpha_{32}^{22} & -\alpha_{12}^{12} + \alpha_{13}^{13} \\ 0 & 0 & \alpha_{32}^{22} \end{pmatrix}, \ \overline{\delta}(e_{32}) = \begin{pmatrix} 0 & -\alpha_{13}^{11} & 0 \\ 0 & -\alpha_{23}^{22} & 0 \\ -\alpha_{21}^{11} & \alpha_{12}^{12} - \alpha_{13}^{13} & \alpha_{23}^{22} \end{pmatrix}.$$

Let

$$a = \begin{pmatrix} 0 & -\alpha_{12}^{11} & -\alpha_{13}^{11} \\ \alpha_{21}^{11} & -\alpha_{12}^{12} & -\alpha_{23}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & -\alpha_{13}^{13} \end{pmatrix} \in \mathrm{M}_3(F).$$

Then $\overline{\delta}(x) = [a, x]$, for all $x \in \mathrm{M}_3(F)$, as desired.

For (ii), assume that $\operatorname{char} F = 2$. Then, by following the same process given in the proof of Lemma 4.12, we have

$$\overline{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{21}^{11} & 0 & 0 \\ \alpha_{31}^{11} & 0 & 0 \end{pmatrix} + \alpha_{11}^{11} I_3,$$

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$$\overline{\delta}(e_{22}) = \begin{pmatrix} 0 & \alpha_{12}^{11} & 0 \\ \alpha_{21}^{11} & 0 & \alpha_{23}^{22} \\ 0 & \alpha_{32}^{22} & 0 \end{pmatrix} + \alpha_{11}^{22} I_3,$$

$$\overline{\delta}(e_{33}) = \begin{pmatrix} 0 & 0 & \alpha_{13}^{11} \\ 0 & 0 & \alpha_{23}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & 0 \end{pmatrix} + (\alpha_{11}^{11} + \alpha_{11}^{22}) I_3,$$

$$\overline{\delta}(e_{12}) = \begin{pmatrix} \alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} \\ 0 & \alpha_{21}^{11} & 0 \\ 0 & \alpha_{31}^{11} & 0 \end{pmatrix} + (\alpha_{11}^{12} + \alpha_{21}^{11}) I_3,$$

$$\overline{\delta}(e_{21}) = \begin{pmatrix} \alpha_{12}^{11} & 0 & 0 \\ \alpha_{12}^{12} & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{32}^{22} & 0 & 0 \end{pmatrix} + (\alpha_{11}^{21} + \alpha_{12}^{11}) I_3,$$

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$$\overline{\delta}(e_{13}) = \begin{pmatrix} \alpha_{31}^{11} & \alpha_{32}^{22} & \alpha_{13}^{13} \\ 0 & 0 & \alpha_{21}^{11} \\ 0 & 0 & \alpha_{31}^{11} \end{pmatrix} + (\alpha_{11}^{13} + \alpha_{31}^{11}) I_3,$$

$$\overline{\delta}(e_{31}) = \begin{pmatrix} \alpha_{13}^{11} & 0 & 0 \\ \alpha_{23}^{22} & 0 & 0 \\ \alpha_{13}^{13} & \alpha_{12}^{11} & \alpha_{13}^{11} \end{pmatrix} + (\alpha_{11}^{31} + \alpha_{13}^{11}) I_3.$$

$$\overline{\delta}(e_{23}) = \begin{pmatrix} 0 & 0 & \alpha_{12}^{11} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & \alpha_{12}^{12} + \alpha_{13}^{13} \\ 0 & 0 & \alpha_{32}^{22} \end{pmatrix} + (\alpha_{22}^{23} + \alpha_{32}^{22}) I_3,$$

$$\overline{\delta}(e_{32}) = \begin{pmatrix} 0 & \alpha_{13}^{11} & 0 \\ 0 & \alpha_{23}^{22} & 0 \\ \alpha_{21}^{11} & \alpha_{12}^{12} + \alpha_{13}^{13} & \alpha_{23}^{22} \end{pmatrix} + (\alpha_{22}^{32} + \alpha_{23}^{22}) I_3.$$

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Let

$$a = \begin{pmatrix} 0 & \alpha_{12}^{11} & \alpha_{13}^{11} \\ \alpha_{21}^{11} & \alpha_{12}^{12} & \alpha_{23}^{22} \\ \alpha_{31}^{11} & \alpha_{32}^{22} & \alpha_{13}^{13} \end{pmatrix} \in \mathrm{M}_3(F).$$

Then $\overline{\delta} = \overline{d} + \overline{\nu}$, where $\overline{d}(x) = [a, x]$ and

$$\begin{aligned} \overline{\nu}(x) \ = \ & \big(\alpha_{11}^{11}x_{11} + \alpha_{11}^{22}x_{22} + (\alpha_{11}^{11} + \alpha_{11}^{22})x_{33} + (\alpha_{11}^{12} + \alpha_{21}^{11})x_{12} + (\alpha_{11}^{21} + \alpha_{12}^{11})x_{21} \\ & + (\alpha_{11}^{13} + \alpha_{31}^{11})x_{13} + (\alpha_{11}^{31} + \alpha_{13}^{11})x_{31} + (\alpha_{22}^{23} + \alpha_{32}^{22})x_{23} + (\alpha_{22}^{32} + \alpha_{23}^{22})x_{32}\big) I_3, \end{aligned}$$

for all $x = (x_{ij}) \in \mathrm{M}_3(F)$, as asserted. □

Lemma 4.14. *Let F be a field and $\overline{\delta}\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ be an F-linear weak Jordan derivation such that $\overline{\delta}(F) = 0$. Then $\overline{\delta} = \overline{d} + L + \zeta$ for some inner derivation $\overline{d}\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ and F-linear maps $L, \zeta\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$. The F-linear maps L and ζ are of the forms*

$$L(x) = \begin{pmatrix} 0 & \beta_4 x_{21} \\ \beta_5 x_{12} + \beta_6 x_{21} & 0 \end{pmatrix}$$

and

$$\zeta(x) = (\beta_1(x_{11} - x_{22}) + \beta_2 x_{12} + \beta_3 x_{21}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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for all $x = (x_{ij}) \in \mathrm{M}_2(F)$, *where*

$$\beta_1 = \alpha_{11}^{11},\ \beta_2 = \alpha_{11}^{12} + \alpha_{21}^{11},\ \beta_3 = \alpha_{11}^{21} + \alpha_{12}^{11},$$

$$\beta_4 = \alpha_{12}^{21},\ \beta_5 = \alpha_{21}^{12},\ \beta_6 = \alpha_{12}^{12} + \alpha_{21}^{21}.$$

Moreover, $\overline{\delta}$ *is a derivation if and only if* $\beta_i = 0$ *for all* $i = 1, ..., 6$.

Proof. We follow the same process step by step as given in the proof of Lemma 4.12.

Step 1. Calculate (14) for $x = e_{ii}$, $i = 1, 2$. For $x = e_{11}$,

$$\overline{\delta}(e_{11}^2) - e_{11}\overline{\delta}(e_{11}) - \overline{\delta}(e_{11})e_{11} = \begin{pmatrix} -\alpha_{11}^{11} & 0 \\ 0 & \alpha_{22}^{11} \end{pmatrix} \in F$$

implies $-\alpha_{11}^{11} = \alpha_{22}^{11}$. Similarly, for $x = e_{22}$, we have $-\alpha_{11}^{22} = \alpha_{22}^{22}$. Thus

$$\overline{\delta}(e_{11}) = \begin{pmatrix} \alpha_{11}^{11} & \alpha_{12}^{11} \\ \alpha_{21}^{11} & -\alpha_{11}^{11} \end{pmatrix},\quad \overline{\delta}(e_{22}) = \begin{pmatrix} \alpha_{11}^{22} & \alpha_{12}^{22} \\ \alpha_{21}^{22} & -\alpha_{11}^{22} \end{pmatrix}.$$

Step 2. Since $\overline{\delta}(I_2) = 0$, we have $\alpha_{11}^{11} + \alpha_{11}^{22} = 0$, $\alpha_{12}^{11} + \alpha_{12}^{22} = 0$, and $\alpha_{21}^{11} + \alpha_{21}^{22} = 0$.

From these, we have

$$\overline{\delta}(e_{22}) = \begin{pmatrix} -\alpha_{11}^{11} & -\alpha_{12}^{11} \\ -\alpha_{21}^{11} & \alpha_{11}^{11} \end{pmatrix}.$$

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Step 3. For each $x = e_{ij}$, $i \neq j$, calculate (14), and then calculate (15) for $y = e_{11}$.

For $x = e_{12}$,

$$e_{12}\overline{\delta}(e_{12}) + \overline{\delta}(e_{12})e_{12} = \begin{pmatrix} \alpha_{21}^{12} & \alpha_{22}^{12} + \alpha_{11}^{12} \\ 0 & \alpha_{21}^{12} \end{pmatrix} \in F$$

implies $\alpha_{22}^{12} = -\alpha_{11}^{12}$. Thus

$$\overline{\delta}(e_{12}) = \begin{pmatrix} \alpha_{11}^{12} & \alpha_{12}^{12} \\ \alpha_{21}^{12} & -\alpha_{11}^{12} \end{pmatrix}.$$

Similarly, for $x = e_{21}$, we have $\alpha_{22}^{21} = -\alpha_{11}^{21}$ and so

$$\overline{\delta}(e_{21}) = \begin{pmatrix} \alpha_{11}^{21} & \alpha_{12}^{21} \\ \alpha_{21}^{21} & -\alpha_{11}^{21} \end{pmatrix}.$$

Note that

$$\overline{\delta}(e_{12}) - e_{12}\overline{\delta}(e_{11}) - \overline{\delta}(e_{11})e_{12} - e_{11}\overline{\delta}(e_{12}) - \overline{\delta}(e_{12})e_{11} = -(\alpha_{11}^{12} + \alpha_{21}^{11})I_2$$

and

$$\overline{\delta}(e_{21}) - e_{21}\overline{\delta}(e_{11}) - \overline{\delta}(e_{11})e_{21} - e_{11}\overline{\delta}(e_{21}) - \overline{\delta}(e_{21})e_{11} = -(\alpha_{11}^{21} + \alpha_{12}^{11})I_2.$$

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Step 4. Calculate (15) for $x = e_{12}$ and $y = e_{21}$. We have

$$e_{12}\overline{\delta}(e_{21}) + \overline{\delta}(e_{21})e_{12} + e_{21}\overline{\delta}(e_{12}) + \overline{\delta}(e_{12})e_{21} = (\alpha_{12}^{12} + \alpha_{21}^{21})I_2.$$

Step 5. In $\mathrm{M}_2(F)$ case, this step does not exist.

Step 6. As a result, we get

$$\overline{\delta}(e_{11}) = \begin{pmatrix} 0 & \alpha_{12}^{11} \\ \alpha_{21}^{11} & 0 \end{pmatrix} + \alpha_{11}^{11} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\overline{\delta}(e_{22}) = \begin{pmatrix} 0 & -\alpha_{12}^{11} \\ -\alpha_{21}^{11} & 0 \end{pmatrix} + (-\alpha_{11}^{11}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\overline{\delta}(e_{12}) = \begin{pmatrix} -\alpha_{21}^{11} & \alpha_{12}^{12} \\ 0 & \alpha_{21}^{11} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_{21}^{12} & 0 \end{pmatrix} + (\alpha_{11}^{12} + \alpha_{21}^{11}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\overline{\delta}(e_{21}) = \begin{pmatrix} -\alpha_{12}^{11} & 0 \\ -\alpha_{12}^{12} & \alpha_{12}^{11} \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{12}^{21} \\ \alpha_{12}^{12} + \alpha_{21}^{21} & 0 \end{pmatrix} + (\alpha_{11}^{21} + \alpha_{12}^{11}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Let

$$a = \begin{pmatrix} 0 & -\alpha_{12}^{11} \\ \alpha_{21}^{11} & -\alpha_{12}^{12} \end{pmatrix} \in \mathrm{M}_2(F)$$

and $\overline{d}(x) = [a, x]$ for $x \in \mathrm{M}_2(F)$. Then $\overline{\delta} = \overline{d} + L + \zeta$, where L and ζ are of the forms as in the description of this Lemma.

Finally we show the last statement. If $\beta_i = 0$ for all $i = 1, ..., 6$, then $\overline{\delta} = \overline{d}$ is a derivation. Conversely, assume that $\overline{\delta}$ is a derivation. In particular, it is a Jordan derivation. Note that $\overline{\delta}$ is a Jordan derivation if and only if

$$\overline{\delta}(x^2) - x\overline{\delta}(x) - \overline{\delta}(x)x = 0$$

and

$$\overline{\delta}(xy + yx) - x\overline{\delta}(y) - y\overline{\delta}(x) - \overline{\delta}(x)y - \overline{\delta}(y)x = 0$$

for all $x, y \in \{e_{ij}\}_{i,j=1,2}$. From the above computations, we see that $\beta_i = 0$ for all $i = 1, ..., 6$. Hence the last statement holds. □

Proof of Theorem 4.5. Recall that, by Proposition 4.10, we can assume that $\dim_C RC \leq 16$ if char $R \neq 2$, and $\dim_C RC \leq 9$ if char $R = 2$. Also, it follows from Lemma 4.11 that we can assume that δ is $Z(R)$-linear and $\delta(Z(R)) = 0$, and let $\overline{\delta}\colon \mathrm{M}_n(F) \to \mathrm{M}_n(F)$ be an extension of δ as above, where $n = \sqrt{\dim_C RC} > 2$. For (i), suppose that $\dim_C RC \leq 16$

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and char $R \neq 2$. Then, by Lemma 4.12 and Lemma 4.13, $\overline{\delta}$ is a derivation, and so is δ. Hence (i) holds.

Now we turn to prove (ii). Assume that $\dim_C RC = 9$ and char $R = 2$. It follows from Lemma 4.13 that there exist an $a \in RC \otimes_C F$ and an F-linear map $\overline{\nu}\colon RC \otimes_C F \to F$ such that $\overline{\delta}(x) = [a, x] + \overline{\nu}(x)$ for all $x \in RC \otimes_C F$. If $F = C$, then $a \in RC$ and $\overline{\nu}$ maps R into C, as asserted. Suppose that $\dim_C F = 3$. Let $\{1, w_2, w_3\}$ be a basis of F over C. Then

$$a = a_1 \otimes 1 + a_2 \otimes w_2 + a_3 \otimes w_3,$$

for some $a_1, a_2, a_3 \in RC$. Thus,

$$[a, x] = [a_1, x] \otimes 1 + [a_2, x] \otimes w_2 + [a_3, x] \otimes w_3$$

for all $x \in R$. Also, we write

$$\overline{\nu}(x) = \nu_1(x) \otimes 1 + \nu_2(x) \otimes w_2 + \nu_3(x) \otimes w_3$$

for all $x \in R$, where $\nu_i : RC \to C$. Thus, we have

$$\delta(x) \otimes 1 = ([a_1, x] + \nu_1(x)) \otimes 1 + ([a_2, x] + \nu_2(x)) \otimes w_2 + ([a_3, x] + \nu_3(x)) \otimes w_3$$

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for all $x \in R$. This means $\delta(x) = [a_1, x] + \nu(x)$ for $x \in R$, where $\nu = \nu_1|_R : R \to C$, as desired. □

Proof of Theorem 4.6. By Lemma 4.11, we can assume that δ is $Z(R)$-linear and $\delta(Z(R)) = 0$, and let $\overline{\delta}\colon \mathrm{M}_2(F) \to \mathrm{M}_2(F)$ be an extension of δ as above. Hence Theorem 4.6 follows directly from Lemma 4.14. □

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