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碩士論文

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Master's Thesis

從古典散射衝量去得出電磁任意子以及三維重力的散射振幅

Use classical scattering impulse to find quantum scattering amplitude of electromagnetic anyon and scattering amplitude of gravity in the (2+1) dimension

莊文瑜

Wen-Yu Chuang

指導教授: 黃宇廷 博士

Advisor: Yu-Ting Huang Ph.D.

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摘要

在量子場論中，我們曾經學過如何計算不同物理場下的散射振幅，同時我們也知道如何從散射振幅去推導出不同物理場下的散射截面。在此篇文章中會介紹如何用散射振幅去推導出古典的散射衝量，並且使用這個方式從古典散射衝量去回推出散射振幅。在 (2+1) 維的時空中，有一些情況我們無法使用費曼規則去得出他們的散射振幅。第一個例子就是重力在 (2+1) 維的情況，由於重力在 (2+1) 維的時候沒有任何動態自由度，所以我們無法畫出費曼圖。但是，重力在 (2+1) 維的散射衝量並不為零，因此我們可以使用上面提到的方法從散射衝量去回推散射振幅。另一個例子是任意子–一種只會出現在 (2+1) 維時空且自旋並非整數或半整數的粒子。由於任意子並非基本粒子，所以我們無法使用費曼圖來算出散射振幅。但是同樣的，由於任意子的散射衝量並不為零，所以我們可以用上述方法得出散射振幅。

關鍵字：散射振幅、散射衝量、任意子

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Abstract

In quantum field theory, we have learned how to calculate the scattering amplitude by drawing Feynman diagrams and then using Feynman rules to calculate the contribution of each diagram to the scattering amplitude. Besides, we have also learned how to use the scattering amplitude to derive the cross section which is a classical observable. However, in the (2+1) dimension, there are some special cases in which we can't define Feynman rules. For example, gravity in the (2+1) dimension has no dynamic degree of freedom. However, we can see there are non-trivial physical observables that reflect the effect of gravity such as the impulse of a particle going through a gravitational source. Another example is the scattering amplitude of anyon which is a special kind of particle in (2+1) dimension. Since the spin of anyon can be an arbitrary number, it is not an elementary particle and we can't define its Feynman rules either. As gravity in (2+1) dimension, the impulse of particles going through a source composed of anyon is also non-trivial. Therefore, we can calculate the scattering impulse of the above two cases and try to use

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the scattering impulse to derive the scattering amplitude of them.

Keywords: scattering amplitude, scattering impulse, anyon

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Chapter 1 Introduction

In this section, we will see (1) the dynamics of gravity in the (2+1) dimension[4] (2) the reason why in the (2+1) dimension, we can have particles with arbitrary spin[9] (3) how to derive the scattering impulse from scattering amplitude[6]

1.1 gravity in the (2+1) dimension

In this part, we can see the dynamics of gravity in (2+1) dimension which is written down in S. Deser, R. Jackiw's "Three-Dimensional Einstein Gravity: Dynamics of Flat Space"[4].

They use Einstein field equation to calculate the metric for a mass particle located at the origin.

The stress energy tensor for a particle located in the origin is

$$T^{00} = m_2 \hat{\delta}^2(0) \tag{1.1}$$

where other components are equal to zero.

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The Einstein field equation is

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.2}$$

where the Einstein field tensor $G_{\mu\nu}$ is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R \tag{1.3}$$

Consider the static case, since the system is rotationally invariant, the metric is

$$g_{00} = A(x) \quad g_{0i} = 0 \quad g_{ij} = B_{ij}(x) \tag{1.4}$$

Plug the stress energy tensor and Riemann tensor into the Einstein field equation, we can find the metric for a particle located in the origin is

$$\begin{aligned} g_{00} &= -1 \\ g_{ij} &= \delta_{ij} r^{-8GM} \end{aligned} \tag{1.5}$$

from this metric, we can see the spacetime of a particle at the origin is flat except there is a conical singularity at the origin. Therefore, we can see that the gravity in the (2+1) dimension is topological, so there are no propagating gravitons and we can't define the Feynman rules.

1.2 anyon

Anyon is a special kind of particle in (2+1) dimension whose spin isn't constrained from being integer or half-integer. We can see why in the (2+1) dimension the particle

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spin can be an arbitrary number but in the (3+1) dimension can only be integer or half-integer by spin-statistics from path dependence, which is introduced in M.D. Schwartz's "Quantum Field Theory and the Standard Model"[9].

Suppose we have two identical particles at positions x_1 and x_2 at time $t = 0$, and then at a later time, they are still at x_1 and x_2. Since the two particles are identical, the trajectories of these two particles during this period may be two particles moving back to their original position respectively, switching their position with each other or moving around each other many times. To characterize the transformation, we can define the angle ϕ, which is the angle that relative position vector of the two particles $(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ turns around. This angle ϕ is frame-independent and is a topological property associated with the path. We illustrate that in Figure 1.1.

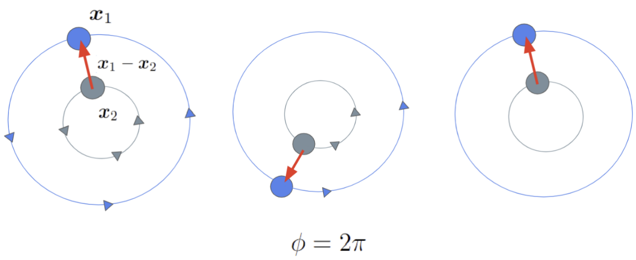

Figure 1.1: The red arrow is the position vector of the two particles[9].

Figure 1.1 is the path of the two particles turning around each other and going back to their original position x_1 and x_2. We can see that the relative vector turns around an angle 2π.

We can also see in Figure 1.2 that the angle ϕ can also be 0 or π for the two identical particles to finally stay at x_1 and x_2 respectively.

Since the two particles are identical, the two-particle state after the transformation of $\phi = 0, \pi, 2\pi$ will only pick up a phase proportional to this angle ϕ. For $\phi = \pi$ (i.e. the

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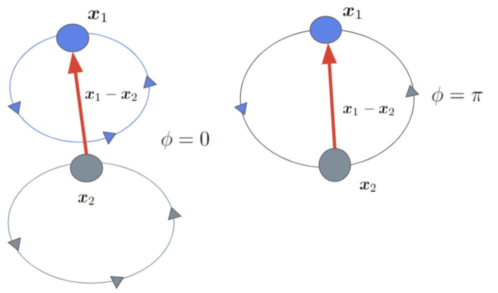

Figure 1.2: the route of $\phi = 0$ and the route of $\phi = \pi$[9]

two particles switch the position with each other),

$$|\phi_1(x_2)\phi_2(x_1)\rangle = e^{i\kappa\pi} |\phi_1(x_1)\phi_2(x_2)\rangle \tag{1.6}$$

where κ is the number characteristic of the particle type.

In (3+1) dimension, the angle ϕ can be defined up to 2π. For example, we can pull out particle 2 in Figure 1.1, and then the angle ϕ can become 0.

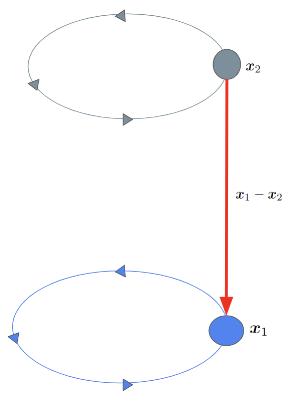

Figure 1.3: The red arrow is perpendicular to the route of the two particles, so it will not turn around. The angle $\phi = 0$[9].

Therefore, in the (3+1) dimension, we can't distinguish the path of angle $\phi = 2\pi$ from the path of angle $\phi = 0$, and thus κ must be integer. For the two particles to switch their position with each other, the two-particle state must satisfy

$$|\phi_1(x_2)\phi_2(x_1)\rangle = \pm |\phi_1(x_1)\phi_2(x_2)\rangle \tag{1.7}$$

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That is, only fermionic and bosonic statistics are possible.

To perform the interchange of the two particles, we can first translate the two particles by $(\boldsymbol{x}_1 - \boldsymbol{x}_2)$, then rotate the whole system by angle π. Under translation, nothing special happens. For rotation, the way to rotate a particle state depends on its spin, that is, depends on the representations of the Lorentz group.

For example, to transform a Dirac spinor which is the solution of the Dirac equation and has spin $\frac{1}{2}$, we use the $(\frac{1}{2}, 0) \bigoplus (0, \frac{1}{2})$ representation of the Lorentz group. A rotation of an angle θ_z is

$$\Lambda_s(\theta_z) = \begin{pmatrix} e^{\frac{i}{2}\theta_z} & & & \\ & e^{\frac{-i}{2}\theta_z} & & \\ & & e^{\frac{i}{2}\theta_z} & \\ & & & e^{\frac{-i}{2}\theta_z} \end{pmatrix} \tag{1.8}$$

Suppose we rotate an electron with spin-up by angle $\theta_z = \pi$,

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} i \\ 0 \\ i \\ 0 \end{pmatrix} \tag{1.9}$$

Therefore, for a two-particle state interchange,

$$|\phi_1(x_2)\phi_2(x_1)\rangle = -\,|\phi_1(x_1)\phi_2(x_2)\rangle \tag{1.10}$$

For another example, a vector, which has spin 1, 0 and transforms under the $(\frac{1}{2}, \frac{1}{2})$ repre-

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sentation of the Lorentz group. A rotation of angle θ_z is

$$\Lambda_s(\theta_z) = \begin{pmatrix} 1 & & & \\ & e^{-i\theta_z} & & \\ & & e^{i\theta_z} & \\ & & & 1 \end{pmatrix} \tag{1.11}$$

A two-particle state will pick up $+1$ for a rotation of angle $\theta_z = \pi$, that is, an interchange of the two particles takes

$$|\phi_1(x_2)\phi_2(x_1)\rangle = |\phi_1(x_1)\phi_2(x_2)\rangle \tag{1.12}$$

The above derivation works for all integers and half integers. Since in the (3+1) dimension, after an interchange, the two identical particle state can only pick up +1 or -1, the particle spin can only be integer or half-integer in the (3+1) dimension.

The above situation will be different in (2+1) dimension. The path of $\phi = 2\pi$ can be distinguished from the path of $\phi = 0$. That is because we can not pull out the diagram, the path in Figure 1.1 can't become the path in Figure 1.3. This means the two-particle state under an interchange will not be constrained to pick up ± 1.

$$|\phi_1(x_2)\phi_2(x_1)\rangle = e^{i\kappa\pi} |\phi_1(x_1)\phi_2(x_2)\rangle \tag{1.13}$$

where κ can be an arbitrary number. We can see that particles in the (2+1) dimension can have spin other than integer and half-integer. The particles which have spin other than integer and half-integer are called anyon.

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1.3 matches scattering amplitude to scattering impulse

In this part, I will show how to calculate the scattering impulse from scattering amplitude derived by David A. Kosower, Ben. Maybee and Donal. O'Connell[6].

In the beginning, we know the expected value of momentum is

$$\langle P^{\mu} \rangle = \langle \psi | \mathbb{P}^{\mu} | \psi \rangle \tag{1.14}$$

We are going to see the impulse of particle 1 probing another particle 2 at the origin. The impulse is the difference of the initial momentum at $\tau = -\infty$ and the momentum after scattering at $\tau = \infty$.

$$\langle \Delta P_1^{\mu} \rangle = \left\langle \psi | S^{\dagger} \mathbb{P}_1^{\mu} S | \psi \right\rangle - \langle \psi | \mathbb{P}_1^{\mu} | \psi \rangle \tag{1.15}$$

where S is the S-matrix defined as $|\psi\rangle_{out} = S\, |\psi\rangle_{in}$ is the time-evolution operator evolving the initial state $|\psi\rangle_{in}$ at $\tau = -\infty$ into the final state $|\psi\rangle_{out}$ at $\tau = \infty$.

And then since we know the scattering amplitude $\mathcal{A}(p_1, p_2 \to p_1', p_2')$ is the matrix elements of transition matrix T, that is $\langle p_1' p_2' | T | p_1 p_2 \rangle = A(p_1, p_2 \to p_1', p_2') \hat{\delta}^{(4)}(p_1' + p_2' - p_1 + p_2)$. Therefore, to make contact with scattering amplitude, we can write down the s-matrix in terms of $S = 1 + iT$.

We then use the unitarity of s-matrix, $S^{\dagger} S = 1$,

$$\begin{aligned}
&(1 - iT^{\dagger})(1 + iT) = 1 \\
&1 + i(T - T^{\dagger}) + T^{\dagger} T = 1 \\
&T^{\dagger} T = i(T^{\dagger} - T)
\end{aligned} \tag{1.16}$$

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We can write scattering impulse as

$$\langle \Delta p_1^\mu \rangle = \langle \psi | i[\mathbb{P}_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\mathbb{P}_1^\mu, T] | \psi \rangle \tag{1.17}$$

We can take a brief look at equation(1.17). The first term contains one transition matrix and the second term has $T^\dagger T$ multiplied, so the first term has only one scattering amplitude may be related to the tree diagram, and the second term has two scattering amplitudes multiplied may be related to the loop diagram which can be decomposed into two tree diagrams.

In the next part, we will use classical methods to calculate the scattering impulse and then find the scattering amplitude in the classical limit. In the classical limit, $\hbar \to 0$, and since the contribution for each Feynman diagram is proportional to $\hbar^{(L-1)}$, where L is the number of loops in the Feynman diagram, when $\hbar \to 0$, only the tree diagrams will contribute to the scattering amplitude.

First, we see the initial state $|\phi_1 \phi_2\rangle_{in}$ is

$$|\psi\rangle_{in} = \int \hat{\mathrm{d}}^4 p_1 \hat{\mathrm{d}}^4 p_2 \hat{\delta}^{(+)}(p_1^2 - m_1^2) \hat{\delta}^{(+)}(p_2^2 - m_2^2) \phi_1(p_1) \phi_2(p_2) e^{\frac{ib \cdot p_1}{\hbar}} |p_1 p_2\rangle_{in} \tag{1.18}$$

where $\hat{\mathrm{d}}^n p \equiv \frac{\mathrm{d}^n p}{(2\pi)^n}$ and $\hat{\delta}^{(+)}(p^2 - m^2) \equiv 2\pi \Theta(p^0) \delta(p^2 - m^2)$.

In the above equation, we have $e^{ib \cdot p_1/\hbar}$ because particle 2 is supposed to be fixed at the origin, and particle 1 is relative to particle 2 by the impact parameter b.

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Next, put the initial state (1.18) into the first term on the right-hand side of (17),

$$\langle\psi|i[\mathbb{P}_1^\mu, T]|\psi\rangle = \int \hat{\mathrm{d}}^4p_1\hat{\mathrm{d}}^4p_2\hat{\mathrm{d}}^4p_1'\hat{\mathrm{d}}^4p_2'\hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2)$$
$$\hat{\delta}^{(+)}(p_1'^2 - m_1'^2)\hat{\delta}^{(+)}(p_2'^2 - m_2'^2)\phi_1(p_1)\phi_1^*(p_1')\phi_2(p_2)\phi_2^*(p_2')$$
$$\times e^{\frac{ib\cdot(p_1-p_1')}{\hbar}} i(p_1'^\mu - p_1^\mu)\langle p_1'p_2'|\, T\, |p_1p_2\rangle \tag{1.19}$$

Then introducing the momentum shift $q_i = p_i' - p_i$, and we can replace the integration over p_i' into the integration over q_i.

$$\langle\psi|i[\mathbb{P}_1^\mu, T]|\psi\rangle = \int \hat{\mathrm{d}}^4p_1\hat{\mathrm{d}}^4p_2\hat{\mathrm{d}}^4q_1\hat{\mathrm{d}}^4q_2\hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2)$$
$$\hat{\delta}^{(+)}((p_1+q_1)^2 - m_1^2)\hat{\delta}^{(+)}((p_2+q_2)^2 - m_2^2)$$
$$\times \phi_1(p_1)\phi_1^*(p_1+q_1)\phi_2(p_2)\phi_2^*(p_2+q_2)e^{\frac{ib\cdot(p_1-q_1)}{\hbar}}$$
$$\times i(p_1'^\mu - p_1^\mu)\mathcal{A}(p_1, p_2 \to p_1+q_1, p_2+q_2)\hat{\delta}^{(4)}(q_1+q_2) \tag{1.20}$$

Integral over q_2, and relabel $q_1 \to q$

$$\langle\psi|i[\mathbb{P}_1^\mu, T]|\psi\rangle = \int \hat{\mathrm{d}}^4p_1\hat{\mathrm{d}}^4p_2\hat{\mathrm{d}}^4q\hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2)$$
$$\hat{\delta}((p_1+q)^2 - m_1^2)\hat{\delta}((p_2-q)^2 - m_2^2)\Theta(p_1^0+q^0)\Theta(p_2^0-q^0)$$
$$\times \phi_1(p_1)\phi_1^*(p_1+q)\phi_2(p_2)\phi_2^*(p_2-q)e^{\frac{ib\cdot q}{\hbar}}$$
$$\times iq^\mu\mathcal{A}(p_1, p_2 \to p_1+q, p_2-q) \tag{1.21}$$

Since we need to take $\hbar \to 0$ to see the classical limit of the impulse, we can't set $\hbar = 1$ and need to restore $\hbar$ in the calculation.

To restore $\hbar$, we need to distinguish between the momentum p^μ of a particle and its wavenumber, which we denote $\bar{p}^\mu$. In quantum mechanics, the relation between the

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wavenumber and its momentum is

$$\bar{p} = \frac{p}{\hbar} \tag{1.22}$$

Because now $\hbar$ doesn't set to 1, the physical quantities which have dimensions of $[L]^{-1}$, for example, the wavenumber, don't have the same dimensions as those quantities which have dimensions of $[M]^1$, for example, the momentum.

For the calculation in scattering impulse, we write the momentum q as $\hbar\bar{q}$, and take the limit $\hbar \to 0$. For particle momentum p_1 and p_2, in the calculation, we treat them as two point-like particles. For this description to be valid, the Compton wavelengths $l_c^{(i)} \equiv \frac{\hbar}{m_i}$ must be very small, so the wavenumber $\frac{2\pi}{\lambda}$ of them will be very large. In the limit of $\hbar \to 0$, it will not approach 0, so we can treat them as genuine momentum.

Despite restoring $\hbar$ in q, we also need to restore the $\hbar$ in the scattering amplitude. We can see the $\hbar$ dependence by the dimensional analysis of equation (1.21).

Since the dimension of the expected value of momentum is $M \cdot L/T$, the dimension of the right-hand side of the equation (1.21) must be equal to $M \cdot L/T$. Then we can use this fact to know the dimensions of scattering amplitude in (3+1) dimension. Before going to check this, let's first see the dimension of $\phi(p)$. Since the normalization condition is

$$\begin{aligned}
1 &= \langle\psi|\psi\rangle \\
&= \int \hat{\mathrm{d}}^4 p_1 \hat{\mathrm{d}}^4 p_2 \hat{\mathrm{d}}^4 p_1' \hat{\mathrm{d}}^4 p_2' \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2) \\
&\times \hat{\delta}^{(+)}(p_1'^2 - m_1^2)\hat{\delta}^{(+)}(p_2'^2 - m_2^2)\phi_1(p_1)\phi_1^*(p_1')\phi_2(p_2)\phi_2^*(p_2') \\
&\times e^{\frac{ib\cdot(p_1 - p_1')}{\hbar}} \langle p_1' p_2' | p_1 p_2 \rangle \\
&= \int \hat{\mathrm{d}}^4 p_1 \hat{\mathrm{d}}^4 p_2 \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2)|\phi_1(p_1)|^2|\phi_2(p_2)|^2
\end{aligned} \tag{1.23}$$

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we can obtain this normalization by requiring $\phi_i(p_i)$ to satisfy

$$\int \hat{\mathrm{d}}^4 p_i \hat{\delta}^{(+)}(p_i^2 - m_i^2)|\phi_i(p_i)|^2 = 1 \tag{1.24}$$

The dimension of $\phi_i(p_i)$ is

$$[\phi_i(p_i)] = M^{-1} \tag{1.25}$$

where we set the speed of light $c = 1$ and then $L = T$.

Now, we can see the dimensions of the scattering amplitude from the dimensional analysis of equation (1.21)

$$M = M^{12} \times M^{-8} \times M^{-4} \times M \times [\mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q)] \tag{1.26}$$

so the dimension of scattering amplitude is 1.

For electromagnetic scattering, the coupling constant is charge q and the dimension of charge can be seen from Coulomb's law. In the CGS unit, the vacuum permittivity is 1 when the speed of light c is equal to 1. So the dimension of the charge q is

$$[q] = (M \cdot L)^{1/2} \tag{1.27}$$

Since the scattering amplitude of electromagnetism is

$$\mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q) = 4e^2 \frac{(P_1 \cdot P_2)}{q^2} \tag{1.28}$$

to make the scattering amplitude in (3+1) dimension dimensionless, the coupling constant is $e^2/\hbar$.

Similarly, for gravitational scattering, the coupling constant is the gravitational con-

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stant G. From Newton's law of gravitation, the dimension of the gravitational constant G is

$$[G] = L/M \tag{1.29}$$

The scattering amplitude of gravity in the (3+1) dimension is

$$\mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q) = 16\pi G \frac{(P_1 \cdot P_2)^2}{q^2} \tag{1.30}$$

to make the scattering amplitude in (3+1) dimension dimensionless, the coupling constant is $G/\hbar$. However, in later calculations, we can see that in the (2+1) dimension, the $\hbar$ dependence of scattering amplitude differs from that in the (3+1) dimension.

Combining all the above discussion, the scattering impulse which restores $\hbar$ is

$$\begin{aligned}\langle\psi|i[\mathbb{P}_1^\mu, T]|\psi\rangle =& \hbar^5 \int \hat{\mathrm{d}}^4 p_1 \hat{\mathrm{d}}^4 p_2 \hat{\mathrm{d}}^4 \bar{q} \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2) \\ & \hat{\delta}((p_1 + \hbar\bar{q})^2 - m_1^2)\hat{\delta}((p_2 + \hbar\bar{q})^2 - m_1^2)\Theta(p_1^0 + \hbar\bar{q}^0)\Theta(p_2^0 - \hbar\bar{q}^0) \\ & \times \phi_1(p_1)\phi_1^*(p_1 + \hbar\bar{q})\phi_2(p_2)\phi_2^*(p_2 - \hbar\bar{q})e^{ib\cdot\bar{q}} \\ & \times i\bar{q}^\mu \mathcal{A}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q})\end{aligned} \tag{1.31}$$

Remind that the scattering amplitude contains couplings $e^2/\hbar$ in the electrodynamics case, $G/\hbar$ in the gravity case, and propagator $1/(\hbar\bar{q})^2$ in both cases, so we can cancel out $\hbar^3$.

$$\begin{aligned}\langle\psi|i[\mathbb{P}_1^\mu, T]|\psi\rangle =& \hbar^2 \int \hat{\mathrm{d}}^4 p_1 \hat{\mathrm{d}}^4 p_2 \hat{\mathrm{d}}^4 \bar{q} \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2) \\ & \hat{\delta}((p_1 + \hbar\bar{q})^2 - m_1^2)\hat{\delta}((p_2 + \hbar\bar{q})^2 - m_1^2)\Theta(p_1^0 + \hbar\bar{q}^0)\Theta(p_2^0 - \hbar\bar{q}^0) \\ & \times \phi_1(p_1)\phi_1^*(p_1 + \hbar\bar{q})\phi_2(p_2)\phi_2^*(p_2 - \hbar\bar{q})e^{ib\cdot\bar{q}} \\ & \times i\bar{q}^\mu \mathcal{A}(p_1, p_2 \to p_1 + \bar{q}, p_2 - \bar{q})\end{aligned} \tag{1.32}$$

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Since we are going to see the scattering impulse in the classical limit, we suppose that the incoming states are point particles that have momentum P_1 and P_2. That is, their wave functions have a peak at $p = P_i$ in momentum space. Therefore, $\phi(p_i) \approx \delta(P_i - p_i)$. On the other hand, the momentum of a point particle P_1 and P_2 also satisfy the classical equations of motion, so $P_i^2 = m_i^2$, and the energy $P_i^0 > 0$.

Combining the above conditions and integral over p_1 and p_2 in the $\hbar \to 0$ limit, the final result will be independent of $\hbar$.

$$
\begin{aligned}
\langle\psi|i[\mathbb{P}_1^\mu, T]|\psi\rangle = &\int \hat{\mathrm{d}}^4\bar{q}\hat{\delta}(2P_1 \cdot \bar{q})\hat{\delta}(2P_2 \cdot \bar{q})e^{ib\cdot\bar{q}}i\bar{q}^\mu \\
&\times \bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + \bar{q}, P_2 - \bar{q}) \\
=&\frac{1}{4m_1m_2}\int \hat{\mathrm{d}}^4\bar{q}\hat{\delta}(u_1 \cdot \bar{q})\hat{\delta}(u_2 \cdot \bar{q})e^{ib\cdot\bar{q}}i\bar{q}^\mu \\
&\times \bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + \bar{q}, P_2 - \bar{q}) \qquad (1.33)
\end{aligned}
$$

where $\bar{\mathcal{A}}^{(0)}$ is the leading order of scattering amplitude. We have seen that the scattering impulse can be derived by the leading order of the scattering amplitude. Now we want to see whether we can extract the leading order of the scattering amplitude from the scattering impulse calculated by the classical method.

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Chapter 2 matches the scattering impulse to the scattering amplitude in the (3+1) dimension

2.1 electromagnetism in the (3+1) dimension

In this section, I will perform the derivation of scattering amplitude from scattering impulse and compare the result with the scattering amplitude calculated by Feynman rules which was written down in D. A. Kosower, B. Maybee, D. O' Connell's "Amplitudes, Observables, and Classical Scattering"[6].

Let's first calculate the impulse of a charged scalar particle moving in an electromagnetic field produced by another charged scalar particle fixed at the origin in (3+1) dimension. The force on a moving charged particle in an electromagnetic field is the Lorentz force,

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{2.1}$$

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Expressing it in the covariant form using the electromagnetic tensor,

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{2.2}$$

$$\frac{\mathrm{d}p^{\mu}}{\mathrm{d}\tau} = qF^{\mu\nu}u_{\nu} \tag{2.3}$$

Suppose a scalar particle 1 with charge e, moving at the velocity u_1 probes another particle 2 with charge e fixed at the origin. The current of particle 2 is

$$J_2^{\mu} = eu_2^{\mu}\delta^{(3)}(r) \tag{2.4}$$

where $u_2 = (1, 0, 0, 0)$.

From the Maxwell equation, the field tensor satisfies

$$\partial_{\mu}F_2^{\mu\nu} = eu_2^{\mu}\delta^{(3)}(r) \tag{2.5}$$

Working in Lorentz gauge $\partial_{\mu}A^{\mu} = 0$, the gauge field satisfies

$$\Box A_2^{\mu}(x) = eu_2^{\mu}\delta^{(3)}(r) \tag{2.6}$$

Write down the above equation in momentum space,

$$-q^2 A_2^{\mu}(q) = \int \hat{\mathrm{d}}^4 x e^{-(iq_0\cdot x_0 + iq_i\cdot x_i)} eu_2^{\mu}\delta^{(3)}(r) \tag{2.7}$$

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And we know

$$q_0 = q \cdot u_2 \tag{2.8}$$

where $u_2 = (1, 0, 0, 0)$. We can now get the gauge field in momentum space

$$A_2^\mu(q) = \frac{\hat{\delta}(q \cdot u_2) e u_2^\mu}{q^2} \tag{2.9}$$

So the field tensor $F^{\mu\nu}$ in position space is

$$F_2^{\mu\nu}(x) = ie \int \hat{\mathrm{d}}q^4 \hat{\delta}(q \cdot u_2) e^{iq\cdot x} \frac{q^\mu u_2^\nu - q^\nu u_2^\mu}{q^2} \tag{2.10}$$

At leading order, we will use the straight line approximation to the trajectory of particle 1, and the change of its momentum in time τ is

$$\frac{\mathrm{d}P_1^\mu(x)}{\mathrm{d}\tau} = ie^2 \int \hat{\mathrm{d}}q^4 \hat{\delta}(q \cdot u_2) e^{iq\cdot(b+u_1\tau)} \frac{q^\mu u_2 \cdot u_1 - u_2^\mu q \cdot u_1}{q^2} \tag{2.11}$$

The scattering impulse at leading order is

$$\Delta P_1^\mu = ie^2 \int \hat{\mathrm{d}}^4 q \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot q) e^{iq\cdot b} \frac{q^\mu}{q^2} (u_1 \cdot u_2) \tag{2.12}$$

Compared with the relation (1.33), the leading order of scattering amplitude derived by scattering impulse is

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 4e^2 \frac{(P_1 \cdot P_2)}{q^2} \tag{2.13}$$

Then we use Feynman rules to calculate the scattering amplitude at the leading order[9], the Feynman diagram in Figure 1,

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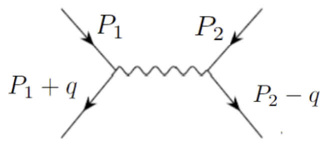

Figure 2.1: the Feynman diagram of 4 external charged scalar with the exchange of a massless photon[9]

The Feynman rule of a three-point vertex is

$$ie(-P_1^\mu - (P_1^\mu + q^\mu)) \tag{2.14}$$

The photon propagator in the Lorentz gauge is

$$\frac{-ig_{\mu\nu}}{q^2} \tag{2.15}$$

Then the scattering amplitude is

$$i\mathcal{A}(P_1, P_2 \to P_1 + q, P_2 - q) = i\frac{e^2(2P_1 + q)(2P_2 + q)}{q^2} \tag{2.16}$$

Finally, we can get the leading order of the electrodynamic scattering amplitude

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 4e^2\frac{(P_1 \cdot P_2)}{q^2} \tag{2.17}$$

which matches the result derived by scattering impulse.

2.2 gravity in the (3+1) dimension

In [6], they calculate the case of electromagnetism in (3+1) dimension. In this part, I will show the process of calculating the scattering amplitude of the gravity in the (3+1) dimension using the scattering impulse which was written down in [7].

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We will calculate the scattering impulse of a scalar particle 1 by probing a static scalar particle 2 at the origin (i.e. $u_2 = (1,0,0,0)$). To calculate the scattering impulse, I use the geodesic equation[8], which is the equation of motion of a point particle in a curved spacetime. The geodesic equation is the equation of motion of the Hilbert-Einstein action. The derivation of the geodesic equation below is from Sean Carroll's "Spacetime and Geometry–An Introduction to General Relativity[3].

$$S = -m\int\sqrt{-\mathrm{d}s^2} = -m\int \mathrm{d}\tau\sqrt{-g_{\mu\nu}(x(\tau))\frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu(\tau)}{\mathrm{d}\tau}} \tag{2.18}$$

And vary the above action with respect to the trajectory $x(\tau)$ (assume τ as the proper time, that is $\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2}$)

$$\begin{aligned}\delta S =& -m\int \mathrm{d}\tau\frac{1}{2\sqrt{-\dot{x}^2}}[(\delta g_{\mu\nu}(x))\dot{x}^\mu\dot{x}^\nu + 2g_{\mu\nu}(x)\dot{x}^\mu(\delta\dot{x}^\nu)]\\ =& -m\int \mathrm{d}\tau(\delta x^\nu[\frac{1}{2}(\partial_\nu g_{\mu\lambda}(x))\dot{x}^\mu\dot{x}^\lambda - \left(\frac{\mathrm{d}}{\mathrm{d}\tau}g_{\mu\nu}(x)\right)\dot{x}^\mu\\ &- g_{\mu\nu}(x)\ddot{x}^\mu] + \frac{\mathrm{d}}{\mathrm{d}\tau}(...))\\ =& -m\int \mathrm{d}\tau(\delta x^\nu[\frac{1}{2}(\partial_\nu g_{\mu\lambda}(x))\dot{x}^\mu\dot{x}^\lambda - (\partial_\lambda g_{\mu\nu}(x))\dot{x}^\lambda\dot{x}^\mu\\ &- g_{\mu\nu}(x)\ddot{x}^\mu] + \frac{\mathrm{d}}{\mathrm{d}\tau}(...))\\ =& m\int \mathrm{d}\tau(\delta x^\nu[g_{\mu\nu}(x)\ddot{x}^\mu + \frac{1}{2}(\partial_\lambda g_{\mu\nu}(x) + \partial_\mu g_{\lambda\nu}(x) - \partial_\nu g_{\mu\lambda}(x))\dot{x}^\mu\dot{x}^\lambda]\\ &+ \frac{\mathrm{d}}{\mathrm{d}\tau}(...))\end{aligned} \tag{2.19}$$

Thus we know the equation of motion of a point particle in a curved spacetime is

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \frac{1}{2}g^{\mu\nu}[\partial_\lambda g_{\rho\nu}(x) + \partial_\rho g_{\lambda\nu}(x) - \partial_\nu g_{\rho\lambda}(x)]\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0 \tag{2.20}$$

and we then define $\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2}g^{\mu\nu}[\partial_\lambda g_{\rho\nu}(x) + \partial_\rho g_{\lambda\nu}(x) - \partial_\nu g_{\rho\lambda}(x)]$ as christoffel symbols.

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Now we can calculate the scattering impulse. For a scalar point particle at the origin, the spacetime metric is the Schwarzschild metric[3].

Treating the Schwarzschild metric perturbatively in G, we can express the derivation from Minkowski as

$$h_{tt} = h_{rr} = \frac{2GM}{r} \tag{2.21}$$

others are equal to zero.

In the non-relativistic limit, $\frac{\mathrm{d}t}{\mathrm{d}\tau}$ dominates over $\frac{\mathrm{d}x^i}{\mathrm{d}\tau}$, so the geodesic equation is approximately given by

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}x^2} = -\Gamma^\mu_{00}(\frac{\mathrm{d}t}{\mathrm{d}\tau})^2 \tag{2.22}$$

$$\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\lambda}\partial_\lambda g_{00} \tag{2.23}$$

To first order in G,

$$\Gamma^\mu_{00} = -\frac{1}{2}\partial^\mu h_{00} \tag{2.24}$$

Fourier transform (2.24) to momentum space,

$$\Gamma^\mu_{00}(q) = \int \hat{\mathrm{d}}^4 x e^{-(iq_0\cdot x_0 + iq_i\cdot x_i)}\Gamma^\mu_{00}(x) \tag{2.25}$$

And we know

$$q_0 = q\cdot u_2 \tag{2.26}$$

where $u_2 = (1, 0, 0, 0)$.

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Then do the integral over the time component τ

$$\Gamma^{\mu}_{00}(q) = \int \mathrm{d}\tau e^{-i(q\cdot u_2)\tau} \int \hat{\mathrm{d}}^3 x e^{-iq_i\cdot x_i} \Gamma^{\mu}_{00}(x)$$
$$= -\hat{\delta}(u_2 \cdot q) \int \hat{\mathrm{d}}^3 x e^{-iq_i \cdot x_i} \Gamma^{\mu}_{00}(x) \tag{2.27}$$

We can get

$$\Gamma^{\mu}_{00}(q) = \int \hat{\mathrm{d}}^4 x e^{-iq\cdot x} \frac{-1}{2} \partial^{\mu} h_{00} = i\hat{\delta}(u_2 \cdot q) q^{\mu} G m_2 \frac{4\pi}{q^2} \tag{2.28}$$

Plugging everything in, we can get the impulse

$$\Delta P^{\mu}_1 = m_1 \int \mathrm{d}\tau \int \hat{\mathrm{d}}^4 q e^{iq\cdot(b+u_1\tau)} i\hat{\delta}(u_2\cdot q) q^{\mu} G m_2 \frac{4\pi}{q^2} (\frac{\mathrm{d}t}{\mathrm{d}\tau})^2$$
$$= m_1 \int \hat{\mathrm{d}}^4 q e^{-iq\cdot b} i\hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2\cdot q) q^{\mu} G m_2 \frac{4\pi}{q^2}\gamma^2 \tag{2.29}$$

where $\gamma^2 = (u_1 \cdot u_2)^2$.

Finally, the scattering amplitude of a scalar particle probing another scalar particle at the origin in the (3+1) dimension deriving from the scattering impulse is,

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 16\pi G \frac{(P_1 \cdot P_2)^2}{q^2} \tag{2.30}$$

And to calculate scattering amplitude in quantum field theory, we will use the formalism developed in [5][1]. The particles can be labeled by their momentum and their little group which leaves the momentum invariant. The little group for massive particles in (3+1) dimension is SO(3)→SU(2), and is SO(2)→U(1) for massless particles. So for massless particles, we can label particles by momentum and its helicity h. For massive particles, we can label particles by spin s.

Then we know that the scattering amplitude $\mathcal{M}$ is composed of objects that carry

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the above labels. We are now going to calculate the scattering amplitude of four massive scalar particles with the exchange of a massless graviton.

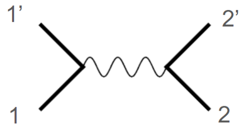

Figure 2.2: The Feynman diagram of four external scalar particles with the exchange of a massless graviton[1].

We know the four-point amplitude $\mathcal{A}_4[1,2,1',2']$ in s-channel can be calculated by[5]

$$\mathcal{A}_4[1,1',\to 2,2'] = \frac{\mathcal{A}_3[1,1',q^+]\mathcal{A}_3[2,2',q^-]}{s} + \frac{\mathcal{A}_3[1,1',q^-]\mathcal{A}_3[2,2',q^+]}{s} \tag{2.31}$$

To calculate the three-point scattering amplitude of two massive legs with mass m and a massless leg. We can define "x" which carries +1 little group weight of massless leg[1]. Use the convention the same as in [1], the momentum can be written down as $p_{\alpha\dot{\alpha}} \equiv p_\mu(\sigma)^\mu_{\alpha\dot{\alpha}}$ where $\sigma^\mu_{\alpha\dot{\alpha}} = (1,\sigma^i)_{\alpha\dot{\alpha}}$ and σ^i are Pauli matrices. Then we can write down $p_{\alpha\dot{\alpha}}$ as the direct product of two, 2-vectors $\lambda, \tilde{\lambda}$

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tag{2.32}$$

The variable "x" is defined as

$$x\lambda_{3\alpha} = \frac{p_{1\alpha\dot{\alpha}}}{m}\tilde{\lambda}^{3\dot{\alpha}}, \quad \frac{\tilde{\lambda}^{3\dot{\alpha}}}{x} = \frac{p_1^{\alpha\dot{\alpha}}\lambda_{3\alpha}}{m} \tag{2.33}$$

Since the scattering amplitude in D spacetime dimension with n external legs has mass dimension $[\mathcal{A}_n] = \frac{n}{2}(2-D)+D$, $[\mathcal{A}_3] = 1$. Because the mass dimension of the "x" variable is 0, the scattering amplitude of two massive scalar particles and one massless

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graviton is

$$\mathcal{A}_3[1, 1', q^+] = m x_{11'}^2 \tag{2.34}$$

The scattering amplitude is

$$\begin{aligned}\mathcal{A}(P_1, P_2 \to P_1 + q, P_2 - q) &= \frac{m^2(\frac{x_{11'}^2}{x_{22'}^2} + \frac{x_{22'}^2}{x_{11'}^2})}{sM_{pl}^2} \\ &= \frac{4G\pi[(2P_2 + q) \cdot P_1]^2}{q^2} \\ &= 4\pi G \frac{(2P_1 \cdot P_2 + q \cdot P_1)^2}{q^2}\end{aligned} \tag{2.35}$$

The leading order of the scattering amplitude is

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 16\pi G \frac{(P_1 \cdot P_2)^2}{q^2} \tag{2.36}$$

which matches the result derived by scattering impulse.

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Chapter 3 matches the scattering impulse to the scattering amplitude in the (2+1) dimension

3.1 Dimensional analysis of the scattering amplitude in the (2+1) dimension

From the examples of electromagnetism and gravity in the (3+1) dimension, we can see that we can derive the scattering amplitude by scattering impulse.

Before using relation (1.33) to calculate the scattering amplitude, we need to note that the $\hbar$ dependence of the scattering amplitude will differ from that in the (3+1) dimension.

To satisfy the normalization condition $1 = \langle\psi|\psi\rangle$, $\phi_i(p_i)$ in (2+1) dimension must satisfy

$$\int \hat{\mathrm{d}}^3 p_i \hat{\delta}^{(+)}(p_i^2 - m_i^2)|\phi_i(p_i)|^2 = 1 \tag{3.1}$$

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The dimension of $\phi_i(p_i)$ in (2+1) dimension is

$$[\phi_i(p_i)] = M^{-1/2} \tag{3.2}$$

Therefore, from the dimensional analysis of equation (1.21) in the (2+1) dimension, the dimension of scattering amplitude in the (2+1) dimension is

$$M = M^9 \times M^{-8} \times M^{-2} \times M \times [\mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q)] \tag{3.3}$$

The dimension of scattering amplitude in the (2+1) dimension is M. In the next sections, we first use relation (33) to find out the scattering amplitude, and then check whether the whole $\hbar$ dependence cancels out.

3.2 scattering amplitude of gravity in the (2+1) dimension

Now we can use this relation (1.33) to calculate the scattering amplitude of some special cases in the (2+1) dimension.

First, I will show the scattering amplitude of gravity in the (2+1) dimension which was derived in [2]. As we mentioned before, in the (2+1) dimension, the gravity is topological, so we can't use the Feynman diagram to calculate the scattering amplitude. Therefore, we calculate the classical scattering impulse and use this impulse to calculate the scattering amplitude.

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In section 2.1, we know the metric for a particle located in the origin is

$$g_{00} = -1$$
$$g_{ij} = \delta_{ij} r^{-8GM} \tag{3.4}$$

from this metric, we can see the spacetime of a particle at the origin is flat except there is a conical singularity at the origin.

So the Christoffel symbols of this system are

$$\Gamma^0_{\mu\nu} = 0 \tag{3.5}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha}(\partial_j g_{\alpha k} + \partial_k g_{\alpha j} - \partial_\alpha g_{jk}) \tag{3.6}$$

Put the metric into the spatial part of Christoffel symbols

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}\delta^{i\alpha} r^{8GM}(\partial_j(\delta_{\alpha k} r^{-8GM}) + \partial_k(\delta_{\alpha j} r^{-8GM}) - \partial_\alpha(\delta_{jk} r^{-8GM}))\\ &= \frac{1}{2}\delta^{i\alpha} r^{8GM}(\delta_{\alpha k}(-8GM) r^{-8GM-1}\frac{x_j}{r} + \delta\alpha j(-8GM) r^{-8GM-1}\frac{x_k}{r}\\ &- \delta_{jk} r^{-8GM-1}(-8GM)\frac{x_\alpha}{r})\\ &= -4GM\delta^i_k \frac{x_j}{r^2} - 4GM\delta^i_j \frac{x_k}{r^2} + 4GM\delta_{jk}\frac{x^i}{r^2}\end{aligned} \tag{3.7}$$

Probe another particle 1 to the spacetime generated by particle 2 at the origin. Then from geodesic equations, we can know the change of velocity of particle 1 per unit time τ is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = 8Gm_2 \frac{x_j}{r^2}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{d\tau} - 4Gm_2 \frac{x^i}{r^2}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x_j}{\mathrm{d}\tau} \tag{3.8}$$

To get the scattering amplitude by the classical impulse, we find the Fourier transformation of the equation (77) using $\partial_i \ln r = \frac{x_i}{r^2}$ and$\mathcal{F}[-\frac{1}{2\pi}\ln r] = \frac{1}{q^2}$, then with a similar

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calculation in the (3+1) dimension case, the classical impulse is

$$\int \mathrm{d}\tau \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = 8\pi G m_2 \int \hat{\mathrm{d}}^3 q e^{iq\cdot b}(-\frac{2iq_j}{q^2} u_1^i u_1^j + \frac{iq^i}{q^2} u_1^2) i\hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot q) \tag{3.9}$$

From the relation (1.33), we can get the classical limit of scattering amplitude for a particle 1 probing another particle 2 is

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 32G\pi \frac{(P_1 \cdot P_2)^2 - (m_1 m_2)^2}{q^2} \tag{3.10}$$

Now we want to see the $\hbar$ dependence of this scattering amplitude. From Gauss's law for gravity in the (2+1) dimension,

$$\oint_c \mathbf{g} \cdot \hat{\mathbf{n}} \mathrm{d}s = -4\pi G m \tag{3.11}$$

where **g** is the gravitational field. From (3.11), we can know the magnitude of the gravitational field **g** in the (2+1) dimension is

$$g(r) = \frac{2Gm}{r} \tag{3.12}$$

Therefore the dimension of G in the (2+1) dimension is

$$[G] = M^{-1} \tag{3.13}$$

In sec.3.1, we see that the dimension of scattering amplitude is M, so there is no need to multiply any $\hbar$ on G.

In the (2+1) dimension, after restoring the $\hbar$ in q, we can get $\hbar^3$ from $\hat{\mathrm{d}}q^3$, $\hbar^{-2}$ from $\hat{\delta}(p_i \cdot q)$, $\hbar$ from iq and $\hbar^{-2}$ from the propagator $1/q^2$ in the scattering amplitude. Finally,

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all $\hbar$ cancel out, so we can indeed use the relation (1.33) to match the scattering amplitude of gravity in the (2+1) dimension from the scattering impulse.

Recall in the (3+1) dimension, the classical limit of scattering amplitude is

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 16\pi G \frac{(P_1 \cdot P_2)^2}{q^2} \tag{3.14}$$

To our surprise, the scattering amplitude in (3+1) dimensions and (2+1) dimensions look similar, where we can just see the $(m_1 m_2)^2$ in (2+1) dimension as the zeroth component of $(P_1 \cdot P_2)^2$ in (3+1) dimension.

3.3 scattering impulse of gravity in position space and the scattering angle

On the other hand, I calculate the classical impulse in position space for (3+1) and (2+1) dimensions, to see if there is a difference between the two cases.

We use the methods developed in [6] for electromagnetic impulses.

First, let's calculate the scattering impulse for gravity in (3+1) dimension and we set $u_1 = (\gamma, 0, 0, \gamma\beta)$, $u_2 = (1, 0, 0, 0)$ and $\gamma^2(1 - \beta^2) = 1$ to simplify the results.

$$\Delta P_1^{\mu} = \frac{1}{4m_1 m_2} \int \hat{\mathrm{d}}^4 q \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot q) e^{-iq\cdot b} i q^{\mu} 16 G\pi \frac{(p_1 \cdot p_2)^2}{q^2} \tag{3.15}$$

$$= i4G\pi m_1 m_2 \int \hat{\mathrm{d}}^4 q \hat{\delta}(q^0) \hat{\delta}(\gamma q^0 - \gamma\beta q^3) e^{-ib\cdot q} \frac{\gamma^2}{q^2} q^{\mu} \tag{3.16}$$

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$$= -i\frac{Gm_1m_2}{\pi|\beta|}\int \mathrm{d}^2 q e^{ib\cdot q_\perp}\frac{\gamma}{q_\perp^2}q_\perp^\mu \tag{3.17}$$

Since we set u_1 in z-direction, the $q_\perp$ is in x-y plane. Let λ be the magnitude of $q_\perp$, therefore, we can set $q_\perp^\mu$=(0,$\lambda\cos\theta$,$\lambda\sin\theta$,0).

$$\begin{aligned}
\Delta P_1^\mu &= -i\frac{Gm_1m_2}{\pi|\beta|}\int_0^\infty \mathrm{d}\lambda\lambda\int_{-\pi}^{\pi}\mathrm{d}\theta e^{i|\boldsymbol{b}|\lambda\cos\theta}\frac{\gamma}{\lambda^2}(0,\lambda\cos\theta,\lambda\sin\theta,0)\\
&= -i\frac{Gm_1m_2\gamma}{\pi|\beta|}\int_0^\infty \mathrm{d}\lambda\int_{-\pi}^{\pi}\mathrm{d}\theta e^{i|\boldsymbol{b}|\lambda\cos\theta}(0,\cos\theta,\sin\theta,0)\\
&= \frac{2Gm_1m_2\gamma}{|\beta|}\int_0^\infty \mathrm{d}\lambda J_1(|\boldsymbol{b}|\lambda)\hat{\boldsymbol{b}}\\
&= \frac{2Gm_1m_2}{|\beta|}\gamma\frac{b^\mu}{b^2}
\end{aligned} \tag{3.18}$$

Before going to see the impulse of gravity in the (2+1) dimension, we use the result (3.18) to calculate the scattering angle, and then compare the result to the scattering angle calculated by the orbital equation.

The scattering angle can be calculated by impulse via $\tan(\Delta\theta) = \frac{|\boldsymbol{\Delta P_1}|}{|\boldsymbol{P_{int}}|}$, where $|\boldsymbol{P_{int}}|$ is the magnitude of the initial momentum of the probing particle.

$$\begin{aligned}
|\boldsymbol{P_{int}}| &= m_1\gamma\beta\\
|\boldsymbol{\Delta P_1}| &= \frac{2Gm_1m_2}{|\beta|}\gamma\frac{1}{b}\\
\Delta\theta &= \tan^{-1}\left(\frac{2Gm_2}{|\beta|^2 b}\right)
\end{aligned} \tag{3.19}$$

Next, we calculate the scattering angle by using the orbital equation mentioned in Goldstein, Poole, and Safko's Classical Mechanics Chapter 3[8]

$$\theta(b) = \pi - 2\int_{r_m}^\infty \frac{b\mathrm{d}r}{r\sqrt{r^2(1-\frac{U(r)}{E})-b^2}} \tag{3.20}$$

where r_m is the closest distance of the two particles, U is the potential energy and E is the

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total energy.

Plugging the potential energy of this system into the equation (3.20), we can get

$$\Delta\theta = \pi - 2\int_{r_m}^{\infty} \frac{b\mathrm{d}r}{r\sqrt{r^2(1+\frac{Gm_1m_2}{rE}) - b^2}}$$
$$= \pi - 2\tan^{-1}\left(\frac{\frac{Gm_1m_2r}{2E} - b^2}{b\sqrt{r^2 + \frac{Gm_1m_2r}{E} - b^2}}\right)\Bigg|_{r_m}^{\infty} \tag{3.21}$$

To find r_m, we need to solve $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r\sqrt{r^2(1+\frac{Gm_1m_2}{rE}) - b^2}}{b} \tag{3.22}$$

r_m satisfies the below equation

$$\frac{r_m\sqrt{r_m^2(1+\frac{Gm_1m_2}{r_mE}) - b^2}}{b} = 0$$
$$\sqrt{r_m^2(1+\frac{Gm_1m_2}{r_mE}) - b^2} = 0 \tag{3.23}$$

Combining equation (3.23) and $E = \frac{1}{2}m_1(\gamma\beta)^2$ (since we suppose at $\tau = 0$, $r = -\infty$, total energy $E \approx$ total kinetic energy.) with equation (3.21), we can get the scattering angle

$$\Delta\theta = 2\pi - 2\tan^{-1}\left(\frac{Gm_2}{|\beta|^2 b}\right) \tag{3.24}$$

The difference between the two results is that they are in different coordinates. The scattering angle calculated by $\tan(\Delta\theta) = \frac{|\boldsymbol{\Delta P_1}|}{|\boldsymbol{P_{int}}|}$ is the angle between the incoming and outgoing momentum of the probing particle, so it is the scattering angle in the laboratory frame. On the other hand, the scattering angle calculated by the orbital equation is the angle between the two particles after scattering, so it is the scattering angle in the center of mass frame.

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Now, we can go back to see the classical impulse of gravity in (2+1) dimension in position space.

The scattering impulse of gravity in the (2+1) dimension is

$$\Delta P_1^\mu = i8\pi G m_1 m_2 \int \hat{\mathrm{d}}^3 q \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot q) e^{iq\cdot b} \frac{q^\mu}{q^2}((u_1 \cdot u_2)^2 - 1) \tag{3.25}$$

As before, we set $u_1 = (\gamma, 0, \gamma\beta)$ and $u_2 = (1, 0, 0)$, where the velocity parameter β satisfying $\gamma^2(1 - \beta^2) = 1$ to simplify the result.

$$\begin{aligned}\Delta P_1^\mu &= i8\pi G m_1 m_2 \int \hat{\mathrm{d}}^3 q \hat{\delta}(q_0)\hat{\delta}(\gamma q_0 - \gamma\beta q_2) e^{iq\cdot b} \frac{q^\mu}{q^2}(\gamma^2 - 1) \\ &= \frac{i4G m_1 m_2}{|\beta|} \int \mathrm{d}q e^{ib\cdot q_\perp} \frac{q_\perp^\mu}{q_\perp^2}(\gamma - \frac{1}{\gamma})\end{aligned} \tag{3.26}$$

Since we set u_1 at y-direction, the $q_\perp$ is at x-direction. Let λ be the magnitude of $q_\perp$, therefore, we can set $q_\perp^\mu = (0, \lambda, 0)$.

$$\begin{aligned}\Delta P_1^\mu &= \frac{i4G m_1 m_2}{|\beta|} \int_{-\infty}^{\infty} \mathrm{d}\lambda e^{i|\boldsymbol{b}|\lambda} \frac{q_\perp^\mu}{\lambda^2}(\gamma - \frac{1}{\gamma}) \\ &= \frac{-4G m_1 m_2 \pi}{|\beta|}(\gamma - \frac{1}{\gamma})\hat{b}\end{aligned} \tag{3.27}$$

We can see that although in momentum space, gravity in (3+1) dimension or in (2+1) looks like they have similar physical properties, in position space, they look different. That is, we can see that for gravity in (3+1) dimension, the scattering impulse is proportional to the inverse of the impact parameter ($\frac{1}{b}$), but for gravity in (2+1) dimension, the scattering impulse is independent of the magnitude of the impact parameter (b).

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Chapter 4 Chern-Simons term and Anyon impulse

Now we are going to see how to calculate the impulse of anyon. For electrodynamic Lagrangian in (2+1) dimension, apart from the Maxwell kinetic term, we can also have another term $k\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}$ added to the Lagrangian, which is called the Chern-Simons term. We will illustrate the solutions of the equations of motion of this Lagrangian has arbitrary spin and use these equations of motion to calculate the scattering impulse which is written in D. J Burger, W. T. Emond, and N. Moynihan's "Anyons and the Double Copy"[2].

The electrodynamic Lagrangian added a Chern-Simons term is

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + k\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \tag{4.1}$$

where k is the Chern-Simons level number. Add $A_\mu J^\mu$ to this Lagrangian, and calculate the equations of motion with source,

$$\partial_\nu F^{\mu\nu} - ke^2\epsilon^{\mu\nu\rho}\partial_\nu A_\rho = J^\mu \tag{4.2}$$

The total charge associated with anyon is

$$Q = \int \mathrm{d}^2x J^0 = \int \mathrm{d}^2x \partial_i E^i - ke^2 \int \mathrm{d}^2x \epsilon^{ij}\partial_i A_j = 4\pi q - ke^2\Phi \tag{4.3}$$

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where Φ is the magnetic flux. We see that the anyon can carry both electric charge and magnetic flux. Therefore, if we add a Chern-Simons term to the Lagrangian, each charged particle will be attached to magnetic flux. Attaching a magnetic flux will infect the angular momentum of particles. In quantum mechanics, the angular momentum operator in an electromagnetic field is

$$\hat{L} = \epsilon^{ij}\hat{r}_i[\hat{p}_j - qA_j] \tag{4.4}$$

where $\hat{r}_i$ and $\hat{p}_i$ are position and momentum operator respectively. And since

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \Phi \tag{4.5}$$

The magnitude of magnetic vector potential is

$$|\boldsymbol{A}| = \frac{\Phi}{2\pi r} \tag{4.6}$$

and the magnetic vector potential is in the direction perpendicular to its position vector $\boldsymbol{r}$.

We can now write down the angular momentum operator in polar coordinate

$$\hat{L} = -i\partial_\theta - q\frac{\Phi}{2\pi} \tag{4.7}$$

The wave function of this system is composed of the eigenfunction of its Hamiltonian operator. Since this system is rotationally invariant, $[\hat{H}, \hat{L}] = 0$, the $\hat{H}$ and $\hat{L}$ share the same eigen function. The angular part of eigenfunctions ψ_n is

$$\psi_n(\theta) \propto e^{in\theta} \tag{4.8}$$

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Since

$$|\psi_n(0)|^2 = |\psi_n(2\pi)|^2 \tag{4.9}$$

n must be an integer or half-integer.

Under a rotation of angle $\theta = \pi$

$$\psi_n(\theta + \pi) = e^{i\pi\hat{L}}\psi_n(\theta) = e^{i(\pi n - \frac{1}{2}q\Phi)}\psi_n(\theta) \tag{4.10}$$

If $\Phi \neq 0$, the interchange of a two-particle state will pick up $e^{-iq\Phi}$ other than ± 1. Therefore the spin of this wave function can be an arbitrary number.

Since we have seen that the solutions of the equations of motion of the Lagrangian containing Chern-Simons term have arbitrary spin, we can then use this equation of motion to calculate the impulse of the anyon[1].

Suppose a scalar particle with charge Q_1 and moving at the velocity u_1 probes an anyon with charge Q_2 and magnetic flux Φ_2. By Ampère's circuital law

$$\nabla \times \boldsymbol{B} = \boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t} \tag{4.11}$$

In this system, there is no change of the electric field term, so the current of anyon is

$$J_2^\mu = Q_2 u_2^\mu \delta^{(2)}(r) + \epsilon^{\mu\nu\rho}\partial_\nu u_{2\rho}\Phi_2\delta^{(2)}(r) \tag{4.12}$$

Put the current (4.12) into the equation (4.2), we can see the equation of $F^{\mu\nu}$ is

$$(\eta^{\mu\rho}\partial^\nu + \frac{ke^2}{2}\epsilon^{\mu\nu\rho})F_{\nu\rho} = Q_2 u_2^\mu \delta^{(2)}(r) + \epsilon^{\mu\nu\rho}\partial_\nu u_{2\rho}\Phi_2\delta^{(2)}(r) \tag{4.13}$$

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Define $m = ke^2$, and write down equation (4.13) in terms of the dual vector field $\tilde{F}^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$

$$(m\eta^{\mu\nu} + \epsilon^{\mu\nu\rho}\partial_\rho)\tilde{F}_\nu = (Q_2 u_2^\mu + \epsilon^{\mu\nu\rho}\partial_\nu u_{2\rho}\Phi_2)\delta^{(2)}(r) \tag{4.14}$$

To derive the electromagnetic tensor in momentum space, we know

$$\begin{aligned}
&(m\eta^{\mu\alpha} + \epsilon^{\mu\alpha\rho}\partial_\rho)(m\eta_{\alpha\nu} + \epsilon_{\alpha\nu\lambda}\partial^\lambda)\tilde{F}^\nu \\
&= (\epsilon^{\mu\alpha\rho}\epsilon_{\alpha\nu\lambda}\partial_\rho\partial^\lambda + m\epsilon^{\mu\alpha\rho}\partial_\rho\eta_{\alpha\nu} + m\eta^{\mu\alpha}\epsilon_{\alpha\nu\lambda}\partial^\lambda + m^2\delta^\mu_\nu)\tilde{F}^\nu
\end{aligned} \tag{4.15}$$

The right hand side of equation (4.15) is

$$\begin{aligned}
&(\epsilon^{\mu\alpha\rho}\epsilon_{\alpha\nu\lambda}\partial_\rho\partial^\lambda + m\epsilon^{\mu\alpha\rho}\partial_\rho\eta_{\alpha\nu} + m\eta^{\mu\alpha}\epsilon_{\alpha\nu\lambda}\partial^\lambda + m^2\delta^\mu_\nu)\tilde{F}^\nu \\
&= [-(\delta^\mu_\nu\delta^\rho_\lambda - \delta^\rho_\nu\delta^\mu_\lambda)\partial_\rho\partial^\lambda + 2m\epsilon^\mu{}_{\nu\rho}\partial^\rho + m^2\delta^\mu_\nu]\tilde{F}^\nu \\
&= [(-\partial^2)\delta^\mu_\nu + \partial_\nu\partial^\mu + 2m\epsilon^\mu{}_{\nu\rho}\partial^\rho + m^2\delta^\mu_\nu]\tilde{F}^\nu
\end{aligned} \tag{4.16}$$

Choose a gauge such that $\partial_\mu\tilde{F}^\mu = 0$

$$\begin{aligned}
&(\epsilon^{\mu\alpha\rho}\epsilon_{\alpha\nu\lambda}\partial_\rho\partial^\lambda + m\epsilon^{\mu\alpha\rho}\partial_\rho\eta_{\alpha\nu} + m\eta^{\mu\alpha}\epsilon_{\alpha\nu\lambda}\partial^\lambda + m^2\delta^\mu_\nu)\tilde{F}^\nu \\
&= [(-\partial^2)\delta^\mu_\nu + 2m\epsilon^\mu{}_{\nu\rho}\partial^\rho + m^2\delta^\mu_\nu]\tilde{F}^\nu
\end{aligned} \tag{4.17}$$

Combine equation (4.17) and (4.14), we can get

$$(\partial^2 + m^2)\tilde{F}^\mu = (m\eta^{\mu\nu} - \epsilon^{\mu\nu\rho}\partial_\rho)J_{2\nu} \tag{4.18}$$

Express the current in momentum space,

$$J_2^\mu(q) = (Q_2 u_2^\mu + \epsilon^{\mu\nu\rho} i q_\nu u_{2\rho}\Phi_2)\delta(q \cdot u_2) \tag{4.19}$$

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Then we can write the dual vector tensor in momentum space as

$$\tilde{F}^{\mu}(q) = \frac{\delta(q\cdot u_2)}{q^2+m^2}(Q_2(mu_2^{\mu} - i\epsilon^{\mu\nu\rho}u_{2\nu}q_{\rho}) + i\Phi_2(q^{\mu}(u_2\cdot q) - u_2^{\mu}q^2) + im\Phi_2\epsilon^{\mu\alpha\beta}q_{\alpha}u_{2\beta}) \tag{4.20}$$

The electromagnetic field tensor is

$$F_{\mu\nu}(q) = \epsilon_{\mu\nu\rho}\tilde{F}^{\rho} = \frac{\delta(q\cdot u_2)}{q^2+m^2}[Q_2(-iu_{2\mu}q_{\nu} + iu_{2\nu}q_{\mu} + m\epsilon_{\mu\nu\rho}u_2^{\rho}) + i\Phi_2\epsilon_{\mu\nu\rho}(q^{\rho}(u_2\cdot q) - u_2^{\rho}q^2) + im\Phi_2(q_{\mu}u_{2\nu} - q_{\nu}u_{2\mu})] \tag{4.21}$$

Plug the $F^{\mu\nu}(q)$ into equation (2.3), the scattering impulse is

$$\Delta P_1^{\mu} = Q_1\int \hat{d}^3q\delta(u_1\cdot q)\delta(u_2\cdot q)e^{iq\cdot b} \times \frac{(iq^{\mu}(Q_2+m\Phi_2)(u_1\cdot u_2) + mQ_2\epsilon^{\mu\nu\rho}u_{1\nu}u_{2\rho} - i\Phi_2\epsilon^{\mu\nu\rho}q^2u_{1\nu}u_{2\rho})}{q^2+m^2} \tag{4.22}$$

Use the Schouten identity[10] with the convention $\epsilon(a,b,c) = \epsilon^{\mu\nu\rho}a_{\mu}b_{\nu}c_{\rho}$ and

$$\epsilon^{\mu}(a,b) = \epsilon^{\mu\nu\rho}a_{\nu}b_{\rho}$$

$$a^{\mu}\epsilon(b,c,d) = \epsilon^{\mu}(c,d)(a\cdot b) - \epsilon^{\mu}(b,d)(a\cdot c) + \epsilon^{\mu}(b,c)(a\cdot d) \tag{4.23}$$

The impulse (4.22) can be written down as

$$\Delta P_1^{\mu} = Q_1\int \hat{d}^3q\delta(u_1\cdot q)\delta(u_2\cdot q)e^{iq\cdot b}iq^{\mu} \times [\frac{((Q_2+m\Phi_2)(u_1\cdot u_2) - \Phi_2\epsilon(u_1,u_2,q))}{q^2+m^2} - \frac{imQ_2\epsilon(u_1,u_2,q)}{q^2(q^2+m^2)}] \tag{4.24}$$

where $q^{\mu}\epsilon(u_1,u_2,q) = \epsilon^{\mu}(u_2,q)(q\cdot u_1) - \epsilon^{\mu}(u_1,q)(u_2\cdot q) + \epsilon^{\mu}(u_1,u_2)q^2$ and in the impulse (107), by delta functions $\delta(u_1\cdot q)$ and $\delta(u_2\cdot q)$, we can neglect the $(u_1\cdot q)$ and $(u_2\cdot q)$

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terms.

Use relation (1.33), the scattering amplitude is

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q)$$
$$= \frac{4((Q_1Q_2 + mQ_1\Phi_2)(P_1 \cdot P_2) - Q_1\Phi_2\epsilon(P_1, P_2, q))}{q^2 + m^2} - \frac{i4mQ_1Q_2\epsilon(P_1, P_2, q)}{q^2(q^2 + m^2)} \quad (4.25)$$

Similar to the calculation of scattering amplitude in the (2+1) dimension, we need to check whether the $\hbar$ dependence will cancel out.

First, we can use Gauss's law to see the dimension of charge Q in the (2+1) dimension.

$$\oint_c \mathbf{E} \cdot \hat{\mathbf{n}} \mathrm{d}s = Q \quad (4.26)$$

where **E** is the electric field. From (4.26), we can know the magnitude of the electric field in the (2+1) dimension is

$$E(r) = \frac{Q}{2\pi r} \quad (4.27)$$

The dimension of charge Q is $M^{1/2}$.

Next, we can use $\Phi = \int \mathrm{d}^2x\epsilon^{ij}\partial_i A_j$ to find the dimension of magnetic flux Φ. The action $\mathcal{S}$ divided by $\hbar$ is dimensionless, so from the action for electromagnetic field in the (2+1) dimension,

$$\mathcal{S} = -\int \mathrm{d}^3x \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \quad (4.28)$$

We can know the dimension of A^μ in the (2+1) dimension is $M^{1/2}$. Therefore, the dimension of magnetic flux Φ is

$$[\Phi] = M^{1/2} \cdot L \quad (4.29)$$

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For the scattering amplitude (4.25), the dimension of the term $\frac{Q_1 Q_2 (P_1 \cdot P_2)}{q^2+m^2}$ and the term $\frac{m Q_1 Q_2 \epsilon(P_1, P_2, q)}{q^2(q^2+m^2)}$ are M, so there is no need to add any $\hbar$. On the other hand, the dimension of the term $\frac{m Q_1 \Phi_2 (P_1 \cdot P_2)}{q^2+m^2}$ and the term $\frac{Q_1 \Phi_2 \epsilon(P_1, P_2, q)}{q^2+m^2}$ are M^2/L, so they must be multiplied by $1/\hbar$. However, remember that we also need to restore $\hbar$ in each q and m, the q and m in the numerator for the two terms will restore a $\hbar$ which cancels out the $1/\hbar$.

Finally, we can see as expected, that all $\hbar$ cancel out, so we can indeed use relation (1.33) to match the scattering amplitude of anyon from the scattering impulse.

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Chapter 5 Discussion

In [6], we see that we can derive the scattering impulse from the scattering amplitude. Next, for electromagnetism and gravity in the (3+1) dimension, we see that we can derive the scattering amplitude from the scattering impulse. Therefore, we can use the relation (5.1) between scattering impulse and scattering amplitude to find the scattering amplitude in two special cases in the (2+1) dimension.

$$\Delta P^{\mu} = \frac{1}{4m_1 m_2} \int \hat{\mathrm{d}}^4 \bar{q} \hat{\delta}(u_1 \cdot \bar{q}) \hat{\delta}(u_2 \cdot \bar{q}) e^{ib\cdot\bar{q}} i\bar{q}^{\mu} \bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + \bar{q}, P_2 - \bar{q}) \tag{5.1}$$

For gravity in (2+1) dimension[4][2], we find that the leading component of scattering amplitude (5.3) looks similar to gravity in (3+1) dimension (5.2).

For gravity in (3+1) dimension

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 16\pi G \frac{(P_1 \cdot P_2)^2}{q^2} \tag{5.2}$$

For gravity in (2+1) dimension

$$\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q) = 32G\pi \frac{(P_1 \cdot P_2)^2 - (m_1 m_2)^2}{q^2} \tag{5.3}$$

If we treat the mass term as the zero component of the momentum in the (3+1) dimension, they look the same.

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However, the scattering impulse in position space in different spacetime dimensions looks different.

The scattering impulse of gravity in (3+1) dimension (5.4) depends on the magnitude of impact parameter by $\frac{1}{|\mathbf{b}|}$.

$$\Delta P^{\mu} = \frac{2Gm_1m_2}{|\beta|}\gamma\frac{b^{\mu}}{b^2} \tag{5.4}$$

But the scattering impulse of gravity in the (2+1) dimension doesn't depend on the magnitude of the impact parameter.

$$\Delta P^{\mu} = \frac{-4Gm_1m_2\pi}{|\beta|}(\gamma - \frac{1}{\gamma})\hat{b} \tag{5.5}$$

In the final part, we use relation (5.1) to match the scattering amplitude of anyon from the classical impulse[2].

$$\begin{aligned}&\bar{\mathcal{A}}^{(0)}(P_1, P_2 \to P_1 + q, P_2 - q)\\&= \frac{4((Q_1Q_2 + mQ_1\Phi_2)(P_1 \cdot P_2) - Q_1\Phi_2\epsilon(P_1, P_2, q))}{q^2 + m^2} - \frac{i4mQ_1Q_2\epsilon(P_1, P_2, q)}{q^2(q^2 + m^2)}\end{aligned} \tag{5.6}$$

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