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碩士論文

Department of Mathematics

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Master's Thesis

Erdős-Ko-Rado 定理的變體：Borg 和 Erdős-Rothschild 問題

Variations of the Erdős-Ko-Rado theorem: Borg and Erdős-Rothschild Problem

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致謝

這篇論文對我而言，是在台大數學系這七年中的種種，緩慢累積而成的果實。為此，我不免俗地需要感謝這幾年以來，與我攜手並進的同學，以及為我指引方向的師長與前輩們。而過去三年，身為指導教授的戴尚年教授，總是給予我很多研究上的幫助，可以說若是沒有他的指導，此篇論文必定無法完成。最後，我需要感謝我的父母，給予我充裕的自由與時間，讓我能夠無後顧之憂地投身這幾年的學習和研究。

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摘要

給定一個整數 t，如果一個集合族中的任意兩個集合的交集大小至少為 t，則稱該集合族為 **t-交叉的**（**t-intersecting**），當 $t = 1$ 時，則簡稱為**交叉的**（**intersecting**）。

考慮 $1, 2, ..., n$ 的所有大小為 k 的集合所形成的集合族。跟據 Erdős–Ko–Rado 定理，如果 n 小於等於 $2k$，那麼在該集合族所有的交叉的集合子族中，平凡的集合族（即包含一個單點的所有集合所形成的集合族）的大小會達到最大值。在本文中，我們介紹了關於該定理的兩類相關推廣問題。

第一類是討論 r-標記集合（r-signed sets）上的問題。給定整數 r，一個 r-標記集合由一個集合 F 和一個定義在其上，取值為 1 至 r 的函數組成。我們將探討 Borg 提出的兩個相關猜想，並解釋其與 Chvátal 和 Kleitman 猜想的關係。

第二部分當中，我們引入了另一個擴展該問題的方向，也就是將 Erdős–Rothschild 問題的核心想法引入交叉集合族中。在這一問題裡，需要最大化的目標從集合族的大小變為集合族的 3-著色（3-colorings）數量，且要求每個種顏色的集合族都是 t-交叉的。

關鍵字： Erdős–Ko–Rado 定理、標記集合、交叉集合族、Erdős–Rothschild 定理、集合著色

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Abstract

Given an integer t, a set family is called t**-intersecting** if any two sets in it have an intersection of size at least t and is abbreviated as **intersecting** if $t = 1$.

Let $\binom{[n]}{k}$ denote all subsets of $\{1, 2, \ldots, n\}$ with size k. The Erdős–Ko–Rado theorem says that if n is at least $2k$, then the size of an intersecting family in $\binom{[n]}{k}$ is maximized by the trivial one, which means an intersecting family consisting of a common element. There are many different ways of generalizing the theorem, and we introduce two related problems in this thesis.

The first is to consider the question on the r-signed sets, which is a pair of a set F and a function on it with values ranging from 1 to r, where r is a given integer. We will discuss two related conjectures proposed by Borg and explain the relation with Chvátal's and Kleitman's Conjectures.

In the second part, we introduce another variant problem extended with the idea of the Erdős–Rothschild problem. In this question, the target that should be maximized is changed from the size of a set family to the number of 3-colorings of a set family such that each monochromatic part is t-intersecting.

Keywords: Erdős–Ko–Rado, signed sets, intersecting family, Erdős–Rothschild, set colouring

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Chapter 1 Introduction

1.1 Basic definition

Let $\mathcal{F}$ be a family of sets, we say that $\mathcal{F}$ is an **intersecting family** if for any $F_1, F_2 \in \mathcal{F}$, $F_1 \cap F_2 \neq \emptyset$. There are many problems related to the intersection of sets, and one of the most frequently asked questions is how large an intersecting subfamily can be in a given family. We begin the discussion with the following example. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ for the integer n.

Theorem 1.1.1. *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting family. Then $|\mathcal{F}| \leq 2^{n-1}$.*

One can prove this result by observing that an intersecting family $\mathcal{F}$ cannot contain a set A and its complement A^c at the same time. Although this example seems a little bit trivial, there is something to note. As can be easily seen, a **star**, which is a special type of set family consisting of all sets containing a fixed element, attains the upper bound. However, this is not the only family (up to permutations of the ground set $[n]$) that satisfies the bound. For example, when n is odd, the family $\mathcal{F}_{\geq n/2} := \{F \in \mathcal{F} : 2 \cdot |F| > n\}$ also has size 2^{n-1} and one can verify that $\mathcal{F}_{\geq n/2}$ is an intersecting family by using the Pigeonhole principle.

Next, we introduce the first cornerstone for intersection-related problems, the Erdős-Ko-Rado theorem.

Theorem 1.1.2 (Erdős-Ko-Rado, Erdős (1961))**.** *Let $n \geq 2k$ and $\binom{[n]}{k}$ denote the family of all subsets in $[n]$ with size k. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

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Moreover, if $n > 2k$, then stars are the only families of sets that reach the bound.

There are many different ways to generalize the result. For instance, Erdős et al. considered and gave some partial result of the following one in the same paper: fixing an integer t, one can define the t-**intersecting** family $\mathcal{F}$ by requiring every pair of sets to have an intersection of size at least t and ask how large a t-intersecting family can be.

As in the intersecting case, we define a t-**star** to the family that contains t common elements. Once again, the t-star family beats all other possible families in terms of size when n is sufficiently large.

Theorem 1.1.3 (Erdős (1961))**.** *Let $\binom{[n]}{k}$ denote the family of all subsets in $[n]$ with size k. There exists an integer $n_0 = n_0(k,t)$ such that if $n \geq n_0$, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Moreover, the equality holds if and only if $\mathcal{F}$ is a family consisting of all k-subsets containing t common elements.

The reason why we said this result is partial is due to the bound n_0 is not optimal. It was until 20 years later when Frankl and Wilson gave the exact bound $n_0 = (k-t+1)(t+1)$ in Frankl and Wilson (1981). In fact, in these EKR type questions, determining the exact bound is much harder than showing the growth rate of a star family beats all other intersecting families in terms of sizes.

The t-intersecting EKR theorem is later generalized to a beautiful result, the complete intersection theorem, proven by Ahlswede and Khachatrian (1997).

Theorem 1.1.4 (Ahlswede and Khachatrian (1997))**.** *For integer $0 \leq r \leq k-t$, let*

$$n_r = (k-t+1) \cdot \frac{t-1}{r},$$

with convention $\frac{t-1}{0} = \infty$. Then, if

$$n_r \leq n \leq n_{r+1},$$

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the size of the largest t-intersecting family in $\binom{[n]}{k}$ is maximized by the family

$$\mathcal{F}_r := \{F \in \binom{[n]}{k} : |F \cap [t+2r]| \geq t+r\}.$$

Although we have this ultimate intersecting theorem for set families, there are still many open problems extended from the EKR problem that remain unsolved. In this thesis, we introduce two related problems of this kind.

1.2 r-signed sets and Borg's Conjecture

One common extension is to ask the EKR type question for different combinatorial objects and here we consider this question on the r-**signed set** family.

Definition 1.2.1. *Let r be an integer, F be a subset of $[n]$. We define the r**-signed set** on F by a pair of set with a function on F with range $[r]$. i.e.*

$$\mathcal{S}_{\{F\},r} := \{(F, f) : \text{ where } f \text{ is a functions from } F \text{ to } [r]\}.$$

Equivalently, given $y_i = f(x_i)$, one can express an r-signed sets in the following form

$$\mathcal{S}_{\{F\},r} := \{(x_1, y_1), \ldots, (x_k, y_k) : \{x_1, \ldots, x_k\} = F, y_i \in [r]\}.$$

And for a family $\mathcal{F} \subseteq 2^{[n]}$, we define the family of r-signed sets on $\mathcal{F}$ by

$$\mathcal{S}_{\mathcal{F},r} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{\{F\},r}.$$

In the second characterization, each signed set is a subset of $[n] \times [r]$ of size k, so the intersection on $[n] \times [r]$ provides a natural definition for the intersection of two r-signed sets. The Erdős–Ko–Rado problem can then be asked on this special set family and is already proven for many particular set families $\mathcal{F}$, which will be introduced in Chapter 2 more explicitly.

One of the main results of this thesis is related to the following conjecture proposed by Borg.

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Conjecture 1.2.2 (Borg (2009))**.** *There exists a constant $r_0(t)$ such that for any $r \geq r_0$, the largest t-intersecting subfamilies of $\mathcal{S}_{\mathcal{F},r}$ are t-stars.*

Although no proof or disproof is given in this thesis, we managed to find a tighter bound for the lower bound of r when the underlying family $\mathcal{F}$ and t is fixed. We let $\alpha(\mathcal{F})$ denotes the size of the largest set in $\mathcal{F}$.

Theorem 1.2.3. *Let $\mathcal{F} \subseteq 2^{[n]}$ be any family. Given integers t, for all $r > r_0 = e \cdot (t+1) \cdot \alpha(\mathcal{F})$, the largest t-intersecting subfamilies of $\mathcal{S}_{\mathcal{F},r}$ are t-stars.*

Two years after Conjecture 1.2.2 was proposed, Borg gave a new related conjecture about the r-signed sets intersecting family.

Conjecture 1.2.4 (Borg, Borg (2011*b*))**.** *Given a weight function $w : 2^{[n]} \to \mathbb{R}$ that satisfies $w(F') \geq w(F)$ for all $F' \subseteq F, F', F \in \mathcal{F}$. The function*

$$w(\mathcal{F}) := \sum_{F \in \mathcal{F}} w(F)$$

is maximized by some star families among all intersecting families.

Conjecture 1.2.4 implies that the largest intersecting r-signed set family is as large as a star family. On top of that, this conjecture also has some relation with Chvátal's and Kleitman's Conjecture.

1.3 Erdős–Rothschild type extension

Definition 1.3.1. *Let $c, t \in \mathbb{N}, c \geq 2$ a **(c, t)-colorings** of a set family $\mathcal{F}$ is a c-coloring on elements of $\mathcal{F}$ such that every color class forms a t-intersecting family.*

Given this definition, the following question is asked. What set family maximizes the number of (c, t)-coloring?

We say a t-intersecting family is **maximal** if we cannot find another set that can be added to the family while preserving the t-intersecting property.

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For $c = 2$, given a set family $\mathcal{F}$ and a maximal t-intersecting subfamily $\mathcal{F}_0$ of it. Since $\mathcal{F}_0$ is maximal, for every $F \notin \mathcal{F}_0$, there exists $G \in \mathcal{F}_0$ such that $|F \cap G| < t$. Because we only have two colors, the color of F is determined by the color of G. Therefore, the 2-coloring on $\mathcal{F}_0$ is actually decided by the 2-coloring on $\mathcal{F}$, which gives an upper bound $2^{|\mathcal{F}_0|}$. On the other hand, taking $\mathcal{F}$ to be one of the largest t-intersecting families and considering all 2-colorings on it gives a construction that matches the bound.

In this thesis, we will focus on the 3 color case and consider the problem with a set family replaced with the multiset family and the r-signed set family on $\binom{[n]}{k}$. We will show that the trivial construction still beats other possible families in these two settings provided some condition on the variables.

Theorem 1.3.2. *The trivial t-intersecting family maximizes the number of $(3, t)$-colorings in multisets family and the r-signed sets family over $\binom{[n]}{k}$ with n, k, r satisfying certain constraints.*

1.4 Outline of this thesis

We explain how we obtain the improved bound on r_0 in Theorem 1.2.3 in Chapter 2 and explain the relation of 1.2.4 with Chvátal's and Kleitman's Conjectures in Chapter 3. In Chapter 4, we will explain more about the results of the Erdős-Rothschild problem and look at the Erdő–Rothschild type extension of the EKR problem on multiset and r-signed set families.

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Chapter 2 The Erdős-Ko-Rado Theorem for signed set family

2.1 Background

We start with some simple EKR type theorem for signed set. In the following, we say that a family has (t-)EKR property if some (t-)star is one of its largest (t-)intersecting subfamily.

Theorem 2.1.1. *The family $\mathcal{S}_{\{F\},r}$ has the EKR property.*

Proof. By construction, we know that a star subfamily in $\mathcal{S}_{F,r}$ has size r^{k-1}, where k is the size of F. On the other hand, let $\mathcal{A}$ be an intersecting subfamily in $\mathcal{F}$. For each $i \in [r]$, let $f + i_F$ be the function obtained from f by adding a constant i to every output of f, where the addition is performed modulo r. Then define

$$\mathcal{A}_i := \{(F, f + i_F) : (F, f) \in \mathcal{A}\}.$$

Claim 2.1.2. *$\mathcal{A}_1, \ldots, \mathcal{A}_r$ are pairwise disjoint.*

Suppose otherwise, then we can find $i, j \in [n]$ and $(F, f_1), (F, f_2) \in \mathcal{A}$ such that $f_1 + i \equiv f_2 + j$. This implies for all $x \in F$, $f_1(x) + i - j = f_2(x) \implies f_1(x) \neq f_2(x)$. Thus, (F, f_1) and (F, f_2) are two disjoint signed sets in the intersecting signed sets family $\mathcal{A}$, contradiction.

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This claim shows that an intersecting family in $\mathcal{S}_{\{F\},r}$ cannot have a size larger than r^{k-1} and thus proves the EKR property of $\mathcal{S}_{\{F\},r}$. □

Now, we look at r-signed sets on a given family $\mathcal{F}$. For the t-EKR problem some special case like $\mathcal{F} = \binom{[n]}{k}$, have been solved(Bey (1999) for $t \leq k < n$, Frankl and Tokushige (1999), Ahlswede and Khachatrian (1998) for $k = n$). In addition, Feghali (2019) gives an injective proof for the intersecting case of $\mathcal{F} = \binom{[n]}{k}$ when $n > 2k$.

In Borg (2007) Borg generalized the EKR problem on r-signed to the general family $\mathcal{F}$ and made the following conjecture.

Conjecture 2.1.3 (Borg (2007))**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be any family, stars are the largest among all intersecting family in $\mathcal{S}_{\mathcal{F},r}$. Moreover, they are the unique structure if $r \geq 3$.*

Remark 2.1.4. *For the case $r = 2$, there exists another construction as large as the star family. The author also conjectured that this is the only one other possible construction for the largest intersecting family in $\mathcal{S}_{\mathcal{F},2}$.*

In the same paper, the author verified the conjecture for the family $\mathcal{F}$ with some property.

Definition 2.1.5. *Given $\mathcal{F} \subseteq 2^{[n]}$ and an element $x \in [n]$, we say that $\mathcal{F}$ is compressed to x if for any set $F \in \mathcal{F}$ and $y \in F$, the set $F \setminus \{y\} \cup \{x\}$ is also in $\mathcal{F}$.*

Theorem 2.1.6 (Borg (2007))**.** *Conjecture 2.1.3 is true if $\mathcal{F}$ is compressed with respect to some element x.*

Two years later, the author showed the r-signed set family $\mathcal{S}_{\mathcal{F},r}$ has t-EKR property provided $r \geq r_0$, which depends on t and the family $\mathcal{F}$.

Theorem 2.1.7 (Borg (2009))**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be any family. Given integers t, for all $r > r_0 = \binom{\alpha(\mathcal{F})}{t}\binom{\alpha(\mathcal{F})}{t+1}$, the largest t-intersecting subfamilies of $\mathcal{S}_{\mathcal{F},r}$ are t-stars.*

Then he conjectured that we can actually give a lower bound r_0 independent of the family $\mathcal{F}$.

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Conjecture 2.1.8 (Borg (2009))**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be any family. There exists a constant $r_0(t)$ such that for any $r \geq r_0$, the largest t-intersecting subfamilies of $\mathcal{S}_{\mathcal{F},r}$ are t-stars.*

Although we are still unsure whether Conjecture 1.2.2 is true or not, we have improved the lower bound of r_0.

2.2 Proof of improved bound on r_0

We first recall the statement of Theorem 1.2.3 and let $\alpha(\mathcal{F})$ denote the size of largest set in $\mathcal{F}$.

Theorem 1.2.3. *Let $\mathcal{F} \subseteq 2^{[n]}$ be any family. Given integers t, for all $r > r_0 = e \cdot (t + 1) \cdot \alpha(\mathcal{F})$, the largest t-intersecting subfamilies of $\mathcal{S}_{\mathcal{F},r}$ are t-stars. Here, $\alpha(\mathcal{F})$ denotes the size of the largest set in $\mathcal{F}$.*

Compared with the exponential bound in Theorem 2.1.7, the lower bound of r is adapted to a function that is linear in $t \cdot \alpha(\mathcal{F})$.

We improve Borg's argument in Borg (2009) by making use of the following definition. Note that for a t-intersecting signed set family $\mathcal{A}$, its **domain** $\mathcal{D}(\mathcal{A}) := \{F : (F, f) \in A \text{ for some } f\}$ is t-intersecting. By considering the same question on all t-intersecting subfamilies of $\mathcal{F}$, we may assume that $\mathcal{F}$ is a t-intersecting family and consider $\mathcal{D}(\mathcal{A}) = \mathcal{F}$ without loss of generality.

Definition 2.2.1. *Let $\mathcal{F}$ be a family, we say that A is a t-**cover** of $\mathcal{F}$ if for any set $F \in \mathcal{F}$, $|F \cap A| \geq t$.*

Let S be a t-cover of $\mathcal{F}$ and $s = |S|$. Fix an order on all subsets of S of size t. For every $F \in \mathcal{F}$, we assign F to be a member of F_T, where T is the first subset of F appearing in the ordering.

This divides $\mathcal{F}$ into $\binom{s}{t}$ disjoint families $\{\mathcal{F}_T : T \subseteq S, |T| = t\}$. Note that each $\mathcal{F}_T$ is t-intersecting and we have a t-star r-singed set family centered at T with size $r^{-t} \cdot |\mathcal{S}_{\mathcal{F}_T,r}|$.

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Notice that

$$\sum_{T\subseteq S,|T|=t} r^{-t}\cdot|\mathcal{S}_{\mathcal{F}_T,r}| = r^{-t}\cdot\sum_{T\subseteq S,|T|=t}|\mathcal{S}_{\mathcal{F}_T,r}| = r^{-t}\cdot|\mathcal{S}_{\mathcal{F},r}|.$$

By pigeonhole, we have the following observation.

Observation 2.2.2. *There exists a t-star in $\mathcal{S}_{\mathcal{F},r}$ with size at least $\binom{s}{t}^{-1}\cdot r^{-t}\cdot|\mathcal{S}_{\mathcal{F},r}|$.*

As smaller s means a larger lower bound, we should pick s as small as possible, we denote the smallest possible s with $\tau_t(\mathcal{F})$. As for upper bound, we use the following lemma proven by Borg with an observation.

Lemma 2.2.3. *For any t-intersecting family $\mathcal{A}\subseteq\mathcal{S}_{\mathcal{F},r}$, there exists an t-intersecting family $\mathcal{A}^*$ such that for any $A_1,A_2\in\mathcal{A}^*$, $|A_1\cap A_2\cap([n]\times[1])|\geq t$ and $|\mathcal{A}|=|\mathcal{A}^*|$.*

Observation 2.2.4. *Let $\mathcal{A}$ be a largest t-intersecting family with the property in Lemma* 2.2.3*. For any r-signed set (A,f) in $\mathcal{A}$, $f^{-1}(\{1\})$ is a t-cover of $\mathcal{A}$.*

We save the proof of the lemma for later and assume that $\mathcal{A}$ is the largest non-trivial t-intersecting family with property in Lemma 2.2.3. When focusing on a set $F\in\mathcal{F}$, one can rewrite the size of $|\mathcal{A}\cap\mathcal{S}_{F,r}|$ by observing that the collection of preimages of f, i.e.

$$\mathcal{B}(\mathcal{A},F) := \{f^{-1}(\{1\}) : (F,f)\in\mathcal{A}\cap\mathcal{S}_{F,r}\},$$

is a t-intersecting family. This yields the following equality

$$|\mathcal{A}\cap\mathcal{S}_{\mathcal{F},r}| = \sum_{B\in\mathcal{B}(\mathcal{A},F)}(r-1)^{|F|-|B|} = \sum_{k=s}^{|F|}|\mathcal{B}_k(\mathcal{A},F)|\cdot(r-1)^{|F|-k},$$

where $\mathcal{B}_k$ denotes sets in $\mathcal{B}$ with size k.

There are two small remarks for above equality. Firstly, the summation starts at s from the definition of t-cover, and secondly, each $\mathcal{B}_k(\mathcal{A},F)$ is a k-uniform $\{t,t+1,\dots,k-1\}$-intersecting, which is defined as follows.

Definition 2.2.5. *Given integers k,n and $L\subseteq\{0,1,\dots,k-1\}$, we say a family $\mathcal{F}$ if $\forall A,B\in\mathcal{F}$ is L-intersecting, $|A\cap B|\in L$.*

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Then we can use the well-known Ray-Chaudhuri–Wilson theorem to get a bound on $\mathcal{B}_k(\mathcal{A}, F)$.

Theorem 2.2.6 (Ray-Chaudhuri–Wilson)**.** *Given integers k, n, $L \subseteq \{0, 1, \ldots, k-1\}$ and a k-uniform L-intersecting family $\mathcal{F}$, then $|\mathcal{F}| \leq \binom{n}{|L|}$.*

Therefore, we now have

$$\begin{aligned}
|\mathcal{A} \cap \mathcal{S}_{\mathcal{F},r}| &\leq \sum_{k=s}^{|F|} \binom{|F|}{k-t} \cdot (r-1)^{|F|-k} \\
&= (r-1)^{-t} \cdot \sum_{k=s}^{|F|} \binom{|F|}{k-t} \cdot (r-1)^{|F|-(k-t)} \\
&\leq (r-1)^{-t} \cdot \binom{|F|}{s-t} \cdot r^{|F|-(s-t)} \\
&= (\frac{r-1}{r})^{-t} \cdot \binom{|F|}{s-t} \cdot r^{|F|-s} \\
&\leq e \cdot \binom{|F|}{s-t} \cdot r^{|F|-s}
\end{aligned}$$

The second inequality holds as follows. Fix a set F, one can see that

- $(r-1)^{|F|-(k-t)} =$ the number of r-signed sets (F, f) for given F for which $f^{-1}(\{1\})$ equals a $(k-t)$-subset of F.
- $r^{|F|-(s-t)} =$ the number of r-signed sets (F, f) with preimage $f^{-1}(1)$ containing a set T of size $s-t$.

Summing the first over all subsets of F with size at least $k-t$, we have LHS of the inequality. On the other hand, summing the second over all $(s-t)$-subsets of F contributes the the RHS of the inequality. The second summation overcounts some elements multiple times, leading to the inequality.

Additionally, the last inequality requires an extra assumption that $r \geq t+1$, but it is irrelevant compared to the bound we are going to get.

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The final step is to compare this upper bound with the lower bound of a t-star in $\mathcal{S}_{\mathcal{F},r}$ and to obtain a bound of r. When the lower bound for a t-star beats this general upper bound for non-trivial t-intersecting signed set subfamily of $\mathcal{S}_{\mathcal{F},r}$ with $\mathcal{D}(\mathcal{A})$, the family $\mathcal{S}_{\mathcal{F},r}$ must have the EKR-property.

In conclusion, we want to find a condition on r such that the following inequality holds.

$$\binom{s}{t}^{-1} \cdot r^{-t} \cdot |\mathcal{S}_{\mathcal{F},r}| = \binom{s}{t}^{-1} \cdot r^{-t} \cdot \sum_{F \in \mathcal{F}} r^{|F|} \geq e \cdot \sum_{F \in \mathcal{F}} \binom{|F|-1}{s-t} \cdot r^{|F|-s}$$

Trivially, this is true if for each $F \in \mathcal{F}$,

$$\binom{s}{t}^{-1} r^{|F|-t} > e \cdot \binom{|F|}{s-t} r^{|F|-s}.$$

Or equivalently,

$$r^{s-t} > e \cdot \binom{s}{t}\binom{|F|}{s-t}.$$

Note that the condition is stronger for larger $|F|$, so we actually require

$$r^{s-t} > e \cdot \binom{s}{t}\binom{\alpha(F)}{s-t}.$$

Because the RHS is smaller than

$$e \cdot \frac{s^{s-t}}{(s-t)!} \cdot \frac{\alpha(F)^{s-t}}{(s-t)!}$$

Using the fact that $(\frac{e}{x! \cdot x!})^{1/x}$ and $(\frac{e}{x! \cdot x!})^{1/x} \cdot x$ are both maximized at 1 with value e among all integers, we obtain a lower bound $r_0 = e \cdot (t+1) \cdot \alpha(\mathcal{F})$.

2.3 Proof of Lemma 2.2.3

The proof of Lemma 2.2.3 uses the shifting technique. Roughly speaking, it works like compressing the set family and forcing its members to be more likely to contain some

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common element.

In this section, it would be more convenient to consider a signed set (F, f) in the form $\{(x_i, y_i) : i = 1, \ldots, |F|\}$ such that $F = \{x_1, \ldots, x_{|F|}\}$ and $y_i = f(x_i)$.

Definition 2.3.1. *For $(a, b) \in [n] \times [2, r]$, we define an operator $\delta_{(a,b)}$ on signed sets $A = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ as follows:*

$$\delta_{(a,b)}(A) := \begin{cases} A \setminus (a,b) \cup (a,1) & \text{if } (a,b) \in A \\ A & \text{otherwise} \end{cases}$$

Then we define an operator $\Delta_{(a,b)}$ on signed set families by

$$\Delta_{(a,b)}(\mathcal{A}) := \{A \in \mathcal{A} : \delta_{(a,b)}(A) \in \mathcal{A}\} \cup \{\delta_{(a,b)}(A) : A \in \mathcal{A}, \delta_{(a,b)}(A) \notin A\}$$

For a t-intersecting signed family $\mathcal{A}$, we define a new family A' by applying $\{\Delta_{(a,b)} : (a, b) \in [n] \times [2, r]\}$ one by one on $\mathcal{A}$

$$\mathcal{A}' = \Delta_{(1,n)} \circ \Delta_{(2,n)} \circ \cdots \Delta_{(n-1,2)} \circ \Delta_{(n,2)}(\mathcal{A}).$$

Intuitively, a signed set (F, f) is "high" if f has many large values in its image, and then applying the operators $\Delta_{(a,b)}$ will make all signed sets as "low" as possible. Now, we claim that $\mathcal{A}'$ is indeed the family we need for Lemma 2.2.3.

Theorem 2.3.2. *The family $\mathcal{A}'$ is still t-intersecting and $|\mathcal{A}'| = |\mathcal{A}|$, and for every $A_1, A_2 \in \mathcal{A}'$, they intersect in at least t elements in $[n] \times [1]$.*

Proof. It is not difficult to see that each operator $\Delta_{(a,b)}$ maintains the size of the family. Assume the final family $\mathcal{A}'$ is not t-intersecting, then there must be a step in the process such that $\Delta_{(a,b)}$ sends a t-intersecting family, say $\mathcal{B}$, to a non t-intersecting one, say $\mathcal{B}'$. Assume $B_1', B_2' \in \mathcal{B}'$ and $|B_1' \cap B_2'| < t$, then we look at their preimages $B_i := \delta_{(a,b)}^{-1}(B_i')$. By definition of $\delta_{(a,b)}$, we know that either $B_1 = B_1'$ or $B_1 = B_1' \setminus \{(a, 1)\} \cup \{(a, b)\}$. In summation, there might be two cases for each and, therefore, four cases in total.

- **Case 1**: $B_1 = B_1'$ and $B_2 = B_2'$

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In this case, $\mathcal{B}$ is not t-intersecting.

- **Case 2**: $B_1 \neq B_1'$ and $B_2 \neq B_2'$
  In this case, $(a, b) \in B_1 \cap B_2$. So the size of intersection is unchanged, which is impossible.

- **Case 3**: $B_1 = B_1'$ and $B_2 \neq B_2'$
  In this case, $|B_1 \cap B_2| = t-1$ and $|B_1 \cap B_2'| = t$ implies $(a, b) \in |B_1 \cap B_2'|$. However, this means $(a, b) \in B_1$, together with the fact B_1 is invariant under $\Delta_{(a,b)}$, we must have $B_1 \setminus \{(a, b)\} \cup \{(a, 1)\} \in \mathcal{B}$. This set intersects B_2' in less than t elements. which contradicts the assumption $\mathcal{B}$ is t-intersecting.

The final case is the same as case 3. Thus, we have proved that $\mathcal{A}'$ is t-intersecting. Now we only need to show that they have an intersection of size at least t in $[n] \times [1]$.

Again, we prove this by contradiction. Suppose $A_1, A_2 \in \mathcal{A}'$ is t-intersecting but $|A_1 \cap A_2 \cap ([n] \times 1)| < t$. Let $A_1 \cap A_2 = \{(a_1, b_1), \ldots, (a_m, b_m)\}$, one can observe that $A_1 \setminus \{(a_1, b_1), ..., (a_m, b_m)\} \cup \{(a_1, 1), \ldots, (a_m, 1)\}$ is in $\mathcal{A}'$. But this set has an intersection of size less than t with A_2, and therefore contradicts that $\mathcal{A}'$ is t-intersecting. $\square$

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Chapter 3 Relation with Chvátal's and Kleitman's Conjectures

3.1 Another related conjecture

In this chapter, we introduce another conjecture that Borg proposed in Borg (2011b) and briefly explain its relation with two other famous EKR-type conjectures.

Conjecture 1.2.4 (Borg (2011b)). *Given a weight function $w : 2^{[n]} \to \mathbb{R}$ that satisfies $w(F') \geq w(F)$ for all $F' \subseteq F, F', F \in \mathcal{F}$. The function*

$$w(\mathcal{F}) := \sum_{F \in \mathcal{F}} w(F)$$

is maximized by some star families among all intersecting families.

The statement is a little different from Borg (2011b) but is equivalent. Now, we show why this statement generalizes the EKR theorem for r-singed sets.

Given a family $\mathcal{F} \subseteq 2^{[n]}$, if we define a weight function on $2^{[n]}$ by

$$w(S) = |\{(F, f) \in \mathcal{S}_{\mathcal{F},r} : f^{-1}(\{1\}) = S\}|$$

Then the conjecture immediately reduces to the first part of Conjecture 1.2.4. But to feel the power of the conjecture, we consider a different weight function w.

We say that a family $\mathcal{F} \subseteq 2^{[n]}$ is a **down set family** if it is closed under subsets. For

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a given down set family $\mathcal{F}$, if let the weight function $w = \mathrm{id}_{\mathcal{F}}$, where

$$\mathrm{id}_{\mathcal{F}}(F) := \begin{cases} 1 & \text{if } F \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases}$$

Then Conjecture 1.2.4 reduces to Chvátal's Conjecture.

Conjecture 3.1.1 (Chvátal (1974))**.** *The largest intersecting family inside a down set family $\mathcal{F}$ is a star.*

Although Chvátal's conjecture is very simple and easy to understand, it has remained unsolved for about half a century. However, there are still some partial progress on this problem; for example, Snevily (1992) verified the conjecture if the down set family is compressed with respect to some element.

Another conjecture that usually appears with Chvátal's Conjecture is the Kleitman's conjecture. It is also a strengthing of for Chvátal's conjecture, proposed by Kleitman (1979), but we follow a formulation given by Friedgut, Kahn, Kalai and Keller in Friedgut et al. (2018).

Definition 3.1.2. *Let $f, g : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be two functions, we say f flows to g if there exists a function $v : 2^{[n]} \times 2^{[n]} \to \mathbb{R}_{\geq 0}$ such that:*

- *for any $A \in 2^{[n]}$, $\sum_{B \in 2^{[n]}} v(A, B) = f(A)$,*
- *for any $B \in 2^{[n]}$, $\sum_{A \in 2^{[n]}} v(A, B) = g(B)$,*
- *If $A \not\subseteq B$, then $v(A, B) = 0$.*

To catch the definition, one can imagine a network on $2^{[n]}$ with edge set $\{(A, B) : A \subseteq B\}$ and v is a function denoting the amount of water / money / whatever you like that would go to B from A in the next minute. Then f would represent the amount of

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something at each point right now, and g would be the one after one minute. The final line is a constraint on what paths can be used for the transmission.

With this definition, the statement of Kleitman's Conjecture can be formulated as follows.

Conjecture 3.1.3 (Kleitman (1979))**.** *For every maximal intersecting family $\mathcal{F} \subseteq 2^{[n]}$, there exists a convex combination of $\mathrm{id}_i := \mathrm{id}_{\mathcal{F}_i}$ that flows to it, where $\mathcal{F}_i = \{F \subseteq 2^{[n]} : i \in F\}$.*

As one might have not been convinced that this is a generalization of Chvátal's conjecture, we show it by proving that this conjecture implies Conjecture 1.2.4.

Theorem 3.1.4. *Kleitman's Conjecture implies Conjecture 1.2.4.*

Proof. Fix a weighted function $w : 2^{[n]} \to \mathbb{R}$, assume $\mathcal{F}$ is an intersecting family that maximizes $w(\mathcal{F})$. We first extend it to a maximal intersecting family $\mathcal{F}' \subseteq 2^{[n]}$. Then by Kleitman's conjecture, there exists a convex combination of id_i flowing to $\mathrm{id}_{\mathcal{F}'}$. Let c_i denote coefficients of the convex combination, we get

$$w(\mathcal{F}) \leq w(\mathcal{F}') = \sum_{F\in 2^{[n]}} w(F)\cdot\mathrm{id}_{\mathcal{F}'}(F) = \sum_{i\in[n]}\sum_{F\in 2^{[n]}} c_i\cdot w(F)\cdot\mathrm{id}_i(F) + \sum_{A\subseteq B\subseteq[n]} [w(B)-w(A)]\cdot v(A,B)$$

Since $A \subseteq B \implies w(B) - w(A) \geq 0$ and $v(A,B) \leq 0$, the second term is negative. Thus

$$w(\mathcal{F}) \leq \sum_{i\in[n]}\sum_{F\in 2^{[n]}} c_i\cdot w(F)\cdot\mathrm{id}_i(F) \leq \sum_{i\in[n]} c_i\cdot w(\mathcal{F}_i) \leq \sum_{i\in[n]} c_i\cdot\max_{i\in[n]}\{w(\mathcal{F}_i)\} = \max_{i\in[n]}\{w(\mathcal{F}_i)\}$$

In other word, a star will maximize the weighted sum. □

Remark 3.1.5. *Last year, J. Cary proved the Kleitman's Conjecture for families that satisify some technique conditions. (See Cary (2024))*

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Chapter 4 The Erdős-Rothschild Type Extension

4.1 Background

How many edges can we add between n vertices without constructing a triangle? In 1907, Mantel answered this question by establishing the following theorem.

Theorem 4.1.1 (Mantel (1907))**.** *A triangle-free graph on n-vertex can have at most $\lfloor \frac{n^2}{4} \rfloor$ edges. The bound is attained by a complete bipartite graph with two balanced parts.*

Mantel's theorem is one of the fundamental theorems in graph theory and has a lot of variations. Turán's theorem should be one of the most well-known extensions since it generalizes the theorem to complete graphs K_t and determines the result asymptotically. They showed the optimal construction for K_t-free n-vertex graph is a balanced $(t-1)$-partite graph. Such graph is constructed by dividing n vertices into $(t-1)$ parts as equal as possible and add edges between two points in different parts. Some people also called this graph the Turá n graph $T(n, t-1)$ and the number of its edges is usually denoted by $\text{ex}(n, K_t)$.

Now, we look at another extension of Mantel's theorem. In Erdős (n.d.), Erdős and Rothschild proposed the following. Given a red / blue edge coloring on a graph $G = (V, E)$, we say that it is **monochromatic-triangle-free** if G_{r}, the subgraph formed by collecting red edges, and G_{b}, the one formed by collecting blue edges, are both triangle free. The Erdős-Rothschild problem asks: What is the graph on n vertices that has the most distinct monochromatic-triangle-free colorings?

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As one may think, a graph has more red/blue-colorings if it has many edges, and for a triangle-free graph, any red/blue-coloring on it will certainly contain no monochromatic triangle.

Erdős and Rothschild conjectured that this lower bound is also the best possible construction. Several years later, Yuster (1996) proved the conjecture to be true. After that, this problem is considered in larger cliques and more colors and that is where the interesting part of this question emerged.

Definition 4.1.2. *Let $c, t \in \mathbb{N}$, a (c, t)-colorings of a graph G is a c-coloring on the edge set of G such that every color class is K_t-free.*

As one might have thought, the question asked what is the maximum number of (c, t)-colorings an n-vertex graph can have?

From previous experience, by coloring the Turán graph $T(n, t-1)$ arbitrary with c colors, it is natural to guess that the answer should be $c^{\mathrm{ex}(n,K_t)}$. Nevertheless, this is far from the truth; in fact, Alon et al. (2004) showed that the bound is only true if $c \leq 3$. There are some further results in this direction, but they are beyond the scope of this thesis. In fact, we want to consider this question for t-intersecting families. With abuse of notation, we define

Definition 4.1.3. *Let $c, t \in \mathbb{N}$, an (c, t)-colorings of a family $\mathcal{F}$ is an c-coloring on elements of $\mathcal{F}$ such that every color class form a t-intersecting family.*

The goal of this section is to determine the family $\mathcal{F}$ maximizes the number of $(3, t)$-colorings in our interested set families.

In Clemens et al. (2018), Clemens, Das and Tran proved a general theorem that solves this problem under some assumptions. Although we focus on the case with 3 colors here, they actually gave some result of the c color version of this problem.

We say a t-intersecting family is **extremal** if it is not only maximal, but also attains the maximum possible size. The following theorem gives a criteria that shows when the idea of coloring a t-intersecting family arbitrary is the optimal one,

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Theorem 4.1.4. *(Clemens et al. (2018)) Let N_0 be the size of the extremal t-intersecting families, N_1 the size of the largest non-trivial t-intersecting families. We further assume that the intersection of 2 t-intersecting families has size at most N_2 and the total number of maximal t-intersecting families is less than M. If*

$$N_0 - \max(N_1, N_2) - \frac{6 \log_2 M}{2 \log_2 3 - 3} > 0$$

, then a t-intersecting family $\mathcal{F}$ can have at most 3^{N_0} $(3, t)$-colorings and the equality holds when $\mathcal{F}$ is an extremal t-intersecting family.

Note that we didn't prescribe the under lying set families in the theorem. That means the criteria can be applied to many different set families. In Clemens et al. (2018), they applied it to the uniform set families $\binom{[n]}{k}$, the permutation families $\mathcal{S}_n$ and the family of k-dimensional subspaces of $\mathbb{F}_q^n$. Note that the size of intersection of 2 permutations is the number of indices that they agree on.

As we said before, the problem of finding N_0 is called the EKR type questions and it has been proven that in many different set families with sufficiently large n, the t-intersecting family is trivial.

Additionally, in these situations, one can ask the so-called Hilton-Milner type problems, which essentially ask what is the largest t-intersecting family that is not contained in a trivial one.

As for the final number M, we require an upper bound for the use of Theorem 4.1.4. For the k-uniform set family setting, we have the following theorem.

Theorem 4.1.5 (Balogh et al. (2015))**.** *The number of maximal t-intersecting k-uniform in $[n]$ is less than*

$$\binom{n}{k}^{\binom{2(k-t)+1}{k-t}}$$

In the following context, we will prove Theorem 4.1.4 and apply it to some set families in which the Erdős–Ko–Rado and Hilton–Milner type results are known.

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4.2 Proof of Theorem 4.1.4

This proof is the same as the one in Clemens et al. (2018), we include it here for completeness.

Fix an arbitrary ground set family equipped with an intersection. Let N_0, N_1, N_2, M be the corresponding number as defined in Theorem 4.1.4. We consider the number of $(3,t)$-colorings of $\mathcal{F}$ and denote it with $c(\mathcal{F})$.

For a t-intersecting subfamily $\mathcal{G}$, we can assign it a maximal t-intersecting family $\tau(\mathcal{G})$ that contains it. Given a $(3,t)$-coloring, let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be its monochromatic parts. We say that this coloring has type $(\tau(\mathcal{F}_1), \tau(\mathcal{F}_2), \tau(\mathcal{F}_3))$. Note that there are at most M^3 types.

Now, we want to find an upper bound for a fixed type $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$. But before that, we first look at the special case $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3 = \mathcal{G}$. In this case, we have $\mathcal{F} \subseteq \mathcal{G}$, which implies $c(\mathcal{F}) \leq 3^{|\mathcal{G}|} \leq 3^{N_0}$.

Now, we assume that these 3 maximal t-intersecting families are not all the same. For each $F \in \mathcal{F}$, let $w(F)$ denote the number of times F appears in these 3 families. This number also means the number of possible colors of F of this type. Therefore, if we let $n_i = |\{F \in \mathcal{F} : w(F) = i\}|$, we have $2^{n_2}3^{n_3}$ as the upper bound of $(3,t)$-colorings of $\mathcal{F}$ in this type.

Since each maximal t-intersecting family cannot have more than N_0 edges, by double counting, it yields inequality $n_1 + 2n_2 + 3n_3 \leq 3N_0$. To optimize $2^{n_2}3^{n_3}$ under $n_1 + 2n_3 + 3n_3 \leq 3N_0$, n_3 should be as large as possible, However, the condition that these 3 are not all the same adds a bound max$\{N_1, N_2\}$ on a_3. Thus, when we sum over all possible types, we will get the following calculation.

$$
\begin{aligned}
c(\mathcal{F}) &\leq M^3 \cdot 2^{\frac{3}{2}(N_0 - n_3)} \cdot 3^{n_3} = M^3 \cdot 2^{\frac{3}{2}N_0} \cdot (2^{-\frac{3}{2}}3)^{n_3} = M^3 \cdot 3^{N_0} \cdot (2^{\frac{3}{2}}3^{-1})^{N_0 - n_3} \\
&= 3^{N_0} \cdot (2^{\frac{3}{2}}3^{-1})^{N_0 - n_3 - \frac{6\log_2 M}{2\log_2 3 - 3}} \leq 3^{N_0} \cdot (2^{\frac{3}{2}}3^{-1})^{N_0 - \max\{N_1, N_2\} - \frac{6\log_2 M}{2\log_2 3 - 3}}
\end{aligned}
$$

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Since $2^{\frac{3}{2}}3^{-1} < 1$ and the exponent is greater than 0, $c(\mathcal{F})$ will be strictly less than 3^{N_0}, which proves Theorem 4.1.4.

4.3 Erdős-Rothschild problem on k-uniform multiset family

We start the discussion with the case of multiset family $\left(\!\binom{[m]}{k}\!\right) := \{\text{multisubsets of } [n] \text{ with size } k\}$. Note that we do count duplicate elements in intersection of two multisets. For example, the intersection of $\{1, 1, 1\}$ and $\{1, 1, 2\}$ has size 2. With this said, we can naturally define the t-intersecting multiset family. We say a t-intersecting multiset family is trivial if every member contains common t elements and non-trivial otherwise.

Theorem 4.3.1. *Given integer k and t. For* ***sufficiently large*** *m, a t-intersecting subfamily of $\left(\!\binom{[m]}{k}\!\right)$ can have at most 3^{N_0} $(3, t)$-colorings, where $N_0 = \left(\!\binom{m}{k-t}\!\right) = \binom{m+k-t-1}{k-t}$. Moreover, the equality holds when $\mathcal{F}$ is a trivial t-intersecting family.*

In Meagher and Purdy (2016), K. Meagher and A. Purdy proved the following Erdős–Ko–Rado and Hilton-Milner type results for t-intersecting multiset families.

Theorem 4.3.2 (Füredi et al. (2015))**.** *Let m, k, t be two integers with $k > t$ and $m + k - 1 > (k - t + 1)(t + 1)$. For every t-intersecting family $\mathcal{F}$ of k-multisets of $[m]$,*

$$|\mathcal{F}| \leq \left(\!\binom{m}{k-t}\!\right) = \binom{m+k-t-1}{k-t}$$

Moreover, the equality holds if and only if $\mathcal{F}$ consists of all k-multisets containing some fixed t elements.

Now, we need an answer for the Hilton-Milner type result. i.e. the size of extremal non-trivial t-intersecting family. Firstly, we consider following two constructions for non-trivial t-intersecting family:

$$\mathcal{G}_1 = \{S \in \left(\!\binom{[m]}{k}\!\right) : |S \cap [t+2]| \geq t+1\}$$
$$\mathcal{G}_2 = \{S \in \left(\!\binom{[m]}{k}\!\right) : [t] \subseteq S, S \cap [t+1, k+1] \neq \emptyset\} \cup \{[k+1] - \{i\} : i = 1, \ldots, t\}$$

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Note that

$$|\mathcal{G}_1| = (t+2)\cdot(m-1)^{k-t-1} + m^{k-t-2}$$
$$|\mathcal{G}_2| = \left(\!\!\binom{m}{k-t}\!\!\right) - \left(\!\!\binom{m-(k-t)-1}{k-t}\!\!\right) + t$$

In fact, these two t-intersecting families are actually the only two possible candidates for the extremal t-intersecting family.

Theorem 4.3.3 (Meagher and Purdy (2016))**.** *Let $m > k \geq t$ be integers such that $m \geq \max\{t(k-t)+3, 2k-t\}$. For every non-trivial t-intersecting family $\mathcal{F}$ of k-multisets of $[m]$,*

$$|\mathcal{F}| \leq \max\{|\mathcal{G}_1|, |\mathcal{G}_2|\}$$

The final ingredient we want is the number of maximal t-intersecting multiset family. As mentioned above, to apply Theorem 4.1.4, we only need an upper bound of this number. What we will do is to project the multiset family to a set family with larger ground set, which reduces the case to Theorem 4.1.5 and provides us an upper bound that is good enough.

Assuming that element x that appears i times in a multiset, we map these i many x to $(x,1),\ldots,(x,i)$ in $[m]\times[k]$. This yield a map σ that sends a t-intersecting k-uniform multiset family to the k-uniform t-intersecting family with ground set $[m]\times[k]$, which is essentially the same as $[mk]$. For a set family $\mathcal{F}$, we let $\Sigma(\mathcal{F}) := \{\sigma(F) : F \in \mathcal{F}\}$ denote the image.

It can be easily seen that this mapping is one-to-one and preserves the intersection. In other words, if $\mathcal{F}$ is a k-uniform t-intersecting multiset family, then its image $\Sigma(\mathcal{F})$ is a k-uniform t-intersecting family with ground set $[mk]$.

One potential problem is that the image $\Sigma(\mathcal{F})$ of a maximal t-intersecting family $\mathcal{F}$ is not guaranteed to be maximal t-intersecting. We deal with this problem in two steps, firstly, we may extend the family $\Sigma(\mathcal{F})$ is a maximal t-intersecting one, call it $\Sigma'(\mathcal{F})$. Secondly, we show that the map Σ' is also one-to-one on maximal t-intersecting families.

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Lemma 4.3.4. *The map Σ' is one-to-one when restricted on maximal t-intersecting families.*

Proof. Suppose otherwise that there exists $\mathcal{F}_1, \mathcal{F}_2$ such that $\Sigma'(\mathcal{F}_1) = \Sigma'(\mathcal{F}_2)$. By definition, we know there exists a set $G \in \Sigma(\mathcal{F}_1) \setminus \Sigma(\mathcal{F}_2)$. Let $F = \sigma^{-1}(G)$. Since G belongs to $\Sigma'(\mathcal{F}_2)$, F intersects every member of $\mathcal{F}_2$ with size at least t. So $\mathcal{F}_2 \cup \{F\}$ is also t-intersecting, which contradicts that $\mathcal{F}_2$ is maximal t-intersecting. □

Thus, the number we want can be bounded by the number of k-uniform t-intersecting families in $[mk]$. Applying the theorem with $n = mk$ gives an easy upper bound.

Note that $N_2 \le \max\{N_1, \left(\!\binom{m}{k-t-1}\!\right)\}$, so we should discuss case by case depending on the output of $\max\{N_1, N_2\}$. We use Δ to denote the formula in Theorem 4.1.4.

Case 1: $\max\{N_1, N_2\} = |\mathcal{G}_1|$

In this case,

$$\begin{aligned}
\Delta &= \left(\!\binom{m}{k-t}\!\right) - (t+2)(m-1)^{k-t-1} - m^{k-t-2} - \frac{6}{2\log_2 3 - 3}\log_2 \binom{mk}{k}^{\binom{2(k-t)+1}{k-t}} \\
&\ge \frac{1}{(k-t)!}\cdot m^{k-t} - (t+2)m^{k-t-1} - \frac{6}{2\log_2 3 - 3}\binom{2(k-t)+1}{k-t}\cdot \log_2 \binom{mk}{k} \\
&\ge (\frac{m}{(k-t)!} - t - 2)\cdot m^{k-t-1} - \frac{6}{2\log_2 3 - 3}\cdot \binom{2(k-t)+1}{k-t}\cdot k\log_2(em) \\
&\ge (\frac{m}{(k-t)!} - t - 2)\cdot m^{k-t-1} - 72k\cdot \binom{2(k-t)+1}{k-t}\cdot \log_2 m
\end{aligned}$$

Notice that when k and t are fixed, the second term has order $O(\log m)$, so if $m > (t-2)(k-t)!$, then $\Delta > 0$ for sufficiently large m.

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Case 2: $\max\{N_1, N_2\} = |\mathcal{G}_2|$

In this case,

$$\begin{aligned}\Delta &= \left(\!\!\binom{m-(k-t)-1}{k-t}\!\!\right) - t - \frac{6}{2\log_2 3 - 3}\log_2 \binom{mk}{k}^{\binom{2(k-t)+1}{k-t}} \\ &\geq \binom{m-2}{k-t} - t - 72k \cdot \binom{2(k-t)+1}{k-t} \cdot \log_2 m\end{aligned}$$

Again, $\Delta > 0$ holds for sufficiently large m.

Case 3: $\max\{N_1, N_2\} = \left(\!\binom{m}{k-t-1}\!\right)$ In this case,

$$\begin{aligned}\Delta &= \left(\!\!\binom{m}{k-t}\!\!\right) - \left(\!\!\binom{m}{k-t-1}\!\!\right) - \frac{6}{2\log_2 3 - 3}\log_2 \binom{mk}{k}^{\binom{2(k-t)+1}{k-t}} \\ &= \binom{m+k-t-1}{k-t} - \binom{m+k-t-2}{k-t-1} - \frac{6}{2\log_2 3 - 3}\log_2 \binom{mk}{k}^{\binom{2(k-t)+1}{k-t}} \\ &= \binom{m+k-t-2}{k-t} - \frac{6}{2\log_2 3 - 3}\log_2 \binom{mk}{k}^{\binom{2(k-t)+1}{k-t}} \\ &\geq \frac{1}{(k-t)!}(m-1)^{k-t} - 72k \cdot \binom{2(k-t)+1}{k-t} \cdot \log_2 m\end{aligned}$$

Once again, $\Delta > 0$ holds for sufficiently large m. Therefore, we obtain the following theorem of Erdős-Rothschild type problem.

Theorem 4.3.5. *Given integers $m > k > t$, there exists $m_0(k,t)$ such that if $m > m_0(k,t)$, then a t-intersecting multiset family $\mathcal{F}$ can have at most $3^{\left(\!\binom{m}{k-t}\!\right)}$ $(3,t)$-colorings. Moreover, the inequality holds if and only if $\mathcal{F}$ is a t-star.*

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4.4 Erdős-Rothschild problem on k-uniform r-signed sets

Now, we consider the same problem on signed set family. Given integers k, r, we define the r-**signed** k-**set family** as follows:

$$\mathcal{S}_{n,k,r} := \{(A, f) : A \in \binom{[n]}{k}, f : A \to [r]\}$$

In other words, $\mathcal{S}_{n,k,r}$ is the special case of previously discussed signed set family $\mathcal{S}_{\mathcal{F},r}$ with $\mathcal{F} = \binom{[n]}{k}$. We first record some EKR and Hilton–Milner type results in this setting. Notice that unlike in section 2, we do not fix the family $\binom{[n]}{k}$. In other words, n can now grow.

The t-EKR result of $\mathcal{S}_{n,n,r}$ was also independently proved in Ahlswede and Khachatrian (1998) and in Frankl and Tokushige (1999). For $k < n$, the t-intersecting $\mathcal{S}_{n,k,r}$ is solved by Bey (1999).

Theorem 4.4.1 (Ahlswede and Khachatrian (1998), Frankl and Tokushige (1999)). *Let $n \geq t$ and r be integers. The largest t-intersecting subfamily of $\mathcal{S}_{n,n,r}$ is trivial if and only if $r \geq t + 1$.*

Theorem 4.4.2 (Bey (1999)). *Let $n \geq k \geq t$ and r be integers. A t-star has largest size among all the largest t-intersecting subfamily of $\mathcal{S}_{n,k,r}$ if and only if $n \geq \frac{k-t+r}{r} \cdot (t+1)$.*

Combining these two, we have the following corollary.

Corollary 4.4.3 (Borg (2011*a*)). *Let $n \geq k \geq t$ and r be integers. The largest t-intersecting subfamily of $\mathcal{S}_{n,k,r}$ is trivial if $r \geq t + 1$.*

As for Hilton–Milner type result, we first look at two constructions. For integer s, M_s denotes the r-signed set $\{(1, 1), (2, 1), \ldots, (s, 1)\}$. We define

$$\mathcal{H}_1 := \{A \in \binom{[n]}{k} : |A \cap M_{t+2}\}| \geq t + 1\}$$

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and

$$\mathcal{H}_2 := \left\{A \in \binom{[n]}{k} : M_t \subset A, |A \cap M_\ell| \geq t+1\right\} \cup \left\{A \in \binom{[n]}{k} : |A \cap M_\ell| \geq \ell - 1\right\},$$

where $\ell = \min\{k+1, n\}$

Theorem 4.4.4 (Yao et al. (2021))**.** *Let $n > k \geq t$ and r be integers and $\mathcal{F}$ be a non-trivial t-intersecting subfamily of $\mathcal{S}_{n,k,r}$. If $r \geq r_0(n,k,t)$, $n \geq t+2 \geq 4$ and $n \geq k \geq t$, then*

$$|\mathcal{F}| \leq \{\mathcal{H}_1, \mathcal{H}_2\},$$

where

$$r_0(n,k,t) = \frac{(k-t+3)(k-t-1)}{n-t-1} \cdot \max\{\binom{t+2}{2}, \frac{k-t+1}{2}\}.$$

Remark 4.4.5. *Although the theorem does not cover the case for $k = t+1$, one can observe that in this special situation, a nontrivial t-intersecting k-uniform family has size at most $k+1$.*

Let n, k, t, r be the numbers defined above, observe that

$$|\mathcal{H}_1| \leq (t+2) \cdot \binom{n-t-1}{k-t-1} \cdot r^{k-t-1},$$
$$|\mathcal{H}_2| \leq (\ell - t) \cdot \binom{n-t-1}{k-t-1} \cdot r^{k-t-1} + (\ell - t) \cdot \binom{n-\ell+1}{k-\ell+1} \cdot r^{k-\ell+1}.$$

Note that $\binom{n-\ell+1}{k-\ell+1}$ is actually bounded by n and when $\ell \geq t+2$, the $r^{k-t+1} > r^{k-\ell+1}$ and the second term is usually the lower order term.

Using this lemma and some calculation, we have

$$N_0 - \max\{N_1, N_2\} \geq \left(\frac{n-t}{k-t} \cdot r - \max\{t+2, k-t+2\}\right) \binom{n-t-1}{k-t-1} \cdot r^{k-t-1}$$

Now we deal with the last term in the discriminant. The idea of finding an upper bound for the number of maximal t-intersecting families in $\mathcal{S}_{n,k,r}$ is the same as in the multiset case, where we ignore the extra structure of the ground set and use Theorem 4.1.5.

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This time, the ground set has size nr, and the exponential term is independent of n and r, so $\log M = O(\log r)$ grows logarithmically in the variables.

$$\log M \leq \binom{2(k-t)+1}{k-t} \cdot \log \binom{nr}{k} \approx O(\log nr)$$

Thus, we know that the last term grows much slower compared to the first one, which shows

$$\Delta \geq \left(\frac{n-t}{k-t} \cdot r - \max\{t+2, k-t+2\}\right) \binom{n-t-1}{k-t-1} \cdot r^{k-t-1} - O(\log nr) > 0$$

for all sufficiently large n and $r \geq \frac{k-t}{n-t} \cdot \max\{t+2, k-t+2\}$. As a conclusion, we have the following theorem.

Theorem 4.4.6. *For fixed integers $k \geq t$ and sufficiently large n and $r \geq \frac{k-t}{n-t} \cdot \max\{t+2, k-t+2\}$, a t-intersecting subfamily of $\mathcal{S}_{n,k,r}$ can have at most 3^{N_0} $(3,t)$-colorings, where $N_0 = \binom{n-t}{k-t} \cdot r^{k-t}$. Moreover, the equality holds when $\mathcal{F}$ is a trivial t-intersecting family.*

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Chapter 5 Conclusion and further extension

We focused on two types of extension of the Erdős–Ko–Rado theorem in this thesis and there are still many unsolved parts.

In the EKR thoerem for r-signed sets $\mathcal{S}_{\mathcal{F},r}$, we find an improved lower bound for r that depends on t and $\alpha(\mathcal{F})$, the size of the largest sets in $\mathcal{F}$, but it still does not answer the conjecture, which demands a lower bound that only depends on t. We then introduced another conjecture proposed by Borg and explain how it ``sits between" the well-known Chvátal's Conjecture and Kleitman's Conjecture and, of course, it will be very intriguing if one proves or disproves these problems.

As for the second topic, the Erdős–Rothschild type extension of the Erdős–Ko–Rado theorem, we only consider the case of 3-coloring. As mentioned previously, the construction for optimal family of more than 4 colors might no longer be trivial.

In addition, finding a tighter upper bound for the number of maximal t-intersecting families is also a potential problem. In this thesis, a very loose estimation is used since its order is not the dominant one, but we reduce it to the set setting by considering it as a large ground set, which might be probably far from the correct order.

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