國立臺灣大學數學研究所

碩士論文

Department of Mathematics

National Taiwan University

Master Thesis

正質量定理的相關研究

Survey on Positive Mass Theorem

王業凱

Ye-Kai Wang

指導教授：李瑩英 博士

Advisor: Yng-Ing Lee, Ph.D.

中華民國 97 年 7 月

July, 2008

致謝

首先要感謝我的家人: 爸爸、媽媽、妹妹。謝謝他們 20 多年來的養育、陪伴、鼓勵。我做過任何一件有意義的事都是可以被輕易取代的, 不足以刻畫我本人, 唯有我的家人 (以及以下的師長朋友) 讓我在這個世界上是獨一無二的。

與這篇 note 有關的。謝謝李瑩英老師兩年來的指導、關心, 鼓勵我們做數學不只是取決於能力, 態度與視野更加重要。我對拿到正質量定理這個題目感到榮幸,R. Schoen 和丘成桐的文章對於非線性方程在幾何上的作用做了清楚的示範, 感謝老師以這個方式引導我認識幾何分析。謝謝王藹農老師接受我無時無刻的打擾, 當我拿著各式希奇古怪的問題敲門時, 老師總是說有空, 很有耐心地跟我一起研究。跟他一起 seminar 複幾何與黎曼面是我在研究所最愉快的事之一。謝謝崔茂培老師的指點以及 Ricci flow 的教學, 他把我們當成是自己的學生照顧。謝謝陳泊寧同學解決我在 quasilocal mass 的疑惑, 憑我自己無法完成這份 note 的第五節。

在數學上對我影響很大的幾個人。齊震宇是我的高微、ODE、代數拓樸、微分幾何的啓蒙老師, 我在一、二年級見識到的熱情, 是我選擇爲分幾何的主因, 也謝謝他每次看到我時都會給予建議。蔡忠潤讓我體會到能做計算的人才是最厲害的, 並在他最近的演講示範了一個研究生可以達到什麼程度以及必須達到什麼程度。陳朝銑總是與我分享數學家的熱血故事與他對數學、物理各個分支的廣泛學習, 跟他聊天一次可以激發我對數學的熱情兩天以上。陳志傑跟我一起殺到中壢旁聽王金龍老師的微分幾何, 考 GRE 時持續激勵我, 不時以他學習的經驗提醒我去體會什麼是好的數學。

接著是三位同窗: 國瑋、志偉哥、呂楊凱。兩年來朝夕相處, 一起吃飯、唸書、偷懶、鬼扯, 我單方面的認爲跟他們建立了相當的友誼, 前兩位是我見過最沒私心、最爲其他人著想的, 有時會把別人的事擺在自己的工作之前。第三位是我見過最會聊天的人。三人皆多才多藝, 但專長與本文無關, 故略。

一些非列出不可的數學系同學: 達叔、李庭諭、林立人、徐義程、杜侑潮。

倒數第二的是爲數不多的朋友們。國中同學: 林文元、賴幹、呂布、謝方穎、林玉淳。跟他們打牌聊天整夜, 輸家請吃早餐是有些莫名奇妙但偶爾得來上一次的樂趣, 期待下一次出遊。科博館高中義工, 軍中的弟兄, 鄉服的同學, 兩次的暑假隊是我研究所最美好的成就。跟以上的朋友在一起代表暫時離開課業, 但不曉得爲什麼, 聚會結束後會彷彿受到鼓舞似的, 有繼續唸下去的動力, 可能是他們是有魔力的人吧!

最後再回到家人。每天晚上回家跟媽媽聊天, 早上吃她準備的早餐; 跟妹妹的作息差了兩個小時, 偶爾才能講到話; 爸爸不時上台北跟我們吃飯。我常會忽略那些習以爲常的關心, 得再三抱歉。無論如何, 我的優點都是爸爸媽媽 (特別是媽媽) 的功勞 (兩歲以後養成的也許跟我妹有關), 謝謝他們。

摘要

這篇文章介紹 quasilocal mass 的相關研究, 主要是 Brown-York and Liu-Yau quasilocal mass 的定義以及如何證明它們是非負的。第一個重要發現是史宇光與譚聯輝證明了 time-symmetric 的情形, 接著劉秋菊與丘成桐給出一般情形下的證明, 最後我們討論王慕道與丘成桐最近提出的 quasilocal mass 的定義。
關鍵字: 正質量定理

Abstract. This note surveys the definition of quasilocal mass and its positivity. In particular, we focus on Brown-York and Liu-Yau quasilocal mass. We first present Shi and Tam's result on the positivity of quasilocal mass in the Riemannian case and then Liu and Yau's approach to the general case. Finally, we mention a modification of Liu-Yau quasilocal mass by Wang and Yau.
Key word: quasilocal mass, positive mass theorem, quasi-spherical, isometric embedding, Jang's equation, static mean curvature

Contents

1 **Introduction** 1

2 **The definition of Brown-York and Liu-Yau quasilocal mass** 4

3 **The Riemannian Case** 5

4 **The General Case** 8

5 **A new quasilocal mass of Wang and Yau** 11

1 Introduction

In this note, we survey the definition of quasilocal mass and its positivity.

We first review the history of the positive mass theorem. In Einstein's theory of general relativity, the spacetime is a 4-dimension Lorentzian manifold (N, g) satisfying Einstein equation $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi G\, T_{\alpha\beta}$, where G is Newtonian constant and $T_{\alpha\beta}$ is the (symmetric)energy-momentum tensor. Since general relativity can be viewed as an extension of classical Newtonian mechanics, it is desirable to define the notion of mass, energy, momentum, and angular momentum.

There are, however, several fundamental difficulties. First, the underlying manifold is unknown. All physical obsevations up to now are only local measurements compared with the scale of the universe and give no information about the topology of the universe. Second, Einstein equation is a nonlinear hyperbolic system of 10 degrees of freedom. The knowledge and techniques for such systems are limited. Third, there is no precise definition on how to relate the distribution of matter to $T_{\alpha\beta}$. It seems impossible to treat the general case directly.

A natural approach is to start from special cases. One case that has the longest history and is most extensively studied is the isolated gravitating system. Its origin could be traced back to Schwarzschild's model of the gravitational field of a single star in 1916. Mathematically, the isolated gravitating system is represented by an asymptotically flat spacelike hypersurface in the spacetime.

Definition. A 3-dimensional manifold $M \subset (N, g)$ is asymptotically flat if for some compact set C, $M\backslash C = \cup_{i=1}^{p} M_i$ such that each M_i is diffeomorphic to $\mathbb{R}^3\backslash B_0(R_i)$. Under this diffeomorphism, the metric is required to be of the form

$$g_{ij} = \delta_{ij} + a_{ij},$$

where $a_{ij} = O(r^{-1}), \partial_k a_{ij} = O(r^{-2}), \partial_k\partial_l a_{ij} = O(r^{-3})$. Moreover, the second fundamental form p_{ij} of M decay as $p_{ij} = O(r^{-2}), \partial_k p_{ij} = O(r^{-3})$. The triple (M, g_{ij}, p_{ij}) is called an *initial data set.*

One usually requires that the energy-momentum tensor satisfies certain energy conditions. We say that $T_{\alpha\beta}$ satisfies the dominant energy condition if for any orthonormal frame $\{e_\alpha | \alpha = 0, 1, 2, 3\}$ at $p \in M$, with e_0 normal to M,

$$T_{00}^2 \geq (\sum_{i=1}^{3} T_{0i}^2), \tag{T1}$$

and

$$T_{00} \geq |T_{\alpha\beta}|. \tag{T2}$$

If only (T1) holds, we say $T_{\alpha\beta}$ satisfies the weak energy condition.

In 1962, Arnowitt, Deser, and Misner defined the total energy and total momentum of an asymptotically flat manifold. They are defined on each asymptotically end M_l

$$E_l = \frac{1}{16\pi G}\lim_{R\to\infty}\int_{S_R}(g_{ij,j} - g_{jj,i})d\Omega^i,$$
$$P_{lk} = \frac{1}{16\pi G}\lim_{R\to\infty}\int_{S_R}2(p_{ik} - \delta_{ik}p_{jj})d\Omega^i.$$

Remark.

1. E_l is called the ADM energy(mass) of that end. In mathematical literature, the name mass is preferred. In physics, however, the meaning of the mass and the energy are different. The mass is a frame-independent quantity while the energy is the time component of a four vector. For the definitions in section 2, the Brown-York one is a quasilocal energy and the Liu-Yau one is a quasilocal mass. For a suitable expression of the Brown-York energy-momentum and further discussion, see [M].
2. In 1986, Bartnik showed that the definition of ADM mass is actually independent of the choice of coordinate system [B].

Although ADM mass is physically a natural candidate representing the mass of a system, the positivity of this quantity is not clear. Many physists and mathematicians proved its positivity under additional assumptions. Finally, this problem was settled by R. Schoen and S.-T. Yau, using geometric analysis, and Witten, using spinors.

Theorem ([SY1, SY2])**.** *Let (M, g_{ij}, p_{ij}) be an asymptotically flat 3-dimensional manifold satisfying $\mu \geq |\mathbf{J}|$ in a spacetime N. Then $E_l \geq 0$ on each end M_l. If $E_l = 0$ for some l then M has only one end and M can be isometrically embedded into four dimensional Minkowski spacetime as a spacelike hypersurface so that p_{ij} is the second fundamental form. In particular M is topologically $\mathbb{R}^3$.*

Theorem ([W], see also [PT])**.** *Let (M, g_{ij}, p_{ij}) be an asymptotically flat 3-dimensional manifold satisfying the dominant energy condition in a spacetime N. Then $E_l \geq |P_l|$ on each end M_l. If $E_l = 0$ for some l then M has only one end and M is flat along N, i.e. $R_{\alpha\beta\gamma\delta}|_M = 0$, where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor of N*

Remark. For the definition of μ and $\mathbf{J}$, see section 4. The condition $\mu \geq |\mathbf{J}|$ is equivalent to the weak energy condition(See Appendix.)

It is worth some discussion on both approaches since they form the basis of the later proofs of various positive mass theorems. Schoen and Yau first prove the Riemannian case. That is, the mass of an asymptotically flat Riemannian three manifolds with nonnegative scalar curvature is nonnegative and zero only if it is isometric to $\mathbb{R}^3$ with Euclidean metric. They reduce the general case to the Riemannian one by constructing a scalar flat Riemannian three manifold. This manifold is obtained by solving the Jang's equation and tends to the original one at the infinity. Therefore, the mass of the two manifolds are the same. Witten divides the proof into three steps. First, he derived a Weitzenböck formula for the spinor

$$\int_M |\nabla\psi|^2 + \langle\psi, \mathscr{R}.\psi\rangle - |\mathscr{D}\psi|^2 = \frac{1}{2}\int_{\partial M}\langle\psi, [e^i, e^j].\nabla_j\psi\rangle e_i \lrcorner \mu \tag{1}$$

The next and usually the hardest step is to prove the existence of asymptotically constant harmonic spinors. Witten's proof in this step is not rigorous. The full proof is given in [PT, section5]. The final step is to apply the Weitzenböck formula to an asymptotically constant spinor and identify the boundary integral with $m_{\mathrm{ADM}}|\psi_0|^2$.

Remark. Spinor is the section of the spinor bundle over M. On the end M_l, the spinor bundle is trivial. Picking one trivialization, the spinor can be viewed as a vector-valued function, so we can define what a constant spinor is. Note that the notion of constant spinor depends on coordinate.

In physics, it is also desirable to find suitable quasilocal notions of energy and momentum. We would like to define an energy-momentum tensor for a compact spacelike two-surface in spacetime. The energy-momentum vector should only depend on the first and second fundamental forms and the connection on normal bundle of the two-surface. According to Christodoulou and S.-T. Yau [CY], Melissa Liu and S.-T. Yau [LY2], the quasilocal mass should also satisfy the following properties.

(1) It should be zero for the flat spacetime.
(2) The quasilocal mass should be equivalent to the standard definition when evaluated on the spheres if the spacetime is spherically symmetric. In particular, for the centered spheres in the Schwarzschild spacetime, the quasilocal mass should be equivalent to the standard mass.
(3) For an asymptotically flat slice, the quasilocal energy-momentum vector of the coordinate sphere should asymptotic to the ADM energy-momentum vector.
(4) For an asymptotically null slice, the quasilocal energy-momentum vector of the coordinate sphere should asymptotic to the Bondi energy-momentum vector.
(5) For an apparent horizon Σ, the quasilocal mass should be no less than a (universal) constant multiple of the irreducible mass which is $\sqrt{\mathrm{Area}(\Sigma)/16\pi}$.

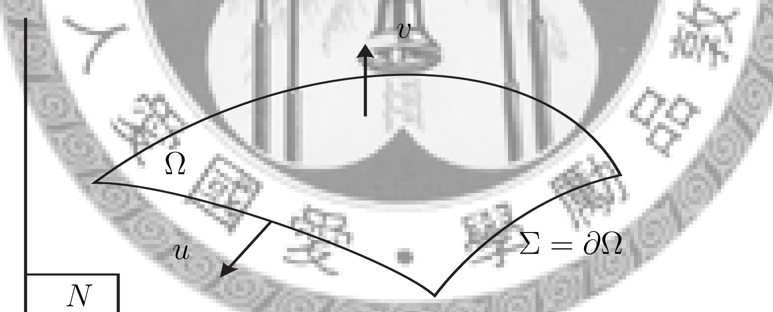
(6) The quasilocal energy-momentum vector should be non-spacelike and the quasilocal mass should be nonnegative.

There have been many attempts to define quasilocal mass(most of them did not give the associated momentum 3-vector). Unfortunately, none of these definition could satisfy all required properties. In this note, we only discuss the Brown-York type quasilocal mass. For other definitions, the readers may consult [Sz].

The rest of the note is organized as follows. In section 2, we recall the definition of Brown-York and Liu-Yau quasilocal mass and the properties of the latter. In section 3, we describe Yu-Guang Shi and Luen-Fai Tam's approach to proving the positivity of quasilocal mass in Riemannian case. In section 4, we discuss how Liu and Yau solved the general case. In section 5, we discuss Wang and Yau's modification of Liu-Yau quasilocal mass.

2 The definition of Brown-York and Liu-Yau quasilocal mass

Let Ω be a compact spacelike hypersurface with boundary in a time-oriented spacetime N with timelike future-directed unit normal v, and let Σ be a connected component of $\partial\Omega$ with outward normal u. We denote the second fundamental form of Ω in N and Σ in Ω by p_{ij} and p_{ab} respectively, and $K = trK_{ij}$, $k = trk_{ab}$.



Remark. In this note, we follow the usual convention. The Greek indices $\alpha, \beta, \ldots = 0, 1, 2, 3$; the Latin indices $i, j, \ldots = 1, 2, 3$; and $a, b, \ldots = 1, 2$.

We need the Weyl embedding theorem:

Theorem (Weyl embedding theorem [Ni, Po]). *Let Σ be a closed surface with a Riemannian metric of positive Gauss curvature, then there exists an isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^3$ that is unique up to Euclidean rigid motion. Furthermore, $i(\Sigma)$ is convex.*

Suppose Σ has positive Gauss curvature. By Weyl embedding theorem, Σ can be isometrically embedded into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. The second fundamental form $(k_0)_{ab}$ of the embedded surface is positive definite and determined by the intrinsic curvature of Σ.

The Brown-York quasilocal mass is defined as

$$E(\Sigma, \Omega) = \frac{1}{8\pi G} \int_\Sigma k_0 - k.$$

If in addition the mean curvature vector $\overrightarrow{\mathbf{H}}$ of $\Sigma \subset N$ is spacelike, the Liu-Yau quasilocal mass is defined as

$$E(\Sigma) = \frac{1}{8\pi G} \int_\Sigma k_0 - |\overrightarrow{\mathbf{H}}|.$$

Remark. Brown and York proposed their definition in 1992 through the Hamilton-Jacobi analysis and verified its properties except positivity. In 2002, Shi and Tam proved the positivity of Brown-York mass in the time-symmetric case(see section 3). In 2003, based on Yau's work on blackholes, Liu and Yau proposed their definition out of the geometric consideration.

Liu-Yau's quasilocal mass is more intrinsic because it is independent of the three manifold Σ encloses. It is also a good candidate in view of the requirements mentioned in the introduction. For (1), see section 5. On the Schwarzchild spacetime, $E(S_r) = r(1 - \sqrt{1 - \frac{2M}{r}})$ for $r > 2M$. Note $E(S_{2M}) = 2M, E(S_\infty) = M$, which is consistent with (2). For (3) and (4), see [Epp]. $E(\Sigma)$ satisfies (5) by Minkowski inequality for convex surfaces. The positivity of $E(\Sigma)$ is discussed in section 4.

3 The Riemannian Case

In this section, we describe Shi and Tam's proof on the positivity of quasilocal mass [ST] for the Riemannian case.

When Ω has zero second fundamental form($p_{ij} = 0$), we say it is time-symmetric. The weak energy condition $\mu \geq |\mathbf{J}|$ implies Ω has nonnegative scalar curvature. The assumption that Σ has spacelike mean curvature vector implies Σ has positive mean curvature in Ω. In this case the Brown-York and Liu-Yau quasilocal mass coincide, and the positivity of quasilocal mass reduces to a problem of Riemannian geometry.

Theorem 1 ([ST, Theorem 4.2.]). *Let (Ω^3, g) be a Riemannian manifold of dimension 3 with compact closure with smooth boundary and with nonnegative scalar curvature. Suppose $\partial\Omega$ has finitely many components Σ_i so that each component has positive Gauss* *curvature and positive mean curvature H with respect to the outward normal. Then for each* Σ_i*,*

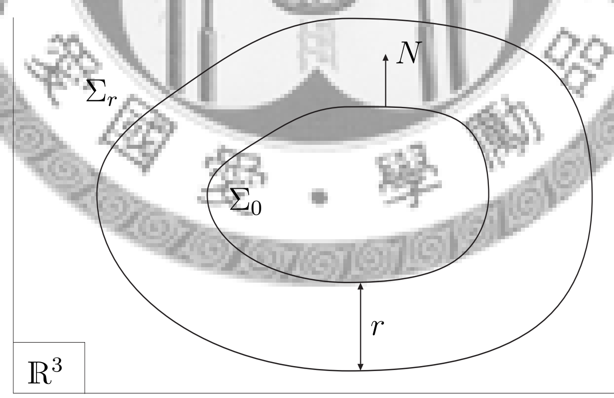
$$\int_{\Sigma_i} H d\sigma \leq \int_{\Sigma_i} H_0^{(i)} d\sigma.$$

where $H_0^{(i)}$ *is the mean curvature of* Σ_i *with respect to the outward normal when it is isometrically embedded into* $\mathbb{R}^3$*. Moreover, if the equality holds for some* Σ_i*, then* $\partial\Omega$ *has only one component and* Ω *is isometric to a domain in* $\mathbb{R}^3$*.*

Proof(sketch)
Step1: The main idea is applying Bartnik's quasi-spherical construction. Roughly speaking, we turn Ω into a complete asymptotically flat manifold by gluing ends to Ω and try to relate the quasilocal mass to the ADM mass of this new manifold. For simplicity, we assume $\partial\Omega$ has only one component in the following. First we isometrically embed Σ into $\mathbb{R}^3$ as a strictly convex hypersurface Σ_0. The position vector of the exterior E of Σ_0 is $\mathbf{Y} = \mathbf{X} + r\mathbf{N}$, where $\mathbf{X}$ is the position vector of Σ_0 and $\mathbf{N}$ is the unit outward normal of Σ_0. Let Σ_r be the convex hypersurface at distance r to Σ_0. The Euclidean space outside Σ_0 can be represented by $(\Sigma_0 \times [0,\infty), dr^2 + g_r)$, where g_r is the induced metric on Σ_r. Next, we solve the prescribed scalar curvature equation

$$\begin{cases} 2H_0 \frac{\partial u}{\partial r} &= 2u^2 \triangle_r u + (u - u^3) R^r \quad \text{on } \Sigma_0 \times [0,\infty) \\ u(x,0) &= u_0(x) \end{cases} \tag{2}$$

where $u_0(x)$ is a positive smooth function on Σ_0, and H_0, R^r are the mean curvature and scalar curvature of Σ_r.



The purpose of the above construction is to deform the Euclidean metric radially to get an asymptotically flat metric while keeping the scalar curvature equal to zero. The asymptotic behavior of u also gives the ADM mass of the metric.

Theorem ([ST, Theorem 2.1.])**.** *The initial value problem (2) has a unique solution u on* $\Sigma_0 \times [0,\infty)$ *such that*

- $u(z) = 1 + \frac{m_0}{\rho^{n-2}} + v$, *where* m_0 *is a constant and* v *satisfies* $|v| = O(\rho^{1-n})$ *and* $|\nabla_0 v| = O(\rho^{-n})$;

- *The metric* $ds^2 = u^2 dr^2 + g_r$ *is asymptotically flat with scalar curvature* $R \equiv 0$ *outside* Σ_0;

- *The ADM mass* m_{ADM} *of* ds^2 *is given by*

$$c(n)m_{\text{ADM}} = (n-1)\omega_{n-1}m_0 = \lim_{r\to\infty}\int_{\Sigma_r} H_0(1-u^{-1})d\sigma_r = \lim_{r\to\infty}\int_{\Sigma_r}(H_0 - H)d\sigma_r,$$

  for some positive constant $c(n)$, *where* H_0 *and* H *are the mean curvatures of* Σ_r *with respect to the Euclidean metric and* ds^2 *respectively.*

Step2: If we view $H(x)$ as a function on Σ_0, by the assumption of Theorem 1, $\frac{H_0(x)}{H(x)}$ is positive on Σ_0. We solve the prescribed scalar curvature equation with initial value $u(x,0) = \frac{H_0(x)}{H(x)}$, and let $ds^2 = u^2dr^2 + g_r$. Let (M, g') be the Riemannian manifold obtained by gluing (Ω, g) and (E, ds^2) along $\Sigma \simeq \Sigma_0$.

Note the following properties of g' :

i) g' is only Lipschitz near $\partial\Omega$.
ii) The mean curvature at $\partial\Omega$ with respect to $g'|_{N\setminus\Omega}$ and $g'|_{\overline{\Omega}}$ coincide.
iii) g' is asymptotically flat.
iv) The scalar curvature of $N\setminus\partial\Omega$ is nonnegative (zero on $N\setminus\overline{\Omega}$).

Shi and Tam are able to prove a positive mass theorem for this type of metric. The Weitzenböck formula remains the same by (ii)([ST, Lemma 3.2.]) The existence of asymptotically constant harmonic spinor is more involved and is treated in [ST, pages 23-27]

Step3: The last step is to prove the monotonicity of mass expression.

Lemma ([ST, Lemma 4.2.]).

$$m(r) = \int_{\Sigma_r} H_0(1-u^{-1})d\sigma_r \quad \textit{is nonincreasing in } r.$$

Since $m(0) = \int_\Sigma H_0 - H$, $m(\infty) = m_{\text{ADM}} \geq 0$, this completes the proof of Theorem 1.

4 The General Case

Recall (Ω, g_{ij}, p_{ij}) refers to a compact spacelike hypersurface with boundary in a time-oriented four dimensional spacetime N, where g_{ij} and p_{ij} are the induced metric and second fundamental form of Ω. The local mass density μ and local current density J^i of Ω are

$$\mu = \frac{1}{2}(R - \sum_{ij} p^{ij} g_{ij} + (\sum_i p_i^i)^2)$$
$$J^i = \sum_j D_j(p^{ij} + (\sum_k p_k^k) g^{ij})$$

where R is the scalar curvature of g_{ij}.

Theorem 2 ([LY2, Theorem 1]). *Suppose μ and J^i satisfies the weak energy condition $\mu \geq \sqrt{J^i J_i}$ and the boundary $\partial\Omega$ has finitely many connected components $\Sigma^1, \ldots, \Sigma^l$, each of which has positive Gaussian curvature and spacelike mean curvature vector in N. Then $E(\Sigma^\alpha) \geq 0$ for $\alpha = 1, \ldots, l$. Moreover, if $E(\Sigma^\alpha) = 0$ for some α, then N is flat along Ω and $\partial\Omega$ is connected.*

Proof(sketch)
Step1: (Construct a scalar flat three manifold)The main idea is to reduce this case into the Riemannian one. Consider the Jang's equation on Ω with Dirichlet boundary condition:

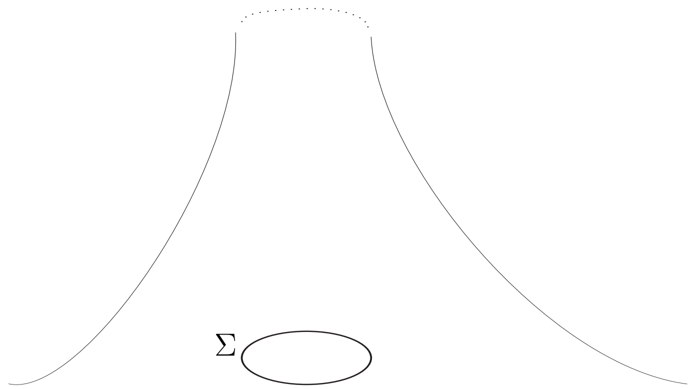
$$\begin{cases} \sum_{i,j=1}^3 (g^{ij} - \frac{f^i f^j}{1+|\nabla f|^2})(\frac{f_{ij}}{1+|\nabla|^2} - p_{ij}) = 0 \\ f|_{\partial\Omega} \equiv 0 \end{cases}$$

Yau showed there exists a solution to this boundary value problem when (Ω, g_{ij}, p_{ij}) contains no apparent horizon [Y](with the main estimates in [SY2, setion 3]). When (Ω, g_{ij}, p_{ij}) has apparent horizons, the solution would blow up around the apparent horizons, but the graph of the solution in $\Omega \times \mathbb{R}$ can be compactified to get a smooth manifold with a discontinuous metric [SY2, p.257].

Let $\bar{g}_{ij} = g_{ij} + f_i f_j$ be a new metric that coincides with g_{ij} on $\partial\Omega$. The scalar curvature of $\bar{g}$ satisfies

$$\bar{R} \geq 2\,|X|^2 - 2\,\mathrm{div} X,$$

for some vector field X(For the explicit form of X, see [LY2, p.7]) This is enough for the existence of a scalar flat metric.



Proposition ([LY2, Proposition 5]). *Suppose the scalar curvature $\bar{R}$ satisfies $\bar{R} \geq c|X|^2 - 2\,\mathrm{div}X$, for some constant $c > \frac{1}{2}$ and some smooth vector field X on Ω. Then there is a unique metric $\hat{g}_{ij}$ on Ω such that*

1. *The metric $\hat{g}_{ij}$ is conformal to $\bar{g}_{ij}$.*
2. *The scalar curvature of $\hat{g}_{ij}$ is zero.*
3. *The metric $\hat{g}_{ij}$ coincides with $\bar{g}_{ij}$ on $\partial\Omega$.*
4. *Let $\bar{H}$ and $\hat{H}$ denote the mean curvature with respect to the metric $\bar{g}$ and $\hat{g}$ respectively, and let $\bar{\nu}$ denote the outward unit normal of $\partial\Omega$ in $(\Omega, \bar{g})$. Then*

$$\int_{\partial\Omega} \hat{H} \geq \int_{\partial\Omega} (\bar{H} - \langle X, \bar{\nu} \rangle),$$

where the equality holds if and only if $\bar{R} = 0, X = 0$, and $\hat{g}_{ij} = \bar{g}_{ij}$.

Step2:(Glue ends to Ω) We have the following computation:

Lemma ([LY2, Lemma 6.]).

$$\bar{H} - \langle X, \bar{\nu} \rangle \geq |\overrightarrow{\mathbf{H}}|$$

Because $\overrightarrow{\mathbf{H}}$ is assumed to be spacelike, $\bar{H} - \langle X, \bar{\nu} \rangle$ is positive. Liu and Yau next modified Shi and Tam's approach by solving the prescribed scalar curvature equation (2) with initial value $h(x, 0) = \frac{H_0^\alpha}{\bar{H} - \langle X, \bar{\nu} \rangle}$ on $E^\alpha \simeq \Sigma^\alpha \times [0, \infty)$. Again $m^\alpha(r) = \frac{1}{8\pi G} \int_{\Sigma_r^\alpha} (H_0 - H) d\sigma_r$ on $(E^\alpha, g^\alpha = h^2 dr^2 + g_r)$ is nonincreasing in r. Together with the previous lemma,

$$m^\alpha(0) = \frac{1}{8\pi G} \int_\Sigma (H_0^\alpha - (\bar{H} - \langle X, \bar{\nu} \rangle)) d\sigma \leq \frac{1}{8\pi G} \int_\Sigma (H_0^\alpha - |\overrightarrow{\mathbf{H}}|) d\sigma = E(\Sigma^\alpha) \quad (3)$$

Let $(M, \tilde{g})$ be the three manifold obtained by gluing (E^α, g^α) to $(\Omega, \hat{g} = u^4 \bar{g})$. $\tilde{g}$ is a continuous Riemannian metric that is

1. smooth on $M\backslash\Omega$ and $\bar{\Omega}$, and is Lipschitz near $\partial\Omega$.

2. asymptotically flat on each end E^α.

3. scalar flat on $M\backslash\partial\Omega$.

Step3: In view of (3), it suffices to prove a positive mass theorem for $(M,\tilde{g})$. However, two difficulties arise because of the discontinuity of mean curvature along $\partial\Omega$. First, a new term appears in the Weitzenböck formula:

Lemma ([LY2, Lemma 11.]). *Let U be an open set of M. For any spinor $\eta \in W_0^{1,2}(U,S)$, $\psi \in W_{loc}^{1,2}(U,S)$, we have*

$$\int_U \langle \mathbf{D}\psi, \mathbf{D}\eta\rangle = \int_U \langle \nabla\psi, \nabla\eta\rangle_{\tilde{g}} + \int_{\partial\Omega\cap U} (2\bar{\nu}(u) + \frac{1}{2}\langle X, n\bar{u}\rangle)\langle\psi,\eta\rangle,$$

where u is the conformal factor of $\hat{g} = u^4\bar{g}$.

Liu and Yau overcome these difficulties by establishing the following inequality

Proposition ([LY2, Proposition 10], see also [WY, Theorem 5.1] for a simpler argument). *For $r > L$ and $\psi \in W_{loc}^{1,2}(M,S)\cap C^\infty(M\backslash M_L, S)$, we have*

$$2\int_{M_r} |\mathbf{D}\psi|^2 \geq \frac{1}{10}\int_{M_r} |\nabla\psi|^2 + \frac{1}{16}\int_\Omega u^{-2}|du|^2|\psi|^2 + \sum_{\alpha=1}^{l}\int_{S_r^\alpha} \langle \frac{H}{2}\psi - c(\nu)\breve{\mathbf{D}}\psi, \psi\rangle$$

where $\breve{\mathbf{D}}$ is the Dirac operator on S_r^α.

Second, the zeroth term of the Dirac operator can be discontinuous along $\partial\Omega$. Liu and Yau modified the argument in [PT, section 5] under weaker regularity. Indeed, the harmonic spinors lie in $W^{1,p}(M)$ instead of $C^\infty(M)$, but such regularity is sufficient to prove the positive mass theorem here.

By a calculation similar to that in [PT, pages 231-232],

$$\lim_{r\to\infty}\int_{S_r^\alpha} \langle \frac{H}{2}\psi - c(\nu)\breve{\mathbf{D}}\psi, \psi\rangle = -m_\infty^\alpha|\psi_0|^2$$

for a spinor asymptotic to a constant spinor ψ_0, where $m_\infty^\alpha = \lim\limits_{r\to\infty} m^\alpha(r)$. This finishes the proof of Theorem 2.

5 A new quasilocal mass of Wang and Yau

In this fianl section, we present the recent work of Mu-Tao Wang and Yau on quasilocal mass [WY].

The Liu-Yau quasilocal mass does not satisfy the required property (1) in the introduction. In [MST], Murchadha, Szabados, and Tod construct some surfaces with strictly positive Liu-Yau mass lying in the lightcone of $\mathbb{R}^{3,1}$. In order to resolve this inconsistency, Wang and Yau proposed to take the momentum information p_{ij} into account. They take the reference to be an isometric embedding into $\mathbb{R}^{3,1}$ instead of $\mathbb{R}^3$. The first task is to show the existence and uniqueness of such isometric embedding with prescribed time function.

Theorem 3 ([WY, Theorem 3.1]). *Let Σ be a two-surface diffeomorphic to S^2 with metric σ , τ be a function on Σ, and T_0 be a fixed timelike vector in $\mathbb{R}^{3,1}$. Suppose*

$$K + (1 + |\nabla\tau|^2)^{-1}\det(\nabla^2\tau) > 0$$

where K is the Gauss curvature of Σ and $\det(\nabla^2\tau)$ is the determinant of the Hessian of τ. Then there exists a unique spacelike embedding $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ with the induced metric σ and $\langle X, T_0\rangle = \tau$.

The new quasilocal mass is defined as the difference of the *static mean curvature* between the two isometric embedding $i : \Sigma \hookrightarrow N$, and $i_0 : \Sigma \hookrightarrow \mathbb{R}^{3,1}$.

Definition.

1. [WY, Definition 2.1] Suppose $i : \Sigma \hookrightarrow N$ is an embedded spacelike two-surface. Given a smooth function τ on Σ and a spacelike normal e_3, the static mean curvature associated with these data is defined to be

$$h(\Sigma, i, \tau, e_3) = -\sqrt{1 + |\nabla\tau|^2}\langle\overrightarrow{\mathbf{H}}, e_3\rangle + \alpha_{e_3}(\nabla\tau)$$

where $\overrightarrow{\mathbf{H}}$ is the mean curvature vector of Σ in N and $\alpha_{e_3}(v) = \langle\nabla^N_v e_3, e_4\rangle$ is the connection form of the normal bundle of Σ in N determined by e_3 and the future-directed timelike unit normal e_4 orthogonal to e_3.

2. [WY, Definition 2.2] Given an isometric embedding $i : \Sigma \hookrightarrow N$ with spacelike mean curvature vector $\overrightarrow{\mathbf{H}}$. Denote

$$\mathfrak{H}(\Sigma, i, \tau) = \int_\Sigma h(\Sigma, i, \tau, \bar{e}_3)dv_\Sigma,$$

where $h(\Sigma, i, \tau, \bar{e}_3) = \min_{e_3}\{h(\Sigma, i, \tau, e_3)\}$.

3. [WY, Definition 5.2] Given a spacelike embedding $i : \Sigma \hookrightarrow N$. Suppose the set of admissible functions is non-empty(See [WY, Definition 5.1]). The quasilocal mass is defined to be the infimum of

$$\mathfrak{H}(\Sigma, i_0, \tau) - \mathfrak{H}(\Sigma, i, \tau)$$

among all admissible τ, where i_0 is the unique spacelike isometric embedding of Σ into $\mathbb{R}^{3,1}$ associated with τ given by Theorem 3.

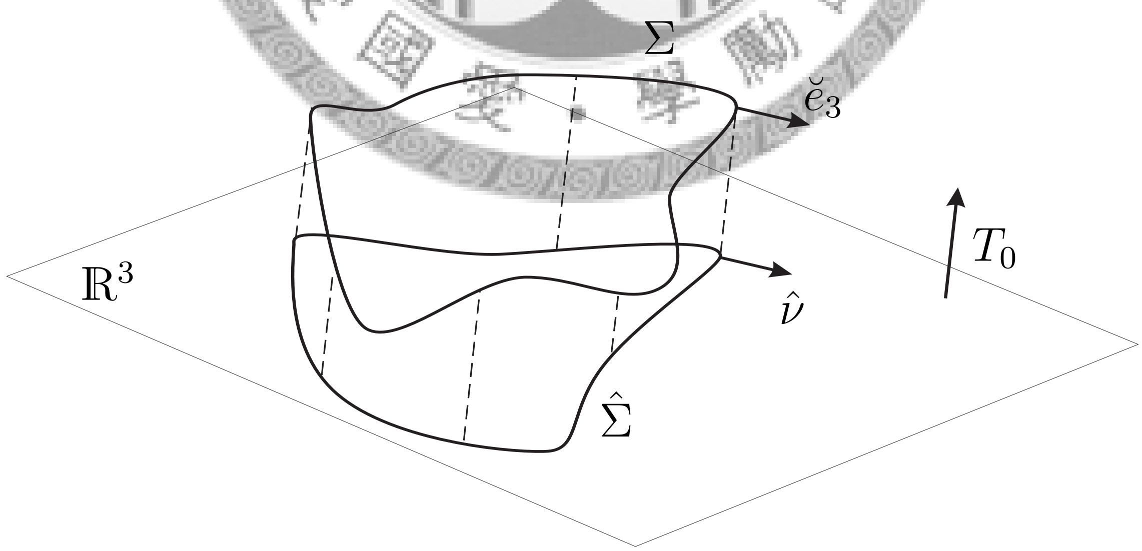
For a two-surface $\Sigma \subset \mathbb{R}^{3,1}$, $i = i_0$. If the projection of Σ along some time direction is a convex surface, then Σ has zero quasilocal mass. This case covers the examples of Murchadha, Szabados, and Tod.

We briefly mention the idea of proving the positivity of this new quasilocal mass. For an embedded two-surface $\Sigma \subset \mathbb{R}^{3,1}$, we denote by $\hat{\Sigma}$ its projection onto $\mathbb{R}^3$. The most important observation of Wang and Yau is to identify the two terms $\int_\Sigma k_0$ and $\int_\Sigma H - \langle X, \nu \rangle$ appearing in Liu and Yau's proof to the integral of some static mean curvature.

Theorem 4 ([WY, Proposition 3.1, 3.2])**.**

$$\int_{\hat{\Sigma}} \hat{k} = \int_\Sigma h(\Sigma, i_0, \tau, \breve{e}_3) dv_\Sigma.$$

where $\hat{k}$ is the mean curvature of $\hat{\Sigma}$ in $\mathbb{R}^3$ with respect to the outward normal $\hat{\nu}$, and $\breve{e}_3$ is obtained by parallel translating $\hat{\nu}$ along T_0. Furthermore, when the mean curvature of Σ in $\mathbb{R}^{3,1}$ is spacelike, $\int_{\hat{\Sigma}} \hat{k} dv_{\hat{\Sigma}} = \mathfrak{H}(\Sigma, i_0, \tau)$.



Theorem 5 ([WY, Theorem 4.1]). *Let $i : \Sigma \hookrightarrow N$ be a spacelike embedding. Given any smooth function τ on Σ and any spacelike hypersurface Ω with $\partial\Omega = \Sigma$. Suppose the Dirichlet problem of the Jang's equation over Ω subject to the boundary condition that $f = \tau$ on Σ is solvable. Then there exists a spacelike unit normal e_3' along Σ in N such that the expression $\tilde{k} - \langle \tilde{\nabla}_{\tilde{e}_4}\tilde{e}_4, \tilde{e}_3\rangle + P(\tilde{e}_4, \tilde{e}_3)$(this is the familiar term $\hat{H} - \langle X, \nu\rangle$ in Liu and Yau's paper) at $\tilde{q} \in \tilde{\Sigma} \subset \Omega \times \mathbb{R}$ is equal to*

$$(1 + |\nabla\tau|^2)^{-1/2} h(\Sigma, i, \tau, e_3') \text{ at } q \in \Sigma,$$

where $\tilde{q} = (q, \tau(q)) \in \tilde{\Sigma}$.

Combining these two theorems and the result of Liu and Yau,

$$\begin{aligned}\mathfrak{H}(\Sigma, i_0, \tau) &= \int_{\hat{\Sigma}} \hat{k} \\ &\geq \int_{\tilde{\Sigma}} \tilde{k} - \langle \tilde{\nabla}_{\tilde{e}_4}\tilde{e}_4, \tilde{e}_3\rangle + P(\tilde{e}_4, \tilde{e}_3) \\ &= \int_{\Sigma} h(\Sigma, i, \tau, e_3') dv_\Sigma \\ &\geq \mathfrak{H}(\Sigma, i, \tau)\end{aligned}$$

Suppose the two-surface Σ bounds a spacelike hypersurface in N, and has positive Gauss curvature and spacelike mean curvature vector. Then the assumptions of Theorem 3 and Theorem 5 are satisfied(For details, see [WY, Theorem 4.2].) We can conclude that

Theorem ([WY, Corollary 5.3]). *Under the assumption of Theorem 2, the new quasilocal mass is nonnegative.*

Appendix

It is a well-known fact that the weak energy condition is equivalent to $\mu \geq |\mathbf{J}|$. We just write it down in this appendix for completion.

We first fix the notation. Let N be a 4-manifold with a Lorentzian metric of signature $(-+++)$. For a point $x \in N$, and an orthonormal frame $\{e_\alpha \mid \alpha = 0, \ldots, 3\}$ near x, we denote the curvature tensor by

$$\begin{aligned}R(e_\alpha, e_\beta)e_\gamma &= \nabla_{e_\alpha}\nabla_{e_\beta}e_\gamma - \nabla_{e_\beta}\nabla_{e_\alpha}e_\gamma - \nabla_{[e_\alpha, e_\beta]}e_\gamma = R_{\alpha\beta\gamma}{}^{\delta} e_\delta, \\ R_{\alpha\beta\gamma\delta} &= g_{\sigma\delta} R_{\alpha\beta\gamma}{}^{\sigma}.\end{aligned}$$

Note $R_{\alpha\beta\gamma 0} = -R_{\alpha\beta\gamma}{}^{0}$. The Ricci curvature and scalar curvature are

$$
\begin{aligned}
R_{\alpha\beta} &= R_{\gamma\alpha\beta}{}^{\gamma} = -R_{0\alpha\beta 0} + R_{1\alpha\beta 1} + R_{2\alpha\beta 2} + R_{3\alpha\beta 3},\\
R &= g^{\alpha\beta} R_{\alpha\beta} = -R_{00} + R_{11} + R_{22} + R_{33}.
\end{aligned}
$$

Suppose we have a spacelike hypersurface $M \subset N$ with the induced metric. We denote the connection and curvature of N and M by $\bar{D}, \bar{R}_{\alpha\beta\gamma\delta}$ and D, R_{ijkl} respectively. Let e_0 be the timelike unit normal of M. In the neighborhood of a fix point $x \in M$, we choose a normal frame $\{e_i \mid i = 1, \ldots, 3\}$ at x diagonalizing the second fundamental form, that is, for any i, j, $D_{e_i} e_j(x) = 0$, $p_{ij}(x) = p(e_i, e_j) = k_i \delta_{ij}$.

We compute the Gauss and Codazzi equations for hypersurfaces:

$$
\begin{aligned}
\langle R(X,Y)Z, W\rangle &= \langle \bar{R}(X,Y)Z, W\rangle - p(X,W)p(Y,Z) + p(X,Z)p(Y,W)\\
-\langle \bar{R}(X,Y)Z, e_0\rangle &= D_X p(Y,Z) - D_Y p(X,Z).
\end{aligned}
$$

where X, Y, Z, W are tangent vectors of M.
Proof. For Gauss equation,

$$
\begin{aligned}
\langle D_X D_Y Z, W\rangle &= \langle \bar{D}_X D_Y Z, W\rangle\\
&= \langle \bar{D}_X(\bar{D}_Y Z + \langle \bar{D}_Y Z, e_0\rangle e_0), W\rangle, \quad \text{since} \langle e_0, e_0\rangle = -1\\
&= \langle \bar{D}_X \bar{D}_Y Z, W\rangle + \langle \bar{D}_Y Z, e_0\rangle \langle \bar{D}_X e_0, W\rangle\\
&= \langle \bar{D}_X \bar{D}_Y Z, W\rangle - p(Y,Z)p(X,W)
\end{aligned}
$$

The computation of $\langle D_X D_Y Z, W\rangle$, $\langle D_{[X,Y]} Z, W\rangle$ is similar.
For Codazzi equation,

$$
\begin{aligned}
D_X p(Y,Z) &= X p(Y,Z) - p(D_X Y, Z) - p(Y, D_X Z)\\
&= -X\langle \bar{D}_Y Z, e_0\rangle - \langle \bar{D}_Z e_0, D_X Y\rangle - \langle \bar{D}_Y e_0, D_X Z\rangle\\
&= -\langle \bar{D}_X \bar{D}_Y Z, e_0\rangle - \langle \bar{D}_Y Z, \bar{D}_X e_0\rangle - \langle \bar{D}_Z e_0, D_X Y\rangle - \langle \bar{D}_Y e_0, D_X Z\rangle\\
-D_Y p(X,Z) &= \langle \bar{D}_Y \bar{D}_X Z, e_0\rangle + \langle \bar{D}_X Z, \bar{D}_Y e_0\rangle + \langle \bar{D}_Z e_0, D_Y X\rangle + \langle \bar{D}_X e_0, D_Y Z\rangle
\end{aligned}
$$

Canceling the second and fourth terms and combining the third term,

$$
\langle \bar{D}_Z e_0, D_Y X - D_X Y\rangle = -\langle \bar{D}_Z e_0, [X,Y]\rangle = \langle e_0, \bar{D}_Z [X,Y]\rangle = \langle e_0, \bar{D}_{[X,Y]} Z\rangle,
$$

we get the desired result.

To verify our claim, it is sufficient to show $\mu = T_{00}$, $J_i = T_{0i}$, up to a constant. From

Einstein equation $T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$(We omit the constant $8\pi G$),

$$
\begin{aligned}
T_{00} &= \bar{R}_{00} + \frac{1}{2}\bar{R} \\
&= \frac{1}{2}(\bar{R}_{00} + \bar{R}_{11} + \bar{R}_{22} + \bar{R}_{33}) \\
&= \frac{1}{2}(2\bar{R}_{1221} + 2\bar{R}_{1331} + 2\bar{R}_{2332}) \\
&= \frac{1}{2}(2R_{1221} + 2p_{11}p_{22} + 2R_{1331} + 2p_{11}p_{33} + 2R_{2332} + 2p_{22}p_{33}) \\
&\qquad\qquad\qquad\qquad \text{by Gauss equation and } p_{ij} = k_i\delta_{ij} \\
&= \frac{1}{2}(R - (p_{11}^2 + p_{22}^2 + p_{33}^2) + (p_{11} + p_{22} + p_{33})^2) \\
J_i &= \sum_{j=1}^{3} D_j p_{ij} - e_i(\sum_{k=1}^{3} p_{kk}) \\
&= \sum_{j=1}^{3} D_i p_{jj} - \bar{R}_{jij0} - e_i(\sum_{k=1}^{3} p_{kk}) \qquad \text{by Codazzi equation} \\
&= \bar{R}_{i0}
\end{aligned}
$$

References

[B] Bartnik, Robert *The mass of an asymptotically flat manifold.* Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693.

[CY] Christodoulou, D.; Yau, S.-T. *Some remarks on the quasi-local mass.* Mathematics and general relativity (Santa Cruz, CA, 1986), 9–14, Contemp. Math., **71**, Amer. Math. Soc., Providence, RI, 1988.

[Epp] Epp, Richard J. Angular momentum and an invariant quasilocal energy in general relativity. Phys. Rev. D (3) **62** (2000), no. 12, 124108

[LY1] Liu, Chiu-Chu Melissa; Yau, Shing-Tung *Positivity of quasilocal mass.* Phys. Rev. Lett. **90** (2003), no. 23, 231102

[LY2] Liu, Chiu-Chu Melissa; Yau, Shing-Tung *Positivity of quasi-local mass. II.* J. Amer. Math. Soc. **19** (2006), no. 1, 181–204

[M] N. Ó. Murchadha *The Liu-Yau mass as a quasi-local energy in general relativity*, arXiv:0706.1166

[MST] N. Ó. Murchadha; L. B. Szabados; K. P. Tod. *Comment on: "Positivity of quasilocal mass"* Phys. Rev. Lett. **92** (2004), no. 25, 259001

[Ni] Nirenberg, Louis *The Weyl and Minkowski problems in differential geometry in the large.* Comm. Pure Appl. Math. **6**, (1953). 337–394.

[Po] Pogorelov, A. V. *Regularity of a convex surface with given Gaussian curvature.* (Russian) Mat. Sbornik N.S. **31(73)**, (1952). 88–103

[PT] Parker, Thomas; Taubes, Clifford Henry *On Witten's proof of the positive energy theorem.* Comm. Math. Phys. **84** (1982), no. 2, 223–238.

[ST] Shi, Yuguang; Tam, Luen-Fai *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature.* J. Differential Geom. **62** (2002), no. 1, 79–125.

[SY1] Schoen, Richard; Yau, Shing-Tung. *On the proof of the positive mass conjecture in general relativity.* Comm. Math. Phys. **65** (1979), no. 1, 45–76.

[SY2] Schoen, Richard; Yau, Shing-Tung. *Proof of the positive mass theorem. II.* Comm. Math. Phys. **79** (1981), no. 2, 231–260.

[Sz] L. B. Szabados. *Quasi-local energy-momentum and angular momentum in GR: a review article* Living Rev. Relativity, **7** (2004), 4.

[W] Witten, Edward *A new proof of the positive energy theorem.* Comm. Math. Phys. **80** (1981), no. 3, 381–402.

[WY] Wang, Mu-Tao; Yau, Shing-Tung. *Isometric embedding into the Minkowski space and new quasi-local mass*, arXiv:0805.1370v2

[Y] Yau, Shing-Tung. *Geometry of three manifolds and existence of black hole due to boundary effect.* Adv. Theor. Math. Phys. 5 (2001), no. 4, 755–767