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Master Thesis

平直時空的宇宙關聯子

Flat Space Cosmological Correlator

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MASTER'S THESIS ACCEPTANCE CERTIFICATE
NATIONAL TAIWAN UNIVERSITY

平直時空的宇宙關聯子

（論文中文題目）(Chinese title of Master's thesis)

Flat Space Cosmological Correlator

（論文英文題目）(English title of Master's thesis)

本論文係__劉友安___(姓名)__R09222019__（學號）在國立臺灣大學__物理系__(系/所/學位學程)完成之碩士學位論文，於民國_112__年_6_月_21_日承下列考試委員審查通過及口試及格，特此證明。

The undersigned, appointed by the Department / Institute of _Physics_______ on ___June____ (month)___2023__ (year) have examined a Master's thesis entitled above presented by____You-An Liu___ (name)___R09222019 _____ (student ID) candidate and hereby certify that it is worthy of acceptance.

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摘要

在本論文中，我們考慮四維平坦空間等時關聯子的自舉。我們在平坦空間相關器中回顧了「宇宙光學定理」(COT)，然後利用總能量為零和部分能量為零極限的約束、顯式局部性以及 Ward-Takahashi 恆等式對樹級關聯子進行約束。為了應用於費米子關聯子，我們推導出半整數算符的 COT，並給出適用於 Dirac 和 Majorana 費米子的獨特規則，適用於宇宙的內部體積。

關鍵字：宇宙學關聯子、半自旋、伴重力子、平直時空、邊界項、自舉

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Abstract

In this thesis, we consider the bootstrap of a four-dimensional flat space equal time correlator. we review the "Cosmological Optical Theorem" (COT) in the context of flat space correlators and proceed to constrain tree-level correlators using the constraints of total energy and partial energy poles, manifest locality, and Ward-Takahashi identities. To apply this to fermionic correlators, we derive the COT for half-integer operators and give distinctive rules suitable for Dirac and Majorana fermions in the bulk.

Keywords: Cosmological Correlator, Spin Half, Gravitino, Flat Space, Boundary term, Bootstrap

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Chapter 1 Introduction

In recent years, a series of studies have been conducted on the bootstrap of physical observables. These studies have employed fundamental principles to impose constraints on physical observables. Early research in this field includes amplitude bootstrap, as discussed in [9] [17], and the bootstrap of Conformal Field Theory (CFT), as outlined in [25][14]. Contemporary investigations have expanded to include inflationary cosmological correlators.

The successful bootstrap program in de Sitter space perturbative cosmological correlators [4], encapsulates the physics of slow-roll inflation. The tree-level structures can be reconstructed from the residues of singularities, de Sitter isometries, unitarity, and locality.

This research follows the massless four-particle test led by McGady DA, Rodina L. and Benincasa P, Cachazo F. McGady DA and Rodina L. demonstrated that the four-particle test in amplitude is enough constrained to necessitate a massless spin 3/2 that respects supersymmetry.[19] On the other hand, Benincasa P and Cachazo F. established that there is no higher spin massless particle if we adhere to the equivalence principle. They further demonstrated that the Yang-Mills color structure constant f^{abc} should respect the Bianchi identity. [5]

The objective of this thesis is to make initial progress towards the massless four-

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particle test of the cosmological correlator on a fixed time spatial slice in flat space. This can be considered a simplified model of the cosmological correlator if we set the fixed time slice at the current time. The time slice will be the boundary of the universe in the past. We demonstrate that in flat space, we can utilize known properties on singularities [20] [4] [18] [26] and unitarity [12] to reconstruct the correlator on the boundary for the massless integer and half-integer spins. In the case of the correlator, the massless spinning particle responds to the conserved operator. It's reasonable that the physical rules are enough constrained like the amplitude case.

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Chapter 2 Review of flat space correlators

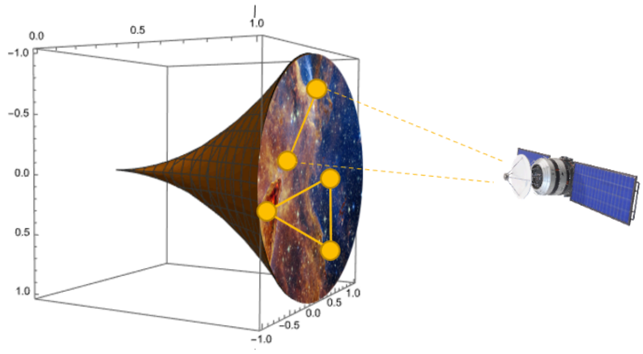

Figure 2.1: The evolution of physical states progresses from the distant past towards the moment we designate as $t = 0$, resulting in what is referred to as the cosmological background. By examining this background spectrum, we can identify N points correlation functions between pairs, triplets, quartets of points, and so on. In cosmology, the measurable quantities resemble the all-in-state version of amplitude, which, as mentioned in certain sources [29], is referred to as the ”in-in formalism.”

In the context of cosmology, the physical observable is the cosmological correlation, which is the expectation value of the product of the boundary field operators. The cosmological correlation can be calculated by Feynman path integral, where the integrand is the wavefunction and the field insertions. Therefore, the wavefunction contains the same information as the cosmological correlation. The wavefunction can be expanded, and the coefficients are referred to as wave function coefficients or correlators. To calculate the tree-level correlator, we need to insert the classical solution of the fields into the action,

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and the classical solutions have to satisfy the Bunch-Davies boundary condition. In this section, we will introduce the concept of cosmological correlation, the correlators, and the Bunch-Davies boundary condition. In addition, we also introduce the lifter to convert a fermionic boundary field to a propagating bulk field.

2.1 Cosmological Correlation Function (in-in correlator)

Let's consider the Minkowski spacetime x^μ with a spatial slice at $x^0 = t = 0$. The bulk field $\phi(x^\mu)$ could have a boundary profile:

$$\phi(t=0,\vec{x}) = \phi_{\mathbf{0}}(\vec{x}), \tag{2.1}$$

We will be interested in the boundary correlation which is the observable we measure on the background $|\Omega_0\rangle$ as we show in Figure 2.1[1],

$$\langle O(\vec{x}_1)O(\vec{x}_2)O(\vec{x}_3)\dots O(\vec{x}_n)\rangle_{\text{in-in}} = \langle \Omega_0|\hat{\phi}(\vec{x}_1)\hat{\phi}(\vec{x}_2)\dots\hat{\phi}(\vec{x}_n)|\Omega_0\rangle \tag{2.2}$$

where $|\Omega_0\rangle$ is the wavefunction encoding the probability of the boundary profile $\phi_{\mathbf{0}}(\vec{x})$. This observable can be also called in-in formalism. For a comparison to the following derivations, in-in formalism could be calculated by the Hamiltonian approach with creation and annihilation operator [29]. In our thesis, we use the wave function method to calculate it. In this method, it will useful to consider particular $\phi_{\mathbf{0}}(\vec{x})$ as the states $|\phi_{\mathbf{0}}(\vec{x})\rangle$ such that

$$\hat{\phi}(\vec{x})\,|\phi_{\mathbf{0}}(\vec{x})\rangle = \phi_{\mathbf{0}}(\vec{x})\,|\phi_{\mathbf{0}}(\vec{x})\rangle \tag{2.3}$$

[1] We use the photo from NASA in this figure.[23]

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Then by inserting a complete basis

$$I = \int d\phi_{\mathbf{0}} \left|\phi_{\mathbf{0}}\right\rangle \left\langle\phi_{\mathbf{0}}\right| \tag{2.4}$$

for the two-point correlation, we could obtain

$$\begin{aligned}
&\langle O(\vec{x}_1)O(\vec{x}_2)\rangle_{\text{in-in}} := \langle\Omega_0|\,\hat{\phi}_{\mathbf{0}}(\vec{x}_1)\hat{\phi}_{\mathbf{0}}(\vec{x}_2)\,|\Omega_0\rangle \\
&= \int \prod_{\vec{x}^{(1)}} d\phi_{\mathbf{0}}^{(1)}(\vec{x}^{(1)}) \int \prod_{\vec{x}^{(2)}} d\phi_{\mathbf{0}}^{(2)}(\vec{x}^{(2)})\, \langle\Omega_0|\,\hat{\phi}_{\mathbf{0}}(\vec{x}_1)\,|\phi_{\mathbf{0}}^{(1)}\rangle \left\langle \phi_{\mathbf{0}}^{(1)} \middle| \phi_{\mathbf{0}}^{(2)} \right\rangle \langle\phi_{\mathbf{0}}^{(2)}|\,\hat{\phi}_{\mathbf{0}}(\vec{x}_2)\,|\Omega_0\rangle \\
&= \int \prod_{\vec{x}^{(1)}} d\phi_{\mathbf{0}}^{(1)}(\vec{x}^{(1)}) \int \prod_{\vec{x}^{(2)}} d\phi_{\mathbf{0}}^{(2)}(\vec{x}^{(2)}) \left\langle \Omega_0 \middle| \phi_{\mathbf{0}}^{(1)} \right\rangle \phi_{\mathbf{0}}^{(1)}(\vec{x}_1)\phi_{\mathbf{0}}^{(2)}(\vec{x}_2)\delta(\phi_{\mathbf{0}}^{(1)} - \phi_{\mathbf{0}}^{(2)}) \left\langle \phi_{\mathbf{0}}^{(2)} \middle| \Omega_0 \right\rangle \\
&= \int \prod_{\vec{x}} d\phi_0(\vec{x})\phi_0(\vec{x}_1)\phi_0(\vec{x}_2)|\Psi[\phi_0]|^2
\end{aligned} \tag{2.5}$$

And for the higher point, we could easily extend the above derivation,

$$\langle\Omega|\,\hat{\phi}(\vec{x}_1)\hat{\phi}(\vec{x}_2)\ldots\hat{\phi}(\vec{x}_n)\,|\Omega\rangle = \int d\phi_{\mathbf{0}}\; \phi_{\mathbf{0}}(\vec{x}_1)\phi_{\mathbf{0}}(\vec{x}_2)\ldots\phi_{\mathbf{0}}(\vec{x}_n)|\,\langle\phi_{\mathbf{0}}|\Omega\rangle\,|^2. \tag{2.6}$$

Thus all correlators can be extracted from the wave function on $\phi_{\mathbf{0}}(\vec{x})$ basis, i.e.

$$\Psi[\phi_{\mathbf{0}}] := \langle\phi_{\mathbf{0}}|\Omega\rangle \tag{2.7}$$

the $\Psi[\phi_{\mathbf{0}}]$ is identified as the path integral,

$$\Psi[\phi_{\mathbf{0}}] = \int_{\phi(t=0,\vec{x})=\phi_{\mathbf{0}}(\vec{x}),\phi(t=-\infty)=\phi_{-\infty}} D\phi(t,\vec{x})\; e^{i\int d^4x\mathcal{L}[\phi(t,\vec{x})]}, \tag{2.8}$$

where we integral over the path $\phi(t,\vec{x})$ subject to the boundary conditions. The path integral could be computed perturbatively by writing

$$\phi(t,\vec{x}) = \phi_{cl}^{BD}(t,\vec{x}) + \tilde{\phi}(t,\vec{x}) \tag{2.9}$$

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where $\tilde{\phi}(t,\vec{x})$ is the perturbation of the field and the $\phi_{cl}^{BD}(t,\vec{x})$ is the background field we perturbed at, called Bunch-Davies vacuum, it's the classical solution of the E.O.M or the saddle point of the path integral under some boundary condition configuration we'll discuss later.

$$\begin{aligned}\Psi[\phi_{\mathbf{0}}] &= e^{iS(\phi_{cl}^{BD})}\left[\int_{0=\tilde{\phi}(t=0,\vec{x})=\tilde{\phi}(t=-\infty,\vec{x})} D\tilde{\phi}(t,\vec{x})e^{i\int d^4x\frac{\partial^2\mathcal{L}}{\partial\phi^2}\tilde{\phi}^2+O(\tilde{\phi}^3)}\right] \\ &:= \Psi_{tree}[\phi_0]+\Psi_{1-loop}[\phi_0]+\Psi_{2-loop}[\phi_0]+\dots\end{aligned} \tag{2.10}$$

where S is the action, $S[\phi(t,\vec{x})]=\int d^4x\mathcal{L}[\phi(t,\vec{x})]$. Then we could find the leading order will be tree-level contribution like amplitude given by classical path, and the loop contribution could be shown that it's from the integration of the perturbation [15]. We could always make the path be normalized such that $\int D\phi = 1$. Then we could identify the tree-level correlator for the zero-order expansion of the quantum correction as

$$\Psi_{tree}[\phi_0] = e^{iS(\phi_{cl}^{BD})}. \tag{2.11}$$

In the thesis, we focus on $\Psi_{tree}[\phi_0]$, and all the tree-level physics comes from the leading classical path contribution. And we'll discuss the classical configuration around $\phi_{cl}^{BD}(t,\vec{x})$ we perturbed at in the section (2.3).

2.2 Boundary Actions

Due to the presence of a boundary, different ways of writing the Lagrangian which are related by integration by parts identities will now differ by boundary terms. The guideline for the correct boundary term is that the stationary solution to the variation of the total action, i.e. $\delta S = 0$, must coincide with the solution to the equation of motion. For the

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case of scalars and vectors, this leads to a unique choice of bulk lagrangian with zero boundary terms. In the case of gravity, one requires the Gibbons–Hawking–York boundary term [11, 31]. For fermions, additional subtlety arises due to its Lagrangian being first order in derivatives, which we now review.

We will use the spin-$\frac{1}{2}$ case as our main example. Let's begin with the usual bulk action:

$$S = \int d^4x \left(\frac{i}{2}\bar{\chi}\partial\!\!\!/ \chi - \frac{i}{2}\bar{\chi}\overleftarrow{\partial\!\!\!/}\chi + m\bar{\chi}\chi \right), \tag{2.12}$$

Its variation can be separated into a bulk term and boundary contributions:

$$\delta S = \int d^4x \left(\delta\bar{\chi}(i\partial\!\!\!/ + m)\chi + \bar{\chi}(-i\overleftarrow{\partial\!\!\!/} + m)\delta\chi \right) + \int_{t=0} d^3x \left(\frac{i}{2}\bar{\chi}_0\gamma^0\delta\chi_0 - \frac{i}{2}\delta\bar{\chi}_0\gamma^0\chi_0 \right). \tag{2.13}$$

To ensure that the result of extremelization leads to the usual equations of motion, i.e. $(-i\partial\!\!\!/ - m)\chi = 0$, the boundary contribution must be zero:

$$\int_{t=0} d^3x \left(\frac{i}{2}\bar{\chi}_0\gamma^0\delta\chi_0 - \frac{i}{2}\delta\bar{\chi}_0\gamma^0\chi_0 \right) = 0 \tag{2.14}$$

A naive way to satisfy the condition is to require χ and $\bar{\chi}$ be fixed on the boundary, hence $\delta\chi_0 = \delta\bar{\chi}_0 = 0$. However, χ and $\bar{\chi}$ are canonical conjugates to each other (similar to x and p in classical mechanics) [2]. Said in another way, the Dirichlet condition on $\delta\chi_{\mathbf{0}}$ is equivalent to a Neumann boundary condition on $\delta\bar{\chi}_0$, and one cannot set both conditions at once given the first derivative nature.

To proceed, we choose to impose the Dirichlet condition on half of the fermions and add an additional "boundary action" engineered such that its variation cancels whatever

[2] While $\bar{\chi}$ is the complex adjoint of χ, on the path integral they are independently complexified, so their boundary conditions are independent.

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boundary contribution remains. Since the boundary is at a fixed time slice, it is natural to separate the 4D spinor into

$$\chi = \chi^{+} + \chi^{-}, \tag{2.15}$$

where $\gamma^0\chi^{\pm} = \pm\chi^{\pm}$. We impose Dirichlet boundary conditions $\delta\chi_{\mathbf{0}}^{+} = \delta\bar{\chi}_{\mathbf{0}}^{-} = 0$, that is,

$$\begin{aligned} \gamma^0 \delta\chi_{\mathbf{0}} &= -\delta\chi_{\mathbf{0}} \\ \gamma^0 \delta\bar{\chi}_{\mathbf{0}} &= \delta\bar{\chi}_{\mathbf{0}}\,. \end{aligned} \tag{2.16}$$

The LHS of (2.14) then becomes

$$\int_{t=0} d^3x \left(\frac{i}{2}\bar{\chi}_0\gamma^0\delta\chi_0 - \frac{i}{2}\delta\bar{\chi}_0\gamma^0\chi_0 \right) = -\frac{i}{2}\delta \int_{t=0} d^3x \bar{\chi}_0\chi_0. \tag{2.17}$$

An appropriate boundary action S_{boundary} can be added to S_0 so as to cancel this term

$$S_{\text{boundary}} = \frac{i}{2}\int_{t=0} d^3x \bar{\chi}_0\chi_0, \tag{2.18}$$

and the full action is $S = S_0 + S_{\text{boundary}}$. Note that if we instead choose $\delta\chi_{\mathbf{0}}^{-} = \delta\bar{\chi}_{\mathbf{0}}^{+} = 0$, the corresponding boundary action is the same as (2.18) with an additional minus sign. However, if we choose $\delta\chi_{\mathbf{0}}^{+} = \delta\chi_{\mathbf{0}}^{-} = 0$ or $\delta\bar{\chi}_{\mathbf{0}}^{+} = \delta\bar{\chi}_{\mathbf{0}}^{-} = 0$, no consistent solution exists.

It is worth noting that after substituting the E.O.M into S, the term $S_0 = 0$, and we are left with the action $S = S_{\text{boundary}}$. Thus for fermions, tree-level correlation functions only receive a contribution from the boundary action.

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2.3 Choice of Boundary conditions

At the boundary, the bulk fields have to satisfy the boundary conditions. The boundary value of the classical solution at $t = 0$ is ϕ_0,

$$\phi_{cl}(\vec{x}, t = 0) = \phi_0(\vec{x}), \tag{2.19}$$

while at $t = -\infty$, the bulk fields have to meet the Bunch-Davies boundary conditions, which require the field to be well-defined

$$\phi_{cl}(\vec{x}, t = -\infty) < \infty. \tag{2.20}$$

For Harmonic fields such as a free scalar, the general of classical solutions for $\Box\phi_{cl} = 0$ is

$$\phi_{cl}(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \left[A(\vec{p})e^{iEt} + B(\vec{p})e^{-iEt}\right] e^{i\vec{p}\cdot\vec{x}} \tag{2.21}$$

in which we define $E = |\vec{p}| > 0, \vec{p}\cdot\vec{x} = x^i p_i, \eta_{ij} = ((-1, 0, 0, 0), (0, -1, 0, 0), (0, 0, 0, -1))$. We call them positive/negative energy mode. After we impose all the boundary conditions including the Bunch-Davies boundary condition, the negative energy mode must be excluded for certain analytical continuation on the energy of the classical field $E - i\epsilon, \epsilon > 0$.

$$\phi_{cl}^{BD}(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \phi_{\mathbf{0}}(\vec{p}) e^{i(E-i\epsilon)t} e^{i\vec{p}\cdot\vec{x}} \tag{2.22}$$

Note that, in Anti-de Sitter (AdS) space, there's no natural choice such as $\phi_{cl}(\vec{x}, t = -\infty) < \infty$ or Bunch Davies boundary condition, because the definition of time is ill-defined in AdS space. [21]

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Photon We now discuss the classical solution of the E.O.M of the free integer spin field. For the massless vector field $A_{\mu,cl}(\vec{x},t)$ in the temporal gauge $A_{0,cl}(\vec{x},t)=0$, the E.O.M is

$$\begin{gathered}\partial_t(\partial^i A_{i,cl}(\vec{x},t))=0\\ \Box A_{i,cl}(\vec{x},t)-\partial_i\partial^j A_{j,cl}(\vec{x},t)=0,\end{gathered}\tag{2.23}$$

which has a general solution

$$\begin{aligned}A_{i,cl}(\vec{x},t)&=\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}A_{i,cl}(\vec{p},t)\\ A_{i,cl}(\vec{p},t)&=[\pi_{ij}e^{iEt}-\hat{p}_i\hat{p}_j]A^j(\vec{p})+[\pi_{ij}e^{-iEt}-\hat{p}_i\hat{p}_j]B^j(\vec{p}),\end{aligned}\tag{2.24}$$

where we define $\pi_{ij}\equiv\eta_{ij}+\hat{p}_i\hat{p}_j, \hat{p}_i=p_i/E$, and A_j and B_j are constant coefficients that will be fixed by the boundary conditions. After we impose the boundary condition $A_{i,cl}(\vec{p},0)=A_{\mathbf{0},i}(\vec{p})$ and $A_{i,cl}(\vec{p},-\infty)<\infty$, the classical solution becomes

$$A^{BD}_{i,cl}(\vec{p},t)=[\pi_{ij}e^{i(E-i\epsilon)t}-\hat{p}_i\hat{p}_j]A^j_0(\vec{p}),\tag{2.25}$$

which is a positive energy mode. The detailed calculations can be found in the App. (C.2).

Graviton For graviton, the solution of the E.O.M in the temporal gauge $h_{\mu 0}=0$ after we apply the boundary condition is

$$\begin{aligned}h^{BD}_{ij,cl}(\vec{x},t)&=\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}h^{BD}_{ij,cl}(\vec{p},t)\\ h^{BD}_{ij,cl}(\vec{p},t)&=[\pi_{ik}\pi_{jl}e^{i(E-i\epsilon)t}+(\eta_{ik}\eta_{jl}-\pi_{ik}\pi_{jl})]h^{kl}_{\mathbf{0}}(\vec{p}).\end{aligned}\tag{2.26}$$

Moreover, we'll find that one of the E.O.M for graviton reads

$$(\eta_{ij}\nabla^2-\partial_i\partial_j)h^{ij}_0(\vec{x})=0.\tag{2.27}$$

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In momentum space, it is

$$\pi_{ij} h_{\mathbf{0}}^{ij}(\vec{p}) = 0, \tag{2.28}$$

from which we see the E.O.M of graviton constrains the boundary profile.

Fermion Given that we only have boundary conditions for $\bar{\chi}_-$ and χ_+, it is natural to decompose the classical solution into $\chi_\pm$ when solving the E.O.M. The goal is to represent χ_- in terms of χ_+, which yields an E.O.M in terms of χ_+. This allows us to use the boundary condition $\chi_{+,\mathbf{0}}$ to derive the classical solution. The E.O.M under decomposition of the fields $\chi^\pm$ satisfy

$$(\pm i\partial_t + m)\,\chi^\pm = \not{\mathbf{p}}\chi^\mp, \tag{2.29}$$

which relates χ^- to χ^+:

$$\chi^- = \left(\frac{\not{\mathbf{p}}}{-E^2+m^2}\right)\left[(i\partial_t + m)\,\chi^+\right]. \tag{2.30}$$

where we use boldsymbol to denote the 3D vector and $\not{\mathbf{p}} := p^i\gamma_i$. Then substituting the equation into $(-i\partial_t + m)\,\chi^- = \not{\mathbf{p}}\chi^+$, we have

$$\left(\partial_t^2 + E^2\right)\chi^+ = 0, \tag{2.31}$$

which is similar to the E.O.M of the scalar field. The solution which that matches the boundary conditions $\chi^+(\vec{p},0) = \chi_{\mathbf{0}}^+$ and $\chi^+(\vec{p},-\infty) < \infty$ is

$$\chi^+(\vec{p},t) = \chi_{\mathbf{0}}^+ e^{i(E-i\epsilon)t}, \tag{2.32}$$

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which is a positive energy mode. Substituting into (2.30), we have

$$\chi_0^- = \left(\frac{\not{p}}{-E^2+m^2}\right)\left[i\left(iE\chi_0^+\right)+m\chi_0^+\right] = (\frac{\not{p}}{E+m})\chi_{0,+}. \tag{2.33}$$

Similarly, for $\bar{\chi}$, we have

$$\bar{\chi}^+ = \left(\frac{-\not{p}}{-E^2+m^2}\right)\left[(i\partial_t+m)\,\bar{\chi}^-\right], \bar{\chi}^-(\mathbf{p},t) = \bar{\chi}_{\mathbf{0}}^- e^{iEt} \tag{2.34}$$

$$\bar{\chi}_0^+ = \left[i\left(iE\bar{\chi}_0^-\right)+m\bar{\chi}_0^-\right]\left(\frac{-\not{p}}{-E^2+m^2}\right) = \bar{\chi}_0^-(\frac{-\not{p}}{E+m}). \tag{2.35}$$

Plugging the classical solution of χ^+ into the action, we can derive the two-point wave function coefficient

$$\begin{aligned} iS_{cl} &= \int (1/2)\,\bar{\chi}_0\chi_0\,d^3x = \int \frac{d^3p}{(2\pi)^3}\,(1/2)\,\bar{\chi}_0(-\vec{p})\chi_0(\vec{p}) \\ &= \int \frac{d^3p}{(2\pi)^3}\,(1/2)\,\bar{\chi}_{0,-}(-\vec{p})(1+\frac{\not{p}}{E+m})(1+\frac{\not{p}}{E+m})\chi_{0,+}(\vec{p}) \\ &= \int \frac{d^3p}{(2\pi)^3}\bar{\chi}_{0,-}(-\vec{p})\frac{\not{p}}{E+m}\chi_{0,+}(\vec{p}). \end{aligned} \tag{2.36}$$

Gravitino For gravitino, similar to the case of the graviton, the boundary condition is constrained from E.O.M:

$$\gamma^i\gamma_0 M_i(t=0,\vec{x}) = \partial_i\psi_{\mathbf{0}}^i(\vec{x}) - (\gamma^j\partial_j)(\gamma^k\psi_{\mathbf{0},k}(\vec{x})) = 0 \tag{2.37}$$

in the momentum space, we have

$$(\gamma_j\psi_{\mathbf{0}}^j(\vec{p})) = -\hat{\not{p}}(\hat{p}_i\psi_{\mathbf{0}}^i(\vec{p})). \tag{2.38}$$

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Under the decomposition $\gamma_0 \psi^i_{\pm,\mathbf{0}} = \pm \psi^i_{\pm,\mathbf{0}}$, we could easily check that

$$(\gamma_j \psi^j_{+,\mathbf{0}}(\vec{p})) = -\not{\boldsymbol{p}}(\hat{p}_i \psi^i_{+,\mathbf{0}}(\vec{p})) = 0. \tag{2.39}$$

So EOM indeed applies a constraint on the boundary condition $\psi_{+,\mathbf{0}}$. Similarly, for the conjugate field, we have boundary condition constraints,

$$(\bar{\psi}^j_{-,\mathbf{0}}(\vec{p})\gamma_j) = -(\hat{p}_i \bar{\psi}^i_{-,\mathbf{0}}(\vec{p}))\not{\boldsymbol{p}}. \tag{2.40}$$

2.4 Wave function coefficients (cosmological correlators)

The correlation function is determined by the Wave Function, so we can calculate the wave function and expand it into

$$\Psi[\phi_0] =: \sum_{n=2} \int \prod_i^n d^3p_i \cdot \psi_n(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n) \cdot (\phi_0(\vec{p}_1)\phi_0(\vec{p}_2)\ldots\phi_0(\vec{p}_n)) \cdot \delta^3\left(\sum_a^n \vec{p}_a\right) \tag{2.41}$$

in which $\psi_n(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n)$ is the n-pt wave function coefficients, or correlator, in momentum space, respectively. To make a distinction we will always refer to the in-in correlator with its full name, while the correlator is a shorthand for wave function coefficient. Sometimes, we use the bracket notation to denote the correlator

$$\langle O(\vec{p}_1)O(\vec{p}_2)\ldots O(\vec{p}_n)\rangle = \langle O_1 O_2 \ldots O_n\rangle := \psi_n(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n) \tag{2.42}$$

We will always refer to the correlators as our observables because the analytical structure of the correlator is more simple than the correlation function. By the relationship [2.6],

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we could calculate the correlation function as [12]

$$
\begin{aligned}
\langle O(-\vec{p})O(\vec{p})\rangle_{\text{in-in}} &= \frac{1}{2Re\langle O(-\vec{p})O(\vec{p})\rangle} \\
\langle O_1O_2O_3\rangle_{\text{in-in}} &= -2\left(\prod_a^3 \frac{1}{2Re\langle O(-\vec{p}_a)O(\vec{p}_a)\rangle}\right)\cdot Re\langle O_1O_2O_3\rangle \\
\langle O_1O_2O_3O_4\rangle_{\text{in-in}} &= -2\left(\prod_a^4 \frac{1}{2Re\langle O(-\vec{p}_a)O(\vec{p}_a)\rangle}\right) \\
&\quad \cdot\left[Re\langle O_1O_2O_3O_4\rangle - \frac{Re\langle O_1O_2O_{-s}\rangle\cdot Re\langle O_3O_4O_s\rangle}{Re\langle O(-\vec{s})O(\vec{s})\rangle} - t - u\right]
\end{aligned}
\tag{2.43}
$$

In this thesis, we pay attention to the flat space correlator, in which, we do gravitational perturbation on the flat Minkowski background.

Feynman Rule of the Correlator For the tree-level correlator, we could plug the classical solution into the action and extract the correlator we defined in the previous section:

$$
\begin{aligned}
&\Psi_{tree}[\phi_0] = e^{iS(\phi_{cl}^{BD}(\phi_0))} \\
&iS(\phi_{cl}^{BD}(\phi_0)) =: \sum_{n=2}\int\prod_i^n d^3p_i\cdot\psi_{n,tree}(\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_n)\cdot(\phi_0(\vec{p}_1)\phi_0(\vec{p}_2)\ldots\phi_0(\vec{p}_n))\cdot\delta^3\left(\sum_a^n\vec{p}_a\right)
\end{aligned}
\tag{2.44}
$$

and the classical action is precisely the action substituted with the solution of the E.O.M.

In general, we could perturbatively solve the E.O.M by what we called the Cosmological Schwinger-Dyson equation (in the rest of the paper, we use ϕ_{cl} to represent ϕ_{cl}^{BD}):

$$
\phi_{cl}(\phi_0,t,\vec{x}) = \int d^3x' K(\vec{x},\vec{x}',t)\phi_0(\vec{x}') + \int d^3x'd^3t' G(\vec{x},\vec{x}',t,t')\left(-\frac{\delta L_{int}}{2\delta\phi(\vec{x}',t')}\right)\Bigg|_{\phi=\phi_{cl}}
\tag{2.45}
$$

We refer to K as the bulk-to-boundary propagator, which is the solution of free EOM with boundary conditions on flat past and current time,

$$
\Box_{\vec{x},t}K(\vec{x},\vec{x}',t) = 0\,;\; K(\vec{x},\vec{x}',t=0) = \delta^3(\vec{x}-\vec{x}')\,;\; K(\vec{x},\vec{x}',t=-\infty) = 0. \tag{2.46}
$$

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In addition, we refer to G as the bulk-to-boundary propagator, which is the solution of the Green equation and boundary condition,

$$\Box_{\vec{x},t}G(\vec{x},\vec{x}',t,t') = \delta^4(x_\mu - x'_\mu)\ ;\ G(\vec{x},\vec{x}',t=0,t'=0) = 0\ ;\ G(\vec{x},\vec{x}',t=-\infty,t') = 0 \tag{2.47}$$

Then we could perturbatively build up the classical solution with two propagators

$$\begin{aligned} \phi_{cl}^{(0)}(\phi_0,t,\vec{x}) &= \int d^3x' K(\vec{x},\vec{x}',t)\phi_0(\vec{x}') \\ \phi_{cl}^{(1)}(\phi_0,t,\vec{x}) &= \int d^3x' d^3t' G(\vec{x},\vec{x}',t,t') \left(-\frac{\delta L_{int}}{2\delta\phi(\vec{x}',t')}\right)\Bigg|_{\phi=\phi_{cl}^{(0)}} \\ &\vdots \end{aligned} \tag{2.48}$$

Because there are spatial translation and rotation invariance (but there's no time translation and boot invariance, for we set $t=0$ as our boundary), the bulk-to-boundary and bulk-to-bulk propagator should only depend on $(\vec{x}-\vec{x}')$, and the Fourier transform of them are

$$\begin{aligned} K(\vec{x},\vec{x}',t) = K(\vec{x}-\vec{x}',t) &= \int \frac{d^3p}{(2\pi)^3} K(\vec{p},t) e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \\ G(\vec{x},\vec{x}',t,t') = G(\vec{x}-\vec{x}',t,t') &= \int \frac{d^3p}{(2\pi)^3} G(\vec{p},t,t') e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}. \end{aligned} \tag{2.49}$$

In the momentum space, (2.46) and (2.47) can be solved

$$\begin{aligned} K(\vec{p},t) &= e^{i(E-i\epsilon)t} \\ G(\vec{p},t,t') &= \frac{i}{2E}\left(e^{i(E-i\epsilon)(t-t')}\theta(t'-t) + e^{-i(E-i\epsilon)(t-t')}\theta(t-t') - e^{i(E-i\epsilon)(t+t')}\right) \end{aligned} \tag{2.50}$$

in which the step function is defined by

$$\theta(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & x < 0 \end{cases}. \tag{2.51}$$

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The correlators in momentum space can be built by the propagators. For the contact diagram, the correlators take the form (for detailed calculations, see C.1)

$$\begin{aligned}\psi_{n,contact} &= \sum_{perm} \int dt\ (ig) V(\vec{p}_1, \vec{p}_2 \ldots, \vec{p}_n, \partial_t) K_1(\vec{p}_1, t) K_2(\vec{p}_2, t) \ldots K_n(\vec{p}_n, t) \\ &= \sum_{perm} \frac{gV(\vec{p}_1, \vec{p}_2 \ldots, \vec{p}_n, E_{1\sim n})}{K_T} + O(K_T^0)\end{aligned} \tag{2.52}$$

in which $K_T := \sum_{a=1}^{n} E_a$ is the total energy, V is the contact vertices, g is the coupling constant and the permutation sum sums over all the contribution of the label permutation for the same field. There will be an extra $n!$ factor if there are n identical external legs. There is a total energy pole for contact diagrams, which will be further discussed in the next section. We could represent the correlator with the Feynman diagram in Fig. 2.2.

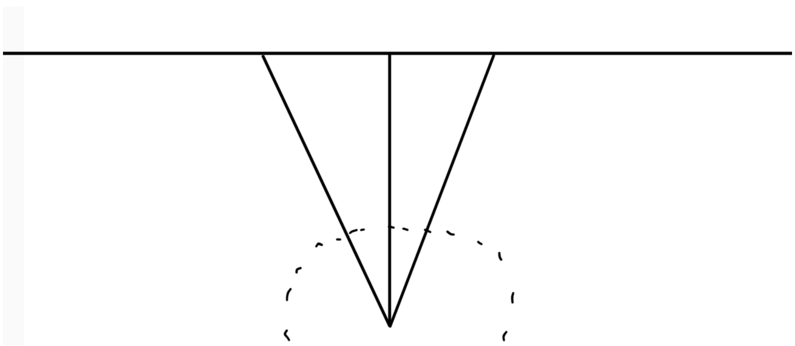

Figure 2.2: The Contact Feynman diagram of the cosmological correlator and the pattern of the singularities. The horizontal line on the top represents the fixed time boundary. The lines stretched below and intersecting to a point represent the bulk-to-boundary propagator. The intersection point represents the vertices.

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For the exchanging diagram, the correlator takes the form

$$
\begin{aligned}
&\psi_{4,exchange} = \\
&\sum_{perm} \int dt \int dt' \, (ig^2) K_1(\vec{p}_1, t) K_2(\vec{p}_2, t) V_L(\vec{p}_1, \vec{p}_2, \vec{p}_s, \partial_t) \\
&\otimes G(\vec{p}_s, E_s, t, t') \otimes V_R(\vec{p}_3, \vec{p}_4, -\vec{p}_s, \partial'_t) K_3(\vec{p}_3, t') K_4(\vec{p}_4, t') \\
&= g^2 \sum_{perm}^{(s)} \frac{V_L(\vec{p}_1, \vec{p}_2, \vec{p}_s, E_{1\sim 4}, E_s) \otimes V_R(\vec{p}_3, \vec{p}_4, -\vec{p}_s, E_{1\sim 4}, E_s)}{K_T E_L E_R} + O(K_T^0 E_L E_R) + (t) + (u)
\end{aligned} \tag{2.53}
$$

in which we define $\otimes$ to be some indice contractions and the partial energy as the energy sum of the external legs in the left (right) sub-diagram $E_L = E_{12s}$ ($E_R = E_{34s}$) where we use the subscript to label the sum of the energy, $E_{abc...} := E_a + E_b + E_c + \dots$. By the structure of the bulk-to-bulk propagator, we see that besides the total energy pole, there are two additional partial energy poles, which will be discussed further in the next chapter. We can draw a Feynman diagram Fig. 2.3 to represent the exchanging correlator.

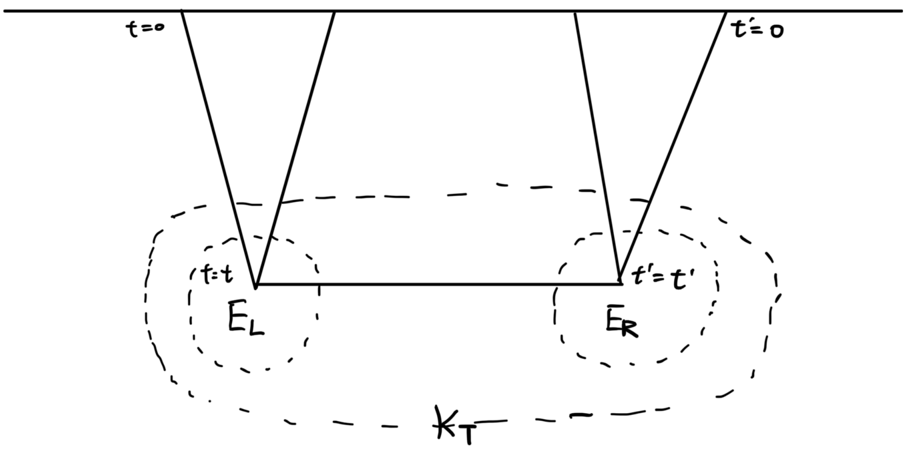

Figure 2.3: The Exchanging Feynman diagram of the cosmological correlator and the pattern of the singularities. The line between vertices represents the bulk-to-bulk propagator.

Notice the structure of the correlators (2.52) and (2.53) are derived from the bosonic fields. However, we can show that the correlators of the fermionic fields are similar, see

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(C.146) and (C.158).

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Chapter 3 On-shell approach to correlators

The idea of the bootstrap program is to fix the correlator by physical constraints. The necessary constraints include the Ward-Takahashi identity (WT identity), the total energy pole constraint, and the partial energy pole constraint. The WT identity reflects the gauge freedom of the boundary fields, the total energy pole restores the flat spacetime amplitude, and the partial energy pole is a direct consequence of the cosmological optical theorem (COT). COT can be derived from either the unitary nature of the theory or from the properties of the correlators, such as discontinuity. In this chapter, we introduce the above concepts and introduce our bootstrap procedure.

3.1 Ward-Takahashi identities (WT identity)

If there are massless spinning fields, the boundary fields will contain extra gauge freedom. For example, scalar QED contains a massless vector field A and a scalar ϕ,

$$\begin{aligned}\Psi[A_{i,0}(\vec{x}), \phi_0(\vec{x})] &= \Psi[A_{i,0}(\vec{x}) + \delta A_{i,0}(\vec{x}), \phi_0(\vec{x}) + \delta\phi_0(\vec{x})] \\ \delta A_{i,0}(\vec{x}) &= \partial_i \alpha(\vec{x}) \\ \delta\phi_0(\vec{x}) &= ie\alpha(\vec{x})\phi_0(\vec{x})\end{aligned} \tag{3.1}$$

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The wave function should be invariant under the gauge transform. Expanding the above equation perturbatively, we can derive the Ward Takahashi Identity of the scalar QED for 3pts and 4pts correlator

$$
\begin{aligned}
p_1^i \langle J_{1,i} O_2^* O_3 \rangle &= -e \, \langle O_{1+2}^* \, O_3 \rangle \, + e \, \langle O_2^* \, O_{1+3} \rangle \, = \, e \, (E_2 - E_3) \\
p_1^i \langle J_{1,i} O_2^* J_{3,j} O_4 \rangle &= -e \, \langle O_{1+2}^* \, J_{3,j} O_4 \rangle \, + e \, \langle O_2^* \, J_{3,j} O_{1+4} \rangle
\end{aligned}
\tag{3.2}
$$

where we abuse our notation and define $O_{a+b} := O(\vec{p}_a + \vec{p}_b)$. The detailed derivation and WT identity of other spinning correlators will be left in the App. E.2. In this paper, we refer $\langle J_{1,i} O_2^* O_3 \rangle$ as the correlators, which has free indices; and we refer $\langle J_1 O_2^* O_3 \rangle := \epsilon_{1,\mathbf{0}}^i \langle J_{1,i} O_2^* O_3 \rangle$ as the contracted correlators in which the boundary polarization $\epsilon_{1,\mathbf{0}}^i$ is the boundary condition of the vector field.

In general, the WT identity will determine the longitudinal parts of the correlator. In the case of $\langle J O^* O \rangle$, we could decompose the boundary polarization with transverse component $\epsilon_{\mathbf{0}}^{i,\mathrm{T}}$ perpendicular to the momentum and the longitudinal component $\epsilon_{\mathbf{0}}^{i,L}$ parallel to the momentum.

$$
\begin{aligned}
\epsilon_0^i(\vec{p}) &= \epsilon_0^{i,T} + \epsilon_0^{i,L} \\
&= \pi^{ij} \epsilon_{j,0}(\vec{p}) - \hat{p}^i \hat{p}^j \epsilon_{j,0}(\vec{p})
\end{aligned}
\tag{3.3}
$$

where we defined $\pi_{ij} \equiv \eta_{ij} + \hat{p}_i \hat{p}_j, \hat{p}_i = p_i/E$. Then we could decompose the contracted correlator accordingly

$$
\begin{aligned}
\langle J_1 O_2^* O_3 \rangle &= \langle J_1^{\mathrm{T}} O_2^* O_3 \rangle + \langle J_1^L O_2^* O_3 \rangle \\
\langle J_1^{\mathrm{T}} O_2^* O_3 \rangle &= \epsilon_{1.i} \pi_{1,j}^i \langle J_1^j O_2^* O_3 \rangle \\
\langle J_1^L O_2^* O_3 \rangle &= -\epsilon_{1,i} \hat{p}_1^i \hat{p}_{1,j} \langle J_1^j O_2^* O_3 \rangle = -\epsilon_{1,i} \frac{\hat{p}_1^i}{E_1} (E_2 - E_3)
\end{aligned}
\tag{3.4}
$$

in which we call the $\langle J_1^{\mathrm{T}} O_2^* O_3 \rangle$ transverse mode of the contracted correlator and $\langle J_1^L O_2^* O_3 \rangle$

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longitudinal mode of the contracted correlator. The latter can be determined by Ward-Takahashi identities.

3.2 Singularities and the cosmological optical theorem for flat space

We define total energy K_T as the sum of the energy of all the external legs. On the total energy pole, which is a single pole[1], the energy conservation is restored, and the residue of the K_T will be the amplitude:

$$\lim_{K_T \to 0} \psi_n = \frac{M_n}{K_T} \tag{3.5}$$

It reflects the fact that in the far past, physics should not be influenced by the boundary at $t = 0$. Therefore, the correlator's path integral formula should get back to amplitude. The detailed derivation is in App. F.3.

From the Feynman rules in sec.2.4 we can derive the cosmological optical theorem(COT) for the correlator given as

$$\begin{aligned}\psi_4^{(\phi/J/T)}(E_{1\sim4}, \boldsymbol{p}_{1\sim4}) + \psi^{*(\phi/J/T)}_4(-E_{1\sim4}, \boldsymbol{p}_{1\sim4}) \\ = \tilde{\psi}_{3,i_1\ldots}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \cdot P^{i_1 j_1 \ldots}_{2,\phi/J/T}(\vec{p}_s) \cdot \tilde{\psi}_{3,j_1\ldots}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4) \\ + \{\text{t-channel}\} + \{\text{u-channel}\},\end{aligned} \tag{3.6}$$

where a non-zero right-hand side corresponds to the contribution from the exchange dia-

[1]For comparison, dS/EAds correlator will have higher order total energy pole.[4]

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gram with

$$P_{2,\phi}(\vec{p}_s) = \psi_{2,\phi}^{in-in}(\vec{p}_s) = \frac{1}{2Re\psi_2(\vec{p}_s)} = \frac{1}{2E_s}$$
$$P_{2,J}^{i_1j_1}(\vec{p}_s) = \frac{\pi_s^{i_1j_1}}{2E_s} \tag{3.7}$$
$$P_{2,T}^{i_1i_2j_1j_2}(\vec{p}_s) = \frac{\Pi_{s,(2,2)}^{i_1i_2j_1j_2}}{2E_s}; \Pi_{s,(2,2)}^{i_1i_2j_1j_2} = \pi_s^{i_1j_1}\pi_s^{i_2j_2} - \frac{1}{2}\pi_s^{i_1i_2}\pi_s^{j_1j_2}$$

and the shifted correlator is defined by,

$$\tilde{\psi}_{3,i_1\ldots}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s) := \psi_{3,i_1\ldots}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s) - \psi_{3,i_1\ldots}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, -E_s).$$

in which $\psi_{3,i_1\ldots}$ is the correlator obtained by pulling out all the internal boundary polarizations from the contracted 3-point correlator. The boundary polarization is constrained for the graviton. We show all of these constraints at (2.28). If we only have contact diagrams then the right-hand side is zero. Note that from the Feynman rules (2.53), we have

$$\begin{aligned}
&\psi_{4,s}^{*,(\phi/J/T)}(-E_{1\sim4}, E_s, \boldsymbol{p}_{1\sim4}) \\
&= \sum_{perm} \int dt \int dt' \; (ig^2)^* K_1^*(-E_1, \vec{p}_1, t) K_2^*(-E_2, \vec{p}_2, t) V_L^*(\vec{p}_1, \vec{p}_2, \vec{p}_s, \partial_t) \\
&\otimes G_{\phi/A/h}^*(\vec{p}_s, E_s, t, t') \otimes V_R^*(\vec{p}_3, \vec{p}_4, -\vec{p}_s, \partial_t') K_3^*(-E_3, \vec{p}_3, t') K_4^*(-E_4, \vec{p}_4, t') \\
&= \sum_{perm} \int dt \int dt' \; (-ig^2) K_1(E_1, \vec{p}_1, t) K_2(E_2, \vec{p}_2, t) V_L(\vec{p}_1, \vec{p}_2, \vec{p}_s, \partial_t) \\
&\otimes G_{\phi/A/h}(\vec{p}_s, -E_s, t, t') \otimes V_R(\vec{p}_3, \vec{p}_4, -\vec{p}_s, \partial_t') K_3(\vec{E}_3, p_3, t') K_4(E_4, \vec{p}_4, t') \\
&= -\psi_{4,s}^{(\phi/J/T)}(E_{1\sim4}, -E_s, \boldsymbol{p}_{1\sim4})
\end{aligned} \tag{3.8}$$

in which we the fact that $V_{R/L} \in \mathbb{R}$, $K_a^*(-E_a, \vec{p}_1, t) = K_a(E_a, \vec{p}_1, t)$ and $G_{\phi/A/h}^*(\vec{p}_s, E_s, t, t') = G_{\phi/A/h}(\vec{p}_s, -E_s, t, t')$. In this form, we can rewrite the COT in an equivalent form where

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for each channel

$$\psi_{4,s}^{(\phi/J/T/\chi/\psi)}(E_{1\sim4}, E_s, \boldsymbol{p}_{1\sim4}) - \psi_{4,s}^{(\phi/J/T/\chi/\psi)}(E_{1\sim4}, -E_s, \boldsymbol{p}_{1\sim4})$$
$$= \tilde{\psi}_{3,i_1\ldots}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \cdot P^{i_1 j_1 \ldots}_{2,\phi/J/T/\chi/\psi}(\vec{p}_s) \cdot \tilde{\psi}_{3,j_1\ldots}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4), \tag{3.9}$$

with the gluing factor of the fermion and gravitino,

$$P_{2,\chi}(\vec{p}_s) = \left(\frac{1+\gamma_0}{2}\right)\frac{\not{\boldsymbol{p}}_s}{2}\left(\frac{1-\gamma_0}{2}\right)$$
$$P^{ij}_{2,\psi}(\vec{p}_s) = \frac{1}{2}\Pi^{ij}_{s,(3/2,3/2)} = \frac{1}{2}\left(\frac{1+\gamma_0}{2}\right)(-\pi^{ij}_s \not{\boldsymbol{p}}_s - \frac{1}{2}\not{\boldsymbol{\pi}}^i \not{\boldsymbol{p}}_s \not{\boldsymbol{\pi}}^j)\left(\frac{1-\gamma_0}{2}\right); \not{\boldsymbol{\pi}}^i = \pi_{ij}\gamma^j = \gamma_i + \hat{p}_i \not{\boldsymbol{p}}\,. \tag{3.10}$$

We should note that the boundary polarization we pulling out for gravitino is constrained by (2.39). For contact diagrams, we can naturally incorporate fermions where the complex conjugate is defined as $C\psi_{n,c} = \psi^{\dagger}_{n,c}|_{\bar{\chi}_{-,\mathbf{0}} \leftrightarrow \chi_{+,\mathbf{0}}}$. Then we simply have

$$\psi_{n,c}(E_{1\sim n}, \vec{p}_{1\sim n}) + C\psi_{n,c}(-E_{1\sim n}, \vec{p}_{1\sim n}) = 0\,. \tag{3.11}$$

We could derive the COT from these two frameworks individually:

1. **Propagator Property** [20] The correlator can be written in terms of the bulk-to-boundary propagator and the bulk-to-bulk propagator. Take the scalar field as an example, the propagators have the following properties

$$K_\phi(\vec{p}, E - i\epsilon, t) = K^*_\phi(\vec{p}, -E - i\epsilon, t) \tag{3.12}$$

and

$$\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} G_\phi(\sqrt{z_s}, t, t') = \frac{1}{2E_s} \cdot \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} K_\phi(\sqrt{z_s}, t) \cdot \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} K_\phi(\sqrt{z_s}, t'). \tag{3.13}$$

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The two properties lead to the cosmological optical theorem in the form of (3.6).

2. **Unitarity** [12]

If we expand the time evolution operator $\hat{U}$ in terms of the correlators, the unitary constraints $\hat{U}\hat{U}^{\dagger} = 1$ will be translated into a constraint on the correlators, which give rise to the cosmological optical theorem in the form of (3.9). The unitary constraint also helps us to relate the contact COT and the exchanging COT.

3.2.1 Bosonic Field

Unitarity There's another equivalent way to get the correlator from the time evolution operator based on similar field and operator construction in [27] Ch.7.2. With this method, we can interpret the unitarity constraint of the time evolution operator to be the COT of the correlator. To demonstrate our analysis of the time evolution operator in the interacting picture, supposed that the interaction Hamiltonian operator is $\hat{H}_{int} = (\hat{H}_A + \hat{H}_B)$ which has two pieces with individual coupling constant $g_{A/B}$.

$$\begin{aligned} \hat{U}_I &= T\exp\left(-i\int_{-\infty}^{0} dt(\hat{H}_A + \hat{H}_B)\right) \\ &= 1 + (g_A\hat{U}_{g_A} + g_B\hat{U}_{g_B}) + g_A g_B \hat{U}_{g_A g_B} + g_A^2\hat{U}_{g_A g_A} + g_B^2\hat{U}_{g_B g_B} + O(g_{A/B}^3). \end{aligned} \tag{3.14}$$

in which T means the time-ordered products and the expansion of the time-evolution operator will be

$$\begin{aligned} U_{g_{A/B}} &= -i\int_{-\infty}^{\eta_0} d\eta \hat{H}^{A/B}(\eta) \\ \hat{U}_{g_{A/B}g_{A/B}} &= -\int_{-\infty}^{\eta_0} d\eta \int_{-\infty}^{\eta_0} d\eta' \hat{H}^{A/B}(\eta)\hat{H}^{A/B}(\eta')\,\theta(\eta-\eta') + \hat{H}^{A/B}(\eta')\hat{H}^{A/B}(\eta)\,\theta(\eta'-\eta) \\ \hat{U}_{g_A g_B} &= -\int_{-\infty}^{\eta_0} d\eta \int_{-\infty}^{\eta_0} d\eta' \hat{H}^{A}(\eta)\hat{H}^{B}(\eta')\,\theta(\eta-\eta') + \hat{H}^{B}(\eta')\hat{H}^{A}(\eta)\,\theta(\eta'-\eta)\,. \end{aligned} \tag{3.15}$$

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We define the operator known as the Hamiltonian, using the familiar form in the context of quantum field theory,

$$\hat{H}_{A/B}(t) = \int d^3x V_A(\partial_{\vec{x}}, \partial_t)\hat{\phi}^3_{cl}(\vec{x}, t) = \prod_a^3 \int \frac{d^3p_a}{(2\pi)^3}\delta(\sum \vec{p}_a)V_{A/B}(\vec{p}_{1\sim 3}, \partial_t)\hat{\phi}_{cl}(\vec{p}_1, t)\hat{\phi}_{cl}(\vec{p}_2, t)\hat{\phi}_{cl}(\vec{p}_3, t). \tag{3.16}$$

We restrict the vertices $V_{A/B}$ to be real. Then the unitarity constraints $\hat{U}_I\hat{U}_I^\dagger = 1$ gives

$$\begin{aligned} \hat{U}_{gA} + \hat{U}_{gA}^\dagger &= 0 \\ \hat{U}_{gB} + \hat{U}_{gB}^\dagger &= 0 \\ \hat{U}_{g_Ag_B} + \hat{U}_{g_Ag_B}^\dagger &= -(\hat{U}_{g_A}\hat{U}_{g_B}^\dagger + \hat{U}_{g_A}^\dagger\hat{U}_{g_B}) \\ &\vdots \end{aligned} \tag{3.17}$$

Unitarity: Contact diagrams The creation operators and the annihilation operators satisfy the commutation relationship

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{q}), \tag{3.18}$$

where the annihilation operators kill the vacuum state in the far past

$$a_{\vec{p}}\,|0\rangle = 0. \tag{3.19}$$

Consider the field $\hat{\phi}_{cl}$ in the interaction picture, which satisfy the E.O.M:

$$\begin{aligned} [\hat{H}_0, \hat{\phi}_{cl}(\vec{x}, t)] &= -i\partial_t\hat{\phi}(\vec{x}, t) \\ \hat{\phi}_{cl}(\vec{x}, t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}e^{i\vec{x}\cdot\vec{p}+iEt}a_{\vec{p}}^\dagger + \frac{1}{\sqrt{2E}}e^{-i\vec{x}\cdot\vec{p}-iEt}a_{\vec{p}} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}K_\phi(E, \vec{p})a_{\vec{p}}^\dagger + \frac{1}{\sqrt{2E}}K_\phi(-E, -\vec{p})a_{\vec{p}} \end{aligned} \tag{3.20}$$

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in which $\hat{H}_0 = \int \frac{d^3p}{(2\pi)^3} E_p a^\dagger_{\vec{p}} a_{\vec{p}}$, where $E_p > 0$. In the last line, we identify the exponential term as the bulk-to-boundary propagator. To get the correlator from the time-evolution operator in the momentum space, we define the momentum basis

$$\begin{aligned} |\vec{p}\rangle &= a^\dagger_{\vec{p}} |0\rangle \\ I &= \int \frac{d^3p}{(2\pi)^3} |\vec{p}\rangle \langle \vec{p}| \end{aligned} \tag{3.21}$$

and the Fock space momentum basis

$$|\vec{p}_1, \vec{p}_2, \ldots\rangle = a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} \ldots |0\rangle \tag{3.22}$$

Then we could easily show that we could extract the correlator from the first order of time evolution operator U_{gA} if we apply (3.20) and (3.22) and do the Wick contraction with momentum basis and the zero-state $|0\rangle$, [2]

$$\langle p_1, p_2, \ldots | \hat{U}_{gA} |0\rangle = -i \langle p_1, p_2, \ldots | \int_{-\infty}^{0} \hat{H}_A(t) dt |0\rangle = -\psi_{n,contact}(E_i) \prod_a^n \frac{1}{\sqrt{2E_a}} \tag{3.23}$$

And for the conjugate of the operator $\hat{U}^\dagger_{gA}$, by the same approach

$$\langle p_1, p_2, \ldots | \hat{U}^\dagger_{gA} |0\rangle = -i \langle p_1, p_2, \ldots | \int_{-\infty}^{0} \hat{H}^\dagger_A(t) dt |0\rangle = -\psi^*_{n,contact}(-E_i - i\epsilon) \cdot \prod_a^n \frac{1}{\sqrt{2E_a}}, \tag{3.24}$$

we could extract the conjugate of the correlator with flipping energy signs of the external energy. We should remark that no boundary condition comes in. Then by the first unitarity constraints on (3.17), we have

$$\langle \phi_1, \phi_2, \ldots | \hat{U}_{gA} |0\rangle + \langle \phi_1, \phi_2, \ldots | \hat{U}^+_{gA} |0\rangle = 0. \tag{3.25}$$

[2] Because $|0\rangle_I = e^{iH_0 \infty} |0\rangle = |0\rangle$ for $H_0 |0\rangle = 0$, so the zero-state in interaction picture will be just normal $|0\rangle$.

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If we take (3.23) and (3.24) into this equation, we'll get the COT as (3.11).

Unitarity: Exchanging Scalar For the scalar exchange diagram, we could derive the COT by the third equation in (3.17):

$$\langle p_1p_2p_3p_4|\hat{U}_{g_Ag_B}+\hat{U}^{\dagger}_{g_Ag_B}|0\rangle = -\langle p_1p_2p_3p_4|(\hat{U}_{g_A}\hat{U}^{\dagger}_{g_B}+\hat{U}^{\dagger}_{g_A}\hat{U}_{g_B})|0\rangle \tag{3.26}$$

in which we extract the scalar correlator for the exchange diagram from the contraction of the zero-state and the momentum basis, for the first term of LHS, if we apply the wick contraction, (3.20) and (3.22)

$$\begin{aligned}
&\langle p_1p_2p_3p_4|\hat{U}_{g_Ag_B}|0\rangle \\
&\quad=\sum_{perm}(\prod_i^4\frac{1}{\sqrt{2E_i}}) \\
&\quad\quad\cdot-\int_{-\infty}^0 dt\int_{-\infty}^0 dt' K_\phi(E_1,t)K_\phi(E_2,t)V_A(t,\partial_t,\vec{p}_1,\vec{p}_2,\vec{p}_s)\cdot T\langle 0|\,\hat{\phi}_{cl}(\vec{p}_s,t)\hat{\phi}_{cl}(-\vec{p}_s,t')\,|0\rangle \\
&\quad\quad\cdot V_B(t',\partial_t,\vec{p}_3,\vec{p}_4,-\vec{p}_s)K_\phi(E_3,t')K_\phi(E_4,t') \\
&\quad=\sum_{perm}(\prod_i^4\frac{1}{\sqrt{2E_i}}) \\
&\quad\quad\cdot-i\int_{-\infty}^0 dt\int_{-\infty}^0 dt' K_\phi(E_1,t)K_\phi(E_2,t)V_A(t,\partial_t,\vec{p}_1,\vec{p}_2,\vec{p}_s)\cdot G_\phi^{(Fey)}(t,t',E_s,\vec{p}_s) \\
&\quad\quad\cdot V_B(t',\partial_t,\vec{p}_3,\vec{p}_4,-\vec{p}_s)K_\phi(E_3,t')K_\phi(E_4,t') \\
&\quad=-(\prod_i^4\frac{1}{\sqrt{2E_i}})\left(\psi_4(E_1,E_2,E_3,E_4,E_s)-\frac{1}{2E_s}(\psi_{3,A}(E_1,E_2,E_s)\psi_{3,B}(E_3,E_4,E_s))\right) \\
&\quad\quad+(t)+(u),
\end{aligned} \tag{3.27}$$

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where $G_\phi^{(Fey)} = T\langle 0|\hat{\phi}_{cl}(t)\hat{\phi}_{cl}(t')|0\rangle$ is the normal Feynman propagator, which has the relation

$$G_\phi(t,t',E_s,\vec{p}_s) = G_\phi^{(Fey)}(t,t',E_s,\vec{p}_s) - \frac{i}{2E_s}K_\phi(E_s,t)K_\phi(E_s,t'). \tag{3.28}$$

Similarly (note that the vertices $V_{A/B}$ are real),

$$\begin{aligned}&\langle p_1p_2p_3p_4|\hat{U}^\dagger_{g_Ag_B}|0\rangle\\ &= -(\prod_i^4 \frac{1}{\sqrt{E_i}})\Big(\psi_4^*(-E_1,-E_2,-E_3,-E_4,E_s) - \frac{1}{2E_s}(\psi_{3,A}(-E_1,-E_2,E_s)\psi_{3,B}(-E_3,-E_4,E_s))\Big)\\ &+(t)+(u).\end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29), we have

$$\begin{aligned}\langle p_1p_2p_3p_4|\hat{U}_{g_Ag_B} + \hat{U}^\dagger_{g_Ag_B}|0\rangle&\\ = -(\prod_i^4 \frac{1}{\sqrt{E_i}})\{&\psi_4(E_1,E_2,E_3,E_4,E_s) + \psi_4^\dagger(-E_1,-E_2,-E_3,-E_4,E_s)\\ &- \langle O_{-s}O_s\rangle_{in-in}\begin{pmatrix}\psi_{3,A}(E_1,E_2,E_s)\psi_{3,B}(E_3,E_4,E_s)\\ +\psi_{3,A}(E_1,E_2,-E_s)\psi_{3,B}(E_3,E_4,-E_s)\end{pmatrix} + (t)+(u)\},\end{aligned} \tag{3.30}$$

which is the LHS of (3.26). The two-point correlation function $\frac{1}{2E_s} = \langle O_{-s}O_s\rangle_{in-in}$ can be identified as the propagator for exchanging vertices. The 3-point correlator products come from the difference between $G_\phi^{(Fey)}$ and G_ϕ.

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On the other hand, the first term of the RHS of (3.26) is

$$
\begin{aligned}
&-\langle p_1p_2p_3p_4|U_{g_A}U_{g_B}^\dagger|0\rangle \\
&\quad=\sum_{perm}(\prod_i^4\frac{1}{\sqrt{2E_i}}) \\
&\quad\cdot-\int_{-\infty}^0 dt\int_{-\infty}^0 dt' K_\phi(E_1,t)K_\phi(E_2,t)V_A(t,\partial_t,\vec{p}_1,\vec{p}_2,\vec{p}_s)\cdot\langle 0|\,\hat{\phi}_{cl}(\vec{p}_s,t)\hat{\phi}^*_{cl}(-\vec{p}_s,t')\,|0\rangle \\
&\quad\cdot V_B^*(t',\partial_t,\vec{p}_3,\vec{p}_4,-\vec{p}_s)K_\phi(-E_3,t')K_\phi(-E_4,t') \\
&\quad=(\prod_i^4\frac{1}{\sqrt{2E_i}})\cdot\langle O_{-s}O_s\rangle_{in-in}\psi_{3,A}(E_1,E_2,E_s)\psi_{3,B}(-E_3,-E_4,E_s)+(t)+(u) \\
&\quad=-(\prod_i^4\frac{1}{\sqrt{2E_i}})\cdot\langle O_{-s}O_s\rangle_{in-in}\psi_{3,A}(E_1,E_2,E_s)\psi_{3,B}(E_3,E_4,-E_s)+(t)+(u).
\end{aligned}
\tag{3.31}
$$

In the second equality we identified $\langle 0|\,\hat{\phi}_{cl}(\vec{p}_s,t)\hat{\phi}^*_{cl}(-\vec{p}_s,t')\,|0\rangle=\frac{1}{2E_s}K_\phi(E_s,t)K_\phi(E_s,t')$. We also identified $\frac{1}{2E_s}$ as the two point in-in formalsim $\langle O_{-s}O_s\rangle_{in-in}$. [3] In the last equality, we used the fact that the correlator is real and the 3-pt COT. The conjugate term can be derived similarly, and we could combine them to get

$$
\begin{aligned}
&\langle p_1p_2p_3p_4|(U_{g_A}U_{g_B}^\dagger+U_{g_A}^\dagger U_{g_B})|0\rangle \\
&\quad=-(\prod_i^4\frac{1}{\sqrt{E_i}}\int\frac{d^3p_i}{(2\pi)^3}) \\
&\quad\cdot-\langle O_{-s}O_s\rangle_{in-in} \\
&\quad\cdot[\psi_{3,A}(E_1,E_2,E_s)\psi_{3,B}(E_3,E_4,-E_s)+\psi_{3,A}(E_1,E_2,-E_s)\psi_{3,B}(E_3,E_4,E_s)]+(t)+(u).
\end{aligned}
\tag{3.32}
$$

Equating (3.30) and (3.32), we can extract the COT channel by channel[4], then the

[3]If we take $t,t'=0$ on $\langle 0|\,\hat{\phi}_{cl}(\vec{p}_s,t)\hat{\phi}^*_{cl}(-\vec{p}_s,t')\,|0\rangle$, then we could identify $\frac{1}{2E_s}=\langle 0|\,\hat{\phi}_0\hat{\phi}_0^*\,|0\rangle$. And by the fact that $U_I(0,-\infty)\,|0\rangle=|\Omega_0\rangle=|0\rangle+O(g_{A/B})$, we could write $\langle 0|\,\hat{\phi}_0\hat{\phi}_0^*\,|0\rangle=\langle\Omega_0|\,\hat{\phi}_0\hat{\phi}_0^*\,|\Omega_0\rangle+O(g_{A/B})$. Then for the tree-level contribution we drop the higher order contribution on the coupling constants, the term $\langle 0|\,\hat{\phi}_0\hat{\phi}_0^*\,|0\rangle$ is exactly two-point in-in correlator by definition. We don't really need the exact expression of the in-in correlator to identify it in the extraction of the correlator from the time-evolution operator.

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COT reads

$$\psi_4^{(\phi)}(E_{1\sim4}, E_s, \boldsymbol{p}_{1\sim4}) + \psi^{*(\phi)}_4(-E_{1\sim4}, E_s, \boldsymbol{p}_{1\sim4}) \quad = \tilde{\psi}_3(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \cdot \langle O_{-\boldsymbol{s}} O_{\boldsymbol{s}} \rangle_{in-in} \cdot \tilde{\psi}_3(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4) \tag{3.33}$$

In (3.26), $U_{gAgB}+U^\dagger_{gAgB}$ contains $G_\phi^{(Fey)}$, but the four-point correlator is written in G_ϕ. The shifted correlator is needed to absorb the differences between $G_\phi^{(Fey)}$ and G_ϕ. Moreover, for the in-in correlator

$$\begin{aligned} &\psi_{4,in-in}(E_{1\sim4}, E_s, E_t, E_u, \boldsymbol{p}_{1\sim4}) - \psi_{4,in-in}(E_{1\sim4}, -E_s, E_t, E_u, \boldsymbol{p}_{1\sim4}) \\ &= \frac{\tilde{\psi}_3^{in-in}\tilde{\psi}_3^{in-in}}{2\psi_{2,s}^{in-in}}. \end{aligned} \tag{3.34}$$

It's the equation suggested by [12], it's the COT written in the in-in correlator. The shifted-in-in correlator is also needed. We cannot write COT without shifted observable because the propagator is still not $G_\phi^{(Fey)}$ even in the case of in-in correlators.

Unitarity: Exchanging Vector/Tensor We can promote the COT of scalar exchanging diagram (3.33) to the general spin. The 4pt correlator (plus its complex conjugate with energy flipping) can be obtained by gluing two 3pt shifted correlators. The free indices of the 3pt shifted correlators will contract with a 2-point in-in correlator. [3]

For example, for the correlator exchanging vector field $\psi_4^{(J)}$, substituting the COT with the 2pt in-in correlator of the massless vector field,

$$\boxed{\begin{aligned} &\psi_4^{(J)}(E_{1\sim4}, E_s, \boldsymbol{p}_{1\sim4}) + \psi_4^{*,(J)}(-E_{1\sim4}, E_s, \boldsymbol{p}_{1\sim4}) \\ &= \tilde{\psi}_{3,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s, E_s) \ \ \langle J_{-\boldsymbol{s}} J_{\boldsymbol{s}} \rangle^{ij}_{in-in} \ \tilde{\psi}_{3,j}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s) \end{aligned}} \tag{3.35}$$

[4] Division of correlator channel by channel will be ambiguous upto a contact term, but we know contact term under the Optical theorem should be 0, so every channel by channel division of correlator will give the same COT.

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in which $\langle J_{-\boldsymbol{s}} J_{\boldsymbol{s}} \rangle^{ij}_{in-in} = \frac{\left(\pi^{ij}_s + \xi\ \hat{p}^i_s \hat{p}^j_s\right)}{2E_s}$. Because the dependence on ξ can be eliminated by WT identity,

$$\begin{aligned} \hat{p}^i_s \psi_{3,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s, E_s) &= e\psi_2(\boldsymbol{p}_2) - e\psi_2(\boldsymbol{p}_1) \\ k^i_s \tilde{\psi}_{3,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s, E_s) &= k^i_s \psi_{3,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s, E_s) - k^i_s \psi_{3,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s, -E_s) = 0, \end{aligned} \tag{3.36}$$

we'll get the COT with the gluing factor $P^{i_1 j_1}_{2,J}(\vec{p}_s) = \frac{\pi^{i_1 j_1}_s}{2E_s}$ in (3.6).

For the correlator exchanging tensor field $\psi^{(T)}_4$, substituting the COT with the 2pt in-in correlator of the graviton,

$$\boxed{\psi^{(T)}_4(E_{1\sim 4}, E_s, \boldsymbol{p}_{1\sim 4}) + \psi^{*,(T)}_4(-E_{1\sim 4}, E_s, \boldsymbol{p}_{1\sim 4}) = \tilde{\psi}_{3,ij}\ \ \langle T_{-\boldsymbol{s}} T_{\boldsymbol{s}} \rangle^{ijkl}_{in-in} \tilde{\psi}_{3,kl}}. \tag{3.37}$$

in which $\langle T_{-\boldsymbol{s}} T_{\boldsymbol{s}} \rangle^{ijkl}_{in-in} = P^{ij}_{s,i'j'} \frac{\left(\pi^{i'k'}_s \pi^{j'l'}_s + \xi\ \hat{p}^{i'}_s \hat{p}^{k'}_s \eta^{j'l'} + \dots\right)}{2E_s} P^{kl}_{s,k'l'}$. The extra projector P^{ijkl}_s we add to the in-in correlator is defined by

$$P^{ijab}_s = \Pi^{ijkl}_{(2,2)} + \pi^{ij}_s \hat{p}^k_s \hat{p}^l_s - \pi^{ik}_s \hat{p}^j_s \hat{p}^l_s + \hat{p}^i_s \hat{p}^j_s \hat{p}^k_s \hat{p}^l_s \tag{3.38}$$

such that $P^{ijab}_s P^{kl}_{s,ab} = P^{ijkl}_s$ and, for the general tesnor $h'_{kl,s,\boldsymbol{0}}$, the $h_{ij,s,\boldsymbol{0}} = P^{ijkl}_s h'_{kl,s,\boldsymbol{0}}$ satisfy the constriant (2.28) . The projector in the 2-point in-in correlator reflects that it comes from the correlator with the constrained polarization.

Because the dependence on ξ can be eliminated by WT identity, we'll get the COT with the gluing factor $P^{ijkl}_{2,T}(\vec{p}_s) = \frac{\Pi^{ikjl}_{s,(2,2)}}{2E_s}$ in (3.6). As a remark, we can note that $\Pi^{ikjl}_{s,(2,2)}$ captures all the physical boundary conditions, which include traceless and transverse.

Bosonic Field: Propagator Property We now derive COT in another approach, which relies on the properties of the propagators, such as discontinuity and conjugation relation-

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ships [12, 20].

Propagator Property: Contact Diagram The flat space scalar bulk-to-boundary propagator, $K_\phi(\vec{p}, E, t) = e^{iEt}$ have the following identity,

$$K_\phi(\vec{p}, E - i\epsilon, t) = K^*_\phi(\vec{p}, -E - i\epsilon, t) \tag{3.39}$$

with

$$\partial_t K_\phi(\vec{p}, E - i\epsilon, t) = (\partial_t K_\phi(\vec{p}, -E - i\epsilon, t))^* \,. \tag{3.40}$$

For integer spin, the above identities still hold, since the only complex term in the bulk-to-boundary operator $K^{l_1 l'_1 \cdots}(\vec{p}, E, t)$ is e^{iEt}, and we assumed that the vertices iV are real functions of momentum and time derivative.[5] The real condition of the vertices is equivalent to requiring the interacting term of the Hamiltonian to be Hermitian

$$H_{int} = iV \rightarrow H_{int} + H^*_{int} = 0 \tag{3.41}$$

According to the Feynman rules (see 2.4 for details), the correlators can be expressed as

$$\psi_c^{l_1 l_2 \cdots}(E_{1\sim n}, \vec{p}_{1\sim n} \dots) = \int d^4x (iV_{l'_1 \dots, l'_2 \dots, \dots}(\vec{p}_{1\sim n}, \partial_t)) K^{l_1 l'_1 \cdots}(\vec{p}_1, E_1 - i\epsilon, t) K^{l_2 l'_2 \cdots}(\vec{p}_2, E_2 - i\epsilon, t) \dots \tag{3.42}$$

According to (3.39) and (3.40), the correlators have the following identity

$$\psi_c^{l_1 l_2 \cdots}(E_{1\sim n}, \vec{p}_{1\sim n} \dots) + \psi_c^{*, l_1 l_2 \cdots}(-E_{1\sim n}, \vec{p}_{1\sim n} \dots) = 0., \tag{3.43}$$

which is the COT for the contact diagram.

[5] ∂_t will apply on any of bulk-to-boundary operator. Be careful, it will bring down iE instead of E.

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Propagator Property: Exchange Scalar There's an alternative way to derive the COT. We could investigate the discontinuity of the complexified exchange correlator [20]. We use $z = \vec{p} \cdot \vec{p}$ to denote the momentum self-contraction, and the on-shell energy E is the square root of z. Then if the momentum is complexified, the function $E = \sqrt{z}$ will have a branch cut on z-plane. If we set the branch cut on the positive real axis, the discontinuity of the correlator reads

$$\begin{aligned} \underset{z=|E|^2 \pm i\epsilon}{\text{Disc}} \psi_c(z) &:= \psi_c(z = |E|^2 - i\epsilon) - \psi_c(z = |E|^2 + i\epsilon) \\ &= \psi_c(E = |E| - i\epsilon) - \psi_c(E = -|E| - i\epsilon) \end{aligned} \tag{3.44}$$

where $|E|^2 = |z| \in \mathbb{R}$. On the E space, the discontinuity is just the difference between the correlator and its energy flipping.

To explain why this setting of the branch cut is natural, we consider the legal domain of the energy is

$$Im(E) < 0. \tag{3.45}$$

such that e^{iEt} in the bulk-to-boundary and the bulk-to bulk propagator converges at the far past $t = -\infty$ (Bunch-Davies condition). Defining the square root like

$$E = \sqrt{z} := |z| e^{i\frac{Arg(z)}{2}} \tag{3.46}$$

where the principal value of the argument is defined by $Arg(z) \in [0, -2\pi)$, we can check that the square root indeed maps the full complex plane to the lower half plane.

It's important to mention that only terms with an odd power of energy survive under the discontinuity defined by (3.44). These terms must come from the e^{iEt} or the derivative of e^{iEt} indicating that the discontinuity, or the constraint (3.45), is a result of the expo-

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nential term e^{iEt}. In the exchange diagram, we can apply the discontinuity on the internal energy square $z_s = E_s^2 = (\vec{p}_1 + \vec{p}_2)^2$, which allows us to extract the exponential parts of the bulk-to-bulk propagator. Especially, for the scalar exchange correlator,

$$\begin{aligned}
&\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \psi_4^{(s)}(E_1, E_2, E_3, E_4, z_s, \dots) \\
&= \int dt \int dt' V_L(t, \vec{p}_a, \partial_t) K(t, \vec{p}_1) K(t, \vec{p}_2) \cdot \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} G_\phi(\sqrt{z_s}, t, t') \cdot V_R(t', \vec{p}_a, \partial_{t'}) K(t', \vec{p}_3) K(t', \vec{p}_4) \\
&= \frac{1}{2E_s} \cdot \int dt \int dt' V_L(t, \vec{p}_a, \partial_t) K(t, \vec{p}_1) K(t, \vec{p}_2) \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \cdot K_\phi(\sqrt{z_s}, t) \\
&\cdot V_R(t', \vec{p}_a, \partial_{t'}) K(t', \vec{p}_3) K(t', \vec{p}_4) \cdot \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} K_\phi(\sqrt{z_s}, t') \\
&= \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \psi^L_{3,i\dots}(E_1, E_2, E_3, E_4, z_s, \dots) \cdot \frac{1}{2E_s} \cdot \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \psi^R_{3,i'\dots}(E_1, E_2, E_3, E_4, z_s, \dots)
\end{aligned} \tag{3.47}$$

in which we apply the factorization property of the discontinuity of the scalar bulk-to-bulk propagator,

$$\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} G_\phi(\sqrt{z_s}, t, t') = \frac{1}{2E_s} \cdot \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} K_\phi(\sqrt{z_s}, t) \cdot \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} K_\phi(\sqrt{z_s}, t'). \tag{3.48}$$

It's just the COT because if we write the discontinuity explicitly, we'll have

$$\begin{aligned}
&\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \psi_4^{(s)}(E_1, E_2, E_3, E_4, z_s, \dots) \\
&= \psi_4^{(s)}(E_1, E_2, E_3, E_4, E_s - i\epsilon, \dots) - \psi_4^{(s)}(E_1, E_2, E_3, E_4, -E_s - i\epsilon, \dots) \\
&\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \psi^{L/R}_{3,i\dots}(E_1, E_2, E_3, E_4, z_s, \dots) \\
&= \psi^{L/R}_{3,i\dots}(E_1, E_2, E_3, E_4, E_s, \dots) - \psi^{L/R}_{3,i\dots}(E_1, E_2, E_3, E_4, -E_s, \dots) \\
&= \tilde{\psi}^{L/R}_{3,i\dots}(E_1, E_2, E_3, E_4, E_s, \dots),
\end{aligned} \tag{3.49}$$

Then substituting the (3.47) with (3.49), we can get the COT written in (3.11).

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Propagator Property: Exchange Vector/Tensor In the Lorentz gauge or the transverse mode of the massless vector and tensor field, A_i^{T} and h_{ij}^{TT}, the bulk-to-bulk propagator of the transverse (and traceless) field is

$$\begin{aligned} G^T_{A,ij}(\sqrt{z_s}, t, t') &= \pi_{s,ij} G_\phi(\sqrt{z_s}, t, t') \\ G^{\mathrm{TT}}_{h,iji'j'}(\sqrt{z_s}, t, t') &= \Pi^{ijkl}_{s,(2,2)} G_\phi(\sqrt{z_s}, t, t'). \end{aligned} \tag{3.50}$$

Then the discontinuities on the z_s of the bulk-to-bulk propagators are

$$\begin{aligned} \operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} G_{A,ij}(\sqrt{z_s}, t, t') &= \frac{\pi_{s,ij}}{2E_s} \cdot \operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} K_\phi(\sqrt{z_s}, t) \cdot \operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} K_\phi(\sqrt{z_s}, t') \\ \operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} G_{h,ii'jj'}(\sqrt{z_s}, t, t') &= \frac{\Pi^{ijkl}_{s,(2,2)}}{2E_s} \cdot \operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} K_\phi(\sqrt{z_s}, t) \cdot \operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} K_\phi(\sqrt{z_s}, t'). \end{aligned} \tag{3.51}$$

Similar to the derivation of the COT exchanging scalar by the discontinuity, substituting the correlator written in the Feynman Rule [6] with (3.51), we could get (3.35) and (**??**) .

3.2.2 Bosonic Partial Energy Pole

The COT gives another constraint on the correlators, which is the partial energy pole constraint. Let's use the scalar exchange COT, i.e., (3.6), for demonstration. The 4pt correlator ψ_4 has energy poles $\frac{1}{E_{12s}}$ and $\frac{1}{E_{34s}}$, while ψ_4^* has energy poles $\frac{1}{-E_{12}+E_s}$ and $\frac{1}{-E_{34}+E_s}$, since the sign of the external energy is flipped. The partial energy pole locates at $\frac{1}{E_{12s}} = 0$, which appears only in ψ_4. On the other hand, ψ_3 also has the energy pole $\frac{1}{E_{12s}}$, which is the sum of the energy of the external legs of the sub-diagram, and the residue is the amplitude. As a result, the residues of

$$\operatorname*{Res}_{E_{12s}\to 0} \psi_4^{(\phi)}(E_{1\sim 4}, E_s, \boldsymbol{p}_{1\sim 4}) = \frac{M_3(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \cdot \tilde{\psi}_3(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4)}{2E_s}$$

[6]Because the correlator should be invariant under the gauge we choose for internal polarization.

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We'll get a similar expression for the E_{34s} pole, whose residue is also the product of amplitude and shift-correlator. It's trivial to promote this result to higher spinning boson correlators:

$$\underset{E_{12s}\to 0}{\text{Res}}\ \psi_4^{(J)} = \tilde{M}_{3,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s)\ \frac{\pi_s^{ij}}{2E_s}\tilde{\psi}_{3,j}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s) \tag{3.52}$$

$$\underset{E_{12s}\to 0}{\text{Res}}\ \psi_4^{(T)} = \tilde{M}_{3,ij}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s)\ \frac{\Pi_{s,(2,2)}^{ijkl}}{2E_s}\tilde{\psi}_{3,kl}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s) \tag{3.53}$$

where $\tilde{M}_{3,i}$, $\tilde{M}_{3,ij}$ is defined by

$$\begin{aligned} M_3(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) &=:\ \tilde{M}_3^i(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s)\ \epsilon_{s,i,\mathbf{0}} \\ M_3(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) &=:\ \tilde{M}_3^{ij}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s)\ \epsilon_{s,i,\mathbf{0}}\epsilon_{s,j,\mathbf{0}}. \end{aligned} \tag{3.54}$$

3.2.3 Dirac Fermion Field

In this section, we only use the Feynman Rule to find the COT of the fermion. To write down the COT, we need to carefully calculate the conjugate of the correlator and examine the discontinuity of the correlator.

Contact diagram The Dirac fermion operators are also complex, for the correlator, the operators are $\chi_{+,p}$ and $\bar{\chi}_{-,p}$. We define the conjugate of the correlator to be the normal conjugation ($\psi_n \to \psi_n^\dagger$) with $\chi_{+,p} \leftrightarrow \bar{\chi}_{-,p}$. [7] In this paper, we use C to label the conjugate

[7] If we examine the conjugate in the COT of the scalar QED correlator. It seems like we also need to interchange the complex operator O_p and its conjugate operator O_p^*. If we use C to label the conjugate of the correlator and under the conjugate correlator will take complex conjugate and make $O \leftrightarrow O^*$, then the COT reads

$$\langle O^*_{1,k_1\ldots} O_{2,l_1\ldots} \ldots\rangle_c(E_{1\sim n}, \vec{p}_{1\sim n}) + C\langle O^*_{1,k_1\ldots} O_{2,l_1\ldots} \ldots\rangle_c(-E_{1\sim n}, \vec{p}_{1\sim n}) = 0 \tag{3.55}$$

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of the correlator. Then, we could propose the contact COT of the fermion,

$$\begin{aligned}&\langle \bar{\chi}_{-,B,1}\chi_{+,A,2}\dots\rangle_c(E_{1\sim n},\vec{p}_{1\sim 3})\\&+C\langle \bar{\chi}_{-,B,1}\chi_{+,A,2}\dots\rangle_c(-E_{1\sim n},\vec{p}_{1\sim 3})=0.\end{aligned} \tag{3.58}$$

where $\langle \bar{\chi}_{-,B,1}\chi_{+,A,2}\dots\rangle_c$ is the contact correlator obtained by pulling out the boundary condition $\bar{\chi}_{-,B,1},\chi_{+,A,2},\dots$ in the expansion of the wavefunction. We could apply the COT to the QED 3-point contact correlator, which is

$$\langle J_1\bar{\chi}_{B,-,2,e}\chi_{A,+,3,-e}\rangle=\frac{e}{K_T}\left[(1-\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{2}})\not{\epsilon}_1(1+\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{3}})\right]_{BA}. \tag{3.59}$$

And the conjugate of the correlator with energy signs flipping is

$$\begin{aligned}C\langle J_1\bar{\chi}_{B,-,2,e}\chi_{A,+,3,-e}\rangle(-E_{1\sim 3})&:=(\langle J_1\chi_{A,+,2,e}\bar{\chi}_{B,+,3,-e}\rangle)^\dagger(-E_{1\sim 3})\\&=\frac{-e}{-K_T}\left[(1+\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{2}})\gamma_0\not{\epsilon}_1\gamma_0(1-\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{3}})\right]_{BA}\\&\sim\frac{e}{-K_T}\left[(1-\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{2}})\not{\epsilon}_1(1+\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{3}})\right]_{BA}.\end{aligned} \tag{3.60}$$

The equivalence of the last line is because the boundary field $\bar{\chi}_-$ and χ_+ absorbs the γ_0 by $\bar{\chi}_-\gamma_0=-\bar{\chi}_-$ and $\gamma_0\chi_-=\chi_+$. Substituting (3.59) and (3.60), we could check that (3.58) is satisfied.

Then we could find the Feynman Rule we use for QED can be proved to be applicable

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Then, if we apply the COT to the 3-point contact correlator (we use the subscripts to label the charge of the particle),

$$\langle J_1O^*_{2,e}O_{3,-e}\rangle=\frac{1}{K_T}e(p_2-p_3)\cdot\epsilon_1. \tag{3.56}$$

The conjugate of the correlator will be

$$C\langle J_1^TO^*_{2,e}O_{3,-e}\rangle:=(\langle J_1^TO_{2,e}O^*_{3,-e}\rangle)^*=\frac{1}{K_T}(-e)(p_3-p_2)\cdot\epsilon_1=(\langle J_1^TO^*_{2,e}O_{3,-e}\rangle)^*. \tag{3.57}$$

The operation, $O\leftrightarrow O^*$, makes $2\leftrightarrow 3$ with $e\to -e$. And, because labels 2 and 3 are anti-commute in the correlator, we find that $C\langle J_1O^*_{2,e}O_{3,-e}\rangle=\langle J_1O^*_{2,e}O_{3,-e}\rangle^*$. The conjugate of the correlator is equivalent to the normal complex conjugate. As a result, the normal Optical theorem, like (3.11), still works for the contact correlator of the scalar QED.

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to the general theory in the (C.146). Then, we could extend (3.11) to the general theory by similar calculations.

Exchange Diagram: Exchange Fermion Because of the difficulties to get the COT in the version including the conjugate of the fermion correlator. [8] We only propose the COT in the version of the discontinuity. (we define $\vec{p_s} = -\vec{p}_1 - \vec{p}_2 = \vec{p}_3 + \vec{p}_4$ here.)

First, we should note that the fermionic COT can't be rigorously derived by promoting the derivation of the COT of the bosonic field. The reason is that the Feynman rule we use might not be applied to the fermionic correlators which is expanded the boundary action S_b. So in App. C.3, we show that, under the discontinuity, the fermionic correlator indeed shares a similar Feynman Rule structure with the bosonic field. Then in (C.157) we find the discontinuity of the bulk-to-bulk propagator gives the discontinuity of the correlator,

$$\begin{aligned}&\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}}\ \psi_4^{(\chi)}(\vec{p}_{1\sim4}, z_s)\\ &= \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}}\ \psi_{3,A}(\vec{p}_1, \vec{p}_2, z_s) \cdot \left[\frac{1+\gamma_0}{2} \cdot \frac{-\not{p}_s}{2E_s} \cdot \frac{1-\gamma_0}{2}\right]^{AB} \underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}}\ \psi_{3,B}(\vec{p}_3, \vec{p}_4, z_s).\end{aligned} \tag{3.62}$$

[8]Because of the C operation, it will mix different channels in the LHS of the COT because we permute the label of the field and its conjugate. Then we cannot make the COT to be equal channel by channel. Even if we sum up all the channels in the COT like

$$\begin{aligned}&\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}}\ \psi_4^{(\chi)}(\vec{p}_{1\sim4}, z_s) + \underset{z_s=|E_t|^2\pm i\epsilon}{\text{Disc}}\ \psi_4^{(\chi)}(\vec{p}_{1\sim4}, z_t) = \psi_4(\vec{p}_{1\sim4}, E_s) - \psi_4^{(\chi)}(\vec{p}_{1\sim4}, -E_s) + (t)\\ &\text{"} = \text{"}\psi_{4,(s)}^{(\chi)}(\vec{p}_{1\sim4}, E_s, E_{1\sim4}) + C\psi_{4,(t)}^{(\chi)}(\vec{p}_{1\sim4}, E_t, -E_{1\sim4})\\ &+ \psi_{4,(t)}^{(\chi)}(\vec{p}_{1\sim4}, E_t, E_{1\sim4}) + C\psi_{4,(s)}^{(\chi)}(\vec{p}_{1\sim4}, E_s, -E_{1\sim4}),\end{aligned} \tag{3.61}$$

. The last equality, "=" should not be correct for the massive and massless fermion correlator $\langle J\bar{\chi}J\chi\rangle$ so the COT version with the conjugate remains unknown.

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in which on the RHS of the above equation,

$$\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \psi_{3,A}(\vec{p}_1,\vec{p}_2,z_s) = \psi_{3,A}(\vec{p}_1,\vec{p}_2,E_s) - \psi_{3,A}(\vec{p}_1,\vec{p}_2,-E_s) = \tilde{\psi}_{3,A}(\vec{p}_1,\vec{p}_2,E_s). \tag{3.63}$$

is identified as the shifted-correlator. Then (3.62) gives the discontinuity version of the fermion COT.

$$\begin{aligned}&\underset{z_s=|E_s|^2\pm i\epsilon}{\text{Disc}} \psi_4^{(\chi)}(\vec{p}_{1\sim 4},z_s) = \psi_4(\vec{p}_{1\sim 4},E_s) - \psi_4(\vec{p}_{1\sim 4},-E_s)\\ &= \tilde{\psi}^+_{3,A}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s)\left(\left(\frac{1+\gamma_0}{2}\right)\frac{-\boldsymbol{\not p}_s}{2E_s}\left(\frac{1-\gamma_0}{2}\right)\right)^{AB}\tilde{\psi}^-_{3,B}(-\boldsymbol{p}_s,\boldsymbol{p}_3,\boldsymbol{p}_4,E_s).\end{aligned} \tag{3.64}$$

We should remark that this form of COT is the same for the massive and the massless fermion. And we could identify the gluing factor, $\left[\frac{1+\gamma_0}{2}\cdot\frac{-\not p_s}{2E_s}\cdot\frac{1-\gamma_0}{2}\right]^{AB}$, as the 2-point in-in correlator

$$\langle\chi_{+,\boldsymbol{s}}\bar{\chi}_{-,-\boldsymbol{s}}\rangle^{AB}_{in-in} = (\frac{1+\gamma_0}{2})(2Re\langle\chi_{+,\boldsymbol{s}}\bar{\chi}_{-,-\boldsymbol{s}}\rangle)^{-1}(\frac{1-\gamma_0}{2}) = (\frac{1+\gamma_0}{2})\frac{-\boldsymbol{\not p}_s}{2E_s}(\frac{1-\gamma_0}{2}) \tag{3.65}$$

in which we the real part of the correlator is

$$\begin{aligned}&Re\langle\chi_{+,\boldsymbol{s}}\bar{\chi}_{-,-\boldsymbol{s}}\rangle = \frac{1}{2}(\langle\chi_{+,\boldsymbol{s}}\bar{\chi}_{-,-\boldsymbol{s}}\rangle + C\langle\chi_{+,\boldsymbol{s}}\bar{\chi}_{-,-\boldsymbol{s}}\rangle)\\ &= \frac{1}{2}(\langle\chi_{+,\boldsymbol{s}}\bar{\chi}_{-,-\boldsymbol{s}}\rangle + (\langle\bar{\chi}_{+,\boldsymbol{s}}\chi_{-,-\boldsymbol{s}}\rangle))^\dagger\\ &= \frac{1}{2}\cdot(\frac{\boldsymbol{\not p}_s}{E_s+m} + (\left(\frac{\boldsymbol{\not p}_s}{E_s+m}\right)^{-1}\frac{\boldsymbol{\not p}_s}{E_s+m}\left(\frac{\boldsymbol{\not p}_s}{E_s+m}\right)^{-1})^\dagger)\\ &\sim \frac{\not p_s}{E_s+m} + \frac{\not p_s}{E_s-m} = \frac{2E_s\not p_s}{E_s^2-m^2}\end{aligned} \tag{3.66}$$

where the equivalance $\sim$ means that γ_0 is absorbed by boundary condition $\bar{\chi}_-/\chi_+$. Then the COT of the correlator exchanging fermions (3.64), is just the discontinuity version of the COT substituting the two-point in-in correlator with the fermionic one.

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Exchange Diagram: Exchange Massless Spin 3/2 Now for spin 3/2 particle, we could easily extend our result by substituting the two-point in-in correlator in the COT exchanging fermion field with a 2-point in-in correlator for a 3/2 particle,

$$\psi_{4,s}^{(\psi)}(E_{1\sim4},E_s)-\psi_{4,s}^{(\psi)}(E_{1\sim4},-E_s)=\tilde{\psi}^{+}_{3,A,i}\langle\psi^{i}_{+,\boldsymbol{s}}\bar{\psi}^{j}_{-,-\boldsymbol{s}}\rangle^{AB}_{in-in}\tilde{\psi}^{-}_{3,B,j} \tag{3.67}$$

in which $\langle\psi_{-,-\boldsymbol{s}}\psi_{+,\boldsymbol{s}}\rangle^{ij}_{in-in}=P^{i}_{s,i'}\frac{1+\gamma_0}{2}\frac{-\pi_{s,i'j'}\hat{\boldsymbol{p}}\!\!\!/_s+\xi\hat{p}^{i'}_s\hat{p}^{j'}_s+\ldots}{2}\frac{1-\gamma_0}{2}\bar{P}^{j}_{s,j'}$.The extra projector P^{ij}_s, $\bar{P}^{ij}_s$ we add to the in-in correlator is defined by

$$\begin{aligned}\bar{P}^{ij}_s&=\left(-\hat{\boldsymbol{p}}\!\!\!/_s\Pi^{ij}_{(3/2,3/2)}+\hat{p}^i_s\hat{\boldsymbol{p}}\!\!\!/_s\gamma^j-\hat{p}^i_s\hat{p}^j_s\right)\frac{1-\gamma_0}{2}\\ P^{ij}_s&=\frac{1+\gamma_0}{2}\left(-\Pi^{ij}_{(3/2,3/2)}\hat{\boldsymbol{p}}\!\!\!/_s+\hat{p}^i_s\gamma^j\hat{\boldsymbol{p}}\!\!\!/_s-\hat{p}^i_s\hat{p}^j_s\right)\end{aligned} \tag{3.68}$$

such that $P^{ia}_sP^{j}_{s,a}=P^{ij}_s$, $\bar{P}^{ia}_s\bar{P}^{j}_{s,a}=\bar{P}^{ij}_s$ and, for the general 3D spin 3/2 field $\psi'_{+,i,s,\boldsymbol{0}}$, $\bar{\psi}'_{-,i,s,\boldsymbol{0}}$, the $\psi_{+,i,s,\boldsymbol{0}}=P_{ij,s}\psi'^{,j}_{+,s,\boldsymbol{0}}$, $\bar{\psi}_{-,i,-s,\boldsymbol{0}}=\bar{P}_{ij,-s}\psi'^{,j}_{-,-s,\boldsymbol{0}}$ satisfy the constriant (2.39) and (2.40). The projector in the 2-point in-in correlator reflects that it comes from the correlator with the constrained polarization.

Similar to the in-in correlator of the photon and the graviton, there's a gauge term proportional to ξ in the 2-point in-in correlator. We could use the WT identity to eliminate ξ dependences. Dropping out the gauge term and simplifying the in-in correlator of gravitino, we could show that the COT could be written with gluing factor $\frac{1}{2}\Pi^{ij}_{s,(3/2,3/2)}$ defined in (3.10). With the property

$$\gamma^i\Pi^{ij}_{s,(3/2,3/2)}=p^i_s\Pi^{ij}_{s,(3/2,3/2)}=\Pi^{ij}_{s,(3/2,3/2)}\gamma_j=0 \tag{3.69}$$

, the $\Pi^{ij}_{s,(3/2,3/2)}$ captures the sum of the physical gravitino boundary condition, which is gamma-traceless and transverse.

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3.2.4 Majorana Fermion Field

We define the B-operator, $B = -i\gamma_2$ and the charge conjugation operator

$$C : \chi \to B\chi^*.$$

Then Majornara condition of the 4D spinor will be

$$C\chi = \chi \tag{3.70}$$

$$\chi = \begin{bmatrix} \upsilon \\ (i\sigma_2)\upsilon^* \end{bmatrix}$$

Now we want the Majornara fermion as the boundary condition of the classical solution, so we split (A.30) into two 3D spinors, $\gamma_0\chi_\pm = \pm\chi_\pm$:

$$(\frac{1+\gamma_0}{2})C(\chi_+ + \chi_-) = \chi_+ = (-i\gamma_2)\chi_-^* = (-B)\bar{\chi}_-^T \tag{3.71}$$

in the last equality, we set $\bar{\chi}_- = \chi_-^\dagger\gamma_0$ again to relate the field and its momentum conjugate then we have the 3D Majornara condition of fermion

$$\bar{\chi}_- = -\chi_+^T B \tag{3.72}$$

so we find that the dual boundary condition of the 3D Dirichlet boundary condition $\bar{\chi}_-$ and χ_+ is related under the Majorana condition just like fermion and anti-fermion related with each other in 4D.

In the case of the amplitude, we could show the Majorana flipping relationship by the property of the B operator, $B\gamma_\mu B = -(\gamma_\mu)^*$, and the operator $D = iB\gamma_0$ with a property

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$-D\gamma^\mu D = (\gamma^\mu)^{\mathrm{T}}, D.D = I.$ [10]

$$Bu_1^* = u_1 = -iD\bar{u}_1^T; \bar{u}_2^*(-B) = \bar{u}_2 = u_2^T(-iD)$$
$$\bar{u}_1\gamma^\mu u_2 = (\bar{u}_1\gamma^\mu u_2)^{\mathrm{T}} = -u_2^T D\gamma^\mu D\bar{u}_1 = \bar{u}_2\gamma^\mu u_1 \tag{3.73}$$

This equality implies that there's no actual difference between $\bar{\chi}$ and χ, like for the amplitude

$$M(h_1\bar{\chi}_2\chi_3) = (\epsilon_{1,\mu}(p_2 - p_3))\bar{u}_2\not{\epsilon}_1 u_3 = -(\epsilon_{1,\mu}(p_3 - p_2))\bar{u}_3\not{\epsilon}_1 u_2 = -M(h_1\bar{\chi}_3\chi_2). \tag{3.74}$$

And this flipping relationship could be extended to the theory with the vertices composed of $\gamma_{\mu\nu\rho\ldots} = \{\gamma_\mu[\gamma_\nu, \{\gamma_\rho \ldots\}]\}$ with a property $-D\gamma_{\mu\nu\rho\ldots}D = (\gamma_{\mu\nu\rho\ldots})^T$. Actually, in our paper, we only discuss this type of vertices.

To check that we have no problem matching the amplitude to the correlator under the total energy pole, we show that if we write the fermion polarization $u, \bar{u}$ in the boundary condition ($u = (1+\not{\boldsymbol{p}})\chi_+, \bar{u} = \bar{\chi}_-(1-\not{\boldsymbol{p}})$), the Majorana flipping relationship still works.

$$\begin{aligned}\bar{u}_1\gamma^\mu u_2 &= \bar{\chi}_{1,-}(1-\hat{\not{\boldsymbol{p}}}_1)\gamma^\mu(1+\hat{\not{\boldsymbol{p}}}_2)\chi_{2,+} = (\bar{\chi}_{1,-}(1-\hat{\not{\boldsymbol{p}}}_1)\gamma^\mu(1+\hat{\not{\boldsymbol{p}}}_2)\chi_{2,+})^\dagger \\ &= -\chi_{2,+}^T D(1+\hat{\not{\boldsymbol{p}}}_2)D^2\gamma^\mu D^2(1-\hat{\not{\boldsymbol{p}}}_1)D\bar{\chi}_{1,-}^T \\ &= \bar{\chi}_{2,-}(1+\hat{\not{\boldsymbol{p}}}_2)\gamma^\mu(1-\hat{\not{\boldsymbol{p}}}_1)\chi_{1,+} = \bar{u}_2\gamma^\mu u_1\end{aligned} \tag{3.75}$$

where for convenience, we write the Majorana condition on boundary condition (**??**) as $\bar{\chi}_- = \chi_+^T(-iD), (-iD)\bar{\chi}_-^T = \chi_+$.

Then although the C-conjugation includes a process that $\bar{\chi}_- \leftrightarrow \chi_+$, for the Majorana correlator, this process won't change the correlator due to the flipping relationship. The

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C-conjugate in the Fermion COT is just equivalent to normal conjugation,

$$C\psi_{n,maj} = (\psi_{n,maj})^{\dagger}|_{\bar{\chi}_{-}\leftrightarrow\chi_{+}} = (\psi_{n,maj})^{\dagger}. \tag{3.76}$$

3.2.5 Fermionic Partial Energy pole

If we take the partial energy pole residue on both sides of the fermion COT,

$$\begin{aligned}\mathrm{Res}_{E_{12s}\to 0}\ \psi_4^{(\chi)} &= M^{+}_{3,A}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s)\left(\left(\frac{1+\gamma_0}{2}\right)\frac{-\hat{\not{\boldsymbol{p}}}_s}{2}\left(\frac{1-\gamma_0}{2}\right)\right)^{AB}\tilde{\psi}_{3,B}(-\boldsymbol{p}_s,\boldsymbol{p}_3,\boldsymbol{p}_4,E_s)\\ = M^{u}_{3,A}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s)\frac{1}{2E_s}&\left(\frac{(-\not{S}-m)(I+(-\gamma_0-I)\frac{E_s}{E_s+m})}{E_{34s}}-\frac{-\not{S}-m}{E_{34}-E_s}\right)^{AB}N^{u}_{3,B}(-\boldsymbol{p}_s,\boldsymbol{p}_3,\boldsymbol{p}_4)\end{aligned} \tag{3.77}$$

in which we define

$$\begin{aligned}M_3(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s)|_{u_s=(1+\frac{\not{\boldsymbol{p}}_s}{E_s+m})\chi_{+,s,\boldsymbol{0}}} &=: M^{+}_{3,A}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s)\chi^{A}_{+,s,\boldsymbol{0}}\\ M_3(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s) &=: M^{u}_{3,A}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s)u^{A}_s\end{aligned} \tag{3.78}$$

and the last equation holds only when we could write the three-point correaltor as

$$\psi_3(E_s,E_3,E_4) = (1+\frac{\hat{p}_s}{E_s+m})\psi_3^u = (1+\frac{\hat{p}_s}{E_s+m})\frac{N_3^u}{E_s+E_3+E_4} \tag{3.79}$$

where N_3^{μ} is independent of the E_s. It's true for ψ_3 is pure transverse and for QED correlator $\langle J\bar{\chi}\chi\rangle$. In these cases, $N_3^u = M_3^u$. Similarly, we have

$$\begin{aligned}&\mathrm{Res}_{E_{34s}\to 0}\ \psi_4^{(\chi)}\\ &= \tilde{\psi}_{3,A}(\boldsymbol{p}_s,\boldsymbol{p}_1,\boldsymbol{p}_2,E_s)\left(\left(\frac{1+\gamma_0}{2}\right)\frac{-\hat{\not{\boldsymbol{p}}}_s}{2}\left(\frac{1-\gamma_0}{2}\right)\right)^{AB}M^{-}_{3,B}(\boldsymbol{p}_3,\boldsymbol{p}_4,-\boldsymbol{p}_s)\\ &= N^{u}_{3,A}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_s)\ \frac{1}{2E_s}\left(\frac{\gamma_0\ (\not{S}_{-})}{E_{12s}}-\frac{\not{S}_{-}}{E_{12}-E_s}\right)^{AB}M^{u}_{3,B}(-\boldsymbol{p}_s,\boldsymbol{p}_3,\boldsymbol{p}_4)\end{aligned} \tag{3.80}$$

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in which we define

$$M_3(\boldsymbol{p}_3, \boldsymbol{p}_4, -\boldsymbol{p}_s)|_{\bar{u}_{-s}=\bar{\chi}_{-,-s,\mathbf{0}}(1-\frac{\not{\boldsymbol{p}}_s}{E_s+m})} =: \bar{\chi}^A_{-,-s,\mathbf{0}} M^+_{3,A}(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_s) \tag{3.81}$$
$$M_3(\boldsymbol{p}_3, \boldsymbol{p}_4, -\boldsymbol{p}_s) =: \bar{u}^A_{-s} M^u_{3,A}(\boldsymbol{p}_3, \boldsymbol{p}_4, -\boldsymbol{p}_s)$$

and the last equation holds only when N could be defined like (3.79). We could check that $\langle J\bar{\chi}J\chi\rangle$ we derived in the Lagrangian approach as shown in (D.199) indeed satisfies these asymmetric residues on the left and right partial energy pole. And we could extend the result to massless spin 3/2 particle:

$$\begin{aligned}
&\mathrm{Res}_{E_{12s}\to 0}\ \psi_4^{(\psi)} \\
&= M^+_{3,A,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \left(\left(\frac{1+\gamma_0}{2}\right) \frac{-\pi_s^{ij}\hat{\not{\boldsymbol{p}}}_s - \frac{1}{2}\not{\boldsymbol{\pi}}^i_s \hat{\not{\boldsymbol{p}}}_s \not{\boldsymbol{\pi}}^j_s}{2} \left(\frac{1-\gamma_0}{2}\right) \right)^{AB} \tilde{\psi}_{3,j,B}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s) \\
&= M^u_{3,i,A}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \frac{\pi_s^{ij}}{2E_s} \left(\frac{-\not{S}}{E_{34}-E_s} - \frac{(-\not{S} - m)\,(-\gamma_0)}{E_3+E_4+E_s} \right)^{AB} N^u_{3,j,B}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4) \\
&+ M^+_{3,A,i}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \left(\left(\frac{1+\gamma_0}{2}\right) \frac{-\not{\boldsymbol{\pi}}^i_s \hat{\not{\boldsymbol{p}}}_s \not{\boldsymbol{\pi}}^j_s}{4} \left(\frac{1-\gamma_0}{2}\right) \right)^{AB} \tilde{\psi}_{3,j,B}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s)
\end{aligned} \tag{3.82}$$

$$\begin{aligned}
&\mathrm{Res}_{E_{34s}\to 0}\ \psi_4^{Exc3/2} \\
&= \tilde{\psi}_{3,A,i}(\boldsymbol{p}_s, \boldsymbol{p}_1, \boldsymbol{p}_2, E_s) \left(\left(\frac{1+\gamma_0}{2}\right) \frac{-\pi_s^{ij}\hat{\not{\boldsymbol{p}}}_s - \frac{1}{2}\not{\boldsymbol{\pi}}^i_s \hat{\not{\boldsymbol{p}}}_s \not{\boldsymbol{\pi}}^j_s}{2} \left(\frac{1-\gamma_0}{2}\right) \right)^{AB} M^-_{3,j,B}(\boldsymbol{p}_3, \boldsymbol{p}_4, -\boldsymbol{p}_s) \\
&= N^u_{3,i,A}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s) \frac{\pi_s^{ij}}{2E_s} \left(-\frac{\not{S}_-}{E_{12}-E_s} + \frac{(\gamma_0)\,(\not{S}_-)}{E_1+E_2+E_s} \right)^{AB} M^u_{3,j,B}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4) \\
&+ \tilde{\psi}_{3,A,i}(\boldsymbol{p}_s, \boldsymbol{p}_1, \boldsymbol{p}_2, E_s) \left(\left(\frac{1+\gamma_0}{2}\right) \frac{-\not{\boldsymbol{\pi}}^i_s \hat{\not{\boldsymbol{p}}}_s \not{\boldsymbol{\pi}}^j_s}{4} \left(\frac{1-\gamma_0}{2}\right) \right)^{AB} M^-_{3,j,B}(\boldsymbol{p}_3, \boldsymbol{p}_4, -\boldsymbol{p}_s)
\end{aligned} \tag{3.83}$$

The last equalities for individual partial energy pole residues are held when n N could be defined like (3.79).

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3.3 The bootstrap program

3.3.1 2pt correlators

By dimensional analysis and Ward Takahashi identity of two-point correlator discussed in App. F.1, we can fix the 2pt correlator of scalar fields O, vector fields J^i, and graviton fields T^{ij} to be

$$\begin{aligned}\langle O_{-\boldsymbol{p}} O_{\boldsymbol{p}} \rangle &= E \\ \langle J^i_{-\boldsymbol{p}} J^j_{\boldsymbol{p}} \rangle &= E\pi_{ij} \\ \langle T^{ij}_{-\boldsymbol{p}} T^{kl}_{\boldsymbol{p}} \rangle &= E\pi_{i,k}\pi_{j,l}\end{aligned} \tag{3.84}$$

where we defined $\pi_{ij} \equiv \eta_{ij} + \frac{p_i p_j}{E_p^2}$. For the graviton two-point correlator we apply the E.O.M that $\pi_{ij} h^{ij}_{\mathbf{0},p} = 0$ then in the correlator we apply $\pi_{ij} \sim 0$. And for the fermionic correlator, by dimension counting of the action, we know it should be a dimensionless factor. And the two-point correlator should be sandwiched with the 3D boundary spinor condition $\bar{\chi}_- / \chi_+$. With the identity

$$\bar{\chi}_{-,-\boldsymbol{p}} I \chi_{+,\boldsymbol{p}} = \bar{\chi}_{-,-\boldsymbol{p}} \gamma_0 \chi_{+,\boldsymbol{p}} = 0. \tag{3.85}$$

the only nonvanishing 3D rotational invariant factor will be

$$\langle \bar{\chi}^-_{-\boldsymbol{p}} \chi^+_{\boldsymbol{p}} \rangle = \frac{\not{\boldsymbol{p}}}{E_{\boldsymbol{p}}} = \hat{\not{\boldsymbol{p}}} \tag{3.86}$$

And for gravitino, the dimension counting is the same as spin half fermion, the two-point correlator should be a dimensionless factor. By the similar argument to the spin half fermion, the only nonvanishing matrice sandwiched between 3D gravitino boundary

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condition $\bar{\psi}^i_-/\psi^j_+$ like §App. F.1 shows will be

$$\langle \bar{\psi}^{i,-}_{-\boldsymbol{p}} \psi^{j,+}_{\boldsymbol{p}} \rangle \propto \hat{\boldsymbol{p}} \tag{3.87}$$

And by the WT identity of the two-point correlator,

$$p_i \langle \bar{\psi}^{i,-}_{-\boldsymbol{p}} \psi^{j,+}_{\boldsymbol{p}} \rangle = 0. \tag{3.88}$$

Now we know that in the two-point correlator, only the transverse part survives. Now we get

$$\langle \bar{\psi}^{i,-}_{-\boldsymbol{p}} \psi^{j,+}_{\boldsymbol{p}} \rangle = \pi_{ij,p} \hat{\boldsymbol{p}} \tag{3.89}$$

in which we set the normalization to 1. For the gravitino two-point correlator we apply the E.O.M that $(\gamma_j \psi^j_{\boldsymbol{0}}(\vec{p})) = -\hat{\boldsymbol{p}}(\hat{p}_i \psi^i_{\boldsymbol{0}}(\vec{p}))$ then in the correlator we apply $\gamma_j \sim -\hat{\boldsymbol{p}} \hat{p}_j$.

3.3.2 3pt correlators

Procedure to obtain the 3pt correlators:

1. Decompose the correlator and use Ward-Takahashi identity (We list all of these in §App. E.2) to determine the longitudinal part. And there should not have a total energy pole in the longitudinal part.

2. Apply total energy pole condition to identify the transversal part correlator to be $\frac{M_{3,Lorentz}}{K_T}$, in which the $M_{3,Lorentz}$ is the amplitude with polarization in the Lorentz gauge, said $\epsilon_i p^i = 0$.

3. Find if we could write an unfix polynomial ansatz by momentum dimension counting.

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3.3.3 4pt correlators

1. Decompose the correlator and use the Ward-Takahashi identity to determine the longitudinal part. And check that it indeed satisfies the partial energy pole's residue. There should not have a total energy pole in this part.

2. We decompose the correspondent amplitude channel by channel, and the numerator in the s, t, u channel is in the certain factorization form we choose and with the respondent contact term we choose.

$$M_4 = \frac{V_{3,L,s} \bigotimes V_{3,R,s}}{S} + \frac{V_{3,L,t} \bigotimes V_{3,R,t}}{T} + \frac{V_{3,L,u} \bigotimes V_{3,R,u}}{U} + (\text{certain contact term, polynomial}) \tag{3.90}$$

in which we use $V_{L/R}$ to denote the left and the right vertices.

3. The partial energy pole residue for the bosonic transverse correlator will be

$$\begin{aligned} \underset{E_L=0}{\text{Res}}\, \psi^T_{4,boson} &= \frac{1}{2E_s} V_{3,L} \otimes' \tilde{\psi}^T_R = V_{3,L} \otimes' V_{3,R} \cdot \frac{1}{2E_s} (\frac{1}{E_L}|^{E_s}_{-E_s}) \\ \underset{E_R=0}{\text{Res}}\, \psi^T_{4,boson} &= \frac{1}{2E_s} \tilde{\psi}^T_L \otimes' V_{3,R} = V_{3,L} \otimes' V_{3,R} \cdot \frac{1}{2E_s} (\frac{1}{E_R}|^{E_s}_{-E_s}) \end{aligned} \tag{3.91}$$

where the $\psi^T_{4,s}$ is the pure transverse correlator. On the other hand, we find the fermionic correlator, the gluing factor of residues of individual partial energy poles are different.

$$\begin{aligned} \underset{E_L=0}{\text{Res}}\, \psi^T_{4,fermion} &= \frac{1}{2E_s} V_{3,L} \otimes'_L \tilde{\psi}^T_R = V_{3,L} \otimes'_L V_{3,R} \cdot \frac{1}{2E_s} (\frac{1}{E_L}|^{E_s}_{-E_s}) \\ \underset{E_R=0}{\text{Res}}\, \psi^T_{4,fermion} &= \frac{1}{2E_s} \tilde{\psi}^T_L \otimes'_R V_{3,R} = V_{3,L} \otimes'_R V_{3,R} \cdot \frac{1}{2E_s} (\frac{1}{E_R}|^{E_s}_{-E_s}) \end{aligned} \tag{3.92}$$

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4. Use the fact that

$$\lim_{K_T\to 0}\frac{1}{E_{12s}E_{34s}} = \frac{1}{(E_s - E_{34})E_{34s}} = -\frac{1}{S}$$
$$\lim_{E_{12s}\to 0}\frac{1}{K_T E_{34s}} = \frac{(-1)}{2E_s}\left[\frac{1}{E_{34s}} - \frac{1}{E_3+E_4-E_s}\right] \quad (3.93)$$
$$\lim_{E_{34s}\to 0}\frac{1}{K_T E_{12s}} = \frac{-1}{2E_s}\left[\frac{1}{E_{12s}} - \frac{1}{E_{12}-E_s}\right].$$

Then we could build up the pure transverse correlator channel by channel,

$$\psi_4^T = \psi_{4,s}^T + \psi_{4,t}^T + \psi_{4,u}^T + \psi_{4,c}^T. \quad (3.94)$$

And for the s-channel part, we could build it by

$$\psi_{4,s}^T = \frac{-N_s}{K_T E_R E_L} \quad (3.95)$$

and for the bosonic correlator, we could only build a N_s such that

$$\lim_{K_T\to 0} N_{s,boson} = V_{3,L,s}\otimes V_{3,R,s}$$
$$\lim_{E_R/E_L\to 0} N_{s,boson} = V_{3,L,s}\otimes' V_{3,R,s} \quad (3.96)$$

we accomplish this by writing a form that

$$N_s = V_{3,L,s}\otimes V_{3,R,s} + n_k K_T\text{"} = \text{"}V_{3,L,s}\otimes' V_{3,R,s} + n_p E_L E_R \quad (3.97)$$

in which the second equality is in general nontrivial but we'll prove in the context. And for the fermionic field, the residues of the two partial energy poles are not

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symmetric. Then we need to find N_s such that

$$\lim_{K_T\to 0} N_s = V_{3,L,s} \otimes V_{3,R,s}$$
$$\lim_{E_R\to 0} N_s = V_{3,L,s} \otimes'_R V_{3,R,s} \tag{3.98}$$
$$\lim_{E_L\to 0} N_s = V_{3,L,s} \otimes'_L V_{3,R,s}$$

we accomplish this by writing a form that

$$\begin{aligned} N_s &= V_{3,L,s} \otimes V_{3,R,s} + n_k K_T \\ \text{"} &= \text{"}V_{3,L,s} \otimes'_R V_{3,R,s} + n_{p,R} E_R\text{"} = \text{"}V_{3,L,s} \otimes'_L V_{3,R,s} + n_{p,L} E_L \end{aligned} \tag{3.99}$$

in which the second and third equality is in general nontrivial but we'll prove it in the next section.

5. Sometimes, the correlator built by the previous step will violate the COT with a polynomial term $\Delta_{COT,poly}$ vanishing on all the residue, and in our case, it's invariant under the flipping sign of external energy. Then we could make

$$\psi^T_{4,s} = \frac{-N_s}{K_T E_R E_L} - \Delta_{COT,poly} \tag{3.100}$$

to restore the COT.

6. Use dimension analysis to find if there's an unfixed polynomial term we could not be constrained by COT and the residue of the singularities. The remaining unfix polynomial term will reflect the fact that we don't add a requirement that the coupling is minimal.

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Chapter 4 Bosonic correlators in Flat Spacetime

In this chapter, we introduce the cosmological correlator for bosonic fields in flat spacetime. We will show how to decompose the correlator into the transverse components and the longitudinal components. The longitudinal part can be fixed by the Ward-Takahashi identity. The transversal part can be fixed by the total energy pole and partial energy pole conditions.

4.1 2pt correlators

By dimensional analysis and Ward Takahashi's identity of two-point correlator discussed in App. F.1, we can fix the 2pt correlator of scalar fields O, vector fields J^i, and graviton fields T^{ij} to be

$$\begin{aligned}\langle O_{-\boldsymbol{p}} O_{\boldsymbol{p}}\rangle &= E \\ \langle J^i_{-\boldsymbol{p}} J^j_{\boldsymbol{p}}\rangle &= E\pi_{ij} \\ \langle T^{ij}_{-\boldsymbol{p}} T^{kl}_{\boldsymbol{p}}\rangle &= E\pi_{i,k}\pi_{j,l}\end{aligned} \tag{4.1}$$

where we defined $\pi_{ij} \equiv \eta_{ij} + \frac{p_i p_j}{E_p^2}$.

4.2 3pt correlators

Procedure to obtain the 3pt correlators:

1. Decompose the correlator and use Ward-Takahashi identity (We list all of these in §App. E.2) to determine the longitudinal part.

2. Apply total energy pole condition to determine the transversal part.

4.2.1 $\langle JO^*O\rangle$

The correlator should satisfy the following two conditions: (we set coupling constant $e = 1$)

$$\begin{aligned} &\operatorname*{Res}_{K_T\to 0} \epsilon_{1,i}\langle J_1^i O_2^* O_3\rangle = \epsilon_{1,\mu}(p_2^\mu - p_3^\mu) = \epsilon_{1,i}\pi^i_{1,j}(p_2^j - p_3^j) \\ &p_i\langle J_1^i O_2^* O_3\rangle = -\langle O^*_{2+1}O_3\rangle + \langle O_2^* O_{1+3}\rangle = (E_2 - E_3) \end{aligned} \tag{4.2}$$

In the first condition, we used (E.229). We then show how to determine the correlator.

1. Determine the longitudinal part by Ward-Takahashi identity

$$\begin{aligned} \epsilon_0^i(\vec{p}) &= \epsilon_0^{i,T} + \epsilon_0^{i,L} \\ &= \pi^{ij}\epsilon_{j,0}(\vec{p}) - \hat{p}^i\hat{p}^j\epsilon_{j,0}(\vec{p}) \end{aligned} \tag{4.3}$$

$$\langle J_1 O_2^* O_3\rangle = \epsilon_i\pi^i_{1,j}\langle J_1^j O_2^* O_3\rangle - \epsilon_i\hat{p}^i\hat{p}_j\langle J_1^j O_2^* O_3\rangle = \epsilon_i\pi^i_{1,j}\langle J_1^j O_2^* O_3\rangle - \epsilon_i\frac{\hat{p}_1^i}{E_1}(E_2 - E_3) \tag{4.4}$$

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2. Determine the transversal part by the total energy pole condition

$$\underset{K_T\to 0}{\text{Res}}\,\epsilon_{1,i}\langle J_1^i O_2^* O_3\rangle = \underset{K_T\to 0}{\text{Res}}\,\epsilon_{1,i}\pi^i_{1,j}\langle J_1^j O_2^* O_3\rangle = \epsilon_{1,i}\pi^i_{1,j}(p_2^j - p_3^j) \tag{4.5}$$

3. Combining the longitudinal part and the transversal part, we get

$$\langle J_1 O_2^* O_3\rangle = \frac{\epsilon_i \pi^i_{1,j}(p_2^j - p_3^j)}{K_T} - \epsilon_i \hat{p}_1^i \frac{(E_2 - E_3)}{E_1} \tag{4.6}$$

And we can check that actually in the longitudinal part the $\frac{1}{E_1}$ co-dimension 1 pole is spurious:

$$\underset{E_1\to 0}{\text{Res}}\,\frac{E_2 - E_3}{E_1} = \frac{E_2 - E_3}{E_1}\frac{K_T}{K_T} = \frac{\hat{p}_{1,i}(p_2 - p_3)^i + (E_2 - E_3)}{K_T} = 0 \tag{4.7}$$

Another way to say this result is : If $E_1 = 0$ then $\boldsymbol{p}_2 \to -\boldsymbol{p}_3$ so we will have $E_2^2 = E_3^2$ that is $E_2 + E_3 = 0$ or $E_2 - E_3 = 0$. For $E_2 + E_3 = 0$, it seems like $\frac{(E_2 - E_3)}{E_1}$ will have a co-dimension 1 residue on E_1 but actually, now we also have $K_T = E_1 + E_2 + E_3 = 0$ so it's co-dimension 2 residue. And for $E_2 - E_3 = 0$, we know there's no residue on E_1 for the numerator is vanishing now.

4.2.2 $\langle TOO\rangle$

The correlator should satisfy the following two conditions:

$$\begin{aligned}
&\underset{K_T\to 0}{\text{Res}}\,\epsilon_i\epsilon_j\langle T_1^{ij} O_2 O_3\rangle = \left[\epsilon_\mu(p_2^\mu - p_3^\mu)\right]^2 = \left[\epsilon_i \pi^i_j (p_2^j - p_3^j)\right]^2 \\
&p_{1,i}\langle T_1^{ij} O_2 O_3\rangle = -\frac{1}{2}\left(\langle O_{1+2} O_3\rangle p_2^j + \langle O_2 O_{1+3}\rangle p_3^j\right) = -\frac{1}{2}\left(E_3 p_2^j + E_2 p_3^j\right)
\end{aligned} \tag{4.8}$$

In the first condition, we used (E.232). We then show how to determine the correlator.

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1. The decomposition of graviton boundary condition We write the boundary condition in the bi-vector form

$$h_{ij,0} = \epsilon_{i,0}\epsilon_{j,0} \tag{4.9}$$

and by the EOM $G_{00} = 0$ on the boundary as (2.28) shows

$$\pi^{ij}h_{ij,0} = 0 \to \eta_{ij}\epsilon_{i,0}\epsilon_{j,0} = -(\hat{p}^i\epsilon_{i,0})^2 \tag{4.10}$$

We can solve that

$$\epsilon_{i,0} = P_{ij}\xi^{j,0} := (\hat{p}_i\hat{p}_j + i\epsilon_{ibc}\hat{p}_b\pi_{cj} + \pi_{ij})\xi^{j,0} \tag{4.11}$$

in which $\xi_{i,0}$ is any vector, ϵ_{ijk} is the Levi-Civita symbol.Then we could decompose the ϵ to be the transverse and longitudinal mode like vector field in 4.2.

2. Determine the longitudinal part by Ward-Takahashi identity

$$\begin{aligned}\langle T_1O_2O_3\rangle &= \epsilon^{\mathrm{T}}_{1,i}\langle T^i_1O_2O_3\rangle - (\epsilon_{1,i}\hat{p}^i_1)\hat{p}_{1,j}\langle T^j_1O_2O_3\rangle \\ &= \epsilon^{\mathrm{T}}_{1,i}\epsilon^{\mathrm{T}}_{1,j}\langle T^{ij}_1O_2O_3\rangle - 2(\epsilon_{1,i}\hat{p}^i_1)(\hat{p}_{i',1}\epsilon^{\mathrm{T}}_{1,j}\langle T^{i'j}_1O_2O_3\rangle) + (\hat{p}_{i',1}\hat{p}_{j',1}\langle T^{i'j}_1O_2O_3\rangle)\end{aligned} \tag{4.12}$$

$$\begin{aligned}\langle T^{\mathrm{TT}}_1O_2O_3\rangle &:= \epsilon^{\mathrm{T}}_{1,i}\epsilon^{\mathrm{T}}_{1,j}\langle T^{ij}_1O_2O_3\rangle \\ \langle T^{\mathrm{T}L}_1O_2O_3\rangle &:= -2(\epsilon_{1,i}\hat{p}^i)(\hat{p}_{i'}\epsilon^{\mathrm{T}}_{1,j}\langle T^{i'j}_1O_2O_3\rangle) = -2(\epsilon_{1,i}\hat{p}^i_1)(\epsilon^{\mathrm{T}}_{1,j})(\frac{-1}{2E_1})(E_3p^j_2 + E_2p^j_3) \\ &= (\epsilon_{1,i}\hat{p}^i_1)(\epsilon^{\mathrm{T}}_{1,j}p^j_3)(\frac{E_2 - E_3}{E_1}) \\ \langle T^{LL}_1O_2O_3\rangle &:= (\epsilon_{1,i}\hat{p}^i_1)^2\hat{p}_{i',1}\hat{p}_{j',1}\langle T^{i'j}_1O_2O_3\rangle = \frac{1}{4}(\epsilon_{1,i}\hat{p}^i_1)^2\left((\frac{E_2 - E_3}{E_1})^2 - 1\right)(E_2 + E_3)\end{aligned} \tag{4.13}$$

by (4.7) we know all the codimension E_1 pole is spurious. And we define the single

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transverse part of the correlator for the latter convenience

$$\epsilon_{1,i,0,L}\epsilon_{1,j,0}\langle T_1^{ij}O_2O_3\rangle = \langle T_1^L O_2O_3\rangle = \frac{(-\epsilon_{1,i}\hat{p}_1^i)}{2E_1}\cdot\left[E_3(\epsilon_{1,j}p_2^j)+E_2(\epsilon_{1,j}p_3^j)\right] \tag{4.14}$$

3. Determine the transversal part by the total energy pole condition

$$\underset{K_T\to 0}{\text{Res}}\langle T_1O_2O_3\rangle = \underset{K_T\to 0}{\text{Res}}\langle T_1^{\text{TT}}O_2O_3\rangle = \left(\epsilon_{1,i}\pi_{1,j}^i(p_2^j-p_3^j)\right)^2 \tag{4.15}$$

so we have

$$\langle T_1^{\text{TT}}O_2O_3\rangle = \frac{\left(\epsilon_{1,i}\pi_{1,j}^i(p_2^j-p_3^j)\right)^2}{K_T} \tag{4.16}$$

Notice we cannot add any subleading terms like

$$(\epsilon_{1,i}^T\epsilon_{1,j}^T)\eta^{ij}\cdot Poly_1(E_1,E_2+E_3) \tag{4.17}$$

because we have (2.28) we could decompose it into longitudinal and transverse parts and for the latter one (κ is the coupling constant.)

$$(\epsilon_{1,i}^T\epsilon_{1,j}^T)\eta^{ij} = O(\kappa)\sim 0 \tag{4.18}$$

So we don't need to consider the η_{ij} or trace term in the same order as other terms in the transverse correlator, our EOM on the boundary makes it vanish. But it's not the case in dS space, because we don't have (2.28), but in addition, we have trace WT identity from Weyl invariance of the dS action [4]

$$\eta_{ij}\langle T^{ij}OO\rangle_{dS} = (3-\Delta_O)\left[\langle O_{2+1}O_3\rangle_{dS}+\langle O_2O_{1+3}\rangle_{dS}\right] \tag{4.19}$$

Δ_O is the conformal dimension of the scalar, so actually, we should decompose the

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h_{ij} more in dS to divide the trace parts determined by conformal WT. Notice now the bivector form is not ok, instead [1]

$$h_{0,ij} = \frac{1}{2}(P^{kl}_{1,1,ij} + P^{kl}_{2,1,ij} + P^{kl}_{1,2,ij} + P^{kl}_{2,2,ij})h_{0,kl} \tag{4.20}$$

in which

$$\begin{aligned}
(\mathcal{P}_{1,1})^{ij}_{kl} &= 2\pi^{(i}_{(k}\pi^{j)}_{l)} - \frac{2}{d-1}\pi^{ij}\pi_{kl} = 2\,(\Pi_{2,2})^{ij}_{kl} \\
(\mathcal{P}_{1,2})^{ij}_{kl} &= 2\pi^{(i}_{(l}\hat{k}^{j)}\hat{k}_{k)} - \frac{2}{d-1}\pi^{ij}\hat{k}_k\hat{k}_l \\
(\mathcal{P}_{2,1})^{ij}_{kl} &= 2\hat{k}^{(i}\hat{k}_{(l}\pi^{j)}_{k)} - \frac{2}{d-1}\hat{k}^i\hat{k}^j\pi_{lm} \\
(\mathcal{P}_{2,2})^{ij}_{kl} &= \frac{2(d-2)}{(d-1)}\hat{k}^i\hat{k}^j\hat{k}_k\hat{k}_l
\end{aligned} \tag{4.21}$$

such that $p^i\,(\Pi_{2,2})^{ij}_{kl} = \eta^{ij}\,(\Pi_{2,2})^{ij}_{kl} = 0$, and we could write the transverse traceless parts in bi-vector form

$$(\epsilon^T_{1,i}\epsilon^T_{1,j}) = (\Pi_{2,2})^{kl}_{ij}\,h^{kl} \tag{4.22}$$

with

$$\epsilon^T_{1,i}\epsilon^T_{1,j}\eta_{ij} = \epsilon^T_{1,i}\epsilon^T_{1,j}p_j = 0 \tag{4.23}$$

now the decomposition of the correlator will be

$$\begin{aligned}
h_{0,ij}\langle T^{ij}\dots\rangle_{dS} &= \epsilon^T_{1,i}\epsilon^T_{1,j}\langle T^{ij}\rangle_{dS} + 2h_{kl,0}\pi^{(i}_l\hat{k}^{j)}\hat{k}_k\langle T^{ij}\dots\rangle_{dS} \\
&+ \left(\frac{2d-3}{d-1}h^{kl,0}\hat{k}_l\hat{k}_k - \frac{1}{d-1}h^{kl,0}\pi^{kl}\right)\hat{k}_i\hat{k}_j\langle T^{ij}\dots\rangle_{dS} \\
&+ \frac{1}{d-1}h^{ij,0}\hat{k}_i\hat{k}_j\delta^{kl}\langle T^{kl}\dots\rangle_{dS} \\
&= \langle T^{TT}\rangle_{dS} + \langle T^{LT}\rangle_{dS} + \langle T^{LL}\rangle_{dS} + \langle T^{Trace}\rangle_{dS}
\end{aligned} \tag{4.24}$$

the last term can be determined by trace WT like (4.19).

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4. Combining the longitudinal part and the transversal part, we get

$$\langle TOO\rangle = \langle T_1^{\mathrm{TT}} O_2 O_3\rangle + \langle T_1^{\mathrm{TL}} O_2 O_3\rangle + \langle T_1^{LL} O_2 O_3\rangle \tag{4.25}$$

4.2.3 $\langle TTT\rangle$

The correlator should satisfy the following two conditions:

$$\begin{aligned}
\operatorname*{Res}_{K_T\to 0}\langle T_1T_2T_3\rangle &= M(h_1h_2h_3)\\
&= \left[(\epsilon_1\cdot\epsilon_2)\left(\epsilon_3\cdot(p_1-p_2)\right)+(\epsilon_2\cdot\epsilon_3)\left(\epsilon_1\cdot(p_2-p_3)\right)+(\epsilon_3\cdot\epsilon_1)\left(\boldsymbol{\epsilon}_2\cdot(p_3-p_1)\right)\right]^2\\
&= \Big((\boldsymbol{\epsilon}_{1,T,0}\cdot\boldsymbol{\epsilon}_{2,T,0})\left(\boldsymbol{\epsilon}_{3,T,0}\cdot(\boldsymbol{p}_1-\boldsymbol{p}_2)\right)+(\boldsymbol{\epsilon}_{2,T,0}\cdot\boldsymbol{\epsilon}_{3,T,0})\left(\boldsymbol{\epsilon}_{1,T,0}\cdot(\boldsymbol{p}_2-\boldsymbol{p}_3)\right)\\
&\quad+(\boldsymbol{\epsilon}_{3,T,0}\cdot\boldsymbol{\epsilon}_{1,T,0})\left(\boldsymbol{\epsilon}_{2,T,0}\cdot(\boldsymbol{p}_3-\boldsymbol{p}_1)\right)\Big)^2\\
p_{1,i}\epsilon_{1,j,0}\langle T_1^{ij}T_2T_3\rangle &= (\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{\epsilon}_{2,0})\,p_{2,k}\epsilon_{2,l}\langle T_{1+2}^{kl}T_3\rangle-\frac{1}{2}(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{p}_2)\epsilon_{2,0,k}\epsilon_{2,0,l}\langle T_{1+2}^{kl}T_3\rangle\\
&\quad+(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{\epsilon}_{3,0})\,p_{3,k}\epsilon_{3,l}\langle T_2T_{3+1}^{kl}\rangle-\frac{1}{2}(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{p}_3)\epsilon_{3,0,k}\epsilon_{3,0,l}\langle T_2T_{3+1}^{kl}\rangle
\end{aligned} \tag{4.26}$$

In the first condition, we used (E.232). We then show the pure transverse correlator will be:

$$\begin{aligned}
\langle T_1^{TT}T_2^{TT}T_3^{TT}\rangle = \frac{1}{K_T}\ &\Big((\boldsymbol{\epsilon}_{1,T,0}\cdot\boldsymbol{\epsilon}_{2,T,0})\left(\boldsymbol{\epsilon}_{3,T,0}\cdot(\boldsymbol{p}_1-\boldsymbol{p}_2)\right)+(\boldsymbol{\epsilon}_{2,T,0}\cdot\boldsymbol{\epsilon}_{3,T,0})\left(\boldsymbol{\epsilon}_{1,T,0}\cdot(\boldsymbol{p}_2-\boldsymbol{p}_3)\right)\\
&+(\boldsymbol{\epsilon}_{3,T,0}\cdot\boldsymbol{\epsilon}_{1,T,0})\left(\boldsymbol{\epsilon}_{2,T,0}\cdot(\boldsymbol{p}_3-\boldsymbol{p}_1)\right)\Big)^2
\end{aligned} \tag{4.27}$$

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And the Longitudinal parts of the correlator in $\langle TTT\rangle$ are determined by WT identity,

$$\epsilon_{1,i,0,L}\epsilon_{1,j,0}\langle T_1^{ij}T_2T_3\rangle = \langle T_1^L T_2 T_3\rangle$$
$$= \frac{(-\epsilon_{1,i}\hat{p}_1^i)}{E_1}\cdot\Big[(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{\epsilon}_{2,0})\,p_{2,k}\epsilon_{2,l}\langle T_{1+2}^{kl}T_3\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{p}_2)\epsilon_{2,0,k}\epsilon_{2,0,l}\langle T_{1+2}^{kl}T_3\rangle$$
$$+(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{\epsilon}_{3,0})\,p_{3,k}\epsilon_{3,l}\langle T_2T_{3+1}^{kl}\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{p}_3)\epsilon_{3,0,k}\epsilon_{3,0,l}\langle T_2T_{3+1}^{kl}\rangle\Big] \tag{4.28}$$

4.3 4pt correlators

Procedure to obtain the 4pt correlators:

1. Decompose the correlator and use the Ward-Takahashi identity (We list all of these in §App. E.2) to determine the longitudinal part. And check that the longitudinal part of the correlator determined by step 1 indeed satisfies the partial energy pole's residue.

2. We decompose the correspondent amplitude channel by channel and the numerator in the s, t, u channel is in the factorization form.

$$M_4 = \frac{M_3\bigotimes M_3}{S} + \frac{M_3\bigotimes M_3}{T} + \frac{M_3\bigotimes M_3}{U} + \text{(contact term(polynomial))} \tag{4.29}$$

in which the $\bigotimes$ means two three-point amplitudes are glued with the sum of the internal polarization vector. It's easier to start from a factorized amplitude to match the partial energy pole residue which is also factorized.

3. Apply total energy pole condition and determine the transverse parts of correlator up to $\mathcal{O}(K_T^0)$.

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4. Apply partial energy pole condition on each channel and determine the transverse parts of the correlator. If the right and left sides of the factorization channel are symmetric, we only need to consider the condition on one side, and the other will be automatically satisfied.

5. Determine the transverse parts of correlator b y requiring $\mathcal{O}(K_T^0)$ and $\mathcal{O}(E_L^0)$ on each channel in the previous steps being consistent.

Notice in the end, by momentum dimension counting (We only count the dimension of " momentum and energy" but do not include the dimension of boundary condition and coupling constant, this counting should be the same for every term in the correlator), we could know is there any term like $\mathcal{O}(K_T^0 E_L^0 E_R^0)$ without any pole so it cannot be constrained by residue. (Or, equivalently, we could say that for the contact term if the $\mathcal{O}(K_T^0)$ term is tolerated by momentum dimension counting?) If there's a unfix term, we need to constrain our correlator more with full Optical theorem, and for minimal gravity and scalar, we need to impose a soft limit for the correlator to fix the remaining unfix terms to ensure minimal coupling theory.

If by the dimension counting, there could be no $\mathcal{O}(K_T^0 E_R^0 E_L^0)$ because a term like this should be composed by momentum norm's contraction and energy, in which only spurious pole is tolerated, its energy dimension should be bigger than zero. So the energy dimension of the correlator is negative, there should not be any unfix term.

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4.3.1 $\langle OO^*OO^* \rangle$ exchanging photon

Step 1

The $\langle OO^*OO^* \rangle$ amplitude in the form of factorization numerator in s, t, u channel:

$$M\left(\phi_1\phi_2^*\phi_3\phi_4^*\right) = \frac{1}{S}(p_2 - p_1)\cdot(p_4 - p_3) + \frac{1}{T}(p_2 - p_3)\cdot(p_4 - p_1)$$

Step 2

To simplify the equations, we consider only the s-channel, $\langle O_1O_2^*O_3O_4^* \rangle = \langle O_1O_2^*O_3O_4^* \rangle_s +$ $\langle O_1O_2^*O_3O_4^* \rangle_t$. The computation of the t-channel is similar. The correlator should satisfy the following conditions: ($K_T = E_{1234}; E_L = E_{12s}; E_R = E_{34s}; \boldsymbol{p}_s = \boldsymbol{p}_3 + \boldsymbol{p}_4$)

$$\begin{aligned}
&\operatorname*{Res}_{K_T\to 0}\langle O_1O_2^*O_3O_4^* \rangle_s = M\left(\phi_1\phi_2^*\phi_3\phi_4^*\right) = \frac{(p_2-p_1)^\mu (p_4-p_3)_\mu}{S} \\
&\operatorname*{Res}_{E_L\to 0}\langle O_1O_2^*O_3O_4^* \rangle_s = \tilde{M}^i(\phi_1\phi_2^*\gamma_{-s})\frac{\pi_{s,ij}}{2E_s}\tilde{\psi}^j_{J_sO_3O_4} = (p_2-p_1)^i\cdot\frac{\pi_{s,ij}}{2E_s}\left[\frac{(p_4-p_3)^j}{E_R} - \frac{(p_4-p_3)^j}{E_3+E_4-E_s}\right],
\end{aligned} \tag{4.30}$$

in which $\frac{\pi_{s,ij}}{2E_s} = \langle J_i(-\boldsymbol{p_s})J_j(\boldsymbol{p_s})\rangle_{in-in}$ (if we drop the vanishing gauge term which won't contribute to the residue). The first condition implies

$$\langle O_1O_2^*O_3O_4^* \rangle_s = -\frac{(p_2-p_1)^\mu (p_4-p_3)_\mu}{K_T E_L E_R} - \sum_{n=0}^{\infty} c_n^{(T)}(K_T)^n \tag{4.31}$$

while the second condition implies

$$\langle O_1O_2^*O_3O_4^* \rangle_s = -\frac{(p_2-p_1)^i \pi_{s,ij}(p_4-p_3)^j}{K_T E_L E_R} - \sum_{n=0}^{\infty} c_n^{(L)}(E_L)^n \tag{4.32}$$

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Subtracting (4.31) and (4.32), we get

$$0 = \frac{-\frac{(E_2-E_1)(E_4-E_3)K_T}{E_s} + \frac{(E_2-E_1)(E_4-E_3)E_L E_R}{E_s^2}}{K_T E_L E_R} + \sum_{n=0}^{\infty} c_n^{(T)}(K_T)^n - \sum_{n=0}^{\infty} c_n^{(L)}(E_L)^n \tag{4.33}$$

where we used the identity

$$\begin{aligned}&(E_2 - E_1)(E_3 - E_4) - ((\mathbf{p}_2 - \mathbf{p}_1)\cdot\hat{\mathbf{s}})((\mathbf{p}_4 - \mathbf{p}_3)\cdot\hat{\mathbf{s}}) \\ &= -\frac{(E_2 - E_1)(E_4 - E_3)K_T}{E_s} + \frac{(E_2 - E_1)(E_4 - E_3)E_L E_R}{E_s^2}\end{aligned} \tag{4.34}$$

From (4.33), we can see that we have to choose

$$\begin{aligned}\sum_{n=0}^{\infty} c_n^{(T)}(K_T)^n &= \frac{(E_2 - E_1)(E_4 - E_3)}{E_s E_L E_R} \\ \sum_{n=0}^{\infty} c_n^{(L)}(E_L)^n &= \frac{(E_2 - E_1)(E_4 - E_3)}{E_s^2 K_T}\end{aligned} \tag{4.35}$$

therefore,

$$\langle O_1 O_2^* O_3 O_4^* \rangle_s = -\frac{(p_2 - p_1)^\mu (p_4 - p_3)_\mu}{K_T E_L E_R} - \frac{(E_2 - E_1)(E_4 - E_3)}{E_s E_L E_R} \tag{4.36}$$

$$= -\frac{(p_2 - p_1)^i \pi_{ij} (p_4 - p_3)^j}{K_T E_L E_R} - \frac{(E_2 - E_1)(E_4 - E_3)}{E_s^2 K_T} \tag{4.37}$$

so we could write down the full answer:

$$\begin{aligned}\langle O_1 O_2^* O_3 O_4^* \rangle = &-\frac{(p_2 - p_1)_\mu \eta^{\mu\nu} (p_4 - p_3)_\nu + K_T \cdot \frac{(E_2-E_1)(E_4-E_3)}{E_s}}{E_{12s} E_{34s} K_T} \\ &- \frac{(p_4 - p_1)_\mu \eta^{\mu\nu} (p_2 - p_3)_\nu + K_T \cdot \frac{(E_4-E_1)(E_2-E_3)}{E_t}}{E_{14t} E_{23t} K_T}\end{aligned} \tag{4.38}$$

Because the energy dimension of $\langle O_1 O_2^* O_3 O_4^* \rangle$ is (-1),there should not be any unfix term

.

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In the end, we'll show that the $\frac{1}{E_s}$ co-dimension 1 pole is spurious (or we'll say it's co-dimension 3 pole with partial energy pole kinematics). Notice, when $E_s \to 0$, we'll have $p_1 + p_2 = 0 = p_3 + p_4$, and we want to avoid partial energy pole kinematics so $E_L = E_{34} \neq 0$ and $E_R = E_{12} \neq 0$ should be satisfied, now we only have $E_1 - E_2 = 0$ with $E_3 - E_4 = 0$ which make $\text{Res}_{E_s \to 0} \frac{(E_2-E_1)(E_3-E_4)}{E_s} = 0$, the co-dimension 1 residue vanishing. And at the same time, we could show that this term only has residue when $E_s = E_L = E_{34} = E_R = E_{12} = 0$. It's a co-dimension 3 residue.

4.3.2 $\langle JOJO^* \rangle$

Step 1

Boundary condition decomposition : we can decompose $\langle JOJO^* \rangle$ like:

$$
\begin{aligned}
\langle J_1 O_2^* J_3 O_4 \rangle &= \epsilon_{1,i}\epsilon_{3,j} \langle J_1^i O_2^* J_3^j O_4 \rangle \\
&= \epsilon_{1,i}^{\mathrm{T}}\epsilon_{3,j}^{\mathrm{T}} \langle J_1^i O_2^* J_3^j O_4 \rangle + \epsilon_{1,i}^{L}\epsilon_{3,j}^{\mathrm{T}} \langle J_1^i O_2^* J_3^j O_4 \rangle + \epsilon_{1,i}^{\mathrm{T}}\epsilon_{3,j}^{L} \langle J_1^i O_2^* J_3^j O_4 \rangle + \epsilon_{1,i}^{L}\epsilon_{3,j}^{L} \langle J_1^i O_2^* J_3^j O_4 \rangle \\
&=: \langle J_1^{\mathrm{T}} O_2^* J_3^{\mathrm{T}} O_4 \rangle + \langle J_1^{L} O_2^* J_3^{\mathrm{T}} O_4 \rangle + \langle J_1^{\mathrm{T}} O_2^* J_3^{L} O_4 \rangle + \langle J_1^{L} O_2^* J_3^{L} O_4 \rangle
\end{aligned}
$$

all the terms except the first one are determined by WT identity. So we try to find the form of the first term in Step 2.

Now because the longitudinal parts are determined by the WT identity. We only need to check these parts indeed satisfy the partial energy pole residue condition. Equivalently, we'll say the WT for photon labeled as 1:

$$p_1^i \langle J_{1,i} O_2^* J_{3,j} O_4 \rangle = -\langle O_{1+2}^* J_{3,j} O_4 \rangle + \langle O_2^* J_{3,j} O_{1+4} \rangle \tag{4.39}$$

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$$p_1^i\langle J_{1,i}O_2^*O_s\rangle = -\langle O_{1+2}^*O_s\rangle + \langle O_2^*O_{1+s}\rangle = (E_2 - E_s) \tag{4.40}$$

is consistent with the partial energy pole residue for the longitudinal parts of the photon labeled as 1:

$$\operatorname*{Res}_{E_{12s}\to 0}\langle J_1^{\mathrm{L}}O_2^*J_3O_4\rangle = M_{\gamma_L\phi\phi^*}\cdot\tilde{\Psi}_{JO^*O} = 0 \tag{4.41}$$

$$\begin{aligned}
&\operatorname*{Res}_{E_{34s}\to 0}\langle J_1^{\mathrm{L}}O_2^*J_3O_4\rangle = \tilde{\Psi}_{J_LO^*O}\cdot M_{\gamma\phi\phi^*}\\
&= \frac{\epsilon_1^i\hat{p}_i\cdot(\frac{-1}{E_1})}{2E_s}\left[p_{1,j}\langle J_1O_2^*O_s(E_1,E_2,E_s)\rangle - p_{1,j}\langle J_1O_2^*O_s(E_1,E_2,-E_s)\rangle\right](-2\epsilon_{3i}p_4^i)\\
&= \frac{(\epsilon_1^i\hat{p}_i)}{E_1}\cdot(-2\epsilon_{3i}p_4^i)
\end{aligned} \tag{4.42}$$

in which we use the Ward identity of the amplitude

$$M_{\gamma_{1,L}\phi\phi^*} = \hat{p}_{1,i}\tilde{M}^i_{\gamma_{1,L}\phi\phi^*} = (E_1)^{-1}p_{\mu,1}M^\mu_{\gamma_{1,L}\phi\phi^*} = 0 \tag{4.43}$$

So now we need to get the longitudinal parts by WT identity and take the partial energy pole residue like

$$\begin{aligned}
\operatorname*{Res}_{E_{12s}\to 0}\langle J_1^{\mathrm{L}}O_2^*J_3O_4\rangle &= \operatorname*{Res}_{E_{12s}\to 0}\left[(\epsilon_1^i\hat{p}_i)\cdot\frac{-1}{E_1}\cdot\left(p_1^i\langle J_{1,i}O_2^*J_{3,j}O_4\rangle\right)\right]\\
&= \operatorname*{Res}_{E_{12s}\to 0}\left[(\epsilon_1^i\hat{p}_i)\cdot\frac{-1}{E_1}\cdot\left(-\langle O_{1+2}^*J_{3,j}O_4\rangle + \langle O_2^*J_{3,j}O_{1+4}\rangle\right)\right]\\
&= 0
\end{aligned}$$

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$$\underset{E_{34s}\to 0}{\text{Res}}\langle J_1^{\text{L}} O_2^* J_3 O_4 \rangle = \underset{E_{34s}\to 0}{\text{Res}} \left[(\epsilon_1^i \hat{p}_i) \cdot \frac{-1}{E_1} \cdot \left(p_1^i \langle J_{1,i} O_2^* J_{3,j} O_4 \rangle \right) \right]$$
$$= \underset{E_{34s}\to 0}{\text{Res}} \left[(\epsilon_1^i \hat{p}_i) \cdot \frac{-1}{E_1} \cdot \left(-\langle O_{1+2}^* J_{3,j} O_4 \rangle + \langle O_2^* J_{3,j} O_{1+4} \rangle \right) \right]$$
$$= \frac{(\epsilon_1^i \hat{p}_i)}{E_1} \cdot (-2\epsilon_{3i} p_4^i)$$

Then we have checked that (4.41) and (4.42) are satisfied.

We notice this consistency is also applied to other partial energy pole residues if we do permutation $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ and it's similar to prove the consistency between longitudinal parts and WT identity of the photon labeled as 3. And it's also easy to see that in the longitudinal part $\frac{1}{E_1}$ co-dimension 1 pole is spurious, notice that when $E_1 \to 0$ then $p_1 \to 0$:

$$\underset{E_1\to 0}{\text{Res}}\langle J_1^{\text{L}} O_2^* J_3 O_4 \rangle = \underset{E_{34s}\to 0}{\text{Res}} \left[(\epsilon_1^i \hat{p}_i) \cdot \frac{-1}{E_1} \cdot \left(-\langle O_2^* J_{3,j} O_4 \rangle + \langle O_2^* J_{3,j} O_4 \rangle \right) \right]$$
$$= 0$$

Step 2

$$\langle J_1^{\text{T}} O_2^* J_3^{\text{T}} O_4 \rangle = \langle J_1^{\text{T}} O_2^* J_3^{\text{T}} O_4 \rangle_s + \langle J_1^{\text{T}} O_2^* J_3^{\text{T}} O_4 \rangle_t + \langle J_1^{\text{T}} O_2^* J_3^{\text{T}} O_4 \rangle_c \tag{4.44}$$

To simplify calculations, we only consider the partial energy pole of the s-channel. The computation for the t-channel is similar. The correlator should satisfy the following con-

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ditions:

$$\underset{K_T\to 0}{\text{Res}}\langle J_1^{\text{T}} O_2^* J_3^{\text{T}} O_4\rangle_s = M_{\gamma\phi\gamma\phi^*} = -4\frac{\epsilon_{1i}^{\text{T}} p_2^i p_4^j \epsilon_{3j}^{\text{T}}}{S} - 4\frac{\epsilon_{1i}^{\text{T}} p_4^i p_2^j \epsilon_{3j}^{\text{T}}}{T} - 2\epsilon_1^{\text{T}i}\epsilon_{3i}^{\text{T}}$$

$$\underset{E_{12s}\to 0}{\text{Res}}\langle J_1^{\text{T}} O_2^* J_3^{\text{T}} O_4\rangle_s = M_{\gamma\phi\phi^*}\cdot\frac{1}{2E_s}\cdot\tilde{\psi}_{JOO^*} = 2\epsilon_{1i}^{\text{T}} p_2^i\cdot\frac{(-2\epsilon_{3j}^{\text{T}} p_4^j)}{2E_s}\left[\frac{1}{E_{34s}} - \frac{1}{E_3+E_4-E_s}\right] \tag{4.45}$$

it's trivial to find that the following form satisfies all the constraints

$$\langle J_1^{\text{T}} O_2^* J_3^{\text{T}} O_4\rangle = 4\frac{\epsilon_{1i}^{\text{T}} p_2^i p_4^j \epsilon_{3j}^{\text{T}}}{E_{12s}E_{34s}K_T} + 4\frac{\epsilon_{1i}^{\text{T}} p_4^i p_2^j \epsilon_{3j}^{\text{T}}}{E_{14t}E_{23t}K_T} - \frac{2\epsilon_1^{\text{T}i}\epsilon_{3i}^{\text{T}}}{K_T} \tag{4.46}$$

if we apply the trivial identity

$$\lim_{K_T\to 0}\frac{1}{E_{12s}E_{34s}} = \frac{1}{(E_s-E_{34})E_{34s}} = -\frac{1}{S} \tag{4.47}$$

$$\lim_{E_{12s}\to 0}\frac{1}{K_T E_{34s}} = \frac{(-1)}{2E_s}\left[\frac{1}{E_{34s}} - \frac{1}{E_3+E_4-E_s}\right] \tag{4.48}$$

and the momentum counting of the $\langle JO^*JO\rangle$ is (-1). So (4.46) is a unique form for the transverse correlator.

4.3.3 $\langle OOOO\rangle$ exchanging graviton

Step 1

The $\langle OOOO\rangle$ amplitude in the form of factorization numerator in s,t,u channel:[1]

$$M_{Gravity}(\phi_1\phi_2\phi_3\phi_4) = \frac{1}{S}[(p_2-p_1)\cdot(p_4-p_3)]^2 + \frac{1}{T}[(p_2-p_3)\cdot(p_4-p_1)]^2 + \frac{1}{U}[(p_2-p_4)\cdot(p_1-p_3)]^2 \tag{4.49}$$

[1] By double copy $M_{Gravity}(\phi\phi\phi\phi) = M_{photon}^2(\phi^*\phi\phi^*\phi)$ or the $V_L^{\mu\nu}\eta_{\mu\nu}\eta_{\mu'\nu'}V_R^{\mu'\nu'}$ term sum on all the channel should be proportional to $S+T+U$ then it's equal to zero and vanishes.

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Step 2

We decompose the correlator to three channel parts like $\langle O_1 O_2^* O_3 O_4^* \rangle = \langle O_1 O_2^* O_3 O_4^* \rangle_s +$ $\langle O_1 O_2^* O_3 O_4^* \rangle_t + \langle O_1 O_2^* O_3 O_4^* \rangle_u$. And without loss of the generality, we demonstrate the calculation for the s-channel. The residue of the total and partial energy pole will be

$$\underset{K_T \to 0}{\text{Res}} \langle O_1 O_2 O_3 O_4 \rangle_s = M(\phi_1 \phi_2 \phi_3 \phi_4) = \frac{((p_2 - p_1)^\mu (p_4 - p_3)_\mu)^2}{S} \tag{4.50}$$

$$\begin{aligned}
\underset{E_{12s} \to 0}{\text{Res}} \langle O_1 O_2 O_3 O_4 \rangle_s &= \tilde{M}^{ii'}(\phi_1 \phi_2 h{-}s) \frac{\Pi^{iji'j'}_{s,(2,2)}}{2E_s} \tilde{\psi}^{jj'}_{T_s O_3 O_4} \\
&= \tilde{M}^{ii'}(\phi_1 \phi_2 h{-}s) \frac{\pi_{s,ij}\pi_{s,i'j'}}{2E_s} \tilde{\psi}^{jj'}_{T_s O_3 O_4} - \frac{1}{2} \tilde{M}^{ii'}(\phi_1 \phi_2 h{-}s) \frac{\pi_{s,ii'}\pi_{s,jj'}}{2E_s} \tilde{\psi}^{jj'}_{T_s O_3 O_4} \\
&= (p_2 - p_1)^i (p_2 - p_1)^{i'} \cdot \frac{\pi_{s,ij}\pi_{s,i'j'}}{2E_s} \\
&\cdot \left[\frac{(p_4 - p_3)^j (p_4 - p_3)^{j'}}{E_{34s}} - \frac{(p_4 - p_3)^j (p_4 - p_3)^{j'}}{E_3 + E_4 - E_s} \right] \\
&- \frac{1}{2} \tilde{M}^{ii'}(\phi_1 \phi_2 h{-}s) \frac{\pi_{s,ii'}\pi_{s,jj'}}{2E_s} \tilde{\psi}^{jj'}_{T_s O_3 O_4} \\
&= \frac{((p_2 - p_1)^i \cdot \pi_{s,ij} \cdot (p_4 - p_3)^j)^2}{2E_s} \left[\frac{1}{E_{34s}} - \frac{1}{E_3 + E_4 - E_s} \right] \\
&- \frac{1}{2} \frac{((p_2 - p_1)^i \cdot \pi_{s,ij} \cdot (p_2 - p_1)^j)((p_4 - p_3)^i \cdot \pi_{s,ij} \cdot (p_4 - p_3)^j)}{2E_s} \\
&\cdot \left[\frac{1}{E_{34s}} - \frac{1}{E_3 + E_4 - E_s} \right] \\
&= \frac{-((p_2 - p_1)^i \cdot \pi_{s,ij} \cdot (p_4 - p_3)^j)^2}{K_T E_{34s}} \\
&- \frac{1}{2} \frac{-((p_2 - p_1)^i \cdot \pi_{s,ij} \cdot (p_2 - p_1)^j)((p_4 - p_3)^i \cdot \pi_{s,ij} \cdot (p_4 - p_3)^j)}{K_T E_{34s}},
\end{aligned} \tag{4.51}$$

We could inspired by the $\langle OO^*OO^* \rangle$ step 2 calculation we could match the first term of the partial energy pole residue to the amplitude by the equivalence from (4.36) and

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(4.37)

$$(p_2-p_1)^\mu(p_4-p_3)_\mu+K_T\frac{(E_2-E_1)(E_4-E_3)}{E_s} \tag{4.52}$$

$$=(p_2-p_1)^i\pi_{s,ij}(p_4-p_3)^j+E_{12s}E_{34s}\frac{(E_2-E_1)(E_4-E_3)}{E_s^2} \tag{4.53}$$

and we're left with the term could be rewrite into the term vanishing at K_T and vanishing at each partial energy pole, by the quick idnetification to the follow contraction

$$\begin{aligned}(p_2-p_1)^\mu(p_2-p_1)_\mu&=-(p_1+p_2)^\mu(p_1+p_2)_\mu=E_{12s}(-K_T+E_{34s})\\&=(E_2-E_1)^2+(p_2-p_1)_i\pi_s^{ij}(p_2-p_1)_j-(p_2-p_1)_i\hat{p}_s^i\hat{p}_s^j(p_2-p_1)_j\\&=(p_2-p_1)_i\pi_s^{ij}(p_2-p_1)_j+[(E_2-E_1)^2-\frac{(E_2^2-E_1^2)^2}{E_s^2}]\\&=(p_2-p_1)_i\pi_s^{ij}(p_2-p_1)_j+(\frac{E_2-E_1}{E_s})^2[E_s^2-E_{12}^2]\end{aligned} \tag{4.54}$$

$$[E_s^2-E_{12}^2][E_s^2-E_{34}^2]=(E_{12s}E_{34s})^2-K_T^2E_s^2+K_TE_s(E_s^2+E_{12}E_{34}) \tag{4.55}$$

$$\begin{aligned}&((p_1-p_2)_i\pi_s^{ij}(p_1-p_2)_j)((p_4-p_3)_i\pi_s^{ij}(p_4-p_3)_j)\\&=\left(E_{12s}(-K_T+E_{34s})-(\frac{E_2-E_1}{E_s})^2[E_s^2-E_{12}^2]\right)\left(E_{34s}(-K_T+E_{12s})-(\frac{E_4-E_3}{E_s})^2[E_s^2-E_{34}^2]\right)\\&=-2E_sK_TE_{12s}E_{34s}+E_{12s}^2E_{34s}^2+K_T[(E_{34s})(\frac{E_2-E_1}{E_s})^2(E_s^2-E_{12}^2)+(E_{12s})(\frac{E_4-E_3}{E_s})^2(E_s^2-E_{34}^2)]\\&-E_{12s}E_{34s}[(\frac{E_2-E_1}{E_s})^2(E_s^2-E_{12}^2)+(\frac{E_4-E_3}{E_s})^2(E_s^2-E_{34}^2)]\\&+(\frac{E_2-E_1}{E_s})^2(\frac{E_4-E_3}{E_s})^2(E_{12s}^2E_{34s}^2-K_T^2E_s^2+K_TE_s(E_s^2+E_{12}E_{34}))\end{aligned} \tag{4.56}$$

we could rewrite to make the match explicit

$$\boxed{\begin{aligned}&((p_1-p_2)_i\pi_s^{ij}(p_1-p_2)_j)((p_4-p_3)_i\pi_s^{ij}(p_4-p_3)_j)\\&+E_{12s}E_{34s}\Pi^C_{1,OOOO}+E_{12s}^2E_{34s}^2\Pi^C_{2,OOOO}=K_TT^C_{OOOO}\end{aligned}} \tag{4.57}$$

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in which we define the completion term

$$\Pi^C_{1,OOOO} = -(\frac{E_2 - E_1}{E_s})^2(E_s^2 - E_{12}^2) - (\frac{E_4 - E_3}{E_s})^2(E_s^2 - E_{34}^2)$$
$$\Pi^C_{2,OOOO} = -\left(1 + (\frac{E_2 - E_1}{E_s})^2(\frac{E_4 - E_3}{E_s})^2\right)$$
$$T^C_{OOOO} = \big[-2E_sE_{12s}E_{34s} + (E_{34s})(\frac{E_2 - E_1}{E_s})^2(E_s^2 - E_{12}^2) + (E_{12s})(\frac{E_4 - E_3}{E_s})^2(E_s^2 - E_{34}^2) + (\frac{E_2 - E_1}{E_s})^2(\frac{E_4 - E_3}{E_s})^2(-K_TE_s^2 + E_s^3 + E_sE_{12}E_{34})\big] \tag{4.58}$$

it's easy to verify following expression,

$$\langle O_1O_2O_3O_4\rangle_s = -\frac{\left((p_2 - p_1)_\mu\eta^{\mu\nu}(p_4 - p_3)_\nu + K_T \cdot \frac{(E_2-E_1)(E_4-E_3)}{E_s}\right)^2 - \frac{1}{2}K_TT^C_{OOOO}}{E_{12s}E_{34s}K_T} \tag{4.59}$$

$$= -\frac{1}{E_{12s}E_{34s}K_T}\{ \left((p_2 - p_1)_i\pi_s^{ij}(p_4 - p_3)_j + E_{12s}E_{34s} \cdot \frac{(E_2 - E_1)(E_4 - E_3)}{E_s^2}\right)^2 \tag{4.60}$$

$$-\frac{1}{2}(E_{12s}E_{34s}\Pi^C_{1,OOOO} + E_{12s}^2E_{34s}^2\Pi^C_{2,OOOO})\}, \tag{4.61}$$

satisfies all the pole residue constraints.

Step 3

Notice now the momentum counting of the correlator is $(+1)$. So there is unfix term on s-channel like:

$$\psi_{unfix} = (a_{T,12}E_{12} + a_{T,34}E_{34} + a_I(E_s)) \tag{4.62}$$

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in (4.61) would be tolerated. So we need to apply the full Optical Theorem as the constraint,

$$\psi_{4,s}(E_{1\sim 4}, E_s, \boldsymbol{p}_{1\sim 4}) + \psi^*{}_{4,s}(-E_{1\sim 4}, E_s, \boldsymbol{p}_{1\sim 4}) = \tilde{\psi}_{3,ij}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_s)\frac{\Pi^{ijkl}_{s,(2,2)}}{2E_s}\tilde{\psi}_{3,kl}(-\boldsymbol{p}_s, \boldsymbol{p}_3, \boldsymbol{p}_4, E_s)$$
$$= 2E_s \cdot \frac{\left((p_2-p_1)^i\pi_{s,ik}(p_3-p_4)^k\right)^2 - \frac{1}{2}\left((p_2-p_1)^i\pi_{s,ik}(p_2-p_1)^k\right)\left((p_4-p_3)^i\pi_{s,ik}(p_4-p_3)^k\right)}{\left(E_{12}^2 - E_s^2\right)\left(E_{34}^2 - E_s^2\right)}$$

To match the Optical theorem, we could expand the (4.61) like

$$\langle O_1O_2O_3O_4\rangle_s$$
$$= -\frac{\left((p_2-p_1)_i\,\pi_s^{ij}(p_4-p_3)_j \;+\; E_{12s}E_{34s}\cdot\frac{(E_2-E_1)(E_4-E_3)}{E_s^2}\right)^2}{E_{12s}E_{34s}K_T} + \frac{\Pi^C_{1,OOOO}}{2K_T} + \frac{E_{12s}E_{34s}\Pi^C_{2,OOOO}}{2K_T}$$
$$= -\frac{\left((p_2-p_1)_i\,\pi_s^{ij}(p_4-p_3)_j\,\right)^2}{E_{12s}E_{34s}K_T} - 2\left((p_2-p_1)_i\,\pi_s^{ij}(p_4-p_3)_j\,\right)\left(\frac{(E_2-E_1)(E_4-E_3)}{K_TE_s^2}\right)$$
$$- \frac{E_{12s}E_{34s}}{K_T}\cdot\left[\left(\frac{(E_2-E_1)(E_4-E_3)}{E_s^2}\right)^2 - \frac{1}{2}\Pi^C_{2,OOOO}\right] + \frac{\Pi^C_{1,OOOO}}{2K_T}$$

Then it should be complemented by a term without changing the poles' residue like (Notice, Π^C is invariant under the flipping energy sign.)

$$\langle O_1O_2O_3O_4\rangle_s \to \langle O_1O_2O_3O_4\rangle_s + E_s\cdot\left[\frac{1}{2} + \frac{3}{2}\left(\frac{(E_1-E_2)(E_3-E_4)}{E_s^2}\right)^2\right]$$

so now the full correlator from 3 different channels will be combined as:

$$\langle O_1O_2O_3O_4\rangle = -\frac{\left((p_2-p_1)_\mu\eta^{\mu\nu}(p_4-p_3)_\nu + K_T\cdot\frac{(E_2-E_1)(E_4-E_3)}{E_s}\right)^2 - \frac{1}{2}K_TT^C_{OOOO}}{E_{12s}E_{34s}K_T}$$
$$+ E_s\cdot\left[\frac{1}{2} + \frac{3}{2}\left(\frac{(E_1-E_2)(E_3-E_4)}{E_s^2}\right)^2\right] + (t) + (u) \tag{4.63}$$

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Now the remaining unfix parameter without changing the Optical Theorem should satisfy the following constraint:

$$Im(a_{T,12}, a_{T,34}) = 0; Re(a_I) = 0 \tag{4.64}$$

The remaining unfix parameters respond to the fact that in the Lagrangian, there is a non-minimal coupling interaction term is unfix:

$$L = \sqrt{g}\left(R + (\partial_\mu \phi)^2 + fR\phi^2\right) \tag{4.65}$$

if we constraint $f = 0$ for minimal coupling scalar, then we could find that there's a shift symmetry of scalar in Lagrangian: under the transform

$$\phi(\vec{x}) \to \phi(\vec{x}) + a \tag{4.66}$$

and in the momentum space

$$\delta\phi(\boldsymbol{p}) = a\delta^3(\boldsymbol{p} - 0) \tag{4.67}$$

in which a is a constant, Lagrangian will be invariant for $f = 0$, and the wave function should also be invariant:

$$\delta L = 0 \to \delta\Psi = \int \frac{d^3p_1}{(2\pi)^3} \sum \langle O(\boldsymbol{p_1}) \dots \rangle (\delta\phi_0(\boldsymbol{p_1}) \dots) \tag{4.68}$$

$$= \int \frac{d^3p_1}{(2\pi)^3} \sum \langle O(\boldsymbol{p_1}) \dots \rangle (a\delta^3(\boldsymbol{p_1} - 0) \dots) = 0 \tag{4.69}$$

so we need to require

$$\lim_{\boldsymbol{p_1} \to 0} \langle O(\boldsymbol{p_1}) \dots \rangle = 0 \tag{4.70}$$

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in $\langle OOOO\rangle$, the shift symmetry constraints will be

$$\lim_{E_a \to 0} \langle O_1 O_2 O_3 O_4 \rangle = 0 \text{ (for a=1}\sim\text{4)} \tag{4.71}$$

then we could fix the parameter in (4.62), because we can check that (4.63) already satisfies the (4.71), notice that $E_1 = 0 = p_s = -p_2$ with $E_s = E_2$ at this limit

$$\lim_{E_1 \to 0} \langle O_1 O_2 O_3 O_4 \rangle = 0 \tag{4.72}$$

and the unfix term under this limit should be

$$\lim_{E_1 \to 0} \psi_{unfix} = (a_{T,12} E_2 + a_{T,34} E_{34} + a_I(E_2))(\dots) = 0 \tag{4.73}$$

the only solution is $a_{T,34} = 0, a_{T,12} = -a_I$ but we know $a_{T,12}$ is pure real and a_I is pure imaginary after they're constrained by Optical Theorem, so we only have $a_{T,34} = a_{T,12} = -a_I = 0$, that is the no unfix term in the correlator in (4.63).

4.3.4 $\langle TOTO\rangle$

Step 1

By the following WT identity of the $\langle TOTO\rangle$:

$$\begin{aligned} p_{1,i}\epsilon_{1,j,0}\langle T_1^{ij} O_2 T_3 O_4\rangle = &-\frac{1}{2}(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1)\langle O_{2+1} T_3 O_4\rangle - \frac{1}{2}(\boldsymbol{p}_4 \cdot \boldsymbol{\epsilon}_1)\langle O_2 T_3 O_{4+1}\rangle \\ &+ (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3) p_{3,a}\langle T^a_{1+3} O_2 O_4\rangle' - \frac{1}{2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3)\langle T_{1+3} O_2 O_4\rangle' \\ &+ \frac{1}{2}(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3)\langle O_{2+1+3} O_4\rangle + \frac{1}{2}(\boldsymbol{p}_4 \cdot \boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3)\langle O_2 O_{4+1+3}\rangle, \end{aligned}$$

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we can determine the Longitudinal parts,

$$
\begin{aligned}
\epsilon_{1,i,0,L}\epsilon_{1,j,0}\langle T_1^{ij}O_2T_3O_4\rangle =: \langle T_1^L O_2 T_3 O_4\rangle \\
= \frac{(-\epsilon_{1,i}\hat{p}_1^i)}{E_1}\cdot\Bigg[-\frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1)\langle O_{2+1}T_3O_4\rangle - \frac{1}{2}(\boldsymbol{p}_4\cdot\boldsymbol{\epsilon}_1)\langle O_2T_3O_{4+1}\rangle \\
+ (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)p_{3,a}\langle T_{1+3}^a O_2O_4\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\langle T_{1+3}O_2O_4\rangle \\
+ \frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\langle O_{2+1+3}O_4\rangle + \frac{1}{2}(\boldsymbol{p}_4\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\langle O_2O_{4+1+3}\rangle\Bigg].
\end{aligned}
$$

then we could check that its partial energy pole residue is indeed consistent like

$$\lim_{E_{12s}\to 0}\langle T_1^L O_2T_3O_4\rangle = 0 = \tilde{M}(h_1^L\phi_2\phi_{-s})\cdot\frac{1}{2E_s}\cdot\tilde{\psi}_{O_sT_3O_4} = 0 \tag{4.74}$$

$$\lim_{E_{34s}\to 0}\langle T_1^L O_2T_3O_4\rangle = \frac{(\boldsymbol{\epsilon_1}\cdot\hat{\boldsymbol{p}}_1)(\boldsymbol{\epsilon_1}\cdot\boldsymbol{p_2})}{2E_1}\cdot\tilde{M}(\phi_s h_3^{\mathrm{TT}}\phi_4) = \tilde{\psi}_{T_1^LO_2O_{-s}}\cdot\frac{1}{2E_s}\cdot\tilde{M}(\phi_s h_3^{\mathrm{TT}}\phi_4) \tag{4.75}$$

$$\lim_{E_{13u}\to 0}\langle T_1^L O_2T_3O_4\rangle = 0 = \tilde{M}(h_1^L h_3 h_{-u}^{ii'})\cdot\frac{\Pi_{s,(2,2)}^{iji'j'}}{2E_u}\cdot\tilde{\psi}_{T_u^{jj'}O_2O_4} = 0 \tag{4.76}$$

$$\lim_{E_{24u}\to 0}\langle T_1^L O_2T_3O_4\rangle = \frac{-(\boldsymbol{\epsilon_1}\cdot\hat{\boldsymbol{p}}_1)}{E_1}\cdot\left[(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)p_{3,a}\epsilon_{3,b} - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\epsilon_{3,a}\epsilon_{3,b}\right]\Pi_{s,(2,2)}^{abcd}\tilde{M}(h_u^{\mathrm{TT},cd}O_2O_4) \tag{4.77}$$

$$= \tilde{\psi}_{T_1^LT_3T_{-u}^{ii'}}\cdot\frac{\Pi_{s,(2,2)}^{ii'jj'}}{2E_u}\cdot\tilde{M}(h_u^{jj'}\phi_2\phi_4) \tag{4.78}$$

Notice that the Ward Identity of amplitude makes $\tilde{M}(h^L) = 0$. And the Longitudinal parts of 3pts are given by (4.14) and (4.28). Notice when pick the total energy pole residue of the longitudinal parts

$$
\begin{aligned}
\operatorname*{Res}_{E_{24u}\to 0}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)p_{3,a}\langle T_{1+3}^a O_2O_4\rangle' &= \operatorname*{Res}_{E_{24u}\to 0}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\epsilon_{3,b}p_{3,a}\langle T_{1+3}^{ab}O_2O_4\rangle' \\
&= \operatorname*{Res}_{E_{24u}\to 0}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\epsilon_{3,b}p_{3,a}\langle T_{1+3}^{ab}O_2O_4\rangle; \qquad (4.79) \\
&= (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\epsilon_{3,b}p_{3,a}\Pi_{(2,2),u}^{abcd}\tilde{M}(h_u^{\mathrm{TT},cd}O_2O_4),
\end{aligned}
$$

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Step 2

The $\langle TOTO\rangle$ amplitude in the form of factorization numerator in s, t, u channel is

$$\tilde{M}(h_1\phi_2h_3\phi_4) = 16\frac{\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_2\right)^2\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_4\right)^2}{S} + 16\frac{\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_4\right)^2\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_2\right)^2}{T}$$
$$+\frac{16}{U}\left[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_3^T\right)-\left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1^T\right)+\frac{(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})}{4}((p_2-p_4)^\mu(p_1-p_3)_\mu)\right]^2$$
$$-8(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})\Bigg(\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_2\right)\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_1\right)+2\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_2\right)\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_2\right)+\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_3\right)\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_2\right)$$
$$+\frac{U}{8}(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})\Bigg)$$

Step 3

Now we only need to match the transverse part to the total energy pole residue and partial energy pole residue applied on individual channels. So we also decompose the correlator into four channels and try to fix them one by one. s, t-channel is exchanging scalar, and the u-channel is exchanging graviton.

$$\langle T_1^{\rm TT}O_2T_3^{\rm TT}O_4\rangle = \langle T_1^{\rm TT}O_2T_3^{\rm TT}O_4\rangle_s+\langle T_1^{\rm TT}O_2T_3^{\rm TT}O_4\rangle_t+\langle T_1^{\rm TT}O_2T_3^{\rm TT}O_4\rangle_u+\langle T_1^{\rm TT}O_2T_3^{\rm TT}O_4\rangle_c \tag{4.80}$$

Without loss of generality, for the scalar exchanging channel, we demonstrate the calculation for the s-channel. The residue of the total and partial energy pole will be

$$\operatorname*{Res}_{K_T\to 0}\langle T_1^{\rm TT}O_2T_3^{\rm TT}O_4\rangle_s = \tilde{M}_s(h_1\phi_2h_3\phi_4) = 16\frac{\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_2\right)^2\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_4\right)^2}{S} \tag{4.81}$$

$$\operatorname*{Res}_{E_{12s}\to 0}\langle T_1^{\rm TT}O_2T_3^{\rm TT}O_4\rangle_s = \tilde{M}(h_1^{\rm TT}\phi_2\phi_{-s})\cdot\frac{1}{2E_s}\cdot\tilde{\psi}_{O_sT_3^{\rm TT}O_4} = -\frac{(2\epsilon_{1,i,T}p_2^i)^2(2\epsilon_{3,i,T}p_4^i)^2}{E_{34}^2-E_s^2} \tag{4.82}$$

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It's easy to see that the following correlator we built from amplitude over K_T and make $S \to -E_L E_R$,

$$\langle T_1^{\mathrm{TT}} O_2 T_3^{\mathrm{TT}} O_4 \rangle_s = -16 \frac{\left(\boldsymbol{\epsilon}_{1,0}^T \cdot \boldsymbol{p}_2\right)^2 \left(\boldsymbol{\epsilon}_{3,0}^T \cdot \boldsymbol{p}_4\right)^2}{K_T E_{12s} E_{34s}}, \tag{4.83}$$

satisfies all the residue constraints. Similarly, for the graviton exchanging channel, or u-channel

$$\underset{K_T \to 0}{\mathrm{Res}} \langle T_1^{\mathrm{TT}} O_2 T_3^{\mathrm{TT}} O_4 \rangle_u = \tilde{M}_u (h_1 \phi_2 h_3 \phi_4) \tag{4.84}$$

$$= \frac{16}{U} \left[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)^\mu (p_1 - p_3)_\mu\right) \right]^2 \tag{4.85}$$

$$\underset{E_{13u} \to 0}{\mathrm{Res}} \langle T_1^{\mathrm{TT}} O_2 T_3^{\mathrm{TT}} O_4 \rangle_u = \tilde{M} (h_1^{\mathrm{TT}} h_3^{\mathrm{TT}} h_{-u}^{ii'}) \cdot \frac{\Pi_{(2,2),u}^{ii'jj'}}{2E_u} \cdot \tilde{\psi}_{T_u^{\mathrm{TT}} O_2 O_4}^{jj'} \tag{4.86}$$

$$= \frac{-16}{E_{24}^2 - E_u^2} \tag{4.87}$$

$$\cdot \left\{ \left[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)_i \pi_u^{ij} (p_1 - p_3)_j\right) \right]^2 \right. \tag{4.88}$$

$$\left. - \frac{1}{2} \left[\boldsymbol{J}_{3,L} \cdot \boldsymbol{\pi}_u \cdot J_{3,L}\right] \left[\frac{1}{4} (\boldsymbol{p}_2 - \boldsymbol{p}_4) \cdot \boldsymbol{\pi}_u \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_4) \right] \right\} \tag{4.89}$$

in which the shorthand of the terms are defined as

$$\boldsymbol{J}_{3,L} = \left(\frac{1}{2} (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) (\boldsymbol{p}_1 - \boldsymbol{p}_3) + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{p}_3)(\boldsymbol{\epsilon}_3^{\mathrm{T}}) - (\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{p}_1)(\boldsymbol{\epsilon}_1^{\mathrm{T}}) \right) \tag{4.90}$$

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$$\underset{E_{24u}\to 0}{\text{Res}} \langle T_1^{\text{TT}} O_2 T_3^{\text{TT}} O_4 \rangle_u = \tilde{\psi}(T_1^{\text{TT}} T_3^{\text{TT}} T_{-u}^{ii'}) \cdot \frac{\Pi_{(2,2),u}^{ii'jj'}}{2E_u} \cdot \tilde{M}_{h_u^{\text{TT}}\phi_2\phi_4}^{jj'} \tag{4.91}$$

$$= \frac{-16}{E_{13}^2 - E_u^2} \tag{4.92}$$

$$\cdot \left\{ \left[(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T)(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T) - (\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T)(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_1^T) + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)_i \pi_u^{ij} (p_1 - p_3)_j\right) \right]^2 \right. \tag{4.93}$$

$$\left. - \frac{1}{2} [\boldsymbol{J}_{3,L} \cdot \boldsymbol{\pi}_u \cdot J_{3,L}] \left[\frac{1}{4} (\boldsymbol{p}_2 - \boldsymbol{p}_4) \cdot \boldsymbol{\pi}_u \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_4) \right] \right\} \tag{4.94}$$

So, if we start from amplitude over K_T as correlator and make $U \to (-E_{13u}E_{24u})$ as u-channel, we'll come up with the problem to match $\eta_{\mu\nu}$ contraction to $\pi_{u,ij}$ contraction of the first term in partial energy pole kinematics. In the first two terms in the square in the amplitude, the match is trivial,

$$\begin{aligned}
&\left[(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T)(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_3^T) - (\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T)(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T)\right] - \left[(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T)(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T) - (\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T)(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_1^T)\right] \\
&= -(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T)(\boldsymbol{p}_2 \cdot \hat{\boldsymbol{p}}_{\boldsymbol{u}})(\hat{\boldsymbol{p}}_{\boldsymbol{u}} \cdot \boldsymbol{\epsilon}_3^T) + (\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T)(\boldsymbol{p}_2 \cdot \hat{\boldsymbol{p}}_{\boldsymbol{u}})(\hat{\boldsymbol{p}}_{\boldsymbol{u}} \cdot \boldsymbol{\epsilon}_1^T) \\
&= -(\boldsymbol{p}_u \cdot \boldsymbol{\epsilon}_1^T)(\boldsymbol{p}_2 \cdot \hat{\boldsymbol{p}}_{\boldsymbol{u}})(\hat{\boldsymbol{p}}_{\boldsymbol{u}} \cdot \boldsymbol{\epsilon}_3^T) + (\boldsymbol{p}_u \cdot \boldsymbol{\epsilon}_3^T)(\boldsymbol{p}_2 \cdot \hat{\boldsymbol{p}}_{\boldsymbol{u}})(\hat{\boldsymbol{p}}_{\boldsymbol{u}} \cdot \boldsymbol{\epsilon}_1^T) \\
&= 0
\end{aligned} \tag{4.95}$$

And for the last term in the square in the amplitude, it's similar to the situation in $\langle OOOO \rangle$. So we could add a term like (4.53) inside the square on the numerator of amplitude to solve the problem. But the second term in the partial energy pole residue we need to match it to be a term vanishing in total energy pole, actually, we could apply the relationship by do

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the 4D trace on the $M(h_1^{\mathrm{TT}} h_3^{\mathrm{TT}} h_u^{\mu\nu})$. Then we found that

$$
\begin{aligned}
M(h_1^{\mathrm{TT}} h_3^{\mathrm{TT}} h_u^{\mu\nu}) &= \frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})^2 (p_1 - p_3) \cdot (p_1 - p_3) \\
&= M(h_1^{\mathrm{TT}} h_3^{\mathrm{TT}} h_u^{00}) + M(h_1^{\mathrm{TT}} h_3^{\mathrm{TT}} h_u^{ij}) \pi_{u,ij} - M(h_1^{\mathrm{TT}} h_3^{\mathrm{TT}} h_u^{ij}) \hat{p}_{u,i} \hat{p}_{u,j} \\
&= \boldsymbol{J}_{3,L} \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{J}_{3,L} + \frac{1}{4}\left(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}\right)^2 \left((E_1 - E_3)^2 - ((\boldsymbol{p}_1 - \boldsymbol{p}_3) \cdot \hat{\boldsymbol{p}_u})^2\right)
\end{aligned} \tag{4.96}
$$

Then we could rexpress the the second term in the partial energy pole residue by with the match we derived in previouschapter (4.54),

$$
\begin{aligned}
& -\frac{1}{2}\left[\boldsymbol{J}_{3,L} \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{J}_{3,L}\right]\left[\frac{1}{4}(\boldsymbol{p}_2 - \boldsymbol{p}_4) \cdot \boldsymbol{\pi}_u \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_4)\right] \\
& = -\frac{1}{32}(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})^2 \left(E_{13u}(-K_T + E_{24u}) - (E_1 - E_3)^2 + ((\boldsymbol{p}_1 - \boldsymbol{p}_3) \cdot \hat{\boldsymbol{p}_u})^2\right) \\
& \cdot \left(E_{13u}(-K_T + E_{24u}) - (E_2 - E_4)^2 + ((\boldsymbol{p}_2 - \boldsymbol{p}_4) \cdot \hat{\boldsymbol{p}_u})^2\right)
\end{aligned} \tag{4.97}
$$

Then we could apply (4.57) we derive in the previous chapter, then we have the match (Notice, the Π^C and T^C here we relabeling the indices of the momentum such that it's the u-channel term.)

$$
\boxed{
\begin{aligned}
& \left[\boldsymbol{J}_{3,L} \cdot \boldsymbol{\pi}_u \cdot J_{3,L}\right]\left[(\boldsymbol{p}_2 - \boldsymbol{p}_4) \cdot \boldsymbol{\pi}_u \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_4)\right] \\
& \qquad + \frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})^2 E_{12s} E_{34s} \Pi^C_{1,OOOO} + \frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})^2 E_{12s}^2 E_{34s}^2 \Pi^C_{2,OOOO} \\
& \qquad = \frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})^2 K_T T^C_{OOOO}
\end{aligned}
} \tag{4.98}
$$

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Then if we substitute the residue of the pole with the match we mentioned before, we would get the correlator

$$\langle T_1^{\mathrm{TT}} O_2 T_3^{\mathrm{TT}} O_4 \rangle_u = \frac{(-16)}{E_{13u} E_{24u} K_T} \Big\{ \Big[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T\right) \\ + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)^\mu (p_1 - p_3)_\mu + K_T \left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u} \right) \right) \Big]^2 . \\ - \frac{1}{32} (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})^2 K_T T_{OOOO}^C \Big\} \tag{4.99}$$

It's trivial to show it satisfies all the residue constraints. In the end, to match the amplitude on the total energy pole we add a contact term as the contact term in amplitude over $-K_T$,

$$\langle T_1^{\mathrm{TT}} O_2 T_3^{\mathrm{TT}} O_4 \rangle_c = \frac{8}{K_T} (\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T}) \\ \cdot \Big(\left(\boldsymbol{\epsilon}_{1,0}^T \cdot \boldsymbol{p}_2\right) \left(\boldsymbol{\epsilon}_{3,0}^T \cdot \boldsymbol{p}_1\right) + 2 \left(\boldsymbol{\epsilon}_{1,0}^T \cdot \boldsymbol{p}_2\right) \left(\boldsymbol{\epsilon}_{3,0}^T \cdot \boldsymbol{p}_2\right) + \left(\boldsymbol{\epsilon}_{1,0}^T \cdot \boldsymbol{p}_3\right) \left(\boldsymbol{\epsilon}_{3,0}^T \cdot \boldsymbol{p}_2\right) \\ + \frac{U}{8} (\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T}) \Big)$$

Now we'll find the exact form of U beyond of total energy pole kinematic is ambiguous, it responds to the fact that some unfixed terms won't change all the residue constraints. So, we need to fix it by full Optical Theorem and the soft limit of scalar energy to ensure minimal coupling.

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Step 4

If we require the correlator to satisfy the full Optical Theorem, the mismatch will only be on the u channel, like

$$\begin{aligned}\Delta_{OPT} &= \langle T_1O_2T_3O_4\rangle_u(E_{1\sim 4}, E_u, \boldsymbol{p}_{1\sim 4}) + \langle T_1O_2T_3O_4\rangle_u(-E_{1\sim 4}, E_u, \boldsymbol{p}_{1\sim 4}) \\ &- (32E_u)\cdot\frac{1}{(E_{12}^2-E_s^2)\,(E_{34}^2-E_s^2)} \\ &\cdot\Big\{\Big[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_3^T\right)-\left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_1^T\right)+\frac{(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})}{4}\left((p_2-p_4)_i\pi_u^{ij}(p_1-p_3)_j\right)\Big]^2 \\ &-\frac{1}{2}\left[\boldsymbol{J}_{3,L}\cdot\boldsymbol{\pi}_u\cdot J_{3,L}\right]\left[\frac{1}{4}(\boldsymbol{p}_2-\boldsymbol{p}_4)\cdot\boldsymbol{\pi}_u\cdot(\boldsymbol{p}_2-\boldsymbol{p}_4)\right]\Big\} \\ &= -32E_u(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2\cdot\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)^2\cdot\left[\frac{1}{2}+\frac{3}{2}\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)^2\right]\end{aligned}$$

So we only need to add this term to the correlator

$$\begin{aligned}&\langle T_1^{\mathrm{TT}}O_2T_3^{\mathrm{TT}}O_4\rangle_u \\ &\to \langle T_1^{\mathrm{TT}}O_2T_3^{\mathrm{TT}}O_4\rangle_u + 16E_u\cdot\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)^2\cdot\left[\frac{1}{2}+\frac{3}{2}\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)^2\right]\end{aligned} \tag{4.100}$$

then the full Optical Theorem is satisfied. Then for minimal coupling scalar, the correlator's soft limit should vanish. The correlator we get at this limit,

$$\lim_{E_2\to 0}\langle T_1^{\mathrm{TT}}O_2T_3^{\mathrm{TT}}O_4\rangle = \lim_{E_2\to 0}\langle T_1^{\mathrm{TT}}O_2T_3^{\mathrm{TT}}O_4\rangle_c = \lim_{E_2\to 0}U(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2 = 0 \tag{4.101}$$

So we need to find a U satisfying the following constraints:

$$\lim_{K_T\to 0}U = E_{13}^2 - E_u^2 \tag{4.102}$$

$$\lim_{E_a\to 0}U = 0\ (\text{for } a = 2, 4) \tag{4.103}$$

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We can start from the Ansatz

$$U(E_a,E_s,E_t,E_u)=E_{13}^2-E_u^2+K_T Poly_1(E_a,E_s,E_t,E_u) \tag{4.104}$$

We can solve that

$$U=E_{13}^2-E_u^2-K_T(E_{13}+aE_u+bE_{24}) \tag{4.105}$$

with the unfix coefficient a,b satisfying $a+b=-1$. But because we also need this term to satisfy the Optical Theorem of the contact term, so we have

$$U(E_a,E_s,E_t,E_u)+U^*(-E_a,E_s,E_t,E_u)=0 \tag{4.106}$$

This constraint that a is pure imaginary while b is purely real. So we must have $a=0,b=-1$. As a result, we have

$$\begin{aligned}
\langle T_1^T O_2 T_3^T O_4\rangle &= -16\frac{\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_2\right)^2\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_4\right)^2}{E_{12s}E_{34s}K_T}-16\frac{\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_4\right)^2\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_2\right)^2}{E_{14t}E_{23t}K_T}\\
&-\frac{16}{E_{13u}E_{24u}K_T}\Big\{\Big[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_3^T\right)-\left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1^T\right)\\
&+\frac{(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})}{4}\left((p_2-p_4)^\mu(p_1-p_3)_\mu+K_T\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u}\right)\right)\Big]^2.\\
&-\frac{1}{32}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2K_T T_{OOOO}^C\Big\}\\
&+16E_u\cdot\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)^2\cdot\left[\frac{1}{2}+\frac{3}{2}\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)^2\right]\\
&+\frac{8(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})}{K_T}\Big(\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_2\right)\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_1\right)+2\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_2\right)\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_2\right)+\left(\boldsymbol{\epsilon}_{1,0}^T\cdot\boldsymbol{p}_3\right)\left(\boldsymbol{\epsilon}_{3,0}^T\cdot\boldsymbol{p}_2\right)\\
&+\frac{E_{13}^2-E_u^2-K_T(E_{13}-E_{24})}{8}(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})\Big).
\end{aligned}$$

Now similar to the case of $\langle OOOO\rangle$ exchanging graviton, eq (4.64) proves that no term won't change the value of pole residue, the soft limit on all the legs' energy, and the full Optical theorem. So we completely fix the correlator.

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Chapter 5 Fermionic correlators in Flat Spacetime

5.1 2pt correlators

5.1.1 Massless Spin Half Fermion

For the massless fermion, by dimension counting of the action, we know it should be a dimensionless factor. And the two-point correlator should be sandwiched with the 3D boundary spinor condition $\bar{\chi}_{-}/\chi_{+}$. With the identity

$$\bar{\chi}_{-,-\boldsymbol{p}} I \chi_{+,\boldsymbol{p}} = \bar{\chi}_{-,-\boldsymbol{p}} \gamma_0 \chi_{+,\boldsymbol{p}} = 0. \tag{5.1}$$

the only nonvanishing 3D rotational invariant factor will be

$$\langle \bar{\chi}^{-}_{-\boldsymbol{p}} \chi^{+}_{\boldsymbol{p}} \rangle = \frac{\not{\boldsymbol{p}}}{E_{\boldsymbol{p}}} = \hat{\not{\boldsymbol{p}}} \tag{5.2}$$

in which we set the normalization to 1. It's a unit vector of the momentum, so it'll be no $1/E_p$ pole on this form. If we want to promote to the massive spin half fermion, if we write the momentum to be

$$\not{\boldsymbol{p}} = \not{\boldsymbol{k}} \sqrt{E_p^2 - m^2} \tag{5.3}$$

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so the dimensionless factor without codimension-one pole will be

$$\psi_{2,m\neq 0} = \frac{\not{\boldsymbol{p}}}{E_{\boldsymbol{p}} \pm m} \tag{5.4}$$

The positive and the minus sign of the mass will depend on our convention to identify the 3D-spinor under the Bunch-Davies vacuum. If we choose $\psi_{2,m\neq 0}\langle = \bar{\chi}^{-}_{-\boldsymbol{p}}\chi^{+}_{\boldsymbol{p}}\rangle$ then $\frac{1}{E_p+m}$ wil match our Lagrangian computation result. And If we choose $\psi_{2,m\neq 0}\langle = \bar{\chi}^{+}_{-\boldsymbol{p}}\chi^{-}_{\boldsymbol{p}}\rangle$ then $\frac{1}{Ep-m}$ wiil match our Lagrangian computation result.

5.1.2 Gravitino

The dimension counting is the same as spin half fermion, the two-point correlator should be a dimensionless factor. By the similar argument to the spin half fermion, the only nonvanishing matrice sandwiched between 3D gravitino boundary condition $\bar{\psi}^i_-/\psi^j_+$ like §App. F.1 shows will be

$$\langle \bar{\psi}^{i,-}_{-\boldsymbol{p}} \psi^{j,+}_{\boldsymbol{p}} \rangle \propto \hat{\not{\boldsymbol{p}}} \tag{5.5}$$

And by the WT identity of the 2-point correlator,

$$p_i \langle \bar{\psi}^{i,-}_{-\boldsymbol{p}} \psi^{j,+}_{\boldsymbol{p}} \rangle = 0. \tag{5.6}$$

Now we know that in the two-point correlator, only the transverse part survives. Now we get

$$\langle \bar{\psi}^{i,-}_{-\boldsymbol{p}} \psi^{j,+}_{\boldsymbol{p}} \rangle = \pi_{ij,p} \hat{\not{\boldsymbol{p}}} \tag{5.7}$$

in which we set the normalization to 1.

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5.2 3pt correlators

The rule is the same as bosonic correlator,

1. Decompose the correlator and use Ward-Takahashi identity (We list all of these in §App. E.2) to determine the longitudinal part.

2. Apply total energy pole condition to determine the transversal part.

5.2.1 $\langle J\bar{\chi}^{-}\chi^{+}\rangle$

The correlator should satisfy the following two conditions: (We already set the coupling constant $e = 1$)

$$\begin{aligned}
\underset{K_T=0}{\text{Res}}\langle J_1\bar{\chi}_2{}^{-}\chi_3^{+}\rangle &= M_3 = \epsilon_{1,\mu}(\bar{u}_2\gamma^{\mu}u_3) \\
&= -\epsilon^{\text{T}}_{1,i}(\bar{u}_2\gamma^{i}u_3) \text{ (Coulomb Gauge, } \vec{p}_1\cdot\vec{\epsilon}_1 = 0) \\
&= -(\bar{\chi}_2^{-}(1-\not{\boldsymbol{p}}_2)\not{\epsilon}_1^{\text{T}}(1+\not{\boldsymbol{p}}_3)\chi_3) \\
p_{1,i}\langle J_1^{i}\bar{\chi}_2{}^{-}\chi_3^{+}\rangle &= (-\hat{\not{\boldsymbol{p}}}_2 - \hat{\not{\boldsymbol{p}}}_3)
\end{aligned} \tag{5.8}$$

in which we use the fact that

$$\begin{aligned}
\not{P}u_{\vec{p}} = 0 &\rightarrow u_{\vec{p}} = (1+\hat{\not{\boldsymbol{p}}})\chi_{\vec{p},+} \\
\bar{u}_{\vec{p}}\not{P} = 0 &\rightarrow \bar{u}_{\vec{p}} = \bar{\chi}_{\vec{p},-}(1-\hat{\not{\boldsymbol{p}}})
\end{aligned} \tag{5.9}$$

to write the 4D on shell spinor $\bar{u}/u$ in 3D spinor boundary condition $\bar{\chi}_{-}/\chi_{+}$.

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1. Determine the longitudinal part by WT identity

$$\langle J_1^L \bar{\chi}_2{}^- \chi_3^+ \rangle = -(\epsilon_{1,i}\hat{p}_1^i)\frac{p_{1,j}}{E_1}\langle J_1^j \bar{\chi}_2{}^- \chi_3^+ \rangle = -(\epsilon_{1,i}\hat{p}_1^i)\bar{\chi}_{2,-}\frac{(-\hat{\not{\boldsymbol{p}}}_2 - \hat{\not{\boldsymbol{p}}}_3)}{E_1}$$
$$= (\epsilon_{1,i}\hat{p}_1^i)\bar{\chi}_{2,-}\frac{(\hat{\not{\boldsymbol{p}}}_2 + \hat{\not{\boldsymbol{p}}}_3)}{E_1}\chi_3^+ \tag{5.10}$$

2. Determine the transverse part by the total energy pole condition

$$\langle J_1^{\mathrm{T}} \bar{\chi}_2^- \chi_3^+ \rangle = \frac{-1}{K_T} \cdot \bar{\chi}_{2,-}(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathrm{T}}(1 + \hat{\not{\boldsymbol{p}}}_3)\chi_{3,+} \tag{5.11}$$

And there is no subleading form we can write proportional to K_T on the total energy pole.

3. Combining the transverse and longitudinal parts, we have

$$\langle J_1 \bar{\chi}_2^- \chi_3^+ \rangle = \langle J_1^{\mathrm{T}} \bar{\chi}_2^- \chi_3^+ \rangle + \langle J_1^L \bar{\chi}_2{}^- \chi_3^+ \rangle$$
$$= \frac{-1}{K_T} \cdot \bar{\chi}_{2,-}(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathrm{T}}(1 + \hat{\not{\boldsymbol{p}}}_3)\chi_{3,+} + (\epsilon_{1,i}\hat{p}_1^i)\bar{\chi}_2^- \frac{(\hat{\not{\boldsymbol{p}}}_2 + \hat{\not{\boldsymbol{p}}}_3)}{E_1}\chi_3^+ \tag{5.12}$$

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5.2.2 $\langle T\bar{\chi}^{-}\chi^{+}\rangle$

The correlator should satisfy the following two conditions: (We already set the coupling constant $\kappa = 1$)

$$\underset{K_T=0}{\text{Res}}\langle T_1\bar{\chi}_2{}^{-}\chi_3^{+}\rangle = M_{3,T\bar{\chi}\chi} = -M_{3,JOO}\cdot M_{3,J\bar{\chi}\chi} \text{ (double copy)}$$

$$= -(\epsilon_{1,\mu}(\bar{u}_2\gamma^{\mu}u_3))(\epsilon_{1,\nu}(p_2-p_3)^{\nu}) \text{ (the only amplitude form satisfying } M|_{\epsilon_1\to p_1}=0)$$

$$= -\epsilon^{\mathrm{T}}_{1,i}(\bar{u}_2\gamma^{i}u_3)(\epsilon^{\mathrm{T}}_{1,j}(p_2-p_3)^{j}) \text{ (Coulomb Gauge, } p_{1,i}h_1^{ij} = p_{1,i}\epsilon_1^{i}\epsilon_1^{j} = 0)$$

$$= -(\epsilon^{\mathrm{T}}_{1,j}(p_2-p_3)^{j})(\bar{\chi}_2^{-}(1-\hat{\boldsymbol{p}}\!\!\!/_2)\not{\epsilon}_1^{\mathrm{T}}(1+\hat{\boldsymbol{p}}\!\!\!/_3)\chi_3)$$

$$p_{1,i}\langle T_1^{i}\bar{\chi}_2{}^{-}\chi_3^{+}\rangle = -\frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1)\hat{\boldsymbol{p}}\!\!\!/_3 + \frac{1}{2}(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1)\hat{\boldsymbol{p}}\!\!\!/_2 + \frac{1}{16}[\boldsymbol{p}\!\!\!/_1,\not{\boldsymbol{\epsilon}}_1]\hat{\boldsymbol{p}}\!\!\!/_3 + \frac{1}{16}\hat{\boldsymbol{p}}\!\!\!/_2[\boldsymbol{p}\!\!\!/_1,\not{\boldsymbol{\epsilon}}_1] \tag{5.13}$$

Then by the first condition, we could show the pure transverse correlator will be

$$\langle T_1^{\mathrm{TT}}\bar{\chi}_2{}^{-}\chi_3^{+}\rangle = \frac{-1}{K_T}(\epsilon^{\mathrm{T}}_{1,j}(p_2-p_3)^{j})(\bar{\chi}_2^{-}(1-\hat{\boldsymbol{p}}\!\!\!/_2)\not{\epsilon}_1^{\mathrm{T}}(1+\hat{\boldsymbol{p}}\!\!\!/_3)\chi_3) \tag{5.14}$$

And the longitudinal parts of the correlator in $\langle T\bar{\chi}\chi\rangle$ are determined by WT identity,

$$\langle T_1^{L}\bar{\chi}_2\chi_3\rangle := -[(\epsilon_1^{i'}\hat{p}_{1,i'})\hat{p}_{1,i}]\epsilon_{1,j}\langle T_1^{ij}\bar{\chi}_2\chi_3\rangle$$
$$= -[(\epsilon_1^{i'}\hat{p}_{1,i'})\frac{1}{E_1}]\left(-\frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1)\hat{\boldsymbol{p}}\!\!\!/_3 + \frac{1}{2}(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1)\hat{\boldsymbol{p}}\!\!\!/_2 + \frac{1}{16}[\boldsymbol{p}\!\!\!/_1,\not{\boldsymbol{\epsilon}}_1]\hat{\boldsymbol{p}}\!\!\!/_3 + \frac{1}{16}\hat{\boldsymbol{p}}\!\!\!/_2[\boldsymbol{p}\!\!\!/_1,\not{\boldsymbol{\epsilon}}_1]\right) \tag{5.15}$$

5.2.3 $\langle T\bar{\psi}^{-}\psi^{+}\rangle$

We use the fermion vector form to describe the boundary condition of the gravitino

$$\bar{\psi}_{-,i} = \epsilon_i\bar{\chi}_{-}$$
$$\psi_{+,i} = \epsilon_i\bar{\chi}_{+} \tag{5.16}$$

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and because we have the boundary EOM

$$\boldsymbol{\gamma} \cdot \boldsymbol{\psi}^{\mathrm{T}} = -\hat{\boldsymbol{p}}\!\!\!/(\hat{\boldsymbol{p}} \cdot \boldsymbol{\psi}^{\mathrm{T}}) = \boldsymbol{\epsilon}\!\!\!/^{\mathrm{T}} \chi_{+} = 0 \tag{5.17}$$

the ϵ and χ are related. And if we write the amplitude polarization in the Coulomb gauge $\vec{\boldsymbol{p}} \cdot \boldsymbol{\psi} = 0$, then we could write down the amplitude polarization in the form of the boundary condition as

$$\begin{aligned} \bar{\psi}_i^u(\vec{p}) &= \epsilon_i^{\mathrm{T}} \bar{\chi}_{-} = \epsilon_i^{\mathrm{T}} \bar{\chi}_{-}(\epsilon_i^{\mathrm{T}}) \cdot (1 - \hat{\boldsymbol{p}}\!\!\!/) \\ \psi_i^u(\vec{p}) &= \epsilon_i^{\mathrm{T}} \chi_{+} = \epsilon_i^{\mathrm{T}} (1 + \hat{\boldsymbol{p}}\!\!\!/) \cdot \chi_{+}(\epsilon_i^{\mathrm{T}}) \end{aligned} \tag{5.18}$$

But the readers should be careful that they're not independent field to the vector component ϵ.

Then the correlator should satisfy the following two conditions: (We already set the

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coupling constant $\kappa = 1$)

$$\underset{K_T=0}{\text{Res}}\langle T_1 \bar{\psi}_2{}^- \psi_3^+\rangle = M_{3,T\bar{\psi}\psi} = M_{3,JJJ,extracted\ color\ algebra} \cdot M_{3,J\bar{\psi}\psi} \text{ (double copy)}$$

$$= [(2\epsilon_{3,\nu}p_{1,\nu})(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_1) + (\epsilon_{2,\nu}\epsilon_3^{\nu})((p_2 - p_3)_\rho \epsilon_1^\rho) - (2\epsilon_{2,\nu}p_1^\nu)(\epsilon_{3,\rho}\epsilon_1^\rho)] (-\epsilon_{1,\mu}(\bar{u}_2\gamma^\mu u_3))$$

(the only amplitude form satisfying $M|_{\epsilon_1 \to p_1} = 0$ and $M|_{\epsilon_{2or3} \to p_{2or3}} = 0$)

$$= -(\bar{u}_2 \not{\boldsymbol{\epsilon}}_1^{\mathrm{T}} u_3) \left[(2\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{p}_1)(\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) + (\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})((\boldsymbol{p}_2 - \boldsymbol{p}_3) \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) - (2\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{p}_1)(\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}})\right]$$

(Coulomb Gauge, $\boldsymbol{p}_a \cdot \boldsymbol{\epsilon}_a = 0,\ a = 1 \sim 3$)

$$= -(\bar{\chi}_2^-(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\boldsymbol{\epsilon}}_1^{\mathrm{T}}(1 + \hat{\not{\boldsymbol{p}}}_3)\chi_3)$$

$$\cdot \left[(2\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{p}_1)(\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) + (\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})((\boldsymbol{p}_2 - \boldsymbol{p}_3) \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) - (2\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{p}_1)(\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}})\right]$$

$$p_{1,k}\langle T_1^k \bar{\psi}_{2,-}\psi_{3,+}\rangle = -\bar{\chi}_{2,-}\hat{\not{p}}_2\chi_{3,+}\left(\boldsymbol{\epsilon}_2^T \cdot \boldsymbol{\epsilon}_3\right)(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1) - \bar{\chi}_{2,-}\hat{\not{p}}_3\chi_{3,+}\left(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_3^T\right)(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1)$$

$$-\ \bar{\chi}_{2,-}\hat{\not{p}}_2\chi_{3,+}\left(\boldsymbol{\epsilon}_2^T \cdot \boldsymbol{p}_1\right)(\boldsymbol{\epsilon}_3 \cdot \boldsymbol{\epsilon}_1) - \bar{\chi}_{2,-}\hat{\not{p}}_3\chi_{3,+}\left(\boldsymbol{\epsilon}_3^T \cdot \boldsymbol{p}_1\right)(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_1)$$

$$-\frac{1}{8}\left(\boldsymbol{\epsilon}_2^T \cdot \boldsymbol{\epsilon}_3\right)(\bar{\chi}_{2,-}\hat{\not{p}}_2[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\chi_{3,+}) + \frac{1}{8}\left(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_3^T\right)(\bar{\chi}_{2,-}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\hat{\not{p}}_3\chi_{3,+})$$

$$p_{2,k}\langle T_1 \bar{\psi}_{2,-}^k\psi_{3,+}\rangle = -\langle T_1 T_{2+3}^{kl}\rangle\, \epsilon_{l,3}\,(\bar{\chi}_{2,-}\gamma_k\chi_{3,+}) - \frac{1}{8}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\left(\epsilon_{1,k}\langle\bar{\psi}_{1+2,-}^k\psi_{3,+}\rangle\right)$$

$$= -\left(\boldsymbol{\epsilon}_1^T \cdot \boldsymbol{\epsilon}_3\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^T\chi_{3,+}\right)E_1 - \frac{1}{8}\left(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3^T\right)(\bar{\chi}_{2,-}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\hat{\not{\boldsymbol{p}}}_3\chi_{3,+})$$

$$p_{3,k}\langle T_1 \bar{\psi}_{2,-}\psi_{3,+}^k\rangle = \langle T_1 T_{2+3}^{kl}\rangle\, \epsilon_{l,2}\,(\bar{\chi}_{2,-}\gamma_k\chi_{3,+}) + \frac{1}{8}\left(\epsilon_{1,k}\langle\bar{\psi}_{2,-}^k\psi_{1+3,+}\rangle\right)[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]$$

$$= \left(\boldsymbol{\epsilon}_1^T \cdot \boldsymbol{\epsilon}_2\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^T\chi_{3,+}\right)E_1 + \frac{1}{8}\left(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2^T\right)(\bar{\chi}_{2,-}\hat{\not{\boldsymbol{p}}}_2[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\chi_{3,+}) \tag{5.19}$$

Notice now besides one diffeomorphism WT identity, we also have two SUSY WT identity. Then by the first condition, we could show the pure transverse correlator will be

$$\langle T_1^{\mathrm{TT}} \bar{\psi}_2{}^{-,\mathrm{T}}\psi_3^{+,\mathrm{T}}\rangle = \frac{-1}{K_T}(\bar{\chi}_2^-(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\boldsymbol{\epsilon}}_1^{\mathrm{T}}(1 + \hat{\not{\boldsymbol{p}}}_3)\chi_3)$$

$$\cdot \left[(2\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{p}_1)(\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) + (\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})((\boldsymbol{p}_2 - \boldsymbol{p}_3) \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) - (2\boldsymbol{\epsilon}_2^{\mathrm{T}} \cdot \boldsymbol{p}_1)(\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}})\right] \tag{5.20}$$

Notice that and for the transverse part of the boundary condition should satisfy the bound-

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ary EOM

$$\boldsymbol{\gamma}\cdot\boldsymbol{\psi}^{\mathrm{T}} = -\hat{\not{\boldsymbol{p}}}(\hat{\boldsymbol{p}}\cdot\boldsymbol{\psi}^{\mathrm{T}}) = \not{\boldsymbol{\epsilon}}^{\mathrm{T}}\chi_{+} = 0 \tag{5.21}$$

then no polynomial ansatz without total energy pole we could write except for the term we discuss in (??). Then we completely fix the pure transverse 3-point correlator.

And we could determine the longitudinal parts by the WT identity. Notice in the §App. (E.2), we already show that the longitudinal parts given by different WT identities are the same.

$$\begin{aligned}
\langle T_1^L\bar{\psi}_{2,-}\psi_{3,+}\rangle &= -[(\epsilon_1^{i'}\hat{p}_{1,i'})\frac{1}{E_1}]\cdot\Big[-\bar{\chi}_{2,-}\hat{\not{p}}_2\chi_{3,+}\left(\boldsymbol{\epsilon}_2^T\cdot\boldsymbol{\epsilon}_3\right)\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1\right)-\bar{\chi}_{2,-}\hat{\not{p}}_3\chi_{3,+}\left(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1\right)\\
&\quad-\bar{\chi}_{2,-}\hat{\not{p}}_2\chi_{3,+}\left(\boldsymbol{\epsilon}_2^T\cdot\boldsymbol{p}_1\right)\left(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_1\right)-\bar{\chi}_{2,-}\hat{\not{p}}_3\chi_{3,+}\left(\boldsymbol{\epsilon}_3^T\cdot\boldsymbol{p}_1\right)\left(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_1\right)\\
&\quad-\frac{1}{8}\left(\boldsymbol{\epsilon}_2^T\cdot\boldsymbol{\epsilon}_3\right)\left(\bar{\chi}_{2,-}\hat{\not{p}}_2\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\chi_{3,+}\right)+\frac{1}{8}\left(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_3^T\right)\left(\bar{\chi}_{2,-}\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\hat{\not{p}}_3\chi_{3,+}\right)\Big]\\
\langle T_1\bar{\psi}_{2,-}^L\psi_{3,+}\rangle &= -[(\epsilon_2^{i'}\hat{p}_{2,i'})\frac{1}{E_2}]\cdot\Big[-\left(\boldsymbol{\epsilon}_1^T\cdot\boldsymbol{\epsilon}_3\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^T\chi_{3,+}\right)E_1\;-\frac{1}{8}\left(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\bar{\chi}_{2,-}\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\hat{\not{\boldsymbol{p}}}_3\chi_{3,+}\right)\Big]\\
\langle T_1\bar{\psi}_{2,-}\psi_{3,+}^L\rangle &= -[(\epsilon_3^{i'}\hat{p}_{3,i'})\frac{1}{E_3}]\cdot\Big[\left(\boldsymbol{\epsilon}_1^T\cdot\boldsymbol{\epsilon}_2\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^T\chi_{3,+}\right)E_1\;+\frac{1}{8}\left(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^T\right)\left(\bar{\chi}_{2,-}\hat{\not{\boldsymbol{p}}}_2\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\chi_{3,+}\right)\Big]
\end{aligned} \tag{5.22}$$

5.3 4pt correlators

The bootstrap rule for the fermionic correlator is the same as the bosonic one. But the two partial energy pole residues of the correlator are not symmetric. So it's not trivial to match them when we want to glue the 3-point into be 4-point correlator.

5.3.1 $\langle J\bar{\chi}^{-}J\chi^{+}\rangle$

To match the amplitude, we need to decompose amplitude channel by channel, and on the individual channel, the numerator of the amplitude is factorized. ($p_s^\mu = p_3^\mu + p_4^\mu; p_t^\mu =$

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$p_1^\mu + p_4^\mu)$

$$
\begin{aligned}
M(\gamma_1\bar{\chi}_3\gamma_3\chi_4) &= M_s(\gamma_1\bar{\chi}_3\gamma_3\chi_4) + M_t(\gamma_1\bar{\chi}_3\gamma_3\chi_4) \\
M_s(\gamma_1\bar{\chi}_3\gamma_3\chi_4) &= M_{A_s}(\gamma_1\bar{\chi}_2\chi_s)\left[\frac{\not{P}_1 + \not{P}_2}{S}\right]_{A_sB_s} M_{B_s}(\gamma_3\bar{\chi}_{-s}\chi_4) \\
&= M_{A_s}(\gamma_1\bar{\chi}_2\chi_s)\left[\frac{-\not{P}_3 - \not{P}_4}{S}\right]_{A_sB_s} M_{B_s}(\gamma_3\bar{\chi}_{-s}\chi_4) \qquad (5.23)\\
&= M_{A_s}(\gamma_1\bar{\chi}_2\chi_s)\left[\frac{-\not{S}}{S}\right]_{A_sB_s} M_{B_s}(\gamma_3\bar{\chi}_{-s}\chi_4) \\
M_t &= (M_s)|_{1\leftrightarrow 3}
\end{aligned}
$$

And we should notice, only on the s-pole, then p_s^μ is on-shell and equal to the sum of the polarization vector.

$$
\lim_{S\to 0} \not{S} = \lim_{S\to 0} \gamma_\mu p_s^\mu = \sum_{(i)} u_s^{(i)} \bar{u}_s^{(i)} \qquad (5.24)
$$

Because we don't have energy conservation in the correlator, our definition of the E_s is already on the shell, $E_s \neq E_3 + E_4$. We can only write the amplitude in the variables we write the correlator as:

$$
\begin{aligned}
\operatorname*{Res}_{K_T\to 0}\langle J_1^{\mathrm{T}}\bar{\chi}_2^- J_3^{\mathrm{T}}\chi_4^+\rangle_s = M_s(\gamma_1\bar{\chi}_3\gamma_3\chi_4) &= M_{A_s}(\gamma_1\bar{\chi}_2\chi_s)\left[\frac{\not{P}_1 + \not{P}_2}{S}\right]_{A_sB_s} M_{B_s}(\gamma_3\bar{\chi}_{-s}\chi_4) \\
&= M_{A_s}(\gamma_1\bar{\chi}_2\chi_s)\left[\frac{-\not{P}_3 - \not{P}_4}{S}\right]_{A_sB_s} M_{B_s}(\gamma_3\bar{\chi}_{-s}\chi_4) \\
&= \bar{\chi}_2{}^-(1 - \hat{\boldsymbol{p}}_2)\not{\epsilon}_1{}^{\mathrm{T}}\left[\frac{\not{P}_3 + \not{P}_4}{(E_{12s}E_{34s})}\right]\not{\epsilon}_3{}^{\mathrm{T}}(1 + \hat{\boldsymbol{p}}_4)\chi_{4,+}
\end{aligned}
\qquad (5.25)
$$

Then we want to write an ansatz for the pure transverse part correlator trivially satisfies

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the partial energy pole residue,

$$\begin{aligned}\operatorname*{Res}_{E_{12s}\to 0}\langle J_1^{\mathrm{T}}\bar{\chi}_2{}^- J_3^{\mathrm{T}}\chi_4^+\rangle_s &= \bar{u}_2\not{\epsilon}_1^{\mathrm{T}}\cdot\frac{-1}{2E_s}\left(\frac{\not{S}(-\gamma_0)}{E_{34s}}-\frac{(\not{S})}{E_3+E_4-E_s}\right)\cdot\not{\epsilon}_3^{\mathrm{T}}u_4\\ &= \bar{\chi}_2{}^-(1-\hat{\not{p}}_2)\not{\epsilon}_1^{\mathrm{T}}\cdot\frac{1}{2E_s}\left(\frac{\not{S}(\gamma_0)}{E_{34s}}+\frac{(\not{S})}{E_3+E_4-E_s}\right)\cdot\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{p}}_4)\chi_{4,+}\\ &= \bar{\chi}_2{}^-(1-\hat{\not{p}}_2)\not{\epsilon}_1^{\mathrm{T}}\cdot\frac{1}{2E_s}\left(\frac{\not{S}(\gamma_0)}{E_{34s}}+\frac{(\not{S})}{K_T}\right)\cdot\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{p}}_4)\chi_{4,+}\\ \operatorname*{Res}_{E_{34s}\to 0}\langle J_1^{\mathrm{T}}\bar{\chi}_2{}^- J_3^{\mathrm{T}}\chi_4^+\rangle_s &= \bar{\chi}_2{}^-(1-\hat{\not{p}}_2)\not{\epsilon}_1^{\mathrm{T}}\cdot\frac{1}{2E_s}\left(\frac{(\gamma_0)\not{S}_-}{E_{12s}}-\frac{(\not{S}_-)}{K_T}\right)\cdot\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{p}}_4)\chi_{4,+},\end{aligned} \tag{5.26}$$

we show in (3.2.5) if we take $m=0$ and the trivial ansatz satisfies the total energy residue.

The ansatz will be the mathch of the form like

$$\begin{aligned}\langle J_1^{\mathrm{T}}\bar{\chi}_2{}^- J_3^{\mathrm{T}}\chi_4^+\rangle_s &= \bar{\chi}_2{}^-(1-\hat{\not{p}}_2)\not{\epsilon}_1^{\mathrm{T}}\cdot\frac{1}{E_{12s}}\left[\frac{K_T\not{S}\gamma_0+E_{34s}\not{S}+\sum_{n=1}^{\infty}c_{R,s,n}E_{12s}^n}{K_T E_{34s}(2E_s)}\right]\cdot\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{p}}_4)\chi_{4,+}\\ &= \bar{\chi}_2{}^-(1-\hat{\not{p}}_2)\not{\epsilon}_1^{\mathrm{T}}\cdot\frac{1}{E_{34s}}\left[\frac{K_T\gamma_0\not{S}_- - E_{12s}\not{S}_- +\sum_{n=1}^{\infty}c_{L,s,n}E_{34s}^n}{K_T E_{12s}(2E_s)}\right]\cdot\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{p}}_4)\chi_{4,+}\\ &= \bar{\chi}_2{}^-(1-\hat{\not{p}}_2)\not{\epsilon}_1^{\mathrm{T}}\cdot\frac{1}{K_T}\left[\frac{(2E_s)(\not{P}_3+\not{P}_4)+\sum_{n=1}^{\infty}c_{T,s,n}K_T^n}{E_{12s}E_{34s}(2E_s)}\right]\cdot\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{p}}_4)\chi_{4,+}\end{aligned} \tag{5.27}$$

And we could focus on the $c_{L,s,n}$, we'll find to match the total energy pole residue

$$\begin{aligned}\sum_{n}^{\infty}c_{L,s,n}E_{34s}^n &= -E_{34s}\not{S}_- + 2E_s(E_{34s}\gamma_0)+\sum_{n=1}^{\infty}c_{LT,s,n}E_{34s}^n\\ &= E_{34s}\not{S}+\sum_{n=1}^{\infty}c_{LT,s,n}E_{34s}^n\end{aligned} \tag{5.28}$$

then to match the other partial energy pole residue set that

$$\sum_{n=1}^{\infty}c_{LT,s,n}E_{34s}^n = 0 \tag{5.29}$$

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Then we have the fixed s-channel transversal correlator

$$\langle J_1^{\mathrm{T}} \bar{\chi}_2{}^- J_3^{\mathrm{T}} \chi_4^+ \rangle_s = \bar{\chi}_2{}^-(1 - \hat{\boldsymbol{p}}\!\!\!/_2) \boldsymbol{\epsilon}\!\!\!/_1^{\mathrm{T}} \cdot \frac{1}{E_{34s}} \left[\frac{K_T \gamma_0 S\!\!\!/_- - E_{12s} \$_- + E_{34s} \$}{K_T E_{12s}(2E_s)} \right] \cdot \boldsymbol{\epsilon}\!\!\!/_3^{\mathrm{T}} (1 + \hat{\boldsymbol{p}}\!\!\!/_4) \chi_{4,+}$$
$$= \bar{\chi}_2{}^-(1 - \hat{\boldsymbol{p}}\!\!\!/_2) \boldsymbol{\epsilon}\!\!\!/_1^{\mathrm{T}} \cdot \left[\frac{(P\!\!\!/_3 + P\!\!\!/_4)}{K_T E_{12s} E_{34s}} - \frac{1 - \gamma_0}{2} \frac{S\!\!\!/_-}{E_s E_{12s} E_{34s}} \right] \cdot \boldsymbol{\epsilon}\!\!\!/_3^{\mathrm{T}} (1 + \hat{\boldsymbol{p}}\!\!\!/_4) \chi_{4,+} \tag{5.30}$$

we could check this match the (D.199) when $m = 0$ and $\epsilon_i = \epsilon_i^{\mathrm{T}}$. And on t-channel transverse correlator, the only difference is $1 \leftrightarrow 3$. And By dimension counting, there's no contact term ansatz we could write. Then cause we know the longitudinal parts are determined by the WT identities, we could check these longitudinal parts are consistent with the partial energy pole residue.

$$\begin{aligned} \langle J_1^L \bar{\chi}_{2,-} J_3 \chi_+ \rangle &= (\frac{-\hat{p}_{1,i} \epsilon_1^i}{E_1}) \cdot p_{1,j} \langle J_1^j \bar{\chi}_{2,-} J_3 \chi_+ \rangle \\ &= (\frac{-\hat{p}_{1,i} \epsilon_1^i}{E_1}) (-\langle \bar{\chi}_{1+2,-} J_3 \chi_{4,+} \rangle + \langle \bar{\chi}_{2,-} J_3 \chi_{1+4,+} \rangle) \end{aligned} \tag{5.31}$$

First we take the E_{12s} pole

$$\operatorname*{Res}_{E_{12s}=0} \langle J_1^L \bar{\chi}_{2,-} J_3 \chi_+ \rangle = 0 = M_{3,A}(\gamma_1^L \bar{\chi}_2 \chi_s) \left((\frac{1+\gamma_0}{2}) \frac{-\hat{p}\!\!\!/_s}{2} (\frac{1-\gamma_0}{2}) \right)^{AB} \tilde{\Psi}_{3,B}(\gamma_3 \bar{\chi}_{-s} \chi_4) \tag{5.32}$$

by the amplitude Ward Identities, we know that $M_3(\gamma_1^L \bar{\chi}_2 \chi_s) = p_1^i \tilde{M}_{3,i}(\gamma_1 \bar{\chi}_2 \chi_s) = p_1^\mu M_{3,\mu}(\gamma_1 \bar{\chi}_2 \chi_s) = 0$. And for the other pole for s-channel

$$\begin{aligned} \operatorname*{Res}_{E_{34s}=0} \langle J_1^L \bar{\chi}_{2,-} J_3 \chi_+ \rangle &= (\frac{\hat{p}_{1,i} \epsilon_1^i}{E_1}) \operatorname*{Res}_{E_{34s}=0} \langle \bar{\chi}_{1+2,-} J_3 \chi_{4,+} \rangle = (\frac{\hat{p}_{1,i} \epsilon_1^i}{E_1}) \bar{\chi}_{2,-,A} M_3^{AB} (\bar{\chi}_{-s} \gamma_3 \chi_4) \chi_{4,B,+} \\ &= \tilde{\Psi}_3^A (J_1^L \bar{\chi}_{2,-} \chi_{s,+}) \left((\frac{1+\gamma_0}{2}) \frac{-\hat{p}\!\!\!/_s}{2} (\frac{1-\gamma_0}{2}) \right)_{AA'} M_3^{A'B} (\bar{\chi}_{-s} \gamma_3 \chi_4) \chi_{4,B,+} \end{aligned} \tag{5.33}$$

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in which we use the 3-point WT identity,

$$\begin{aligned}\langle J_1^L \bar{\chi}_{2,-}\chi_{s,+}\rangle^A &= \bar{\chi}_{2,-,C}(\frac{-\hat{p}_{1,i}\epsilon_1^i}{E_1})(-\hat{\boldsymbol{p}}\!\!\!/_{\boldsymbol{2}} - \hat{\boldsymbol{p}}\!\!\!/_{\boldsymbol{s}})^{CA} \\ \tilde{\Psi_3}^A(J_1^L \bar{\chi}_{2,-}\chi_{s,+}) &= \langle J_1^L \bar{\chi}_{2,-}\chi_{s,+}\rangle^A - \langle J_1^L \bar{\chi}_{2,-}\chi_{s,+}\rangle^A|_{E_s\to -E_s} \\ &= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})(2\bar{\chi}_{2,-}\hat{\boldsymbol{p}}\!\!\!/_{\boldsymbol{s}})^A,\end{aligned} \tag{5.34}$$

and the identity,

$$\begin{aligned}\bar{\chi}_{2,-}(2\hat{\boldsymbol{p}}\!\!\!/_{\boldsymbol{s}})\left((\frac{1+\gamma_0}{2})\frac{-\hat{p}\!\!\!/_s}{2}(\frac{1-\gamma_0}{2})\right) &= \bar{\chi}_{2,-}(2\hat{\boldsymbol{p}}\!\!\!/_{\boldsymbol{s}})\left((\frac{1+\gamma_0}{2})\frac{-\hat{p}\!\!\!/_s}{2}\right) \\ &= \bar{\chi}_{2,-}(\frac{1-\gamma_0}{2})(2\hat{\boldsymbol{p}}\!\!\!/_{\boldsymbol{s}})\frac{-\hat{p}\!\!\!/_s}{2} \\ &= \bar{\chi}_{2,-}.\end{aligned} \tag{5.35}$$

Then the last equality is trivially satisfied. For the pole of E_{32t} and E_{14t}, the consistency could be check if we take $1 \to 3$ for all the above derivation.

5.3.2 $\langle T\bar{\chi}^- T\chi^+\rangle$

First, we need to decompose the amplitude of $M(h\bar{\chi}h\chi)$ channel by channel. We could refer to [7] and change the momentum and overall normalization convention to what we use in the thesis. For

$$\begin{aligned}p_s^\mu &= p_3^\mu + p_4^\mu \\ p_t^\mu &= p_1^\mu + p_4^\mu \\ p_u^\mu &= p_2^\mu + p_4^\mu\end{aligned} \tag{5.36}$$

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We could write down the amplitude (We already take $\kappa = 1$)

$$
\begin{aligned}
M(h_1\bar{\chi}_2h_3\chi_4) &= M_s(h_1\bar{\chi}_2h_3\chi_4) + M_t(h_1\bar{\chi}_2h_3\chi_4) + M_u(h_1\bar{\chi}_2h_3\chi_4) + M_c(h_1\bar{\chi}_2h_3\chi_4) \\
M_s(h_1\bar{\chi}_2h_3\chi_4) &= \frac{4}{S}(p_4\cdot\epsilon_3)(p_2\cdot\epsilon_1)\bar{u}_2\not{\epsilon}_1(\not{P}_3+\not{P}_4)\not{\epsilon}_3u_4 \\
M_t(h_1\bar{\chi}_2h_3\chi_4) &= \frac{4}{T}(p_4\cdot\epsilon_1)(p_2\cdot\epsilon_3)\bar{u}_2\not{\epsilon}_1(\not{P}_1+\not{P}_4)\not{\epsilon}_3u_4 \\
M_u(h_1\bar{\chi}_2h_3\chi_4) &= \frac{1}{U}\left[(2p_3\cdot\epsilon_1)(2p_2\cdot\epsilon_3)-(2p_1\cdot\epsilon_3)(2p_2\cdot\epsilon_1)+(p_1-p_3)\cdot(p_2-p_4)(\epsilon_3\cdot\epsilon_1)\right] \\
&\quad\cdot\left[(2p_3\cdot\epsilon_1)(-\bar{u}_2\not{\epsilon}_3u_4)-(2p_1\cdot\epsilon_3)(-\bar{u}_2\not{\epsilon}_3u_4)+(\epsilon_1\cdot\epsilon_3)(-\bar{u}_2(\not{P}_1-\not{P}_3)u_4)\right] \\
&= \frac{1}{U}\left[(2p_3\cdot\epsilon_1)(2p_2\cdot\epsilon_3)-(2p_1\cdot\epsilon_3)(2p_2\cdot\epsilon_1)+(-2T-U)(\epsilon_3\cdot\epsilon_1)\right] \\
&\quad\cdot\left[(2p_3\cdot\epsilon_1)(-\bar{u}_2\not{\epsilon}_3u_4)-(2p_1\cdot\epsilon_3)(-\bar{u}_2\not{\epsilon}_3u_4)+(\epsilon_1\cdot\epsilon_3)(-\bar{u}_2(-2\not{P}_3)u_4)\right] \\
&\quad\text{(we use } \bar{u}_2\not{P}_2=\not{P}_4u_4=0\text{)} \\
M_c(h_1\bar{\chi}_2h_3\chi_4) &= 2(\epsilon_1\cdot\epsilon_3)\left[(p_3\cdot\epsilon_1)\bar{u}_2\not{\epsilon}_3u_4-(p_1\cdot\epsilon_3)\bar{u}_2\not{\epsilon}_3u_4-(\epsilon_1\cdot\epsilon_3)\bar{u}_2(\not{P}_3)u_4\right] \\
&\quad+2(\epsilon_1\cdot\epsilon_3)\bar{u}_2\not{\epsilon}_1(\not{P}_1+\not{P}_4)\not{\epsilon}_3u_4
\end{aligned}
\tag{5.37}
$$

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which makes Ward Identity of the amplitude be satisfied. Actually, we could identify the amplitude in a double-copy and moreover factorized form.

$$
\begin{aligned}
M_s(h_1\bar{\chi}_2 h_3\chi_4) &= SM_s(\gamma_1\phi_2^*\gamma_3\phi_4)\cdot M_s(\gamma_1\bar{\chi}_2\gamma_3\chi_4) \\
&= (M(\gamma_1\phi_2^*\phi_s)M(\gamma_1\bar{\chi}_2\chi_s))\frac{(-\not{P}_3-\not{P}_4)}{S}(M(\gamma_3\phi_{-s}^*\phi_4)M(\gamma_1\bar{\chi}_{-s}\chi_4)) \\
&= M(h_1\bar{\chi}_2\chi_s)\frac{(-\not{P}_3-\not{P}_4)}{S}M(h_3\bar{\chi}_{-s}\chi_4) \\
M_t(h_1\bar{\chi}_2 h_3\chi_4) &= TM_t(\gamma_1\phi_2^*\gamma_3\phi_4)\cdot M_t(\gamma_1\phi_2^*\gamma_3\phi_4) \\
&= M(h_3\bar{\chi}_2\chi_t)\frac{(-\not{P}_1-\not{P}_4)}{T}M(h_1\bar{\chi}_{-t}\chi_4) \\
M_u(h_1\bar{\chi}_2 h_3\chi_4) &= U(M_{\mu_u}(\gamma_1\gamma_3\gamma_u)M_{\mu'_u}(\gamma_1\gamma_3\gamma_u))(M^{\mu_u}(\gamma_{-u}\phi_2^*\phi_4)M^{\mu'_u}(\gamma_{-u}\bar{\chi}_2\chi_4)) \\
&= M_{\mu_u\mu'_u}(h_1h_3h_u)\frac{\eta^{\mu_u,\nu_u}\eta^{\mu'_u,\nu'_u}}{U}M_{\nu_u\nu'_u}(h_{-u}\bar{\chi}_2\chi_4) \\
M_c(h_1\bar{\chi}_2 h_3\chi_4) &= M_c(\gamma_1\phi_2^*\gamma_3\phi_4)\left[\frac{1}{2}M_{\mu'_u}(\gamma_1\gamma_3\gamma_u)M^{\mu'_u}(\gamma_{-u}\bar{\chi}_2\chi_4) - T\cdot M_t(\gamma_1\bar{\chi}_2\gamma_3\chi_4)\right]
\end{aligned} \tag{5.38}
$$

In which $M(\gamma\gamma\gamma)$ is Yang-Mills 3-pt amplitude with color factor extracted. Then for the s-channel pure transverse correlator, the constraints of the correlator will be

$$
\begin{aligned}
\operatorname*{Res}_{K_T\to 0}\langle T_1^{\mathrm{TT}}\bar{\chi}_2^- T_3^{\mathrm{TT}}\chi_4^+\rangle_s &= M_{A_s}(h_1\bar{\chi}_2\chi_s)\left[\frac{-\not{P}_3-\not{P}_4}{S}\right]_{A_sB_s}M_{B_s}(h_3\bar{\chi}_{-s}\chi_4) \\
&= \bar{\chi_2}^-(1-\hat{\not{\boldsymbol{p}}}_2)(2\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{p}_2)\left[\frac{\not{P}_3+\not{P}_4}{(E_{12s}E_{34s})}\right](-2\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{p}_4)(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}
\end{aligned} \tag{5.39}
$$

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$$\underset{E_{12s}\to 0}{\text{Res}}\langle T_1^{\text{TT}}\bar{\chi}_2{}^- T_3^{\text{TT}}\chi_4^+\rangle_s$$

$$= \bar{u}_2 \not{\epsilon}_1^{\text{T}} \cdot (2\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{p}_2) \cdot \frac{-1}{2E_s} \left(\frac{\not{S}(-\gamma_0)}{E_{34s}} - \frac{(\not{S})}{E_3 + E_4 - E_s} \right) \cdot (-2\boldsymbol{\epsilon}_3^{\text{T}} \cdot \boldsymbol{p}_4) \cdot \not{\epsilon}_3^{\text{T}} u_4$$

$$= \bar{\chi}_2{}^-(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\text{T}} \cdot \frac{1}{2E_s} \cdot (2\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{p}_2) \left(\frac{\not{S}(\gamma_0)}{E_{34s}} + \frac{(\not{S})}{E_3 + E_4 - E_s} \right) \cdot (-2\boldsymbol{\epsilon}_3^{\text{T}} \cdot \boldsymbol{p}_4) \cdot \not{\epsilon}_3^{\text{T}}(1 + \hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}$$

$$= (2\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{p}_2) \cdot (-2\boldsymbol{\epsilon}_3^{\text{T}} \cdot \boldsymbol{p}_4) \cdot \bar{\chi}_2{}^-(1 - \hat{\not{\boldsymbol{p}}}_2) \cdot \not{\epsilon}_1^{\text{T}} \cdot \frac{1}{2E_s} \left(\frac{\not{S}(\gamma_0)}{E_{34s}} + \frac{(\not{S})}{K_T} \right) \cdot \not{\epsilon}_3^{\text{T}}(1 + \hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}$$

$$\underset{E_{34s}\to 0}{\text{Res}}\langle T_1^{\text{TT}}\bar{\chi}_2{}^- T_3^{\text{TT}}\chi_4^+\rangle_s$$

$$= (2\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{p}_2) \cdot (-2\boldsymbol{\epsilon}_3^{\text{T}} \cdot \boldsymbol{p}_4) \cdot \bar{\chi}_2{}^-(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\text{T}} \cdot \frac{1}{2E_s} \left(\frac{(\gamma_0)\not{S}_-}{E_{12s}} - \frac{(\not{S}_-)}{K_T} \right) \cdot \not{\epsilon}_3^{\text{T}}(1 + \hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}, \tag{5.40}$$

So all the constants of transverse $\langle T_1^{\text{TT}}\bar{\chi}_2{}^- T_3^{\text{TT}}\chi_4^+\rangle_s$ are the the same $\langle J_1^{\text{T}}\bar{\chi}_2{}^- J_3^{\text{T}}\chi_4^+\rangle_s$ except the factor $(2\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{p}_2) \cdot (-2\boldsymbol{\epsilon}_3^{\text{T}} \cdot \boldsymbol{p}_4)$. Then we could easily extend the bootstrapped result in (5.30) to

$$\langle T_1^{\text{TT}}\bar{\chi}_2{}^- T_3^{\text{TT}}\chi_4^+\rangle_s = -4(\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{p}_2) \cdot (\boldsymbol{\epsilon}_3^{\text{T}} \cdot \boldsymbol{p}_4)$$
$$\cdot\, \bar{\chi}_2{}^-(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\text{T}} \cdot \left[\frac{(\not{P}_3 + \not{P}_4)}{K_T E_{12s} E_{34s}} - \frac{1-\gamma_0}{2} \frac{\not{S}_-}{E_s E_{12s} E_{34s}} \right] \cdot \not{\epsilon}_3^{\text{T}}(1 + \hat{\not{\boldsymbol{p}}}_4)\chi_{4,+} \tag{5.41}$$

Similar, with $1 \to 3$, we have the t-channel transverse correlator

$$\langle T_1^{\text{TT}}\bar{\chi}_2{}^- T_3^{\text{TT}}\chi_4^+\rangle_t = -4(\boldsymbol{\epsilon}_3^{\text{T}} \cdot \boldsymbol{p}_2) \cdot (\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{p}_4)$$
$$\cdot\, \bar{\chi}_2{}^-(1 - \hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_3^{\text{T}} \cdot \left[\frac{(\not{P}_1 + \not{P}_4)}{K_T E_{32t} E_{14t}} - \frac{1-\gamma_0}{2} \frac{\not{T}_-}{E_t E_{32t} E_{14t}} \right] \cdot \not{\epsilon}_1^{\text{T}}(1 + \hat{\not{\boldsymbol{p}}}_4)\chi_{4,+} \tag{5.42}$$

For u-channel, it's graviton exchanging in the internal leg. The constraints wiil be

$$\underset{K_T\to 0}{\text{Res}}\langle T_1^{\text{TT}}\bar{\chi}_2^- T_3^{\text{TT}}\chi_4^+\rangle_u = M_{\mu_u\mu_u'}(h_1 h_3 h_u)\frac{\eta^{\mu_u,\nu_u}\eta^{\mu_u',\nu_u'}}{U} M_{\nu_u\nu_u'}(h_{-u}\bar{\chi}_2\chi_4)$$

$$= \frac{4}{E_{13u}E_{24u}} \left[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4}((p_2 - p_4)^\mu (p_1 - p_3)_\mu) \right]$$

$$\cdot\, \bar{\chi}_2{}^-(1 - \hat{\not{\boldsymbol{p}}}_2)\left[(2\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\text{T}})\not{\epsilon}_3^{\text{T}} - (2\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\text{T}})\not{\epsilon}_3^{\text{T}} + (\boldsymbol{\epsilon}_1^{\text{T}} \cdot \boldsymbol{\epsilon}_3^{\text{T}})(p_1 - p_3)_\mu\gamma^\mu\right](1 + \hat{\not{\boldsymbol{p}}}_4)\chi_{4,+} \tag{5.43}$$

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$$\underset{E_{13u}\to 0}{\text{Res}}\langle T_1^{\text{TT}}\bar{\chi}_2^- T_3^{\text{TT}}\chi_4^+\rangle_u = \tilde{M}_{i_u i'_u}(h_1 h_3 h_u)\frac{\Pi_{u,(2,2)}^{i_u i'_u j_u j'_u}}{2E_u}\tilde{\Psi}_{j_u j'_u,\langle T_{-u}^{\text{TT}}\bar{\chi}_{2,-}\chi_{4,+}\rangle}$$

$$= \frac{4}{E_{24u}^2 - E_u^2}\{$$

$$\left[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})}{4}\left((p_2-p_4)^i\pi_u^{ij}(p_1-p_3)_j\right)\right]$$

$$\cdot\,\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\text{T}})(\epsilon_{3,i}^{\text{T}}\pi_u^{ij}\gamma_j) - (2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\text{T}})(\epsilon_{3,i}^{\text{T}}\pi_u^{ij}\gamma_j) + (\boldsymbol{\epsilon}_1^{\text{T}}\cdot\boldsymbol{\epsilon}_3^{\text{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$

$$-\frac{1}{2}(\boldsymbol{J}_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}_{3,L})\cdot[(p_2-p_4)^i\pi_{u,ij}\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)\gamma^j(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}]\} \tag{5.44}$$

$$\underset{E_{24u}\to 0}{\text{Res}}\langle T_1^{\text{TT}}\bar{\chi}_2^- T_3^{\text{TT}}\chi_4^+\rangle_u = \tilde{\Psi}_{i_u i'_u,\langle T_1^{\text{TT}}T_3^{\text{TT}}T_u^{\text{TT}}\rangle}\frac{\Pi_{u,(2,2)}^{i_u i'_u j_u j'_u}}{2E_u}M_{j_u j'_u}(h_{-u}\bar{\chi}_2\chi_4)$$

$$= \frac{4}{E_{13u}^2 - E_u^2}\{$$

$$\left[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})}{4}\left((p_2-p_4)^i\pi_u^{ij}(p_1-p_3)_j\right)\right]$$

$$\cdot\,\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\text{T}})(\epsilon_{3,i}^{\text{T}}\pi_u^{ij}\gamma_j) - (2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\text{T}})(\epsilon_{3,i}^{\text{T}}\pi_u^{ij}\gamma_j) + (\boldsymbol{\epsilon}_1^{\text{T}}\cdot\boldsymbol{\epsilon}_3^{\text{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$

$$-\frac{1}{2}(\boldsymbol{J}_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}_{3,L})\cdot[(p_2-p_4)^i\pi_{u,ij}\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)\gamma^j(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}]\} \tag{5.45}$$

In which the $J_{3,L}$ is defined in the previous chapter. Then we need to match the η_{ij} contraction on total energy pole kinematics to $\pi_{u,ij}$ contraction on the first term of the partial energy pole kinematics. Actually, we can do that term by term. We already successfully match the following two term in $\langle TOTO\rangle$ and relabeled $\langle O^*OO^*O\rangle_{Exc\ J}$:

$$(p_1-p_3)^\mu(p_2-p_4)_\mu + K_T\frac{(E_1-E_3)(E_2-E_4)}{E_u} \tag{5.46}$$

$$= (p_1-p_3)^i\pi_{u,ij}(p_2-p_4)^j + E_{13u}E_{24u}\frac{(E_1-E_3)(E_2-E_4)}{E_u^2} \tag{5.47}$$

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$$
\begin{aligned}
&\left[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_3^T\right)-\left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1^T\right)\right]-\left[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_3^T\right)-\left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_1^T\right)\right]\\
&=-\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_3^T\right)+\left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_1^T\right)\\
&=-\left(\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_3^T\right)+\left(\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_1^T\right)\\
&=0
\end{aligned}
\tag{5.48}
$$

Now additionally we have the matches:

$$
\begin{aligned}
&\bar{\chi_2}^-(1-\not{\boldsymbol{p}}_2)\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\left((p_1-p_3)_\mu\gamma^\mu+K_T\frac{E_1-E_3}{E_u}\gamma_0\right)\right](1+\not{\boldsymbol{p}}_4)\chi_{4,+}\\
&=\bar{\chi_2}^-(1-\not{\boldsymbol{p}}_2)\\
&\cdot\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(E_1-E_3)\gamma_0-\frac{1}{E_u^2}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_i p_u^i p_u^j\gamma_j\right.\\
&\left.+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})K_T\frac{E_1-E_3}{E_u}\gamma_0\right]\\
&\cdot(1+\not{\boldsymbol{p}}_4)\chi_{4,+}\\
&=\bar{\chi_2}^-(1-\not{\boldsymbol{p}}_2)\\
&\cdot\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(E_1-E_3)\gamma_0+\frac{(E_1^2-E_3^2)(E_2+E_4)\gamma_0}{E_u^2}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\right.\\
&\left.+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})K_T\frac{E_1-E_3}{E_u}\gamma_0\right]\\
&\cdot(1+\not{\boldsymbol{p}}_4)\chi_{4,+}\\
&=\bar{\chi_2}^-(1-\not{\boldsymbol{p}}_2)\\
&\cdot\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\frac{E_1-E_3}{E_u^2}\left(E_u^2+E_{13}E_{24}+K_TE_u\right)\gamma_0\right]\\
&\cdot(1+\not{\boldsymbol{p}}_4)\chi_{4,+}\\
&=\bar{\chi_2}^-(1-\not{\boldsymbol{p}}_2)(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\left[(p_1-p_3)_i\pi_u^{ij}\gamma_j+E_{13u}E_{24u}\frac{E_1-E_3}{E_u^2}\gamma_0\right](1+\not{\boldsymbol{p}}_4)\chi_{4,+}
\end{aligned}
\tag{5.49}
$$

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and

$$
\begin{aligned}
&\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\eta^{ij}\gamma_j)-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\eta^{ij}\gamma_j)\right]-\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)\right]\\
&=-(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)+(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)\\
&=-(2\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)+(2\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)\\
&=0.
\end{aligned}
\tag{5.50}
$$

In which we use $\bar{\chi}_{2,-}(1-\hat{\boldsymbol{p}}\!\!\!/_2)(E_2\gamma_0+\boldsymbol{p}\!\!\!/_2)=(E_4\gamma_0+\boldsymbol{p}\!\!\!/_4)(1+\hat{\boldsymbol{p}}\!\!\!/_4)\chi_{4,+}=0$. Then every term of the amplitude and the first term of the residue of the partial energy pole will be matched. For the second term of the residue of the partial energy pole. By the 4D trace of the amplitude

$$
\begin{aligned}
-\eta^{\mu\nu}M(h_{-u,\mu\nu}\bar{\chi}_2\chi_4)&=\bar{u}_2(P\!\!\!/_2-P\!\!\!/_4)u_4=0\\
&=\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}\!\!\!/_2)[(E_2-E_4)\gamma_0+(p_2-p_4)^i\pi_{u,ij}\gamma^j-[(p_2-p_4)^i\hat{p}_u^i]\hat{\boldsymbol{p}}\!\!\!/_u](1+\hat{\boldsymbol{p}}\!\!\!/_4)\chi_{4,+}\\
&=\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}\!\!\!/_2)[(p_2-p_4)^i\pi_{u,ij}\gamma^j+(\frac{E_2-E_4}{E_u^2})\gamma_0\left(E_u^2-E_{24}^2\right)](1+\hat{\boldsymbol{p}}\!\!\!/_4)\chi_{4,+},
\end{aligned}
\tag{5.51}
$$

in the last line, we use the trick that

$$
\begin{aligned}
\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}\!\!\!/_2)\hat{\boldsymbol{p}}\!\!\!/_u(1+\hat{\boldsymbol{p}}\!\!\!/_4)\chi_{4,+}&=\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}\!\!\!/_2)\frac{\boldsymbol{p}\!\!\!/_2+\boldsymbol{p}\!\!\!/_4}{E_u}(1+\hat{\boldsymbol{p}}\!\!\!/_4)\chi_{4,+}\\
&=\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}\!\!\!/_2)\frac{(-E_{24})\gamma_0}{E_u}(1+\hat{\boldsymbol{p}}\!\!\!/_4)\chi_{4,+}.
\end{aligned}
\tag{5.52}
$$

in the last equation, we use the Dirac equation.

Then we could rewrite the second term of the partial energy pole residue by the rele-

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beling of (4.55),

$$(\boldsymbol{J}_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}_{3,L})\cdot[(p_2-p_4)^i\pi_{u,ij}\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\gamma^j(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}]$$
$$=\frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2\left(E_{13u}(-K_T+E_{24u})-(\frac{E_1-E_3}{E_u})^2[E_u^2-E_{13}^2]\right)$$
$$\cdot\left((\frac{E_2-E_4}{E_u^2})\left(E_u^2-E_{24}^2\right)\right)\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\gamma_0(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$
$$=\frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\gamma_0(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$
$$\cdot\Big\{K_T\cdot\left[-E_{13u}\left((\frac{E_2-E_4}{E_u^2})\left(E_u^2-E_{24}^2\right)\right)\right]+E_{13u}E_{24u}\cdot\left[\left((\frac{E_2-E_4}{E_u^2})\left(E_u^2-E_{24}^2\right)\right)\right]$$
$$-(\frac{E_1-E_3}{E_u})^2(\frac{E_2-E_4}{E_u^2})\left[(E_{13u}E_{24u})^2-K_T^2E_s^2+K_TE_s^3+K_TE_sE_{12}E_{34}\right]\Big\}. \tag{5.53}$$

Then we could identify the match of the second term of the total energy pole residue and the term vanishing at the total energy pole,

$$\boxed{\begin{aligned}&(\boldsymbol{J}_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}_{3,L})\cdot[(p_2-p_4)^i\pi_{u,ij}\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\gamma^j(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}]\\&\qquad+E_RE_L\Pi^C_{1,T\bar{\chi}T\chi}+E_R^2E_L^2\Pi^C_{2,T\bar{\chi}T\chi}=K_TT^c_{T\bar{\chi}T\chi}.\end{aligned}} \tag{5.54}$$

in which we define

$$\Pi^C_{1,T\bar{\chi}T\chi}=-\left[\left((\frac{E_2-E_4}{E_u^2})\left(E_u^2-E_{24}^2\right)\right)\right]\cdot\frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\gamma_0(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$
$$\Pi^C_{2,T\bar{\chi}T\chi}=(\frac{E_1-E_3}{E_u})^2(\frac{E_2-E_4}{E_u^2})\cdot\frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\gamma_0(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$
$$T^c_{T\bar{\chi}T\chi}=\left[-E_{13u}\left((\frac{E_2-E_4}{E_u^2})\left(E_u^2-E_{24}^2\right)\right)+(K_TE_s^2-E_s^3-E_sE_{12}E_{34})(\frac{E_1-E_3}{E_u})^2(\frac{E_2-E_4}{E_u^2})\right]$$
$$\cdot\frac{1}{4}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\gamma_0(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}. \tag{5.55}$$

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then by all of the above matching we mentioned, we could find

$$
\begin{aligned}
&\langle T_1^{\mathrm{TT}} \bar{\chi}_2^- T_3^{\mathrm{TT}} \chi_4^+ \rangle_u \\
&= \frac{4}{K_T E_{13u} E_{24u}} \\
&\cdot \Big\{ \big[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_3^T\right) \\
&- \left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)^\mu (p_1 - p_3)_\mu + K_T \frac{(E_1 - E_3)(E_2 - E_4)}{E_u} \right) \big] \\
&\cdot \bar{\chi_2}^- (1 - \not{\boldsymbol{p}}_2) \\
&\cdot \left[(2\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) \not{\epsilon}_3^{\mathrm{T}} - (2\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \not{\epsilon}_3^{\mathrm{T}} + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \left((p_1 - p_3)_\mu \gamma^\mu + K_T \frac{E_1 - E_3}{E_u} \gamma_0 \right) \right] \\
&\cdot (1 + \not{\boldsymbol{p}}_4) \chi_{4,+} - \frac{1}{2} K_T T^c_{T\bar{\chi}T\chi} \Big\}
\end{aligned} \tag{5.56}
$$

trivially satisfy all the pole residue constraints. Moreover, if we require the correlator to satisfy the full Optical Theorem for the current correlator we have, we could find the mismatch will only be on the u-channel cause the square of the term we use to match of the η^{ij} contraction and the π_u^{ij} contraction. To see the mismatch explicitly we could write

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the correlator in the form of

$$
\begin{aligned}
&\langle T_1^{\mathrm{TT}} \bar{\chi}_2^- T_3^{\mathrm{TT}} \chi_4^+ \rangle_u \\
&= \frac{4}{K_T E_{13u} E_{24u}} \cdot \Big[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T\right) \\
&+ \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)^i \pi_u^{ij} (p_1 - p_3)_j\right) \Big] \\
&\cdot \bar{\chi_2}^- (1 - \hat{\boldsymbol{p}}_2) \left[(2\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}} \pi_u^{ij} \gamma_j) - (2\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}} \pi_u^{ij} \gamma_j) + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1 - p_3)_i \pi_u^{ij} \gamma_j\right] (1 + \hat{\boldsymbol{p}}_4) \chi_{4,+} \\
&+ \frac{4}{K_T} \cdot \left[\frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u^2}\right)\right] \\
&\cdot \bar{\chi_2}^- (1 - \hat{\boldsymbol{p}}_2) \left[(2\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}} \pi_u^{ij} \gamma_j) - (2\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}} \pi_u^{ij} \gamma_j) + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1 - p_3)_i \pi_u^{ij} \gamma_j\right] (1 + \hat{\boldsymbol{p}}_4) \chi_{4,+} \\
&+ \frac{4}{K_T} \cdot \Big\{ \left[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)^i \pi_u^{ij} (p_1 - p_3)_j\right) \right] \\
&- \frac{1}{2} \Pi^C_{1,T\bar{\chi}T\chi} \Big\} \\
&\cdot \bar{\chi_2}^- (1 - \hat{\boldsymbol{p}}_2) \left[(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \frac{E_1 - E_3}{E_u^2} \gamma_0\right] (1 + \hat{\boldsymbol{p}}_4) \chi_{4,+} \\
&+ \frac{4 E_{24u} E_{13u}}{K_T} \cdot \Big\{ \left[\frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u^2}\right)\right] \cdot \bar{\chi_2}^- (1 - \hat{\boldsymbol{p}}_2) \left[(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \frac{E_1 - E_3}{E_u^2} \gamma_0\right] \\
&\cdot (1 + \hat{\boldsymbol{p}}_4) \chi_{4,+} - \frac{1}{2} \Pi^C_{2,T\bar{\chi}T\chi} \Big\}
\end{aligned}
\tag{5.57}
$$

Notice in the RHS of the Fermion Optical theorem in (3.64), the flipping external energy term has the C-conjugate which is the conjugate of the correlator with $2 \leftrightarrow 4$ in the gravitational interaction case. In this case, the C-operation will give the additional negative

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sign for the third term with γ_0 sandwiched between the lifters. Then we'll have

$$\Delta_{OPT} = \langle T_1 \bar{\chi}_{2,-,A} T_3 \chi_{4,B,+} \rangle_u (E_{1\sim4}, E_u, \boldsymbol{p}_{1\sim4}) + C \langle T_1 \bar{\chi}_{2,-,A} T_3 \chi_{4,+,B} \rangle_u (-E_{1\sim4}, E_u, \boldsymbol{p}_{1\sim4})$$

$$- \frac{(-8E_u)}{(E_{13}^2 - E_u^2)(E_{24}^2 - E_u^2)}$$

$$\cdot \left[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{\epsilon}_3^T\right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T\right) + \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)^i \pi_u^{ij} (p_1 - p_3)_j\right) \right]$$

$$\cdot \left(\left[(2\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}} \pi_u^{ij} \gamma_j) - (2\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}} \pi_u^{ij} \gamma_j) + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1 - p_3)_i \pi_u^{ij} \gamma_j \right] (1 + \hat{\boldsymbol{p}}_4) \right)_{AB}$$

$$= 4E_u \left[\frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u^2} \right) \right] \cdot \left((1 - \hat{\boldsymbol{p}}_2) \left[(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \frac{E_1 - E_3}{E_u^2} \gamma_0 \right] (1 + \hat{\boldsymbol{p}}_4) \right)_{AB}$$

So we only need to shift the correlator with the term vanishing on all the pole residue

$$\langle T_1 \bar{\chi}_{2,-,A} T_3 \chi_{4,B,+} \rangle_u$$

$$\to \langle T_1 \bar{\chi}_{2,-,A} T_3 \chi_{4,B,+} \rangle_u$$

$$- 2E_u \left[\frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u^2} \right) \right] \cdot \left((1 - \hat{\boldsymbol{p}}_2) \left[(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \frac{E_1 - E_3}{E_u^2} \gamma_0 \right] (1 + \hat{\boldsymbol{p}}_4) \right)_{AB} . \tag{5.58}$$

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Then the u channel correlator,

$$
\begin{aligned}
&\langle T_1^{\mathrm{TT}} \bar{\chi}_2^- T_3^{\mathrm{TT}} \chi_4^+ \rangle_u \\
&= \frac{4}{K_T E_{13u} E_{24u}} \\
&\cdot \{ \left[\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^T \right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_3^T \right) - \left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T \right) \left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1^T \right) \right. \\
&+ \left. \frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left((p_2 - p_4)^\mu (p_1 - p_3)_\mu + K_T \frac{(E_1 - E_3)(E_2 - E_4)}{E_u} \right) \right] \\
&\cdot \bar{\chi_2}^- (1 - \not{\boldsymbol{p}}_2) \\
&\cdot \left[(2\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) \not{\epsilon}_3^{\mathrm{T}} - (2\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \not{\epsilon}_3^{\mathrm{T}} + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \left((p_1 - p_3)_\mu \gamma^\mu + K_T \frac{E_1 - E_3}{E_u} \gamma_0 \right) \right] \\
&\cdot (1 + \not{\boldsymbol{p}}_4) \chi_{4,+} - \frac{K_T}{2} T^C_{T\bar{\chi}T\chi} \} \\
&- 2E_u \left[\frac{(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})}{4} \left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u^2} \right) \right] \\
&\cdot \bar{\chi}_2^- \left((1 - \not{\boldsymbol{p}}_2)(2\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{p}_2) \left[(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \frac{E_1 - E_3}{E_u^2} \gamma_0 \right] \cdot (1 + \not{\boldsymbol{p}}_4) \right) \chi_{4,+}.
\end{aligned}
\tag{5.59}
$$

satisfies the full Optical theorem for exchanging gravitons. As a remark, if we write the COT in discontinuity form, in this form we don't need to do C conjugate (with $2 \leftrightarrow 4$) but flipping the internal energy E_u instead, we could still get the same shift term that makes COT be satisfied. Then for the contact diagram, we could write it as the contact amplitudes over K_T that trivially match the total energy pole residue, like

$$
\begin{aligned}
\langle T_1 \bar{\chi}_{2,-,A} T_3 \chi_{4,B,+} \rangle_c = \frac{1}{K_T} \bar{\chi}_2^- (1 - \not{\boldsymbol{p}}_2) \{ & 2(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \\
& \left[(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}}) \not{\boldsymbol{\epsilon}}_3^{\mathrm{T}} - (\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \not{\boldsymbol{\epsilon}}_3^{\mathrm{T}} - (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) ((P_3 - P_1)^\mu \eta_{\mu\nu} \gamma^\nu) \right] \\
& + 2(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}}) \not{\boldsymbol{\epsilon}}_1^{\mathrm{T}} (p_1 + p_4)_\mu \eta^{\mu\nu} \gamma_\nu \not{\boldsymbol{\epsilon}}_3^{\mathrm{T}} \} (1 + \not{\boldsymbol{p}}_4) \chi_{4,+}.
\end{aligned}
\tag{5.60}
$$

We'll come up with a problem the for the first $\eta_{\mu\nu}$ contraction, we already find an equivalent expression on the total energy pole in (5.49) (But we have no known equivalent

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expression for the second $\eta_{\mu\nu}$ that differs a term vanishing on the total energy pole.),

$$\left((p_1-p_3)_\mu\gamma^\mu + K_T\frac{E_1-E_3}{E_u}\gamma_0\right) \sim \left[(p_1-p_3)_i\pi_u^{ij}\gamma_j + E_{13u}E_{24u}\frac{E_1-E_3}{E_u^2}\gamma_0\right], \tag{5.61}$$

when this term is sandwiched between the lifters and the boundary condition of the spinor. But because we could easily check the equivalent expression or the RHS one won't satisfy the Fermion COT for the contact term. So the LHS is the only form that satisfies all the constants for the contact term. Actually with the (5.60), we already fix the full $\langle T^{\text{TT}}\bar{\chi}T^{\text{TT}}\chi\rangle$. We could not write down any additional polynomial ansatz that is vanishing at all the pole residue and the Fermion COT.

Now we need to check the consistency between the partial energy pole residue and the longitudinal correlator determined by the WT identities.

$$\begin{aligned}
\langle T_1^L\bar{\chi}_{2,-}T_3\chi_+\rangle &= \left(\frac{-\hat{p}_{1,i}\epsilon_1^i}{E_1}\right)\cdot p_{1,j}\epsilon_{1,j'}\langle T_1^{jj'}\bar{\chi}_{2,-}J_3\chi_+\rangle \\
&= \left(\frac{-\hat{p}_{1,i}\epsilon_1^i}{E_1}\right)\cdot\Big\{-\frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_2)\langle\bar{\chi}_{2+1,-}T_3\chi_{4,+}\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_4)\langle\bar{\chi}_{2,-}T_3\chi_{4+1,+}\rangle \\
&+ \frac{1}{16}\bar{\chi}_{2,-,A}([\not{\boldsymbol{p}}_1,\not{\epsilon}_1])^{AB}\langle\bar{\chi}_{2+1,B,-}T_3\chi_{4,+}\rangle' - \frac{1}{16}\langle\bar{\chi}_{2,-}T_3\chi_{4+1,+,A}\rangle'([\not{\boldsymbol{p}}_1,\not{\epsilon}_1])^{AB}\chi_{4,+,B} \\
&+ (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\boldsymbol{p}_{3,a}\langle T_{3+1}^a\bar{\chi}_{2,-}\chi_{4,+}\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\langle T_{3+1}\bar{\chi}_{2,-}\chi_{4,+}\rangle \\
&+ \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{p}_2)\langle\bar{\chi}_{2+3+1,-}\chi_{4,+}\rangle + \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{p}_4)\langle\bar{\chi}_{2,-}\chi_{4+3+1,+}\rangle \\
&+ \frac{1}{32}(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3)\bar{\chi}_{2,-,A}([\not{\epsilon}_3,\not{\epsilon}_1])^{AB}\langle\bar{\chi}_{2+3+1,-,B}\chi_{4,+,C}\rangle\chi_{4,+}^C \\
&- \frac{1}{32}(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3)\bar{\chi}_{2,-,A}\langle\bar{\chi}_{2,-}^A\chi_{4+3+1,+}^B\rangle([\not{\epsilon}_3,\not{\epsilon}_1])_{BC}\chi_{4,+}^C\Big\}
\end{aligned} \tag{5.62}$$

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First we take the E_{12s} pole

$$\underset{E_{12s}=0}{\text{Res}}\langle T_1^L \bar{\chi}_{2,-} T_3 \chi_+ \rangle = 0 = M_{3,A}(h_1^L \bar{\chi}_2 \chi_s) \left((\frac{1+\gamma_0}{2}) \frac{-\hat{\not{p}}_s}{2} (\frac{1-\gamma_0}{2}) \right)^{AB} \tilde{\Psi}_{3,B}(h_3 \bar{\chi}_{-s} \chi_4) \tag{5.63}$$

by the amplitude Ward Identities, we know that $M_3(h_1^L \bar{\chi}_2 \chi_s) = p_1^i \epsilon_1^j \tilde{M}_{3,ij}(h_1 \bar{\chi}_2 \chi_s) = p_1^\mu \epsilon_1^\nu M_{3,\mu\nu}(\gamma_1 \bar{\chi}_2 \chi_s) = 0$. And for the other pole for s-channel

$$\begin{aligned}
\underset{E_{34s}=0}{\text{Res}}\langle T_1^L \bar{\chi}_{2,-} T_3 \chi_+ \rangle &= (-\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) \bar{\chi}_{2,-,A} \left(-\frac{1}{2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_2) + \frac{1}{16}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \right)^{AB} \underset{E_{34s}=0}{\text{Res}} \langle \bar{\chi}_{1+2,-,B} T_3 \chi_{4,+} \rangle \\
&= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) \bar{\chi}_{2,-,A} \left(\frac{1}{2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_2) - \frac{1}{16}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \right)^{AB} M_{BC,3}(\bar{\chi}_{-s} h_3 \chi_4) \chi_{4,+}^C \\
&= \tilde{\Psi}_3^A (T_1^L \bar{\chi}_{2,-} \chi_{s,+}) \left((\frac{1+\gamma_0}{2}) \frac{-\hat{\not{p}}_s}{2} (\frac{1-\gamma_0}{2}) \right)_{AA'} M_3^{A'B} (\bar{\chi}_{-s} h_3 \chi_4) \chi_{4,B,+}
\end{aligned} \tag{5.64}$$

in which we use the 3-point WT identity,

$$\begin{aligned}
&\langle T_1^L \bar{\chi}_{2,-} \chi_{s,+}^A \rangle \\
&= (\frac{-\hat{p}_{1,i}\epsilon_1^i}{E_1}) \bar{\chi}_{2,-,C} \left(-\frac{1}{2}(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1) \hat{\not{\boldsymbol{p}}}_{\boldsymbol{s}} + \frac{1}{2}(\boldsymbol{p}_s \cdot \boldsymbol{\epsilon}_1) \hat{\not{\boldsymbol{p}}}_{\boldsymbol{2}} + \frac{1}{16}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \hat{\not{\boldsymbol{p}}}_{\boldsymbol{s}} + \frac{1}{16} \hat{\not{\boldsymbol{p}}}_{\boldsymbol{2}} [\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \right)^{CA} \\
&\tilde{\Psi}_3^A (T_1^L \bar{\chi}_{2,-} \chi_{s,+}) = \langle T_1^L \bar{\chi}_{2,-} \chi_{s,+}^A \rangle - \langle T_1^L \bar{\chi}_{2,-} \chi_{s,+}^A \rangle |_{E_s \to -E_s} \\
&= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) (\bar{\chi}_{2,-,C}) \left((\frac{1}{2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_2) - \frac{1}{16}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1])(2\hat{\not{\boldsymbol{p}}}_s) \right)^{CA}
\end{aligned} \tag{5.65}$$

and the identity,

$$\begin{aligned}
&\tilde{\Psi}_3^A (T_1^L \bar{\chi}_{2,-} \chi_{s,+}) \left((\frac{1+\gamma_0}{2}) \frac{-\hat{\not{p}}_s}{2} (\frac{1-\gamma_0}{2}) \right) \\
&= \bar{\chi}_{2,-} \left(\frac{1}{2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_2) - \frac{1}{16}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \right) (2\hat{\not{\boldsymbol{p}}}_{\boldsymbol{s}}) \left((\frac{1+\gamma_0}{2}) \frac{-\hat{\not{p}}_s}{2} (\frac{1-\gamma_0}{2}) \right) \\
&= \bar{\chi}_{2,-} \left(\frac{1}{2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_2) - \frac{1}{16}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \right) (2\hat{\not{\boldsymbol{p}}}_{\boldsymbol{s}}) \left((\frac{1+\gamma_0}{2}) \frac{-\hat{\not{p}}_s}{2} \right) \\
&= \bar{\chi}_{2,-} (\frac{1-\gamma_0}{2}) \left(\frac{1}{2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_2) - \frac{1}{16}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \right) (2\hat{\not{\boldsymbol{p}}}_{\boldsymbol{s}}) \frac{-\hat{\not{p}}_s}{2} \\
&= \bar{\chi}_{2,-}.
\end{aligned} \tag{5.66}$$

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Then the last equality is trivially satisfied. For the pole of E_{32t} and E_{14t}, the consistency could be checked if we take $1 \to 3$ for all the above derivation. Now we check the partial energy pole residue of the u-channel correlator. First we take the E_{13u} pole

$$\underset{E_{13u}=0}{\text{Res}} \langle T_1^L \bar{\chi}_{2,-} T_3 \chi_+ \rangle = 0 = \tilde{M}_{i_u i'_u}(h_1^L h_3 h_u) \frac{\Pi_{u,(2,2)}^{i_u i'_u j_u j'_u}}{2E_u} \tilde{\Psi}_{j_u j'_u, \langle T_{-u}^{\text{TT}} \bar{\chi}_{2,-} \chi_{4,+} \rangle} \tag{5.67}$$

by the amplitude Ward Identities, we know that $M_3(h_1 h_3 h_u) = p_1^i \epsilon_1^j \tilde{M}_{3,ij}(h_1 h_3 h_u) = p_1^\mu \epsilon_1^\nu M_{3,\mu\nu}(h_1 h_3 h_u) = 0$. And for the other pole for u-channel

$$\begin{aligned}
\underset{E_{24u}=0}{\text{Res}} \langle T_1^L \bar{\chi}_{2,-} T_3 \chi_+ \rangle &= (-\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) \left((\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3) p_{3,a} - \frac{1}{2} (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \epsilon_{3,a} \right) \epsilon_{3,b} \Pi_{u,(2,2)}^{abcd} \underset{E_{24u}=0}{\text{Res}} \langle \hat{T}_{cd,1+3} \bar{\chi}_{2,-} \chi_{4,+} \rangle \\
&= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) \left(-(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3) p_{3,a} + \frac{1}{2} (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \epsilon_{3,a} \right) \epsilon_{3,b} \Pi_{u,(2,2)}^{abcd} \tilde{M}_{3,cd}(h_{-u} \bar{\chi}_2 \chi_4) \\
&= \tilde{\Psi}_3^{i_u i'_u}(T_1^L T_3 T_u) \frac{\Pi_{u,(2,2)}^{i_u i'_u j_u j'_u}}{2E_u} \tilde{M}_3^{j_u j'_u}(h_{-u} \bar{\chi}_2 \chi_4)
\end{aligned} \tag{5.68}$$

in the last line we use the 3-point WT identity,

$$\begin{aligned}
\langle T_1^L T_3 T_u \rangle^{i_u, i'_u} &= \frac{(-\epsilon_{1,i} \hat{p}_1^i)}{E_1} \cdot \Big[(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3)\, p_{3,k} \epsilon_{3,l} E_u \pi_u^{k(i_u} \pi_u^{l i'_u)} - \frac{1}{2} (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \epsilon_{3,k} \epsilon_{3,l} E_u \pi_u^{k(i_u} \pi_u^{l\, i'_u)} \\
&\quad + \epsilon_1^{(i_u} p_{u,k} \pi_3^{ka} \pi_3^{i'_u)b} \epsilon_{3,a} \epsilon_{3,b} E_3 - \frac{1}{2} (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_u) \pi_3^{i_u a} \pi_3^{i'_u b} \epsilon_{3,a} \epsilon_{3,b} E_3 \Big] \\
\tilde{\Psi}_3{}^{i_u, i'_u}(T_1^L T_3 T_u) &= \langle T_1^L T_3 T_u \rangle^{i_u, i'_u} - \langle T_1^L T_3 T_u \rangle^{i_u, i'_u}|_{E_s \to -E_s} \\
&= 2E_u (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) \left[-(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3)\, p_{3,k} \epsilon_{3,l} \pi_u^{k(i_u} \pi_u^{l i'_u)} + \frac{1}{2} (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \epsilon_{3,k} \epsilon_{3,l} \pi_u^{k(i_u} \pi_u^{l\, i'_u)} \right],
\end{aligned} \tag{5.69}$$

and with the identity,

$$\begin{aligned}
&\tilde{\Psi}_3^{i_u i'_u}(T_1^L T_3 T_u) \frac{\Pi_{u,(2,2)}^{i_u i'_u j_u j'_u}}{2E_u} \tilde{M}_3^{j_u j'_u}(h_{-u} \bar{\chi}_2 \chi_4) \\
&= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) \left[-(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3)\, p_{3,k} \epsilon_{3,l} E_u + \frac{1}{2} (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \epsilon_{3,k} \epsilon_{3,l} \right] \Pi_{u,(2,2)}^{klmn} \tilde{M}_{3,mn}(h_{-u} \bar{\chi}_2 \chi_4) \\
&= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1}) \left[-(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3)\, p_{3,k} E_u + \frac{1}{2} (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \epsilon_{3,k} \right] \epsilon_{3,l} \Pi_{u,(2,2)}^{klmn} \tilde{M}_{3,mn}(h_{-u} \bar{\chi}_2 \chi_4),
\end{aligned} \tag{5.70}$$

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in which we use the fact that amplitude is symmetric.

5.3.3 $\langle T\bar{\psi}^- T\psi^+ \rangle$

First, we need to decompose the amplitude of $M(h\bar{\psi}h\psi)$ channel by channel. We could refer to [6] and change the momentum and overall normalization convention to what we use in the thesis. For

$$\begin{aligned} p_s^\mu &= p_3^\mu + p_4^\mu \\ p_t^\mu &= p_1^\mu + p_4^\mu \\ p_u^\mu &= p_2^\mu + p_4^\mu \end{aligned} \tag{5.71}$$

We could write down the amplitude (We already take $\kappa = 1$.)

$$\begin{aligned} M(h_1\bar{\psi}_2h_3\psi_4) &= M_s(h_1\bar{\psi}_2h_3\psi_4) + M_t(h_1\bar{\psi}_2h_3\psi_4) + M_u(h_1\bar{\psi}_2h_3\psi_4) + M_c(h_1\bar{\psi}_2h_3\psi_4) \\ M_s(h_1\bar{\psi}_2h_3\psi_4) &= \frac{1}{S}\left[M_{\mu_s}(\gamma_1\gamma_2\gamma_s)M^{\mu_s}(\gamma_{-s}\gamma_3\gamma_4)\right]\bar{u}_2\not{\epsilon}_1(-\not{P}_3-\not{P}_4)\not{\epsilon}_3 u_4 \\ M_t(h_1\bar{\psi}_2h_3\psi_4) &= \frac{1}{T}\left[M_{\mu_t}(\gamma_3\gamma_2\gamma_t)M^{\mu_t}(\gamma_{-t}\gamma_1\gamma_4)\right]\bar{u}_2\not{\epsilon}_1(-\not{P}_1-\not{P}_4)\not{\epsilon}_3 u_4 \\ M_u(h_1\bar{\psi}_2h_3\psi_4) &= \frac{1}{U}\left[M_{\mu'_u}(\gamma_1\gamma_3\gamma_u)M^{\mu'_u}(\gamma_{-u}\gamma_2\gamma_4)\right]\cdot\left[M_{\mu_u}(\gamma_1\gamma_3\gamma_u)\cdot(-\bar{u}_2\gamma^{\mu_u}u_4)\right] \\ M_c(h_1\bar{\psi}_2h_3\psi_4) &= \frac{1}{2}\left[M_c(\gamma_3\gamma_2\gamma_4\gamma_1)-M_c(\gamma_3\gamma_4\gamma_2\gamma_1)\right]\cdot\left[M_{\mu_u}(\gamma_1\gamma_3\gamma_u)\cdot(-\bar{u}_2\gamma^{\mu_u}u_4)\right] \\ &+\frac{1}{2}\left[M_c(\gamma_2\gamma_3\gamma_4\gamma_1)\right]\left[\bar{u}_2\not{\epsilon}_1(-\not{P}_3-\not{P}_4)\not{\epsilon}_3 u_4+\bar{u}_2\not{\epsilon}_1(-\not{P}_1-\not{P}_4)\not{\epsilon}_3 u_4\right] \end{aligned} \tag{5.72}$$

where we define the amplitude with Yang-Mills 3-pt and 4-pt contact amplitude with color factor extracted and polarization extracted as

$$\begin{aligned} M(\gamma_1\gamma_2\gamma_3) &= \left[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot(p_1-p_2))+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot(p_2-p_3))+(\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot(p_3-p_1))\right]. \\ M_{\mu_3}(\gamma_1\gamma_2\gamma_3) &= \left[(\epsilon_1\cdot\epsilon_2)(p_1-p_2)_{\mu_3}+(\epsilon_{2,\mu_3})(2\epsilon_1\cdot p_2)+(\epsilon_{1,\mu_3})(-2\epsilon_2\cdot p_1)\right] \\ M_c(\gamma_1\gamma_2\gamma_3\gamma_4) &= \left[2(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)-(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)\right]. \end{aligned} \tag{5.73}$$

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which makes Ward Identity of the amplitude satisfied. (We already check that with Mathematica.) Actually, we could identify the amplitude in a double-copy and moreover factorized form.

$$
\begin{aligned}
M_s(h_1\bar{\psi}_2 h_3\psi_4) &= (M_{\mu_s}(\gamma_1\gamma_2\gamma_s)M(\gamma_1\bar{\psi}_2\psi_s))\frac{(-\not{P}_3-\not{P}_4)}{S}(M^{\mu_s}(\gamma_3\gamma_4\gamma_{-s})M(\gamma_1\bar{\psi}_{-s}\psi_4)) \\
&= M_{\mu_s}(h_1\bar{\psi}_2\psi_s)\frac{\eta^{\mu_s\nu_s}\cdot(-\not{P}_3-\not{P}_4)}{S}M_{\nu_s}(h_3\bar{\psi}_s\psi_4) \\
M_t(h_1\bar{\psi}_2 h_3\psi_4) &= M_{\mu_t}(h_3\bar{\psi}_2\psi_t)\frac{\eta^{\mu_t\nu_t}\cdot(-\not{P}_1-\not{P}_4)}{T}M_{\nu_t}(h_1\bar{\psi}_t\psi_4) \\
M_u(h_1\bar{\psi}_2 h_3\psi_4) &= M_{\mu_u\mu'_u}(h_1h_3h_u)\frac{\eta^{\mu_u,\nu_u}\eta^{\mu'_u,\nu'_u}}{U}M_{\nu_u\nu'_u}(h_{-u}\bar{\psi}_2\psi_4) \\
M_c(h_1\bar{\psi}_2 h_3\psi_4) &= \frac{1}{2}\left[M_c(\gamma_3\gamma_2\gamma_4\gamma_1)-M_c(\gamma_3\gamma_4\gamma_2\gamma_1)\right]\cdot\left[U\cdot M_u(\gamma_1\bar{\chi}_2\gamma_3\bar{\chi}_4)\right] \\
&\quad+\frac{1}{2}\left[M_c(\gamma_2\gamma_3\gamma_4\gamma_1)\right]\left[S\cdot M_s(\gamma_1\bar{\chi}_2\gamma_3\bar{\chi}_4)+T\cdot M_t(\gamma_1\bar{\chi}_2\gamma_3\bar{\chi}_4)\right]
\end{aligned}
\tag{5.74}
$$

Moreover in the transverse part bootstrap the amplitude, and the polarization is written in the Coulomb gauge, especially we write

$$
M_{\mu_3}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_3) = \left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})(p_1-p_2)_{\mu_3}+(\boldsymbol{\epsilon}_2^{\mathrm{T},i}\eta_{i,\mu_3})(2\epsilon_1\cdot p_2)+(\boldsymbol{\epsilon}_1^{\mathrm{T},i}\eta_{i,\mu_3})(-2\epsilon_2\cdot p_1)\right]. \tag{5.75}
$$

s/t channel Then for the s-channel pure transverse correlator, the constraints of the correlator will be

$$
\begin{aligned}
\underset{K_T\to 0}{\mathrm{Res}}\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_s &= M_{\mu_s,A_s}(h_1\bar{\psi}_2\psi_s)\eta^{\mu_s\nu_s}\left[\frac{-\not{P}_3-\not{P}_4}{S}\right]_{A_sB_s}M_{B_s,\nu_s}(h_3\bar{\psi}_{-s}\psi_4) \\
&= \left[M_{\mu_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)M^{\mu_s}(\gamma_{-s}\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}})\right]\cdot\bar{\chi}_2^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\left[\frac{\not{P}_3+\not{P}_4}{(E_{12s}E_{34s})}\right](1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}
\end{aligned}
\tag{5.76}
$$

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$$\operatorname*{Res}_{E_{12s}\to 0}\langle T_1^{\mathsf{TT}}\bar{\psi}_2^{\mathsf{T},-}T_3^{\mathsf{TT}}\psi_4^{\mathsf{T},+}\rangle_s$$
$$= \bar{u}_2\not{\epsilon}_1^{\mathsf{T}}\cdot\tilde{M}_{i_s}(\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_s)\cdot\frac{-\pi_s^{i_sj_s}}{2E_s}\left(\frac{\not{S}(-\gamma_0)}{E_{34s}}-\frac{(\not{S})}{E_3+E_4-E_s}\right)\cdot\tilde{M}_{j_s}(\gamma_3^{\mathsf{T}}\gamma_4^{\mathsf{T}}\gamma_{-s})\cdot\not{\epsilon}_3^{\mathsf{T}}u_4-\frac{1}{4}P_{L,3/2}$$
$$= (\tilde{M}_{i_s}(\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_s)\pi_s^{i_sj_s}\tilde{M}_{j_s}(\gamma_3^{\mathsf{T}}\gamma_4^{\mathsf{T}}\gamma_{-s}))$$
$$\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\cdot\not{\epsilon}_1^{\mathsf{T}}\cdot\frac{1}{2E_s}\left(\frac{\not{S}(\gamma_0)}{E_{34s}}+\frac{(\not{S})}{K_T}\right)\cdot\not{\epsilon}_3^{\mathsf{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}-\frac{1}{4}P_{L,3/2}$$
$$\operatorname*{Res}_{E_{34s}\to 0}\langle T_1^{\mathsf{TT}}\bar{\psi}_2^{\mathsf{T},-}T_3^{\mathsf{TT}}\psi_4^{\mathsf{T},+}\rangle_s$$
$$= (\tilde{M}_{i_s}(\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_t)\pi_s^{i_sj_s}\tilde{M}_{j_s}(\gamma_3^{\mathsf{T}}\gamma_4^{\mathsf{T}}\gamma_{-s}))\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathsf{T}}\cdot\frac{1}{2E_s}\left(\frac{(\gamma_0)\not{S}_-}{E_{12s}}-\frac{(\not{S}_-)}{K_T}\right)\cdot\not{\epsilon}_3^{\mathsf{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}$$
$$-\frac{1}{4}P_{R,3/2}, \tag{5.77}$$

in which

$$P_{L,3/2} := \tilde{M}_{i_s}(\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_s)\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathsf{T}}(1+\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}\cdot\not{\pi}_s^{i_s}\hat{\not{\boldsymbol{p}}}_s\not{\pi}_s^{j_s}\cdot\frac{1-\gamma_0}{2}$$
$$\cdot\left[(1+\hat{\not{\boldsymbol{p}}}_s)\not{\epsilon}_3^{\mathsf{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}\cdot\frac{1}{E_{34s}}\cdot\tilde{M}_{j_s}(\gamma_3^{\mathsf{T}}\gamma_4^{\mathsf{T}}\gamma_{-s})\right]\Bigg|_{-E_s}^{E_s}$$
$$P_{R,3/2} := \left[\tilde{M}_{i_s}(\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_s)\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathsf{T}}(1+\hat{\not{\boldsymbol{p}}}_s)\right]\Bigg|_{-E_s}^{E_s}\cdot\frac{1+\gamma_0}{2}\cdot\not{\pi}_s^{i_s}\hat{\not{\boldsymbol{p}}}_s\not{\pi}_s^{j_s}\cdot\frac{1-\gamma_0}{2}$$
$$\cdot(1+\hat{\not{\boldsymbol{p}}}_s)\not{\epsilon}_3^{\mathsf{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}\cdot\frac{1}{E_{34s}}\cdot\tilde{M}_{j_s}(\gamma_3^{\mathsf{T}}\gamma_4^{\mathsf{T}}\gamma_{-s}) \tag{5.78}$$

in which we define $\not{\pi}_s^i := \pi^{ij}\gamma_j$.

So all the constants of transverse $\langle T_1^{\mathsf{TT}}\bar{\psi}_2^{\mathsf{T},-}T_3^{\mathsf{TT}}\psi_4^{\mathsf{T},+}\rangle_s$ are the same $\langle J_1^{\mathsf{T}}\bar{\psi_2}^{-}J_3^{\mathsf{T}}\psi_4^{+}\rangle_s$ except that we need to match the prefactor of η-contraction and the π_s contraction in $(\tilde{M}_{i_s}(\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_t)\pi_s^{i_sj_s}\tilde{M}_{j_s}(\gamma_3^{\mathsf{T}}\gamma_4^{\mathsf{T}}\gamma_{-s}))$ and $M_{\mu_s}(\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_s)M^{\mu_s}(\gamma_{-s}\gamma_3^{\mathsf{T}}\gamma_4^{\mathsf{T}})$ to match the first term of the partial energy pole residue and the amplitude. We could use (5.46) and (5.48) to

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prove that

$$
\begin{aligned}
&M_{\mu_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)M^{\mu_s}(\gamma_{-s}\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}) + (\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})K_T\frac{(E_1-E_2)(E_3-E_4)}{E_s} \\
&= (\tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_t)\pi_s^{i_s j_s}\tilde{M}_{j_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})) + (\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})E_{12s}E_{34s}\frac{(E_1-E_2)(E_3-E_4)}{E_s^2}
\end{aligned}
\tag{5.79}
$$

But to match the second term $P_{R,3/2}/P_{L,3/2}$ of the partial energy pole we need to make it to match a term vanishing on the total energy pole. And match $P_{R,3/2}/P_{L,3/2}$ with the term vanishing on the E_R/E_L pole individually. First we need to use the 4D γ-trace of the 3 point amplitude to rexpress the $\not{\pi}_s^i$ trace term in the $P_{R,3/2}/P_{L,3/2}$.

$$
\begin{aligned}
M(T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T}}\psi_{s,\mu})\gamma^\mu &= -\bar{u}_2\not{\boldsymbol{\epsilon}}_1^{\mathrm{T}}\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})((-\not{P}_s)-2\not{P}_2)+\not{\boldsymbol{\epsilon}}_2^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{p}_2)-2\not{\boldsymbol{\epsilon}}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_2^{\mathrm{T}}\cdot\boldsymbol{p}_1)\right] \\
&= -\bar{u}_2\not{\boldsymbol{\epsilon}}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})(-\not{P}_s) \\
&= M(T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T}}\psi_{s,0})\gamma_0 + M(T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T}}\psi_{s,i})\pi_s^{ij}\gamma_j - M(T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T}}\psi_{s,i})\hat{p}_s^i\hat{p}_s^j\gamma_j \\
&= -\bar{u}_2\not{\boldsymbol{\epsilon}}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})[(E_1-E_2)\gamma_0-(p_1-p_2)_i\hat{p}_s^i\hat{\not{\boldsymbol{p}}}_s] + \tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)\cdot\bar{u}_2\not{\boldsymbol{\epsilon}}_1^{\mathrm{T}}\not{\boldsymbol{\pi}}_s^{i_s} \\
\gamma^\mu M(T_3^{\mathrm{TT}}\bar{\psi}_{-u,\mu}\psi_4^{\mathrm{T}}) &= -(\not{P}_s)(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_3^{\mathrm{T}}u_4 \\
&= -[(E_3-E_4)\gamma_0-(p_3-p_4)_i\hat{p}_s^i\hat{\not{\boldsymbol{p}}}_s](\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_3^{\mathrm{T}}u_4 + \tilde{M}_{i_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_3^{\mathrm{T}}u_4
\end{aligned}
\tag{5.80}
$$

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then we could first simplify then rexpress the $P_{L,3/2}$,

$$
\begin{aligned}
P_{L,3/2} &= \tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathrm{T}}\not{\pi}_s^{i_s}(1-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1-\gamma_0}{2}\cdot\hat{\not{\boldsymbol{p}}}_s\cdot\frac{1+\gamma_0}{2}\\
&\cdot\left[(1-\hat{\not{\boldsymbol{p}}}_s)\not{\pi}_s^{j_s}\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}\cdot\frac{1}{E_{34s}}\cdot\tilde{M}_{j_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})\right]\Bigg|_{-E_s}^{E_s}\\
&= \tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathrm{T}}\not{\pi}_s^{i_s}(\hat{\not{\boldsymbol{p}}}_s)^2(1-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1-\gamma_0}{2}\cdot\hat{\not{\boldsymbol{p}}}_s\cdot\frac{1+\gamma_0}{2}\\
&\cdot\left[(1-\hat{\not{\boldsymbol{p}}}_s)(\hat{\not{\boldsymbol{p}}}_s)^2\not{\pi}_s^{j_s}\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}\cdot\frac{1}{E_{34s}}\cdot\tilde{M}_{j_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})\right]\Bigg|_{-E_s}^{E_s}\\
&= \tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathrm{T}}\not{\pi}_s^{i_s}(\hat{\not{\boldsymbol{p}}}_s)(1-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}\cdot(\hat{\not{\boldsymbol{p}}}_s)^3\cdot\frac{1-\gamma_0}{2}\\
&\cdot\left[(1-\hat{\not{\boldsymbol{p}}}_s)(\hat{\not{\boldsymbol{p}}}_s)\not{\pi}_s^{j_s}\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}\cdot\frac{1}{E_{34s}}\cdot\tilde{M}_{j_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})\right]\Bigg|_{-E_s}^{E_s}\\
&= \tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathrm{T}}\not{\pi}_s^{i_s}(1+\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}\cdot(-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1-\gamma_0}{2}\\
&\cdot\left[(1+\hat{\not{\boldsymbol{p}}}_s)\not{\pi}_s^{j_s}\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}\cdot\frac{1}{E_{34s}}\cdot\tilde{M}_{j_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})\right]\Bigg|_{-E_s}^{E_s}\\
&= \tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_s)\cdot\bar{\chi_2}^{-}(1-\hat{\not{\boldsymbol{p}}}_2)\not{\epsilon}_1^{\mathrm{T}}\not{\pi}_s^{i_s}(1+\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}\cdot(-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1-\gamma_0}{2}\\
&\cdot\left[\frac{1}{E_{34s}}(1+\hat{\not{\boldsymbol{p}}}_s)\not{\pi}_s^{j_s}\not{\epsilon}_3^{\mathrm{T}}(1+\hat{\not{\boldsymbol{p}}}_4)\chi_{4,+}\cdot\tilde{M}_{j_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})\right]\Bigg|_{-E_s}^{E_s}\\
&= \bar{u}_2\not{\epsilon}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})[(E_1-E_2)\gamma_0+(p_1-p_2)_i\hat{p}_s^i\gamma_0]\cdot(1+\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}\cdot(-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1-\gamma_0}{2}\\
&\cdot\left[\frac{1}{E_{34s}}\cdot(1+\hat{\not{\boldsymbol{p}}}_s)\cdot[(E_3-E_4)\gamma_0-(p_3-p_4)_i\hat{p}_s^i\gamma_0](\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_{\boldsymbol{3}}{}^{\mathrm{T}}u_4\right]\Bigg|_{-E_s}^{E_s}\\
&= \bar{u}_2\not{\epsilon}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\cdot[E_{12s}]\cdot\gamma_0\cdot(1+\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}\cdot(-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1-\gamma_0}{2}\\
&\cdot\left[\frac{1}{E_{34s}}\cdot(1+\hat{\not{\boldsymbol{p}}}_s)\cdot[E_{34s}]\cdot\gamma_0(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_{\boldsymbol{3}}{}^{\mathrm{T}}u_4\right]\Bigg|_{-E_s}^{E_s}\\
&= \bar{u}_2\not{\epsilon}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\cdot[E_{12s}]\cdot\gamma_0\cdot(1+\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}\cdot(-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1-\gamma_0}{2}\cdot(2\hat{\not{\boldsymbol{p}}}_s)\cdot\gamma_0(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_{\boldsymbol{3}}{}^{\mathrm{T}}u_4\\
&= 2E_{12s}\bar{u}_2\not{\epsilon}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\cdot(1-\hat{\not{\boldsymbol{p}}}_s)\cdot\frac{1+\gamma_0}{2}(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_{\boldsymbol{3}}{}^{\mathrm{T}}u_4
\end{aligned}
\tag{5.81}
$$

in which we apply $(\not{\epsilon}^{\mathrm{T}})^2=\not{\epsilon}^{\mathrm{T}}\cdot\not{\epsilon}^{\mathrm{T}}=0$ from the amplitude polarization constraints $\epsilon\cdot\epsilon=0$, $\bar{u}_2\not{\epsilon}_2^{\mathrm{T}}=0$ and the Dirac equation with their version for the conjugate field. And we should notice $\not{P}_s(1+\hat{\not{\boldsymbol{p}}}_s)\frac{1+\gamma_0}{2}=(\not{P}_s)^2\frac{1+\gamma_0}{2}=0$ and $\not{P}_{-s}(1+\hat{\not{\boldsymbol{p}}}_s)\frac{1-\gamma_0}{2}=-(\not{P}_{-s})^2\frac{1+\gamma_0}{2}=0$.

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Similar, we'll have

$$P_{L,3/2} = 2E_{34s}\bar{u}_2 \not{\boldsymbol{\epsilon}}_1^{\mathrm{T}} (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_2^{\mathrm{T}}) \cdot (1 - \hat{\not{\boldsymbol{p}}}_s) \cdot \frac{1+\gamma_0}{2} (\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_4^{\mathrm{T}}) \not{\boldsymbol{\epsilon}}_3^{\mathrm{T}} u_4 \tag{5.82}$$

So indeed, these two terms won't contribute to the partial energy pole we don't need to consider them. Then we could easily extend the bootstrapped result in (5.30) to find

$$\begin{aligned}
&\langle T_1^{\mathrm{TT}} \bar{\psi}_2^{\mathrm{T},-} T_3^{\mathrm{TT}} \psi_4^{\mathrm{T},+} \rangle_s \\
&= \left[(\tilde{M}_{i_s}(\gamma_1^{\mathrm{T}} \gamma_2^{\mathrm{T}} \gamma_t) \pi_s^{i_s j_s} \tilde{M}_{j_s}(\gamma_3^{\mathrm{T}} \gamma_4^{\mathrm{T}} \gamma_{-s})) + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_2^{\mathrm{T}})(\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_4^{\mathrm{T}}) E_{12s} E_{34s} \frac{(E_1 - E_2)(E_3 - E_4)}{E_s^2} \right] \\
&\cdot \bar{\chi}_2{}^{-} (1 - \hat{\not{\boldsymbol{p}}}_2) \not{\boldsymbol{\epsilon}}_1^{\mathrm{T}} \cdot \left[\frac{(\not{P}_3 + \not{P}_4)}{K_T E_{12s} E_{34s}} - \frac{1-\gamma_0}{2} \frac{\not{S}_-}{E_s E_{12s} E_{34s}} \right] \cdot \not{\boldsymbol{\epsilon}}_3^{\mathrm{T}} (1 + \hat{\not{\boldsymbol{p}}}_4) \chi_{4,+}
\end{aligned} \tag{5.83}$$

satisfying all the residue constraints. Similarly, with $1 \leftrightarrow 3$, we'll have the t-channel transverse correlator.

Now to match the full Optical theorem, with a similar calculation to (5.81), we'll find (We could do it in the discontinuity version such that it's channel by channel and easier to do and write down.)

$$\Delta_{s,disc-COT} = \text{COT RHS} - \text{COT LHS} = -\bar{u}_2 \not{\boldsymbol{\epsilon}}_1^{\mathrm{T}} (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_2^{\mathrm{T}}) \cdot \hat{\not{\boldsymbol{p}}}_s \cdot \frac{1+\gamma_0}{2} (\boldsymbol{\epsilon}_3^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_4^{\mathrm{T}}) \not{\boldsymbol{\epsilon}}_3^{\mathrm{T}} u_4 \tag{5.84}$$

it's the polynomial term that won't affect each of the partial energy pole residues. So we

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could add this term back to the above correlator. Now we find the correlator like

$$
\begin{aligned}
&\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_s \\
&= \left[(\tilde{M}_{i_s}(\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_t)\pi_s^{i_s j_s}\tilde{M}_{j_s}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-s})) + (\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})E_{12s}E_{34s}\frac{(E_1-E_2)(E_3-E_4)}{E_s^2}\right] \\
&\cdot\bar{\chi}_2{}^{-}(1-\not{\boldsymbol{p}}_2)\not{\epsilon}_1^{\mathrm{T}}\cdot\left[\frac{(\not{P}_3+\not{P}_4)}{K_T E_{12s}E_{34s}}-\frac{1-\gamma_0}{2}\frac{\not{S}_-}{E_s E_{12s}E_{34s}}\right]\cdot\not{\epsilon}_3^{\mathrm{T}}(1+\not{\boldsymbol{p}}_4)\chi_{4,+} \\
&+\frac{1}{2}\bar{u}_2\not{\epsilon}_1^{\mathrm{T}}(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\cdot\not{\boldsymbol{p}}_s\cdot\frac{1+\gamma_0}{2}(\boldsymbol{\epsilon}_3^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\not{\epsilon}_3{}^{\mathrm{T}}u_4
\end{aligned}
\tag{5.85}
$$

fix all the constraints including the full COT. Similarly, with $1\leftrightarrow 3$, we'll have the t-channel transverse correlator.

u**-channel** For u-channel, it's graviton exchanging in the internal leg. The constraints will be

$$
\begin{aligned}
&\operatorname*{Res}_{K_T\to 0}\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_u = M_{\mu_u\mu_u'}(h_1h_3h_u)\frac{\eta^{\mu_u,\nu_u}\eta^{\mu_u',\nu_u'}}{U}M_{\nu_u\nu_u'}(h_{-u}\bar{\psi}_2\psi_4) \\
&= \frac{1}{E_{13u}E_{24u}}\left[M_{\mu_u'}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)M^{\mu_u'}(\gamma_{-u}\gamma_2^{\mathrm{T}}\gamma_4^{\mathrm{T}})\right] \\
&\cdot\bar{\chi}_2{}^{-}(1-\not{\boldsymbol{p}}_2)\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})\not{\epsilon}_3^{\mathrm{T}}-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\not{\epsilon}_3^{\mathrm{T}}+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_\mu\gamma^\mu\right](1+\not{\boldsymbol{p}}_4)\chi_{4,+}
\end{aligned}
\tag{5.86}
$$

$$
\begin{aligned}
&\operatorname*{Res}_{E_{13u}\to 0}\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_u = \tilde{M}_{i_u i_u'}(h_1h_3h_u)\frac{\Pi_{u,(2,2)}^{i_u i_u' j_u j_u'}}{2E_u}\tilde{\Psi}_{j_u j_u',\langle T_{-u}^{\mathrm{TT}}\bar{\psi}_{2,-}\psi_{4,+}\rangle} \\
&= \frac{1}{E_{24u}^2-E_u^2}\Big\{\left[\tilde{M}_{i_u'}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)\pi_{u,i'j'}\tilde{M}^{j_u'}(\gamma_{-u}\gamma_2^{\mathrm{T}}\gamma_4^{\mathrm{T}})\right] \\
&\cdot\bar{\chi}_2{}^{-}(1-\not{\boldsymbol{p}}_2)\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j\right](1+\not{\boldsymbol{p}}_4)\chi_{4,+} \\
&-\frac{1}{2}\left(\boldsymbol{J}_{3,L}'\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}_{3,L}'\right)\left(\boldsymbol{J}_{3,R}'\cdot\boldsymbol{\pi}_u\cdot\left[\bar{\chi}_2{}^{-}(1-\not{\boldsymbol{p}}_2)(-\boldsymbol{\gamma})(1+\not{\boldsymbol{p}}_4)\chi_{4,+}\right]\right)\Big\}
\end{aligned}
\tag{5.87}
$$

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$$\underset{E_{24u}\to 0}{\text{Res}}\langle T_1^{\text{TT}}\bar{\psi}_2^{\text{T},-}T_3^{\text{TT}}\psi_4^{\text{T},+}\rangle_u = \tilde{\Psi}_{i_u i'_u,\langle T_1^{\text{TT}}T_3^{\text{TT}}T_u^{\text{TT}}\rangle}\frac{\Pi_{u,(2,2)}^{i_u i'_u j_u j'_u}}{2E_u}M_{j_u j'_u}(h_{-u}\bar{\psi}_2\psi_4)$$

$$= \frac{1}{E_{13u}^2 - E_u^2}\Big\{\Big[\tilde{M}_{i'_u}(\gamma_1^{\text{T}}\gamma_3^{\text{T}}\gamma_u)\pi_{u,i'j'}\tilde{M}^{j'_u}(\gamma_{-u}\gamma_2^{\text{T}}\gamma_4^{\text{T}})\Big]$$

$$\cdot\,\bar{\chi}_2{}^-(1-\hat{\not{p}}_2)\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\text{T}})(\epsilon_{3,i}^{\text{T}}\pi_u^{ij}\gamma_j) - (2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\text{T}})(\epsilon_{3,i}^{\text{T}}\pi_u^{ij}\gamma_j) + (\boldsymbol{\epsilon}_1^{\text{T}}\cdot\boldsymbol{\epsilon}_3^{\text{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j\right](1+\hat{\not{p}}_4)\chi_{4,+}$$

$$-\frac{1}{2}\left(\boldsymbol{J}'_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}'_{3,L}\right)\left(\boldsymbol{J}'_{3,R}\cdot\boldsymbol{\pi}_u\cdot\left[\bar{\chi}_2{}^-(1-\hat{\not{p}}_2)(-\boldsymbol{\gamma})(1+\hat{\not{p}}_4)\chi_{4,+}\right]\right)\Big\} \tag{5.88}$$

in which we define the

$$\begin{aligned} J'_{3,L} &= \left[-2(\boldsymbol{\epsilon}_3^{\text{T}}\cdot\boldsymbol{p}_1)\boldsymbol{\epsilon}_1^{\text{T}} + 2(\boldsymbol{\epsilon}_1^{\text{T}}\cdot\boldsymbol{p}_3)\boldsymbol{\epsilon}_3^{\text{T}} + (\boldsymbol{\epsilon}_3^{\text{T}}\cdot\boldsymbol{\epsilon}_1^{\text{T}})(\boldsymbol{p}_1-\boldsymbol{p}_3)\right] \\ J'_{3,R} &= \left[-2(\boldsymbol{\epsilon}_4^{\text{T}}\cdot\boldsymbol{p}_2)\boldsymbol{\epsilon}_2^{\text{T}} + 2(\boldsymbol{\epsilon}_2^{\text{T}}\cdot\boldsymbol{p}_4)\boldsymbol{\epsilon}_4^{\text{T}} + (\boldsymbol{\epsilon}_4^{\text{T}}\cdot\boldsymbol{\epsilon}_2^{\text{T}})(\boldsymbol{p}_2-\boldsymbol{p}_4)\right] \end{aligned} \tag{5.89}$$

which is indeed the $J_{3,L}$ defined in the chapter of $\langle TOTO\rangle$ with different normalization with its relebeling.

Then we need to match the η_{ij} contraction on total energy pole kinematics to $\pi_{u,ij}$ contraction on the first term of the partial energy pole kinematics. Actually, we can do that term by term. We already successfully match the following two term in $\langle TOTO\rangle$ and relabeled $\langle O^*OO^*O\rangle_{Exc\ J}$:

$$(p_1-p_3)^\mu(p_2-p_4)_\mu + K_T\frac{(E_1-E_3)(E_2-E_4)}{E_u} \tag{5.90}$$

$$= (p_1-p_3)^i\pi_{u,ij}(p_2-p_4)^j + E_{13u}E_{24u}\frac{(E_1-E_3)(E_2-E_4)}{E_u^2} \tag{5.91}$$

$$\begin{aligned} &\left[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1^T\right)\right] - \left[\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_3^T\right) - \left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{\epsilon}_1^T\right)\right] \\ &= -\left(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_3^T\right) + \left(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_1^T\right) \\ &= -\left(\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_1^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_3^T\right) + \left(\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_3^T\right)\left(\boldsymbol{p}_2\cdot\hat{\boldsymbol{p}}_{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{p}}_{\boldsymbol{u}}\cdot\boldsymbol{\epsilon}_1^T\right) \\ &= 0 \end{aligned} \tag{5.92}$$

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and the matches we derive in $\langle T\bar{\chi}T\chi\rangle$ in (5.49):

$$\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\left((p_1-p_3)_\mu\gamma^\mu+K_T\frac{E_1-E_3}{E_u}\gamma_0\right)\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$
$$=\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\left[(p_1-p_3)_i\pi_u^{ij}\gamma_j+E_{13u}E_{24u}\frac{E_1-E_3}{E_u^2}\gamma_0\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+} \tag{5.93}$$

and

$$\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\eta^{ij}\gamma_j)-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\eta^{ij}\gamma_j)\right]-\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)\right]$$
$$=-(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)+(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)$$
$$=-(2\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)+(2\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\hat{p}_u^i)(\hat{p}_u^j\gamma_j)$$
$$=0. \tag{5.94}$$

In which we use $\bar{\chi}_{2,-}(1-\hat{\boldsymbol{p}}_2)(E_2\gamma_0+\not{\boldsymbol{p}}_2)=(E_4\gamma_0+\not{\boldsymbol{p}}_4)(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}=0$. Then for the second term of the partial energy pole residue, we can use the 4D trace of the amplitude to transform the second term to the term vanishing at the total energy pole. Like (4.96) and

$$\eta^{\mu\nu}M(h_{-u,\mu\nu}\bar{\psi}_2^{\mathrm{T}}\psi_4^{\mathrm{T}})=-\bar{u}_2\left(-2(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{p}_2)\not{\boldsymbol{\epsilon}}_2^{\mathrm{T}}+2(\boldsymbol{\epsilon}_2^{\mathrm{T}}\cdot\boldsymbol{p}_4)\not{\boldsymbol{\epsilon}}_4^{\mathrm{T}}+(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})(\not{P}_2-\not{P}_4)\right)u_4=0$$
$$=M(h_{-u,00}\bar{\psi}_2^{\mathrm{T}}\psi_4^{\mathrm{T}})+\pi_u^{ij}M(h_{-u,ij}\bar{\psi}_2^{\mathrm{T}}\psi_4^{\mathrm{T}})-\hat{p}_u^i\hat{p}_u^jM(h_{-u,ij}\bar{\psi}_2^{\mathrm{T}}\psi_4^{\mathrm{T}})$$
$$=(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)[(E_2-E_4)(-\gamma_0)-(p_2-p_4)_i(\hat{p}_u)^i(-\hat{\boldsymbol{p}}_u)](1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$
$$+\boldsymbol{J}'_{3,R}\cdot\boldsymbol{\pi}_u\cdot\left[\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)(-\boldsymbol{\gamma})(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}\right]$$
$$=(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)(\frac{E_2-E_4}{E_u^2})(-\gamma_0)\left(E_u^2-E_{24}^2\right)(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}$$
$$+\boldsymbol{J}'_{3,R}\cdot\boldsymbol{\pi}_u\cdot\left[\bar{\chi_2}^-(1-\hat{\boldsymbol{p}}_2)(-\boldsymbol{\gamma})(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}\right] \tag{5.95}$$

in which we use the amplitude polarization condition $\gamma\cdot\psi=0,\psi^\mu=\epsilon^\mu u$ such that for the external field $\bar{u}_2\not{\boldsymbol{\epsilon}}_2^{\mathrm{T}}=0$ with $\not{\boldsymbol{\epsilon}}_4^{\mathrm{T}}u_4=0$, the Dirac equation $\not{P}_4u_4=\bar{u}_2\not{P}_2=0$,and

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the similar trick we do at (5.51). Then the second term of the partial energy pole will be

$$\begin{aligned}
&\left(\boldsymbol{J}'_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}'_{3,L}\right)\left(\boldsymbol{J}'_{3,R}\cdot\boldsymbol{\pi}_u\cdot\left[\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)(-\boldsymbol{\gamma})(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}\right]\right)\\
&=(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})^2(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\left(E_{13u}(-K_T+E_{24u})-(\frac{E_1-E_3}{E_u})^2[E_u^2-E_{13}^2]\right)\\
&\cdot\left((\frac{E_2-E_4}{E_u^2})\left(E_u^2-E_{24}^2\right)\right)\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)(-\gamma_0)(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}
\end{aligned}\tag{5.96}$$

We found it's just the (5.53) with different normalization and now we have an addition factor $(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})$. So, we could easily identify the match

$$\boxed{\begin{aligned}
&\left(\boldsymbol{J}'_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}'_{3,L}\right)\left(\boldsymbol{J}'_{3,R}\cdot\boldsymbol{\pi}_u\cdot\left[\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)(-\boldsymbol{\gamma})(1+\hat{\boldsymbol{p}}_4)\chi_{4,+}\right]\right)\\
&\qquad-4(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})E_RE_L\Pi^C_{1,T\bar{\chi}T\chi}-4(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})E_R^2E_L^2\Pi^C_{2,T\bar{\chi}T\chi}=-4(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})K_TT^c_{T\bar{\chi}T\chi}.
\end{aligned}}\tag{5.97}$$

Then applying all the matches we mentioned before, we could find

$$\begin{aligned}
&\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_u\\
&=\frac{1}{K_TE_{13u}E_{24u}}\\
&\cdot\Big\{\left[(\tilde{M}_{i_u}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)\pi_u^{i_uj_u}\tilde{M}_{j_u}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-u}))+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\boldsymbol{\epsilon}_2^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})E_{13u}E_{24u}\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right]\\
&\cdot\bar{\chi}_2{}^-(1-\hat{\boldsymbol{p}}_2)\\
&\cdot\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_3^{\mathrm{T}}-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_3^{\mathrm{T}}+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\left((p_1-p_3)_\mu\gamma^\mu+K_T\frac{E_1-E_3}{E_u}\gamma_0\right)\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+}\\
&+2(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})K_TT^c_{T\bar{\chi}T\chi}\Big\}
\end{aligned}\tag{5.98}$$

trivially satisfy all the pole residue constraints. Moreover, if we require the correlator to satisfy the full Optical Theorem for the current correlator we have, we could find the mismatch will only be on the u-channel cause the square of the term we use to match the η^{ij} contraction and the π_u^{ij} contraction. To see the mismatch explicitly we could write the

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correlator in the form of

$$
\begin{aligned}
&\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_u \\
&= \frac{1}{K_T E_{13u}E_{24u}}\cdot\left[(\tilde{M}_{i_u}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)\pi_u^{i_u j_u}\tilde{M}_{j_u}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-u}))\right] \\
&\cdot\bar{\chi_2}^{-}(1-\hat{\boldsymbol{p}}_2)\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+} \\
&+\frac{1}{K_T}\cdot\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\boldsymbol{\epsilon}_2^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right] \\
&\cdot\bar{\chi_2}^{-}(1-\hat{\boldsymbol{p}}_2)\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j)+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1-p_3)_i\pi_u^{ij}\gamma_j\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+} \\
&+\frac{1}{K_T}\cdot\left[(\tilde{M}_{i_u}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)\pi_u^{i_u j_u}\tilde{M}_{j_u}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-u}))\right]\cdot\bar{\chi_2}^{-}(1-\hat{\boldsymbol{p}}_2)(2\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{p}_2)\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\frac{E_1-E_3}{E_u^2}\gamma_0\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+} \\
&+\frac{2}{K_T}(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\Pi^c_{1,T\bar{\chi}T\chi} \\
&+\frac{E_{24u}E_{13u}}{K_T}\cdot\left[(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})(\boldsymbol{\epsilon}_{2,T}\cdot\boldsymbol{\epsilon}_{4,T})\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)\right] \\
&\cdot\bar{\chi_2}^{-}(1-\hat{\boldsymbol{p}}_2)(2\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{p}_2)\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\frac{E_1-E_3}{E_u^2}\gamma_0\right](1+\hat{\boldsymbol{p}}_4)\chi_{4,+} \\
&+\frac{2E_{24u}E_{13u}}{K_T}\cdot(\boldsymbol{\epsilon}_4^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_2^{\mathrm{T}})\Pi^c_{2,T\bar{\chi}T\chi}
\end{aligned}
\tag{5.99}
$$

Notice in the RHS of the Fermion Optical theorem in (3.64), the flipping external energy term has the C-conjugate which is the conjugate of the correlator with $2\leftrightarrow 4$ in the gravitational interaction case. In this case the C-operation will give the additional negative sign

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for the third term with γ_0 sandwiched between the lifters. Then we have

$$\Delta_{OPT} = \langle T_1^{\mathrm{TT}}\bar{\psi}_{2,A}^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_{4,B}^{\mathrm{T},+}\rangle_u(E_{1\sim 4}, E_u, \boldsymbol{p}_{1\sim 4}) + C\langle T_1^{\mathrm{TT}}\bar{\psi}_{2,A}^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_{4,B}^{\mathrm{T},+}\rangle_u(-E_{1\sim 4}, E_u, \boldsymbol{p}_{1\sim 4})$$
$$- \frac{(-2E_u)}{(E_{13}^2 - E_u^2)(E_{24}^2 - E_u^2)}$$
$$\cdot \Big\{ \Big[(\tilde{M}_{i_u}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)\pi_u^{i_u j_u}\tilde{M}_{j_u}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-u}))\Big]$$
$$\cdot \Big(\big[(2\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j) - (2\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(\epsilon_{3,i}^{\mathrm{T}}\pi_u^{ij}\gamma_j) + (\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})(p_1 - p_3)_i\pi_u^{ij}\gamma_j\big]\,(1 + \hat{\boldsymbol{p}}_4)\Big)_{AB}$$
$$- \frac{1}{2}\left(\boldsymbol{J}'_{3,L} \cdot \boldsymbol{\pi}_u \cdot \boldsymbol{J}'_{3,L}\right)\left(\boldsymbol{J}'_{3,R} \cdot \boldsymbol{\pi}_u \cdot [(1 - \hat{\boldsymbol{p}}_2)(-\boldsymbol{\gamma})(1 + \hat{\boldsymbol{p}}_4)]_{AB}\right)\Big\}$$
$$= E_u\left[(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})(\boldsymbol{\epsilon}_{2,T} \cdot \boldsymbol{\epsilon}_{4,T})\left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u^2}\right)\right]$$
$$\cdot \left((1 - \hat{\boldsymbol{p}}_2)(2\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{p}_2)\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})\frac{E_1 - E_3}{E_u^2}\gamma_0\right](1 + \hat{\boldsymbol{p}}_4)\right)_{AB}$$

So we only need to shift the correlator with the term vanishing on all the pole residue

$$\begin{aligned}
&\langle T_1^{\mathrm{TT}}\bar{\psi}_{2,A}^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_{4,B}^{\mathrm{T},+}\rangle \\
&\to \langle T_1^{\mathrm{TT}}\bar{\psi}_{2,A}^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_{4,B}^{\mathrm{T},+}\rangle_u \\
&- \frac{1}{2}E_u\left[(\boldsymbol{\epsilon}_{3,T} \cdot \boldsymbol{\epsilon}_{1,T})(\boldsymbol{\epsilon}_{2,T} \cdot \boldsymbol{\epsilon}_{4,T})\left(\frac{(E_1 - E_3)(E_2 - E_4)}{E_u^2}\right)\right] \\
&\cdot \left((1 - \hat{\boldsymbol{p}}_2)(2\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{p}_2)\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}} \cdot \boldsymbol{\epsilon}_3^{\mathrm{T}})\frac{E_1 - E_3}{E_u^2}\gamma_0\right](1 + \hat{\boldsymbol{p}}_4)\right)_{AB}
\end{aligned} \tag{5.100}$$

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Then the u channel correlator,

$$
\begin{aligned}
&\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_u \\
&=\frac{1}{K_T E_{13u}E_{24u}} \\
&\cdot\Big\{\left[(\tilde{M}_{i_u}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)\pi_u^{i_u j_u}\tilde{M}_{j_u}(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_{-u}))+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})(\boldsymbol{\epsilon}_2^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_4^{\mathrm{T}})E_{13u}E_{24u}\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right] \\
&\cdot\bar{\chi_2}^-(1-\boldsymbol{\not{p}}_2) \\
&\cdot\left[(2\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_3^{\mathrm{T}}-(2\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\not{\boldsymbol{\epsilon}}_3^{\mathrm{T}}+(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\left((p_1-p_3)_\mu\gamma^\mu+K_T\frac{E_1-E_3}{E_u}\gamma_0\right)\right](1+\boldsymbol{\not{p}}_4)\chi_{4,+} \\
&-\frac{1}{2}\left(\boldsymbol{J}'_{3,L}\cdot\boldsymbol{\pi}_u\cdot\boldsymbol{J}'_{3,L}\right)\left(\boldsymbol{J}'_{3,R}\cdot\boldsymbol{\pi}_u\cdot\left[\bar{\chi_2}^-(1-\boldsymbol{\not{p}}_2)(-\boldsymbol{\gamma})(1+\boldsymbol{\not{p}}_4)\chi_{4,+}\right]\right)\Big\} \\
&-\frac{1}{2}E_u\left[(\boldsymbol{\epsilon}_{3,T}\cdot\boldsymbol{\epsilon}_{1,T})(\boldsymbol{\epsilon}_{2,T}\cdot\boldsymbol{\epsilon}_{4,T})\left(\frac{(E_1-E_3)(E_2-E_4)}{E_u^2}\right)\right] \\
&\cdot\left(\bar{\chi}_{2,-}(1-\boldsymbol{\not{p}}_2)(2\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{p}_2)\left[(\boldsymbol{\epsilon}_1^{\mathrm{T}}\cdot\boldsymbol{\epsilon}_3^{\mathrm{T}})\frac{E_1-E_3}{E_u^2}\gamma_0\right](1+\boldsymbol{\not{p}}_4)\chi_{4,+}\right)
\end{aligned}
\tag{5.101}
$$

satisfies the full Optical theorem for exchanging gravition. As a remark, if we write the COT in discontinuity form, in this form we don't need to do C conjugate (with $2\leftrightarrow 4$) but flipping the internal energy E_u instead, we could still get the same shift term that makes COT be satisfied. Then for the contact diagram we could write it as the contact amplitudes over K_T that trivially matches the total energy pole residue, like

$$
\begin{aligned}
\langle T_1^{\mathrm{TT}}\bar{\psi}_2^{\mathrm{T},-}T_3^{\mathrm{TT}}\psi_4^{\mathrm{T},+}\rangle_c=\frac{1}{K_T}\cdot\Big\{&\frac{1}{2}\left[M_c(\gamma_3^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_1^{\mathrm{T}})-M_c(\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_2^{\mathrm{T}}\gamma_1^{\mathrm{T}})\right]\cdot\left[M_{\mu_u}(\gamma_1^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_u)\cdot(-\gamma^{\mu_u})\right] \\
&+\frac{1}{2}\left[M_c(\gamma_2^{\mathrm{T}}\gamma_3^{\mathrm{T}}\gamma_4^{\mathrm{T}}\gamma_1^{\mathrm{T}})\right]\left[\not{\epsilon_1}^{\mathrm{T}}(-p_3-p_4)_\mu\gamma^\mu\not{\epsilon_3}^{\mathrm{T}}+\not{\epsilon_1}^{\mathrm{T}}(-p_1-p_4)_\mu\gamma^\mu\not{\epsilon_3}^{\mathrm{T}}\right]\Big\}
\end{aligned}
\tag{5.102}
$$

We'll come up with a problem the for the first $\eta_{\mu\nu}$ contraction, we already find an equivalent expression on the total energy pole in (5.49) (But we have no known equivalent expression for the second and the third $\eta_{\mu\nu}$ that differs a term vanishing on the total energy

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pole.),

$$\left((p_1-p_3)_\mu\gamma^\mu + K_T\frac{E_1-E_3}{E_u}\gamma_0\right) \sim \left[(p_1-p_3)_i\pi_u^{ij}\gamma_j + E_{13u}E_{24u}\frac{E_1-E_3}{E_u^2}\gamma_0\right], \tag{5.103}$$

when this term is sandwiched between the lifters and the boundary condition of the spinor. But because we could easily check the equivalent expression or the RHS one won't satisfy the Fermion COT for the contact term. So the LHS is the only form that satisfies all the constants for the contact term. Actually with the (5.102), we already fix the full $\langle T^{\mathrm{TT}}\bar{\psi}^{\mathrm{T}}T^{\mathrm{TT}}\psi^{\mathrm{T}}\rangle$. We could not write down any additional polynomial ansatz that is vanishing at all the pole residue and the Fermion COT.

WT consistency Now we need to check the consistency between the partial energy pole residue and the longitudinal correlator determined by the WT identities. There are two consistent longitudinal parts of the correlator given by diffeomorphism WT identity and by SUSY WT identity.

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$$
\begin{aligned}
\langle T_1^L \bar{\psi}_{2,-} T_3 \psi_{4,+} \rangle &= \left(\frac{-\hat{p}_{1,i}\epsilon_1^i}{E_1}\right) \cdot p_{1,j}\epsilon_{1,j'} \langle T_1^{jj'} \bar{\psi}_{2,-} J_3 \psi_+ \rangle \\
&= \left(\frac{-\hat{p}_{1,i}\epsilon_1^i}{E_1}\right) \cdot \Big\{ - \bar{\chi}_{2,-} \left(\gamma_i \epsilon_{4,j} \langle T_{2+4}^{ij} T_1 T_3 \rangle' \right) \chi_{4,+} - \frac{1}{8} \bar{\chi}_{2,-,A} \left([\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\right)^{AB} \left(\epsilon_{1,i} \langle \bar{\psi}_{2+1,-,B}^i T_3 \psi \right. \\
&- \frac{1}{8} \bar{\chi}_{2,-,A} \left([\not{\boldsymbol{p}}_3, \not{\boldsymbol{\epsilon}}_3]\right)^{AB} \left(\epsilon_{3,i} \langle \bar{\psi}_{2+3,-,B}^i T_1 \psi_{4,+} \rangle' \right) \\
&- \frac{1}{2} \left(\bar{\chi}_{2,-} \not{\boldsymbol{\epsilon}}_1 \chi_{4,+}\right) \epsilon_{1,j,0} \epsilon_{4,i,0} \langle T_{2+4+1}^{ij} T_3 \rangle' - \frac{1}{2} \left(\bar{\chi}_{2,-} \not{\boldsymbol{\epsilon}}_3 \chi_{4,+}\right) \epsilon_{3,j,0} \epsilon_{4,i,0} \langle T_{2+4+3}^{ij} T_1 \rangle' \\
&+ \frac{1}{16} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_1, \not{\boldsymbol{p}}_3] \left(\epsilon_{3,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3) \\
&+ \frac{1}{32} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_1, \not{\boldsymbol{\epsilon}}_3] \left(\epsilon_{3,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \\
&+ \frac{1}{16} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_3, \not{\boldsymbol{p}}_1] \left(\epsilon_{1,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3) \\
&+ \frac{1}{32} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_3, \not{\boldsymbol{\epsilon}}_1] \left(\epsilon_{1,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \Big\}
\end{aligned}
\tag{5.104}
$$

$$
\begin{aligned}
\langle T_1 \bar{\psi}_{2,-}^L T_3 \psi_{4,+} \rangle &= \left(\frac{-\hat{p}_{2,i}\epsilon_2^i}{E_2}\right) \cdot p_{2,j} \langle T_1 \bar{\psi}_{2,-}^j J_3 \psi_+ \rangle \\
&= \left(\frac{-\hat{p}_{2,i}\epsilon_2^i}{E_2}\right) \cdot \Big\{ - \bar{\chi}_{2,-} \left(\gamma_i \epsilon_{4,j} \langle T_{2+4}^{ij} T_1 T_3 \rangle \right) \chi_{4,+} \\
&- \frac{1}{8} \bar{\chi}_{2,-,A} \left([\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\right)^{AB} \left(\epsilon_{1,i} \langle \bar{\psi}_{2+1,-,B}^i T_3 \psi_{4,+} \rangle \right) \\
&- \frac{1}{8} \bar{\chi}_{2,-,A} \left([\not{\boldsymbol{p}}_3, \not{\boldsymbol{\epsilon}}_3]\right)^{AB} \left(\epsilon_{3,i} \langle \bar{\psi}_{2+3,-,B}^i T_1 \psi_{4,+} \rangle \right) \\
&- \frac{1}{2} \left(\bar{\chi}_{2,-} \not{\boldsymbol{\epsilon}}_1 \chi_{4,+}\right) \epsilon_{1,j,0} \epsilon_{4,i,0} \langle T_{2+4+1}^{ij} T_3 \rangle - \frac{1}{2} \left(\bar{\chi}_{2,-} \not{\boldsymbol{\epsilon}}_3 \chi_{4,+}\right) \epsilon_{3,j,0} \epsilon_{4,i,0} \langle T_{2+4+3}^{ij} T_1 \rangle \\
&+ \frac{1}{16} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_1, \not{\boldsymbol{p}}_3] \left(\epsilon_{3,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3) \\
&+ \frac{1}{32} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_1, \not{\boldsymbol{\epsilon}}_3] \left(\epsilon_{3,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \\
&+ \frac{1}{16} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_3, \not{\boldsymbol{p}}_1] \left(\epsilon_{1,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_3) \\
&+ \frac{1}{32} \left(\bar{\chi}_{2,-} [\not{\boldsymbol{\epsilon}}_3, \not{\boldsymbol{\epsilon}}_1] \left(\epsilon_{1,i,0} \langle \bar{\psi}_{2+1+3,-}^i \psi_{4,+} \rangle \right) \chi_{4,+}\right) (\boldsymbol{\epsilon}_1 \cdot \boldsymbol{p}_3) \Big\}
\end{aligned}
\tag{5.105}
$$

For the longitudinal part $\langle T_1^L \bar{\psi}_{2,-} T_3 \psi_{4,+} \rangle$, first we take the E_{12s} pole

$$
\operatorname*{Res}_{E_{12s}=0} \langle T_1^L \bar{\psi}_{2,-} T_3 \psi_{4,+} \rangle = 0 = \tilde{M}_{3,i_s,A} (h_1^L \bar{\psi}_2 \psi_s) \frac{\Pi_{s,(3/2,3/2)}^{i_s j_s}}{2} \tilde{\Psi}_{3,j_s,B} (h_3 \hat{\bar{\psi}}_{-s} \psi_4) \tag{5.106}
$$

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by the amplitude Ward Identities, we know that $M_3(h_1^L\bar{\psi}_2\psi_s) = p_1^i\epsilon_1^j\tilde{M}_{3,ij}(h_1\bar{\psi}_2\psi_s) = p_1^\mu\epsilon_1^\nu M_{3,\mu\nu}(\gamma_1\bar{\psi}_2\psi_s) = 0$. And for the other pole for s-channel. Notice the dressed projector for the $\bar{\psi}_-^{\mathrm{T},i}$ extracted boundary condition will be $\not{\boldsymbol{p}}_s\pi^{ij}_{s,(3/2,3/2)}\frac{1-\gamma_0}{2} = \frac{1-\gamma_0}{2}\not{\boldsymbol{p}}_s\pi^{ij}_{s,(3/2,3/2)}$.

$$
\begin{aligned}
&\underset{E_{34s}=0}{\mathrm{Res}}\langle T_1^L\bar{\psi}_{2,-}T_3\psi_{4,+}\rangle \\
&= (-\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\bar{\chi}_{2,-,A}\left(-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_2)\epsilon_{2,k}-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)p_{1,k}+\frac{1}{8}[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1]\epsilon_{2,k}\right)^{AB} \\
&\cdot(\not{\boldsymbol{p}}_s\pi^{i_sj_s}_{s,(3/2,3/2)}\frac{1-\gamma_0}{2})_{BC}\underset{E_{34s}=0}{\mathrm{Res}}\langle\hat{\tilde{\psi}}^C_{1+2,-,j_s}T_3\psi_{4,+}\rangle \\
&= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\bar{\chi}_{2,-,A}\left([(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_2)\epsilon_{2,k_s}+(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)p_{1,k_s}-\frac{1}{8}[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1]\epsilon_{2,k_s}]\not{\boldsymbol{p}}_s\right)^{AB} \\
&\cdot\Pi^{i_sk_s',B'}_{s,(3/2,3/2),B}M^{k_s'}_{3,B'C}(\bar{\psi}_{-s}h_3\psi_4)\chi^C_{4,+} \\
&= \tilde{\hat{\Psi}}^A_{3,i_s}(T_1^L\bar{\psi}_{2,-}\psi_{s,+})\frac{\Pi^{k_sk_s'}_{s,(3/2,3/2),AA'}}{2}M^{A'B}_{3,j_s}(\bar{\psi}_{-s}h_3\psi_4)\psi_{4,B,+}
\end{aligned}
\tag{5.107}
$$

in which in the third equality we use the fact that in amplitude only the transverse part survives and in the last equality we use the 3-point WT identity.

$$
\begin{aligned}
&\langle T_1^L\bar{\psi}_{2,-}\hat{\psi}^A_{s,+,i_s}\rangle = \bar{\chi}_{2,-,C}(\frac{-\hat{p}_{1,i}\epsilon_1^i}{E_1}) \\
&\cdot\Big(\not{\boldsymbol{p}}_2\left(\boldsymbol{\epsilon}^T_{2,i_s}\right)(\boldsymbol{p}_s\cdot\boldsymbol{\epsilon}_1)-\not{\boldsymbol{p}}_s\left(\boldsymbol{\epsilon}_2^j\pi_{s,ji_s'}\right)(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1)-\not{\boldsymbol{p}}_2\left(\boldsymbol{\epsilon}_2^T\cdot\boldsymbol{p}_1\right)(\boldsymbol{\epsilon}_{1,i_s}) \\
&-\not{\boldsymbol{p}}_s\left(\pi_{s,i_sj}\boldsymbol{p}_1^j\right)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_1)-\frac{1}{8}\left(\boldsymbol{\epsilon}^T_{2,i_s}\right)\left(\not{\boldsymbol{p}}_2\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\right)+\frac{1}{8}\left(\boldsymbol{\epsilon}_2^j\pi_{s,ji_s}\right)\left(\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\not{\boldsymbol{p}}_2\right)\Big)^{CA}
\end{aligned}
\tag{5.108}
$$

Then

$$
\begin{aligned}
\tilde{\hat{\Psi}}^A_{3,i_s}(T_1^L\bar{\psi}_{2,-}\psi_{s,+}) &= \langle T_1^L\bar{\psi}_{2,-}\hat{\psi}^A_{s,+,i_s}\rangle - \langle T_1^L\bar{\psi}_{2,-}\hat{\psi}^A_{s,+,i_s}\rangle|_{E_s\to-E_s} \\
&= 2(\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\left[\bar{\chi}_{2,-}\left((\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_2)\epsilon_{2,k_s}+(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)p_{1,k_s}-\frac{1}{8}[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1]\epsilon_{2,k_s}\right)\not{\boldsymbol{p}}_s\pi^{k_s}_{s,i_s}\right]^A.
\end{aligned}
\tag{5.109}
$$

For the pole of E_{32t} and E_{14t}, the consistency could be check if we take $1\leftrightarrow 3$ for all the above derivation. Now we check the partial energy pole residue of the u-channel

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correlator. First we take the E_{13u} pole

$$\underset{E_{13u}=0}{\text{Res}}\langle T_1^L\bar{\psi}_{2,-}T_3\psi_{4,+}\rangle = 0 = \tilde{M}_{i_ui'_u}(h_1^Lh_3h_u)\frac{\Pi_{u,(2,2)}^{i_ui'_uj_uj'_u}}{2E_u}\tilde{\hat{\Psi}}_{j_uj'_u,\langle T_{-u}^{\text{TT}}\bar{\psi}_{2,-}\psi_{4,+}\rangle} \tag{5.110}$$

by the amplitude Ward Identities, we know that $M_3(h_1^Lh_3h_u) = p_1^i\epsilon_1^j\tilde{M}_{3,ij}(h_1h_3h_u) = p_1^\mu\epsilon_1^\nu M_{3,\mu\nu}(h_1h_3h_u) = 0$. And for the other pole for u-channel

$$\begin{aligned}\underset{E_{24u}=0}{\text{Res}}\langle T_1^L\bar{\psi}_{2,-}T_3\psi_{4,+}\rangle &= (-\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\left((\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)p_{3,a} - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\epsilon_{3,a}\right)\epsilon_{3,b}\Pi_{u,(2,2)}^{abcd}\underset{E_{24u}=0}{\text{Res}}\langle\hat{T}_{1+3}^{cd}\bar{\psi}_{2,-}\psi_{4,+}\rangle\\ &= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\left(-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)p_{3,i_u} + \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\epsilon_{3,i_u}\right)\epsilon_{3,b}\Pi_{u,(2,2)}^{abcd}\tilde{M}_{3,cd}(h_{-u}\bar{\psi}_2\psi_4)\\ &= \tilde{\Psi}_3^{i_ui'_u}(T_1^LT_3T_u)\frac{\Pi_{u,(2,2)}^{i_ui'_uj_uj'_u}}{2E_u}\tilde{M}_3^{j_uj'_u}(h_{-u}\bar{\psi}_2\psi_4)\end{aligned} \tag{5.111}$$

in the last line we use the 3-point WT identity,

$$\begin{aligned}\langle T_1^LT_3T_u\rangle^{i_u,i'_u} = \frac{(-\epsilon_{1,i}\hat{p}_1^i)}{E_1}\cdot\Big[&(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\,p_{3,k}\epsilon_{3,l}E_u\pi_u^{k(i_u}\pi_u^{li'_u)} - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\epsilon_{3,k}\epsilon_{3,l}E_u\pi_u^{k(i_u}\pi_u^{l\,i'_u)}\\ &+ \epsilon_1^{(i_u}p_{u,k}\pi_3^{ka}\pi_3^{i'_u)b}\epsilon_{3,a}\epsilon_{3,b}E_3 - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_u)\pi_3^{i_ua}\pi_3^{i'_ub}\epsilon_{3,a}\epsilon_{3,b}E_3\Big]\end{aligned}$$

$$\begin{aligned}\tilde{\Psi}_3^{\;i_u,i'_u}(T_1^LT_3T_u) &= \langle T_1^LT_3T_u\rangle^{i_u,i'_u} - \langle T_1^LT_3T_u\rangle^{i_u,i'_u}|_{E_s\to -E_s}\\ &= 2E_u(\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\left[-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\,p_{3,k}\epsilon_{3,l}\pi_u^{k(i_u}\pi_u^{li'_u)} + \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\epsilon_{3,k}\epsilon_{3,l}\pi_u^{k(i_u}\pi_u^{l\,i'_u)}\right],\end{aligned} \tag{5.112}$$

and with the identity,

$$\begin{aligned}&\tilde{\Psi}_3^{i_ui'_u}(T_1^LT_3T_u)\frac{\Pi_{u,(2,2)}^{i_ui'_uj_uj'_u}}{2E_u}\tilde{M}_3^{j_uj'_u}(h_{-u}\bar{\psi}_2\psi_4)\\ &= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\left[-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\,p_{3,k}\epsilon_{3,l}E_u + \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\epsilon_{3,k}\epsilon_{3,l}\right]\epsilon_{3,l}\Pi_{u,(2,2)}^{klmn}\tilde{M}_{3,mn}(h_{-u}\bar{\psi}_2\psi_4)\\ &= (\frac{\hat{p}_{1,i}\epsilon_1^i}{E_1})\left[-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\,p_{3,k}E_u + \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\epsilon_{3,k}\right]\epsilon_{3,l}\Pi_{u,(2,2)}^{klmn}\tilde{M}_{3,mn}(h_{-u}\bar{\psi}_2\psi_4),\end{aligned} \tag{5.113}$$

in which we use the fact that amplitude is symmetric and only transverse amplitude survives , such that in this case we can replace π_u^{ij} with η^{ij}.

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Then for the longitudinal part $\langle T_1\bar{\psi}^L_{2,-}T_3\psi_{4,+}\rangle$, first we take the E_{12s} pole

$$\underset{E_{12s}=0}{\text{Res}}\langle T_1\bar{\psi}^L_{2,-}T_3\psi_{4,+}\rangle = 0 = \tilde{M}_{3,i_s,A}(h_1\bar{\psi}^L_2\psi_s)\left(\Pi^{i_sj_s}_{s,(3/2,3/2)}\right)^{AB}\tilde{\Psi}_{3,j_s,B}(h_3\bar{\psi}_{-s}\psi_4) \tag{5.114}$$

by the amplitude Ward Identities, we know that $M_3(h_1\bar{\psi}^L_2\psi_s) = p^{i_2}_2\tilde{M}_{3,i_2}(h_1\bar{\psi}_2\psi_s) = p^{\mu_2}_2 M_{3,\mu_2}(\gamma_1\bar{\psi}_2\psi_s) = 0$. And for the other pole for s-channel

$$\begin{aligned}
&\underset{E_{34s}=0}{\text{Res}}\langle T_1\bar{\psi}^L_{2,-}T_3\psi_{4,+}\rangle \\
&= (-\frac{\hat{p}_{2,i}\epsilon^i_2}{E_2})\bar{\chi}_{2,-,A}\left(-\frac{1}{8}[\not{\boldsymbol{p}}_1,\not{\epsilon}_1]\epsilon_{1,k}\not{\boldsymbol{p}}_s\pi^{k,k'}_{s,(3/2,3/2)}\frac{1-\gamma_0}{2}\right)^{AB}\underset{E_{34s}=0}{\text{Res}}\langle\hat{\bar{\psi}}_{1+2,-,B,k'}T_3\psi_{4,+}\rangle \\
&= (\frac{\hat{p}_{1,i}\epsilon^i_1}{E_1})\bar{\chi}_{2,-,A}\left(\frac{1}{8}[\not{\boldsymbol{p}}_1,\not{\epsilon}_1]\epsilon_{1,k_s}\not{\boldsymbol{p}}_s\Pi^{k_s}_{s,k'_s,(3/2,3/2)}\right)^{AB} M^{k'_s}_{3,BC}(\bar{\psi}_{-s}h_3\psi_4)\chi^C_{4,+} \\
&= \tilde{\Psi}^A_{3,i_s}(T^L_1\bar{\psi}_{2,-}\psi_{s,+})\left(\Pi^{i_sj_s}_{s,(3/2,3/2)}\right)_{AA'} M^{A'B}_{3,j_s}(\bar{\psi}_{-s}h_3\psi_4)\psi_{4,B,+}
\end{aligned} \tag{5.115}$$

in which in the third equality we use the fact that in amplitude only the transverse part survives and in the last equality we use the 3-point WT identity,

$$\langle T_1\bar{\psi}^L_{2,-}\hat{\psi}^A_{s,+,i_s}\rangle = \bar{\chi}_{2,-,C}(\frac{-\hat{p}_{2,i}\epsilon^i_2}{E_2})\cdot\Big(-\left(\boldsymbol{\epsilon}^T_{1,i_s}\right)\left(\not{\boldsymbol{\epsilon}}^T_1\right)E_1 - \frac{1}{8}\left(\boldsymbol{\epsilon}^j_1\pi_{ji_s}\right)\left(\bar{\chi}_{2,-}[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1]\,\not{\boldsymbol{p}}_s\right)\Big)^{CA} \tag{5.116}$$

$$\begin{aligned}
\hat{\tilde{\Psi}}^A_{3,i_s}(T^L_1\bar{\psi}_{2,-}\psi_{s,+}) &= \langle T^L_1\bar{\psi}_{2,-}\hat{\psi}^A_{s,+,i_s}\rangle - \langle T^L_1\bar{\psi}_{2,-}\hat{\psi}^A_{s,+,i_s}\rangle|_{E_s\to -E_s} \\
&= (\frac{\hat{p}_{1,i}\epsilon^i_1}{E_1})\left[\bar{\chi}_{2,-}\left(\frac{1}{8}[\not{\boldsymbol{p}}_1,\not{\epsilon}_1]\epsilon_{1,k_s}\right)(2\not{\boldsymbol{p}}_s\pi^{k_s}_{s,i_s})\right]^A.
\end{aligned} \tag{5.117}$$

For the pole of E_{32t} and E_{14t}, the consistency could be checked if we take $1\leftrightarrow 3$ for all the above derivation. Now we check the partial energy pole residue of the u-channel correlator. First we take the E_{13u} pole

$$\underset{E_{24u}=0}{\text{Res}}\langle T_1\bar{\psi}^L_{2,-}T_3\psi_{4,+}\rangle = 0 = \tilde{\Psi}^{i_ui'_u}_3(T_1T_3T_u)\frac{\Pi^{i_ui'_uj_uj'_u}_{u,(2,2)}}{2E_u}\tilde{M}^{j_uj'_u}_3(h_{-u}\bar{\psi}^L_2\psi_4) \tag{5.118}$$

by the amplitude Ward Identities, we know that $M_3(h_1\bar{\psi}^L_2\psi_s) = p^{i_2}_2\tilde{M}_{3,i_2}(h_1\bar{\psi}_2\psi_s) =$

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$p_2^{\mu_2} M_{3,\mu_2}(h_1 \bar{\psi}_2 \psi_s) = 0$. And for the other pole for u-channel

$$\begin{aligned}\operatorname*{Res}_{E_{13u}=0} \langle T_1 \bar{\psi}^L_{2,-} T_3 \psi_{4,+}\rangle &= (-\frac{\hat{p}_{2,i}\epsilon^i_2}{E_2}) \operatorname*{Res}_{E_{13u}=0} \langle T_1 T_3 \hat{T}^{i_u i'_u}_u \rangle \Pi^{i_u i'_u j_u j'_u}_{u,(2,2)} \cdot \bar{\chi}_{2,-} \left(-\gamma_{j_u} \epsilon_{4,j'_u}\right) \chi_{4,+} \\ &= (\frac{\hat{p}_{2,i}\epsilon^i_2}{E_2}) \tilde{M}^{i_u i'_u}_3 (h_1 h_3 h_u) \cdot (\Pi^{i_u i'_u j_u j'_u}_{u,(2,2)}) \cdot \bar{\chi}_{2,-} \left(\gamma_{(j_u} \epsilon_{4,j'_u)}\right) \chi_{4,+} \\ &= \tilde{M}_{i_u i'_u}(h_1 h_3 h_u) \frac{\Pi^{i_u i'_u j_u j'_u}_{u,(2,2)}}{2E_u} \tilde{\Psi}_{j_u j'_u, \langle T_{-u} \bar{\psi}^L_{2,-} \psi_{4,+}\rangle}\end{aligned} \tag{5.119}$$

in which in the second equality we use the fact that in amplitude only the transverse part survives and its indices are symmetric and in the last equality we use the 3-point WT identity,

$$\begin{aligned}&\langle T^{j_u, j'_u}_{-u} \bar{\psi}^L_{2,-} \psi_{4,+}\rangle \\ &= \frac{(-\epsilon_{2,i}\hat{p}^i_2)}{E_2} \cdot \left[-\left(\pi_{u,(j_u j} \cdot \boldsymbol{\epsilon}^j_3\right) \left(\bar{\chi}_{2,-} \gamma^k \pi_{u,j'_u)k} \chi_{3,+}\right) E_u + \frac{1}{8} \left(\boldsymbol{\epsilon}^{(j_u,T}_3\right) \left(\bar{\chi}_{2,-} \left[\boldsymbol{\not p_u}, \boldsymbol{\gamma^{j'_u)}}\right] \boldsymbol{\not p}_3 \chi_{3,+}\right)\right] \\ &\tilde{\Psi}_3^{\ i_u, i'_u}(T^L_1 T_3 T_u) = \langle T^L_1 T_3 T_u\rangle^{i_u, i'_u} - \langle T^L_1 T_3 T_u\rangle^{i_u, i'_u}|_{E_s \to -E_s} \\ &= 2E_u (\frac{\hat{p}_{2,i}\epsilon^i_1}{E_2}) \left[\bar{\chi}_{2,-} \left(\gamma_{(j_u} \epsilon_{4,j'_u)}\right) \chi_{4,+}\right].\end{aligned} \tag{5.120}$$

5.4 SUSY Ward Identity Bootstrap

Because we know that SUSY transformation is just an extension of the diffeomorphism. So we could bootstrap it from the WT identity of the diffeomorphism.

5.4.1 $\langle T\bar{\psi}\psi\rangle$

The general ansatz of the SUSY WT of the three-point correlator will be:

$$p_{2,i} \langle T_1 \bar{\psi}^i_{2,-,A} \psi_{3,+,B}\rangle = A_{ij,AB}(\vec{p}_3) \langle T_1 T^{ij}_{2+3}\rangle + B_{i,AA'}(\vec{p}_1, \vec{p}_3) \langle \bar{\psi}^{i,A'}_{1+2,-} \psi_{3,+,B}\rangle \tag{5.121}$$

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We require it must be no residue contribution on the total energy pole. And by the energy dimension counting we know that $A \in O(E^0), B \in O(E^1)$. The counting of the polarization vector on both sides must match too. The indices of the two-point function are symmetric so we don't need to permute the indices and transverse to the momentum. We also don't include the unit vector which is a non-local operator in our ansatz, and because $\epsilon_1 \cdot \epsilon_1 = (\epsilon \cdot \hat{p}_1)^2$, we also don't include the self-contraction of the polarization vector of the graviton. Because we have $\chi_{2,-}\chi_{1,+} = 0$. A should have an odd number of spatial gamma matrices and B should have even, then counting of gamma matrices make the term not vanishing. Then

$$\begin{aligned} A_{ij,AB}(\vec{p}_3) &= a\epsilon_{3,i}\gamma_j \\ B_{i,AB}(\vec{p}_1,\vec{p}_3) &= b_1\epsilon_{1,i}(p_{1,k}\gamma^k)(\epsilon_{1,j}\gamma^j) + b_2\epsilon_{1,i}(p_{1,k}\epsilon_1^k) + b_3\epsilon_{1,i}(p_{3,k}\gamma^k)(\epsilon_{1,j}\gamma^j) + b_4\epsilon_{1,i}(p_{3,k}\epsilon_1^k) \end{aligned} \tag{5.122}$$

And we require the consistency condition that the $\langle T^L \bar{\psi}^L \psi \rangle$ got from SUSY WT

$$\begin{aligned} \langle T_1^L \bar{\psi}_{2,-}^L \psi_{3,+} \rangle = \frac{\epsilon_{1,L}\epsilon_{2,L}}{E_1 E_2} \bar{\chi}_{2,-} \big[& b_1 p_{1,i} \not{\boldsymbol{p}}_1 \not{\boldsymbol{\epsilon}}_1 + b_1 \epsilon_{1,i}(-E_1^2) \\ & + b_2 p_{1,i}(p_{1,k}\epsilon_1^k) + b_2\epsilon_{1,i}(-E_1^2) + b_3 p_{1,i} \not{\boldsymbol{p}}_3 \not{\boldsymbol{\epsilon}}_1 + b_3\epsilon_{1,i} \not{\boldsymbol{p}}_3 \not{\boldsymbol{p}}_1 \\ & + b_4\epsilon_{1,i}(p_{3,k}p_1^k) + b_4 p_{1,i}(p_{3,k}\epsilon_1^k) \big] \epsilon_3^{\mathrm{T},i} \hat{\not{\boldsymbol{p}}}_3 \chi_{3,+} \end{aligned} \tag{5.123}$$

should match the one got from diffeomorphism WT like (F.251):

$$\langle T_1^L \bar{\psi}_{2,-,A}^L \psi_{3,+,B} \rangle = -\frac{1}{8} \left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}_3^T \right) \left(\bar{\chi}_{2,-} [\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1] \, \hat{\not{\boldsymbol{p}}}_3 \chi_{3,+} \right) \frac{\left(\epsilon_1^L \epsilon_2^L \right)}{E_1 E_2} \tag{5.124}$$

Then by the vanishing of $\epsilon_{1,i}\epsilon_3^{\mathrm{T},i}$:

$$b_1 = -b_2, b_3 = 0, b_4 = 0 \tag{5.125}$$

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then

$$\langle T_1^L \bar{\psi}_{2,-}^L \psi_{3,+} \rangle = b_1 \frac{\epsilon_{1,L}\epsilon_{2,L}}{E_1 E_2} \bar{\chi}_{2,-} \left[p_{1,i} \not{\boldsymbol{p}}_{\mathbf{1}} \not{\boldsymbol{\epsilon}}_{\mathbf{1}} - (p_{1,k}\epsilon_1^k) \right] \epsilon_3^{\mathrm{T},i} \hat{\not{\boldsymbol{p}}}_{\mathbf{3}} \chi_{3,+}$$
$$= b_1 \frac{\epsilon_{1,L}\epsilon_{2,L}}{2E_1 E_2} (p_{1,i}\epsilon_3^{\mathrm{T},i}) \bar{\chi}_{2,-} [\not{\boldsymbol{p}}_{\mathbf{1}}, \not{\boldsymbol{\epsilon}}_{\mathbf{1}}] \hat{\not{\boldsymbol{p}}}_{\mathbf{3}} \chi_{3,+} \tag{5.126}$$

Now we could constrain $b_1 = -b_2 = -\frac{1}{4}$, with $b_3, b_4 = 0$. But we won't have a constraint on a.

$$A_{ij,AB}(\vec{p}_3) = a\epsilon_{3,i}\gamma_j$$
$$B_{i,AB}(\vec{p}_1, \vec{p}_3) = B_{i,AB}(\vec{p}_1) := -\frac{1}{8}\epsilon_{1,i}[\not{\boldsymbol{p}}_{\mathbf{1}}, \not{\boldsymbol{\epsilon}}_{\mathbf{1}}] \tag{5.127}$$

5.4.2 $\langle T\bar{\psi}T\psi\rangle$

If we suppose that we could use the same A_{ij} and B_{ij} to write the SUSY WT of the four-point correlator like

$$p_{2,i}\langle T_1 \bar{\psi}_{2,-,A}^i T_3 \psi_{4,+,B}\rangle = A_{ij,AB}(\vec{p}_4)\langle T_1 T_{2+4}^{ij} T_3\rangle + B_{i,AA'}(\vec{p}_1)\langle \bar{\psi}_{1+2,-}^{i,A'} T_3 \psi_{4,+,B}\rangle$$
$$+ B_{i,AA'}(\vec{p}_3)\langle \bar{\psi}_{3+2,-}^{i,A'} T_1 \psi_{4,+,B}\rangle + C_{i,AA'}(\vec{p}_1, \vec{p}_2, \vec{p}_4)\langle \bar{\psi}_{1+2+3,-}^{i,A'} \psi_{4,+,B}\rangle \tag{5.128}$$

in the previous section, we found, we only need to fix a in the $A_{ij,AB}$. We could do that by the match of the correlator longitudinal on leg 1, leg 2, and leg 4 and transverse on the other legs derived by SUSY WT identities and diffeomorphism WT identities. We only show the term with E_u in the following equations. We have

$$\langle T_1^{L\mathrm{T}} \bar{\psi}_{2,-,A}^L T_3^{\mathrm{TT}} \psi_{4,+,B}^L \rangle_{SUSY} = (\frac{-\epsilon_{2,i}\hat{p}_2^i}{E_2}) \cdot A_{ij,AB}(\vec{p}_4) \cdot \langle T_1^{LT} T_u^{ij} T_3^{TT}\rangle + \dots \tag{5.129}$$

and

$$\langle T_1^{L\mathrm{T}} \bar{\psi}_{2,-,A}^L T_3^{\mathrm{TT}} \psi_{4,+,B}^L \rangle_{Diffeo}$$
$$= (\frac{-\epsilon_{1,i}\hat{p}_1^i}{E_1}) \cdot [\langle \bar{\psi}_2^L T_u^{ij} \psi_4^L\rangle (\epsilon_{3,i}^{\mathrm{T}} p_{3,j})(\epsilon_{1,k}^{\mathrm{T}} \epsilon_3^{k,\mathrm{T}}) - \frac{1}{2}(p_{3,i}\epsilon_{1,i}^{\mathrm{T}})\langle \bar{\psi}_2^L T_u^{\mathrm{TT}} \psi_4^L\rangle] + \dots . \tag{5.130}$$

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Then we could require the term with E_u matched under any kinematics, we could choose the kinematic to be $\vec{p}_1 \to 0$ to simplify the expression.

$$\lim_{E_1\to 0}(\frac{-\epsilon_{2,i}\hat{p}_2^i}{E_2}) \cdot A_{ij,AB}(\vec{p}_4) \cdot \langle T_1^{LT} T_u^{ij} T_3^{TT}\rangle = (\frac{\epsilon_{2,i}\hat{p}_2^i}{E_2})(\frac{\epsilon_{1,i}\hat{p}_1^i}{E_1}) \cdot (a\epsilon_{4,i}^L\gamma_j)(-\frac{1}{2}(\epsilon_{1,k}^{\mathrm{T}} p_3^k)\epsilon_3^{\mathrm{T},i}\epsilon_3^{\mathrm{T},j})E_3 \tag{5.131}$$

and

$$\begin{aligned}
&\lim_{E_1\to 0}(\frac{-\epsilon_{1,i}\hat{p}_1^i}{E_1}) \cdot [\langle\bar{\psi}_2^L T_u^{ij}\psi_4^L\rangle(\epsilon_{3,i}^{\mathrm{T}} p_{3,j})(\epsilon_{1,k}^{\mathrm{T}}\epsilon_3^{k,\mathrm{T}}) - \frac{1}{2}(p_{3,i}\epsilon_{1,i}^{\mathrm{T}})\langle\bar{\psi}_2^L T_u^{\mathrm{TT}}\psi_4^L\rangle] \\
&= (\frac{-\epsilon_{1,i}\hat{p}_1^i}{E_1}) \cdot [\langle\bar{\psi}_2^L T_3^{ij}\psi_4^L\rangle(\epsilon_{3,i}^{\mathrm{T}} p_{3,j})(\epsilon_{1,k}^{\mathrm{T}}\epsilon_3^{k,\mathrm{T}}) - \frac{1}{2}(p_{3,i}\epsilon_{1,i}^{\mathrm{T}})\langle\bar{\psi}_2^L T_3^{\mathrm{TT}}\psi_4^L\rangle] \\
&= (\frac{\epsilon_{2,i}\hat{p}_2^i}{E_2})(\frac{\epsilon_{1,i}\hat{p}_1^i}{E_1})(\frac{1}{2}(p_{3,k}\epsilon_{1,\mathrm{T}}^k)(\epsilon_{3,i}^{\mathrm{T}}\epsilon_4^{L,i})(\gamma_j\epsilon_3^{\mathrm{T},j})).
\end{aligned} \tag{5.132}$$

Then comparing the above two expressions, we could find $a = -1$. Now plug this into (5.121) and (5.127), we recover the SUSY WT identity,

$$\begin{aligned}
p_{2,k}\langle T_1\bar{\psi}_{2,-}^k\psi_{3,+}\rangle &= -\langle T_1 T_{2+3}^{kl}\rangle\ \epsilon_{l,3}\ (\bar{\chi}_{2,-}\gamma_k\chi_{3,+}) - \frac{1}{8}\,[\not{p}_1, \not{\epsilon}_1]\ \big(\epsilon_{1,k}\langle\bar{\psi}_{1+2,-}^k\psi_{3,+}\rangle\big) \\
&= -\left(\boldsymbol{\epsilon}_1^T\cdot\boldsymbol{\epsilon}_3\right)\left(\bar{\chi}_{2,-}\not{\epsilon}_1^T\chi_{3,+}\right) E_1\ - \frac{1}{8}\left(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3^T\right)(\bar{\chi}_{2,-}[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1]\,\not{\boldsymbol{p}}_3\chi_{3,+})
\end{aligned} \tag{5.133}$$

it's the same as (F.248) we derived by the supersymmetry transform on the boundary condition.

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5.5 $\langle\bar{\psi}\psi\bar{\psi}\psi\rangle$ and Majorana condition

First, referring to [2], we decompose the amplitude into s, t, u channel with the contact term.

$$
\begin{aligned}
M(\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4) = & \frac{1}{S}\left(M_{\mu_s}(\gamma_1\gamma_2\gamma_s)M_{\nu_s}(\bar{\chi}_1\chi_2\gamma_s)\right)\eta^{\mu_s(\mu_s'}\eta^{\nu_s\nu_s')}\left(M_{\mu_s'}(\gamma_3\gamma_4\gamma_{-s})M_{\nu_s'}(\bar{\chi}_3\chi_4\gamma_{-s})\right) \\
& -\frac{1}{T}\left(M_{\mu_t}(\gamma_3\gamma_2\gamma_t)M_{\nu_t}(\bar{\chi}_3\chi_2\gamma_t)\right)\eta^{\mu_t(\mu_t'}\eta^{\nu_t\nu_t')}\left(M_{\mu_t'}(\gamma_1\gamma_4\gamma_{-t})M_{\nu_t'}(\bar{\chi}_1\chi_4\gamma_{-t})\right) \\
& -\frac{1}{U}\left(M_{\mu_s}(\gamma_1\gamma_3\gamma_u)M_{\nu_u}(\bar{\chi}_1\chi_3\gamma_u)\right)\eta^{\mu_u(\mu_u'}\eta^{\nu_u\nu_u')}\left(M_{\mu_u'}(\gamma_2\gamma_4\gamma_{-u})M_{\nu_u'}(\bar{\chi}_2\chi_4\gamma_{-u})\right) \\
& +\frac{1}{2}\left(\begin{array}{l}
2(\epsilon_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)M_{\mu_t}(\bar{\chi}_3\chi_2\gamma_t)M^{\mu_t}(\bar{\chi}_1\chi_4\gamma_{-t}) \\
+\left[6(\epsilon_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)-2(\epsilon_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)\right]M_{\mu_u}(\bar{\chi}_1\chi_3\gamma_u)M^{\mu_u}(\bar{\chi}_2\chi_4\gamma_{-u}) \\
+\frac{3}{2}(\epsilon_2\cdot\epsilon_3)\cdot\epsilon_4^{\mu_s}M_{\mu_s}(\bar{\chi}_1\chi_2\gamma_s)\cdot\epsilon_1^{\nu_s}M_{\nu_s}(\bar{\chi}_3\chi_4\gamma_{-s}) \\
+\frac{3}{2}(\epsilon_1\cdot\epsilon_3)\cdot\epsilon_4^{\mu_s}M_{\mu_s}(\bar{\chi}_1\chi_2\gamma_s)\cdot\epsilon_2^{\nu_s}M_{\nu_s}(\bar{\chi}_3\chi_4\gamma_{-s}) \\
+\frac{3}{2}(\epsilon_3\cdot\epsilon_4)\cdot\epsilon_1^{\mu_t}M_{\mu_t}(\bar{\chi}_3\chi_2\gamma_t)\cdot\epsilon_2^{\nu_t}M_{\nu_t}(\bar{\chi}_1\chi_4\gamma_{-t}) \\
-\frac{5}{2}(\epsilon_2\cdot\epsilon_4)\cdot\epsilon_1^{\mu_t}M_{\mu_t}(\bar{\chi}_3\chi_2\gamma_t)\cdot\epsilon_3^{\nu_t}M_{\nu_t}(\bar{\chi}_1\chi_4\gamma_{-t}) \\
-4(\epsilon_1\cdot\epsilon_3)\cdot\epsilon_4^{\mu_t}M_{\mu_t}(\bar{\chi}_3\chi_2\gamma_t)\cdot\epsilon_2^{\nu_t}M_{\nu_t}(\bar{\chi}_1\chi_4\gamma_{-t}) \\
+4(\epsilon_1\cdot\epsilon_2)\cdot\epsilon_4^{\mu_t}M_{\mu_t}(\bar{\chi}_3\chi_2\gamma_t)\cdot\epsilon_3^{\nu_t}M_{\nu_t}(\bar{\chi}_1\chi_4\gamma_{-t}) \\
-\frac{1}{2}(\epsilon_1\cdot\epsilon_4)\cdot\epsilon_2^{\mu_u}M_{\mu_u}(\bar{\chi}_1\chi_3\gamma_u)\cdot\epsilon_3^{\nu_u}M_{\nu_u}(\bar{\chi}_2\chi_4\gamma_{-u}) \\
-2(\epsilon_2\cdot\epsilon_3)\cdot\epsilon_4^{\mu_u}M_{\mu_u}(\bar{\chi}_1\chi_3\gamma_u)\cdot\epsilon_1^{\nu_u}M_{\nu_u}(\bar{\chi}_2\chi_4\gamma_{-u}) \\
-\frac{2}{5}(\epsilon_1\cdot\epsilon_2)\cdot\epsilon_4^{\mu_u}M_{\mu_u}(\bar{\chi}_1\chi_3\gamma_u)\cdot\epsilon_3^{\nu_u}M_{\nu_u}(\bar{\chi}_2\chi_4\gamma_{-u})
\end{array}\right)
\end{aligned}
\tag{5.134}
$$

in which where we define

$$
\begin{aligned}
M_{\mu_3}(\gamma_1\gamma_2\gamma_3) &= \left[(\epsilon_1\cdot\epsilon_2)(p_1-p_2)_{\mu_3}+(\epsilon_{2,\mu_3})(2\epsilon_1\cdot p_2)+(\epsilon_{1,\mu_3})(-2\epsilon_2\cdot p_1)\right] \\
M_{\nu_3}(\bar{\chi}_1\chi_2\gamma_3) &= \bar{\chi}_1\gamma_{\nu_3}\chi_2.
\end{aligned}
\tag{5.135}
$$

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The Ward identity of the amplitude is satisfied only when the Majorana condition is applied. Then because the correlator $\langle \bar{\psi}\psi\bar{\psi}\psi \rangle$ must match the amplitude on the total energy pole. The majorana condition for the bulk field imply the relationship (A.35) for the boundary condition $\bar{\chi}_{-,\mathbf{0}}, \chi_{+,\mathbf{0}}$ for fermion field. So the total energy pole residue point out the Majorana condition is necessary for the gravitino correlator.

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Chapter 6 Conclusion

As we've shown in our work, the on-shell bootstrap approach to constraining equal-time correlators is sufficient to fix the four-point function up to terms that can be associated with field redefinitions. The constraints include:

1. The residue of the total energy pole must be the amplitude.
2. Cosmological Optical theorems, which lead to constraints such as the residue of parietal energy poles must be the amplitude times the shifted amplitude, manifest locality, and the absence of certain invariants.
3. The Ward Takashi identity.

There are many interesting and important questions to pursue. First, so far we have used the WT identities as derived from how the operator (or fields) transform under the underlying symmetry. It is interesting to ask if one did not know what precisely is the symmetry, just the fact that there is a conserved current (or tensor), does the above consistency conditions are sufficient to reconstruct the symmetry. This question will be important when we extend the analysis to the de Sitter bootstrap, where it is known that supersymmetry does not allow for de Sitter spacetime. It will be interesting to show that no Ward-Takahashi Identity for a spin-3/2 current exists that is consistent with the de Sit-

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ter bootstrap. On the other hand, in the thesis, we raise the issue that even for flat space correlators there are still some unfix terms probably related to the boundary E.O.M or the field redefinition ambiguity of the correlator. We leave the study of this issue to future work.

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Appendix A — Notation

A.1 Commutation

$$A_{(i}B_{j)} = \frac{1}{2} \cdot (A_iB_j + A_jB_i) \tag{A.1}$$

$$A_{[i}B_{j]} = \frac{1}{2} \cdot (A_iB_j - A_jB_i) \tag{A.2}$$

for the matrix/operator A and matrix/operator B

$$[A, B] = AB - BA \tag{A.3}$$

$$\{A, B\} = AB + BA \tag{A.4}$$

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A.2 Vector Indices

A.2.1 4D metric

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

A.2.2 4D vector

$$A_\mu = \eta_{\mu\nu} A^\nu$$

$$A \cdot B = A^\mu B_\mu$$

A.2.3 3D vector

$$A_i = \eta_{ij} A^i$$

$$A^i B_i = \eta^{ij} A_i B_j = \boldsymbol{A} \cdot \boldsymbol{B}$$

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A.3 Spin 1/2 polarization and classical field

A.3.1 Gamma Matrices

$$\gamma_{0,AB} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}; \sigma_{1,\alpha\dot{\alpha}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_{2,\alpha\dot{\alpha}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma_{3,\alpha\dot{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\gamma_{i,AB} := \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}$$

$$\gamma_{5,AB} := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

$$\gamma^{\mu\nu} := \frac{1}{2}[\gamma^\mu, \gamma^\nu]$$

$$\gamma^{\mu\nu\rho} := \frac{1}{2}\{\gamma^\mu, \gamma^{\nu\rho}\}$$

A.3.2 4D spin 1/2 representation

$$\chi^A = \begin{bmatrix} \lambda_+^\alpha + \lambda_-^\alpha \\ \lambda_+^{\dot{\alpha}} - \lambda_-^{\dot{\alpha}} \end{bmatrix} \overline{\chi}^A = \begin{bmatrix} \lambda_+^{*\,\alpha} - \lambda_-^{*\,\alpha} & \lambda_+^{*\,\dot{\alpha}} + \lambda_-^{*\,\dot{\alpha}} \end{bmatrix} = \left(\chi^A\right)^+ \gamma_0 \tag{A.5}$$

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A.3.3 Momentum notation

- Mom convention : for every boundary condition $B_0(\boldsymbol{x})$, it's Fourier transform is defined by

$$B_0(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} B_0(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{A.6}$$

and we will use subscripts to label the indices of the momentum like

$$B_1 := B(\boldsymbol{p}_1) \tag{A.7}$$

and for the composite momentum we slightly abuse our notation

$$B_{1+2} := B(\boldsymbol{p}_1 + \boldsymbol{p}_2) \tag{A.8}$$

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$$p_\mu = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix}$$

$$E = |p| = \sqrt{-p \cdot p} \neq p_0$$

$$\hat{p} = p/E_p$$

$$K_t := \sum_{i \in external} E_i$$

$$\sigma_0 = I$$

$$p \cdot \overline{\sigma} := E - p^i \sigma_i$$

$$p \cdot \sigma := E + p^i \sigma_i$$

$$\not{p} := \eta^{ij} p_i \gamma_j = \begin{bmatrix} 0 & p^i \sigma_i \\ -p^i \sigma_i & 0 \end{bmatrix} =: \boldsymbol{\not{p}}$$

$$\not{P}_\pm := \not{E} \pm p^i \gamma_i$$

$$\not{P}_+ = \not{P} =: \begin{bmatrix} 0 & p \cdot \sigma \\ p \cdot \overline{\sigma} & 0 \end{bmatrix}$$

$$\not{P}_- =: \begin{bmatrix} 0 & p \cdot \overline{\sigma} \\ p \cdot \sigma & 0 \end{bmatrix}$$

$$E_{a+b+c+\dots} := E_a + E_b + E_c + \dots \tag{A.9}$$

$$E_{s,t,u} = |\boldsymbol{p_{s,t,u}}| \tag{A.10}$$

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$$p_s = p_3 + p_4 \tag{A.11}$$

$$p_t = p_1 + p_4 \tag{A.12}$$

$$p_u = p_2 + p_4 \tag{A.13}$$

$$S = (p_3 + p_4)^\mu (p_3 + p_4)_\mu \tag{A.14}$$

$$T = (p_1 + p_4)^\mu (p_1 + p_4)_\mu \tag{A.15}$$

$$U = (p_2 + p_4)^\mu (p_2 + p_4)_\mu \tag{A.16}$$

(A.17)

A.3.4 Dirac Equation

$$\not{\partial} := \gamma_0 \partial_0 + \gamma^i \partial_i$$

$$(-i \not{\partial} - m)\psi = 0$$

$$\bar{\psi}(i \overleftarrow{\not{\partial}} - m) = 0$$

A.3.5 Dirac Equation Solution (4D fermionic polarization)

$$\psi^{(i)}(\vec{x}, t) = \psi^{u,(i)}(\vec{x}) + \psi^{v,(i)}(\vec{x})$$

$$\psi^{u,(i)}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} u_{\vec{p}}^{(i)} e^{ip^i x_i} e^{iEt}$$

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$$\psi^{v,(i)}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}} v_{-\vec{p}}^{(i)} e^{ip^i x_i} e^{-iEt}$$

$$\bar{\psi}^{(i)}(\vec{x}) = \bar{\psi}^{u,(i)}(\vec{x}) + \bar{\psi}^{v,(i)}(\vec{x})$$

$$\bar{\psi}^{v,(i)}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}} \bar{v}_{-\vec{p}}^{(i)} e^{ip^i x_i} e^{-iEt}$$

$$\bar{\psi}^{u,(i)}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}} \bar{u}_{\vec{p}}^{(i)} e^{ip^i x_i} e^{iEt}$$

$$\not{P}_+ u_{\vec{p}} = \not{P}_- v_{-\vec{p}} = 0 = \bar{v}_{-\vec{p}} \not{P}_- = \bar{u}_{\vec{p}} \not{P}_+$$

A.3.6 3D Boundary Field (3D asymptotic state for correlator, 3D polarization) and Related 4D Classical Solution

We could decompose the boundary condition of the 4D bulk filed into two 3D spinor fields,

$$\gamma_0 \chi_\pm = \chi_\pm$$

$$\gamma_0 \bar{\chi}_\pm = \bar{\chi}_\pm$$

, like

$$\chi = \chi_+ + \chi_-; \chi_+ = \begin{bmatrix} \lambda_+ \\ \lambda_+ \end{bmatrix}; \chi_- = \begin{bmatrix} \lambda_- \\ -\lambda_- \end{bmatrix} \quad \text{(A.18)}$$

$$\overline{\chi} = \bar{\chi}_+ + \bar{\chi}_-; \overline{\chi}_+ = \begin{bmatrix} \lambda_+^* & \lambda_+^* \end{bmatrix}; \overline{\chi}_- = \begin{bmatrix} -\lambda_-^* & \lambda_-^* \end{bmatrix} \quad \text{(A.19)}$$

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Then under the conjugation relationship

$$\begin{aligned}\chi_+/\lambda_+ &\leftarrow conjugate \rightarrow \overline{\chi_+}/\lambda_+^* \\ \overline{\chi_-}/\lambda_-^* &\leftarrow conjugate \rightarrow \chi_-/\lambda_-\end{aligned} \tag{A.20}$$

we could choose $(\chi_+, \bar{\chi}_-)$ (equivalently λ_+, λ_-^*) to impose the Dirichlet boundary condition without certaining field and its conjugate at the same time.

Similarly, the 4D polarization spinors could also be decomposed into two 3D spinors, like for positive energy mode u,

$$u_{\vec{p}} = u_{+,\vec{p}} + u_{-,\vec{p}}. \tag{A.21}$$

If we insert this into the Dirac equation we'll find these two 3D spinor components related by

$$u_{-,\vec{p}} = (\frac{\not{\boldsymbol{p}}}{E_p + m})u_{+,\vec{p}}. \tag{A.22}$$

Similarly, we have

$$\bar{u}_{-,\vec{p}} = \bar{u}_{-,\vec{p}}(\frac{-\not{\boldsymbol{p}}}{E_p + m}). \tag{A.23}$$

Thus it's straightforward to match the flat space amplitude with the 3D boundary condition $(\chi_+, \bar{\chi}_-)$

$$\begin{aligned}u_{\vec{p}} &= (1 + \frac{\not{\boldsymbol{p}}}{E_p + m})u_{+,\vec{p}} \\ \bar{u}_{-,\vec{p}} &= \bar{u}_{-,\vec{p}}(1 - \frac{\not{\boldsymbol{p}}}{E_p + m}).\end{aligned} \tag{A.24}$$

And for the negative energy mode v, we could find

$$\begin{aligned}v_{-\vec{p}} &= (1 - \frac{\not{\boldsymbol{p}}}{E_p + m})v_{+,-\vec{p}} \\ \bar{v}_{-,-\vec{p}} &= \bar{v}_{-,-\vec{p}}(1 + \frac{\not{\boldsymbol{p}}}{E_p + m}).\end{aligned} \tag{A.25}$$

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A.3.7 Majorana fermions and its images in 3D boundary

We define the B-operator

$$B = -i\gamma_2 \tag{A.26}$$

in which satisfy the identity

$$B = B^T \tag{A.27}$$

$$B^2 = 1 \tag{A.28}$$

$$B\gamma_\mu B = -\gamma_\mu^T \tag{A.29}$$

and we could define charge conjugate under B

$$C : \chi \to B\chi^*$$

then Majornara condition of the 4D spinor will be

$$C\chi = \chi \tag{A.30}$$

$$\chi = \begin{bmatrix} v \\ (i\sigma_2)v^* \end{bmatrix}$$

Now we want the Majornara fermion as the boundary condition of the classical solution, we want to split it into two 3D spinors:

$$\chi = \chi_+ + \chi_- \tag{A.31}$$

$$\gamma_0\chi_\pm = \pm\chi_\pm \tag{A.32}$$

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and we should be careful that

$$\gamma_0 \chi_\pm^* = \pm \chi_\pm^* \tag{A.33}$$

so we put this decomposition into Majornara condition (A.30):

$$\begin{aligned}(\frac{1+\gamma_0}{2})C(\chi_+ + \chi_-) &= (\frac{1+\gamma_0}{2})(\chi_+ + \chi_-) = \chi_+ \\ &= (\frac{1+\gamma_0}{2})(-i\gamma_2)(\chi_+^* + \chi_-^*) = (-i\gamma_2)\chi_-^* = (-B)\bar{\chi}_-^T\end{aligned} \tag{A.34}$$

then we have 3D Majornara condition of fermion

$$\bar{\chi}_- = -\chi_+^T B \tag{A.35}$$

so we find that the dual boundary condition of the 3D Dirichlet boundary condition $\bar{\chi}_-$ and χ_+ is related under the Majorana condition just like fermion and anti-fermion related with each other in 4D. And we know there's no pseudo-Majorana fermion for $SO(1,3)$ spinor [24], so it's only a representation of Majorana spinor.

A.4 Amplitude notation

M in our thesis means the amplitude and the field in the parentheses means the external field's spin

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(field's type)	notation	polarization
scalar	O	
fermion	χ	u_A
fermion conjugate	$\overline{\chi}$	$\overline{u_A}$
vector (photon)	γ	ϵ_μ
spin 3/2 particle (gravitino)	ψ	$u_A\epsilon_\mu$
spin 3/2 particle conjugate	$\overline{\psi}$	$\overline{u_A}\epsilon_\mu$
tensor (graviton)	h	$\epsilon_\mu\epsilon_\nu$

and the subscripts mean the label of their momentum, like

$$\phi(\vec{p_1}) = \phi_1 \tag{A.36}$$

so take QED Compton Amplitude as an example, we have$M(\gamma_1\bar{\chi}_2\chi_3) = \bar{u}_2 \not{\epsilon}_1 u_3$ and we define the amplitude with the polarization of the field extracted (the indices on the amplitude M is the same order as the field if not all the polarization field is extracted then the indices with the same scripted number as its field)

$$M(\gamma_1\bar{\chi}_2\chi_3) = \bar{u}_2 \not{\epsilon}_1 u_3 = \epsilon_1^{\mu_1} M_{\mu_1}(\gamma_1\bar{\chi}_2\chi_3) = \epsilon_1^{\mu}\bar{u}_{2,A} M_\mu^{AB}(\gamma_1\bar{\chi}_2\chi_3) u_{3,B} \tag{A.37}$$

with

$$M_{\mu_1}(\gamma_1\overline{\chi_2}\chi_3) = \bar{u}_2\gamma_{\mu_1}u_3 \tag{A.38}$$

$$M_\mu^{AB}(\gamma_1\overline{\chi_2}\chi_3) = \gamma_\mu^{A,B} \tag{A.39}$$

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and because the polarization of amplitude has the following property

$$p^\mu \epsilon_\mu = 0 \rightarrow \epsilon_0 = \hat{p} \cdot \epsilon = -\hat{p}^i \epsilon_i \tag{A.40}$$

so we could one step further extract the 3D polarization vector and have vector indices amplitude

$$M(\gamma_1 \bar{\chi}_2 \chi_3) = \epsilon_1^{\mu_1} M_{\mu_1}(\gamma_1 \bar{\chi}_2 \chi_3) = \epsilon_1^{i_1} \tilde{M}_{i_1}(\gamma_1 \bar{\chi}_2 \chi_3) = \epsilon_1^{i_1} \tilde{M}_{i_1}(\gamma_1 \bar{\chi}_2 \chi_3) \tag{A.41}$$

in explicit form

$$\tilde{M}_{i_1}(\gamma_1 \bar{\chi}_2 \chi_3) = -\hat{p}_{i_1} M_0(\gamma_1 \bar{\chi}_2 \chi_3) + M_{i_1}(\gamma_1 \bar{\chi}_2 \chi_3) = -\hat{p}_{i_1} \bar{u}_2 \gamma_0 u_3 + \bar{u}_2 \gamma_{i_1} u_3 \tag{A.42}$$

A.5 (Uncontracted) Cosmological Correlator, Contracted Cosmological Correlator and in-in correlator

We label the correlator with the wedge bracket and the boundary field (as Dirichlet boundary condition of Equation of Motion) inside the bracket:

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(Field's Type)	Operator	Boundary Condition
scalar	O	ϕ_0
fermion	χ_+	$\chi_{+0,A}$
fermion conjugate	$\overline{\chi_-}$	$\overline{\chi}_{-0.A}$
vector (photon)	J	$\epsilon_{\mathbf{0,i}}$
spin 3/2 particle (gravitino)	ψ_+	$\chi_{+,A}\epsilon_{\mathbf{0,i}}$
spin 3/2 particle conjugate	$\overline{\psi_-}$	$\overline{\chi}_{-.A}\epsilon_{\mathbf{0,i}}$
tensor (graviton)	T	$\epsilon_{\mathbf{0,i}}\epsilon_{0,j}$

(Uncontracted) Cosmological Correlator

In the main context, we define the correlator with the expansion of the wavefunction with the boundary condition extracted. So the (Uncontracted) Cosmological Correlator will have free indices like the QED Compton correlator will be labeled as

$$\langle J_1^i \overline{\chi}_{-,2,A}\chi_{+,3,B}\rangle = \left((1-\hat{\not{p}}_2)\frac{-\gamma_0\hat{p}_1^i+\gamma^i}{K_t}(1+\hat{\not{p}}_3)\right)_{AB} \tag{A.43}$$

Contracted Cosmological Correlator

For convenience, we could define the correlator contracted with the boundary field and then just omit the respondent indices

$$\langle J_1 \overline{\chi}_{-,2,A}\chi_{+,3,B}\rangle = \left((1-\hat{\not{p}}_2)\frac{-\gamma_0\hat{p}_{1,i}\epsilon_1^i+\not{\epsilon}_1}{K_t}(1+\hat{\not{p}}_3)\right)_{AB} \tag{A.44}$$

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if every boundary filed is contracted inside the correlator:

$$\langle J_1^i \overline{\chi}_{-,2}\chi_{+,3}\rangle = \overline{\chi}_{2,-}\left((1-\not{\hat{p}}_2)\frac{-\gamma_0\hat{p}_1^i+\gamma^i}{K_t}(1+\not{\hat{p}}_3)\right)\chi_{3,+} \tag{A.45}$$

Notice for the fermionic correlator, the four-component fermionic 3D field $\chi_+/\bar{\chi}_-$ is exactly the two-component 3D fermionic field's embedding, as (A.18),(A.19) shows. So we also have a two-component form that can be transformed by four-component respondent ones.

$$\begin{aligned}
\langle \boldsymbol{J}_1\overline{\chi}_{-,2}\chi_{+,3}\rangle &= \overline{\chi}_{2,-}\left((1-\not{\hat{\boldsymbol{p}}}_{\boldsymbol{2}})\frac{-\gamma_0\hat{\boldsymbol{p}}_1\cdot\boldsymbol{\epsilon}_1+\not{\boldsymbol{\epsilon}}_{\boldsymbol{1}}}{K_t}(1+\not{\hat{\boldsymbol{p}}}_3)\right)\chi_{3+} \\
&= \begin{bmatrix}-\lambda_{2,-}^* & \lambda_{2,-}^*\end{bmatrix}\left((1-\not{\hat{\boldsymbol{p}}}_{\boldsymbol{2}})\frac{-\gamma_0\hat{\boldsymbol{p}}_1\cdot\boldsymbol{\epsilon}_1+\not{\boldsymbol{\epsilon}}_{\boldsymbol{1}}}{K_t}(1+\not{\hat{\boldsymbol{p}}}_3)\right)\begin{bmatrix}\lambda_{3,+}\\ \lambda_{3,+}\end{bmatrix} \\
&= \lambda_{2,-}^*\{\frac{2}{K_t}\cdot[-(\hat{\boldsymbol{p}}_2\cdot\boldsymbol{\sigma})(\hat{\boldsymbol{p}}_1\cdot\boldsymbol{\epsilon}_1)-(\hat{\boldsymbol{p}}_3\cdot\boldsymbol{\sigma})(\hat{\boldsymbol{p}}_2\cdot\boldsymbol{\epsilon}_1) \\
&\quad -(\boldsymbol{\epsilon}_{\boldsymbol{1}}\cdot\boldsymbol{\sigma})+(\hat{\boldsymbol{p}}_2\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_{\boldsymbol{1}}\cdot\boldsymbol{\sigma})(\hat{\boldsymbol{p}}_1\cdot\boldsymbol{\sigma})]\}\lambda_{3,+} \\
&= \langle \boldsymbol{J}_1\lambda_{2,-}^*\lambda_{3,+}\rangle
\end{aligned}$$

$$\langle \boldsymbol{J}_1^i\lambda_{2,-,\alpha}^*\lambda_{3,+,\dot{\alpha}}\rangle = \frac{2\left[-(\hat{\boldsymbol{p}}_2\cdot\boldsymbol{\sigma})\left(\hat{\boldsymbol{p}}_1^i\right)-(\hat{\boldsymbol{p}}_3\cdot\boldsymbol{\sigma})\left(\hat{\boldsymbol{p}}_2^i\right)-\left(\boldsymbol{\sigma}^i\right)+(\hat{\boldsymbol{p}}_2\cdot\boldsymbol{\sigma})\left(\boldsymbol{\sigma}^i\right)(\hat{\boldsymbol{p}}_1\cdot\boldsymbol{\sigma})\right]_{\alpha\dot{\alpha}}}{K_t} \tag{A.46}$$

in-in correlator

For in-in correlator respondent to correlator we label the in-in correlation function with

$$\langle\overline{\chi}_{-,-p,A}\chi_{+,p,B}\rangle_{in-in} = \frac{(1-\gamma_0)}{2}\frac{1}{2\langle\overline{\chi}_{-,-p,A}\chi_{+,p,B}\rangle}\frac{(1+\gamma_0)}{2} = \frac{(1-\gamma_0)}{2}\frac{(-\not{\hat{\boldsymbol{p}}})}{2}\frac{(1+\gamma_0)}{2} \tag{A.47}$$

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Appendix B — Cosmological Background and Wave Function

We can construct a physical state as [30] do. And define the state at the current time as $\langle\phi_0|$. Then, we can date back the $\langle\phi_0|$ from current time $t = 0$ to past time $t = \eta < 0$ by the equation 9.1.34 in [30].

$$\begin{aligned}
&\langle\phi_n, -nd\eta|\phi_{n+1}, (-(n+1)d\eta)\rangle \\
&= \int d\pi_{n+1}\langle\phi_n, -(n+1)d\eta|exp(-iH(\hat{\phi}(-(n+1)d\eta), \hat{\pi}(-(n+1)d\eta))d\eta)|\pi_{n+1}, -(n+1)d\eta\rangle \\
&\cdot \langle\pi_{n+1}, -(n+1)d\eta|\phi_{n+1}, -(n+1)d\eta\rangle \\
&= \int \frac{d\pi_{n+1}}{2\pi} exp[-iH(\phi_n, \pi_{n+1})d\eta + i(\phi_n - \phi_{n+1})\pi_{n+1}] \\
&\langle\pi_{n+1}, -(n+1)d\eta|\phi_{n+1}, -(n+1)d\eta\rangle = \frac{1}{2\pi} exp(i\phi_{n+1}\pi_{n+1}).
\end{aligned} \tag{B.48}$$

and if we take $d\eta \to 0$ and write $\phi_n(\vec{x}) \to \phi(t, \vec{x}), \pi_{n+1}(\vec{x}) \to \pi(t, \vec{x})$

$$\begin{aligned}
&\int d\phi_n(\vec{x})\,\langle\phi_n, -nd\eta|\phi_{n+1}, (-(n+1)d\eta)\rangle \to \\
&\int d\phi(t, \vec{x}) \int \frac{d\pi(t, \vec{x})}{2\pi} exp\{[-iH(\phi(t, \vec{x}), \pi(t, \vec{x})) + i\partial_t\phi(t, \vec{x})\pi_{n+1}]dt\}.
\end{aligned} \tag{B.49}$$

then we'll have

$$
\begin{aligned}
\langle\phi_0| &= \int \prod_{\vec{x}} d\phi_1(\vec{x}) \langle\phi_0, t=0|\phi_1, t=-d\eta\rangle \langle\phi_1, t=-d\eta| \\
&= \int \prod_{\vec{x}} d\phi_2(\vec{x}) \int \prod_{\vec{x}} d\phi_1(\vec{x}) \langle\phi_0, t=0|\phi_1, t=-d\eta\rangle \langle\phi_1, t=-d\eta|\phi_2, t=-2d\eta\rangle \langle\phi_2, t=-2d\eta| \\
&= \int_{\phi_0=\phi'(0)} \prod_{t,\vec{x}} d\phi'(t,\vec{x}) \int \prod_{t,\vec{x}} \frac{d\pi'(t,\vec{x})}{2\pi} \\
&\exp\left[i\int_{-\infty}^{\eta_0} d^4x \left\{\partial_t\phi'(t,\vec{x})\pi'(t,\vec{x}) - H(\phi'(t,\vec{x}),\pi'(t,\vec{x}))\right\}\right] \langle\phi'(\eta), -\infty| \\
&= \int_{\phi_0=\phi(0)} D\phi \int \frac{D\pi}{2\pi} \exp\left[i\int_{-\infty}^{\eta_0} d^4x \left\{\partial_t\phi(t,\vec{x})\pi(t,\vec{x}) - H(\phi(t,\vec{x}),\pi(t,\vec{x}))\right\}\right] \langle\phi(\eta), -\infty| \,.
\end{aligned}
\tag{B.50}
$$

if the boundary field is originated by some field we certain in the far past $\langle\phi_{-\infty}|$, then we could write

$$
\langle\phi_0| = \int_{\phi_0=\phi(0),\phi_{-\infty}=\phi(-\infty)} D\phi \int \frac{D\pi}{2\pi} \exp\left[i\int_{-\infty}^{\eta_0} d^4x \left\{\partial_t\phi(t,\vec{x})\pi(t,\vec{x}) - H(\phi(t,\vec{x}),\pi(t,\vec{x}))\right\}\right] \langle\phi_{-\infty}| \,.
\tag{B.51}
$$

And because the saddle point of the conjugate field is

$$
\pi(t,\vec{x}) = \pi_0(t,\vec{x}) := -\partial_t\phi(t,\vec{x}).
\tag{B.52}
$$

suggested by [30] for quadratic Hamiltonian for the kinematic term. Then if we integral out the conjugate field π, we'll have

$$
\begin{aligned}
\langle\phi_0| &= \int_{\phi_0=\phi(0),\phi_{-\infty}=\phi(-\infty)} D\phi \exp\left[i\int_{-\infty}^{\eta_0} d^4x \left\{-(\partial_t\phi(t,\vec{x}))^2 - H(\phi(t,\vec{x}),\pi(t,\vec{x}))\right\}\right] \langle\phi_{-\infty}| \\
&= \int_{\phi_0=\phi(0),\phi_{-\infty}=\phi(-\infty)} D\phi \exp\left[i\int_{-\infty}^{\eta_0} d^4x S[\phi(t,\vec{x})]\right] \langle\phi_{-\infty}| \,.
\end{aligned}
\tag{B.53}
$$

in which the action defined as $S[\phi(t,\vec{x})] := \int_{-\infty}^{\eta_0} d^4x \left\{-(\partial_t\phi(t,\vec{x}))^2 - H(\phi(t,\vec{x}),\pi(t,\vec{x}))\right\}$.

It's the Path Integral formalism of the field with ϕ_0 as a boundary condition in current time

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$t = 0$. And the ϕ_0 spectrum is decided by the same Path Integral

$$\begin{aligned}\Psi[\phi_0] = \langle\phi_0|\Omega_0\rangle &= \int_{\phi_0=\phi(0),\phi_{-\infty}=\phi(-\infty)} D\phi \exp\left[i\int_{-\infty}^{\eta_0} d^4xS[\phi(t,\vec{x})]\right]\langle\phi_{-\infty}|\Omega_0\rangle \\ &= N\int_{\phi_0=\phi(0),\phi_{-\infty}=\phi(-\infty)} D\phi \exp\left[i\int_{-\infty}^{\eta_0} d^4xS[\phi(t,\vec{x})]\right]\end{aligned} \tag{B.54}$$

in which $N = \langle\phi_-\infty|\Omega_0\rangle$ is a constant in the path integral and independent of ϕ_0 and related to our setting at the far past vacuum. Without loss of generality, in this paper, we set $N = 1$ in this thesis. As [12] and [4] paper do, we define the ϕ_0 component of the background as the ”Wave function.”

It can be interpreted as the wave function of the universe, called Hartle-Hawking Wave Function $\Psi[h_{0,ij}, \phi_0]$ when the metric $h_{0,ij}$ is metric on the boundary in our Lagrangian. And the Einstein equation $G_{00} = 0$ will be interpreted as Wheeler-Dewitt equation, $H\Psi[h_{0,ij}, \phi_0] = 0$.[13]

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Appendix C — Feynman rules from $e^{iS_{cl}}$

In this section, we apply the discussion in the section (2.4) into the most simple ϕ^3 theory to derive the Feynman Rule by expanding the classical action. And because of the nontrivial boundary action of the fermion, we derive the Feynman Rule for some of the fermion correlators by expanding the boundary action of the fermion.

C.1 Feynman Rule from Bulk Action

Then we demonstrate how we get the Feynman rule of the bosonic field by ϕ^3-theory,

$$iS_{cl} = \frac{i}{2}\int d^4x(-\partial_\mu\phi_{cl}\partial^\mu\phi_{cl}) + \frac{2g}{3}\phi_{cl}^3 =: s_{cl}^{(0)} + gs_{cl}^{(1)} + g^2 s_{cl}^{(2)} + \dots. \tag{C.55}$$

The E.O.M reads (Notice coupling constant is normalized in E.O.M.)

$$\Box\phi_{cl} = -g\phi_{cl}^2 \tag{C.56}$$

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with the classical solution perturbatively expand by the Schwinger-Dyson equation (2.45)

$$\begin{aligned}
\phi_{cl}(t,\vec{x}) &= \phi_{cl}^{(0)}(\phi_0,t,\vec{x}) + g\phi_{cl}^{(1)}(\phi_0,t,\vec{x}) + \dots \\
\phi_{cl}^{(0)}(t,\vec{x}) &= \int d^3x' K(\vec{x},\vec{x}',t)\phi_0(\vec{x}') = \int \frac{d^3p}{(2\pi)^3} K(\vec{p},t)\phi_0(\vec{p})\ ;\ \Box_{t,\vec{x}}\phi_{cl}^{(0)}(t,\vec{x}) = 0 \\
\phi_{cl}^{(1)}(t,\vec{x}) &= -\int d^3x' d^3t' G(\vec{x},\vec{x}',t,t')\cdot(\phi_{cl}^{(0)}(\vec{x}',t'))^2 \\
&= -\int \frac{d^3p_1}{(2\pi)^3}\int \frac{d^3p_2}{(2\pi)^3}\int \frac{d^3p_3}{(2\pi)^3} d^3t' e^{i\vec{p}_1\cdot\vec{x}}\cdot G(\vec{p}_1,t,t')\phi_{cl}^{(0)}(\vec{p}_2,t')\phi_{cl}^{(0)}(\vec{p}_3,t')\delta^3\left(\sum_a^3 \vec{p}_a)\right)
\end{aligned} \tag{C.57}$$

and zero-order expansion will be the two-point correlator will be

$$\begin{aligned}
s_{cl}^{(0)} &= -\frac{i}{2}\int d^4(\partial_\mu\phi_{cl})(\partial^\mu\phi_{cl}) = -\frac{i}{2}\int d^4x\partial_\mu(\phi_{cl}^{(0)}\partial^\mu\phi_{cl}^{(0)}) - \phi_{cl}^{(0)}\Box\phi_{cl}^{(0)} = -\frac{i}{2}\int d^3x\phi_{cl}^{(0)}\partial^0\phi_{cl}^{(0)} \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} E\cdot\phi_0(-\vec{p})\phi_0(\vec{p}) = \int \frac{d^3p}{(2\pi)^3}\langle O_{-p}O_p\rangle\cdot\phi_0(-\vec{p})\phi_0(\vec{p}) \\
\langle O_{-p}O_p\rangle &= \sum_{perm.}\frac{E}{2} = E.
\end{aligned} \tag{C.58}$$

The first-order expansion will be the 3-point correlator

$$\begin{aligned}
s_{cl}^{(1)} &= \frac{i}{2}\int d^4x(-2)\partial_\mu\phi_{cl}^{(1)}\partial^\mu\phi_{cl}^{(0)} + \frac{2}{3}(\phi_{cl}^{(0)})^3 \\
&= \prod_a^3\left(\int \frac{d^3p_a}{(2\pi)^3}\phi_0(\vec{p}_a)\right)\delta^3\left(\sum_a^4 \vec{p}_a\right)\langle O_1O_2O_3\rangle\frac{1}{g} \\
\langle O_1O_2O_3\rangle &= \frac{2g}{(K_T - i\epsilon)}
\end{aligned} \tag{C.59}$$

in which we apply the integral by parts and the fact that we have three equivalent ϕ fields so when identifying the correlator we should include all the permutated indices of the momentum. And we take the boundary value of the perturbative classical solution with $\phi_{cl}^{(0)} = \phi_0(\vec{x})$ and

$$\phi_0(\vec{x}) = \phi_{cl}(0,\vec{x}) = \phi_{cl}^{(0)}(0,\vec{x}) + g\cdot\phi_{cl}^{(1)}(0,\vec{x}) + \dots \tag{C.60}$$

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so

$$\phi_{cl}^{(n\geq 1)}(0,\vec{x}) = 0. \tag{C.61}$$

The second-order expansion will be the 4-point correlator

$$\begin{aligned} s_{cl}^{(2)} &= \frac{i}{2}\int d^4x(-2)\partial_\mu\phi_{cl}^{(2)}\partial^\mu\phi_{cl}^{(0)} - \partial_\mu\phi_{cl}^{(1)}\partial^\mu\phi_{cl}^{(1)} + 2(\phi_{cl}^{(1)})(\phi_{cl}^{(0)})^2 \\ &= \prod_a^4\left(\int\frac{d^3p_a}{(2\pi)^3}\phi_0(\vec{p}_a)\right)\delta^3\left(\sum_a^4\vec{p}_a\right)\langle O_1O_2O_3O_4\rangle\frac{1}{g^2} \\ \langle O_1O_2O_3O_4\rangle &= (-\frac{i}{2}g^2)\sum_{perm.}\int_{-\infty}^0 dt\int_{-\infty}^0 dt' e^{i(E_1+E_2)t'}G(\vec{p}_1+\vec{p}_2,t,t')e^{i(E_3+E_4)t} \\ &= \frac{-4g^2}{E_{12s}E_{34s}K_T} + (t) + (u) \end{aligned} \tag{C.62}$$

in which we apply the first-order expansion of EOM

$$\Box\phi_{cl}^{(1)} = -(\phi_{cl}^{(0)})^2. \tag{C.63}$$

Notice the correct normalization of the correlator will make the Optical theorem (3.33) be satisfied.

In this most simple example, we can find the singularity identity is trivially true for this example. And the correlator could only have total and partial energy singularities.

$$\underset{K_T\to 0}{\text{Res}}\langle O_1O_2O_3\rangle = M(O_1O_2O_3) = 2 \tag{C.64}$$

$$\underset{K_T\to 0}{\text{Res}}\langle O_1O_2O_3O_4\rangle_s = M_s(O_1O_2O_3O_4) = \frac{4}{(E_{12})^2 - E_s^2} = \frac{4}{S} \tag{C.65}$$

$$\begin{aligned} \underset{E_{12s}\to 0}{\text{Res}}\langle O_1O_2O_3O_4\rangle_s &= M_s(O_1O_2O_s)\cdot\frac{1}{2E_s}(\langle O_3O_4O_s\rangle - \langle O_3O_4O_{-s}\rangle) \\ &= \frac{-4}{E_{34}^2 - E_s^2} \end{aligned} \tag{C.66}$$

In (C.59) and (C.62), we show that the correlator from expansion should satisfy the Feyn-

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man Rule we suggest in the (2.52) and (2.53). And for other theories, the calculations are similar to the ϕ^3-theory that we show in App. C.3. So the (2.52) and (2.53) are generally correct for the correlator expanded from the classical bulk action.

C.2 Bounary Action

To get the correct correlator with correct relative normalization and WT identity, we need to be careful to add the correct boundary action, carefully expand the classical action and when identifying the correlator we should be careful to the permutation of the same field. And, for the fermion case, we need to consider how to match the 4D classical solution of the E.O.M to the 3D Dirichlet boundary condition. In general, the action of massless scalar respecting 3D rotation invariance when there's a boundary at $t = 0$ is

$$S = -\frac{1}{2}\int_{-\infty}^{t=0} d^4x(\partial_\mu\phi)^2 + c\int_{t=0} d^3x\phi\partial_0\phi \tag{C.67}$$

in which c is a unfix coefficient which will cause a relative coefficient between 2pts and higher points correlators if we put the Schwinger Dyson equation in. To fix this coefficient we need to consider the boundary condition setting when we solve the EOM of scalar

$$\Box\phi = 0 \tag{C.68}$$

It's the second derivative differential equation on time, so we know we need to set the Dirichlet boundary condition "or" Neumann Boundary condition, for our definition of Cosmological correlator we choose Dirichlet so we fix the boundary value on $\phi(\vec{x}, t =$

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$0) = \phi_0$ but not on its derivative $\partial_0\phi(\vec{x},t)|_{t=0}$:

$$\delta\phi(\vec{x},0) = 0; \delta(\partial_0\phi(\vec{x},0)) \neq 0 \text{ (Dirichlet)} \tag{C.69}$$

and do the variation on our action and require it vanish at EOM, $\Box\phi = 0$:

$$\delta S = c\int_{t=0} d^3x\phi\delta(\partial_0\phi) = 0 \tag{C.70}$$

To render the equation of motion a legitimate saddle point of the Lagrangian, it is necessary to establish $c = 0$. Thus, the accurate imposition of boundary conditions necessitates an appropriate formulation of the kinetic Lagrangian, which generally comprises both bulk and boundary components. Then we get the scalar kinematic action we use to derive the Feynman rule in the previous section.

There's another perspective to set the $c = 0$, it's the Hamiltonian approach, we view the Lagrangian as a canonical formalism from some legal Hamiltonian $H(\phi, \pi = -\partial_0\phi)$. Notice it should not have a dependency on $\partial_0\pi = -\partial_0^2\phi$. Then we rewrite the Lagrangian to absorb the boundary action in the bulk through the integral by parts

$$S = -\int_{-\infty}^{t=0} d^4x(1-c)(\partial_\mu\phi)^2 - c\phi\Box\phi \tag{C.71}$$

and it should be derived from the Hamiltonian approach

$$\begin{aligned} S &= \int_{-\infty}^{t=0} d^4x\pi\partial_0\phi + H(\phi, \pi = -\partial_0\phi) \\ &= \int_{-\infty}^{t=0} d^4x - (\partial_\mu\phi)^2 + H'(\phi, \pi = -\partial_0\phi) \end{aligned} \tag{C.72}$$

but

$$\phi\Box\phi = \phi\partial_0\pi - \phi\nabla^2\phi \tag{C.73}$$

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So any Lagrangian with $c \neq 0$ could not be derived from any legal Hamiltonian. We need to set $c = 0$. And we would know the unitary identity of the correlator relies on the legal Hamiltonian, so the 4pt COT setting the relative coefficient between 2pts and higher points correlator, equivalently, it set $c = 0$ here.

After we set a proper Hamiltonian, we put the free-field solution

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \cdot \phi_0(\vec{p}) e^{iEt} \tag{C.74}$$

Notice Bunch-Davies vacuum requires the field to be convergent at far past. Then, We have

$$iS = -\frac{i}{2}\int d^4x (\partial_\mu \phi)^2 = -\frac{i}{2}\int d^3x \phi_0(-\vec{p})(\partial_0\phi(\vec{p}))|_{t=0} = \int \frac{d^3p}{(2\pi)^3}\langle O_{-p}O_p\rangle \cdot \phi_0(-\vec{p})\phi_0(\vec{p}) \tag{C.75}$$

$$\langle O_{-p}O_p\rangle = \sum_{perm} \frac{E}{2} = E. \tag{C.76}$$

We note that the boundary action for the scalar field is trivial if we write the bulk field in the quadratic and Lorentz invariant form without the second derivative in time. And by a similar approach, we could show for the massless vector field, the boundary action is still trivial

$$S = -\frac{1}{2}\int_{-\infty}^{0} d^4x \frac{1}{4}(F_{\mu\nu})^2 + \int_{t=0} d^3x \left(b \cdot A^i\partial_0 A_i + c \cdot A^i \partial_i A^0 + d \cdot A^0 \partial_0 A^0\right). \tag{C.77}$$

- Temporal Gauge: $A_0(t, \vec{x}) = 0$.

  Then the variation on the boundary will all be Dirichlet like $\delta(A_i(t = 0, \vec{x})) = \delta A_0(t = 0, \vec{x}) = 0$. We could fix the action (C.77) to be

$$b = 0, c = 0, d = 0 \tag{C.78}$$

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by the requirement that the variation of action at $\partial_\mu F^{\mu\nu} = 0$ vanishing. (Before we take the gauge condition.)

$$\begin{aligned} \partial_t(\partial^i A_i(\vec{x}, t)) = 0 \\ \Box A_i(\vec{x}, t) - \partial_i \partial^j A_j(\vec{x}, t) = 0 \end{aligned} \tag{C.79}$$

The free field equation will be

$$A_i(\vec{p}, t) = [\pi_{ij} e^{iEt} - \hat{p}_i \hat{p}_j] A_{j,0}(\vec{p}) \tag{C.80}$$

in the momentum space. Notice Bunch-Davies vacuum requires the field to be convergent at far past.

- Lorentz Gauge: $\partial_0 A^0(\vec{x}, t) = -\partial_i A^i(\vec{x}, t)$.

The boundary condition will be $\delta(A_i(t=0, \vec{x})) = \delta(\partial_0 A_0)(t=0, \vec{x}) = 0$. For A_0 it's the Neumann Boundary condition. We could fix the action (C.77) to be

$$b = 0, c = 0, d = 0 \tag{C.81}$$

by the requirement that the variation of action at $\partial_\mu F^{\mu\nu} = 0$ vanishing.(Notice, the variation $\delta A_\mu(t, \vec{x})$ is vanishing on the boundary but not on the bulk.)

$$\begin{aligned} A_i(\vec{p}, t) = e^{iEt} A_{i,0}(\vec{p}) \\ A_0(\vec{p}, t) = -\hat{p}_i A_i(\vec{p}, t) \end{aligned} \tag{C.82}$$

Notice Bunch-Davies vacuum requires the field to be convergent at far past.

No matter what boundary condition setting (with the respondent form of action) and

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the gauge condition we take, the correlator we expand from the classical action is the same

$$iS[A_{\mu,cl}] = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\langle J^i J^j\rangle A_{i,0}(-\vec{p})A_{j,0}(\vec{p}) \tag{C.83}$$

$$\langle J^i J^j\rangle = \sum_{perm.} \frac{E}{2}(\pi_{ij}) = E(\pi_{ij}) \tag{C.84}$$

Notice this result is satisfied and could be bootstrapped by the 2-point WT identity of the photon correlator.

Similar to the massless photon, we need to add the boundary action of graviton

$$S = S_{bulk} + S_{bdry} \tag{C.85}$$

$$S_{bulk} = \int_{-\infty}^{t=0} d^4x\sqrt{-detg}R \tag{C.86}$$

we fix the normalization to be 2 to the 2pt normalization we want. If want to fix the boundary term of the graviton, we need to apply the proper boundary condition consistent with the gauge we choose.

- Temporal Gauge : $h_{0\mu} = 0$

  The consistent boundary condition variation should be $\delta h_{0\mu}(t=0,\vec{x}) = \delta h_{ij}(t=0,\vec{x}) = 0$, then the S_{bdry} could be fixed as [28] showed, first proposed by Hawking and Gibbens.

$$S_{bdry} = \int_{t=0} d^3x\sqrt{-detg}K \tag{C.87}$$

  in which the extrinsic curvature K is defined as

$$K = 2\nabla^\mu n_\mu \tag{C.88}$$

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$n^\mu = (1,0,0,0)$ is the normal vector of our boundary. This boundary action is called Hawking-Gibbens boundary action. Then we could solve the EOM

$$\begin{aligned}
&G_{00} = 0 = (\eta_{ij}\nabla^2 - \partial_i\partial_j)h^{ij} \\
&G_{0i} = 0 = \partial_0(-\partial^j h_{ij} + \partial_i h) \\
&G_{ij} = 0 = -\partial_k\partial_{(i}h_{j)k} - \frac{1}{2}(\Box h_{ij} + \partial_i\partial_j h) + \frac{\eta_{ij}}{2}(\Box h - \partial_k\partial_l h^{kl}) = 0
\end{aligned} \tag{C.89}$$

and we can solve EOM with the aid of

$$\eta^{ij}G_{ij} = \partial_0^2 h = 0 \tag{C.90}$$

and because we require $\lim_{t\to-\infty} h < \infty$ then $\partial_0 h = 0 = \partial_0 - \partial^j h_{ij}$, so $G_{ij} = 0$ could be simplified as

$$(\partial_0^2 + E^2)(\pi_{ik}\pi_{jl})h_{ij} = 0 \tag{C.91}$$

in the momentum space. So the solution will be

$$h_{ij}(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \left((\pi_{ik}\pi_{jl})e^{iEt} + (\eta_{(ik}\eta_{j)l} - \pi_{ik}\pi_{jl})\right) h_0^{kl}(\vec{p}) \tag{C.92}$$

If we take this back to the Einstein action with the Hawking-Gibbens boundary term to the second order

$$\Psi[h_{ij}(\vec{x},t)] = iS^{(2)} = \int \frac{d^3p}{(2\pi)^3} \langle T_{ij,-p}T_{kl,p}\rangle h_{\mathbf{0}}^{ij}(-\vec{p}) h_{\mathbf{0}}^{kl}(\vec{p}) + O(h^3) \tag{C.93}$$

we could get the 2pts correlation function

$$\langle T_{ij,-p}T_{kl,p}\rangle = \sum_{perm.} \frac{E}{2}(\pi_{ik}\pi_{jl}) = E(\pi_{ik}\pi_{jl}) \tag{C.94}$$

- Lorentz Gauge : $\partial^\mu h_{\mu\nu}$ For the Lorentz gauge the consistent field variation on the

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boundary must be $\delta\partial_0 h_{0i}(t = 0, \vec{x}) = \delta h_{00}(t = 0, \vec{x}) = \delta h_{ij}(t = 0, \vec{x}) = 0$. There is some Neumann condition on the gauge component, so we cannot apply the Hawking Gibbens boundary term. Actually, we can expand the action in the Lorentz gauge and bootstrap the boundary term with the ansatz of the boundary term then we'll get the action (We have checked this action automatically respects the remanent gauge invariance.)

$$S^{(2)} = \int_{t=-\infty}^{t=0} d^4x - \frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{2}\partial^\rho h_{\mu\nu}\partial^\nu h^\mu_\rho + \int_{t=0} d^3x 2h_{0i}\partial_0 h^{0i} + h^0_i \partial^i h_{00} \tag{C.95}$$

such that $\delta S^{(2)} = 0$ under the harmonic EOM

$$\Box h_{\mu\nu} = 0 \tag{C.96}$$

with the solution

$$h_{\mu\nu}(\vec{p}, t) = h_{\mu\nu,0}(\vec{p})e^{iEt} \tag{C.97}$$

and we take back the classical action and replace the $h_{00} = \hat{p}_i\hat{p}_j h^{ij}$ and $h_{0i} = -\hat{p}_j h^{ij}$ then we would get the same 2-point correlation as the temporal gauge

$$\Psi[h_{ij}(\vec{x}, t)] = iS^{(2)} = \int \frac{d^3p}{(2\pi)^3}\langle T_{ij,-p}T_{kl,p}\rangle h^{ij}_{\mathbf{0}}(-\vec{p})h^{kl}_{\mathbf{0}}(\vec{p}) + O(h^3) \tag{C.98}$$

then

$$\langle T_{ij,-p}T_{kl,p}\rangle = \sum_{perm.} \frac{E}{2}(\pi_{ik}\pi_{jl}) = E(\pi_{ik}\pi_{jl}) \tag{C.99}$$

Just as the bosonic action, we improve the boundary term discussion of AdS fermion in [16] and adapt it to flat space, we propose that the action of the massless fermion is also

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should be completed with a boundary term which is 3D rotation invariance.

$$S = \int \frac{1}{2}\,\bar{\chi}(-i\,\overrightarrow{\partial\!\!\!/})\chi + \frac{1}{2}\bar{\chi}(-i\,\overleftarrow{\partial\!\!\!/})\chi - m\bar{\chi}\chi\;\; d^4x \tag{C.100}$$

$$+\int b\,\bar{\chi}_0\chi_0 + c\,\bar{\chi}_0\gamma^0\chi_0\, d^3x \tag{C.101}$$

Notice the EOM won't be influenced by the boundary term:

$$(-i\,\overrightarrow{\partial\!\!\!/} - m)\chi = 0 \tag{C.102}$$

$$\bar{\chi}(i\,\overleftarrow{\partial\!\!\!/} - m) = 0 \tag{C.103}$$

and for a nature decomposition on half-space, we define

$$\chi = \chi_+ + \chi_- \tag{C.104}$$

$$\bar{\chi} = \bar{\chi}_+ + \bar{\chi}_- \tag{C.105}$$

$$\gamma_0\chi_\pm = \pm\chi_\pm \tag{C.106}$$

$$\bar{\chi}_\pm\gamma_0 = \pm\bar{\chi}_\pm \tag{C.107}$$

we can choose $(\chi_+, \bar{\chi}_-)$ (individual have 2 d.o.f) as our field and their conjugate momentum will be $(\chi_-, \bar{\chi}_+)$. Notice for our free EOM has a general solution composed of positive and negative energy modes with 2 coefficients for every momentum, then we can only have "Two" boundary conditions. And for a well define wave function correlator depending on the boundary field, we naturally choose the following Two Dirichlet boundary conditions:

$$\begin{aligned}(\chi_+(t=0,\vec{x}), \bar{\chi}_-(t=0,\vec{x})) &= (\chi_{+,0}, \bar{\chi}_{-,0})\\ (\chi_+(t=-\infty(1-i\epsilon),\vec{x}), \bar{\chi}_-(t=-\infty(1-i\epsilon),\vec{x})) &= (0,0)\end{aligned} \tag{C.108}$$

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to select the positive energy mode and it's easy to check the boundary condition at tilt negative infinity will set the coefficient of negative energy mode to be zero.

The Dirichlet boundary condition requires that if our EOM solution (classical solution) is the saddle point of the Lagrangian which respect Lorentz invariant in bulk among all the function respecting Dirichlet ONLY:

$$\delta S|_{\chi_{cl},\bar{\chi}_{cl}} = 0 \ \ \text{(under } \delta\chi_{+,0}(p) = 0, \delta\bar{\chi}_{-,0}(p)) = 0) \tag{C.109}$$

We can use this condition to constraint the coefficients in our action to be (we assume $S_{int}$is simple or it won't have any derivative on the fermion field)

$$\delta S|_{\chi_{cl},\bar{\chi}_{cl}} = \int d^3x(i\frac{1}{2})\delta\bar{\chi}_0\gamma^0\chi_0 - i\frac{1}{2}\bar{\chi}_0\gamma^0\delta\chi_0 + b\delta\bar{\chi}_0\chi + b\bar{\chi}_0\delta\chi_0 + c\delta\bar{\chi}_0\gamma_0\chi + c\bar{\chi}_0\gamma_0\delta\chi_0 \tag{C.110}$$

$$= \int d^3x(i\frac{1}{2} + c)\delta\bar{\chi}_{+,0}\chi_0 - (i\frac{1}{2} - i + c)\bar{\chi}_0\delta\chi_{0,-} + b\delta\bar{\chi}_{0,+}\chi_0 + b\bar{\chi}_0\delta\chi_{0,-} \tag{C.111}$$

$$= 0 \tag{C.112}$$

$$\text{(under } \delta\chi^0_{+,0}(p) = 0, \delta\bar{\chi}^0_{-,0}(p)) = 0) \tag{C.113}$$

so we have $i\frac{1}{2} + c = i\frac{1}{2} - c = -b$, then we have

$$c = 0; b = -\frac{i}{2} \tag{C.114}$$

and because the definition of the correlator is related to the wave function

$$\Psi[(\chi_{+,0}, \bar{\chi}_{-,0})] = e^{iS(\chi_{cl}(\chi_{+,0}),\bar{\chi}_{cl}(\bar{\chi}_{-,0}))} \tag{C.115}$$

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So we start all the theory from the action:

$$S = \int (1/2)\, \bar{\chi}(-i\, \overrightarrow{\not\partial})\chi - (1/2)\bar{\chi}(-i\, \overleftarrow{\not\partial})\chi - m\bar{\chi}\chi \;\, d^4x \tag{C.116}$$

$$+ \int (-i/2)\, \bar{\chi}_0\chi_0 \, d^3x \tag{C.117}$$

And by this, if we take the classical solution, (2.34), and (2.35) into the action, we would find that the bulk term is just vanishing because it's proportional to E.O.M. As a remark, it's also the reason why completing the bulk action with the boundary term is necessary. In the end, we would get the two-point correlator:

$$\begin{aligned} iS_{cl} &= \int (1/2)\, \bar{\chi}_0\chi_0 \, d^3x = \int \frac{d^3p}{(2\pi)^3}\, (1/2)\, \bar{\chi}_0(-\vec{p})\chi_0(\vec{p}) \\ &= \int \frac{d^3p}{(2\pi)^3}\, (1/2)\, \bar{\chi}_{0,-}(-\vec{p})(1 + \frac{\not{\boldsymbol{p}}}{E+m})(1 + \frac{\not{\boldsymbol{p}}}{E+m})\chi_{0,+}(\vec{p}) \\ &= \int \frac{d^3p}{(2\pi)^3} \bar{\chi}_{0,-}(-\vec{p}) \frac{\not{\boldsymbol{p}}}{E+m} \chi_{0,+}(\vec{p}) \\ &= \int \frac{d^3p}{(2\pi)^3} \bar{\chi}_{0,-}(-\vec{p}) \langle \chi_0^-(-\vec{p})\chi_0^+(\vec{p})\rangle \chi_{0,+}(\vec{p}) \\ \langle \chi_0^-(-\vec{p})\chi_0^+(\vec{p})\rangle &= \frac{\not{\boldsymbol{p}}}{E+m} \end{aligned} \tag{C.118}$$

for massless fermion what we do for the bootstrap

$$\langle \chi_0^-(-\vec{p})\chi_0^+(\vec{p})\rangle_{m=0} = \frac{\not{\boldsymbol{p}}}{E} = \hat{\not{\boldsymbol{p}}} \tag{C.119}$$

Similarly, we start with the action for massless gravitino with the boundary term, then constrain the boundary term by the correct variation of the field on the boundary, which is up to our choice of gauge. Similar discussion for EAdS we can view [8].

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- Lorentz gauge: $\gamma^\mu\psi_\mu = \bar{\psi}_\mu\gamma^\mu = 0$

$$S = \int_{t=-\infty}^{t=0} d^4x\bar{\psi}_\mu(i\gamma^{\mu\nu\rho})(-\frac{1}{2}\partial_\nu + \frac{1}{2}\overleftarrow{\partial_\nu})\psi_\rho + \int_{t=0} d^3xb\bar{\psi}^i_{\mathbf{0}}\psi_{i,\mathbf{0}} + c(\bar{\psi}^i_{\mathbf{0}}\gamma_i)(\gamma^j\psi_{\mathbf{0},j})$$
$$+ d\bar{\psi}^i_{\mathbf{0}}\gamma_0\psi_{i,\mathbf{0}} + e\bar{\psi}^0_{\mathbf{0}}\gamma_0\psi_{0,\mathbf{0}} \tag{C.120}$$

The bold symbol 0 means the boundary condition to distinguish the boundary condition and 0 vector component. Notice for the Lorentz gauge

$$\gamma_0\psi_{0,\pm} = -\gamma_i\psi^i_\mp$$
$$\bar{\psi}_{0,\pm}\gamma_0 = -\psi^i_\mp\gamma_i \tag{C.121}$$

Then, we could fix the ansatz by the variation of the field on the boundary, we have the consistent variation $\delta\psi_{+,i}(\vec{x}, t = 0) = \delta\bar{\psi}_{-,i}(\vec{x}, t = 0) = 0$ and $\delta\psi_{-,0}(\vec{x}, t = 0) = -\delta(\gamma_0\gamma_i\psi^i_+(\vec{x}, t = 0)) = 0 = \delta\bar{\psi}_{+,0}(\vec{x}, t = 0) = -\delta(\bar{\psi}^i_-(\vec{x}, t = 0)\gamma_i\gamma_0)$. In contrast,$\delta\psi_{-,i}(\vec{x}, t = 0) \neq 0 \neq \delta\bar{\psi}_{+,i}(\vec{x}, t = 0)$ and $\delta\psi_{+,0}(\vec{x}, t = 0) = -\delta(\gamma_0\gamma_i\psi^i_-(\vec{x}, t = 0)) \neq 0 \neq \delta\bar{\psi}_{-,0}(\vec{x}, t = 0) = -\delta(\bar{\psi}^i_+(\vec{x}, t = 0)\gamma_i\gamma_0)$.

$$\delta(S) = 0 = \int_{t=0} d^3x\frac{i}{2}\delta\bar{\psi}_{\mathbf{0},+,i}\psi^i_{\mathbf{0},+} + \frac{i}{2}\bar{\psi}_{\mathbf{0},-,i}\delta\psi^i_{\mathbf{0},-} + \frac{i}{2}\delta\bar{\psi}_{\mathbf{0},-,0}\psi^0_{\mathbf{0},-} + \frac{i}{2}\bar{\psi}_{\mathbf{0},+,0}\delta\psi^0_{\mathbf{0},+}$$
$$+ b(\delta\bar{\psi}_{\mathbf{0},+,i}\psi^i_{\mathbf{0},+} + \bar{\psi}^i_-\delta\psi_{\mathbf{0},-,i}) + c(\delta\bar{\psi}_{\mathbf{0},-,0}\psi^0_{\mathbf{0},-} + \bar{\psi}^0_{\mathbf{0},+}\delta\psi_{\mathbf{0},+,0})$$
$$+ d(\delta\bar{\psi}_{\mathbf{0},+,i}\psi^i_{\mathbf{0},+} - \bar{\psi}^i_-\delta\psi_{\mathbf{0},-,i}) + e(-\delta\bar{\psi}_{\mathbf{0},-,0}\psi^0_{\mathbf{0},-} + \bar{\psi}^0_{\mathbf{0},+}\delta\psi_{\mathbf{0},+,0}) \tag{C.122}$$

The coefficient should be fixed as $i(b+d) = i(b-d) = i(c-e) = i(c+e) = \frac{1}{2}$,then

$$c = e = 0$$
$$b = c = \frac{-i}{2} \tag{C.123}$$

$$S = \int_{t=-\infty}^{t=0} d^4x\bar{\psi}_\mu(i\gamma^{\mu\nu\rho})(-\frac{1}{2}\partial_\nu + \frac{1}{2}\overleftarrow{\partial_\nu})\psi_\rho + \int_{t=0} d^3x\frac{-i}{2}\bar{\psi}^\mu_{\mathbf{0}}\psi_{\mu,\mathbf{0}} \tag{C.124}$$

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Similar to fermion E.O.M, we could solve the E.O.M

$$\gamma^{\nu}(\partial_{[\nu}\psi_{\rho]}) = M_{\rho} = 0 \tag{C.125}$$

after we apply the gauge condition we'll have the simplified E.O.M.

$$\not{\partial}\psi_{\rho} = 0 \tag{C.126}$$

and under the Lorentz gauge, this also implies $\partial_{\rho}\psi^{\rho} = 0$. Then it will be like the case of massless fermion. And if take the classical solution to the boundary $t = 0$,

$$\psi^{-}_{\mathbf{0},\mu} = (\frac{\not{\boldsymbol{p}}}{E+m})\psi_{\mathbf{0},+,\mu} \tag{C.127}$$

$$\bar{\psi}^{+}_{\mathbf{0},\mu} = \bar{\psi}^{-}_{\mathbf{0},\mu}(\frac{-\not{\boldsymbol{p}}}{E+m}). \tag{C.128}$$

Notice that if we take the classical solution into action, the bulk term should be vanishing too, then the two-point function is determined by the boundary term:

$$\begin{aligned} iS_{cl} &= \int \frac{1}{2}\bar{\psi}^{\mu}_{\mathbf{0}}\psi_{\mathbf{0},\mu} = \int \frac{1}{2}\bar{\psi}^{\mu}_{\mathbf{0},-}\psi^{-}_{\mathbf{0},\mu} + \frac{1}{2}\bar{\psi}^{\mu}_{\mathbf{0},+}\psi^{+}_{\mathbf{0},\mu} = \int \frac{d^3p}{(2\pi)^3}\bar{\psi}^{\mu}_{\mathbf{0},-,-\vec{p}}\hat{\not{\boldsymbol{p}}}\psi^{+}_{\mathbf{0},\mu} \\ &= \int \frac{d^3p}{(2\pi)^3}\bar{\psi}_{\mathbf{0},i,-,-\vec{p}}(\eta_{ij} + \hat{p}_i\hat{p}_j)\hat{\not{\boldsymbol{p}}}\psi^{+}_{\mathbf{0},j,\mu} \\ &= \int \frac{d^3p}{(2\pi)^3}\bar{\psi}_{\mathbf{0},i,-,-\vec{p}}(\pi_{ij}\hat{\not{\boldsymbol{p}}})\psi^{+}_{\mathbf{0},j,\mu} = \int \frac{d^3p}{(2\pi)^3}\bar{\psi}_{\mathbf{0},i,-,-\vec{p}}\langle\psi^{i}_{-,-p}\psi^{j}_{+,p}\rangle\psi^{+}_{\mathbf{0},j,\mu} \end{aligned} \tag{C.129}$$

$$\langle\psi^{i}_{-,-p}\psi^{j}_{+,p}\rangle = \pi_{ij}\hat{\not{\boldsymbol{p}}}. \tag{C.130}$$

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C.3 Fermionic Feynman Rule from Boundary Action

And all the correlators are not expanded from the bulk action but the boundary action S_b. Then we need to provide some rigorous proof to extract the similar Feynman Rule structure from the boundary action S_b. We should notice like the bosonic, the proof is independent of the vertices and the interaction type of the fermion. We only use the property of bulk-to-boundary and bulk-to-bulk propagator of the fermion which reflects the kinematic of the fermion.

Because we know the bosonic COT for exchanging diagrams depends on the structure of the bulk-to-bulk propagator, G_ϕ. So we use the 3D Schwinger-Dyson equation, whose bulk-to-bulk boundary propagator of the χ_+ is just G_ϕ, and we should notice the E.O.M of χ_+ has a dressed interaction term as (D.208) shows.

$$\left(\partial_t^2 + E^2\right)\chi^+ = \left(\frac{1+\gamma^0}{2}\right)\left(\not{\mathbf{p}} - i\partial_t + m\right)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}. \tag{C.131}$$

Similarly we could derive

$$\left(\partial_t^2 + E^2\right)\bar{\chi}^- = \frac{\delta S_{\text{int}}}{\delta\chi}(i\overleftarrow{\partial_t} - m + \not{\mathbf{p}})\left(\frac{1-\gamma_0}{2}\right). \tag{C.132}$$

And if we reexpress the interaction term as the vertices:(g is the coupling constant)

$$\begin{aligned}\frac{\delta S_{\text{int}}}{\delta\bar{\chi}} &=: gV\chi \\ \frac{\delta S_{\text{int}}}{\delta\chi} &=: g\bar{\chi}\bar{V}\end{aligned} \tag{C.133}$$

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and because they're deduced from the same Lagrangian, we require that

$$S_{int} = \int d^4x \bar{\chi} \frac{\delta L_{int}}{\delta \bar{\chi}} = \int d^4x \bar{\chi} \bar{V} \chi = \int d^4x \frac{\delta L_{int}}{\delta \bar{\chi}} \chi = \int d^4x \bar{\chi} V \chi \tag{C.134}$$

then the E.O.M could be rewritten as

$$\left(\partial_t^2 + E^2\right) \chi^+ = g \left(\frac{1+\gamma^0}{2}\right) \left(\not{\mathbf{p}} - i\partial_t + m\right) V \chi. \tag{C.135}$$

it's just the same E.O.M of the scalar with dressed interacting term. Then $\vec{p}_s = \vec{p}_3 + \vec{p}_4$ (We already label the momentum to match our convention for $\langle J_1 \bar{\chi}_2 J_3 \chi_4 \rangle_s$)

$$\begin{aligned} \chi_+^{(0)}(\vec{p}_4, t) &= e^{iE_4 t} \chi_{\mathbf{0},+}(\vec{p}_4) \\ \chi_+^{(1)}(\vec{p}_s, t) &= g \int dt' \int \frac{d^3 p_4}{(2\pi)^3} G_\phi(\vec{p}_s, t, t') \cdot \frac{1+\gamma_0}{2} \cdot (\not{\mathbf{p}}_{\mathbf{s}} - i\partial_{t'} + m) \\ &\left[V(\vec{p}_3, t')(1 + \frac{\not{\mathbf{p}}_{\mathbf{4}}}{E_4 + m}) \chi_{\mathbf{0},+}(\vec{p}_4) e^{iE_4 t'} \right] \end{aligned} \tag{C.136}$$

and we could get the χ_- spinor as (D.207) shows by

$$\chi^- = \left(\frac{\not{\mathbf{p}}}{-E^2 + m^2}\right) \left[\left(i\partial_t + m\right) \chi^+ - g \left(\frac{1+\gamma^0}{2}\right) V \chi \right], \tag{C.137}$$

Then we have

$$\begin{aligned} \chi_-^{(0)}(\vec{p}_4, t) &= \left(\frac{\not{\mathbf{p}}_4}{E+m}\right) e^{iE_4 t} \chi_{\mathbf{0},+}(\vec{p}_4) \\ \chi_-^{(1)}(\vec{p}_s, t) &= \left(\frac{\not{\mathbf{p}}_{\mathbf{s}}}{-E_s^2 + m^2}\right) \\ &\left[\left(i\partial_t + m\right) \chi_+^{(1)}(\vec{p}_s, t) - g \left(\frac{1+\gamma^0}{2}\right) \int \frac{d^3 p_4}{(2\pi)^3} V(\vec{p}_3, t)(1 + \frac{\not{\mathbf{p}}_{\mathbf{4}}}{E_4 + m}) \chi_{\mathbf{0},+}(\vec{p}_4) e^{iE_4 t} \right] \end{aligned} \tag{C.138}$$

and we should take care of that, on the boundary only the χ_+^0 match the boundary condition

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for the Dirichlet requirement, so

$$\chi_+^{(1)}(t=0,\vec{p}) =: \chi_{+,\mathbf{0}}^{(1)} = 0 \quad ; \quad \chi_-^{(1)}(t=0,\vec{p}) =: \chi_{-,\mathbf{0}}^{(1)} \neq 0 \tag{C.139}$$

Similarly, we have

$$\bar{\chi}_-^{(1)}(t=0,\vec{p}) =: \bar{\chi}_{+,\mathbf{0}}^{(1)} = 0 \quad ; \quad \bar{\chi}_+^{(1)}(t=0,\vec{p}) =: \bar{\chi}_{+,\mathbf{0}}^{(1)} \neq 0 \tag{C.140}$$

Now we expand the classical boundary action: (The bulk term is proportional to the E.O.M and vanishes.)

$$iS_{b,cl}^{(1)} = \frac{1}{2}\int \bar{\chi}_{\mathbf{0}}^{(1)}\chi_{\mathbf{0}}^{(0)} + \bar{\chi}_{\mathbf{0}}^{(0)}\chi_{\mathbf{0}}^{(1)} d^3x = \frac{1}{2}\int \bar{\chi}_{+,\mathbf{0}}^{(1)}\chi_{+,\mathbf{0}}^{(0)} + \bar{\chi}_{-,\mathbf{0}}^{(0)}\chi_{-,\mathbf{0}}^{(1)} d^3x \tag{C.141}$$

To calculate this, we can exploit the integral by parts of the E.O.M

$$\begin{aligned}\int d^4x\bar{\chi}^{(0)}(-i\gamma_\mu\partial^\mu - m)\chi^{(1)} &= -\int d^4x g\bar{\chi}^{(0)}V\chi^{(0)} \\ &= \int d^4x\bar{\chi}^{(0)}(i\gamma_\mu\overleftarrow{\partial}^\mu - m)\chi^{(1)} + \int d^4x\partial^\mu\left(\bar{\chi}^{(0)}(-i\gamma_\mu)\chi^{(1)}\right) \\ &= \int d^3x\bar{\chi}_{-,\mathbf{0}}^{(0)}(i)\chi_{-,\mathbf{0}}^{(1)}\end{aligned} \tag{C.142}$$

Then we could write

$$\int d^3x\bar{\chi}_{-,\mathbf{0}}^{(0)}\chi_{-,\mathbf{0}}^{(1)} = i\int d^4x g\bar{\chi}^{(0)}V\chi^{(0)} \tag{C.143}$$

Now for the $\bar{\chi}$ E.O.M we could have

$$\begin{aligned}\int d^4x\bar{\chi}^{(1)}(i\gamma_\mu\overleftarrow{\partial}^\mu - m)\chi^{(1)} &= -\int d^4x g\bar{\chi}^{(0)}\bar{V}\chi^{(0)} \\ &= \int d^4x\bar{\chi}^{(1)}(-i\gamma_\mu\partial^\mu - m)\chi^{(0)} + \int d^4x\partial^\mu\left(\bar{\chi}^{(1)}(i\gamma_\mu)\chi^{(0)}\right) \\ &= \int d^3x\bar{\chi}_{+,\mathbf{0}}^{(1)}(i)\chi_{+,\mathbf{0}}^{(0)}\end{aligned} \tag{C.144}$$

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Then we could write

$$\int d^3x\bar{\chi}^{(1)}_{+,\mathbf{0}}\chi^{(0)}_{+,\mathbf{0}} = i\int d^4xg\bar{\chi}^{(0)}\bar{V}\chi^{(0)} \tag{C.145}$$

Then if we take (C.143) and (C.145) into the boundary action expansion (C.141) then we have (We define $\tilde{V} = \frac{V+\bar{V}}{2}$)

$$\boxed{iS^{(1)}_{cl} = \int d^4x\bar{\chi}^{(0)}ig(\tilde{V})\chi^{(0)} = \prod_a^3\int\frac{d^3p_a}{(2\pi)^3}\delta^3(\sum_a\vec{p}_a)\psi_3(\vec{p}_{1\sim3})} \tag{C.146}$$

$\psi_3(\vec{p}_{1\sim3})$ is the correlator contracted with the boundary condition. Here we prove the Feynman Rule struct for the contact diagram. Then we could recover (D.197) for $gV = g\bar{V} = -e\not{A}$

$$\begin{aligned}\psi_3(\vec{p}_{1\sim3}) = \langle J_1\bar{\chi}_{2,-}\chi_{3,+}\rangle &= \int dt\bar{\chi}^{(0)}(\vec{p}_2)(igV(\vec{p}_1,t))\chi^{(0)}(\vec{p}_4,t) \\ &= \bar{\chi}_{2,-}\left[\left(1+\frac{\not{\boldsymbol{p}}_2}{E_2+m}\right)\frac{(\epsilon_{1,i}\gamma^i_1)}{K_T}\left(1+\frac{\not{\boldsymbol{p}}_3}{E_3+m}\right)\right]\chi_{3,+}\end{aligned} \tag{C.147}$$

The Feynman Rule structure of the spinor is similar to the bosonic one. However, for the exchanging diagrams, we need to consider the second-order expansion of the boundary action. Actually, it's too complicated, we could only use the discontinuity to extract the Feynman Rule structure we could use to build the COT for the diagrams exchanging Fermions. First, we need to know that similar to the first-order expansion, because the zero-order expansion of χ_+ and $\bar{\chi}_-$ already satisfy the boundary condition. Then

$$\chi^{(2)}_+(t=0,\vec{p}) =: \chi^{(2)}_{+,\mathbf{0}} = 0 \quad ; \quad \chi^{(2)}_-(t=0,\vec{p}) =: \chi^{(2)}_{-,\mathbf{0}} \neq 0 \tag{C.148}$$

$$\bar{\chi}^{(2)}_-(t=0,\vec{p}) =: \bar{\chi}^{(2)}_{+,\mathbf{0}} = 0 \quad ; \quad \bar{\chi}^{(2)}_+(t=0,\vec{p}) =: \bar{\chi}^{(2)}_{+,\mathbf{0}} \neq 0 \tag{C.149}$$

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Then the expansion of the boundary action will be

$$iS^{(2)}_{b,cl} = \frac{1}{2}\int \bar{\chi}^{(2)}_{\mathbf{0}}\chi^{(0)}_{\mathbf{0}} + \bar{\chi}^{(0)}_{\mathbf{0}}\chi^{(2)}_{\mathbf{0}} + \bar{\chi}^{(1)}_{\mathbf{0}}\chi^{(1)}_{\mathbf{0}} d^3x = \frac{1}{2}\int \bar{\chi}^{(2)}_{+,\mathbf{0}}\chi^{(0)}_{+,\mathbf{0}} + \bar{\chi}^{(0)}_{-,\mathbf{0}}\chi^{(2)}_{-,\mathbf{0}} d^3x \tag{C.150}$$

Then we apply the integral by parts of the E.O.M again, we could get

$$\begin{aligned} \int d^3x\bar{\chi}^{(0)}_{-,\mathbf{0}}\chi^{(2)}_{-,\mathbf{0}} &= i\int d^4xg\bar{\chi}^{(0)}V\chi^{(1)} \\ \int d^3x\bar{\chi}^{(2)}_{+,\mathbf{0}}\chi^{(0)}_{+,\mathbf{0}} &= i\int d^4xg\bar{\chi}^{(1)}\bar{V}\chi^{(0)} \end{aligned} \tag{C.151}$$

Then if we take these equations back to boundary action expansion,

$$iS^{(2)}_{cl} = \frac{1}{2}\int d^4x\bar{\chi}^{(0)}(igV)\chi^{(1)} + \bar{\chi}^{(1)}(ig\bar{V})\chi^{(0)} = \prod_a^4 \int \frac{d^3p_a}{(2\pi)^3}\delta^3(\sum_a^4 \vec{p}_a)\psi_4(\vec{p}_{1\sim 4}) \tag{C.152}$$

$$\psi_4(\vec{p}_{1\sim 4}, z_s) = \frac{1}{2}\int dt\bar{\chi}^{(0)}(\vec{p}_2,t)(igV(\vec{p}_1,t))\chi^{(1)}(\vec{p}_s,t) + \bar{\chi}^{(1)}(-\vec{p}_s,t)(ig\bar{V}(\vec{p}_3,t))\chi^{(0)}(\vec{p}_4,t). \tag{C.153}$$

Notice that the $\mathrm{Disc}_{z_s=|E_s|^2\pm i\epsilon}$ or the energy flipping differences should only extract the term with $e^{iE_st} = e^{i\sqrt{z_s}t}$ dependence. Then we could apply the discontinuity on the correlator (contracted with the boundary condition) $\psi_4(\vec{p}_{1\sim 4}, z_s = -\vec{p}_s\cdot\vec{p}_s + m^2)$ with complexified momentum on the bulk-to-bulk propagator. (We write the energy for the bulk to bulk propagator as $\sqrt{z_s}$).

$$\begin{aligned} &\underset{z_s=|E_s|^2\pm i\epsilon}{\mathrm{Disc}}\psi_4(\vec{p}_{1\sim 4}, z_s) = \\ &\frac{1}{2}\int dt \quad \bar{\chi}^{(0)}(\vec{p}_2,t)(igV(\vec{p}_1,t))\left(\underset{z_s=|E_s|^2\pm i\epsilon}{\mathrm{Disc}}\chi^{(1)}(\vec{p}_s,z_s,t)\right) \\ &+\left(\underset{z_s=|E_s|^2\pm i\epsilon}{\mathrm{Disc}}\bar{\chi}^{(1)}(-\vec{p}_s,z_s,t)\right)(ig\bar{V}(\vec{p}_3,t))\chi^{(0)}(\vec{p}_4,t) \end{aligned} \tag{C.154}$$

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And by (C.136) and (C.138) we have

$$\begin{aligned}
&\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}\chi^{(1)}(\vec{p}_s,t,z_s)\\
&=g\int dt'\left(1+\frac{\not{\boldsymbol{p}}_{\boldsymbol{s}}}{-z_s+m^2}(i\partial_t+m)\right)\left(\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}G_\phi(\vec{p}_s,\sqrt{z_s},t,t')\right)\\
&\cdot\frac{1+\gamma_0}{2}\cdot(\not{\boldsymbol{p}}_{\boldsymbol{s}}-i\partial_{t'}+m)\left[V(\vec{p}_3,t')\chi^{(0)}(\vec{p}_4,t')\right]\\
&=g\int dt'\left(\frac{1}{2E_s}\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}K_\phi(\vec{p}_s,\sqrt{z_s},t)\left(1+\frac{\not{\boldsymbol{p}}_{\boldsymbol{s}}}{\sqrt{z_s}+m}\right)\right)\\
&\cdot\frac{1}{2E_s}\cdot\frac{1+\gamma_0}{2}\cdot\not{\boldsymbol{p}}_{\boldsymbol{s}}\\
&\cdot\left(\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}K_\phi(\vec{p}_s,\sqrt{z_s},t')\left(1+\frac{\not{\boldsymbol{p}}_{\boldsymbol{s}}}{\sqrt{z_s}+m}\right)\right)\left[V(\vec{p}_3,t')\chi^{(0)}(\vec{p}_4,t')\right]
\end{aligned}\tag{C.155}$$

Similarly,

$$\begin{aligned}
&\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}\bar{\chi}^{(1)}(-\vec{p}_s,t,z_s)\\
&=g\int dt'\left[\bar{\chi}^{(0)}(\vec{p}_2,t)\bar{V}(\vec{p}_1,t)\right](-\not{\boldsymbol{p}}_{\boldsymbol{s}}+i\overleftarrow{\partial_{t'}}-m)\left(\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}G_\phi(\vec{p}_s,\sqrt{z_s},t,t')\right)\\
&\cdot\frac{1-\gamma_0}{2}\cdot\left(1+\frac{\not{\boldsymbol{p}}_{\boldsymbol{s}}}{-z_s+m^2}(i\overleftarrow{\partial_t}+m)\right)\\
&=\int dt'\left[\bar{\chi}^{(0)}(\vec{p}_2,t')\bar{V}(\vec{p}_1,t')\right]\left(\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}K_\phi(\vec{p}_s,\sqrt{z_s},t)(1+\frac{\not{\boldsymbol{p}}_{\boldsymbol{s}}}{\sqrt{z_s}+m})\right)\\
&\cdot\frac{-\not{\boldsymbol{p}}_{\boldsymbol{s}}}{2E_s}\cdot\frac{1-\gamma_0}{2}\\
&\cdot\left(\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon}K_\phi(\vec{p}_s,\sqrt{z_s},t')(1+\frac{\not{\boldsymbol{p}}_{\boldsymbol{s}}}{\sqrt{z_s}+m})\right)
\end{aligned}\tag{C.156}$$

where we use the integral by parts with the fact that $G_\phi(t'=0,t)=G_\phi(t',t=0)=0$. And we should notice that under the discontinuity we find a structure similar to the scalar Feynman rule with propagator G_ϕ hidden in the above two equations. Then we could

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show that

$$
\begin{aligned}
&\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} \psi_4(\vec{p}_{1\sim 4}, z_s) \\
&= \frac{1}{2}\int dt \left[\bar{\chi}^{(0)}(\vec{p}_2,t)V(\vec{p}_1,t)\right] \left(\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} K_\phi(\vec{p}_s,\sqrt{z_s},t)(1+\frac{\boldsymbol{\not p_s}}{\sqrt{z_s}+m})\right) \\
&\cdot \frac{1+\gamma_0}{2}\cdot\frac{-\boldsymbol{\not p_s}}{2E_s}\cdot\frac{1-\gamma_0}{2} \\
&\int dt'\cdot\left(\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} K_\phi(\vec{p}_s,\sqrt{z_s},t')(1+\frac{\boldsymbol{\not p_s}}{\sqrt{z_s}+m})\right)\left[\bar{V}(\vec{p}_3,t')\chi^{(0)}(\vec{p}_4,t')\right] + (\bar{V}\leftrightarrow V)
\end{aligned}
\tag{C.157}
$$

Notice the factor $(1+\frac{\boldsymbol{\not p_s}}{\sqrt{z_s}+m})$ is just the lifter of $\bar{\chi}^0(-\vec{p}_s)$ and $\chi^0(\vec{p}_s)$. With (C.134) in zero-order expansion, actually, we can replace the $\tilde{V}$ with V or $\bar{V}$ in (C.146). Then the left and the right term are the ψ_3 with the rightest and the most left boundary condition extracted, denoted as $\psi_{3,A}$. We could write down the discontinuity version of the COT

$$
\boxed{
\begin{aligned}
&\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} \psi_4(\vec{p}_{1\sim 4}, z_s) \\
&= \operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} \psi_{3,A}(\vec{p}_1,\vec{p}_2,z_s)\cdot\left[\frac{1+\gamma_0}{2}\cdot\frac{-\boldsymbol{\not p_s}}{2E_s}\cdot\frac{1-\gamma_0}{2}\right]^{AB}\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} \psi_{3,B}(\vec{p}_3,\vec{p}_4,z_s)
\end{aligned}
}
\tag{C.158}
$$

in which on the RHS of the above equation,

$$
\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} \psi_{3,A}(\vec{p}_1,\vec{p}_2,z_s) = \psi_{3,A}(\vec{p}_1,\vec{p}_2,E_s) - \psi_{3,A}(\vec{p}_1,\vec{p}_2,-E_s) = \tilde{\psi}_{3,A}(\vec{p}_1,\vec{p}_2,E_s)
\tag{C.159}
$$

so the RHS of the COT indeed gives factorization of shift correlators. And the LHS of the discontinuity

$$
\operatorname*{Disc}_{z_s=|E_s|^2\pm i\epsilon} \psi_4(\vec{p}_{1\sim 4}, z_s) = \psi_4(\vec{p}_{1\sim 4}, E_s) - \psi_4(\vec{p}_{1\sim 4}, -E_s)
\tag{C.160}
$$

We should notice it's not the RHS of the normal COT. This disc version of the COT shares a similar form to the scalar one in (3.9). It's because, under the discontinuity, we could

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extract a similar Feynman Rule structure with propagator G_ϕ in the above calculation.

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Appendix D — Correlator Calculation by the Lagrangian Approach

To get the correct correlator with correct relative normalization and WT identity, we need to be careful about adding the correct boundary action, carefully expand the classical action and when identifying the correlator we should be careful to the permutation of the same field. And, for the fermion case, we need to consider how to match the 4D classical solution of the E.O.M to the 3D Dirichlet boundary condition.

D.1 scalar QED : $\langle JO^*O\rangle$ and $\langle JO^*JO\rangle$

We calculate the correlator of scalar QED in the Lorentz gauge by the Lagrangian approach, $p^\mu A_\mu(\vec{p}) = 0$ in the momentum space. For the bulk to boundary propagator of the photon in the Lorentz gauge, we can refer (C.82). The calculation will be similar to the previous section. We'll write the action like

$$iS_{cl} = i\int d^4x(-(\partial_\mu - ieA_\mu)\phi^*_{cl} \cdot (\partial^\mu + ieA_\mu)\phi_{cl}) \tag{D.161}$$

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D.1.1 $\langle JO^*O\rangle$

Just like what we do for cubic scalar correlator, if we take the Schwinger-Dyson equation into action, just as we show in the previous section, the kinetic term won't contribute to 3pts. And because every field in $\langle JOO^*\rangle$ is different from each other.

$$iS_{cl}^{(1)} = eA_\mu^{(0)}(\phi^{(0)}\partial^\mu\phi^{(0),*} - \phi^{*,(0)}\partial^\mu\phi^{(0)}) \tag{D.162}$$

The identification of the correlator in momentum space won't have permutation on the label of the momentum like ϕ^3 theory. Then the correlator will be

$$\begin{aligned}\langle J_1O_2^*O_3\rangle &= \int dt\prod_a^3\int\frac{d^3p_a}{(2\pi)^3}(ie)\epsilon_{1,\mu}(p_2-p_3)^\mu e^{iK_Tt} = e\frac{\epsilon_{1,\mu}(p_2-p_3)^\mu}{K_T}\\ &= e\frac{(-\epsilon_1^i\hat{p}_{1,i})(E_2-E_3)+\epsilon_{1,i}(p_2-p_3)^i}{K_T}\\ &= \epsilon_{1,i}\langle J_1^iO_2^*O_3\rangle\end{aligned} \tag{D.163}$$

In which, we replace that $\epsilon_0 = -\epsilon^i\hat{p}_i$. It's trivial that it satisfies the constrain that total energy pole residue is amplitude, and we can check the WT identity as

$$\begin{aligned}p_{1,i}\langle J_1^iO_2^*O_3\rangle &= e\frac{(E_1)(E_2-E_3)-(p_2+p_3)_i(p_2-p_3)^i}{K_T}\\ &= e\frac{(E_1)(E_2-E_3)+E_2^2-E_3^2}{K_T} = e(E_2-E_3)\\ &= e\langle O_2^*O_{1+3}\rangle - e\langle O_{2+1}^*O_1\rangle\end{aligned} \tag{D.164}$$

So our unique bootstrap result in the context should match this result.

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D.1.2 $\langle JO^*JO\rangle$

Just like what we do for cubic scalar correlator, if we take the Schwinger-Dyson equation into the action, just as we show in the previous section, the second order expansion of the action will be (Our expansion here includes the coupling coefficient, like $S = S^{(0)} + S^{(1)} + S^{(2)} + O(e^3)$)

$$\begin{aligned} iS^{(2)}_{cl,\langle JO^*JO\rangle} = \int d^4x \quad & i(-\partial_\mu \phi^{*,(1)}\partial^\mu \phi^{(1)}) \\ & + eA^{(0)}_\mu (\phi^{(1)}\partial^\mu \phi^{(0),*} - \phi^{*,(1)}\partial^\mu \phi^{(0)} + \phi^{(0)}\partial^\mu \phi^{(1),*} - \phi^{*,(0)}\partial^\mu \phi^{(1)}) \\ & - ie^2 A^{(0)}_\mu A^{\mu,(0)}\phi^{*,(0)}\phi^{(0)} \end{aligned} \tag{D.165}$$

Notice that the $eA^{(1)}_\mu(\phi^{(0)}\partial^\mu\phi^{(0),*} - \phi^{*,(0)}\partial^\mu\phi^{(0)})$ will contribute to $\langle O^*OO^*O\rangle$ so we won't include this term here. We leave this calculation to readers for exercise. Then we use the E.O.M and integral by parts to re-express the kinetic term contribution

$$\begin{aligned} \int d^4x \quad i(-\partial_\mu \phi^{*,(1)}\partial^\mu \phi^{(1)}) &= \int_{t=0} d^3x \quad i\partial_\mu(-\phi^{*,(1)}\partial^\mu\phi^{(1)}) - \int d^4x \quad i(-\phi^{*,(1)}\Box\phi^{(1)}) \\ &= \int d^4x e\phi^{*,(1)}A^{(0)}_\mu(2\partial^\mu\phi^{(0)}) \end{aligned} \tag{D.166}$$

and we use the fact that $\phi^{(1)}(t=0,\vec{x}) = 0$ and E.O.M $\Box\phi^{(1)} = (ie)A^{(0)}_\mu(-2\partial^\mu\phi^{(0)})$. Then

$$\begin{aligned} iS^{(2)}_{cl,\langle JO^*JO\rangle} &= eA^{(0)}_\mu(\phi^{(1)}\partial^\mu\phi^{(0),*} + \phi^{*,(1)}\partial^\mu\phi^{(0)} + \phi^{(0)}\partial^\mu\phi^{(1),*} - \phi^{*,(0)}\partial^\mu\phi^{(1)}) \\ & + ie^2A^{(0)}_\mu A^{\mu,(0)}\phi^{*,(0)}\phi^{(0)} \\ &= eA^{(0)}_\mu(\partial^\mu\phi^{(0),*}\cdot\phi^{(1)} + \phi^{*,(1)}\partial^\mu\phi^{(0)} - \phi^{(1),*}\partial^\mu\phi^{(0)} + \partial^\mu\phi^{*,(0)}\cdot\phi^{(1)}) \\ & - ie^2A^{(0)}_\mu A^{\mu,(0)}\phi^{*,(0)}\phi^{(0)} \\ &= eA^{(0)}_\mu(2\partial^\mu\phi^{(0),*}\cdot\phi^{(1)}) - ie^2A^{(0)}_\mu A^{\mu,(0)}\phi^{*,(0)}\phi^{(0)} \end{aligned} \tag{D.167}$$

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in which we use integral by parts with $\phi^{(1)}(t=0,\vec{x})=0$ making the boundary term vanishing and we use the fact that $\partial_\mu A_0^\mu=0$. And we insert the Singer-Dyson equation in the first order.

$$\phi_{cl}^{(1)}(t,\vec{x}) = \int d^3x'd^3t'G(\vec{x},\vec{x}',t,t')\cdot(i)A_\mu^{(0)}(-2\partial^\mu\phi^{(0)})$$
$$= \int\frac{d^3p_s}{(2\pi)^3}\int\frac{d^3p_3}{(2\pi)^3}\int\frac{d^3p_4}{(2\pi)^3}d^3t'e^{i\vec{p}_s\cdot\vec{x}}\cdot G(\vec{p}_s,t,t')(2\epsilon_{3,\mu}p_4^\mu)e^{iE_{34}t'}\delta^3\left(\vec{p}_s+\vec{p}_3+\vec{p}_4\right)\phi_0(\vec{p}_4) \tag{D.168}$$

so if we label $\vec{p}_s=-\vec{p}_3-\vec{p}_4=\vec{p}_1+\vec{p}_2$

$$iS^{(2)}_{cl,\langle JO^*JO\rangle} = \left(\prod_a^4\int\frac{d^3p_a}{(2\pi)^3}\delta^3(\sum_b^4\vec{p}_b)\right)\left[\int dt\int dt'ie^2e^{iE_{12}t}(2\epsilon_{1,\nu}p_2^\nu)G(\vec{p}_s,t,t')(2\epsilon_{3,\mu}p_4^\mu)e^{iE_{34}t'}\right.$$
$$\left.-\,ie^2\int dte^{iK_Tt}\epsilon_{1,\mu}\epsilon_3^\mu\right]\phi_{\mathbf{0}}^*(\vec{p}_2)\phi_0(\vec{p}_4)$$
$$= \left(\prod_a^4\int\frac{d^3p_a}{(2\pi)^3}\delta^3(\sum_b^4\vec{p}_b)\right)\langle J_1^iO_2^*J_3^jO_4\rangle\epsilon_{1,i}\epsilon_{3,j}\phi_{\mathbf{0}}^*(\vec{p}_2)\phi_0(\vec{p}_4) \tag{D.169}$$

then we could identify the correlator, and notice two photo field is the same, so we need to permute the label of the momentum of them

$$\langle J_1O_2^*J_3O_4\rangle = \sum_{1\leftrightarrow3}\left[\int dt\int dt'ie^2e^{iE_{12}t}(2\epsilon_{1,\nu}p_2^\nu)G(\vec{p}_s,t,t')(2\epsilon_{3,\mu}p_4^\mu)e^{iE_{34}t'}-ie^2\int dte^{iK_Tt}\epsilon_{1,\mu}\epsilon_3^\mu\right]$$
$$= \left[\int dt\int dt'ie^2e^{iE_{12}t}(2\epsilon_{1,\nu}p_2^\nu)G(\vec{p}_s,t,t')(2\epsilon_{3,\mu}p_4^\mu)e^{iE_{34}t'}+(t)\right.$$
$$\left.-\,2\,ie^2\int dte^{iK_Tt}\epsilon_{1,\mu}\epsilon_3^\mu\right]$$
$$= e^2\frac{(2\epsilon_{1,\nu}p_2^\nu)(2\epsilon_{3,\mu}p_4^\mu)}{K_TE_{12s}E_{34s}}+(t)-e^2\frac{(2\epsilon_{1,\mu}\epsilon_3^\mu)}{K_T} \tag{D.170}$$

And it's trivial to check that the total energy pole residue is the amplitude and the transverse parts of the amplitude are indeed what we get at (4.46). The WT identity check will be

$$\epsilon_1^\mu=(-\epsilon_{1,i}\hat{p}_1^i,\epsilon_1^i) \tag{D.171}$$

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and when WT-identity we will replace $\epsilon_1^i \to p_1^i$, we'll find in the Lorentz gauge it equivalent to $\epsilon_1^\mu \to (E_1, \vec{p}_i) = p_1^\mu$. So we can write

$$\begin{aligned}
p_{1,i}\langle J_1^i O_2^* J_3 O_4\rangle &= e^2\frac{(2p_{1,\nu}p_2^\nu)(2\epsilon_{3,\mu}p_4^\mu)}{K_T E_{12s}E_{34s}} + e^2\frac{(2\epsilon_{3,\nu}p_2^\nu)(2p_{1,\mu}p_4^\mu)}{K_T E_{32t}E_{14t}} - e^2\frac{(2p_{1,\mu}\epsilon_3^\mu)}{K_T} \\
&= e^2\frac{(2\epsilon_{3,\mu})(-p_4^\mu - p_2^\mu - p_1^\mu)}{K_T} + e^2\frac{(2\epsilon_{3,\mu}p_4^\mu)}{E_{34s}} + e^2\frac{(2\epsilon_{3,\mu}p_2^\mu)}{E_{32t}} \\
&= (-2e^2\epsilon_{3,0}) + e^2\frac{(2\epsilon_{3,\mu}p_4^\mu)}{E_{34s}} + e^2\frac{(2\epsilon_{3,\mu}p_2^\mu)}{E_{32t}} \\
-e\langle O_{2+1}^* J_3 O_4\rangle + e\langle O_2^* J_3 O_{4+1}\rangle &= -e^2\frac{\epsilon_{3,\mu}(p_s^\mu - p_4^\mu)}{E_{34s}} + e^2\frac{\epsilon_{3,\mu}(p_2^\mu - \bar{p}_t^\mu)}{E_{32t}} \\
&= -e^2\frac{\epsilon_{3,\mu}((p_3+p_4+p_s)^\mu - 2p_4^\mu)}{E_{34s}} + e^2\frac{\epsilon_{3,\mu}(2p_2^\mu - (p_2+p_3+\bar{p}_t)^\mu)}{E_{32t}} \\
&= -e^2\frac{\epsilon_{3,\mu}((p_3+p_4+p_s)^\mu - 2p_4^\mu)}{E_{34s}} + e^2\frac{\epsilon_{3,\mu}(2p_2^\mu - (p_2+p_3+\bar{p}_t)^\mu)}{E_{32t}} \\
&= (-2e^2\epsilon_{3,0}) + e^2\frac{(2\epsilon_{3,\mu}p_4^\mu)}{E_{34s}} + e^2\frac{(2\epsilon_{3,\mu}p_2^\mu)}{E_{32t}}
\end{aligned} \tag{D.172}$$

in which $\bar{p}_t = (E_t, -\vec{p}_t)$, $p_t^i = p_2^i + p_3^i = -p_1^i - p_4^i$ and

$$S = (p_1+p_2)^\mu(p_1+p_2)_\mu = 2(p_1^\mu p_{2,\mu}) = -E_{12s}(-E_{12}+E_s) = -E_{12s}E_{34s} + K_T E_{12s} \tag{D.173}$$

Indeed, the WT identity is satisfied. Because the transverse parts and WT which means the longitudinal parts are matched to the bootstrap result in our main context. Then we should know the partial energy pole residues are also satisfied like what we do in the main context.

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D.2 TOO

We refer to the notes of Austin Joyce, we use the temporal gauge to calculate the $\langle TOO\rangle$ correlator. First, the action of the scalar coupling to graviton will be

$$iS = -i\int d^4x\sqrt{-detg}D_\mu\phi\cdot D^\mu\phi = -i\int d^4x\sqrt{-detg}\partial_\mu\phi\cdot\partial^\mu\phi \tag{D.174}$$

and notice we expand the metric like

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{D.175}$$

and for the temporal guage, we set $h_{0\mu} = 0$. There's still a guage freemdom $h_{ij} \sim h_{ij} + \partial_{(i}\xi_{j)}$. So we could have WT identity in this gauge. We use κ as our coupling constant to expand the SD equation and the action. As we know, three points correlator will come from the first-order expansion of the graviton in the action (Notice it's just the stress tensor $T^{(0)}_{ij}$ contracted with $g^{(1),ij} = -\kappa h^{(0),ij}$)

$$\begin{aligned} iS^{(1)}_{cl} &= -i\kappa T^{(0)}_{ij}h^{(0),ij} = i\kappa\left(\partial_i\phi^{(0)}\partial_j\phi^{(0)} - \frac{1}{2}\eta_{ij}(\partial_\mu\phi^{(0)})^2\right)h^{(0),ij} \\ T_{ij} &= -\sqrt{-detg}\left(\partial_i\phi\partial_j\phi - \frac{1}{2}g_{ij}(\partial_\mu\phi)^2\right) \end{aligned} \tag{D.176}$$

Notice we should include the expansion of the determinant.

$$\delta\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}g^{\mu\nu}\delta h_{\mu\nu} \to \left(\sqrt{-\det g}\right)^{(0)} = \frac{1}{2}\eta^{ij}h_{ij} \tag{D.177}$$

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Then we plug in the bulk-to-boundary propagator, especially for the temporal gauge graviton, we could identify it in (C.92)

$$h_{ij}^{(0)}(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \left((\pi_{ik}\pi_{jl})e^{iEt} + (\eta_{(ik}\eta_{j)l} - \pi_{ik}\pi_{jl}) \right) h_0^{kl}(\vec{p}) \tag{D.178}$$

as

$$K_{ijkl}(\vec{p},t) = \left((\pi_{ik}\pi_{jl})e^{iEt} + (\eta_{(ik}\eta_{j)l} - \pi_{ik}\pi_{jl}) \right). \tag{D.179}$$

Then we would have

$$\begin{aligned}
\langle T_{1,kl}O_2O_3\rangle &= \sum_{2\leftrightarrow 3}\int dt(i\kappa)\left(-p_2^i p_3^j + \frac{1}{2}\eta^{ij}(p_{2,\mu}p_3^{\mu})\right) e^{iE_{23}t}K_{ijkl}(\vec{p}_1,t) \\
&= \int dt(i\kappa)\left(-2p_2^{(i}p_3^{j)} + \eta^{ij}(E_2E_3 + p_{2,m}p_3^m)\right) e^{iE_{23}t}K_{ijkl}(\vec{p}_1,t) \\
&= \int dt(i\kappa)\left(-2p_2^{(i}p_3^{j)}(\pi_{1,ik}\pi_{1,jl}) + \pi_1^{kl}(E_2E_3 + p_{2,m}p_3^m)\right) e^{iK_Tt} \\
&+ \int dt(i\kappa)\left(-2p_2^{(i}p_3^{j)} + \eta^{ij}(E_2E_3 + p_{2,m}p_3^m)\right) e^{iE_{23}t}(\eta_{(ik}\eta_{j)l} - \pi_{ik}\pi_{jl}) \\
&= (\frac{\kappa}{K_T})\left(-2p_2^ip_3^j(\pi_{1,ik}\pi_{1,jl}) + 2\hat{p}_{1,(k}p_{2,l)}(E_1 - E_2 + E_3)\right. \\
&\left. + \hat{p}_{1,k}\hat{p}_{1,l}(\hat{p}_{1,m}p_2^m)(-E_3 + E_2) + \frac{1}{2}\hat{p}_{1,k}\hat{p}_{1,l}(-E_3^2 - E_2^2 + E_1^2 - 2E_3E_2 - 2E_1E_2)\right)
\end{aligned} \tag{D.180}$$

in the last step, we use Mathematica to calculate it in the kinematic and bootstrap algebraic form then we simplify it. We should remember that in the calculation from boundary E.O.M of the graviton like (2.28). We could identify $\pi_{1,ij}$ as 0.

And we also use Mathematica to check that (D.180) indeed satisfies that total energy pole is amplitude and WT identity. So it should be equal to the unique result in the main context we bootstrap from these two constraints.

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D.3 QED : $\langle J\bar{\chi}\chi\rangle$ and $\langle J\bar{\chi}J\chi\rangle$

The Lagrangian approach to derive the fermionic correlator is similar to what we do in C.2 to derive the two-point correlator of fermion. We improve the calculation in [22] and adapt it to flat space.

$$\begin{aligned} S = &\int (1/2)\,\bar{\chi}(-i\,\partial\!\!\!/)\chi - (1/2)\bar{\chi}(-i\,\overleftarrow{\partial\!\!\!/})\chi - m\bar{\chi}\chi\, + \mathcal{L}_{int}\, d^4x \\ &+ \int (-i/2)\,\bar{\chi}_0\chi_0\, d^3x \end{aligned} \tag{D.181}$$

where we define the boundary action as $S_b = \int (-i/2)\,\bar{\chi}_0\chi_0\, d^3x$. Then our E.O.M will be

$$(-i\,\partial\!\!\!/ - m)\chi = -\frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}} \tag{D.182}$$

$$\bar{\chi}(i\,\overleftarrow{\partial\!\!\!/} - m) = -\frac{\delta\mathcal{L}_{int}}{\delta\chi} \tag{D.183}$$

We propose two equivalent approaches to get the Schwinger-Dyson equation to get the solution of E.O.M under the boundary condition

$$\begin{aligned} (\chi_+(t=0,\vec{x}), \bar{\chi}_-(t=0,\vec{x})) &= (\chi_{+,0}, \bar{\chi}_{-,0}) \\ (\chi_+(t=-\infty(1-i\epsilon),\vec{x}), \bar{\chi}_-(t=-\infty(1-i\epsilon),\vec{x})) &= (0,0) \end{aligned} \tag{D.184}$$

The first approach is the Schwinger-Dyson equation of the full spinor, and get the relationship of $\chi_{-,0}(\chi_{+,0})$ and $\bar{\chi}_{+,0}(\bar{\chi}_{-,0})$ by Schwinger-Dyson equation, we propose the Schwinger-Dyson equation like

$$\chi(p,t) = K(p,t)\chi_0(p) + \int dt' G(p,t,t')\left(-\frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}(p,t')}\right) \tag{D.185}$$

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where

$$K(p,t) = e^{iE_p\, t}\frac{\not{P}+m}{2E_p}\gamma_0 \tag{D.186}$$

$$G(p,t,t') = i\theta(t-t')e^{-iE_p\,(t-t')}\frac{\not{P}_- - m}{2E_p} - i\theta(t'-t)e^{iE_p\,(t'-t)}\frac{\not{P}+m}{2E_p} \tag{D.187}$$

to match the full spinor boundary condition

$$\begin{gathered}(\chi(t=0,\vec{x}),\bar{\chi}(t=0,\vec{x})) = (\chi_0,\bar{\chi}_0)\\ (\chi(t=-\infty(1-i\epsilon),\vec{x}),\bar{\chi}(t=-\infty(1-i\epsilon),\vec{x})) = (0,0)\end{gathered} \tag{D.188}$$

notice in this form of Schwinger-Dyson, unlike the bosonic one, the propagator is Feynman without homogeneous term, and we could perturbatively get the relationship of the boundary condition to make the boundary condition only have 3D spinor degree of freedom, like $\chi_{-,0}(\chi_{+,0})$ and $\bar{\chi}_{+,0}(\bar{\chi}_{-,0})$. We could take the Schwinger-Dyson equation to $t=0$, Then

$$\chi_0(p) = \frac{\not{P}+m}{2E_p}\gamma^0\chi_0(p) + i\int dt' e^{iE_pt'}\frac{\not{P}_- - m}{2E_p}\left(-\frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}}(\vec{p},t')\right) \tag{D.189}$$

and we could use the trick to rewrite by $\chi_0 = \chi_{0,+} + \chi_{0,-}$

$$\frac{\not{P}+m}{2E_p}\gamma^0\chi_0(p) = \frac{1}{2}\chi_0(p) + \frac{\not{P}+m}{2E_p}\left(2\chi_0^+(p) - \chi_0(p)\right) \tag{D.190}$$

then we obtain the recurrence relationship

$$\begin{aligned}\chi_0(p) &= \left(1-\frac{\not{\boldsymbol{p}}}{E_p+m}\right)\left[\frac{\not{\boldsymbol{p}}+m}{E_p}\chi_0^+(p) + i\int dt' e^{iE_pt'}\frac{\not{P}_- - m}{2E_p}\left(-\frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}}(\vec{p},t')\right)\right]\\ &= \left(1+\frac{\not{\boldsymbol{p}}}{E_p+m}\right)\chi_0^+(p) + i\left(1-\frac{\not{\boldsymbol{p}}}{E_p+m}\right)\int dt' e^{iE_pt'}\frac{\not{P}_- - m}{2E_p}\left(-\frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}}(\vec{p},t')\right)\end{aligned} \tag{D.191}$$

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Similar derivation for $\bar{\chi}_0(p)$ yields

$$\bar{\chi}_0(p) = \left[\bar{\chi}_0^- \left(-\frac{\not{p} - m}{E_p}\right) - i\int dt' \left(-\frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}}(\vec{p}, t')\right) e^{iE_pt'} \frac{\not{P}_- + m}{2E_p}\right] \left(1 + \frac{\not{p}}{E_p + m}\right) \tag{D.192}$$

Then we could perturbatively expand the boundary field in the 3D spinor boundary degree of freedom $\chi_{+,0}, \bar{\chi}_{-,0}$ by the lifter $\mathbb{P}$

$$\begin{aligned} \chi_0(p) &= \mathbb{P}^{(e^0)}(\vec{p})\chi_0^+(p) + \int \frac{d^3q}{(2\pi)^3}\mathbb{P}^{(e^1)}(\vec{p}, \vec{q})\chi_0^+(q) + \cdots \\ \bar{\chi}_0(p) &= \bar{\chi}_0^-(p)\bar{\mathbb{P}}^{(e^0)}(\vec{p}) + \int \frac{d^3q}{(2\pi)^3}\chi_0^-(q)\bar{\mathbb{P}}^{(e^1)}(\vec{p}, \vec{q}) + \cdots \end{aligned} \tag{D.193}$$

For the case of QED in the Lorentz gauge, we set $p^\mu A_\mu(\vec{p}) = 0$ in the momentum space. For the bulk to boundary propagator of the photon in the Lorentz gauge, we can refer (C.82). sAnd the interaction term of the Lagrangian will be $\mathcal{L}_{int}(x_\mu) = -e\bar{\chi}(x_\mu)\not{A}(x_\mu)\chi(x_\mu)$, $\not{A} := A_\nu\gamma^\nu$, and the Fourier transforms of the interaction term we use in the Schwinger-Dyson equation are

$$\begin{aligned} \frac{\delta\mathcal{L}_{int}}{\delta\chi}(\vec{p}, t') &= e\int \frac{d^3q}{(2\pi)^3}\not{A}(\vec{p} - \vec{q}, t')\chi(\vec{q}, t') \\ \frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}}(\vec{p}, t') &= e\int \frac{d^3q}{(2\pi)^3}\bar{\chi}(\vec{q}, t')\not{A}(\vec{p} - \vec{q}, t') \end{aligned} \tag{D.194}$$

Then in $O(e^0)$ level the expansion of the boundary field reads

$$\begin{aligned} \chi_{\mathbf{0}}^{(0)}(p) &= \mathbb{P}^{(e^0)}(\vec{p})\chi_0^+(p) = \left(1 + \frac{\not{p}}{E_p + m}\right)\chi_{\mathbf{0}}^+(p) \\ \bar{\chi}_{\mathbf{0}}^{(0)}(p) &= \bar{\chi}_0^-(p)\bar{\mathbb{P}}^{(e^0)}(\vec{p}) = \chi_{\mathbf{0}}^-(p)\left(1 - \frac{\not{p}}{E_p + m}\right) \end{aligned} \tag{D.195}$$

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we recover the (2.33) and (2.35) where we use the free classical solution of half spinor to get it. In $O(e^1)$ level, we have

$$\begin{aligned}
\mathbb{P}^{(e^1)} &= \frac{e}{E_{\text{tot}}}\left(1-\frac{\not{\boldsymbol{p}}}{E_p+m}\right)\left(\frac{\not{P}_- - m}{2E_p}\right)\left(-\gamma^0\frac{p^i-q^i}{E_{p-q}}+\gamma^i\right) \\
&\cdot\left(\frac{\not{q}+m}{2E_q}\gamma^0\right)\left(1+\frac{\not{\boldsymbol{q}}}{E_q+m}\right)A_{0i}(\vec{p}-\vec{q}) \\
&\equiv \mathbb{P}^{(e^1)i}(\vec{p},\vec{q})A_{0i}(\vec{p}-\vec{q}) \\
\bar{\mathbb{P}}^{(e^1)} &= -\frac{e}{E_{\text{tot}}}\left(1-\frac{\not{\boldsymbol{q}}}{E_q+m}\right)\left(\gamma^0\frac{\not{Q}-m}{2E_q}\right)\left(-\gamma^0\frac{p^i-q^i}{E_{p-q}}+\gamma^i\right) \\
&\cdot\left(\frac{\not{\boldsymbol{p}}_- + m}{2E_p}\right)\left(1-\frac{\not{\boldsymbol{p}}}{E_p+m}\right)A_{0i}(\vec{p}-\vec{q}) \\
&\equiv \bar{\mathbb{P}}^{(e^1)i}(\vec{p},\vec{q})A_{0i}(\vec{p}-\vec{q})
\end{aligned} \tag{D.196}$$

where $E_{\text{tot}} \equiv E_p + E_{p-q} + E_q$. Substitute into iS_b, we get

$$\begin{aligned}
\langle J^i_{\vec{q}-\vec{p}}\bar{\chi}_{-,-\vec{q},\alpha}\chi_{+,\vec{p},\dot{\alpha}}\rangle &= \frac{1}{2}\left(\bar{\mathbb{P}}^{(e^0)}(-\vec{q})\mathbb{P}^{(e^1)i}(\vec{q},\vec{p})+\bar{\mathbb{P}}^{(e^1)i}(-\vec{p},-\vec{q})\mathbb{P}^{(e^0)}(\vec{p})\right) \\
&= \frac{-e}{K_T}\left[\left(1+\frac{\not{\boldsymbol{q}}}{E_q+m}\right)\gamma^i_{q-p}\left(1+\frac{\not{\boldsymbol{p}}}{E_p+m}\right)\right]_{\alpha\dot{\alpha}}
\end{aligned} \tag{D.197}$$

where $\gamma^i_{q-p} \equiv -\gamma^0\frac{q^i-p^i}{E_{q-p}}+\gamma^i$. We can see that the residue of the total energy pole is the 3pt amplitude of QED sandwiched by $\mathbb{P}^{(e^0)} = \left(1+\frac{\not{\boldsymbol{p}}}{E_p+m}\right)$ which represents we map the 4D spinor or free solution of E.O.M $u_{\vec{p}} = \chi^{(0)}_{\mathbf{0}}, \bar{u}_{\vec{p}} = \bar{\chi}^{(0)}_{\mathbf{0}}$ to the 3D spinor $\chi_+, \bar{\chi}_-$. We could check that indeed the WT identity is satisfied.

$$\begin{aligned}
(q-p)_i\langle J^i_{\vec{q}-\vec{p}}\bar{\chi}_{-,-\vec{q},\alpha}\chi_{+,\vec{p},\dot{\alpha}}\rangle &= \frac{-e}{K_T}\bar{\chi}_{-,-\vec{q}}\left(1+\frac{\not{\boldsymbol{q}}}{E_q+m}\right)(K_T\gamma_0)\left(1+\frac{\not{\boldsymbol{p}}}{E_p+m}\right)\chi_{+,\vec{p}} \\
&= e\bar{\chi}_-(-\vec{q})\frac{\not{\boldsymbol{q}}}{E_q+m}\chi_+(\vec{q}) - e\bar{\chi}_-(-\vec{p})\frac{\not{\boldsymbol{p}}}{E_p+m}\chi_+(\vec{p}) \\
&= e\langle\bar{\chi}_{-,-\vec{q}}\chi_{+,\vec{p}+\vec{q}-\vec{p}}\rangle - e\langle\bar{\chi}_{-,-\vec{q}+\vec{q}-\vec{p}}\chi^*_{+,\vec{p}}\rangle
\end{aligned} \tag{D.198}$$

where we use E.O.M $E_p\gamma_0\chi^{(0)}_{\mathbf{0}}(\vec{p}) = -\not{\boldsymbol{p}}\chi^{(0)}_{\mathbf{0}}(\vec{p})$ and $\bar{\chi}^{(0)}_{\mathbf{0}}(-\vec{q})E_q\gamma_0 = \chi^{(0)}_{\mathbf{0}}(-\vec{q})\not{\boldsymbol{q}}$. So We know this result must match the unique bootstrap result by WT-identity and total energy

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pole amplitude in the main context.

Similarly, we could get the four-point correlator $\langle J\bar{\chi}_- J\chi_+\rangle$ by the $O(e^2)$ expansion of the lifter $\mathbb{P}^{(e^2)}$, $\bar{\mathbb{P}}^{(e^2)}$ and $\bar{\mathbb{P}}^{(e^1)}\mathbb{P}^{(e^1)}$. (For comparison, $\langle\bar{\chi}_-\bar{\chi}_-\chi_+\chi_+\rangle$ will be got from the $O(e^2)$ contribution composed by expansion the photon field $A^{(1)}$ multiplied with the first order lifter $\mathbb{P}^{(e^1)}$)($\vec{p}_s = \vec{p}_3 + \vec{p}_4$) ($\not{S} = \not{E}_s + \not{\boldsymbol{p}}_s$, $\not{S}_- = \not{E}_s - \not{\boldsymbol{p}}_s$)

$$\begin{aligned}
&\langle J_1^j \bar{\chi}_{-,2} J_3^i \chi_{+,4}\rangle \\
&= -\left(1+\frac{\not{\boldsymbol{p}}_2}{E_{p_2}+m}\right)\gamma^j_{p_1}\left[-\frac{\not{P}_4+\not{P}_3+m}{K_T E_{12s}E_{34s}}+\frac{1}{E_{12s}E_{34s}}\left(\frac{1-\gamma_0}{2}\right)\frac{\not{S}_- - m}{E_s+m}\right]\gamma^i_{p_3}\left(1+\frac{\not{\boldsymbol{p}}_4}{E_4+m}\right) \\
&-\left(1+\frac{\not{\boldsymbol{p}}_2}{E_2+m}\right)\gamma^i_{p_3}\left[-\frac{\not{P}_4+\not{P}_1+m}{K_T E_{23t}E_{14t}}+\frac{1}{E_{23t}E_{14t}}\left(\frac{1-\gamma_0}{2}\right)\frac{\not{T}_- - m}{E_t+m}\right]\gamma^j_{p_1}\left(1+\frac{\not{\boldsymbol{p}}_4}{E_4+m}\right)
\end{aligned} \tag{D.199}$$

we already use Mathematica to check its WT identity, Total Energy Pole residue, and Parital Energy Pole residue. So this result will be the unique result we bootstrap in the context. Notice $\vec{p}_s$ is the internal momentum. And the factor $\frac{\not{P}_{s-}-m}{E_s+m} = -1+\not{\boldsymbol{p}}_s/(E_s+m)$ and $\not{\boldsymbol{p}}_s/(E_s+m)$ factor seems like co-dimension 1 pole. But it's not because

$$\frac{\not{\boldsymbol{p}}_s}{(E_s+m)} = (\sqrt{(E_s+m)}\sqrt{(E_s-m)})\cdot\frac{\hat{\not{\boldsymbol{p}}}_s}{(E_{p_2}+m)} = \frac{\sqrt{(E_s-m)}\hat{\not{\boldsymbol{p}}}_s}{\sqrt{(E_s+m)}} \tag{D.200}$$

or

$$\operatorname*{Res}_{(E_s+m)=0}\frac{\not{\boldsymbol{p}}_s}{(E_s+m)} = \lim_{(E_s+m)\to 0}(E_s+m)\frac{\not{\boldsymbol{p}}_s}{(E_s+m)} = \lim_{(E_s+m)\to 0}\not{\boldsymbol{p}}_s = 0(\because |\vec{p}_s| = E_s^2-m^2=0) \tag{D.201}$$

It's just a reflection of the gluing factor, the two-point correlation $\frac{\not{\boldsymbol{p}}_s}{2(E_s+m)}$ on the Fermion Exchange Optical theorem and the respondent partial energy pole.

Actually, it's easy to analytically check the partial energy pole residue in (3.2.5). By

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the following calculation

$$
\begin{aligned}
&\operatorname*{Res}_{E_{34s}\to 0}\left(\frac{\not{P}_4+\not{P}_3+m}{K_T E_{12s}E_{34s}}-\frac{1}{E_{12s}E_{34s}}\left(\frac{1-\gamma_0}{2}\right)\frac{\not{S}_- - m}{E_s+m}\right)\\
&=\left(\frac{-\not{S}_- + m}{K_T E_{12s}}-\frac{1}{E_{12s}}\left(\frac{1-\gamma_0}{2}\right)\frac{\not{S}_- - m}{E_s+m}\right)\\
&=\frac{1}{2E_s}\left((\not{S}_- - m)(\frac{1}{E_{12s}}-\frac{1}{E_{12}-E_s})-\frac{E_s}{E_{12s}}(1-\gamma_0)\frac{\not{S}_- - m}{E_s+m}\right)\\
&=\frac{1}{2E_s}\left(\frac{(I+(\gamma_0-I)\frac{E_s}{E_s+m})\,(\not{S}_- - m)}{E_{12s}}-\frac{\not{S}_- - m}{E_{12}-E_s}\right)
\end{aligned}
\tag{D.202}
$$

and the following calculation

$$
\begin{aligned}
&\operatorname*{Res}_{E_{12s}\to 0}\left(\frac{\not{P}_4+\not{P}_3+m}{K_T E_{12s}E_{34s}}-\frac{1}{E_{12s}E_{34s}}\left(\frac{1-\gamma_0}{2}\right)\frac{\not{S}_- - m}{E_s+m}\right)\\
&=\frac{-\not{P}_1-\not{P}_2+K_T\gamma_0+m}{K_T E_{34s}}-\frac{1}{E_{34s}}\left(\frac{1-\gamma_0}{2}\right)\frac{\not{S}_- - m}{E_s+m}\\
&=\left(\frac{\not{S}+m}{K_T E_{34s}}+\frac{\gamma_0}{E_{34s}}-\frac{1}{E_{34s}}\left(\frac{1-\gamma_0}{2}\right)\frac{\not{S}_- - m}{E_s+m}\right)\\
&=\frac{1}{2E_s}\\
&\cdot\left((-\not{S}-m)(\frac{1}{E_{34s}}-\frac{1}{E_{34}-E_s})+\frac{2E_s}{E_{34s}}\frac{(E_s+m)\gamma_0}{E_s+m}\left(\frac{1+\gamma_0}{2}+\frac{1-\gamma_0}{2}\right)-\frac{2E_s}{E_{34s}}\frac{(1-\gamma_0)}{2}\frac{\not{S}_- - m}{E_s+m}\right)\\
&=\frac{1}{2E_s}\left((-\not{S}-m)(\frac{1}{E_{34s}}-\frac{1}{E_{34}-E_s})+\frac{2E_s}{E_{34s}}\frac{(E_s\gamma_0+m)}{E_s+m}\frac{1+\gamma_0}{2}+\frac{2E_s}{E_{34s}}\frac{(E_s\gamma_0-m)}{E_s+m}\frac{1-\gamma_0}{2}\right.\\
&\left.-\frac{2E_s}{E_{34s}}\frac{(1-\gamma_0)}{2}\frac{\not{S}_- - m}{E_s+m}\right)\\
&=\frac{1}{2E_s}\left((-\not{S}-m)(\frac{1}{E_{34s}}-\frac{1}{E_{34}-E_s})-\frac{1}{E_{34s}}\frac{E_s}{E_s+m}(-\not{S}-m)\,(1+\gamma_0)\right)\\
&=\frac{1}{2E_s}\left(\frac{(-\not{S}-m)(I+(-\gamma_0-I)\frac{E_s}{E_s+m})}{E_{34s}}-\frac{-\not{S}-m}{E_{34}-E_s}\right)
\end{aligned}
\tag{D.203}
$$

They're indeed the gluing factor on the partial energy pole.

There is an alternative way to get the same correlator, by the generalization of the method in App.C.2. We could express the E.O.M by two 3-D spinors χ_+/χ_- and see they are related by E.O.M directly, so we only need to solve the harmonic E.O.M of χ_+ like

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a scalar perturbatively. Then the E.O.M gives us the solution of χ_- perturbatively. The E.O.M of 3D-spinor will be the (**??**) added with interaction terms.

It's the more natural way to extract the Optical theorem from our calculation of the Fermion exchange correlator in §(3.2.3).

$$(\pm i\partial_t + m)\,\chi^{\pm} = \not{\mathbf{p}}\chi^{\mp} + \left(\frac{1\pm\gamma^0}{2}\right)\left(\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right). \tag{D.204}$$

with the independent Dirichlet boundary condition,

$$\chi_+(t=0,\vec{p}) = \chi_{+,0}(\vec{p}) \quad ;\bar{\chi}_-(t=0,\vec{p}) = \bar{\chi}_{-,0}(\vec{p}) \tag{D.205}$$

and Bunch-Davies vacuum state constraints

$$\chi_+(t=-\infty,\vec{p}) = 0 \quad ;\bar{\chi}_-(t=-\infty,\vec{p}) = 0 \tag{D.206}$$

From the equations above, we can then yield,

$$\chi^- = \left(\frac{\not{\mathbf{p}}}{-E^2+m^2}\right)\left[(i\partial_t+m)\,\chi^+ - \left(\frac{1+\gamma^0}{2}\right)\left(\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right)\right], \tag{D.207}$$

If we substitute (D.207) into the E.O.M of χ^-, we would get a second order equation of χ^+,

$$\left(\partial_t^2+E^2\right)\chi^+ = \left(\frac{1+\gamma^0}{2}\right)\left(\not{\mathbf{p}} - i\partial_t + m\right)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}. \tag{D.208}$$

Then we could find the χ_+ could be viewed as a scalar with the dressed interaction term, then it's trivial to write the Schwinger-Dyson equation of χ_+ by the scalar SD equation.

$$\chi^+(\mathbf{p},t) = K_\phi\left(\mathbf{p},t\right)\chi_{\mathbf{0}}^+ + \int dt' G_\phi(\mathbf{p};t,t')\left(\frac{1+\gamma^0}{2}\right)\left(\not{\mathbf{p}} - i\partial_{t'} + m\right)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}. \tag{D.209}$$

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and the scalar propagator is the same as (2.50)

$$K_\phi(\vec{p},t) = e^{i(E-i\epsilon)t}$$
$$G_\phi(\vec{p},t,t') = \frac{i}{2E}\left(e^{i(E-i\epsilon)(t-t')}\theta(t'-t) + e^{-i(E-i\epsilon)(t-t')}\theta(t-t') - e^{i(E-i\epsilon)(t+t')}\right) \tag{D.210}$$

Next, we can substitute the SD equation of χ^+ (D.209) back to (**??**), tuning the time coordinate to 0 on both sides would lead us the explicit dependence of $\chi_{\mathbf{0}}^-$ in terms of $\chi_{\mathbf{0}}^+$,

$$\chi_0^- = \left(\frac{\not{p}}{-E^2+m^2}\right)\left[i\left(iE\chi_0^+ + \int dt' e^{iEt'}\left(\frac{1+\gamma^0}{2}\right)(\not{p} - i\partial_{t'} + m)\left(\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right)\right)\right.$$
$$\left. + m\chi_0^+ - \left(\frac{1+\gamma^0}{2}\right)\left(\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right)_{t=0}\right], \tag{D.211}$$

where we have used $(\partial_t G_\phi)_{t=0} = e^{iEt'}$. Then we'll get the recurrence equation if we notice that

$$\begin{aligned}
\chi_0^- &\sim \left(\frac{\not{p}}{-E^2+m^2}\right)\left[i\left(iE\chi_0^+ + \int dt' e^{iEt'}(\not{p} - i\partial_{t'} + m)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right)\right. \\
&\left. + m\chi_0^+ - \left(\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right)_{t=0}\right] \\
&= \left(\frac{\not{p}}{-E^2+m^2}\right)\left[(-E+m)\chi_0^+ + i\int dt' e^{iEt'}(\not{p} - E + m)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right] \\
&(integral\ by\ parts) \\
&= \left[\frac{\not{p}}{(E+m)}\chi_0^+ + i\int dt' e^{iEt'}\left(1 + \frac{\not{p}}{E+m}\right)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right] \\
&\sim \left[\frac{\not{p}}{(E+m)}\chi_0^+ + i\int dt' e^{iEt'}\left(\frac{-\not{P}_- + m}{E+m}\right)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right] \\
&\sim \left[\frac{\not{p}}{(E+m)}\chi_0^+ + i\int dt' e^{iEt'}\frac{\not{P}_- - 2\not{E} + m}{E+m}\left(\frac{-\not{P}_- + m}{2(E)}\right)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right] \\
&\sim \left[\frac{\not{p}}{(E+m)}\chi_0^+ + i\left(1 - \frac{\not{p}}{E+m}\right)\int dt' e^{iEt'}\left(\frac{-\not{P}_- + m}{2(E)}\right)\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right] \\
\chi_0 &= \left[\left(1 + \frac{\not{p}}{(E+m)}\right)\chi_0^+ + i\left(1 - \frac{\not{p}}{E+m}\right)\int dt' e^{iEt'}\left(\frac{\not{P}_- - m}{2(E)}\right)\left(-\frac{\delta S_{\text{int}}}{\delta\bar{\chi}}\right)\right]
\end{aligned} \tag{D.212}$$

The $\sim$ means that the following expression taken into Lagrangian S_b will have the same

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value, notice that in Lagrangian's classical value, only boundary term S_b will contribute and

$$\int d^3x \bar{\chi}_0 \chi_0 = \bar{\chi}_{0,-}\chi_{0,-} + \bar{\chi}_{0,+}\chi_{0,+} \tag{D.213}$$

so in the $\chi_{0,-}$ the prefix $\frac{1-\gamma_0}{2}$ always be absorbed into $\bar{\chi}_{0,-}$. Actually, the (D.212) is the same as (D.192). And, by a similar approach, we can derive the equation (D.192) again. They are the starting point to get the lifter and compose the correlator. So, in this way, we can get the same result as the previous way using the 4D spinor SD equation.

D.4 $T\bar{\chi}\chi$

The Lagrangian of the graviton interacting with the massless fermions will be the curved space extension of the free fermion action in flat space. So, by (D.181), we have

$$\begin{aligned} S = &\int \sqrt{-detg}\left[(1/2)\,\bar{\chi}(-i\not{D})\chi - (1/2)\bar{\chi}(-i\overleftarrow{\not{D}})\chi\right]\, d^4x \\ &+ \int (-i/2)\,\bar{\chi}_0\chi_0\, d^3x \end{aligned} \tag{D.214}$$

in which

$$\not{D} = \gamma^a e_a^\mu \nabla_\mu \tag{D.215}$$

the e_a^μ is vielbein defined as

$$\begin{aligned} g^{\mu\nu} &= e_a^\mu e_b^\nu \eta^{ab} \\ e_\mu^a e_b^\mu &= \delta_b^a \\ e_a^\mu e_\nu^a &= \delta_\nu^\mu. \end{aligned} \tag{D.216}$$

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And the covariant derivative of the fermion is defined by ($\gamma_{ab} := \frac{1}{2}[\gamma_a, \gamma_b]$)

$$\nabla_\mu \chi = \partial_\mu \chi - \frac{1}{4} w_\mu^{ab} \gamma_{ab} \chi$$
$$\nabla_\mu \bar{\chi} = \partial_\mu \bar{\chi} + \frac{1}{4} \bar{\chi} \gamma_{ab} w_\mu^{ab} \tag{D.217}$$

in which the $w_\mu^{ab}(e_a^\mu)$ is the torsion-free spin connection dependent on the vielbein, [10]

$$w_\mu^{ab} = 2e^{\nu[a}\partial_{[\mu]} e_{\nu]}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\nu e_\sigma^c. \tag{D.218}$$

And we perturb the metric in the temporal gauge like

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \; ; \; h_{\mu 0} = 0 \tag{D.219}$$

and the respondent vielbein perturbation will be

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu} \\ e_\mu^a &= \delta_\mu^a + \frac{\kappa}{2} h_\mu^a + O(\kappa^2) \\ e_a^\mu &= \delta_a^\mu - \frac{\kappa}{2} h_a^\mu + O(\kappa^2) \\ e^{a\nu} &= \eta^{a\nu} - \frac{\kappa}{2} h^{a\nu} + O(\kappa^2) \\ e_{a\nu} &= \eta_{a\nu} + \frac{\kappa}{2} h_{a\nu} + O(\kappa^2) \end{aligned} \tag{D.220}$$

with the spin connection perturbation

$$w_\mu^{(1),ab} = -\kappa \partial^{[a} h_\mu^{b]} \tag{D.221}$$

To get the 3pt correlator $\langle T \bar{\chi} \chi \rangle$, from the E.O.M like

$$\sqrt{-detg}(-i\not{D})\chi = 0$$
$$\sqrt{-detg}(\bar{\chi}(i\overleftarrow{\not{D}})) = 0, \tag{D.222}$$

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we need to extract the 3pt vertices from first-order expansion of the E.O.M,

$$
\begin{aligned}
-i\partial\!\!\!/\chi^{(1)} &= -\frac{\delta\mathcal{L}_{int}}{\delta\bar{\chi}} =: -\kappa V_3\chi^{(0)} = \frac{\kappa}{2}(\eta^{ij}h_{ij})(i\partial\!\!\!/\chi^{(0)}) + i\frac{\kappa}{4}\gamma^c w_c^{(1),ab}\gamma_{ab}\chi^{(0)} + i\frac{\kappa}{2}\gamma^a h_{aj}\partial^j\chi^{(0)} \\
&= i\frac{\kappa}{4}\gamma^c w_c^{(1),ab}\gamma_{ab}\chi^{(0)} + i\frac{\kappa}{2}\gamma^a h_{aj}\partial^j\chi^{(0)} \\
\bar{\chi}^{(1)}(i\overleftarrow{\partial\!\!\!/}) &= -\frac{\delta\mathcal{L}_{int}}{\delta\chi} =: -\kappa\bar{\chi}^{(0)}\bar{V}_3 = -\frac{\kappa}{2}(\eta^{ij}h_{ij})(\bar{\chi}^{(0)}(i\overleftarrow{\partial\!\!\!/})) + i\frac{\kappa}{4}\bar{\chi}^{(0)}\gamma_{ab}\gamma^c w_c^{(1),ab} - i\frac{\kappa}{2}\bar{\chi}^{(0)}\overleftarrow{\partial}^j\gamma^a h_{aj} \\
&= i\frac{\kappa}{4}\bar{\chi}^{(0)}\gamma_{ab}\gamma^c w_c^{(1),ab} - i\frac{\kappa}{2}\bar{\chi}^{(0)}\overleftarrow{\partial}^j\gamma^a h_{aj}.
\end{aligned}
\tag{D.223}
$$

Notice that $D\!\!\!\!/^{(1)}\chi^{(0)} = \frac{\kappa}{2}\gamma^a h_a^\mu\partial_\mu\chi^{(0)} - \frac{\kappa w_c^{(1),ab}}{4}\gamma^c\gamma_{ab}\chi^{(0)}$. Because we have volume unit $\sqrt{-detg}$ in the action and the usual Feynman rule, like (C.147), we have

$$
\begin{aligned}
iS_b^{(1)} &= \frac{1}{2}\int d^3x(\bar{\chi}_{\mathbf{0}}\chi_{\mathbf{0}})^{(1)} + \frac{1}{2}\int d^3x(\sqrt{-detg})^{(1)}(\bar{\chi}_{\mathbf{0}}\chi_{\mathbf{0}})^{(0)} \\
&= i\kappa\int d^4x\bar{\chi}^{(0)}(\frac{V_3+\bar{V}_3}{2})\chi^{(0)} + \frac{1}{2}\int d^3x(\sqrt{-detg})^{(1)}(\bar{\chi}_{\mathbf{0}}\chi_{\mathbf{0}})^{(0)} \\
&= \left(\prod_a^3\int\frac{d^3p_a}{(2\pi)^3}\delta(\sum_a^3\vec{p}_a)\right)\frac{\kappa}{4}(\eta_{ij}h_{3,\mathbf{0}}^{ij})\bar{\chi}_{2,-,\mathbf{0}}\left(1-\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{2}}\right)\left(1+\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{1}}\right)\chi_{1,+,\mathbf{0}} \\
&+ \int dt\bar{\chi}_{2,-,\mathbf{0}}\left(1-\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{2}}\right)\left(\frac{-i\kappa e^{iE_{12}t}}{2}\right)\left(\gamma^a h_{aj}^{(0)}(\vec{p}_3,t)\frac{(p_1-p_2)^j}{2}\right)\left(1+\hat{\boldsymbol{p}}\!\!\!/_{\mathbf{1}}\right)\chi_{1,+,\mathbf{0}}.
\end{aligned}
\tag{D.224}
$$

in which we use the fact that

$$
\begin{aligned}
w_c^{(1),ab}\{\gamma^c,[\gamma_a,\gamma_b]\} &= -\kappa\partial^a h_c^b\{\gamma^c,[\gamma_a,\gamma_b]\} = -\kappa\partial^a h^{bc}(2\gamma_a\gamma_b\gamma_c - 2\gamma_c\gamma_b\gamma_a) \\
&= -\kappa\partial^a h^{bc}(2\gamma_a\eta_{bc} - 2\eta_{cb}\gamma_a) = 0.
\end{aligned}
\tag{D.225}
$$

Then we plug the classical solution of the temporal guage (C.92) into (D.224), we could identify the correlator

$$
\begin{aligned}
\langle T_{3,ij}\bar{\chi}_{2,-}\chi_{1,+}\rangle &= \frac{\kappa}{K_T}\bar{\chi}_{2,-,\mathbf{0}}(1-\hat{\boldsymbol{p}}\!\!\!/_2)\left(\frac{1}{4}(E_1-E_2)\gamma_{(i}\hat{p}_{3,j)} - \frac{1}{4}p_{3,(i}\gamma_{j)} - \frac{1}{4}p_{3,(i}\hat{p}_{3,j)}\right. \\
&\left.-\frac{1}{2}p_{1,(i}\gamma_{j)} - \frac{1}{4}(E_1+E_2)\hat{p}_3^i\hat{p}_3^j + \frac{1}{2}p_{1,(i}\hat{p}_{3,j)}\gamma_0 + \frac{1}{2}[p_{1,k}\hat{p}_3^k]\hat{p}_{3,(i}\hat{p}_{3,j)}\gamma_0\right)(1+\hat{\boldsymbol{p}}\!\!\!/_1)\chi_{1,+,\mathbf{0}}
\end{aligned}
\tag{D.226}
$$

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We use Mathematica to do the integral under random kinematics and use the bootstrap program to write the answer in the algebraic form. And we also use random kinematics to check that the algebraic form satisfies the total energy pole residue is the amplitude and WT identity. So, our unique bootstrap result in the main context should be equal to this result.

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Appendix E — Ward Identity of the Amplitude

For a general amplitudes with integer spin s, the amplitude can be written as $\epsilon_{\mu_1\cdots\mu_s}M^{\mu_1\cdots\mu_s}$. In this section, we show how to rewrite the amplitudes as $\epsilon_{i_1\cdots i_s}M^{i_1\cdots i_s}$. This will be useful to determine the transversal part of the correlator.

E.1 Spin 1 field

A scattering amplitude containing a spin 1 field can be written as

$$M = \epsilon^\mu M_\mu \tag{E.227}$$

Ward identity states that we have the residual gauge freedom to transform $\epsilon^\mu \to \epsilon^\mu + \alpha p^\mu$. Take $\alpha \equiv -\frac{p^\mu}{E_p}$, we have

$$\begin{aligned} &\epsilon^0 \to 0 \\ &\epsilon^i \to \epsilon^i - \frac{p^i}{E_p}\epsilon^0 = \epsilon^i + \frac{p^i p^j}{E_p^2}\epsilon_j = \pi^{ij}\epsilon_j \end{aligned} \tag{E.228}$$

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Therefore, we can rewrite the amplitude as

$$M = \epsilon_i \pi^{ij} M_j \tag{E.229}$$

Note that this equation holds only if the total four-momentum is conserved.

E.2 Spin 2 field

A scattering amplitude containing a spin 2 field can be written as

$$M = \epsilon^\mu \epsilon^\nu M_{\mu\nu} \tag{E.230}$$

Ward identity states that we have the residual gauge freedom to transform $\epsilon^\mu \to \epsilon^\mu + \alpha p^\mu$. Take $\alpha \equiv -\frac{p^\mu}{E_p}$, we have

$$\begin{aligned} &\epsilon^0 \to 0 \\ &\epsilon^i \to \epsilon^i - \frac{p^i}{E_p}\epsilon^0 = \epsilon^i + \frac{p^i p^j}{E_p^2}\epsilon_j = \pi^{ij}\epsilon_j \end{aligned} \tag{E.231}$$

Therefore, we can rewrite the amplitude as

$$M = \epsilon_i \pi^{ii'} \epsilon_{j'} \pi^{jj'} M_{i'j'} \tag{E.232}$$

Note that this equation holds only if the total four-momentum is conserved.

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Appendix F — Ward Takahashi Identities of the Correlator

F.1 2pt WT identity

Notice there's still gauge freedom in Kinetic action, so there's a Ward Takahashi Identity to ensure the gauge invariance. Because there's no one-point correlator, so no lower point contribution in 2-pts WT identity.

- $\langle JJ \rangle$

  For U(1) symmetry parametrized as $\delta\epsilon_{i,0}(\boldsymbol{x}) = \partial_i\alpha(\boldsymbol{x})$ then the $\delta\Psi(A_{i,0}) = 0$ tell us

  $$\delta\Psi(A_{i,0}) = 2\int \frac{d^3p}{(2\pi)^3}\langle J_i J_j\rangle A_0^i(-\vec{p})\delta A_0^j(\vec{p}) = 2i\int \frac{d^3p}{(2\pi)^3}(p^j\langle J_i J_j\rangle)A_0^i(-\vec{p})\alpha_0(\vec{p}) = 0 \tag{F.233}$$

  So we get

  $$p^j\langle J_i J_j\rangle = 0 \tag{F.234}$$

- $\langle TT \rangle$

  For the diffeomorphism parametrized as $\delta h_{0,ij} = 2\partial_{(i}\xi_{j),0} + O(h)$, then similar to

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(F.233), the invariance of wavefunction $\delta\Psi(h_{0,ij}) = 0$ tell us

$$p^i\langle T_{ij}T_{kl}\rangle = 0 \tag{F.235}$$

We could know the longitudinal parts of the two-point function should be zero and by the EOM $\pi_{ij}h_{\mathbf{0}}^{ij} = 0$ the term of π_{ij} should not contribute.

- $\langle\bar{\psi}\psi\rangle$

For the SUSY parametrized as $\delta\psi_{0,+,i} = \partial_i\epsilon_+ + O(h)$ and $\delta\bar{\psi}_{0,-,i} = \partial_i\bar{\epsilon}_- + O(h)$, then similar to (F.233), the invariance of wavefunction $\delta\Psi(\psi_{+,0,i}, \bar{\psi}_{-,0,i}) = 0$ tell us

$$p^i\langle\bar{\psi}_{-,i}\psi_{+,j}\rangle = p^j\langle\bar{\psi}_{-,i}\psi_{+,j}\rangle = 0 \tag{F.236}$$

.

F.2 3pt WT identity

- Scalar theory:

For U(1) symmetry parametrized as

$$\delta\phi_0(\boldsymbol{x}) \;=\; (-\,ie\alpha(\boldsymbol{x}))\,\phi_0(\boldsymbol{x}) \quad ; \quad \delta\phi_0^* \;=\; (ie\alpha(\boldsymbol{x}))\,\phi_0(\boldsymbol{x}) \;;\; \delta\epsilon_{i,0}(\boldsymbol{x}) \;=\; \partial_i\alpha(\boldsymbol{x}) \tag{F.237}$$

and in momentum space will be

$$\delta\phi_0(\boldsymbol{p}_1 + \boldsymbol{p}_2) \;=\; -\,ie\,\alpha(\boldsymbol{p}_1)\;\phi_0(\boldsymbol{p}_1)$$

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$$\delta\phi_0^*(\boldsymbol{p}_1+\boldsymbol{p}_2) \ = \ ie\,\alpha(\boldsymbol{p}_1)\,\phi_0(\boldsymbol{p}_1) \tag{F.238}$$

$$\delta\epsilon_{i,0}(\boldsymbol{p}_1) \ = i\,p_{1,i}\,\alpha(\boldsymbol{p}_1)$$

in position space, then the WT-identity will be directly from the invariance of wavefunction under the boundary condition' s decomposition

$$\begin{aligned}\delta\Psi(A_{i,0},\phi_0) = \prod_a^3 \int \frac{d^3p_a}{(2\pi)^3}\delta^3\left(\sum_a^3 \vec{p_a}\right) \\ \big\{\langle O_2^* O_{1+3}\rangle\phi_{\mathbf{0}}^*(\vec{p_2})\delta\phi_0(\vec{p_1}+\vec{p_3}) + \langle O_{2+1}^* O_3\rangle\phi_{\mathbf{0}}^*(\vec{p_2}+\vec{p_1})\delta\phi_0(\vec{p_3}) \\ + \langle J_{1,i}O_2^*O_3\rangle\delta A_0^i(\vec{p_1})\phi_{\mathbf{0}}^*(\vec{p_2})\phi_0(\vec{p_3})\big\} \\ = 0\end{aligned} \tag{F.239}$$

$$p_1^i\langle J_{1,i}O_2^*O_3\rangle = -e\,\langle O_{1+2}^*\,O_3\rangle\ + e\,\langle O_2^*\,O_{1+3}\rangle \ = \ e\,(E_2-E_3) \tag{F.240}$$

and for the diffeomorphism

$$\delta h_{0,ij} = 2\,\partial_{(i}\xi_{j),0}\ + O(h)\ \ ;\ \delta\phi_0 = \xi_0^i\partial_i\phi_0 + O(h) \tag{F.241}$$

Similar to (F.239), the invariance of wavefunction tells us the WT-identity will be

$$p_{1,i}\langle T_1^{ij}O_2O_3\rangle = -\frac{1}{2}\left(\langle O_{1+2}O_3\rangle p_2^j + \langle O_2O_{1+3}\rangle p_3^j\right) = -\frac{1}{2}\left(E_3p_2^j + E_2p_3^j\right). \tag{F.242}$$

- Pure Gravity: (double-checked in momentum space and coordinate space)

$$\delta h_{0,ij} = 2\,\nabla_{(i}\xi_{j),0} = 2\,\partial_{(i}\xi_{j),0} - 2\Gamma_{ij}^m\xi_m = 2\,\partial_{(i}\xi_{j),0} - 2\xi_{(i,0}^m\partial h_{j)m,0} + \xi^m\partial_m h_{ij} + O\left(h^2\right) \tag{F.243}$$

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Similar to (F.239), the invariance of wavefunction tells us the WT identity will be

$$p_{1,i}\epsilon_{1,j,0}\langle T_1^{ij}T_2T_3\rangle = (\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{\epsilon}_{2,0})\,p_{2,k}\epsilon_{2,l}\langle T_{1+2}^{kl}T_3\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{p}_2)\epsilon_{2,0,k}\epsilon_{2,0,l}\langle T_{1+2}^{kl}T_3\rangle$$
$$+ (\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{\epsilon}_{3,0})\,p_{3,k}\epsilon_{3,l}\langle T_2T_{3+1}^{kl}\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_{1,0}\cdot\boldsymbol{p}_3)\epsilon_{3,0,k}\epsilon_{3,0,l}\langle T_2T_{3+1}^{kl}\rangle \tag{F.244}$$

- Fermion Theory: for the U(1) symmetry parametrized as:

$$\delta\chi_{+,0}(\boldsymbol{x}) = (-ie\alpha(\boldsymbol{x}))\,\chi_{+,0}(\boldsymbol{x}) \;;\; \delta\bar{\chi}_{-,0}(\boldsymbol{x}) = (ie\alpha(\boldsymbol{x}))\,\bar{\chi}_{-,0}(\boldsymbol{x}) \;;\; \delta\epsilon_{i,0}(\boldsymbol{x}) = \partial_i\alpha(\boldsymbol{x}) \tag{F.245}$$

Similar to (F.239), the invariance of wavefunction tells us the WT identity will be

$$p_1^i\langle J_{1,i}\bar{\chi}_{2,-}\chi_{3,+}\rangle = -e\,\langle\bar{\chi}_{-,1+2}\,\chi_{+,3}\rangle + e\,\langle\bar{\chi}_{-,2}\,\chi_{+,1+3}\rangle = e\,(-\not{\boldsymbol{p}}_2 - \not{\boldsymbol{p}}_3) \tag{F.246}$$

The last equation is true for the massless spinor. For the diffeomorphism

$$\delta h_{0,ij} = 2\,\partial_{(i}\lambda_{j),0} + O(h)$$
$$\delta\chi_{0,+} = \lambda^m\partial_m\chi_{0,+} - \frac{1}{8}\partial_a\lambda_b\left[\gamma^a,\gamma^b\right]\chi_{0,+} + O(h)$$
$$\delta\bar{\chi}_{0,-} = \bar{\chi}_{0,-}\overleftarrow{\partial}_m\lambda^m + \frac{1}{8}\bar{\chi}_{0,-}\left[\gamma^a,\gamma^b\right]\partial_a\lambda_b + O(h)$$

Similar to (F.239), the invariance of wavefunction tells us the WT identity will be

$$p_{1,i}\epsilon_{1,j}\langle T_1^{ij}\bar{\chi}_{-,2}\chi_{+,3}\rangle = -\frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1)\langle\bar{\chi}_{-,1+2}\chi_{+,3}\rangle - \frac{1}{2}(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1)\langle\bar{\chi}_{-,2}\chi_{+,1+3}\rangle$$
$$+ \frac{1}{16}\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\langle\bar{\chi}_{-,1+2}\chi_{+,3}\rangle - \frac{1}{16}\langle\bar{\chi}_{-,2}\chi_{+,1+3}\rangle\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]$$
$$= -\frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1)\not{\boldsymbol{p}}_3 + \frac{1}{2}(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1)\not{\boldsymbol{p}}_2 + \frac{1}{16}\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\not{\boldsymbol{p}}_3 + \frac{1}{16}\not{\boldsymbol{p}}_2\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right] \tag{F.247}$$

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The last equaily is true for the massless spinor.

- N=1 Pure Supergravity

for the supersymmetry transform

$$
\begin{aligned}
\bar{\epsilon}_{-} &= \epsilon_{+}^{T}\, B \\
\delta_{\bar{\epsilon}_{-}} h_{ij} &= \bar{\epsilon}_{-}\gamma_{(i}\psi_{j,+)} + O(h) = \delta_{\epsilon_{+}} h_{ij} = -\bar{\psi}_{-(j}\,\gamma_{i)}\,\epsilon_{+} + O(h) \\
\delta\psi_i^{+} &= \partial_i\,\epsilon_{+} + \frac{1}{8}\,w_{iab}\left[\gamma^a,\gamma^b\right]\epsilon_{+} = \partial_i\,\epsilon_{+} - \frac{1}{8}\,\partial_a h_{bi}\left[\gamma^a,\gamma^b\right]\epsilon_{+} + O\left(h^2\right) \\
\delta\bar{\psi}_i^{-} &= \partial_i\,\bar{\epsilon}_{-} - \frac{1}{8}\bar{\epsilon}_{-}\,w_{iab}\left[\gamma^a,\gamma^b\right] = \partial_i\,\bar{\epsilon}_{-} + \frac{1}{8}\bar{\epsilon}_{-}\;\partial_a h_{bi}\left[\gamma^a,\gamma^b\right] + O\left(h^2\right)
\end{aligned}
$$

Similar to (F.239), the invariance of wavefunction tells us the gravitino WT-identity will be

$$
\begin{aligned}
p_{2,k}\langle T_1\bar{\psi}_{2,-}^{k}\psi_{3,+}\rangle &= -\langle T_1 T_{2+3}^{kl}\rangle\,\epsilon_{l,3}\,\left(\bar{\chi}_{2,-}\gamma_k\chi_{3,+}\right) - \frac{1}{8}\left[\not{\boldsymbol{p}}_1,\not{\epsilon}_1\right]\left(\epsilon_{1,k}\langle\bar{\psi}_{1+2,-}^{k}\psi_{3,+}\rangle\right) \\
&= -\left(\boldsymbol{\epsilon}_1^{T}\cdot\boldsymbol{\epsilon}_3\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^{T}\chi_{3,+}\right)E_1 - \frac{1}{8}\left(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3^{T}\right)\left(\bar{\chi}_{2,-}\left[\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1\right]\hat{\not{\boldsymbol{p}}}_3\chi_{3,+}\right)
\end{aligned} \tag{F.248}
$$

it will be equivalent to identity ($2\ \leftrightarrow 3,\ \ \bar{\chi}_{-} = \chi_{+}^{T}\ B$)

$$
\begin{aligned}
p_{3,k}\langle T_1\bar{\psi}_{2,-}\psi_{3,+}^{k}\rangle &= \langle T_1 T_{2+3}^{kl}\rangle\,\epsilon_{l,2}\,\left(\bar{\chi}_{2,-}\gamma_k\chi_{3,+}\right) + \frac{1}{8}\left(\epsilon_{1,k}\langle\bar{\psi}_{2,-}^{k}\psi_{1+3,+}\rangle\right)\left[\not{\boldsymbol{p}}_1,\not{\epsilon}_1\right] \\
&= \left(\boldsymbol{\epsilon}_1^{T}\cdot\boldsymbol{\epsilon}_2\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^{T}\chi_{3,+}\right)E_1 + \frac{1}{8}\left(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^{T}\right)\left(\bar{\chi}_{2,-}\hat{\not{\boldsymbol{p}}}_2\left[\not{\boldsymbol{p}}_1,\not{\epsilon}_1\right]\chi_{3,+}\right)
\end{aligned} \tag{F.249}
$$

there’ s a consistency check from different WT identities (Notice it’ s still holding for Dirac Fermion)

$$
\langle T_1\bar{\psi}_{2,-}^{L}\psi_{3,+}^{L}\rangle = -\left(\boldsymbol{\epsilon}_1^{T}\cdot\boldsymbol{p}_3\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^{T}\chi_{3,+}\right)\frac{E_1}{E_2E_3}\epsilon_{2,L}\epsilon_{3,L} = \left(\boldsymbol{\epsilon}_1^{T}\cdot\boldsymbol{p}_2\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1^{T}\chi_{3,+}\right)\frac{E_1}{E_2E_3}\epsilon_{2,L}\epsilon_{3,L}
$$

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and for the consisent parametrization of diffeomorphism transform

$$
\begin{aligned}
\delta h_{0,ij} &= \partial_{(i}\lambda_{j),0} + O(h) \\
\delta \psi^i_{0,+} &= \lambda^m \partial_m \psi^i_{0,+} + (\partial_i \lambda_m)\psi^m_{0,+} - \frac{1}{8}\partial_a \lambda_b \left[\gamma^a, \gamma^b\right] \psi^i_{0,+} + O(h) \\
\delta \bar{\psi}^i_{0,-} &= \bar{\psi}^i_{0,-} \overleftarrow{\partial}_m \lambda^m + (\partial_i \lambda_m)\bar{\psi}^m_{0,-} + \frac{1}{8}\bar{\psi}^i_{0,-} \left[\gamma^a, \gamma^b\right] \partial_a \lambda_b + O(h)
\end{aligned}
$$

Similar to (F.239), the invariance of wavefunction tells us the graviton WT-identity will be

$$
\begin{aligned}
p_{1,k}\langle T^k_1 \bar{\psi}_{2,-}\psi_{3,+}\rangle = &-\bar{\chi}_{2,-}\not{p}_2\chi_{3,+}\left(\boldsymbol{\epsilon}^T_2 \cdot \boldsymbol{\epsilon}_3\right)\left(\boldsymbol{p}_3 \cdot \boldsymbol{\epsilon}_1\right) - \bar{\chi}_{2,-}\not{p}_3\chi_{3,+}\left(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}^T_3\right)\left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}_1\right) \\
&- \bar{\chi}_{2,-}\not{p}_2\chi_{3,+}\left(\boldsymbol{\epsilon}^T_2 \cdot \boldsymbol{p}_1\right)\left(\boldsymbol{\epsilon}_3 \cdot \boldsymbol{\epsilon}_1\right) - \bar{\chi}_{2,-}\not{p}_3\chi_{3,+}\left(\boldsymbol{\epsilon}^T_3 \cdot \boldsymbol{p}_1\right)\left(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_1\right) \\
&- \frac{1}{8}\left(\boldsymbol{\epsilon}^T_2 \cdot \boldsymbol{\epsilon}_3\right)\left(\bar{\chi}_{2,-}\not{p}_2\left[\not{\boldsymbol{p}}_1, \not{\epsilon}_1\right]\chi_{3,+}\right) + \frac{1}{8}\left(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}^T_3\right)\left(\bar{\chi}_{2,-}\left[\not{\boldsymbol{p}}_1, \not{\epsilon}_1\right]\not{p}_3\chi_{3,+}\right)
\end{aligned}
\tag{F.250}
$$

and the consistency check from graviton and gravitino will be

$$
\langle T^L_1 \bar{\psi}^L_{2,-}\psi_{3,+}\rangle = -\frac{1}{8}\left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}^T_3\right)\left(\bar{\chi}_{2,-}\left[\not{\boldsymbol{p}}_1, \not{\boldsymbol{\epsilon}}_1\right]\not{\boldsymbol{p}}_3\chi_{3,+}\right)\frac{\left(\epsilon^L_1\epsilon^L_2\right)}{E_1E_2} = \frac{1}{8}\left(\boldsymbol{p}_2 \cdot \boldsymbol{\epsilon}^T_3\right)\left(\bar{\chi}_{2,-}\left[\not{\boldsymbol{p}}_1, \not{\epsilon}_1\right]\not{p}_3\chi_{3,+}\right)\frac{\left(\epsilon^L_1\epsilon^L_2\right)}{E_1E_2}
\tag{F.251}
$$

$$
\langle T^L_1 \bar{\psi}_{2,-}\psi^L_{3,+}\rangle = \frac{1}{8}\left(\boldsymbol{p}_1 \cdot \boldsymbol{\epsilon}^T_2\right)\left(\bar{\chi}_{2,-}\not{\boldsymbol{p}}_2\left[\not{\boldsymbol{p}}_1, \not{\epsilon}_1\right]\chi_{3,+}\right)\frac{\left(\epsilon^L_1\epsilon^L_3\right)}{E_1E_3} = -\frac{1}{8}\left(\boldsymbol{\epsilon}^T_2 \cdot \boldsymbol{p}_3\right)\left(\bar{\chi}_{2,-}\not{p}_2\left[\not{\boldsymbol{p}}_1, \not{\epsilon}_1\right]\chi_{3,+}\right)\frac{\left(\epsilon^L_1\epsilon^L_3\right)}{E_1E_3}
\tag{F.252}
$$

F.3 4pt WT identity

- Scalar theory:

$$
p^i_1\langle J_{1,i}O^*_2 J_{3,j}O_4\rangle = -e\,\langle O^*_{1+2}\,J_{3,j}O_4\rangle + e\,\langle O^*_2\,J_{3,j}O_{1+4}\rangle
\tag{F.253}
$$

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and for the gravity (double-checked in momentum space and coordinate space)

$$\delta h_{0,ij} = 2\,\partial_{(i}\xi_{j),0} - 2\xi^m_{(i,0}\partial h_{j)m,0} + \xi^m\partial_m h_{ij} + O\left(h^2\right) \quad ; \quad \delta\phi_0 = \xi_{0,i}\partial^i\phi_0 - \xi_{i,0}h_0^{ij}\partial_j\phi_0 \tag{F.254}$$

Similar to (F.239), the invariance of wavefunction tells us the WT identity will be

$$\begin{aligned} p_{1,i}\epsilon_{1,j,0}\langle T_1^{ij}O_2T_3O_4\rangle = & -\frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1)\langle O_{2+1}T_3O_4\rangle - \frac{1}{2}(\boldsymbol{p}_4\cdot\boldsymbol{\epsilon}_1)\langle O_2T_3O_{4+1}\rangle \\ & + (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)p_{3,a}\langle T^a_{1+3}O_2O_4\rangle' \; - \frac{1}{2}\,(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\langle T_{1+3}O_2O_4\rangle' \\ & + \frac{1}{2}(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\langle O_{2+1+3}O_4\rangle + \frac{1}{2}(\boldsymbol{p}_4\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\langle O_2O_{4+1+3}\rangle \end{aligned}$$

Because we don't apply the constraint (2.28) of the boundary condition $h_{kl,\mathbf{0}}$ on the gauge transformation, so the correlator with free indices is produced by pulling out the unconstrained boundary condition $h'_{ij,\mathbf{0}}$. We call this unconstrined correlator and define it by

$$\begin{aligned} \langle T^a_{1+3}O_2O_4\rangle' &:= \epsilon_{3,b}\langle T^{ab}_{1+3}O_2O_4\rangle' \\ \langle T^{ab}_{1+3}O_2O_4\rangle' &:= P^{kl}_{2,ab}\langle T^{ab}_{1+3}O_2O_4\rangle. \end{aligned} \tag{F.255}$$

such that

$$h_{1+3,kl,\mathbf{0}}\langle T^{kl}_{1+3}O_2O_4\rangle' := h'_{1+3,kl,\mathbf{0}}P^{kl}_{2,k'l'}\langle T^{kl}_{1+3}O_2O_4\rangle. \tag{F.256}$$

Similarly, we should write the 3-point WT identity like (F.244), (F.248), and (F.249) by the two-point unconstrained correlator, but actually, in these case the unconstrained correlator is equal to the original correlator due to the two-point correlator is pure transverse.

The projector P^{ijkl} we define in the (**??**).

- Fermion theory:

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$$p_1^i\langle J_{1,i}\bar{\chi}_{2,-}J_{3,j}\chi_{4,+}\rangle = -e\left\langle\bar{\chi}_{1+2,-}J_{3,j}\chi_{4,+}\right\rangle + e\left\langle\bar{\chi}_{2,-}J_{3,j}\chi_{1+4,+}\right\rangle \tag{F.257}$$

and for gravity, the diffeomorphism is

$$\begin{aligned}
\delta h_{0,ij} &= 2\,\partial_{(i}\lambda_{j),0} - 2\lambda^m_{(i,0}\partial h_{j)m,0} + \lambda^m\partial_m h_{ij} + O\left(h^2\right)\\
\delta\chi_{0,+} &= \lambda^a e^i_a\nabla_i\chi_{0,+} - \frac{1}{8}e^i_a\nabla_i\lambda_b\left[\gamma^a,\gamma^b\right]\chi_{0,+}\\
&= \lambda^m\partial_m\chi_{0,+} - \frac{1}{8}\partial_a\lambda_b\left[\gamma^a,\gamma^b\right]\chi_{0,+}\\
&\quad - \frac{1}{2}\lambda_a h^{ab}\partial_b\chi_{0,+} + \frac{1}{16}h_{ca}\partial^c\lambda_b\left[\gamma^a,\gamma^b\right]\chi_{0,+} + O\left(h^2\right)\\
\delta\bar{\chi}_{0,-} &= \bar{\chi}_{0,-}\overleftarrow{\nabla}_i e^i_a\lambda^a + \frac{1}{8}\bar{\chi}_{0,-}\left[\gamma^a,\gamma^b\right]\nabla_i\lambda_b\, e^i_a\\
&= \bar{\chi}_{0,-}\overleftarrow{\partial}_m\lambda^m + \frac{1}{8}\bar{\chi}_{0,-}\left[\gamma^a,\gamma^b\right]\partial_a\lambda_b\\
&\quad - \frac{1}{2}\lambda_a h^{ab}\partial_b\bar{\chi}_{0,-} - \frac{1}{16}\bar{\chi}_{0,-}\left[\gamma^a,\gamma^b\right]h_{ca}\partial^c\lambda_b + O\left(h^2\right)
\end{aligned}$$

Similar to (F.239), the invariance of wavefunction tells us the WT identity will be

$$\begin{aligned}
&p_{1,i}\epsilon_{1,j,0}\langle T^{ij}_1\bar{\chi}_{2,-}T_3\chi_{4,+}\rangle\\
&= -\frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_2)\langle\bar{\chi}_{2+1,-}T_3\chi_{4,+}\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_4)\langle\bar{\chi}_{2,-}T_3\chi_{4+1,+}\rangle\\
&+ \frac{1}{16}\bar{\chi}_{2,-,A}([\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1])^{AB}\langle\bar{\chi}_{2+1,B,-}T_3\chi_{4,+}\rangle' - \frac{1}{16}\langle\bar{\chi}_{2,-}T_3\chi_{4+1,+,A}\rangle'([\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1])^{AB}\chi_{4,+,B}\\
&+ (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\boldsymbol{p}_{3,a}\langle T^a_{3+1}\bar{\chi}_{2,-}\chi_{4,+}\rangle - \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\langle T_{3+1}\bar{\chi}_{2,-}\chi_{4,+}\rangle\\
&+ \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{p}_2)\langle\bar{\chi}_{2+3+1,-}\chi_{4,+}\rangle + \frac{1}{2}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{p}_4)\langle\bar{\chi}_{2,-}\chi_{4+3+1,+}\rangle\\
&+ \frac{1}{32}(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3)\bar{\chi}_{2,-,A}([\not{\boldsymbol{\epsilon}}_3,\not{\boldsymbol{\epsilon}}_1])^{AB}\langle\bar{\chi}_{2+3+1,-,B}\chi_{4,+,C}\rangle\chi^C_{4,+}\\
&- \frac{1}{32}(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3)\bar{\chi}_{2,-,A}\langle\bar{\chi}^A_{2,-}\chi^B_{4+3+1,+}\rangle([\not{\boldsymbol{\epsilon}}_3,\not{\boldsymbol{\epsilon}}_1])_{BC}\chi^C_{4,+}
\end{aligned} \tag{F.258}$$

- N=1 Pure Supersymmetry

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The diffeomorphism will be parametrized by

$$\begin{aligned}\delta h_{0,ij} =& \ \partial_{(i}\lambda_{j)} - \lambda^m_{(i}\partial h_{j)m,0} + \frac{1}{2}\lambda^m\partial_m h_{ij} + O\left(h^2\right)\\ \delta\psi^i_{0,+} =& \ \lambda^m\partial_m\psi^i_{0,+} + (\partial_i\lambda_m)\psi^m_{0,+} - \frac{1}{8}\partial_a\lambda_b\left[\gamma^a,\gamma^b\right]\psi^i_{0,+}\\ &+ \lambda^a h_{0,ab}\partial^b\psi^i_{0,+} + (\partial_i\lambda^a)\,h_{0,ab}\psi^b_{0,+} - \frac{1}{16}h_{ca,0}\partial^c\lambda_b\left(\left[\gamma^a,\gamma^b\right]\psi^i_{0,+}\right) + O\left(h^2\right)\\ \delta\bar\psi^i_{0,-} =& \ \bar\psi^i_{0,-}\overleftarrow{\partial}_m\lambda^m + (\partial_i\lambda_m)\bar\psi^m_{0,-} + \frac{1}{8}\bar\psi^i_{0,-}\left[\gamma^a,\gamma^b\right]\partial_a\lambda_b + O(h)\\ &+ \lambda^a h_{0,ab}\partial^b\bar\psi^i_{0,-} + (\partial_i\lambda^a)\,h_{0,ab}\bar\psi^b_{0,-} + \frac{1}{16}h_{ca,0}\partial^c\lambda_b\left(\bar\psi^i_{0,-}\left[\gamma^a,\gamma^b\right]\right) + O\left(h^2\right)\end{aligned}$$

Similar to (F.239), the invariance of wavefunction tells us the WT identity will be

$$\begin{aligned}&p_{1,i}\epsilon_{1,j}\langle T^{ij}_1\bar\psi_{2,-}T_3\psi_{4,+}\rangle =\\ &-\langle\bar\psi_{2+1,-}T_3\psi_{4,+}\rangle'(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_1) - \bar\chi_{2,-,A}\left(p_{1,k}\langle\bar\psi^{k,A}_{2+1,-}T_3\psi_{4,+}\rangle'\right)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_1)\\ &+\frac{1}{8}\bar\chi_{2,-,A}\left[\not{\boldsymbol{p}}_1,\not{\epsilon}_1\right]^{AB}\langle\bar\psi_{2+1,-,B}T_3\psi_{4,+}\rangle'\\ &-\langle\bar\psi_{2,-}T_3\psi_{4+1,+}\rangle'(\boldsymbol{p}_4\cdot\boldsymbol{\epsilon}_1) - (\boldsymbol{\epsilon}_4\cdot\boldsymbol{\epsilon}_1)\left(p_{1,k}\langle\bar\psi_{2+1,-}T_3\psi^{k,B}_{4+1,+}\rangle'\right)\chi_{4,B,+}\\ &-\frac{1}{8}\langle\bar\psi_{2,-}T_3\psi_{4+1,+,A}\rangle'\left(\left[\not{\boldsymbol{p}}_1,\not{\epsilon}_1\right]\right)^{AB}\chi_{4,+,B}\\ &+\left(\langle\bar\psi_{2,-}T^{ij}_{3+1}\psi_{4,+}\rangle'\epsilon_{3,i}\,p_{3,j}\right)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3) - \frac{1}{2}(\boldsymbol{p}_3\cdot\boldsymbol{\epsilon}_1)\langle\bar\psi_{2,-}T_{3+1}\psi_{4,+}\rangle'\\ &-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{p}_2\cdot\boldsymbol{\epsilon}_3)\langle\bar\psi_{2+1+3,-}\psi_{4,+}\rangle - (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_2)\left(p_{1,k}\bar\chi_{2,-,A}\langle\bar\psi^{k,A}_{2+1+3,-}\psi_{4,+}\rangle\right)\\ &+\frac{1}{16}(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3)\bar\chi_{2,-,A}(\left[\not{\epsilon}_3,\not{\epsilon}_1\right])^{AB}\langle\bar\psi_{2+1+3,-,B}\psi_{4,+,C}\rangle\chi^C_{4,+}\\ &-(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{p}_4\cdot\boldsymbol{\epsilon}_3)\langle\bar\psi_{2,-}\psi_{4+1+3,+}\rangle - (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_4)\left(p_{1,k}\langle\bar\psi_{2,-}\psi^{k,B}_{4+1+3,+}\rangle\chi_{4,+,B}\right)\\ &-\frac{1}{16}(\boldsymbol{p}_1\cdot\boldsymbol{\epsilon}_3)\bar\chi_{2,-,A}\langle\bar\psi^A_{2,-}\psi^B_{4+1+3,+}\rangle(\left[\not{\epsilon}_3,\not{\epsilon}_1\right])_{BC}\chi^C_{4,+}\end{aligned} \tag{F.259}$$

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and the Supersymmetry transform is

$$
\begin{aligned}
\bar{\epsilon}_- &= \epsilon_+^T\, B \\
\delta_{\bar{\epsilon}_-} h_{ij} &= \bar{\epsilon}_- \gamma_{(i}\psi_{j,+)} + \frac{1}{2}\bar{\epsilon}_- \gamma^a h_{a(i}\psi_{j,+)} + O\left(h^2\right) = \delta_{\epsilon_+} h_{ij} = -\bar{\psi}_{-(j}\,\gamma_{i)}\,\epsilon_+ - \bar{\psi}_{-(j}\,h^a_{i)}\gamma_a\,\epsilon_+ + O\left(h^2\right) \\
\delta\psi_i^+ &= \partial_i\,\epsilon_+ \,+\, \frac{1}{8}\,w_{iab}\left[\gamma^a,\gamma^b\right]\epsilon_+ = \partial_i\,\epsilon_+ \,-\, \frac{1}{8}\,\partial_a h_{bi}\left[\gamma^a,\gamma^b\right]\epsilon_+ \\
&\quad + \frac{1}{16}h^{aj}\partial^b h_{ij}[\gamma_a,\gamma_b]\epsilon_+ + \frac{1}{32}h^{ja}\partial_j h_i^b[\gamma_a,\gamma_b]\epsilon_+ + O\left(h^3\right) \\
\delta\bar{\psi}_i^- &= \partial_i\,\bar{\epsilon}_- - \frac{1}{8}\bar{\epsilon}_-\,w_{iab}\left[\gamma^a,\gamma^b\right] = \partial_i\,\bar{\epsilon}_- + \frac{1}{8}\bar{\epsilon}_-\ \partial_a h_{bi}\left[\gamma^a,\gamma^b\right] \\
&\quad - \frac{1}{16}h^{aj}\partial^b h_{ij}\ \left(\bar{\epsilon_-}[\gamma_a,\gamma_b]\right) - \frac{1}{32}h^{ja}\partial_j h_i^b\left(\bar{\epsilon_-}[\gamma_a,\gamma_b]\right) + O\left(h^3\right)
\end{aligned}
$$

In above equations, because we don't apply the constraint (2.39) and (2.39) of the boundary condition $\psi_{+,i,\mathbf{0}}$ on the gauge transformation, so the correlator with free indices is produced by pulling out the unconstrained boundary condition $\psi'_{+,i,\mathbf{0}}$. We call this unconstrined correlator and define it by

$$
\begin{aligned}
\langle\bar{\psi}_{2,-}T_3\psi_{4+1,+,A}\rangle' &:= \langle\bar{\psi}_{2,-}T_3\psi_{4+1,+,A,i}\rangle\epsilon_4^i \\
\langle\bar{\psi}_{2,-}T_3\psi_{4+1,+,iA}\rangle &:= \langle\bar{\psi}_{2,-}T_3\psi^{B,a}_{4+1,+}\rangle P_{ia,AB}.
\end{aligned}
\tag{F.260}
$$

such that

$$
\langle\bar{\psi}_{2,-}T_3\psi_{4+1,+,iA}\rangle\psi_{\mathbf{0}}^{\prime,iA} = \langle\bar{\psi}_{2,-}T_3\psi^{B,a}_{4+1,+}\rangle P_{ia,AB}\psi_{\mathbf{0}}^{iA}. \tag{F.261}
$$

Similarly, we define

$$
\langle\bar{\psi}^{k,A}_{2+1,-}T_3\psi_{4,+}\rangle' := P_{ka,AB}\langle\bar{\psi}_{2+1,-,a,B}T_3\psi_{4,+}\rangle. \tag{F.262}
$$

The projector P^{ijAB}, $\bar{P}^{ijAB}$ we define in the (3.68).

Actually, we should write 3-point WT identity like (F.248), and (F.249) by the two-point unconstrained correlator, but actually, in these cases the unconstrained correlator is

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equal to the original correlator due to the two-point correlator is pure transverse.

Then similar to (F.239), the invariance of wavefunction tells us the WT identity will be

$$
\begin{aligned}
p_{2,i}\langle T_1\bar{\psi}^i_{2,-}T_3\psi_{4,+}\rangle = &-\bar{\chi}_{2,-}\left(\gamma_i\epsilon_{4,j}\langle T^{ij}_{2+4}T_1T_3\rangle'\right)\chi_{4,+} - \frac{1}{8}\bar{\chi}_{2,-,A}\left([\not{\boldsymbol{p}}_1,\not{\boldsymbol{\epsilon}}_1]\right)^{AB}\left(\epsilon_{1,i}\langle\bar{\psi}^i_{2+1,-,B}T_3\psi_{4,+}\rangle'\right)\\
&-\frac{1}{8}\bar{\chi}_{2,-,A}\left([\not{\boldsymbol{p}}_3,\not{\boldsymbol{\epsilon}}_3]\right)^{AB}\left(\epsilon_{3,i}\langle\bar{\psi}^i_{2+3,-,B}T_1\psi_{4,+}\rangle'\right)\\
&-\frac{1}{2}\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_1\chi_{4,+}\right)\epsilon_{1,j,0}\epsilon_{4,i,0}\langle T^{ij}_{2+4+1}T_3\rangle' - \frac{1}{2}\left(\bar{\chi}_{2,-}\not{\boldsymbol{\epsilon}}_3\chi_{4,+}\right)\epsilon_{3,j,0}\epsilon_{4,i,0}\langle T^{ij}_{2+4+3}T_1\rangle'\\
&+\frac{1}{16}\left(\bar{\chi}_{2,-}[\not{\boldsymbol{\epsilon}}_1,\not{\boldsymbol{p}}_3]\left(\epsilon_{3,i,0}\langle\bar{\psi}^i_{2+1+3,-}\psi_{4,+}\rangle\right)\chi_{4,+}\right)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\\
&+\frac{1}{32}\left(\bar{\chi}_{2,-}[\not{\boldsymbol{\epsilon}}_1,\not{\boldsymbol{\epsilon}}_3]\left(\epsilon_{3,i,0}\langle\bar{\psi}^i_{2+1+3,-}\psi_{4,+}\rangle\right)\chi_{4,+}\right)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)\\
&+\frac{1}{16}\left(\bar{\chi}_{2,-}[\not{\boldsymbol{\epsilon}}_3,\not{\boldsymbol{p}}_1]\left(\epsilon_{1,i,0}\langle\bar{\psi}^i_{2+1+3,-}\psi_{4,+}\rangle\right)\chi_{4,+}\right)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\\
&+\frac{1}{32}\left(\bar{\chi}_{2,-}[\not{\boldsymbol{\epsilon}}_3,\not{\boldsymbol{\epsilon}}_1]\left(\epsilon_{1,i,0}\langle\bar{\psi}^i_{2+1+3,-}\psi_{4,+}\rangle\right)\chi_{4,+}\right)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{p}_3)
\end{aligned}
\tag{F.263}
$$

$$
p_{1,i}\langle\bar{\psi}^i_1\psi_2\bar{\psi}_3\psi_4\rangle = -\bar{\chi}_{1,-}\left(\gamma_i\epsilon_{4,j}\langle\psi_2\bar{\psi}_3T^{ij}_{4+1}\rangle'\right) - \bar{\chi}_{1,-}\left(\gamma_i\epsilon_{2,j}\langle T^{ij}_{2+1}\bar{\psi}_3\psi_4\rangle'\right) - \bar{\chi}_{1,-}\left(\gamma_i\epsilon_{3,j}\langle T^{ij}_{3+1}\bar{\psi}_2\psi_4\rangle'\right)
\tag{F.264}
$$

in which, for the Majorana spinor, there's no actual difference of the $\bar{\psi}$ and ψ, or $\langle\bar{\psi}^i_1\psi_2\bar{\psi}_3\psi_4\rangle = -\langle\bar{\psi}_2\psi^i_1\bar{\psi}_3\psi_4\rangle$.

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Appendix G — Total Energy Pole

We can compare the definition of flat space amplitude M and correlator ψ

$$S = 1 + iM = \langle 0, f|0, i\rangle = \int D\phi(t)\ exp(i \int_{-\infty}^{\infty} d^4x\ \mathcal{L}(\phi(t))$$

$$\Psi = \langle 0, \Omega|0, i\rangle = \int D\phi(t)\ exp(i \int_{-\infty}^{0} d^4x\ \mathcal{L}(\phi(t))\ \sim \int d\phi_0\ exp(i \int_{-\infty}^{0} d^4x\ \mathcal{L}(\phi_{cl}(t, \phi_0))$$

$$= \int d\phi_0\ exp\left(\prod_i^n \phi_{i,0}\ \cdot \psi_n\right)$$

We notice the only difference besides the integral region is conventionally we absorb i in correlator but don't do that for amplitude so the total energy pole residue reads, the imaginary comes from $\delta(\Sigma E) \to \frac{1}{i\,K_T}$ cancels the imaginary rescaling between correlator and amplitude:

$$\lim_{K_T \to 0} \psi_n \ = \ \frac{M_n}{K_T} \tag{G.265}$$

If we encounter the spinning field, we now define ψ as the correlator contracted with boundary condition for convenience and ψ_i for the correlator whose boundary condition

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is extracted

$$\Psi = \int d\epsilon_0 \; exp\left(\prod_i^n \epsilon_{i,0,j_i} \; \cdot \psi_n^{j_1 \dots j_n}\right) = exp\left(\int d\epsilon_0 \; \psi_n\right)$$

we should take amplitude's polarization ϵ vector as the boundary condition ϵ_0 for correlator

$$\lim_{K_T \to 0} \psi_n\left(\epsilon_0^i = \epsilon^i\right) \; = \; \frac{M_n\left(\epsilon^i\right)}{K_T} \qquad \text{(G.266)}$$

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