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碩士論文

Department of Finance

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National Taiwan University

Master Thesis

一般化 Heath-Jarrow-Morton 利率模型
對利率衍生性金融商品定價

Using General Heath-Jarrow-Morton Model to
Price Interest Rate Derivatives

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摘要

本文提供了一個靈活的多因子隨機波動度 Heath–Jarrow–Morton 模型，此模型讓遠期利率與其波動度具有相關性，且有 N 個隨機因子會影響利率結構，另有額外 N 個隨機因子會只會影響波動度(及利率衍生性商品)。此模型改進了 Trolle and Schwartz (2009)的模型，讓即期利率(instantaneous spot rate) $f(t,t)$ 也會影響利率波動度。此模型能夠轉換成有限狀態變數(finite number of state variables)的馬可夫表現(Markov representation)系統，故能輕易地使用蒙地卡羅模擬法來評價各種利率衍生性產品。本文也應證了此模型符合馬可夫性質。在此應用了有限狀態變數(finite number of state variables)導出風險中立下的瞬間遠期利率 $f(t,T)$、零息債券價格。此動態過程符合 Duffie, Pan and Singleton (2000)(簡稱 DPS)提出的 Affine Jump-Diffusions 的條件，能獲得債券選擇權評價公式的解析解。

關鍵詞：Heath–Jarrow–Morton 模型、隨機波動度、狀態依賴波動度、有限狀態變數、馬可夫性質、債券選擇權評價

Abstract

We provide a flexible stochastic volatility multifactor model of the term structure of interest rates in the Heath-Jarrow-Morton (HJM) framework. This model features unspanned stochastic volatility factors and correlation between forward rates and their volatilities. Moreover, let instantaneous spot rate affect the interest rate volatility. This model can be converted into finite number of state variables and reduced to a Markovian system. Moreover, This dynamic process consistent with Affine Jump-Diffusions Process, which is proposed in Duffie, Pan and Singleton (2000), and consequently we can obtain bond option prices with analytic form.

Keywords: Heath–Jarrow–Morton model, stochastic volatility, state dependent volatility, Markovian, bond option prices

目錄

一、 簡介

1. 文獻回顧

在 Heath, Jarrow and Morton(1992)之前所發展的利率模型，主要為有限維度(finite dimensional)且具有馬可夫性質的模型，其利率大多由即期利率(instantaneous spot rate)或再多加一到兩個狀態變數(state variables)所決定。有馬可夫性質的好處在於，對衍生性金融商品定價的時候，我們可以利用建樹的方式，較有效率快速的來計算衍生性金融商品價格。然而當我們拿市場資料去預估模型參數，並利用模型模擬卻不能很準確的描述市場殖利率曲線(initial yield curve)，市場觀察參數以及模型參數也無法清楚地解釋其關係。此外這些模型也無法描述利率的性質，例如：像利率波動度結構具有駝峰的性質(hump volatility curve)。

然而，在 Heath, Jarrow and Morton(1992) (簡稱 HJM)的利率模型推出後，改善了早期利率模型的這些缺點，對利率模型研究有非常大的貢獻。第一，它是一個遠期利率曲線模型(forward rate curve model)，利率曲線會受到無套利條件(no arbitrage condition)所限制；第二，它能夠很容易地擴充到多因子模型，來捕捉不同的市場特性；第三，它能夠涵蓋大部份的傳統模型，如 Vasicek (1977)、Cox-Ingersoll-Ross (1985)、Ho and Lee (1986) 和 Hull and White (1990)等都是 HJM 的子集合。

但是 HJM 模型的最大缺點在於他沒有符合馬可夫性質。所以通常 HJM 模型都是使用蒙地卡羅模擬法來計算出債券衍生性金融商品的價格，但這會耗費很多的電腦運算時間。有非常多文獻討論如何 HJM 模型轉換成符合馬可夫性質的方法，包括 Carverhill(1994)，Ritchken and Sankarasubramanian(1995)，Bhar and Chiarella(1997)，Chiarella and kwon(1998a)，Chiarella and kwon(1998b)，和 Bhar，Chiarella，El-Hassan and Zheng(1999)。有了這些轉換方式，我們保留了早期利率

模型有的馬可夫性質以及 HJM 模型的優點,更有利於利率衍生性金融商品的研究。

HJM 模型在無套利條件(no arbitrage condition)下,遠期利率的漂浮項(drift term)是被波動度(volatility term)唯一決定的。所以波動度的設定,成為了 HJM 模型最重要的研究對象。近年不少研究均針對 HJM 的波動度作探討,希望能捕捉市場實況的資訊,並盡可能得到評價衍生性商品的封閉解。例如 Bhar and Chiarella (1997)考慮了波動度與短期利率成正比關係,Mercurio and Moraleda (2000)和(2001)考慮了波動度的駝峰現象,Collin-Dufresne and Goldstein (2003)和 Trolle and Schwartz (2009)考慮隨機波動度,利用 Duffie, Pan and Singleton (2000)的 Affine Jump-Diffusions 得出評價利率商品的封閉解。

最近文獻提出關於利率波動度有下列特性:第一,利率波動度是隨機的。第二,利率波動度內含非利率期限結構因子 (unspanned stochastic volatility factor)[1],利率衍生性商品受某些隨機因子影響,但這些因子並不影響利率期間結構,所以稱它為非利率期限結構因子(unspanned stochastic volatility factors)。在 Andersen and benzoni(2005)的文獻中也確實驗證了在真實的利率波動度內,確實含有非利率期限結構因子(unspanned stochastic volatility factors)在其中。第三,利率波動度的變化和利率的變化具有相關的現象。Andersen and Lund(1997)和 Ball and Torous(1999),以上兩篇作者皆研究短期利率動態過程,其中他們發現,相對利率波動度和利率是呈現負相關的關係;而絕對利率波動和利率成正相關的關係。第四,非條件下(unconditional=realized and implied)波動度結構具有駝峰(hump)的性質。此特性在 Dai and Singleton(2003)的文獻中有所討論。

Trolle and Schwartz (2009)提出一個多因子 HJM 模型,此模型特別在於有考慮到利率波動度的四個特性。模型內含 N 個會影響利率期限結構的因子以及 N 個非利率期限結構因子(unspanned stochastic volatility factors),其特性為只會影響利率衍生性金融商品而不會影響利率期限結構的因子。此外,模型也有描述出利率波

[1] 詳見 Collin-Dufresne and Goldstein (2002)、Heidari and Wu (2003)、Casassus, Collin-Dufresne, and Goldstein (2005)和 Li and Zhao (2006)。

動度的變化和利率的變化具有相關性以及波動度結構具有駝峰(hump)的性質(隨機因子對不同天期的遠期利率影響程度不同，短天期及長天期受影響較小，而中期受影響較大)。

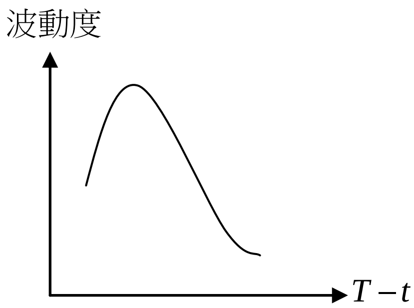

圖 1　駝峰形波動度

2. 研究動機與目的

Trolle and Schwartz (2009)提出一個靈活的多因子 HJM 模型，且非條件波動度有駝峰現象，並受非利率期限結構因子(unspanned stochastic volatility factors)影響：

(1.1) $$df(t,T)=\mu_f(t,T)dt+\sum_{i=1}^{N}\sigma_{f,i}(t,T)\sqrt{\upsilon_i(t)}dW_i^Q(t)$$

(1.2) $$d\upsilon_i(t)=\kappa_i(\theta_i-\upsilon_i(t))dt+\sigma_i\sqrt{\upsilon_i(t)}\left(\rho_i dW_i^Q(t)+\sqrt{1-\rho_i^2}dZ_i^Q(t)\right)$$

$i=1,\dots,N$，其中 $\sigma_{f,i}(t,T)=(\alpha_{0,i}+\alpha_{1,i}(T-t))e^{-\gamma_i(T-t)},\frac{\alpha_{1,i}}{\alpha_{0,i}}>\gamma_i$，$dW_i^Q(t)$ 和 $dZ_i^Q(t)$ 是在風險中立測度(risk neutral measure) Q 下的獨立標準布朗運動。此模型是具隨機波動度的 HJM 模型，我們稱為 $\upsilon_i(t)$ 隨機波動因子[2]。其優點如下：第一，波動度是隨機的；第二，波動度具有非利率期限結構因子(unspanned stochastic volatility factors) $dZ_i^Q(t)$；此模型有 N 個因子(利率結構因子 $dW_i^Q(t)$)會影響利率結構，另有額外 N 個因子(非利率期限結構因子(unspanned stochastic volatility factors) $dZ_i^Q(t)$)

[2] $\upsilon_i(t)$ 是 Cox-Ingersoll-Ross 過程，具有平均反轉(mean reverting)特性，詳見 Heston (1993)的股價隨機波動度模型。

會影響波動度(故亦影響利率衍生性商品)；第三，可以描述出 $f(t,T)$ 與 $\upsilon_i(t)$ 具有相關性；第四，非條件波動度有駝峰現象；第五，而且此動態過程符合 Duffie, Pan and Singleton (2000) (簡稱 DPS)提出的 Affine Jump-Diffusions (簡稱 AJD)的條件，能獲得債券選擇權評價公式的解析解。

然而，Chan et al. (1992)的實證結果發現，即期利率 $f(t,t)$ 波動度是 $f(t,t)$ 的遞增函數。但是 Trolle and Schwartz (2009)為了要符合 AJD，所以放棄了波動度與即期利率(spot rate) $f(t,t)$ 的遞增關係。Trolle and Schwartz (2009)這篇文章裡面對於模型的設定並沒有考慮到即期利率(spot rate) $f(t,t)$ 也會影響利率波動度。因此，本文針對此問題作改良，本文將模仿 Trolle and Schwartz (2009)的設定再加入即期利率(spot rate)的狀態變數，令波動度 $\sigma = \sigma\left(t, T, \upsilon(t), f(t,t)\right)$，讓波動度同時受 unspanned 因子和 $f(t,t)$ 的直接影響，我們稱它為狀態依賴波動度 (state dependent volatility)，藉此探討更一般化 HJM 利率模型之特性，以及評價結果。

3. 研究架構

本文共計三章，其架構說明如下：

第一章為簡介，首先我們會回顧利率模型以及利率性質探討方面的文獻，以及說明利率衍生性金融商品隨著利率模型發展上所面臨的問題以及考量；進而闡述本文研究想要用更一般化 HJM 利率模型，去對利率衍生性商品作訂價的研究動機與研究架構。

第二章主要是簡單介紹 HEATH-JARROW-MORTON 模型基本架構，將會介紹風險中立下 HJM 模型的建構過程以及討論 HJM 利率模型的優點。

第三章為隨機波動模型之一般化 HJM 模型的基本假設與模型設定。在此主要是改良 Trolle and Schwartz (2009)提出的多因子 HJM 模型。第一節先給定風險中立下的瞬間遠期利率 $f(t,T)$ 的隨機過程。第二節推出風險中立下的瞬間遠期利率 $f(t,T)$。第三節驗證我們的模型符合馬可夫性質。第四節推導出零息債券價格。

第五節再利用 Bhar and Chiarella (1997)的轉換，讓此動態過程符合 Duffie, Pan and Singleton (2000)(簡稱 DPS)提出的 Affine Jump-Diffusions 的條件，能獲得債券選擇權評價公式的解析解。

第四章為結論。

二、 HEATH-JARROW-MORTON 模型基本架構

HJM 利率模型的創新之一，就是以遠期利率隨機過程來建構無套利機會的利率模型。以下將介紹風險中立下 HJM 模型的建構過程以及討論 HJM 利率模型的優點。

1. 連續時間下風險中立之 HJM 利率模型—由遠期利率隨機過程建構起

下列式子為 HJM 所假設，在時間點 t 對時間點 T，風險中立下瞬間遠期利率 $f(t,T)$ 的隨機過程：

(2.1) $$f(t,T)=f(0,T)+\sum_{i=1}^{N}\int_0^t \mu_{f,i}(s,T)ds+\sum_{i=1}^{N}\int_0^t \sigma_{f,i}(s,T)dW_i^Q(s)$$

其中 $0\le t\le T\le \tau,\ \mu_{f,i}(s,T)=\sigma_{f,i}(s,T)\int_s^T \sigma_{f,i}(u,T)du$。上式代表期初給定的遠期利率曲線，非隨機的固定值。等式(2.1)的最後一項即是瞬間遠期利率的隨機來源(random sources)，從時間點 0 之後，此 N 項獨立的布朗運動(brownian motion)即決定整條遠期利率曲線的隨機震動；遠期利率對 N 個布朗運動變動的敏感度，反應於不同的波動(volatiltiy)函數之上。

當瞬間遠期利率的過程被指定之後，即期利率(instantaneous spot rate)與債券價格的動態過程，隨機應孕而生。

(2.2) $$r(t)=f(t,t)=f(0,t)+\sum_{i=1}^{N}\int_0^t \mu_{f,i}(s,t)ds+\sum_{i=1}^{N}\int_0^t \sigma_{f,i}(s,t)dW_i^Q(s)$$

由式(2.2)可發現，原本出現的瞬間即期利率隨機過程被一群不同時間點下的遠期利率波動，亦即遠期利率波動結構(term structure of volatility)所取代。這代表著在 HJM

利率模型架構下，能擺脫與風險之市場價格相依的關係。自此之後，HJM 利率模型對衍生性金融商品的評價，在概念上，給定一期出的遠期利率曲線，只需針對不同波動結構設定，便能完成評價。

下列式子為等式(2.1)和等式(2.2)的動態過程如下:

$$(2.3) \qquad df(t,T)=\sum_{i=1}^{N}\mu_{f,i}(t,T)dt+\sum_{i=1}^{N}\sigma_{f,i}(t,T)dW_i^Q(t)$$

$$(2.4) \qquad dr(t)=df(t,t)=\begin{bmatrix}\dfrac{\partial f(0,t)}{\partial t}+\sum_{i=1}^{N}\int_0^t\sigma_{f,i}^2(s,t)ds\\ +\sum_{i=1}^{N}\int_0^t\dfrac{\partial\sigma_{f,i}(s,t)}{\partial t}\int_s^t\sigma_{f,i}(s,u)duds\\ +\sum_{i=1}^{N}\int_0^t\dfrac{\partial\sigma_{f,i}(s,t)}{\partial t}dW_i^Q(s)\end{bmatrix}dt+\sum_{i=1}^{N}\sigma_{f,i}(t,t)dW_i^Q(t)$$

接著應用伊藤定理(Ito’s Lemma)，可推出債券價格的動態過程：

$$(2.5) \qquad \frac{dP(t,T)}{P(t,T)}r=r(t)dt+\sum_{i=1}^{n}\sigma_i^P(t,T)dW_i^Q(t)$$

其中 $\sigma_i^P(t,T)=-\int_t^T\sigma_{f,i}(t,u)du$

2. HJM 利率模型之創新

HJM 在利率模型上的創新有三：(1)直接由遠期利率曲線著手，描述利率期間結構變動；(2)瞬間即期利率的隨機過程，是由多個影響利率期限結構的隨機因子所組成；(3)此模型不需透過期間結構倒推法(inversion of the term structure)來移除風險之市場價格。

總而言之，給定一條期初的遠期利率曲線，HJM(1992)按照 Harrison & Kreps(1979)之論述，對遠期利率隨機過程之漂移項予以設定，以保證一風險中立之機率測度存在，再對利率衍生商品作定價。

三、 隨機波動模型之一般化 HJM 模型的基本假設與模型設定

1. 隱含隨機波動模型下的 HJM 架構

結合隱含隨機波動模型的 HJM 模型架構下，在時間點 t 對時間點 T，風險中立下的瞬間遠期利率 $f(t,T)$ 的隨機過程如下：

(3.1) $$df(t,T) = \mu_f(t,T)dt + \sum_{i=1}^{N}\sigma_{f,i}(t,T)\sqrt{af(t,t)+b\upsilon_i(t)}dW_i^Q(t)$$

隱含隨機波動的隨機過程如下：

(3.2) $$d\upsilon_i(t) = \kappa_i(\theta_i - \upsilon_i(t))dt + \sigma_i\sqrt{af(t,t)+b\upsilon_i(t)}(\rho dW_i^Q(t) + \sqrt{1-\rho^2}dZ_i^Q(t))$$

$i=1,....,N$，$W_i^Q(t)$ 和 $Z_i^Q(t)$ 為在風險中立測度(risk neutral measure) Q 下的獨立標準布朗運動。此模型是具隨機波動度的HJM模型，我們稱為 $\upsilon_i(t)$ 隨機波動因子[3]。我們的模型設定優點如下：第一，波動度是隨機的；第二，波動度具有非利率期限結構因子(unspanned stochastic volatility factors) $dZ_i^Q(t)$；此模型有 N 個因子(利率結構因子 $dW_i^Q(t)$)會影響利率結構，另有額外 N 個因子(非利率期限結構因子(unspanned stochastic volatility factors) $dZ_i^Q(t)$)會影響波動度(故亦影響利率衍生性商品)；第三，可以描述出 $f(t,T)$ 與 $\upsilon_i(t)$ 具有相關性；第四，非條件波動度有駝峰現象；第五，考慮到即期利率(instantaneous spot rate) $f(t,t)$ 也會影響利率波動度。第六；而且此動態過程符合Duffie, Pan and Singleton (2000) (簡稱DPS)提出的Affine Jump-Diffusions (簡稱AJD)的條件，能獲得債券選擇權評價公式的解析解。

由式(3.1)&(3.2)可知，影響利率期限結構的因子有 N 個，每一個因子都有它特有的波動結構，因此有 N 個波動結構，而且每個波動結構受到到期期限函數

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[3] $\upsilon_i(t)$ 是 Cox-Ingersoll-Ross 過程，具有平均反轉(mean reverting)特性，詳見 Heston (1993)的股價隨機波動度模型。

$\sigma_{f,i}(t,T)$、狀態函數即期利率 $f(t,t)$ 和對應的隨機波動率 $\upsilon_i(t)$、以及 $f(t,t)$&$\upsilon_i(t)$ 兩者的線性組合的影響。再者，每一個波動率 $\upsilon_i(t)$ 都是隨機過程，總共有 N 個隨機過程方程式來表達 N 個波動率，所以式(3.1)&(3.2)是一個 $N+1$ 的方程式系統。另外，因為總共有 N 個隨機波動率 $\upsilon_i(t)$，而且每一個波動率 $\upsilon_i(t)$ 受到一個對應自己的隨機因子影響，所以隨機因子個數有 $N\times 2$ 個；特別的是，每一個隨機波動度的隨機項可以被分解成獨立的兩個驅動力量(driving forces)：一個是影響利率期限結構的驅動力量(driving force)、另一個則是與利率期限結構的驅動力量(driving force)獨立的驅動力量(driving force)，我們稱後者這種驅動力量(driving force)為非利率期限結構因子(unspanned stochastic volatility factors)，本文即是透過這種分解方式來建構遠期利率與其波動率的相關性。

在我們的模型設定下，我們結合了傳統的HJM模型和隨機波動模型。遠期利率曲線有 N 個驅動因子(driving forces)，而除了 $\rho_i=-1$ 和 $\rho_i=1$ 的情況下，遠期利率波動率有 $N+N$ 個驅動因子(driving forces)。特別的是，我們的遠期利率模型比 Trolle & Schwartz (2008)的模型多考慮了一個因子:即期利率 $f(t,t)$，這使得我們的模型更一般化。在 $a=1,b=0$ 時，我們的模型可視為 Heath, Jarrow, and Morton的模型。在 $a=0,b=1$ 時，我們的模型可視為Trolle & Schwartz (2008)的模型.在 $a=0,b=1,N=1$ 時，我們的模型可視為在股票衍生性金融商品常用到的 Heston(1993)模型。

參數設定	模型
$a=1,b=0$	Heath, Jarrow, and Morton的模型
$a=0,b=1$	Trolle & Schwartz (2008)的模型
$a=0,b=1,N=1$	股票衍生性金融商品常用的Heston(1993)模型

在Heath, Jarrow, and Morton(1992)文章中提到，若要我們的模型符合無套利的情況，我們模型等式(3.1)的漂移項必須滿足下列條件:

(3.3) $$\mu_f(t,T)=\sum_{i=1}^{N}\left[af(t,t)+b\upsilon_i(t)\right]\sigma_{f,i}(t,T)\int_t^T\sigma_{f,i}(t,u)du$$

簡而言之，在我們的模型下，在時間點t對時間點T，風險中立下的瞬間遠期利率$f(t,T)$是由即期利率$f(t,t)$、遠期利率波動函式(the forward rate volatility) functions $\sum_{i=1}^{N}\sigma_{f,i}(t,T)\sqrt{af(t,t)+b\upsilon_i(t)}$和波動率狀態變數(the volatility state variable)$\upsilon_i(t)$。

一般來說，$\sigma_{f,i}(t,T)$的設定會影響到遠期利率曲線是否有路徑相依。假使遠期利率曲線有路徑相依，會使我們在對衍生性金融商品定價時複雜許多。有很多利率期限結構的文獻都有在探討在什麼樣的情況下才能使HJM模型路徑不相依，也就是對HJM的有限維馬可夫過程轉換作探討。例如: Carverhill (1994)和Jeffrey (1995)提供了充份必要條件；Ritchken and Sankarasubramaniam(1995)；Bhar and Chiarella(1997)；Inui and Kijima(1998)；de Jong and Santa-Clara(1999)；Ritchken and Chuang(1999)； Bhar, Chiarella, El-Hassan and Zheng (2000)；Björk and Svensson (2001)；Chiarella and Kwon(2003)；和Björk, Landén and Svensson (2004)。Bhar and Chiarella (1997)指出，若波動度期間結構呈$\sigma_{f,i}(t,T)=p_n(T-t)e^{-\gamma_i(T-t)}$，其中$p_n(\tau)$是一個$n$次多項式，則HJM模型能轉換成有限狀態變數(finite number of state variables)的系統，且波動度結構有時間齊一性(time-homogeneous)的性質。這樣的轉換可以使遠期利率曲線符合馬可夫性質。所以在我們的模型設定中，讓遠期利率波動率限制如下：

$$\sigma_{f,i}(t,T)=(\alpha_{0,i}+\alpha_{1,i}(T-t))e^{-\gamma_i(T-t)},\frac{\alpha_{1,i}}{\alpha_{0,i}}>\gamma_i \quad (3.4)$$

在這樣設定下，可以描繪遠期利率波動對到期時間有駝峰形波動結構(hump-shaped shocks)，換句話說，等式(3.1)中的隨機項，因為等式(3.4)的設定，而使$dW_i^Q(t)$對$df(t,T)$有不同的影響，此影響即是：短期$dW_i^Q(t)$對$df(t,T)$影響程度小；中期$dW_i^Q(t)$對$df(t,T)$影響程度變大；長期$dW_i^Q(t)$對$df(t,T)$影響程度又變小。此外，當$a=0,b=1,N=1$和$\alpha_{1,1}=0$時，我們的模型可視為隨機波動模型下 Hull and

White(1990)的模型。當 $a=0, b=1, N=1, \alpha_{1,1}=0$ 和 $\gamma_1=0$ 時，我們的模型可視為隨機波動模型下 Ho and Lee(1986)的模型。

2. 風險中立下的瞬間遠期利率 $f(t,T)$

(3.5) $$f(t,T)=f(0,T)+\sum_{i=1}^{N}\mathrm{B}_{x_i}(T-t)x_i(t)+\sum_{i=1}^{N}\sum_{j=1}^{6}\mathrm{B}_{\phi_{j,i}}(T-t)\phi_{j,i}(t)$$

其中

(3.6) $$\mathrm{B}_{x_i}(\tau)=\left(\alpha_{0,i}+\alpha_{1,i}\tau e^{-\gamma_i\tau)}\right)e^{-\gamma_i\tau}$$

(3.7) $$\mathrm{B}_{\phi_{1,i}}(\tau)=\alpha_{1,i}e^{-\gamma_i\tau}$$

(3.8) $$\mathrm{B}_{\phi_{2,i}(t)}(\tau)=\frac{\alpha_{1,i}}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(\alpha_{0,i}+\alpha_{1,i}\tau\right)e^{-\gamma_i\tau}$$

(3.9) $$\mathrm{B}_{\phi_{3,i}(t)}(\tau)=-\left(\frac{\alpha_{0,i}\alpha_{1,i}}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)+\frac{\alpha_{1,i}}{\gamma_i}\left(\frac{\alpha_{1,i}}{\gamma_i}+2\alpha_{0,i}\right)\tau+\frac{\alpha_{1,i}^2}{\gamma_i}\tau^2\right)e^{-2\gamma_i\tau}$$

(3.10) $$\mathrm{B}_{\phi_{4,i}(t)}(\tau)=\frac{\alpha_{1,i}^2}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)e^{-\gamma_i\tau}$$

(3.11) $$\mathrm{B}_{\phi_{5,i}(t)}(\tau)=-\frac{\alpha_{1,i}}{\gamma_i}\left(2\alpha_{0,i}+\frac{\alpha_{1,i}}{\gamma_i}+2\alpha_{1,i}\tau\right)e^{-2\gamma_i\tau}$$

(3.12) $$\mathrm{B}_{\phi_{6,i}(t)}(\tau)=\frac{-\alpha_{1,i}^2}{\gamma_i}e^{-2\gamma_i\tau}$$

等是(3.5)的狀態變數如下：

(3.13) $$dx_i(t)=-\gamma_i x_i(t)dt+\sqrt{af(t,t)+b\upsilon_i(t)}e^{-\gamma_i(t-s)}dW_i^Q(t)$$

(3.14) $$d\phi_{1,i}(t)=\left(x_i(t)-\gamma_i\phi_{1,i}(t)\right)dt$$

(3.15) $$d\phi_{2,i}(t)=\left(\left[af(t,t)+b\upsilon_i(t)\right]-\gamma_i\phi_{2,i}(t)\right)dt$$

(3.16) $$d\phi_{3,i}(t)=\left(\left[af(t,t)+b\upsilon_i(t)\right]-2\gamma_i\phi_{3,i}(t)\right)dt$$

(3.17) $$d\phi_{4,i}(t)=\left(\phi_{2,i}(t)-\gamma_i\phi_{4,i}(t)\right)dt$$

(3.18) $$d\phi_{5,i}(t)=\left(\phi_{3,i}(t)-2\gamma_i\phi_{5,i}(t)\right)dt$$

(3.19) $$d\phi_{6,i}(t)=\left(2\phi_{5,i}(t)-2\gamma_i\phi_{6,i}(t)\right)($$

限制式： $x_i(0)=\phi_{1,i}(0)=\phi_{2,i}(0)=\phi_{3,i}(0)=\phi_{4,i}(0)=\phi_{5,i}(0)=\phi_{6,i}(0)=0$

證明：請參考附錄一。

從式(3.13)至式(3.19)可見，增加$7\times N$個狀態變數，使模型能表示成馬可夫表現(Markov representation)。其中$\phi_{1,i}(t),\ldots,\phi_{6,i}(t)$是局部確定的(locally deterministic)，它們負責收集$x_i(t)$和$\upsilon_i(t)$的歷史資訊，所以我們稱為“輔助的”狀態變數，此讓我們的模型有馬可夫性質。

本文的瞬間遠期利率隨機過程沒有直接與即期利率狀態變數$f(t,t)$和波動率狀態變數$\upsilon_i(t)$相關，而且也是由影響利率期間結構的 N 個因子、以及 9 個狀態變數$\phi_{1,i}(t), \ldots, \phi_{6,i}(t), f(t,t), \upsilon_i(t)$ & $x_i(t)$ 共同遵循一個 affine 的隨機過程而決定之。本文模型的$\phi_{1,i}(t), \ldots, \phi_{6,i}(t)$是“輔助的”狀態變數，而且是區域性的(locally)非隨機狀態變數，主要是用來反映另外三個隨機狀態變數，即
$f(t,t), \upsilon_i(t)$ & $x_i(t)$，從零時點到 t 時點的路徑。另外值得注意的是，本文的$\phi_{1,i}(t), \ldots, \phi_{6,i}(t)$比 Trolle & Schwartz 者多受到了即期利率狀態變數$f(t,t)$的影響。

3. 馬可夫性質

為了要驗證我們的模型符合馬可夫性質，我們參照了 Bhar and Chiarella(1997) 引進輔助的隨機變數。在此我們只要驗證即期利率$f(t,t)=r(t)$的隨機過程只有時間點t的變數有關，而與時間點t之前的變數無關。由等式(3.5)可以推導出即期利率$f(t,t)$的隨機過程如下：

(3.20)
$$df\left(t,t\right)=\left[\frac{\partial}{\partial t}f\left(0,t\right)+\sum_{i=1}^{N}\left(\gamma_i f\left(0,t\right)-\gamma_i f\left(t,t\right)+\alpha_{1,i}x_i\left(t\right)+\sum_{j=2}^{6}D_{j,i}\phi_{j,i}\right)\right]dt$$
$$+\sum_{i=1}^{N}\alpha_{0,i}\sqrt{af(t,t)+b\upsilon_i(t)}dW_i^Q\left(t\right)$$

(3.21) $$D_{2,i}=\frac{\alpha_{1,i}^{2}+\alpha_{0,i}\alpha_{1,i}\gamma_i}{\gamma_i^2}$$

(3.22) $$D_{3,i}=\frac{\alpha_{0,i}^{2}\gamma_i^{2}-\alpha_{1,i}^{2}-\alpha_{0,i}\alpha_{1,i}\gamma_i}{\gamma_i^2}$$

(3.23) $$D_{4,i}=\frac{\alpha_{0,i}\alpha_{1,i}\gamma_i^2+\alpha_{1,i}^{2}\gamma_i-\alpha_{1,i}^{2}-\alpha_{1,i}\alpha_{0,i}\gamma_i}{\gamma_i^2}$$

(3.24) $$D_{5,i}=2\alpha_{0,i}\alpha_{1,i}-\frac{\alpha_{1,i}^{2}}{\gamma_i}$$

(3.25) $$D_{6,i}=\alpha_{1,i}^{2}$$

其中狀態變數為$S\left(t\right)\equiv\left[f\left(t,t\right),x_i\left(t\right),\phi_{2,i}\left(t\right),\phi_{3,i}\left(t\right),\phi_{4,i}\left(t\right),\phi_{5,i}\left(t\right),\phi_{6,i}\left(t\right)\right]$

證明：請參考附錄二。

等式(3.20)裡面，$x_i\left(t\right),\phi_{2,i}\left(t\right),\phi_{3,i}\left(t\right),\phi_{4,i}\left(t\right),\phi_{5,i}\left(t\right),\phi_{6,i}\left(t\right)$為輔助的狀態變數，這些輔助變數的作用是把時間點$t$之前的全部歷史資訊結合在一起，使得即期利率$f(t,t)$的隨機過程只由時間點$t$的變數有關，用這樣的技巧讓我們的模型符合馬可夫性質。此外，我們發現$f(t,t)$具有平均反轉的特性，漂浮項會有力量讓$f(t,t)$回到平均值。這裡的平均值是修正後的遠期利率，會隨著時間而改變，另外，短期利率波動度也會與當期的利率水準成正比。此兩項特性都類似於 Cox-Ingersill-Ross(1985)。

4. 零息債券價格

在時間點 t 對到期時間點 T 的零息債券價格 $P(t,T)$ 如下:

(3.26)
$$P(t,T) = \exp\left\{-\int_t^T f(t,u)\,du\right\}$$
$$= \frac{P(0,T)}{P(0,t)}\exp\left\{\sum_{i=1}^{N}\beta_{x_i}(T-t)x_i(t)+\sum_{i=1}^{N}\sum_{j=1}^{6}\beta_{\phi_{j,i}}(T-t)\phi_{j,i}(t)\right\}$$

其中:

(3.27)
$$\beta_{x_i}(\tau) = \frac{\alpha_{1,i}}{\gamma_i}\left(\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(e^{-\gamma_i\tau}-1\right)+\tau e^{-\gamma_i\tau}\right)$$

(3.28)
$$\beta_{\phi_{1,i}}(\tau) = \frac{\alpha_{1,i}}{\gamma_i}(e^{-\gamma_i\tau}-1)$$

(3.29)
$$\beta_{\phi_{2,i}}(\tau)=\left(\frac{\alpha_{1,i}}{\gamma_i}\right)^2\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(e^{-\gamma_i\tau}-1\right)+\tau e^{-\gamma_i\tau}\right)$$

(3.30)
$$\beta_{\phi_{3,i}}(\tau) = -\frac{\alpha_{1,i}}{\gamma_i^2}\left(\begin{array}{l}\left(\frac{\alpha_{1,i}}{2\gamma_i^2}+\frac{\alpha_{0,i}}{\gamma_i}+\frac{\alpha_{0,i}^2}{2\alpha_{1,i}}\right)\left(e^{-2\gamma_i\tau}-1\right)\\ +\left(\frac{\alpha_{1,i}}{\gamma_i}+\alpha_{0,i}\right)\tau e^{-2\gamma_i\tau}+\frac{\alpha_{1,i}}{2}\tau^2 e^{-2\gamma_i\tau}\end{array}\right)$$

(3.31)
$$\beta_{\phi_{4,i}}(\tau)=\left(\frac{\alpha_{1,i}}{\gamma_i}\right)^2\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(e^{-\gamma_i\tau}-1\right)$$

(3.32)
$$\beta_{\phi_{5,i}}(\tau)=-\frac{\alpha_{1,i}}{\gamma_i^2}\left(\left(\frac{\alpha_{1,i}}{\gamma_i}+\alpha_{0,i}\right)\left(e^{-2\gamma_i\tau}-1\right)+\alpha_{1,i}\tau e^{-2\gamma_i\tau}\right)$$

(3.33)
$$\beta_{\phi_{6,i}}(\tau)= -\frac{1}{2}\left(\frac{\alpha_{1,i}}{\gamma_i}\right)^2\left(e^{-2\gamma_i\tau}-1\right)$$

證明：請參考附錄三。

而 $P(t,T)$ 的隨機過程如下:

(3.34)
$$\frac{dP(t,T)}{P(t,T)}=r(t)\,dt+\beta_{x_i}(T-t)\sqrt{a(t,f(t))}\,\sigma_i\,b(t)\,d$$

5. 對零息債券之歐式選擇權定價

我們的模型也符合 Duffin and Kan(1996)所提出的 Affine 的性質(the affine class of dynamic term structure models of Duffie and Kan(1996))，有此種 Affine 性質的好處在於，可以使我們在對利率衍生性金融商品訂價時有解析解。

為了要對零息債券之歐式選擇權定價，我們延續 Trolle and Schwartz(2008)的定價方法。T&S 定價方法是參照 Duffie, Pan, and Singleton (2000) 所提出的，這個方式的優點是可以讓零息債券之歐式選擇權有解析解。首先，我們先算出 Duffie, Pan, and Singleton (2000)中的轉換式:

(3.35) $$\psi(u,t,T_0,T_1) = E_t^Q[e^{-\int_t^{T_0} r_s ds} e^{u\log(P(T_0,T_1))}]$$

接著，我們再對零息債券之歐式選擇權作定價。

定理 1. Duffie, Pan, and Singleton (2000)中的轉換式(3.35)解為：

(3.36) $$\psi(u,t,T_0,T_1) = e^{\left[\begin{array}{l} M(T_0-t)+\sum_{i=1}^{N} N_i(T_0-t)v_i(t)+O(T_0-t)f(t,t)+\sum_{i=1}^{N} R_i(T_0-t)x_i(t)+\sum_{i=1}^{N}\sum_{j=1}^{6} Q_{j,i}(T_0-t)\phi_{j,i}(t) \\ +u\log(P(t,T_1))+(1-u)\log(P(t,T_0)) \end{array}\right]}$$

其中 $M(\tau)$, $N_i(\tau)$, $O(\tau)$, $R_i(\tau)$ 及 $Q_{j,i}(\tau)$ 的 ODE 解如下：

(3.37) $$\frac{dM(\tau)}{d\tau} = \sum_{i=1}^{N} N_i(\tau)\kappa_i\theta_i + O(\tau)\left[\frac{\partial}{\partial t} f(0,t) + \sum_{i=1}^{N} \gamma_i f(0,t)\right]$$

(3.38)

$$\frac{dN_i(\tau)}{d\tau} = -N_i(\tau)\kappa_i + b\left\{\begin{array}{l} Q_{2,i}(\tau)+Q_{3,i}(\tau) \\ +\frac{1}{2}\left[\begin{array}{l} N_i^2(\tau)\sigma_i^2\rho_i + R_i^2(\tau)+O^2(\tau)\alpha_{0,i}^2 \\ +\left(u^2-u\right)B_{x,i}{}^2(T_1-T_0+\tau)+\left(\left(1-u\right)^2-\left(1-u\right)\right)B_{x,i}{}^2(\tau) \end{array}\right] \\ \\ +N_i(\tau)R_i(\tau)\sigma_i\rho_i + N_i(\tau)O(\tau)\sigma_i\rho_i\alpha_{0,i} \\ +N_i(\tau)u\mathrm{B}_{x,i}(T_1-T_0+\tau)\sigma_i\rho_i + N(\tau)(1-u)\mathrm{B}_{x,i}(\tau)\sigma_i\rho_i \\ +O(\tau)R_i(\tau)\alpha_{0,i} + uO(\tau)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-T_0+\tau) \\ +(1-u)O(\tau)\alpha_{0,i}\mathrm{B}_{x,i}(\tau)+uR_i(\tau)\mathrm{B}_{x,i}(T_1-T_0+\tau) \\ +(1-u)R_i(\tau)\mathrm{B}_{x,i}(\tau)+u(1-u)\mathrm{B}_{x,i}(T_1-T_0+\tau)\mathrm{B}_{x,i}(\tau) \end{array}\right\}$$

(3.39)

$$\frac{d\mathrm{O}(\tau)}{d\tau} = -\gamma_i O(\tau) + a\sum_{i=1}^{N}\left\{\begin{array}{l} Q_{2,i}(\tau)+Q_{3,i}(\tau) \\ +\frac{1}{2}\left[\begin{array}{l} N_i^2(\tau)\sigma_i^2\rho_i + R_i^2(\tau)+O^2(\tau)\alpha_{0,i}^2 \\ +\left(u^2-u\right)B_{x,i}{}^2(T_1-T_0+\tau)+\left(\left(1-u\right)^2-\left(1-u\right)\right)B_{x,i}{}^2(\tau) \end{array}\right] \\ \\ +N_i(\tau)R_i(\tau)\sigma_i\rho_i + N_i(\tau)O(\tau)\sigma_i\rho_i\alpha_{0,i} \\ +N_i(\tau)u\mathrm{B}_{x,i}(T_1-T_0+\tau)\sigma_i\rho_i + N(\tau)(1-u)\mathrm{B}_{x,i}(\tau)\sigma_i\rho_i \\ +O(\tau)R_i(\tau)\alpha_{0,i} + uO(\tau)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-T_0+\tau) \\ +(1-u)O(\tau)\alpha_{0,i}\mathrm{B}_{x,i}(\tau)+uR_i(\tau)\mathrm{B}_{x,i}(T_1-T_0+\tau) \\ +(1-u)R_i(\tau)\mathrm{B}_{x,i}(\tau)+u(1-u)\mathrm{B}_{x,i}(T_1-T_0+\tau)\mathrm{B}_{x,i}(\tau) \end{array}\right\}$$

(3.40) $$\frac{dR_i(\tau)}{d\tau} = Q_{1,i}(\tau)+\alpha_i O(\tau)-\gamma_i R_i(\tau)$$

(3.41) $$\frac{dQ_{1,i}(\tau)}{d\tau} = -\gamma_i Q_{1,i}(\tau)$$

(3.42) $$\frac{dQ_{2,i}(\tau)}{d\tau} = -\gamma_i Q_{2,i}(\tau)+Q_{4,i}(\tau)+O(\tau)D_i$$

(3.43) $$\frac{dQ_{3,i}(\tau)}{d\tau} = -2\gamma_i Q_{3,i}(\tau)+Q_{5,i}(\tau)+O(\tau)D_i$$

(3.44) $$\frac{dQ_{4,i}(\tau)}{d\tau} = -\gamma_i Q_{4,i}(\tau)+O(\tau)D_i$$

(3.45) $$\frac{dQ_{5,i}(\tau)}{d\tau} = -2\gamma_i Q_{5,i}(\tau)+2Q_{6,i}(\tau)+O(\tau)D_i$$

(3.46) $$\frac{dQ_{6,i}(\tau)}{d\tau} = -2\gamma_i Q_{6,i}(\tau)+O(\tau)D_i$$

其中，式子(3.37)至(3.46)條件限制式為 $M(0)=0,\ N_i(0)=0,\ O(0)=0,\ R_i(0)=0$ 和

$Q_{j,i}(0)=0.$

證明：請參考附錄四。

定理 2. 零息債券之歐式選擇權作定價

根據 Duffin, Pan, and singleton(2000)和 Collin-Dufresne and Goldstein(2003)，我們若要對零息債券之歐式選擇權作定價，我們可以應用反傅立葉轉換得到其解析解。在時間t對到期執行時間T_0的歐式賣權$Put\ (t,T_0,T_1,K)$作定價，執行價格K，其中標的資產為到期時間T_1的零息債券$P(t,T_1)$。則

(3.47) $$Put(t,T_0,T_1,K) = K\,G_{0,1}(-\log K) - G_{1,1}(-\log K)$$

其中式子(3.47)中的$G_{a,b}\left(y\right)$定義如下：

(3.48) $$G_{a,b}\left(y\right)=\frac{\psi(a,t,T_0,T_1)}{2}-\frac{1}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\left[\psi(a+iub,t,T_0,T_1)e^{iuy}\right]}{u}du$$

其中$i=\sqrt{-1}$。

證明：請參考附錄五。

四、 結論

本文提出一個的隨機波動度的多因子利率模型，此模型不僅可以描述利率的特性：第一，利率波動度是隨機的。第二，利率波動度內含非利率期限結構因子(unspanned stochastic volatility factors)。第三，利率波動度的變化和利率的變化具有相關的現象。第四，非條件下(unconditional=realized and implied)波動度結構具有駝峰(hump)的性質。此外，此模型更多考慮了即期利率(instantaneous spot rate) $f(t,t)$ 也會影響利率波動度。

本文也應證了此模型符合馬可夫性質。在此應用了輔助變數導出風險中立下的瞬間遠期利率 $f(t,T)$、零息債券價格,。此動態過程符合 Duffie, Pan and Singleton (2000)(簡稱 DPS)提出的 Affine Jump-Diffusions 的條件，能獲得債券選擇權評價公式的解析解。

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附錄

附錄一

首先，我們要先算出 $\mu_f(t,T)$ ，所以將式子(3.4)帶入式子(3.3)

$$\mu_f(t,T)=\sum_{i=1}^{N}\left[af(t,t)+b\upsilon_i(t)\right]\sigma_{f,i}(t,T)\int_t^T\sigma_{f,i}(t,u)du$$

$$=\sum_{i=1}^{N}\left[af(t,t)+b\upsilon_i(t)\right]\begin{bmatrix} \frac{\alpha_{0,i}\alpha_{1,i}}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(e^{-\gamma_i(T-t)}-e^{-2\gamma_i(T-t)}\right)-\left(\frac{\alpha_{0,i}\alpha_{1,i}}{\gamma_i}\right)(\mathrm{T}-t)e^{-2\gamma_i(T-t)} \\ +\frac{\alpha_{1,i}^2}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)(T-t)\left(e^{-\gamma_i(T-t)}-e^{-2\gamma_i(T-t)}\right)-\left(\frac{\alpha_{1,i}^2}{\gamma_i}\right)(T-t)^2e^{-2\gamma_i(T-t)} \end{bmatrix}$$

接著，將算出來的 $\mu_f(t,T)$ 及 $\sigma_{f,i}(t,T)$ 帶入式子(3.1)積分

$$f(t,T)=f(0,T)+\int_0^t\mu_f(s,T)ds+\sum_{i=1}^{N}\int_0^t\sigma_{f,i}(s,T)\sqrt{af(s,s)+b\upsilon_i(s)}dW_i^Q(s)$$

$$=f(0,T)+\sum_{i=1}^{N}\mathrm{B}_{x_i}(T-t)x_i(t)+\sum_{i=1}^{N}\sum_{j=1}^{6}\mathrm{B}_{\phi_{j,i}}(T-t)\phi_{j,i}(t)$$

$$\mathrm{B}_{x_i}(\tau)=\left(\alpha_{0,i}+\alpha_{1,i}\tau e^{-\gamma_i\tau)}\right)e^{-\gamma_i\tau}$$

$$\mathrm{B}_{\phi_{1,i}}(\tau)=\alpha_{1,i}e^{-\gamma_i\tau}$$

$$\mathrm{B}_{\phi_{2,i}(t)}(\tau)=\frac{\alpha_{1,i}}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(\alpha_{0,i}+\alpha_{1,i}\tau\right)e^{-\gamma_i\tau}$$

$$\mathrm{B}_{\phi_{3,i}(t)}(\tau)=-\left(\frac{\alpha_{0,i}\alpha_{1,i}}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)+\frac{\alpha_{1,i}}{\gamma_i}\left(\frac{\alpha_{1,i}}{\gamma_i}+2\alpha_{0,i}\right)\tau+\frac{\alpha_{1,i}^2}{\gamma_i}\tau^2\right)e^{-2\gamma_i\tau}$$

$$\mathrm{B}_{\phi_{4,i}(t)}(\tau)=\frac{\alpha_{1,i}^2}{\gamma_i}\left(\frac{1}{\gamma_i}+\frac{\alpha_{0,i}}{\alpha_{1,i}}\right)e^{-\gamma_i\tau}$$

$$\mathrm{B}_{\phi_{5,i}(t)}(\tau)=-\frac{\alpha_{1,i}}{\gamma_i}\left(2\alpha_{0,i}+\frac{\alpha_{1,i}}{\gamma_i}+2\alpha_{1,i}\tau\right)e^{-2\gamma_i\tau}$$

$$B_{\phi_{6,i}(t)}(\tau)= \frac{-\alpha_{1,i}^2}{\gamma_i} e^{-2\gamma_i \tau}$$

狀態變數如下：

$$x_i(t)=\int_0^t \sqrt{af(s,s)+b\upsilon_i(s)}e^{-\gamma_i(t-s)}dW_i^Q(s)$$

$$\phi_{1,i}(t)=\int_0^t \sqrt{af(s,s)+b\upsilon_i(s)}(t-s)e^{-\gamma_i(t-s)}dW_i^Q(s)$$

$$\phi_{2,i}(t)=\int_0^t \left[af(s,s)+b\upsilon_i(s)\right]e^{-\gamma_i(t-s)}ds$$

$$\phi_{3,i}(t)=\int_0^t \left[af(s,s)+b\upsilon_i(s)\right]e^{-2\gamma_i(t-s)}ds$$

$$\phi_{4,i}(t)=\int_0^t \left[af(s,s)+b\upsilon_i(s)\right](t-s)e^{-\gamma_i(t-s)}ds$$

$$\phi_{5,i}(t)=\int_0^t \left[af(s,s)+b\upsilon_i(s)\right](t-s)e^{-2\gamma_i(t-s)}ds$$

$$\phi_{6,i}(t)=\int_0^t \left[af(s,s)+b\upsilon_i(s)\right](t-s)^2 e^{-2\gamma_i(t-s)}ds$$

$$dx_i(t)=-\gamma_i x_i(t)dt+\sqrt{af(t,t)+b\upsilon_i(t)}e^{-\gamma_i(t-s)}dW_i^Q(t)$$

$$d\phi_{1,i}(t)=\left(x_i(t)-\gamma_i\phi_{1,i}(t)\right)dt$$

$$d\phi_{2,i}(t)=\left(\upsilon_i(t)-\gamma_i\phi_{2,i}(t)\right)dt$$

$$d\phi_{3,i}(t)=\left(\upsilon_i(t)-2\gamma_i\phi_{3,i}(t)\right)dt$$

$$d\phi_{4,i}(t)=\left(\phi_{2,i}(t)-\gamma_i\phi_{4,i}(t)\right)dt$$

$$d\phi_{5,i}(t)=\left(\phi_{3,i}(t)-2\gamma_i\phi_{5,i}(t)\right)dt$$

$$d\phi_{6,i}(t)=\left(2\phi_{5,i}(t)-2\gamma_i\phi_{6,i}(t)\right)dt$$

附錄二

定義 $\sigma_i(t,T)=(\alpha_{0,i}+\alpha_{1,i}(T-t))e^{-\gamma_i(T-t)}$

$\Rightarrow \sigma_{f,i}(t,T)=\sigma_i(t,T)\left[af(t,t)+b\upsilon_i(t)\right]$ 代入式 (2.4)

$$df(t,t)=\begin{bmatrix}\frac{\partial}{\partial t}f\left(0,t\right)+\sum_{i=1}^{N}\frac{\partial}{\partial t}\int_0^t\left[af(u,u)+b\upsilon_i(u)\right]\sigma_i\left(u,t\right)\int_u^t\sigma_i\left(u,y\right)dydu\\+\sum_{i=1}^{N}\int_0^t\sqrt{af(u,u)+b\upsilon_i(u)}\frac{\partial}{\partial t}\sigma_i\left(u,t\right)dW_i^Q\left(u\right)\end{bmatrix}dt$$

$$+\sum_{i=1}^{N}\left[af(t,t)+b\upsilon_i(t)\right]\sigma_i(t,t)dW_i^Q\left(t\right)$$

$$=\begin{bmatrix}\frac{\partial}{\partial t}f\left(0,t\right)+\sum_{i=1}^{N}\int_0^t\left[af(u,u)+b\upsilon_i(u)\right]\frac{\partial}{\partial t}\left[\sigma_i\left(u,t\right)\int_u^t\sigma_i\left(u,y\right)dy\right]du\backslash\\+\sum_{i=1}^{N}\int_0^t\sqrt{af(u,u)+b\upsilon_i(u)}\frac{\partial}{\partial t}\sigma_i\left(u,t\right)dW_i^Q\left(u\right)\end{bmatrix}dt$$

$$+\sum_{i=1}^{N}\left[af(t,t)+b\upsilon_i(t)\right]\sigma_i\left(t,t\right)dW_i^Q\left(t\right)$$

其中 $\frac{\partial}{\partial T}\sigma_i\left(t,T\right)=\alpha_{1,i}e^{-\gamma_i(T-t)}-\gamma_i\left(\alpha_{0,i}+\alpha_{1,i}\left(T-t\right)\right)e^{-\gamma_i(T-t)}$

$$df(t,t)=\begin{bmatrix}\frac{\partial}{\partial t}f\left(0,t\right)\\+\sum_{i=1}^{N}\int_0^t\left[af(u,u)+b\upsilon_i(u)\right]\left(\alpha_{1,i}e^{-\gamma_i(t-u)}-\gamma_i\left(\alpha_{0,i}+\alpha_{1,i}\left(t-u\right)\right)e^{-\gamma_i(t-u)}\right)\int_u^t\sigma_i\left(u,y\right)dydu\\+\sum_{i=1}^{N}\int_0^t\left[af(u,u)+b\upsilon_i(u)\right]\sigma_i\left(u,t\right)^2du\\+\sum_{i=1}^{N}\int_0^t\sqrt{af(u,u)+b\upsilon_i(u)}\left(\alpha_{1,i}e^{-\gamma_i(t-u)}-\gamma_i\left(\alpha_{0,i}+\alpha_{1,i}\left(t-u\right)\right)e^{-\gamma_i(t-u)}\right)dW_i^Q\left(u\right)\end{bmatrix}dt$$

$$+\sum_{i=1}^{N}\alpha_{0,i}\sqrt{af(t,t)+b\upsilon_i(u)}dW_i^Q\left(t\right)$$

根據 Bhar and Chiarella(1997)裡，Appendix 3 的計算過程

$$df(t,t)=\begin{bmatrix}\frac{\partial}{\partial t}f(0,t)+\sum_{i=1}^{N}\gamma_i f(0,t)\\ -\sum_{i=1}^{N}\gamma_i f(t,t)+\sum_{i=1}^{N}\alpha_{1,i}x_i(t)+\sum_{i=1}^{N}\sum_{j=2}^{6}D_{j,i}\phi_{j,i}\end{bmatrix}dt$$

$$+\sum_{i=1}^{N}\alpha_{0,i}\sqrt{af(t,t)+b\upsilon_i(t)}dW_i^{Q}(t)$$

$$D_{2,i}=\frac{\alpha_{1,i}{}^2+\alpha_{0,i}\alpha_{1,i}\gamma_i}{\gamma_i^2}$$

$$D_{3,i}=\frac{\alpha_{0,i}{}^2\gamma_i^2-\alpha_{1,i}{}^2-\alpha_{0,i}\alpha_{1,i}\gamma_i}{\gamma_i^2}$$

$$D_{4,i}=\frac{\alpha_{0,i}\alpha_{1,i}\gamma_i^2+\alpha_{1,i}{}^2\gamma_i-\alpha_{1,i}{}^2-\alpha_{0,i}\alpha_{1,i}\gamma_i}{\gamma_i^2}$$

$$D_{5,i}=2\alpha_{0,i}\alpha_{1,i}-\frac{\alpha_{1,i}{}^2}{\gamma_i}$$

$$D_{6,i}=\alpha_{1,i}{}^2$$

其中狀態變數為 $S(t)\equiv\left[r(t),x_i(t),\phi_{2,i}(t),\phi_{3,i}(t),\phi_{4,i}(t),\phi_{5,i}(t),\phi_{6,i}(t)\right]$

狀態變數只和 t 時點有關，所以符合馬可夫性質。

附錄三

$$P(t,T) \equiv \exp\left\{-\int_t^T f(t,u)du\right\}$$

$$= \exp\left\{-\int_t^T \left[f(0,u) + \sum_{i=1}^{N} \mathrm{B}_{x_i}(u-t)x_i(t) + \sum_{i=1}^{N}\sum_{j=1}^{6} \mathrm{B}_{\phi_{j,i}}(u-t)\phi_{j,i}(t)du \right]\right\}$$

$$= \exp\left\{-\int_t^T f(0,u)du\right\} \exp\left\{-\int_t^T \left[\sum_{i=1}^{N} \mathrm{B}_{x_i}(u-t)x_i(t) + \sum_{i=1}^{N}\sum_{j=1}^{6} \mathrm{B}_{\phi_{j,i}}(u-t)\phi_{j,i}(t)du \right]\right\}$$

$$= \frac{P(0,T)}{P(0,t)} \exp\left\{ \sum_{i=1}^{N} \mathrm{B}_{x_i}(T-t)x_i(t) + \sum_{i=1}^{N}\sum_{j=1}^{6} \mathrm{B}_{\phi_{j,i}}(T-t)\phi_{j,i}(t) \right\}$$

在此$\mathrm{B}_x(T-t)$和$\mathrm{B}_{\phi_j}(T-t)$將重新定義。

$$\mathrm{B}_{x_i}(\tau) = \frac{\alpha_{1,i}}{\gamma_i}\left(\left(\frac{1}{\gamma_i} + \frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(e^{-\gamma_i\tau} - 1\right) + \tau e^{-\gamma_i\tau} \right)$$

$$\mathrm{B}_{\phi_{1,i}}(\tau) = \frac{\alpha_{1,i}}{\gamma_i}(e^{-\gamma_i\tau} - 1)$$

$$\mathrm{B}_{\phi_{2,i}}(\tau) = \left(\frac{\alpha_{1,i}}{\gamma_i}\right)^2 \left(\frac{1}{\gamma_i} + \frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(\left(\frac{1}{\gamma_i} + \frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(e^{-\gamma_i\tau} - 1\right) + \tau e^{-\gamma_i\tau} \right)$$

$$\mathrm{B}_{\phi_{3,i}}(\tau) = -\frac{\alpha_{1,i}}{\gamma_i^2}\left(\begin{array}{l} \left(\dfrac{\alpha_{1,i}}{2\gamma_i^2} + \dfrac{\alpha_{0,i}}{\gamma_i} + \dfrac{\alpha_{0,i}^2}{2\alpha_{1,i}}\right)\left(e^{-2\gamma_i\tau} - 1\right) \\ + \left(\dfrac{\alpha_{1,i}}{\gamma_i} + \alpha_{0,i}\right)\tau e^{-2\gamma_i\tau} + \dfrac{\alpha_{1,i}}{2}\tau^2 e^{-2\gamma_i\tau} \end{array} \right)$$

$$\mathrm{B}_{\phi_{4,i}}(\tau) = \left(\frac{\alpha_{1,i}}{\gamma_i}\right)^2 \left(\frac{1}{\gamma_i} + \frac{\alpha_{0,i}}{\alpha_{1,i}}\right)\left(e^{-\gamma_i\tau} - 1\right)$$

$$\mathrm{B}_{\phi_{5,i}}(\tau) = -\frac{\alpha_{1,i}}{\gamma_i^2}\left(\left(\frac{\alpha_{1,i}}{\gamma_i} + \alpha_{0,i}\right)\left(e^{-2\gamma_i\tau} - 1\right) + \alpha_{1,i}\tau e^{-2\gamma_i\tau} \right)$$

$$\mathrm{B}_{\phi_{6,i}}(\tau) = -\frac{1}{2}\left(\frac{\alpha_{1,i}}{\gamma_i}\right)^2 \left(e^{-2\gamma_i\tau} - 1\right)$$

附錄四

$$\psi(u,t,T_0,T_1) = E_t^Q[e^{-\int_t^{T_0} r_s ds} e^{u\log(P(T_0,T_1))}]$$

$$e^{-\int_0^t r_s ds}\psi(u,t,T_0,T_1) = E_t^Q[e^{-\int_0^{T_0} r_s ds}\psi(u,T_0,T_0,T_1)]$$

由於$\psi(u,T_0,T_0,T_1) = e^{u\log(P(T_0,T_1))}$，所以我們接下來只需要證明$\eta(\mathrm{t}) \equiv e^{-\int_0^t r_s ds}\psi(u,t,T_0,T_1)$在風險中立機率測度 Q 下，符合平賭的性質。

根據經驗，我們猜測$\psi(u,t,T_0,T_1)$的解如下：

$$\psi(u,t,T_0,T_1) = e^{\left\{\begin{array}{l} M(T_0-t)+\sum_{i=1}^{N} N_i(T_0-t)\nu_i(t)+\mathrm{O}(T_0-t)f(t,t) \\ +\sum_{i=1}^{N} R_i(T_0-t)x_i(t)+\sum_{i=1}^{N}\sum_{j=1}^{6} Q_{j,i}(T_0-t)\phi_{j,i}(t) \\ +u\log(P(t,T_1))+(1-u)\log(P(t,T_0)) \end{array}\right\}}$$

接著，應用伊藤定理計算。因為$\eta(\mathrm{t}) \equiv e^{-\int_0^t r_s ds}\psi(u,t,T_0,T_1)$符合平賭性質，所以可以利用$\eta(\mathrm{t}) \equiv e^{-\int_0^t r_s ds}\psi(u,t,T_0,T_1)$的飄移項為零求解。

定義：$\beta(t) = e^{-\int_0^t r(s)ds} \Rightarrow \quad d\beta(t) = -\beta(t)\cdot r(t)dt$

$$
\begin{aligned}
& d\psi\left(u,t,T_0,T_1\right) \\
& =\frac{\partial\psi}{\partial t}dt+\sum_{i=1}^{N}\frac{\partial\psi}{\partial\nu_i}d\nu_i\left(t\right)+\frac{\partial\psi}{\partial r}dr\left(t\right)+\sum_{i=1}^{N}\frac{\partial\psi}{\partial x_i}dx_i\left(t\right) \\
& \quad+\sum_{i=1}^{N}\sum_{\mathrm{j}=1}^{6}\frac{\partial\psi}{\partial\phi_{j,i}}d\phi_{j,i}(t)+\frac{\partial\psi}{\partial P\left(t,T_1\right)}dP\left(t,T_1\right)+\frac{\partial\psi}{\partial P\left(t,T_0\right)}dP\left(t,T_0\right) \\
& \quad+\frac{1}{2}\left[\begin{array}{l}\displaystyle\sum_{i=1}^{N}\frac{\partial\psi}{\partial\nu_i^2}\left(d\nu_i\left(t\right)\right)^2+\sum_{i=1}^{N}\frac{\partial\psi}{\partial x_i^2}\left(dx_i\left(t\right)\right)^2+\frac{\partial^2\psi}{\partial P^2\left(t,T_1\right)}\left(dP\left(t,T_1\right)\right)^2 \\ \displaystyle+\frac{\partial^2\psi}{\partial P^2\left(t,T_0\right)}\left(dP\left(t,T_0\right)\right)^2+\frac{\partial^2\psi}{\partial r^2}\left(dr\left(t\right)\right)^2\end{array}\right] \\
& \quad+\left[\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial\nu_i\partial x_i}d\nu_i\left(t\right)dx_i\left(t\right)\right]+\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial\nu_i\partial r}d\nu_i\left(t\right)dr\left(t\right)+\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial\nu_i\partial P\left(t,T_1\right)}d\nu_i\left(t\right)dP\left(t,T_1\right) \\
& \quad+\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial\nu_i\partial P\left(t,T_0\right)}d\nu_i\left(t\right)dP\left(t,T_0\right)+\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial x_i\partial r}dx_i\left(t\right)dr\left(t\right)+\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial x_i\partial P\left(t,T_1\right)}dx_i\left(t\right)dP\left(t,T_1\right) \\
& \quad+\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial x_i\partial P\left(t,T_0\right)}dx_i\left(t\right)dP\left(t,T_0\right)+\frac{\partial^2\psi}{\partial r\partial P\left(t,T_1\right)}dr\left(t\right)dP\left(t,T_1\right)+\frac{\partial^2\psi}{\partial r\partial P\left(t,T_0\right)}dr\left(t\right)dP\left(t,T_0\right) \\
& \quad+\frac{\partial^2\psi}{\partial P\left(t,T_0\right)\partial P\left(t,T_1\right)}dP\left(t,T_0\right)dP\left(t,T_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \psi\left(u,t,T_0,T_1\right)\Bigg\{ \Bigg[-M'\left(T_0-t\right)-\sum_{i=1}^{N} N_i'\left(T_0-t\right)\nu\left(t\right)-O'\left(T_0-t\right) f\left(t,t\right) \\
&\quad -\sum_{i=1}^{N} R_i'(T_0-t)x(t)-\sum_{i=1}^{N}\sum_{j=1}^{6} Q_{j,i}'(T_0-t)\phi_{j,i}(t) \\
&\quad +\sum_{i=1}^{N}\left\{\begin{array}{l} Q_{1,i}\left(T_0-t\right)\left(x_i(t)-\gamma_i\phi_{1,i}(t)\right)+Q_{2,i}\left(T_0-t\right)\left(\left[af(t,t)+b\upsilon_i(t)\right]-\gamma_i\phi_{2,i}(t)\right) \\ +Q_{3,i}\left(T_0-t\right)\left(\left[af(t,t)+b\upsilon_i(t)\right]-2\gamma_i\phi_{3,i}(t)\right)+Q_{4,i}\left(T_0-t\right)\left(\phi_{2,i}(t)-\gamma_i\phi_{4,i}(t)\right) \\ +Q_{5,i}\left(T_0-t\right)\left(\phi_{3,i}(t)-2\gamma_i\phi_{5,i}(t)\right)+Q_{6,i}\left(T_0-t\right)\left(2\phi_{5,i}(t)-2\gamma_i\phi_{6,i}(t)\right) \end{array}\right\} \\
&\quad +\sum_{i=1}^{N}\left[\begin{array}{l} \dfrac{1}{2}\left[\begin{array}{l} N_i^2\left(T_0-t\right)\sigma_i^2\rho_i+R_i^2\left(T_0-t\right)+O^2\left(T_0-t\right)\alpha_{0,i}^2 \\ +\left(u^2-u\right)B_{x,i}{}^2\left(T_1-t\right)+\left(\left(1-u\right)^2-\left(1-u\right)\right)B_{x,i}{}^2\left(T_0-t\right) \end{array}\right] \\ +N_i\left(T_0-t\right)R_i\left(T_0-t\right)\sigma_i\rho_i+N_i\left(T_0-t\right)O\left(T_0-t\right)\sigma_i\rho_i\alpha_{0,i} \\ +N_i\left(T_0-t\right)u\mathrm{B}_{x,i}(T_1-t)\sigma_i\rho_i \\ +N\left(T_0-t\right)\left(1-u\right)\mathrm{B}_{x,i}(T_0-t)\sigma_i\rho_i \\ +O\left(T_0-t\right)R_i\left(T_0-t\right)\alpha_{0,i}+uO\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-t) \\ +\left(1-u\right)O\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_0-t)+uR_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_1-t) \\ +(1-u)R_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_0-t)+u(1-u)\mathrm{B}_{x,i}(T_1-t)\mathrm{B}_{x,i}(T_0-t) \end{array}\right]\left[af(t,t)+b\upsilon_i(t)\right] \\
&\quad +\sum_{i=1}^{N}\left\{N_i\left(T_0-t\right)\left[\kappa_i(\theta_i-\upsilon_i(t))\right]\right\} \\
&\quad +O\left(T_0-t\right)\left[\begin{array}{l} \dfrac{\partial}{\partial t} f\left(0,t\right)+\sum_{i=1}^{N}\gamma_i f\left(0,t\right)-\sum_{i=1}^{N}\gamma_i f\left(t,t\right) \\ +\sum_{i=1}^{N}\alpha_{1,i}x_i\left(t\right)+\sum_{i=1}^{N}\sum_{j=2}^{6} D_{j,i}\phi_{j,i} \end{array}\right] \\
&\quad +\sum_{i=1}^{N}\left[-\gamma_i x_i(t)R_{i,i}\left(T_0-t\right)\right]+ur(t)+(1-u)r(t) \Bigg] dt \\
&\quad +\left[\begin{array}{l} \sum_{i=1}^{N} N_i\left(T_0-t\right)\sigma_i\rho_i\sqrt{af(t,t)+b\upsilon_i(t)}+O\left(T_0-t\right)\sum_{i=1}^{N}\alpha_{0,i}\sqrt{af(t,t)+b\upsilon_i(t)} \\ +\sum_{i=1}^{N} R_{i,i}\left(T_0-t\right)\sqrt{af(t,t)+b\upsilon_i(t)} \\ +u\sum_{i=1}^{N}\mathrm{B}_{x,i}(T_1-t)\sqrt{af(t,t)+b\upsilon_i(t)}+\left(1-u\right)\sum_{i=1}^{N}\mathrm{B}_{x,i}(T_2-t)\sqrt{af(t,t)+b\upsilon_i(t)} \end{array}\right] dW_i^Q\left(t\right) \\
&\quad +\left[\sum_{i=1}^{N} N_i\left(T_0-t\right)\sigma_i\sqrt{af(t,t)+b\upsilon_i(t)}\sqrt{1-\rho_i^2}\right] dZ_i^Q(t) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
d\eta(t) &= d\beta(t)\psi(t) \\
&= \beta(t)d\psi(t) + \psi(t)d\beta(t) + d\beta(t)d\psi(t) \\
&\qquad\qquad\qquad\qquad ^{**}d\beta(t)d\psi(t)=0 \\
&= \beta(t)d\psi(t) - \psi(t)\beta(t)\cdot r(t)dt
\end{aligned}
$$

$$
d\eta(t)
$$

$$
= \eta(t)\left\{
\begin{aligned}
&\left[
\begin{aligned}
&-M'(T_0-t) - \sum_{i=1}^{N} N_i'(T_0-t)\nu(t) - O'(T_0-t)f(t,t) \\
&-\sum_{i=1}^{N} R_i'(T_0-t)x(t) - \sum_{i=1}^{N}\sum_{j=1}^{6} Q_{j,i}'(T_0-t)\phi_{j,i}(t) \\
&+\sum_{i=1}^{N}\left\{
\begin{aligned}
&Q_{1,\mathrm{i}}(T_0-t)\left(x_i(t)-\gamma_i\phi_{1,i}(t)\right) + Q_{2,\mathrm{i}}(T_0-t)\left(\left[af(t,t)+b\upsilon_i(t)\right]-\gamma_i\phi_{2,i}(t)\right) \\
&+Q_{3,i}(T_0-t)\left(\left[af(t,t)+b\upsilon_i(t)\right]-2\gamma_i\phi_{3,i}(t)\right) + Q_{4,i}(T_0-t)\left(\phi_{2,i}(t)-\gamma_i\phi_{4,i}(t)\right) \\
&+Q_{5,i}(T_0-t)\left(\phi_{3,i}(t)-2\gamma_i\phi_{5,i}(t)\right) + Q_{6,i}(T_0-t)\left(2\phi_{5,i}(t)-2\gamma_i\phi_{6,i}(t)\right)
\end{aligned}
\right\} \\
&+\sum_{i=1}^{N}\left[
\begin{aligned}
&\frac{1}{2}\left[
\begin{aligned}
&N_i^2(T_0-t)\sigma_i^2\rho_i + R_i^2(T_0-t) + O^2(T_0-t)\alpha_{0,i}^2 \\
&+\left(u^2-u\right)\mathrm{B}_{x,i}{}^2(T_1-t) + \left((1-u)^2-(1-u)\right)B_{x,i}{}^2(T_0-t)
\end{aligned}
\right] \\
&+N_i(T_0-t)R_i(T_0-t)\sigma_i\rho_i + N_i(T_0-t)O(T_0-t)\sigma_i\rho_i\alpha_{0,i} \\
&+N_i(T_0-t)u\mathrm{B}_{x,i}(T_1-t)\sigma_i\rho_i + N(T_0-t)(1-u)\mathrm{B}_{x,i}(T_0-t)\sigma_i\rho_i \\
&+O(T_0-t)R_i(T_0-t)\alpha_{0,i} + uO(T_0-t)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-t) \\
&+(1-u)O(T_0-t)\alpha_{0,i}\mathrm{B}_{x,i}(T_0-t) + uR_i(T_0-t)\mathrm{B}_{x,i}(T_1-t) \\
&+(1-u)R_i(T_0-t)\mathrm{B}_{x,i}(T_0-t) + u(1-u)\mathrm{B}_{x,i}(T_1-t)\mathrm{B}_{x,i}(T_0-t)
\end{aligned}
\right]\left[af(t,t)+b\upsilon_i(t)\right] \\
&+\sum_{i=1}^{N}\left\{N_i(T_0-t)\left[\kappa_i(\theta_i-\upsilon_i(t))\right]\right\} \\
&+O(T_0-t)\left[
\begin{aligned}
&\frac{\partial}{\partial t}f(0,t) + \sum_{i=1}^{N}\gamma_i f(0,t) \\
&-\sum_{i=1}^{N}\gamma_i f(t,t) + \sum_{i=1}^{N}\alpha_{1,i}x_i(t) + \sum_{i=1}^{N}\sum_{j=2}^{6}D_{j,i}\phi_{j,i}
\end{aligned}
\right] + \sum_{i=1}^{N}\left[-\gamma_i x_i(t)R_{i,i}(T_0-t)\right]
\end{aligned}
\right] dt \\
&+\left[
\begin{aligned}
&\sum_{i=1}^{N}N_i(T_0-t)\sigma_i\rho_i\sqrt{af(t,t)+b\upsilon_i(t)} + O(T_0-t)\sum_{i=1}^{N}\alpha_{0,i}\sqrt{af(t,t)+b\upsilon_i(t)} \\
&+\sum_{i=1}^{N}R_{i,i}(T_0-t)\sqrt{af(t,t)+b\upsilon_i(t)} \\
&+u\sum_{i=1}^{N}\mathrm{B}_{x,i}(T_1-t)\sqrt{af(t,t)+b\upsilon_i(t)} + (1-u)\sum_{i=1}^{N}\mathrm{B}_{x,i}(T_2-t)\sqrt{af(t,t)+b\upsilon_i(t)}
\end{aligned}
\right] dW_i^{Q}(t) \\
&+\left[\sum_{i=1}^{N}N_i(T_0-t)\sigma_i\sqrt{af(t,t)+b\upsilon_i(t)}\sqrt{1-\rho_i^2}\right] dZ_i^{Q}(t)
\end{aligned}
\right\}
$$

接著，將飄移項內的元素分成十種類別，有 $\upsilon_i(t)$ 、有 $f\left(t,t\right)$ 、有 $x_i(t)$ 、有 $\phi_{1,i}(t)$ 、有 $\phi_{2,i}(t)$ 、有 $\phi_{3\,i}(t)$ 、有 $\phi_{4,i}(t)$ 、有 $\phi_{5,i}(t)$ 、有 $\phi_{6,i}(t)$ 以及剩下的分成一類，總共十類。每一類的係數加總和為零求解。

1. 剩下

$$
\begin{aligned}
&-M'\left(T_0-t\right)+\sum_{i=1}^{N} N_i\left(T_0-t\right)\kappa_i\theta_i+O\left(T_0-t\right)\left[\frac{\partial}{\partial t} f\left(0,t\right)+\sum_{i=1}^{N}\gamma_i f\left(0,t\right)\right]=0 \\
&\Rightarrow M'\left(T_0-t\right)=\sum_{i=1}^{N} N_i\left(T_0-t\right)\kappa_i\theta_i+O\left(T_0-t\right)\left[\frac{\partial}{\partial t} f\left(0,t\right)+\sum_{i=1}^{N}\gamma_i f\left(0,t\right)\right]
\end{aligned}
$$

2. 有 $\upsilon_i(t)$

$$
\sum_{i=1}^{N}\left[-N_i'\left(T_0-t\right)-N_i\left(T_0-t\right)\kappa_i+b\left[\begin{array}{l}
Q_{2,\mathrm{i}}\left(T_0-t\right)+Q_{3,i}\left(T_0-t\right) \\
+\dfrac{1}{2}\left[\begin{array}{l}
N_i^2\left(T_0-t\right)\sigma_i^2\rho_i+R_i^2\left(T_0-t\right) \\
+O^2\left(T_0-t\right)\alpha_{0,i}^2+\left(u^2-u\right)B_{x,i}{}^2\left(T_1-t\right) \\
+\left(\left(1-u\right)^2-\left(1-u\right)\right)B_{x,i}{}^2\left(T_0-t\right)
\end{array}\right] \\
+N_i\left(T_0-t\right)R_i\left(T_0-t\right)\sigma_i\rho_i \\
+N_i\left(T_0-t\right)O\left(T_0-t\right)\sigma_i\rho_i\alpha_{0,i} \\
+N_i\left(T_0-t\right)u\mathrm{B}_{x,i}(T_1-t)\sigma_i\rho_i \\
+N\left(T_0-t\right)\left(1-u\right)\mathrm{B}_{x,i}(T_0-t)\sigma_i\rho_i \\
+O\left(T_0-t\right)R_i\left(T_0-t\right)\alpha_{0,i}+uO\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-t) \\
+\left(1-u\right)O\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_0-t)+uR_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_1-t) \\
+(1-u)R_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_0-t) \\
+u(1-u)\mathrm{B}_{x,i}(T_1-t)\mathrm{B}_{x,i}(T_0-t)
\end{array}\right]\right]\nu_i\left(t\right)=0
$$

$$\Rightarrow N_i'\left(T_0-t\right)=-N_i\left(T_0-t\right)\kappa_i+b\begin{bmatrix} Q_{2,\mathrm{i}}\left(T_0-t\right)+Q_{3,i}\left(T_0-t\right) \\ +\dfrac{1}{2}\begin{bmatrix} N_i^2\left(T_0-t\right)\sigma_i^2\rho_i+R_i^2\left(T_0-t\right) \\ +O^2\left(T_0-t\right)\alpha_{0,i}^2+\left(u^2-u\right)B_{x,i}{}^2\left(T_1-t\right) \\ +\left(\left(1-u\right)^2-\left(1-u\right)\right)B_{x,i}{}^2\left(T_0-t\right) \end{bmatrix} \\ +N_i\left(T_0-t\right)R_i\left(T_0-t\right)\sigma_i\rho_i \\ +N_i\left(T_0-t\right)O\left(T_0-t\right)\sigma_i\rho_i\alpha_{0,i} \\ +N_i\left(T_0-t\right)u\mathrm{B}_{x,i}(T_1-t)\sigma_i\rho_i \\ +N\left(T_0-t\right)\left(1-u\right)\mathrm{B}_{x,i}(T_0-t)\sigma_i\rho_i \\ +O\left(T_0-t\right)R_i\left(T_0-t\right)\alpha_{0,i}+uO\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-t) \\ +\left(1-u\right)O\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_0-t)+uR_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_1-t) \\ +(1-u)R_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_0-t) \\ +u(1-u)\mathrm{B}_{x,i}(T_1-t)\mathrm{B}_{x,i}(T_0-t) \end{bmatrix}$$

3. 有 $f\left(t,t\right)$

$$\left[-O'\left(T_0-t\right)+a\sum_{i=1}^{N}\begin{bmatrix} Q_{2,\mathrm{i}}\left(T_0-t\right)+Q_{3,i}\left(T_0-t\right) \\ +\dfrac{1}{2}\begin{bmatrix} N_i^2\left(T_0-t\right)\sigma_i^2\rho_i+R_i^2\left(T_0-t\right) \\ +O^2\left(T_0-t\right)\alpha_{0,i}^2+\left(u^2-u\right)B_{x,i}{}^2\left(T_1-t\right) \\ +\left(\left(1-u\right)^2-\left(1-u\right)\right)B_{x,i}{}^2\left(T_0-t\right) \end{bmatrix} \\ +N_i\left(T_0-t\right)R_i\left(T_0-t\right)\sigma_i\rho_i \\ +N_i\left(T_0-t\right)O\left(T_0-t\right)\sigma_i\rho_i\alpha_{0,i} \\ +N_i\left(T_0-t\right)u\mathrm{B}_{x,i}(T_1-t)\sigma_i\rho_i \\ +N\left(T_0-t\right)\left(1-u\right)\mathrm{B}_{x,i}(T_0-t)\sigma_i\rho_i \\ +O\left(T_0-t\right)R_i\left(T_0-t\right)\alpha_{0,i}+uO\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-t) \\ +\left(1-u\right)O\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_0-t)+uR_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_1-t) \\ +(1-u)R_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_0-t) \\ +u(1-u)\mathrm{B}_{x,i}(T_1-t)\mathrm{B}_{x,i}(T_0-t) \end{bmatrix}\right]f\left(t,t\right)=0$$

$$\Rightarrow O'\left(T_0-t\right)=-\gamma_i O\left(T_0-t\right)+a\sum_{i=1}^{N}\begin{bmatrix} Q_{2,\mathrm{i}}\left(T_0-t\right)+Q_{3,i}\left(T_0-t\right) \\ +\frac{1}{2}\begin{bmatrix} N_i^2\left(T_0-t\right)\sigma_i^2\rho_i+R_i^2\left(T_0-t\right) \\ +O^2\left(T_0-t\right)\alpha_{0,i}^2+\left(u^2-u\right)B_{x,i}{}^2\left(T_1-t\right) \\ +\left(\left(1-u\right)^2-\left(1-u\right)\right)B_{x,i}{}^2\left(T_0-t\right) \end{bmatrix} \\ +N_i\left(T_0-t\right)R_i\left(T_0-t\right)\sigma_i\rho_i \\ +N_i\left(T_0-t\right)O\left(T_0-t\right)\sigma_i\rho_i\alpha_{0,i} \\ +N_i\left(T_0-t\right)u\mathrm{B}_{x,i}(T_1-t)\sigma_i\rho_i \\ +N\left(T_0-t\right)\left(1-u\right)\mathrm{B}_{x,i}(T_0-t)\sigma_i\rho_i \\ +O\left(T_0-t\right)R_i\left(T_0-t\right)\alpha_{0,i}+uO\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_1-t) \\ +\left(1-u\right)O\left(T_0-t\right)\alpha_{0,i}\mathrm{B}_{x,i}(T_0-t)+uR_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_1-t) \\ +(1-u)R_i\left(T_0-t\right)\mathrm{B}_{x,i}(T_0-t) \\ +u(1-u)\mathrm{B}_{x,i}(T_1-t)\mathrm{B}_{x,i}(T_0-t) \end{bmatrix}$$

4. 有 $x_i(t)$

$$\sum_{i=1}^{N}\left[-R_i'(T_0-t)+Q_{1,\mathrm{i}}\left(T_0-t\right)+\alpha_{1,i}O\left(T_0-t\right)-\gamma_i R_{i,i}\left(T_0-t\right)\right]x_i(t)=0$$
$$\Rightarrow R_i'(T_0-t)=Q_{1,\mathrm{i}}\left(T_0-t\right)+\alpha_{1,i}O\left(T_0-t\right)-\gamma_i R_{i,i}\left(T_0-t\right)$$

5. 有 $\phi_{1,i}(t)$

$$\sum_{i=1}^{N}\left[-Q_{1,i}'(T_0-t)-\gamma_i Q_{1,\mathrm{i}}\left(T_0-t\right)\right]\phi_{1,i}(t)=0$$
$$\Rightarrow Q_{1,i}'(T_0-t)=-\gamma_i Q_{1,\mathrm{i}}\left(T_0-t\right)$$

6. 有 $\phi_{2,i}(t)$

$$\sum_{i=1}^{N}\left[-Q_{2,i}'(T_0-t)-\gamma_i Q_{2,\mathrm{i}}\left(T_0-t\right)+Q_{4,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{2,i}\right]\phi_{2,i}(t)=0$$
$$\Rightarrow Q_{2,i}'(T_0-t)=-\gamma_i Q_{2,\mathrm{i}}\left(T_0-t\right)+Q_{4,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{2,i}$$

7. 有 $\phi_{3,i}(t)$

$$\sum_{i=1}^{N}\left[-Q_{3,i}'(T_0-t)-2\gamma_i Q_{3,i}\left(T_0-t\right)+Q_{5,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{3,i}\right]\phi_{3,i}(t)=0$$
$$\Rightarrow Q_{3,i}'(T_0-t)=-2\gamma_i Q_{3,i}\left(T_0-t\right)+Q_{5,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{3,i}$$

8. 有 $\phi_{4,i}(t)$

$$\sum_{i=1}^{N}\left[-Q'_{4,i}(T_0-t)-\gamma_i Q_{4,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{4,i}\right]\phi_{4,i}(t)=0$$
$$\Rightarrow Q'_{4,i}(T_0-t)=-\gamma_i Q_{4,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{4,i}$$

9. 有 $\phi_{5,i}(t)$

$$\sum_{i=1}^{N}\left[-Q'_{5,i}(T_0-t)-2\gamma_i Q_{5,i}\left(T_0-t\right)+2Q_{6,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{5,i}\right]\phi_{5,i}(t)=0$$
$$\Rightarrow Q'_{5,i}(T_0-t)=-2\gamma_i Q_{5,i}\left(T_0-t\right)+2Q_{6,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{5,i}$$

10. 有 $\phi_{6,i}(t)$

$$\sum_{i=1}^{N}\left[-Q'_{6,i}(T_0-t)-2\gamma_i Q_{6,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{6,i}\right]\phi_{6,i}(t)=0$$
$$\Rightarrow Q'_{6,i}(T_0-t)=-2\gamma_i Q_{6,i}\left(T_0-t\right)+O\left(T_0-t\right)D_{6,i}$$

附錄五

根據 Duffin, Pan, and singleton(2000)和 Collin-Dufresne and Goldstein(2003)。在時間t對到期執行時間T_0的歐式賣權$Put\ (t,T_0,T_1,K)$作定價，執行價格K，其中標的資產為到期時間T_1的零息債券$P(t,T_1)$。則

$$\begin{aligned}
Put\ (t,T_0,T_1,K) &= E_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\left(K-P(T_0,T_1)\right)1_{P(T_0,T_1)<K}\right] \\
&= \mathrm{K}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds}1_{\log(P(T_0,T_1))<\log(K)}\right]-E_t^Q\left[e^{-\int_t^{T_0} r(s)ds}e^{\log(P(T_0,T_1))}1_{\log(P(T_0,T_1))<\log(K)}\right] \\
&= KG_{0,1}\left(\log K\right)-G_{1,1}\left(\log K\right)
\end{aligned}$$

其中$G_{a,b}\left(y\right)=E_t^Q\left[e^{-\int_t^{T_0} r(s)ds}e^{a\log(P(T_0,T_1))}1_{b\log(P(T_0,T_1))<y}\right]$

若要計算$G_{a,b}\left(y\right)$，我們將應用傅立葉轉換

$$\begin{aligned}
\mathcal{G}_{a,b}(y) &= \int_{\mathbb{R}} e^{iuy}dG_{a,b}\left(y\right) \\
&= E_t^Q\left[e^{-\int_t^{T_0} r_s ds}e^{a+\mathrm{i}ub\log(P(T_0,T_1))}\right] \\
&= \psi(a+\mathrm{i}ub,t,T_0,T_1)
\end{aligned}$$

其中$i=\sqrt{-1}$。接下來應用反傅立葉轉換(the Fourier inversion theorm)，

$$G_{a,b}\left(y\right)=\frac{\psi(a,t,T_0,T_1)}{2}-\frac{1}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\left[\psi(a+iub,t,T_0,T_1)e^{iuy}\right]}{u}du$$