國立臺灣大學理學院數學所

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master Thesis

實二次域上的 p-進 L-函數的模符號構造

Modular Symbols Construction of the p-adic L-functions over Real Quadratic Fields

張誌麟

Zhi-Lin Zhang

指導教授：謝銘倫 教授

Advisor: Ming-Lun Hsieh, Professor

中華民國 111 年 6 月

June 2022

doi:10.6342/NTU202201144

Acknowledgements

My deepest gratitude goes first and foremost to my advisor, Professor Ming-Lun Hsieh, for his invaluable assistance and guidance. He motivated and inspired me to have the right attitude as a researcher. He provided enlightening, informative instruction and critically read the thesis. Second, I would like to express my heartfelt gratitude to Professor Pin-Chi Hung for his numerous constructive comments and helpful discussion. Last but not least, my thanks would go to Yi-Jing Tseng for his support on numerical verification.

doi:10.6342/NTU202201144

摘要

在 [DD06] 中，作者給出了在 $\mathbf{Q}_p^2 - \{(0,0)\}$ 上的一個 p 進測度來定義 Λ 進艾森斯坦模符號，並當 p 是慣性於 F 時，通過計算 Λ 進艾森斯坦模符號在附加於理想類群的閉鏈上的值來構造實二次域 F 上的 p 進 L 函數。我們將此結果推廣到 p 是分裂於 F 的情況，並藉由計算艾森斯坦級數的周期積分來獲得 p 進 L 函數的插值公式。

Abstract

In [DD06], the authors proposed a family of p-adic measures on $\mathbf{Q}_p^2 - \{(0,0)\}$ to define the Λ-adic Eisenstein modular symbol, and constructed the p-adic L-function for a real quadratic field F by evaluating the Λ-adic Eisenstein modular symbol at cycles attached to ideal classes of F for p inert in F. We generalize this result to include the case that p is split in F, and to provide explicit interpolation formulae of the p-adic L-function for F by evaluating explicitly the period integral of Eisenstein series over real quadratic fields.

doi:10.6342/NTU202201144

Contents

doi:10.6342/NTU202201144

1 Introduction

1.1 Motivation

Consider the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges absolutely for $\Re(s) > 1$ and admits a meromorphic continuation to $\mathbf{C}$ with only a simple pole at $s = 1$. The values of $\zeta(s)$ at non-positive integers $1 - n \leq 0$ are given by

$$\zeta(1-n) = -\frac{B_n}{n} \in \mathbf{Q},$$

where B_n is the n-th Bernoulli number. In 1960s, Kubota and Leopoldt showed that these values can be p-adically interpolated:

Theorem 1.1.1. There exists a unique continuous function $\zeta_p(s) : \mathbf{Z}_p \to \mathbf{Q}_p$ that is analytic, except at $s = 1$, such that

$$\zeta_p(1-n) = (1 - p^{n-1}) \cdot \zeta(1-n)$$

for every positive integer n with $n \equiv 0 \pmod{p-1}$.

The function $\zeta_p(s)$ is called the Kubota-Leopoldt p-adic zeta function. Generalizations of the above result to Dedekind zeta function for totally real fields are given by [DR80], [Bar78], and [CN79] in the 1970s, and more recently given by [CD14], [Spi14], and [BKL18]. We remark that the above results used different constructions of the p-adic Dedekind zeta function for totally real fields and different

doi:10.6342/NTU202201144

constructions should give different arithmetic applications. For example, [DK] used their construction to give a p-adic analytic solution to the Hilbert 12th problem.

When F is a real quadratic field, [DD06] consider the modular symbols construction of the p-adic L-function. To be more precise, they constructed Λ-adic modular symbols of Eisenstein series and the p-adic L-function is obtained by evaluating at the Λ-adic modular symbols at cycles corresponding to the ideal classes of F. In their construction, they assume p is inert in the real quadratic field F. The purpose of this thesis is to generalize the result in [DD06]

1. to include the case that p is split in F, and
2. to provide explicit interpolation formulae of the p-adic L-function for F.

We briefly outline the construction of the p-adic L-function. Let χ be an odd ring class character of F unramified everywhere. Let

$$E_k(z) := \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \ q := e^{2\pi i z}$$

be the classical Eisenstein series of even weight k and level $\mathrm{SL}_2(\mathbf{Z})$. Choosing an auxiliary positive integer N and a formal divisor with coefficients in $\mathbf{Z}$

$$\sum_{d|N} n_d [d]$$

such that $\sum_{d|N} n_d = 0$ and $\sum_{d|N} n_d d = 0$, we consider, for each even integers $k \geq 2$, the modified Eisenstein series

$$F_k(z) := -24 \sum_{d|N} n_d \cdot d \cdot E_k(dz)$$

doi:10.6342/NTU202201144

attached to $\sum_{d|N} n_d[d]$. This modified Eisenstein series are of weight k and level $\Gamma_0(N)$ and have no constant term at the cusp ∞, so they are holomorphic at the cusp ∞. For each positive even integer k we will define

$$P(k,\chi) = \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{k-2} \int_{\infty}^{r\infty} F_k^{**}(z) Q(z,1)^{\frac{k-2}{2}} dz$$

where r is an element in $\Gamma_0(N)$, Q is a homogeneous polynomial of degree 2 with coefficients in $\mathbf{Q}$, and F_k^{**} is a modification of F_k such that the integral converges. By using the Λ-adic modular symbol in [DD06], for each a in the narrow class group $\mathrm{Cl}(F)^+$, there exists a p-adic analytic function $\zeta_p(s,a)$ such that

$$\zeta_p(k,a) = \int_{\infty}^{r\infty} F_k^{**}(z) Q(z,1)^{\frac{k-2}{2}} dz$$

for every positive even integer k with $\frac{k-2}{2} \equiv 0 \pmod{p-1}$. Let

$$\mathbf{P}(s,\chi) = \sum_{a\in \mathrm{Cl}^+(F)} \chi(a) \langle \mathrm{N}(a)\rangle^{s-2} \zeta_p(s,a).$$

The following theorem is our main result:

Theorem. Let χ be an odd ring class character of F that is unramified everywhere, and let $L(s,\chi)$ be the complex L-function of χ. Then

$$\mathbf{P}(s,\chi) \sim L_p(1-\frac{s}{2},\chi).$$

where $L_p(s,\chi)$ is the Kubota-Leopoldt p-adic L-function associated to χ.

The proof of this main result is to interpolate $P(s,\chi)$ as an period integral of

doi:10.6342/NTU202201144

Eisentein series over real quadratic fields. To this end, we need to use the language of automorphic forms. Let φ_k be the adelic lift of F_k^{**}, namely, φ_k is an automorphic form $\mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ such that for any z in the upper half-plane,

$$\varphi_k \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = F_k^{**}(z), \quad z = x + iy.$$

Fix a suitable $\mathbf{Q}$-embedding $\Psi : F \to M_2(\mathbf{Q})$. We show that the *period integral*

$$\int_{\mathbf{A}^\times F^\times \backslash \mathbf{A}_F^\times} \varphi_k(\Psi(t))\chi(t) d^\times t$$

equals to $\mathbf{P}(k, \chi)$. On the other hand, by standard unfolding, the period integral is a product of local Tate integrals on $F_v^\times$. We carry out the explicit computation of each local integral and conclude that

$$\int_{\mathbf{A}^\times F^\times \backslash \mathbf{A}_F^\times} \varphi_k(\Psi(t))\chi(t) d^\times t \sim L(1 - \frac{k}{2}, \chi).$$

for any $\frac{k-2}{2} \equiv 0 \pmod{p-1}$.

1.2 Notation and Definitions

In this subsection, we set up the notation for a general totally real number field E that will be frequently used. After this, we fix the real quadratic field F, the ring class character χ of F and the auxiliary prime q throughout the contexts.

doi:10.6342/NTU202201144

1.2.1 Number fields

We fix some notation and definitions for E. For a place v of E, let E_v be the v-completion of E, $\mathcal{O}_{E_v}$ the ring of integer of E_v and q_{E_v} the order of the residue field of $\mathcal{O}_{E_v}$. Let $\mathbf{A}_E$ be the adele of E. Denote $x \mapsto x_v$ and $x \mapsto (x_v)_v$ the embeddings $E \hookrightarrow E_v$ and $E \hookrightarrow \mathbf{A}_E$, respectively. For a positive integer m, denote $S(E_v^m)$, $S((E_v^\times)^m)$, $S(\mathbf{A}_E^m)$ and $S((\mathbf{A}_E^\times)^m)$ the spaces of Bruhat-Schwartz functions on E_v^m, $(E_v^\times)^m$, $\mathbf{A}_E^m$ and $(\mathbf{A}_E^\times)^m$, respectively.

1.2.2 Matrix groups

Define the compact subgroup $K = \prod_v K_v$ of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ by $K_\infty = \mathrm{O}_2(\mathbf{R})$ and $K_v = \mathrm{GL}_2(\mathbf{Z}_v)$. Let B and B_v be the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q})$ and $\mathrm{GL}_2(\mathbf{Q}_v)$ respectively. Let $\widehat{\mathbf{Z}} = \prod_v \mathbf{Z}_v$ be the profinite integer. For a positive integer N, we define the following compact open subgroups of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ by

$$U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : N \text{ divides } c \right\},$$

$$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : N \text{ divides } c \text{ and } d-1 \right\},$$

$$U_2(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : N \text{ divides } b,\ c \text{ and } d-1 \right\}.$$

We have the strong approximation theorem $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}) = \mathrm{GL}_2(\mathbf{Q})^+ \,\mathrm{GL}_2(\mathbf{R}) U_2(N)$ for every positive integer N (cf. [KL06, Theorem 6.8]).

doi:10.6342/NTU202201144

1.2.3 Additive characters

Let $\psi : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$ be the additive character whose archimedean component is $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$ and whose local component at v is denoted by $\psi_v : \mathbf{Q}_v \to \mathbf{C}^\times$. We define the additive character $\psi_E = \prod_v \psi_{E_v} : \mathbf{A}_E/E \to \mathbf{C}^\times$ by setting $\psi_E := \psi \circ \mathrm{Tr}_{E/\mathbf{Q}}$.

1.2.4 Haar measures

We shall normalize the Haar measures on E_v and $E_v^\times$ as follows. Let dx_v be the self-dual Haar measures of E_v with respect to ψ_{E_v}. Put $d^\times x_v = \zeta_{E_v}(1)\frac{dx_v}{|x_v|_{E_v}}$. The Tamagawa measure of $\mathbf{A}_F$ is $dx = \prod_v dx_v$ while the Tamagawa measure of $\mathbf{A}_E^\times$ is defined by $d^\times x = \prod_v dx_v^\times$.

1.2.5 The real quadratic field

We fix the following notation throughout the contexts. Let $F = \mathbf{Q}(\sqrt{D})$ be a real quadratic field with D a square-free positive integer. Let p be a rational prime split or inert in F. We fix an auxiliary prime q different from p and split in F. We fix a formal divisor $\sum_{d|q^2} n_d[d]$ such that

$$\sum_{d|q^2} n_d = 0 \quad \text{and} \quad \sum_{d|q^2} n_d d = 0. \tag{1.2.6}$$

We remark that q^2 plays the role of the auxiliary integer N introduced in (1.1). Let $\chi = \prod_v \chi_v : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ be an *odd ring class character*, that is, a finite order idele class character with its two archimedean components χ_{∞_1}, χ_{χ_2} both equal to the sign character. We assume χ is unramified everywhere. We fix a positive integer k such that $k \equiv 2 \pmod 4$. Let $\theta = \frac{D+\sqrt{D}}{2}$, so $\mathcal{O}_F = \mathbf{Z}[\theta]$. Let $\Psi : F \to M_2(\mathbf{Q})$ be

doi:10.6342/NTU202201144

the $\mathbf{Q}$-embedding defined by

$$\Psi(\theta) = \begin{pmatrix} \mathrm{Tr}(\theta) & -N(\theta) \\ 1 & 0 \end{pmatrix}$$

and is extended $\mathbf{Q}$-linearly. We fix a field isomorphism $\iota_p : \mathbf{C} \to \mathbf{C}_p$.

1.3 Automorphic Forms and Modular Forms

In this subsection, we recall the basic definitions and relations between of automorphic forms and modular forms.

1.3.1 Automorphic forms

A function $\varphi : \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}) \to \mathbf{C}$ is an *automorphic form* of weight k and level N if φ is a smooth, K-finite, $Z(U(\mathfrak{g}))$-finite, slowly increasing function such that

$$\varphi(z\gamma g k_\theta u_{\mathrm{f}}) = \varphi(g)e^{\sqrt{-1}k\theta}, \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for $z \in \mathbf{A}^\times, \gamma \in \mathrm{GL}_2(\mathbf{Q}), \theta \in \mathbf{R}$ and $u_{\mathrm{f}} \in U_2(N)$. Denote $\mathcal{A}_k(N)$ the space of all automorphic forms of weight k and level N.

1.3.2 Adelic modular forms

The group $\mathrm{GL}_2(\mathbf{R})^+ := \{g \in \mathrm{GL}_2(\mathbf{R}) \mid \det g > 0\}$ acts on the complex upper half-plane $\mathbf{H}$ and the automorphy factor is given by

$$\gamma(z) = \frac{az+b}{cz+d}, \quad J(\gamma, z) = cz + d$$

doi:10.6342/NTU202201144

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})^+$ and $z \in \mathbf{H}$. A function $f : \mathbf{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}) \to \mathbf{C}$ is a *modular form* of weight k and level N if f is a smooth, slowly increasing function such that

$$f(\gamma z, \gamma g_{\mathrm{f}} u_{\mathrm{f}}) = \det(\gamma)^{-1} J(\gamma, z)^k f(z, g_{\mathrm{f}})$$

for $\gamma \in \mathrm{GL}_2(\mathbf{Q})^+$, $z \in \mathbf{H}$, $g_{\mathrm{f}} \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ and $u_{\mathrm{f}} \in U_2(N)$. Let $\mathcal{N}_k(N)$ be the space of all modular forms of weight k and level N. To every modular form $f \in \mathcal{N}_k(N)$ we associate a unique automorphic form $\varphi_f \in \mathcal{A}_k(N)$ defined by the formula

$$\varphi_f(g) \coloneqq f\left(g_\infty(\sqrt{-1}), g_{\mathrm{f}}\right) J\left(g_\infty, \sqrt{-1}\right)^{-k} (\det g_\infty) \, |\det g|_{\mathbf{A}}^{\frac{k}{2}-1} \tag{1.3.3}$$

for $g = g_\infty g_{\mathrm{f}} \in \mathrm{GL}_2(\mathbf{R})\mathrm{GL}_2(\widehat{\mathbf{Q}})$ (cf. [Cas73, §3]). Conversely, we can recover the form f from φ_f by

$$f\left(x + \sqrt{-1}y, g_{\mathrm{f}}\right) = y^{-k/2} \varphi_f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g_{\mathrm{f}}\right) |\det g_{\mathrm{f}}|_{\mathbf{A}}^{1-\frac{k}{2}}. \tag{1.3.4}$$

for $z = x + \sqrt{-1}y \in \mathbf{H}$ and $g_{\mathrm{f}} \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$. We call φ_f the *adelic lift* of f.

1.3.5 Classical modular forms

Let $f(z) : \mathbf{H} \to \mathbf{C}$ be a classical holomorphic modular form of weight k and level N. Then $f(z)$ extends to an element in $\mathcal{N}_k(N)$ by setting

$$f(z, \gamma u_{\mathrm{f}}) = f(\gamma^{-1} z) J(\gamma^{-1}, z)^{-k} \det(\gamma)^{-1}, \quad z \in \mathbf{H}, \quad \gamma \in \mathrm{GL}_2(\mathbf{R})^+, \quad u_{\mathrm{f}} \in U_2(N).$$

doi:10.6342/NTU202201144

Conversely, if $f(z, g_f)$ is an element in $\mathcal{N}_k(N)$, then $f(z, 1)$ is a classical holomorphic modular form of weight k and level N.

1.4 Eisenstein Series

In this subsection, we recall the well-known construction of adelic Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$. Almost all of the material in this subsection is standard; we refer the reader to [Bum97].

1.4.1 Induced representations

Let v be a place of $\mathbf{Q}$. The *induced representation* $I(s, 1_v, 1_v)$ consists of all smooth K_v-finite functions $f : \mathrm{GL}_2(\mathbf{Q}_v) \to \mathbf{C}$ such that

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k\right) = \left|\frac{a}{d}\right|^{s+\frac{1}{2}} f(k), \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_v, \quad k \in K_v.$$

$\mathrm{GL}_2(\mathbf{Q}_v)$ acts on $I(s, 1_v, 1_v)$ by the right multiplication ρ. For a compact open subgroup U of $\mathrm{GL}_2(\mathbf{Q}_v)$, we denote $I(s, 1_v, 1_v)^U$ the space of elements in $I(s, 1_v, 1_v)$ fixed by U. By a *spherical vector* in $I(s, 1_v, 1_v)$ we mean a non-zero element in the 1-dimensional subspace $I(s, 1_v, 1_v)^{K_v}$ of $I(s, 1_v, 1_v)$.

1.4.2 Hecke operators

Let v be a place of $\mathbf{Q}$. The *Hecke operator* attached to a compact open subgroup U of G and $g \in \mathrm{GL}_2(\mathbf{Q}_v)$ is defined by

$$[UgU] : I(s, 1_v, 1_v)^U \to I(s, 1_v, 1_v)^U, \quad f \mapsto \frac{1}{\mathrm{vol}(U)} \int_{UgU} \rho(h) f dh.$$

doi:10.6342/NTU202201144

If $UgU = \bigsqcup_{i=1}^n g_iU$ with $g_i \in \mathrm{GL}_2(\mathbf{Q}_v)$ then $[UgU]f = \sum_{i=1}^n \rho(g_i)f$. Define the compact open subgroup

$$I_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_v) : v \text{ divides } c \right\}$$

of $\mathrm{GL}_2(\mathbf{Q}_v)$. Define the $\mathbf{U}_v$-operator and T_v-operator by

$$\mathbf{U}_v = \left[I_v \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} I_v \right] = \sum_{x=1}^{v-1} \rho \begin{pmatrix} v & x \\ 0 & 1 \end{pmatrix}, \; T_p = \left[K_v \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} K_v \right] = \mathbf{U}_p + \rho \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}. \tag{1.4.3}$$

1.4.4 Godement sections

Let v be a place of $\mathbf{Q}$. Define $S_0(\mathbf{Q}_v^2) = S(\mathbf{Q}_v^2)$ if $v < \infty$, and

$$S_0(\mathbf{R}^2) = \{P(x,y)e^{-\pi(x^2+y^2)} \in S(\mathbf{R}^2) : P(x,y) \in \mathbf{C}[x,y]\}.$$

For $\Phi_v \in S_0(\mathbf{Q}_v^2)$, we defined the associated Godement section $f_{\Phi_v,s} : \mathrm{GL}_2(\mathbf{Q}_v) \to \mathbf{C}$ by

$$f_{\Phi_v,s}(g) = |\det g|_v^{s+\frac{1}{2}} \int_{\mathbf{Q}_v^\times} \Phi_v((0,t_v)g)|t_v|_v^{2s+1} d^\times t_v, \quad g \in \mathrm{GL}_2(\mathbf{Q}_v),$$

which converges absolutely for $2s+1>0$. In this case, the map

$$S_0(\mathbf{Q}_v^2) \to I(s, 1_v, 1_v), \quad \Phi_v \mapsto f_{\Phi_v,s}$$

is surjective.

doi:10.6342/NTU202201144

1.4.5 Adelic Eisenstein series

For $\Phi = \bigotimes_v \Phi_v \in S(\mathbf{A}_{\mathbf{Q}}^2)$ with $\Phi_v \in S_0(\mathbf{Q}_v^2)$ for each place v of $\mathbf{Q}$, define $f_{\Phi,s} = \bigotimes_v f_{\Phi_v,s}$ and

$$E_{\Phi,s} : \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}) \to \mathbf{C}, \quad E_{\Phi,s}(g) = \sum_{\gamma \in B(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{Q})} f_{\Phi,s}(\gamma g), \quad g \in \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}),$$

which converges absolutely for $\Re s \gg 0$ and has meromorphic continuation to $s \in \mathbf{C}$ with at most a simple pole at $s = \pm\frac{1}{2}$. For $\alpha \in \mathbf{Q}$, we define the Whittaker function

$$W_\alpha(g, f_{\Phi,s}) = f(g) + \int_{\mathbf{A}_{\mathbf{Q}}} f_{\Phi,s}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\psi(-\alpha x)dx$$

which converges for $\Re s \gg 0$, where the term $f(g)$ is omitted if $\alpha \neq 0$. We have the Fourier expansion

$$E_{\Phi,s}(g) = \sum_{\alpha \in \mathbf{Q}} W_\alpha(g, f_{\Phi,s}), \quad g \in \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}) \tag{1.4.6}$$

for $\Re s \gg 0$.

2 Period Integral of Eisenstein series

2.1 Period integral as a Product of Local Tate Integrals

In this subsection, we introduce the period integral of Eisenstein series, which can be written as a product of local Tate integrals. We will compute these locals integral in the next subsection.

doi:10.6342/NTU202201144

Definition 2.1.1. Let $\Phi \in S_0(\mathbf{A}_{\mathbf{Q}}^2)$. Define the *period integral*

$$P_{\Phi,s,\chi}(g) = \int_{F^\times \mathbf{A}^\times \backslash \mathbf{A}_F^\times} E_{\Phi,s}(\Psi(z)g)\chi(z)d^\times z, \quad g \in \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}),$$

which converges absolutely for $\Re s \gg 0$ by the following lemma (2.1.2):

Lemma 2.1.2. $F^\times \mathbf{A}_{\mathbf{Q}}^\times \backslash \mathbf{A}_F^\times$ is compact.

Proof. Let $(a_v)_v \in \mathbf{A}_K^\times$, so $c \coloneqq |(a_v)_v|_{\mathbf{A}_K} > 0$ and

$$(a_v)_v = (a_{\infty,1}, a_{\infty,2}, a_{\mathrm{f}}) = (\frac{a_{\infty,1}}{\sqrt{c}}, \frac{a_{\infty,2}}{\sqrt{c}}, a_{\mathrm{f}}) \cdot (\sqrt{c}, \sqrt{c}, 1) \in \mathbf{I}_F^1 \mathbf{A}_{\mathbf{Q}}^\times,$$

where $\mathbf{I}_F^1$ is the norm one idele ring of K. Hence $F^\times \mathbf{A}_{\mathbf{Q}}^\times \setminus \mathbf{A}_F^\times = F^\times \mathbf{A}_{\mathbf{Q}}^\times \setminus \mathbf{I}_F^1 \mathbf{A}_{\mathbf{Q}}^\times$. The quotient map

$$\mathbf{I}_F^1 \mathbf{A}_{\mathbf{Q}}^\times \to F^\times \mathbf{A}_{\mathbf{Q}}^\times \setminus \mathbf{I}_F^1 \mathbf{A}_{\mathbf{Q}}^\times$$

is restricted to the continuous function $\mathbf{I}_F^1 \to F^\times \mathbf{A}_{\mathbf{Q}}^\times \setminus \mathbf{I}_F^1 \mathbf{A}_{\mathbf{Q}}^\times$. By the universal property of quotient space, the induced function

$$F^\times \setminus \mathbf{I}_F^1 \to F^\times \mathbf{A}_{\mathbf{Q}}^\times \setminus \mathbf{I}_F^1 \mathbf{A}_{\mathbf{Q}}^\times$$

is continuous. Thus $F^\times \mathbf{A}_{\mathbf{Q}}^\times \setminus \mathbf{I}_F^1 \mathbf{A}_{\mathbf{Q}}^\times$ is a continuous image of a compact set and so compact. $\square$

Definition 2.1.3. Let $\Phi \in S_0(\mathbf{A}_{\mathbf{Q}}^2)$. For each place v of $\mathbf{Q}$ and $g_v \in \mathrm{GL}_2(\mathbf{Q}_v)$, define the *local integral*

$$l_{\Phi_v,s,\chi_v}(g_v) = \int_{F_v^\times} \Phi_v((0,1)\Psi_v(z_v)g_v)\chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v$$

doi:10.6342/NTU202201144

which is a Tate integral and so converges absolutely for $\Re(s) \gg 0$.

Proposition 2.1.4. Let $\Phi \in S_0(\mathbf{A}_{\mathbf{Q}}^2)$ and $g \in \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$. Then

$$P_{\Phi,s,\chi}(g) = \prod_v l_{\Phi_v,s,\chi_v}(g_v)$$

for $\Re(s) \gg 0$.

Proof. This is proved in [Hsi12, §5.1]. □

2.2 Computation of Local Integrals

In this subsection, we compute the local integrals $l_{\Phi_v,s,\chi_v}(\varsigma_v)$ attached to a distinguished element $\varsigma \in \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ and Bruhat-Schwartz function Φ on $\mathbf{A}_{\mathbf{Q}}^2$. To compute the local integrals, we first fix an integral basis for $\mathcal{O}_{F_v}$ for each place v.

2.2.1 Integral Bases

Let v be a finite place of $\mathbf{Q}$. Let $F_v = F \otimes_{\mathbf{Q}} \mathbf{Q}_v$. We fix an ordered basis $\{e_{1,v}, e_{2,v}\}$ for $\mathcal{O}_{F_v} = \mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_v$ over $\mathbf{Z}_v$ as follows: If v is non-split, then $\{e_{1,v}, e_{2,v}\} = \{\theta_v, 1\}$, where $\theta = (D + \sqrt{D})/2 \in F$. If v is split in F then we may write $v = \overline{w}w$ for some places $\overline{w}$, w of F, and there is an $\mathbf{Q}_v$-algebra isomorphism defined by $F \otimes_{\mathbf{Q}} \mathbf{Q}_v \to F_{\overline{w}} \oplus F_w$, $a_v\theta + b_v \mapsto (a_v\overline{\theta} + b_v, a_v\theta + b_v)$ for a_v, $b_v \in \mathbf{Q}_v$. Denote $e_{\overline{w}}$ and e_w the elements in $F \otimes_{\mathbf{Q}} \mathbf{Q}_v$ such that $e_{\overline{w}} \mapsto (1, 0)$, $e_w \mapsto (0, 1)$. We choose $\{e_{1,v}, e_{2,v}\} = \{e_{\overline{w}}, e_w\}$ if v is split. For $z_v \in F \otimes_{\mathbf{Q}} \mathbf{Q}_v$, there is a unique expression $z_v = z_{\overline{w}}e_{\overline{w}} + z_we_w$ with $z_{\overline{w}}$, $z_w \in \mathbf{Q}_v$. The conjugate map on F is extended to $F \otimes_{\mathbf{Q}} \mathbf{Q}_v \to F \otimes_{\mathbf{Q}} \mathbf{Q}_v$ by $\overline{z}_v = z_we_{\overline{w}} + z_{\overline{w}}e_w$.

doi:10.6342/NTU202201144

2.2.2 Choice of ς

Define

$$\varsigma = \prod_v \varsigma_v \in \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}), \quad \varsigma_v = \begin{cases} \dfrac{-1}{2\theta}\begin{pmatrix} \overline{\theta} & \theta \\ 1 & 1 \end{pmatrix} & \text{if } v \text{ is split,} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

The choice of ς is motivated by the fact that if $v = \overline{w}w$ is a place of $\mathbf{Q}$ split in F, then

$$\varsigma_v^{-1}\Psi_v(ae_{\overline{w}} + be_w)\varsigma_v = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbf{Q}_v. \tag{2.2.3}$$

2.2.4 Choice of the Bruhat-Schwartz functions Φ on $\mathbf{A}_\mathbf{Q}^2$

Define the subset $\mathbf{X}_p$ of $\mathbf{Q}_p^2$ by

$$\mathbf{X}_p = \begin{cases} \mathbf{Z}_p^\times \oplus \mathbf{Z}_p^\times & \text{if } p \text{ is split,} \\ \mathbf{Z}_p \oplus \mathbf{Z}_p^\times \sqcup \mathbf{Z}_p^\times \oplus p\mathbf{Z}_p & \text{if } p \text{ is inert.} \end{cases}$$

Let v be a place of $\mathbf{Q}$. Define the functions Φ_v in $S_0(\mathbf{Q}_v^2)$ by

$$\Phi_v(x, y) = \begin{cases} 2^{-k}(x + \sqrt{-1}y)^k e^{-\pi(x^2+y^2)} & \text{if } v = \infty, \\ \mathbb{I}_{\mathbf{X}_p}(x, y) & \text{if } v = p, \\ -24\displaystyle\sum_{d|q^2} dn_d|d|_v \mathbb{I}_{\mathbf{Z}_q\oplus\mathbf{Z}_q}(x, dy) & \text{if } v = q, \\ \mathbb{I}_{\mathbf{Z}_v\oplus\mathbf{Z}_v}(x, y) & \text{otherwise.} \end{cases}$$

doi:10.6342/NTU202201144

Define $\Phi = \prod_v \Phi_v \in S_0(\mathbf{A}_{\mathbf{Q}}^2)$.

Proposition 2.2.5. Let v be a place of $\mathbf{Q}$. Then

$$\begin{aligned} & l_{\Phi_v,s,\chi_v}(\varsigma_v)|_{s=\frac{1-k}{2}} \\ &= (|2\theta|_v^{2-k})^{\epsilon_v} \times \begin{cases} 2^{-1}(\sqrt{-1})^{\frac{k}{2}}. & \text{if } v = \infty, \\ 1 & \text{if } v = p, \\ L(1-\frac{k}{2},\chi_v)\left(-24\sum_{d|q^2} dn_d\chi_{\overline{w}}(d^{-1})|d^{-1}|_v^{1-\frac{k}{2}}\right) & \text{if } v = q = \overline{w}w, \\ L(1-\frac{k}{2},\chi_v) & \text{otherwise,} \end{cases} \end{aligned}$$

where $\epsilon_v = 0$ if v is non-split, and 1 if v is split.

Proof. We will use the integral basis chosen in (2.2.1) to compute the local integral. More precisely, we will use the fact that if $z_v = x_v\theta + y_v \in F_v^\times$ with x_v, $y_v \in \mathbf{Q}_v$, then

$$(0,1)\Psi_v(z_v)\varsigma_v = \begin{cases} \dfrac{-1}{2\theta}(z_{\overline{w}}, z_w) & \text{if } v = \overline{w}w \text{ is split,} \\ (x_v, y_v) & \text{otherwise.} \end{cases}$$

Let v be a finite place of $\mathbf{Q}$. If $v \neq p$ is split then

$$\begin{aligned} & \int_{F_v^\times} \mathbb{I}_{\mathbf{Z}_v\oplus\mathbf{Z}_v}((0,1)\Psi_v(z_v)\varsigma_v)\chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v \\ &= |2\theta|_v^{2s+1} \int_{F_v^\times} \mathbb{I}_{\mathcal{O}_{F_v}}\chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v \\ &= |2\theta|_v^{2s+1} L(s+\frac{1}{2},\chi_v). \end{aligned}$$

doi:10.6342/NTU202201144

If $v \neq p$ is non-split then

$$\begin{aligned}
&\int_{F_v^\times} \mathbb{I}_{\mathbf{Z}_v \oplus \mathbf{Z}_v}((0,1)\Psi_v(z_v)\varsigma_v)\chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v \\
=& \int_{F_v^\times} \mathbb{I}_{\mathcal{O}_{F_v}} \chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v \\
=& L(s+\frac{1}{2},\chi_v).
\end{aligned}$$

If $v = p$ is split then

$$\begin{aligned}
l_{\Phi_v,s,\chi_v} &= \int_{F_v^\times} \Phi_v^{(d)}((0,1)\Psi_v(z_v)\varsigma_v)\chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v \\
&= \int_{\mathbf{Q}_v^\times}\int_{\mathbf{Q}_v^\times} \Phi_v(\frac{-1}{2\theta}(z_{\overline{w}}, z_w))\chi_{\overline{w}}(z_{\overline{w}})\chi_w(z_w)|z_{\overline{w}}z_w|_v^{s+\frac{1}{2}} d^\times z_{\overline{w}} d^\times z_w \\
&= |2\theta|_v^{2s+1}\int_{\mathbf{Q}_v^\times}\int_{\mathbf{Q}_v^\times} \Phi_v(z_{\overline{w}}, z_w)\chi_{\overline{w}}(z_{\overline{w}})\chi_w(z_w)|z_{\overline{w}}z_w|_v^{s+\frac{1}{2}} d^\times z_{\overline{w}} d^\times z_w \\
&= |2\theta|_v^{2s+1}.
\end{aligned}$$

If $v = p$ is inert then

$$\begin{aligned}
&\int_{F_v^\times} \mathbb{I}_{p\mathbf{Z}_p \oplus p\mathbf{Z}_p}((0,1)\Psi_v(z_v)\varsigma_v)\chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v \\
=& \chi_v(p)|p\overline{p}|_p^{s+\frac{1}{2}}\int_{F_v^\times} \mathbb{I}_{\mathbf{Z}_p \oplus \mathbf{Z}_p}((0,1)\Psi_v(z_v)\varsigma_v)\chi_v(z_v)|z_v\overline{z}_v|_v^{s+\frac{1}{2}} d^\times z_v \\
=& \chi_v(p)|p|_{F_v}^{s+\frac{1}{2}} L(s+\frac{1}{2},\chi_v),
\end{aligned}$$

so

$$l_{\Phi_v,s,\chi_v} = L(s+\frac{1}{2},\chi_v) - \chi_v(p)|p|_{F_v}^{s+\frac{1}{2}} L(s+\frac{1}{2},\chi_v) = 1.$$

Now assume $v = q$. Let $\Phi_q^{(d)}(x,y) = \Phi_q^{(0)}(x,dy)$. Since $q = \overline{w}w$ is split by assump-

doi:10.6342/NTU202201144

tion,

$$
\begin{aligned}
& l_{\Phi_q^{(d)},s,\chi_q} \\
&= \int_{F_q^\times} \Phi_q^{(d)}((0,1)\Psi_q(z_q)\varsigma_q)\chi_q(z_q)|z_q\overline{z}_q|_q^{s+\frac{1}{2}} d^\times z_q \\
&= \int_{\mathbf{Q}_q^\times}\int_{\mathbf{Q}_q^\times} \Phi_q^{(d)}(\frac{-1}{2\theta}(z_{\overline{w}}, z_w))\chi_{\overline{w}}(z_{\overline{w}})\chi_w(z_w)|z_{\overline{w}}z_w|_q^{s+\frac{1}{2}} d^\times z_{\overline{w}} d^\times z_w \\
&= |2\theta|_q^{2s+1}\int_{\mathbf{Q}_q^\times}\int_{\mathbf{Q}_q^\times} \Phi_q^{(d)}(z_{\overline{w}}, z_w)\chi_{\overline{w}}(z_{\overline{w}})\chi_w(z_w)|z_{\overline{w}}z_w|_q^{s+\frac{1}{2}} d^\times z_{\overline{w}} d^\times z_w \\
&= |2\theta|_q^{2s+1}\int_{\mathbf{Q}_q^\times} \mathbb{I}_{\mathbf{Z}_q}(z_{\overline{w}})\chi_{\overline{w}}(z_{\overline{w}})|z_{\overline{w}}|_q^{s+\frac{1}{2}} d^\times z_{\overline{w}} \int_{\mathbf{Q}_q^\times} \mathbb{I}_{\mathbf{Z}_q}(dz_w)\chi_{\overline{w}}(z_w)|z_w|_q^{s+\frac{1}{2}} d^\times z_w \\
&= |2\theta|_q^{2s+1}(\chi_{\overline{w}}(d)|d|_q^{s+\frac{1}{2}})^{-1}\int_{\mathbf{Q}_q^\times} \mathbb{I}_{\mathbf{Z}_q}(z_{\overline{w}})\chi_{\overline{w}}(z_{\overline{w}})|z_{\overline{w}}|_q^{s+\frac{1}{2}} d^\times z_{\overline{w}} \int_{\mathbf{Q}_q^\times} \mathbb{I}_{\mathbf{Z}_q}(z_w)\chi_{\overline{w}}(z_w)|z_w|_q^{s+\frac{1}{2}} d^\times z_w \\
&= |2\theta|_q^{2s+1}(\chi_{\overline{w}}(d)|d|_q^{s+\frac{1}{2}})^{-1} L(s+\frac{1}{2}, \chi_q).
\end{aligned}
$$

Hence

$$
l_{\Phi_q,s,\chi_q} = -24\sum_{d|q^2} dn_d l_{\Phi_q^{(d)},s,\chi_q} = |2\theta|_q^{2s+1} L(s+\frac{1}{2},\chi_q)\left(-24\sum_{d|q^2} dn_d\chi_{\overline{w}}|\cdot|_q^{s+\frac{1}{2}}(d^{-1})\right).
$$

For $v = \infty$,

$$
\begin{aligned}
& |2\theta|^{-2s} l_{\Phi_\infty,s-1/2,\chi_\infty} \\
&= \int_{\mathbf{R}^\times}\int_{\mathbf{R}^\times} 2^{-k}(x+\sqrt{-1}y)^k e^{-\pi(x^2+y^2)}\operatorname{sgn}(xy)|xy|^s d^\times x d^\times y \\
&= \sum_{m=0}^{k}\binom{k}{m} 2^{-k}(\sqrt{-1})^{k-m}\int_{\mathbf{R}^\times}\int_{\mathbf{R}^\times} x^m y^{k-m} e^{-\pi(x^2+y^2)}\operatorname{sgn}(xy)|xy|^s d^\times x d^\times y \\
&= \sum_{\substack{m=0 \\ m \text{ is odd}}}^{k}\binom{k}{m} 2^{-k}(\sqrt{-1})^{k-m}\int_{\mathbf{R}^\times}\int_{\mathbf{R}^\times} x^m y^{k-m} e^{-\pi(x^2+y^2)}\operatorname{sgn}(xy)|xy|^s d^\times x d^\times y \\
&= \sum_{\substack{m=0 \\ m \text{ is odd}}}^{k}\binom{k}{m} 2^{-k}(\sqrt{-1})^{k-m}\int_{\mathbf{R}^\times}\int_{\mathbf{R}^\times} |x|^m |y|^{k-m} e^{-\pi(x^2+y^2)}|xy|^s d^\times x d^\times y
\end{aligned}
$$

doi:10.6342/NTU202201144

$$= \sum_{\substack{m=0 \\ m \text{ is odd}}}^{k} \binom{k}{m} 2^{-k} (\sqrt{-1})^{k-m} \int_{\mathbf{R}^\times} \int_{\mathbf{R}^\times} |x|^{m+s} |y|^{k-m+s} e^{-\pi(x^2+y^2)} d^\times x d^\times y$$

$$= \sum_{\substack{m=0 \\ m \text{ is odd}}}^{k} \binom{k}{m} 2^{-k} (\sqrt{-1})^{k-m} \int_{\mathbf{R}^\times} |x|^{m+s} e^{-\pi x^2} d^\times x \int_{\mathbf{R}^\times} |y|^{k-m+s} e^{-\pi y^2} d^\times y$$

$$= \sum_{\substack{m=0 \\ m \text{ is odd}}}^{k} \binom{k}{m} 2^{-k} (\sqrt{-1})^{k-m} \Gamma_{\mathbf{R}}(m+s) \Gamma_{\mathbf{R}}(k-m+s)$$

$$= 2^{-k} (\sqrt{-1})^{k} \pi^{-s-\frac{k}{2}} \sum_{\substack{m=0 \\ m \text{ is odd}}}^{k} \binom{k}{m} (\sqrt{-1})^{-m} \Gamma(\frac{m+s}{2}) \Gamma(\frac{k-m+s}{2}).$$

Write $k = 2k'$. Then

$$\sum_{\substack{m=0 \\ m \text{ is odd}}}^{k} \binom{k}{m} (\sqrt{-1})^{-m} \Gamma(\frac{m+s}{2}) \Gamma(\frac{k-m+s}{2})$$
$$= -\sqrt{-1} \sum_{m'=0}^{\infty} \binom{2k'}{2m'+1} (-1)^{m'} \Gamma(\frac{2m'+1+s}{2}) \Gamma(\frac{k-2m-1+s}{2}).$$

Define

$$(x)_{\overline{n}} = (x)(x+1)\cdots(x+n-1), \quad (x)_{\underline{n}} = (x)(x-1)\cdots(x-n+1), \quad x \in \mathbf{C}, \quad n \in \mathbf{N}.$$

Note that

$$(x)_{\underline{n}} = (-1)^n (-x)_{\overline{n}}, \quad x \in \mathbf{C}, \quad n \in \mathbf{N} \tag{2.2.6}$$

and

$$(x)_{\overline{n}} = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad (x)_{\underline{n}} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}, \quad x \in \mathbf{C}, \quad n \in \mathbf{N} \tag{2.2.7}$$

if both sides of the equations are well-defined. By (2.2.7),

$$\Gamma(\frac{1+s}{2} + m') = \Gamma(\frac{1+s}{2})(\frac{1+s}{2})_{\overline{m'}}.$$

doi:10.6342/NTU202201144

By (2.2.6) and (2.2.7),

$$\Gamma(\frac{2k'-1+s}{2}-m') = \frac{\Gamma(\frac{2k'-1+s}{2})}{(\frac{2k'-1+s}{2}-1)_{\underline{m'}}} = (-1)^{m'}\frac{\Gamma(\frac{2k'-1+s}{2})}{(-\frac{2k'-1+s}{2}+1)_{\overline{m'}}}.$$

Write

$$\binom{2k'}{2m'+1} = \frac{1}{(2m'+1)!}\frac{(2k')!}{(2k'-2m'-1)!}.$$

Since

$$\begin{aligned}(2m'+1)! &= (2m'+1)(2m')\cdots(2)(1)\\ &= (m'+\frac{1}{2})(m')\cdots(1)(\frac{1}{2})2^{2m'+1}\\ &= (\frac{3}{2})_{\overline{m'}}(m')!2^{2m'}\end{aligned}$$

and

$$\begin{aligned}\frac{(2k')!}{(2k'-2m'-1)!} &= (2k')(2k'-1)\cdots(2k'-2m'+1)(2k'-2m')\\ &= (k')(k'-\frac{1}{2})\cdots(k'-m'+\frac{1}{2})(k'-m')2^{2m'+1}\\ &= (k'-1)_{\underline{m'}}(k'-\frac{1}{2})_{\underline{m'}}k'2^{2m'+1}\\ &= (1-k')_{\overline{m'}}(\frac{1}{2}-k')_{\overline{m'}}k'2^{2m'+1},\end{aligned}$$

we have

$$\begin{aligned}&\sqrt{-1}\sum_{m'=0}^{\infty}\binom{2k'}{2m'+1}(-1)^{m'}\Gamma(\frac{2m'+1+s}{2})\Gamma(\frac{k-2m-1+s}{2})\\ =&\sqrt{-1}\sum_{m'=0}^{\infty}\frac{(1-k')_{\overline{m'}}(\frac{1}{2}-k')_{\overline{m'}}k'2^{2m'+1}}{(\frac{3}{2})_{\overline{m'}}(m')!2^{2m'}}(-1)^{m'}\Gamma(\frac{1+s}{2})(\frac{1+s}{2})_{\overline{m'}}(-1)^{m'}\frac{\Gamma(\frac{2k'-1+s}{2})}{(-\frac{2k'-1+s}{2}+1)_{\overline{m'}}}\end{aligned}$$

doi:10.6342/NTU202201144

$$=2k'\sqrt{-1}\Gamma(\frac{1+s}{2})\Gamma(\frac{2k'-1+s}{2})\sum_{m'=0}^{\infty}\frac{(1-k')_{\overline{m'}}(\frac{1}{2}-k')_{\overline{m'}}(\frac{1+s}{2})_{\overline{m'}}}{(\frac{3}{2})_{\overline{m'}}(-\frac{2k'-1+s}{2}+1)_{\overline{m'}}}\cdot\frac{1}{(m')!}$$
$$=2k'\sqrt{-1}\Gamma(\frac{1+s}{2})\Gamma(\frac{2k'-1+s}{2})\,{}_3F_2\begin{pmatrix}1-k' & \frac{1}{2}-k' & \frac{1}{2}+\frac{s}{2} & \\ & & & ;1\\ \frac{3}{2} & \frac{3}{2}-k'-\frac{s}{2} & & \end{pmatrix}$$

where ${}_3F_2$ is the generalized hypergeometric defined by

$${}_3F_2\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \\ & & & ;z\\ \beta_1 & \beta_2 & & \end{pmatrix}=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{\overline{n}}\,(\alpha_2)_{\overline{n}}\,(\alpha_3)_{\overline{n}}}{(\beta_1)_{\overline{n}}\,(\beta_2)_{\overline{n}}}\frac{z^n}{n!}.$$

Let

$$a=1-k',\quad b=\frac{1}{2}-k',\quad c=\frac{1}{2}+\frac{s}{2}.$$

By Dixon's identity [Dix03], for $\Re(1+a/2-b-c)=\Re((1+k'-s)/2)>0$,

$${}_3F_2\begin{pmatrix}1-k' & \frac{1}{2}-k' & \frac{1}{2}+\frac{s}{2} & \\ & & & ;1\\ \frac{3}{2} & \frac{3}{2}-k'-\frac{s}{2} & & \end{pmatrix}=\frac{\Gamma(\frac{3}{2}-\frac{k'}{2})\Gamma(\frac{1}{2}+\frac{k'}{2}-\frac{s}{2})\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}-k'-\frac{s}{2})}{\Gamma(2-k')\Gamma(1-\frac{s}{2})\Gamma(1+\frac{k'}{2})\Gamma(1-\frac{k'}{2}-\frac{s}{2})}$$

where the indetermined forms appear in the expression are understood by meromorphic continuation. Hence

$$|2\theta|^{-2s}l_{\Phi_\infty,s-1/2,\chi_\infty}$$
$$=-2^{-k}(\sqrt{-1})^k\pi^{-s-\frac{k}{2}}k\sqrt{-1}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{k-1+s}{2})\Gamma(\frac{3}{2}-\frac{k}{4})\Gamma(\frac{1}{2}+\frac{k}{4}-\frac{s}{2})\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}-\frac{k}{2}-\frac{s}{2})}{\Gamma(2-\frac{k}{2})\Gamma(1-\frac{s}{2})\Gamma(1+\frac{k}{4})\Gamma(1-\frac{k}{4}-\frac{s}{2})}$$
$$=-2^{-k}(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}}k\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{k-1+s}{2})\Gamma(\frac{3}{2}-\frac{k}{4})\Gamma(\frac{1}{2}+\frac{k}{4}-\frac{s}{2})\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}-\frac{k}{2}-\frac{s}{2})}{\Gamma(2-\frac{k}{2})\Gamma(1-\frac{s}{2})\Gamma(1+\frac{k}{4})\Gamma(1-\frac{k}{4}-\frac{s}{2})}.$$

Now we try to eliminate the indetermined forms. By Euler's reflection formula and

doi:10.6342/NTU202201144

Legendre's duplication formula,

$$
\begin{aligned}
&|2\theta|^{-(2s+1)}l_{\Phi_\infty,s,\chi_\infty}\\
=&-2^{-k}(\sqrt{-1})^{k+1}\pi^{-s-\frac{1}{2}-\frac{k}{2}}k\frac{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{k}{2}+\frac{s}{2}-\frac{1}{4})\Gamma(\frac{3}{2}-\frac{k}{4})\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\frac{\sqrt{\pi}}{2}\Gamma(-\frac{k}{2}-\frac{s}{2}+\frac{5}{4})}{\Gamma(2-\frac{k}{2})\Gamma(\frac{3}{4}-\frac{s}{2})\Gamma(\frac{k}{4}+1)\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})}\\
=&-2^{-k-1}(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}}k\frac{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{k}{2}+\frac{s}{2}-\frac{1}{4})\Gamma(\frac{3}{2}-\frac{k}{4})\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\Gamma(-\frac{k}{2}-\frac{s}{2}+\frac{5}{4})}{\Gamma(2-\frac{k}{2})\Gamma(\frac{3}{4}-\frac{s}{2})\Gamma(\frac{k}{4}+1)\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})}\\
=&-2^{-k-1}(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}}k\frac{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{k}{2}+\frac{s}{2}-\frac{1}{4})\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\Gamma(-\frac{k}{2}-\frac{s}{2}+\frac{5}{4})}{\Gamma(\frac{3}{4}-\frac{s}{2})\Gamma(\frac{k}{4}+1)\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})}\frac{\sqrt{\pi}2^{\frac{k}{2}-1}}{\Gamma(1-\frac{k}{4})}\\
=&-2^{-k-1}(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}}k\frac{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{k}{2}+\frac{s}{2}-\frac{1}{4})\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\Gamma(-\frac{k}{2}-\frac{s}{2}+\frac{5}{4})}{\Gamma(\frac{3}{4}-\frac{s}{2})\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})}\frac{\sqrt{\pi}2^{\frac{k}{2}-1}}{\frac{1}{4}\pi k\csc(\frac{\pi k}{4})}\\
=&-2^{-\frac{k}{2}}(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}}k\frac{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{k}{2}+\frac{s}{2}-\frac{1}{4})\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\Gamma(-\frac{k}{2}-\frac{s}{2}+\frac{5}{4})}{\Gamma(\frac{3}{4}-\frac{s}{2})\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})}\sqrt{\pi}\frac{1}{\pi k\csc(\frac{\pi k}{4})}\\
=&-(2^{-\frac{k}{2}})(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}-\frac{1}{2}}\frac{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{k}{2}+\frac{s}{2}-\frac{1}{4})\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\Gamma(-\frac{k}{2}-\frac{s}{2}+\frac{5}{4})}{\Gamma(\frac{3}{4}-\frac{s}{2})\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})\csc(\frac{\pi k}{4})}\\
=&(2^{-\frac{k}{2}})(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}-\frac{1}{2}}\frac{\Gamma(\frac{k}{2}+\frac{s}{2}-\frac{1}{4})\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\Gamma(-\frac{k}{2}-\frac{s}{2}+\frac{5}{4})}{\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})\csc(\frac{\pi k}{4})}\frac{2^{-s-\frac{1}{2}}\cos(\frac{\pi}{4}-\frac{\pi s}{2})\Gamma(s+\frac{3}{2})}{\sqrt{\pi}(-\frac{s}{2}-\frac{1}{4})}\\
=&-(2^{-\frac{k}{2}})(\sqrt{-1})^{k+1}\pi^{-s-\frac{k}{2}-\frac{1}{2}}\frac{\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})\pi\csc(\frac{\pi}{4}-\frac{k\pi}{2}-\frac{\pi s}{2})}{\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})\csc(\frac{\pi k}{4})}\frac{2^{-s-\frac{1}{2}}\cos(\frac{\pi}{4}-\frac{\pi s}{2})\Gamma(s+\frac{3}{2})}{\sqrt{\pi}(-\frac{s}{2}-\frac{1}{4})}.
\end{aligned}
$$

Put the condition $k \equiv 2 \pmod 4$, we have

$$
|2\theta|^{-(2s+1)}l_{\Phi_\infty,s,\chi_\infty}=-(2^{-\frac{k}{2}-s-\frac{1}{2}})\pi^{-s-\frac{k}{2}}\frac{\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})}{\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})}\frac{\Gamma(s+\frac{3}{2})}{(-\frac{s}{2}-\frac{1}{4})}(-1)^{k/4}\tan(\frac{2\pi s+\pi}{4})
$$

Therefore

$$
\begin{aligned}
&l_{\Phi_\infty,s,\chi_\infty}|_{s=\frac{1-k}{2}}\\
=&-|2\theta|^{2s+1}(2^{-\frac{k}{2}-s-\frac{1}{2}})\pi^{-s-\frac{k}{2}}\frac{\Gamma(\frac{k}{4}-\frac{s}{2}+\frac{1}{4})}{\Gamma(-\frac{k}{4}-\frac{s}{2}+\frac{3}{4})}\frac{\Gamma(s+\frac{3}{2})}{(-\frac{s}{2}-\frac{1}{4})}(-1)^{k/4}\tan(\frac{2\pi s+\pi}{4})\Bigg|_{s=\frac{1-k}{2}}\\
=&|2\theta|^{2-k}\cdot\frac{1}{2}(-1)^{k/4}\\
=&|2\theta|^{2-k}2^{-1}(\sqrt{-1})^{k/2}.
\end{aligned}
$$

doi:10.6342/NTU202201144

3 Modular Symbol of Eisenstein Series

3.1 The Modified Eisenstein Series

In this subsection, we introduce the modified Eisenstein series $F_k(z, g_{\mathrm{f}})$, $F_k^*(z, g_{\mathrm{f}})$, $F_k^{**}(z, g_{\mathrm{f}})$ and prove that the associated Bruhat-Schwartz functions Φ of $F_k^{**}(z, g_{\mathrm{f}})$ is the one which is introduced in §2.2.4.

Definition 3.1.1. Let $f(z, g_{\mathrm{f}}) \in \mathcal{N}_k(N)$. Define

$$\mathbf{U}_p f(z, g_{\mathrm{f}}) = \sum_{x=0}^{p-1} f\left(z, g_{\mathrm{f}} \begin{pmatrix} p & x \\ 0 & 1 \end{pmatrix}_p\right),$$

$$T_p f(z, g_{\mathrm{f}}) = \mathbf{U}_p f(z, g_{\mathrm{f}}) + f\left(z, g_{\mathrm{f}} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}_p\right),$$

$$f^*(z, g_{\mathrm{f}}) = f(z, g_{\mathrm{f}}) - f\left(z, g_{\mathrm{f}} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}_p\right),$$

$$f^{**}(z, g_{\mathrm{f}}) = \begin{cases} f^*(z, g_{\mathrm{f}}) + f^*\left(z, g_{\mathrm{f}} \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_p\right) & \text{if } p \text{ is inert,} \\ f^*(z, g_{\mathrm{f}}) - f^*\left(z, g_{\mathrm{f}} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p\right) & \text{if } p \text{ is split.} \end{cases}$$

doi:10.6342/NTU202201144

3.1.2 Classical and modified Eisenstein series

Recall that for positive even integer k, the classical Eisenstein series of weight k is defined by

$$E_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

By 1.3.5, we can extend $E_k(z)$ to an adelic Eisenstein series $E_k(z, g_{\mathrm{f}})$. Define the modified Eisenstein series

$$F_k(z, g_{\mathrm{f}}) = \sum_{d|q^2} dn_d E_k(dz, g_{\mathrm{f}}).$$

It is not difficult to see that $F_k(z, g_{\mathrm{f}}) \in \mathcal{N}_k(q^2)$ and $F_k^*(z, g_{\mathrm{f}})$, $F_k^{**}(z, g_{\mathrm{f}}) \in \mathcal{N}_k(pq^2)$.

Lemma 3.1.3. Let $E_{\Phi,s}$ be the Eisenstein series induced by Φ. Then $E_{\Phi,s}|_{s=\frac{1-k}{2}}$ is the adelic lift of $F_k^{**}(z, g_{\mathrm{f}})$.

Proof. We first show that $E_{\Phi^{(0)},s}|_{s=\frac{1-k}{2}}$ is the adelic lift of $F_k(z, g_{\mathrm{f}})$, where

$$\Phi_v^{(0)}(x, y) = \begin{cases} 2^{-k}(x + \sqrt{-1}y)^k e^{-\pi(x^2+y^2)} & \text{if } v = \infty, \\ -24 \sum_{d|q^2} dn_d |d|_v \mathbb{I}_{\mathbf{Z}_q \oplus \mathbf{Z}_q}(x, dy) & \text{if } v = q, \\ \mathbb{I}_{\mathbf{Z}_v \oplus \mathbf{Z}_v}(x, y) & \text{otherwise.} \end{cases}$$

We will compute the Fourier expansion for Eisenstein series (1.4.6). We first claim that

$$W_0\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, f_{\Phi^{(0)},s}\right)$$

doi:10.6342/NTU202201144

$$:= f_{\Phi^{(0)},s}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} + \int_{\mathbf{A}_{\mathbf{Q}}} f_{\Phi^{(0)},s}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) dw$$

equals to zero. In fact,

$$\begin{aligned}
& f_{\Phi_q^{(0)},s}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
=& \int_{\mathbf{Q}_q^\times} \sum_{d|q^2} dn_d |d|_q \Phi_q^{(0)}(0, dt)|t|^{2s+1} d^\times t \\
=& \int_{\mathbf{Q}_q} \sum_{d|q^2} dn_d \Phi_q^{(0)}(0, t)|t|^{2s+1} d^\times t \\
=& 0
\end{aligned}$$

and similarly

$$\begin{aligned}
& \int_{\mathbf{Q}_q} f_{\Phi_q^{(0)},s}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) dw \\
=& \int_{\mathbf{Q}_q} \int_{\mathbf{Q}_q^\times} \sum_{d|q^2} dn_d |d|_q \Phi_q^{(0)}(t, tdw)|t|^{2s+1} d^\times t dw \\
=& \int_{\mathbf{Q}_q^\times} \int_{\mathbf{Q}_q} \sum_{d|q^2} dn_d |d|_q \Phi_q^{(0)}(t, tdw)|t|^{2s+1} dw d^\times t \\
=& \int_{\mathbf{Q}_q^\times} \int_{\mathbf{Q}_q} \sum_{d|q^2} dn_d \Phi_q^{(0)}(t, tw)|t|^{2s+1} dw d^\times t \\
=& 0,
\end{aligned}$$

doi:10.6342/NTU202201144

as desired. Now fix $\alpha \in \mathbf{Q}^\times$ and write

$$W_\alpha\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, f_{\Phi^{(0)},s}\right) = \prod_v W_\alpha\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, f_{\Phi_v^{(0)},s}\right)$$

as a product of local integrals. If $v = q$, then

$$\begin{aligned}
& W_\alpha(1, f_{\Phi_q^{(0)},s}) \\
=& \int_{\mathbf{Q}_q}\int_{\mathbf{Q}_q^\times} \Phi_{q,s}(t, tw)|t|_q^{2s+1} d^\times t \psi_q(-\alpha w)dw \\
=& \sum_{d|q^2} dn_d|d|_q \int_{\mathbf{Q}_q}\int_{\mathbf{Q}_q^\times} \Phi_q^{(0)}(t, dtw)|t|_q^{2s+1} d^\times t \psi_q(-\alpha w)dw \\
=& \sum_{d|q^2} dn_d|d|_q \int_{\mathbf{Q}_q^\times}\int_{\mathbf{Q}_q} \mathbb{I}_{\mathbf{Z}_q}(t)\mathbb{I}_{\mathbf{Z}_q}(dtw)|t|_q^{2s+1} \psi_q(-\alpha w)dwd^\times t \\
=& \sum_{d|q^2} dn_d|d|_q \int_{\mathbf{Q}_q^\times}\int_{\mathbf{Q}_q} \mathbb{I}_{\mathbf{Z}_q}(t)\mathbb{I}_{\mathbf{Z}_q}(w)|t|_q^{2s+1} \psi_q(-\frac{\alpha w}{dt})d\frac{w}{|dt|_q}d^\times t \\
=& \sum_{d|q^2} dn_d \int_{\mathbf{Q}_q^\times}\int_{\mathbf{Q}_q} \mathbb{I}_{\mathbf{Z}_q}(t)\mathbb{I}_{\mathbf{Z}_q}(w)|t|_q^{2s} \psi_q(-\frac{\alpha w}{dt})dwd^\times t \\
=& \sum_{d|q^2} dn_d \int_{\mathbf{Z}_q-\{0\}}|t|_q^{2s}\int_{\mathbf{Z}_q} \psi(-\frac{\alpha w}{dt})dwd^\times t.
\end{aligned}$$

Since $\int_{\mathbf{Z}_q}\psi_q(\beta w)dw = \mathbb{I}_{\mathbf{Z}_q}(\beta)$, we have

$$\begin{aligned}
& \sum_{d|q^2} dn_d \int_{\mathbf{Z}_q-\{0\}}|t|_q^{2s}\int_{\mathbf{Z}_q} \psi(-\frac{\alpha w}{dt})dwd^\times t \\
=& \sum_{d|q^2} dn_d \sum_{l=0}^{\mathrm{ord}_q(\alpha/d)} \int_{q^l\mathbf{Z}_q^\times} |t|^{2s} d^\times t \\
=& \sum_{d|q^2} dn_d \sum_{l=0}^{\mathrm{ord}_q(\alpha/d)} q^{-2ls}.
\end{aligned}$$

doi:10.6342/NTU202201144

Similarly, if $v < \infty$, $v \neq q$, then

$$W_\alpha(1, f_{\Phi_v^{(0)},s}) = \int_{\mathbf{Q}_v}\int_{\mathbf{Q}_v^\times} \Phi_{v,s}(t, tw)|t|_v^{2s+1} d^\times t \psi_v(-\alpha w) dw = \sum_{l=0}^{\mathrm{ord}_v(\alpha)} v^{-2ls}.$$

If $v = \infty$, then

$$\begin{aligned}
&|y|^{s+\frac{1}{2}} W_\alpha\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, f_{\Phi_\infty^{(0)},s}\right) \\
=&|y|^{s+\frac{1}{2}} \int_{\mathbf{R}}\int_{\mathbf{R}^\times} \Phi_{\infty,s}(t, t(x+r))|t|^{2s+1} d^\times t \psi_\infty(-\alpha r) dr \\
=&|y|^{s+\frac{1}{2}} 2^{-k} \int_{\mathbf{R}}\int_{\mathbf{R}^\times} ((ty) + \sqrt{-1}t(x+r))^k e^{-\pi((ty)^2+(t(x+r))^2} t^{2s+1} d^\times t \psi_\infty(-\alpha r) dr \\
=&|y|^{s+\frac{1}{2}} 2^{-k} \int_{\mathbf{R}} (y + \sqrt{-1}(x+r))^k \int_{\mathbf{R}^\times} e^{-\pi t^2(y^2+(x+r)^2)} |t|^{k+2s+1} d^\times t \psi_\infty(-\alpha r) dr \\
=&|y|^{s+\frac{1}{2}} 2^{-k} \int_{\mathbf{R}} (y + \sqrt{-1}(x+r))^k \frac{\Gamma_{\mathbf{R}}(k+2s+1)}{(y^2+(x+r)^2)^{\frac{k+2s+1}{2}}} \psi_\infty(-\alpha r) dr \\
=&|y|^{s+\frac{1}{2}} 2^{-k} (y + \sqrt{-1}(x+r))^k \Gamma_{\mathbf{R}}(k+2s+1)(-2\pi\sqrt{-1}) e^{2\pi\sqrt{-1}\alpha z} \\
&\times (-2\pi\sqrt{-1}) \operatorname*{Res}_{r=-x-\sqrt{-1}y} \frac{1}{(y^2+(x+r)^2)^{\frac{k+2s+1}{2}}} \mathbb{I}_{\mathbf{R}_+}(\alpha) \\
\to& y^{k/2} e^{2\pi\sqrt{-1}\alpha z} \mathbb{I}_{\mathbf{R}_+}(\alpha)
\end{aligned}$$

as $y \to \frac{1-k}{2}$. Hence for $\alpha \neq 0$,

$$\begin{aligned}
&W^\alpha_{\varphi_s^{(0)}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\Bigg|_{s=\frac{1-k}{2}} \\
=&y^{k/2} e^{2\pi\sqrt{-1}\alpha z} \mathbb{I}_{\mathbf{R}_+}(\alpha) \left(\sum_{d|q^2} dn_d \sum_{l=0}^{\mathrm{ord}_q(\alpha/d)} q^{l(k-1)}\right) \prod_{v<\infty, v\neq q} \sum_{l=0}^{\mathrm{ord}_v(\alpha)} v^{l(k-1)} \\
=&y^{k/2} e^{2\pi\sqrt{-1}\alpha z} \mathbb{I}_{\mathbf{R}_+}(\alpha) \sum_{d|q^2} dn_d \left(\sum_{l=0}^{\mathrm{ord}_q(\alpha/d)} q^{l(k-1)} \prod_{v<\infty, v\neq q} \sum_{l=0}^{\mathrm{ord}_v(\alpha)} v^{l(k-1)}\right)
\end{aligned}$$

doi:10.6342/NTU202201144

$$=y^{k/2}e^{2\pi\sqrt{-1}\alpha z}\mathbb{I}_{\mathbf{R}_+}(\alpha)\sum_{d|q^2} dn_d\sigma_{k-1}(\frac{\alpha}{d})\mathbb{I}_{\mathbf{Z}}(\alpha).$$

Therefore

$$\begin{aligned}
E_{\Phi^{(0)},s}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}|_{s=\frac{1-k}{2}} &= \sum_{\alpha\in\mathbf{Q}^\times} y^{k/2}e^{2\pi\sqrt{-1}\alpha z}\mathbb{I}_{\mathbf{R}_+}(\alpha)\sum_{d|q^2} dn_d\sigma_{k-1}(\frac{\alpha}{d})\mathbb{I}_{\mathbf{Z}}(\alpha) \\
&= y^{k/2}\sum_{d|q^2} dn_d\sum_{n=1}^{\infty}\sigma_{k-1}(\frac{n}{d})e^{2\pi\sqrt{-1}nz} \\
&= y^{k/2}\sum_{d|q^2} dn_d\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi\sqrt{-1}(dn)z} \\
&= y^{k/2}\sum_{d|q^2} dn_d E_k(dz,1) \\
&= y^{k/2}F_k(z,1),
\end{aligned}$$

that is, $E_{\Phi^{(0)},s}(g)|_{s=\frac{1-k}{2}}$ is the adelic lift of $F_k(z,g_{\mathrm{f}})$, as desired. Now we prove that $E_{\Phi,s}(g)|_{s=\frac{1-k}{2}}$ is the adelic lift of $F_k^{**}(z,g_{\mathrm{f}})$. Assume p is split. Write

$$F_k^{**}(z,1) = y^{-\frac{k}{2}}\varphi_{k,s}^{**}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\Bigg|_{s=\frac{1-k}{2}}$$

where

$$\begin{aligned}
&\varphi_{k,s}^{**}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\Bigg|_{s=\frac{1-k}{2}} \\
&= E_{\Phi^{(0)},s}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} - p^{\frac{k}{2}-1}E_{\Phi^{(0)},s}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)
\end{aligned}$$

doi:10.6342/NTU202201144

$$-p^{\frac{k}{2}-1}E_{\Phi^{(0)},s}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p\right)+p^{k-2}E_{\Phi^{(0)},s}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p\right)\Bigg|_{s=\frac{1-k}{2}}$$

$$= \sum_{\gamma\in B\backslash G}\prod_{v\neq p} f_{\Phi_v,s}\left(\gamma\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\Bigg[f_{\Phi_p^{(0)},s}\left(\gamma\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_p\right)-p^{\frac{k}{2}-1}f_{\Phi_p^{(0)},s}\left(\gamma\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}_p\right)$$

$$-p^{\frac{k}{2}-1}f_{\Phi_p^{(0)},s}\left(\gamma\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p\right)+p^{k-2}f_{\Phi_p^{(0)},s}\left(\gamma\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p\right)\Bigg]\Bigg|_{s=\frac{1-k}{2}}$$

$$= \sum_{\gamma\in B\backslash G}\prod_{v\neq p} f_{\Phi_v,s}\left(\gamma\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)f_{\Phi_p,s}(\gamma)\Bigg|_{s=\frac{1-k}{2}}$$

$$= E_{\Phi,s}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\Bigg|_{s=\frac{1-k}{2}},$$

as desired. Similarly if p is inert. □

3.2 Partial Modular Symbols and p-adic Measures

In this subsection, we review the theory of partial modular symbols. Following the discussion in [Kit94, p.95], we will associate a partial modular symbol for each modular form, and use this modular symbol to construct a family of p-adic measures.

3.2.1 Partial modular symbol

We review the theory of partial modular symbols in the *semi-adelic* language. For each $r \in \mathbf{P}^1(\mathbf{Q})$ we denote by (r) the image in the divisor group of $\mathbf{P}^1(\mathbf{Q})$. Let $\gamma \in \mathrm{GL}_2(\mathbf{Q})$ and $u \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ act on $D = ((r)-(s), g_{\mathrm{f}}) \in \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q})) \times \mathrm{GL}_2(\widehat{\mathbf{Q}})$ by

$$\gamma D u := ((\gamma r)-(\gamma s), \gamma g_{\mathrm{f}} u).$$

doi:10.6342/NTU202201144

For a ring R, let $L_n(R)$ be the space of two-variable homogeneous polynomials of degree n with coefficients in R. For $h = h(X,Y) \in L_n(R)$ and $g \in \mathrm{GL}_2(R)$, define

$$(h|g)(X,Y) = h(aX+bY, cX+dY), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $L_n^*(R) = \mathrm{Hom}_R\left(L_n(R), R\right)$. Moreover, if R is a $\mathbf{Z}_p$-algebra, let $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ acts on $L_n^*(R)$ by $\left(\rho_n(u)\xi\right)(h) = \xi\left(h|_{u_p}\right)$. Let $\mathfrak{D}$ be a subset of $\mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q})) \times \mathrm{GL}_2(\widehat{\mathbf{Q}})$. We denote by $\mathcal{MS}_k(N, \mathfrak{D})$ the space of *p-adic partial modular symbols* of weight k and level N with respect to $\mathfrak{D}$, consisting of functions $\xi : \mathfrak{D} \to L_{k-2}^*(R)$ such that

$$\xi(\gamma D u) = \rho_{k-2}\left(u_p^{-1}\right)\xi(D)$$

for all $\gamma \in \Gamma_0(N)$, $D \in \mathfrak{D}$ and $u \in U_2(N)$ satisfying $\gamma D u \in \mathfrak{D}$.

3.2.2 Partial modular symbols associated with modular forms

To each modular form $f(z, g_{\mathrm{f}}) \in \mathcal{N}_k(N)$, we associate a *complex partial modular symbol* $\eta_f : \mathfrak{D}_f \to L_{k-2}^*(\mathbf{C})$ defined by

$$\eta_f\left((r)-(s), g_{\mathrm{f}}\right)(h) := \int_r^s f\left(z, g_{\mathrm{f}}\right) h(z,1)\mathrm{d}z \tag{3.2.3}$$

where $\mathfrak{D}_f = \bigcup_{g_{\mathrm{f}} \in \mathrm{GL}_2(\widehat{\mathbf{Q}})} D_{g_{\mathrm{f}}} \times \{g_{\mathrm{f}}\}$ and $D_{g_{\mathrm{f}}}$ is the subset of $\mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ spanned over $\mathbf{Z}$ by all of $((r)-(s)) \in \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ such that the integral (3.2.3) converges for all $h \in L_{k-2}(\mathbf{C})$. It is easy to see that for $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$, $D \in \mathfrak{D}_f$ and $u \in U_2(N)$,

$$\eta_f(\alpha D u) = \rho_{k-2}(\alpha)\eta_f(D)$$

doi:10.6342/NTU202201144

whenever $\alpha D u \in \mathfrak{D}_f$. The associated p-adic modular symbol $\xi_f \in \mathcal{MS}_k(N, \mathfrak{D}_f)$ is defined by $\xi_f : \mathfrak{D}_f \to L^*_{k-2}(\mathbf{C}_p)$,

$$\xi_f(D)(h) \coloneqq \eta_f(D)(h \mid g_p^{-1}) \coloneqq \iota_p(\eta_f(D)(h^{\iota_p^{-1}} \mid g_p^{-1})) \tag{3.2.4}$$

where $h^{\iota_p^{-1}} \in \mathbf{C}[x, y]$ is obtain by applying ι_p^{-1} to the coefficients of $h \in \mathbf{C}_p[x, y]$.

3.2.5 Partial modular symbols associated with the modified Eisenstein series

We exhibit concrete elements in $\mathfrak{D}_{F_k}$, $\mathfrak{D}_{F_k^*}$ and $\mathfrak{D}_{F_k^{**}}$. Let Γ be the subgroup

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}[1/p]) : q^2 \mid c \right\}$$

of $\mathrm{GL}_2(\mathbf{Z}[1/p])$. Denote $\mathrm{GL}_2(\mathbf{Q})(\mathrm{Div}^0(\Gamma\infty) \times \{1\}) \, \mathrm{GL}_2(\widehat{\mathbf{Q}})$ the orbit of

$$\mathrm{Div}^0(\Gamma\infty) \times \{1\} \subseteq \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q})) \times \mathrm{GL}_2(\widehat{\mathbf{Q}})$$

under the left action of $\mathrm{GL}_2(\mathbf{Q})$ and the right action of $\mathrm{GL}_2(\widehat{\mathbf{Q}})$. We remark that since q^2 is prime to p, we have $\Gamma\infty = \Gamma_0(q^2)\infty$.

Lemma 3.2.6. $\mathrm{GL}_2(\mathbf{Q})(\mathrm{Div}^0(\Gamma\infty) \times \{1\}) \, \mathrm{GL}_2(\widehat{\mathbf{Q}})$ is contained in $\mathfrak{D}_{F_k}$, $\mathfrak{D}_{F_k^*}$ and $\mathfrak{D}_{F_k^{**}}$.

Proof. Let $D = ((r) - (s), 1) \in \mathrm{Div}^0(\Gamma\infty) \times \{1\}$ and $h \in L_{k-2}(\mathbf{C}_p)$. The condition (1.2.6) implies that $F_k(z, 1)$ is holomorphic at ∞. Hence

$$\xi_{F_k}(D)(h) = \int_r^s F_k(z, 1) h(z, 1) dz$$

doi:10.6342/NTU202201144

converges and $D \in \mathfrak{D}_{F_k}$. Let $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ -p & 0 \end{smallmatrix}\right) \in \Gamma$. Then

$$\xi_{F_k^*}(D)(h) = \xi_{F_k}(D)(h) - \xi_{F_k}((\gamma^{-1}r) - (\gamma^{-1}s), 1)(h|\gamma)$$

converges and $D \in \mathfrak{D}_{F_k^*}$. Similarly, $D \in \mathfrak{D}_{F_k^{**}}$. □

3.3 Period Integral as a Linear Combination of Complex Modular Symbols

In this subsection, we prove that the distinguished period integral $P_{\Phi,s,\chi}(\varsigma)$ can be written as a linear combination of complex modular symbols.

Proposition 3.3.1.

$$\begin{aligned} P_{\Phi,s,\chi}(\varsigma) = &\operatorname{vol}(\widehat{\mathcal{O}}_F^\times)(2\sqrt{-1})^{-1}\delta(\sqrt{-1})^{\frac{k-2}{2}} \\ &\times \sum_{a \in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}} \int_{\gamma_a\infty}^{\Psi(\varepsilon)\gamma_a\infty} F_k^{**}(z, \Psi(a)\varsigma_{\mathrm{f}}) Q_\theta(z,1)^{\frac{k-2}{2}}\, dz \end{aligned}$$

where

- $\delta = \overline{\theta} - \theta$,
- $F_+ \cap \widehat{\mathcal{O}}_F^\times = \varepsilon^{\mathbf{Z}}$,
- $Q_\theta(x,y) = (x - \theta y)(x - \overline{\theta} y)$,
- $\Psi(a)\varsigma_{\mathrm{f}} = \gamma_a u_a \in \mathrm{GL}_2(\mathbf{Q})^+ U_2(pq^2)$ for $a \in \mathrm{Cl}^+(F)$.

Proof. $E_{\Phi,s}$ has the analytic continuation to the entire $s \in \mathbf{C}$. By the identity

doi:10.6342/NTU202201144

$\mathbf{A}^\times = \mathbf{Q}^\times \mathbf{R}_+ \widehat{\mathbf{Z}}^\times$,

$$\begin{aligned}
&P_{\Phi,s,\chi}(\varsigma)\\
=&\mathrm{vol}(\widehat{\mathcal{O}}_F^\times) \int_{F^\times \mathbf{A}^\times \backslash \mathbf{A}_F^\times / \widehat{\mathcal{O}}_F^\times} E_{\Phi,s}(\Psi(y)\varsigma)\chi(y) d^\times y\\
=&\mathrm{vol}(\widehat{\mathcal{O}}_F^\times) \int_{F^\times \backslash \mathbf{A}_F^\times / \mathbf{R}_+ \widehat{\mathcal{O}}_F^\times} E_{\Phi,s}(\Psi(y)\varsigma)\chi(y) d^\times y.
\end{aligned}$$

Define $h : \mathbf{R}_+ \to \mathbf{A}_F^\times$ by $h(x) = ((x, x^{-1}), 1)$. Then we have the exact sequence

$$0 \to \mathbf{R}_+ / \operatorname{Ker}(h) \xrightarrow{h} F^\times \backslash \mathbf{A}_F^\times / \mathbf{R}_+ \widehat{\mathcal{O}}_F^\times \to F^\times \backslash \mathbf{A}_F^\times / (F \otimes_{\mathbf{Q}} \mathbf{R})_+ \widehat{\mathcal{O}}_F^\times \to 0.$$

Regard $F_+ \cap \mathcal{O}_F^\times$ as a subgroup of $F_+ \subseteq \mathbf{R}_+$. Then $\operatorname{Ker}(h) = F_+ \cap \mathcal{O}_F^\times = \varepsilon^{\mathbf{Z}}$, so

$$\begin{aligned}
&P_{\Phi,s,\chi}(\varsigma)\\
=&\mathrm{vol}(\widehat{\mathcal{O}}_F^\times) \int_{F^\times \backslash \mathbf{A}_F^\times / (F\otimes_{\mathbf{Q}}\mathbf{R})_+ \widehat{\mathcal{O}}_F^\times} \int_{\mathbf{R}_+/\varepsilon^{\mathbf{Z}}} E_{\Phi,s}(\Psi(ah(y))\varsigma)\chi(ah(y)) d^\times a d^\times y\\
=&\mathrm{vol}(\widehat{\mathcal{O}}_F^\times) \sum_{a \in \mathrm{Cl}^+(F)} \chi(a) \int_{\mathbf{R}_+/\varepsilon^{\mathbf{Z}}} E_{\Phi,s}(\Psi(a)\Psi_\infty(h(y))\varsigma)\chi(h(y)) d^\times y.
\end{aligned}$$

By (2.2.3),

$$\begin{aligned}
&P_{\Phi,s,\chi}(\varsigma)\\
=&\mathrm{vol}(\widehat{\mathcal{O}}_F^\times) \sum_{a \in \mathrm{Cl}^+(F)} \chi(a) \int_{\mathbf{R}_+/\varepsilon^{\mathbf{Z}}} E_{\Phi,s}\left(\Psi(a)\varsigma_{\mathrm{f}}\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\right) \chi(y) d^\times y.
\end{aligned}$$

Put $s = \frac{1-k}{2}$. Then by Lebesgue's dominated convergence theorem,

$$P_{\Phi,s,\chi}(\varsigma)|_{s=\frac{1-k}{2}}$$

doi:10.6342/NTU202201144

$$= \mathrm{vol}(\widehat{\mathcal{O}}_F^\times) \sum_{a \in \mathrm{Cl}^+(F)} \chi(a) \int_{\mathbf{R}_+/\varepsilon^{\mathbf{Z}}} E_{\Phi,s}\big|_{s=\frac{1-k}{2}} \left(\Psi(a)\varsigma_{\mathrm{f}}\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \right) \chi(y) d^\times y.$$

By lemma (3.1.3) and equation (1.3.3),

$$\begin{aligned} &P_{\Phi,s,\chi}(\varsigma)\big|_{s=\frac{1-k}{2}} \\ =& \mathrm{vol}(\widehat{\mathcal{O}}_F^\times) \sum_{a \in \mathrm{Cl}^+(F)} \chi(a)\, \mathrm{N}(a)^{\frac{k-2}{2}} \\ &\times \int_{y_0}^{\varepsilon y_0} F_k^{**} \left(\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \sqrt{-1}, \Psi(a)\varsigma_{\mathrm{f}} \right) J \left(\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \sqrt{-1} \right)^{-k} d^\times y \end{aligned}$$

where $y_0 > 0$ is arbitrary. By the change of variable

$$z = \varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \sqrt{-1} = \frac{\overline{\theta} y^2 \sqrt{-1} + \theta}{y^2 \sqrt{-1} + 1} =: l(y),$$

we have

$$J \left(\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \sqrt{-1} \right)^{-k} = \left(\frac{\delta y}{y^2 \sqrt{-1} + 1} \right)^k$$

and

$$dz = \frac{2i}{\delta} J \left(\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \sqrt{-1} \right)^{-2} d^\times y$$

and

$$\sqrt{-1}(z - \theta)(z - \theta) = J \left(\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \sqrt{-1} \right)^{-2}.$$

Set

$$c_k = (2\sqrt{-1})^{-1} \delta (\sqrt{-1})^{\frac{k-2}{2}}.$$

doi:10.6342/NTU202201144

Then

$$J\left(\varsigma_\infty \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \sqrt{-1}\right)^{-k} d^\times y = c_k Q_\theta(z,1)^{\frac{k-2}{2}}$$

and

$$\begin{aligned}
&P_{\Phi,s,\chi}(\varsigma)|_{s=\frac{1-k}{2}} \\
&= -\operatorname{vol}(\widehat{\mathcal{O}}_F^\times)c_k \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}} \int_{l(y_0)}^{l(\varepsilon y_0)} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-1}{2}}dz \\
&= -\operatorname{vol}(\widehat{\mathcal{O}}_F^\times)c_k \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}} \int_{l(y_0)}^{\Psi(\bar\varepsilon)l(y_0)} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz \\
&= -\operatorname{vol}(\widehat{\mathcal{O}}_F^\times)c_k \sum_{a\in \mathrm{Cl}^*(F)} \chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}} \int_{z_0}^{\Psi(\bar\varepsilon)z_0} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz \\
&= \operatorname{vol}(\widehat{\mathcal{O}}_F^\times)c_k \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}} \int_{z_0}^{\Psi(\varepsilon)z_0} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz
\end{aligned}$$

where $z_0 = l(y_0) \in \mathbf{H}$ and $y_0 > 0$ is arbitrary. For $a \in \mathrm{Cl}^+(F)$, write $\Psi(a)\varsigma_{\mathrm{f}} = \gamma_a u_a$ with $\gamma_a \in \mathrm{GL}_2(\mathbf{Q})^+$ and $u_{\mathrm{f}} \in U_2(pq^2)$. Then

$$\begin{aligned}
&\int_{\Psi(\varepsilon)z_0}^{\Psi(\varepsilon)\gamma_a\infty} F_k^{**}(y,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(y,1)^{\frac{k-2}{2}}dy, \\
&= \int_{z_0}^{\gamma_a\infty} F_k^{**}(\Psi(\varepsilon)y,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(\Psi(\varepsilon)y,1)^{\frac{k-2}{2}}d\Psi(\varepsilon)y \\
&= \int_{z_0}^{\gamma_a\infty} \det(\Psi(\varepsilon))^{-1}J(\Psi(\varepsilon),y)^k F_k^{**}(y,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(\Psi(\varepsilon)y,1)^{\frac{k-2}{2}}d\Psi(\varepsilon)y \\
&= \int_{z_0}^{\gamma_a\infty} F_k^{**}(y,\Psi(a)\varsigma_{\mathrm{f}})J(\Psi(\varepsilon),y)^{k-2}Q_\theta(\Psi(\varepsilon)y,1)^{\frac{k-2}{2}}dy \\
&= \int_{z_0}^{\gamma_a\infty} F_k^{**}(y,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta|_{\Psi(\varepsilon)}(y,1)^{\frac{k-2}{2}}dy \\
&= \int_{z_0}^{\gamma_a\infty} F_k^{**}(y,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(y,1)^{\frac{k-2}{2}}dy.
\end{aligned}$$

doi:10.6342/NTU202201144

Hence

$$\begin{aligned}
&\int_{z_0}^{\Psi(\varepsilon)z_0} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz\\
&=\left(\int_{z_0}^{\gamma_a\infty}+\int_{\gamma_a\infty}^{\Psi(\varepsilon)\gamma_a\infty}+\int_{\Psi(\varepsilon)\gamma_a\infty}^{\Psi(\varepsilon)z_0}\right)F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz\\
&=\int_{\gamma_a\infty}^{\Psi(\varepsilon)\gamma_a\infty} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz,
\end{aligned}$$

where all of the terms involved are convergent by (3.2.6). Therefore

$$\begin{aligned}
&P_{\Phi,s,\chi}(\varsigma)\big|_{s=\frac{1-k}{2}}\\
=&\operatorname{vol}(\widehat{\mathcal{O}}_F^\times)c_k\sum_{a\in\mathrm{Cl}^+(F)}\chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}}\int_{\gamma_a\infty}^{\Psi(\varepsilon)\gamma_a\infty} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz,
\end{aligned}$$

as desired. □

4 p-adic L-function for Real Quadratic Field

4.1 Integration on $\mathbf{X}_p$ as a p-adic Modular Symbol

In [DD06, Theorem 4.2], the author constructed a collection of p-adic measures $\mu((r)-(s))$ on $\mathbf{Q}_p^2-\{(0,0)\}$ indexed by r, $s\in\Gamma\infty$ such that for every homogeneous polynomial $h(x,y)\in\mathbf{Z}[x,y]$ of degree $k-2$ and r, $s\in\Gamma\infty$,

$$\int_{\mathbf{X}_p} h(x,y)d\mu((r)-(s)) = \int_r^s h(z,1)F_k^{**}(z)dz$$

provided p is inert in F. In this subsection, we generalize this formula to include the case that p is split in F (4.1.4).

doi:10.6342/NTU202201144

Lemma 4.1.1. For each positive even integer k, $F_k^*(z, g_{\mathrm{f}})$ is a $\mathbf{U}_p$-eigenvector with eigenvalue 1.

Proof. We have seem in the proof of (3.1.3) that $E_{\Phi^{(0)},s}|_{s=\frac{1-k}{2}}$ is the adelic lift of $F_k(z, g_{\mathrm{f}})$, where

$$\Phi_v^{(0)}(x,y) = \begin{cases} 2^{-k}(x+\sqrt{-1}y)^k e^{-\pi(x^2+y^2)} & \text{if } v=\infty, \\ -24\sum_{d|q^2} dn_d|d|_v \mathbb{I}_{\mathbf{Z}_q\oplus\mathbf{Z}_q}(x,dy) & \text{if } v=q, \\ \mathbb{I}_{\mathbf{Z}_v\oplus\mathbf{Z}_v}(x,y) & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} T_p f_{\Phi_p^{(0)},s}\begin{pmatrix}1&0\\0&1\end{pmatrix} &= \sum_{x=0}^{p-1} f_{\Phi_p^{(0)}}\begin{pmatrix}p&x\\0&1\end{pmatrix} + \sum_{x=0}^{p-1} f_{\Phi_p^{(0)}}\begin{pmatrix}1&0\\0&p\end{pmatrix} \\ &= \sum_{x=0}^{p-1}|p|_p^{s+\frac{1}{2}}\int_{\mathbf{Q}_p^\times}\Phi_p^{(0)}\left((0,t)\begin{pmatrix}p&x\\0&1\end{pmatrix}\right)|t|^{2s+1}d^\times t \\ &\quad + |p|_p^{s+\frac{1}{2}}\int_{\mathbf{Q}_p^\times}\Phi_p^{(0)}\left((0,t)\begin{pmatrix}1&0\\0&p\end{pmatrix}\right)|t|^{2s+1}d^\times t \\ &= \sum_{x=0}^{p-1}|p|_p^{s+\frac{1}{2}}\int_{\mathbf{Q}_p^\times}\Phi_p^{(0)}(0,t)|t|^{2s+1}d^\times t \\ &\quad + |p|_p^{s+\frac{1}{2}}\int_{\mathbf{Q}_p^\times}\Phi_p^{(0)}(0,pt)|t|^{2s+1}d^\times t \\ &= (p|p|^{s+\frac{1}{2}} + |p|^{-s-\frac{1}{2}})f_{\Phi_p^{(0)},s}\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ &= (p^{-s+\frac{1}{2}} + p^{s+\frac{1}{2}})f_{\Phi_p^{(0)},s}\begin{pmatrix}1&0\\0&1\end{pmatrix}. \end{aligned}$$

doi:10.6342/NTU202201144

Then since $f_{\Phi_p^{(0)},s}$ is a spherical vector in $I(s, 1_p, 1_p)$, we have

$$\mathbf{U}_p f_{\Phi_p^{(0)},s}|_{s=\frac{1-k}{2}} = (p^{\frac{k}{2}} + p^{1-\frac{k}{2}}) f_{\Phi_p^{(0)},s}|_{s=\frac{1-k}{2}}$$

and hence by (1.3.4),

$$T_p F_k(z, g_\mathrm{f}) = p^{\frac{k}{2}-1}(p^{\frac{k}{2}} + p^{1-\frac{k}{2}}) F_k(z, g_\mathrm{f}) = (1 + p^{k-1}) F_k(z, g_\mathrm{f}).$$

It follows that

$$\begin{aligned}
&\mathbf{U}_p F_k^*(z, g_\mathrm{f}) \\
&= \mathbf{U}_p F_k(z, g_\mathrm{f}) - \mathbf{U}_p F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) \\
&= T_p F_k(z, g_\mathrm{f}) - F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) - \mathbf{U}_p F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) \\
&= (1 + p^{k-1}) F_k(z, g_\mathrm{f}) - F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) - \sum_{x=0}^{p-1} F_k \left(z, g_\mathrm{f} \begin{pmatrix} p & px \\ 0 & p \end{pmatrix} \right) \\
&= (1 + p^{k-1}) F_k(z, g_\mathrm{f}) - F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) - p^{k-2} \sum_{x=0}^{p-1} F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \\
&= (1 + p^{k-1}) F_k(z, g_\mathrm{f}) - F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) - p^{k-2} \sum_{x=0}^{p-1} F_k(z, g_\mathrm{f}) \\
&= F_k(z, g_\mathrm{f}) - F_k \left(z, g_\mathrm{f} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) \\
&= F_k^*(z, g_\mathrm{f}),
\end{aligned}$$

doi:10.6342/NTU202201144

as desired. ◻

Lemma 4.1.2. Let $D = ((r)-(s), g_{\mathrm{f}}) \in \mathrm{GL}_2(\mathbf{Q})(\mathrm{Div}^0(\Gamma\infty)\times\{1\})\,\mathrm{GL}_2(\widehat{\mathbf{Q}})$. Then the assignment

$$\mu_{F_k^*}(D)(b+p^n\mathbf{Z}_p) = \xi_{F_k^*}\left(D\begin{pmatrix} p^n & b \\ 0 & 1\end{pmatrix}\right)(Y^{k-2})$$

defines a family of p-adic measures

$$\int_{b+p^n\mathbf{Z}_p} P(x,1)d\mu_f(D)(x) = \xi_{F_k^*}\left(D\begin{pmatrix} p^n & b \\ 0 & 1\end{pmatrix}\right)\left(P\middle|\begin{pmatrix} p^n & b \\ 0 & 1\end{pmatrix}\right)$$

for any $P \in L_{k-2}(\overline{\mathbf{Z}}_p)$, any non-negative integer n and any $b \in (\mathbf{Z}_p/p^n\mathbf{Z}_p)^\times$, where $\xi_{F_k^*}$ is the p-adic modular symbol defined by (3.2.4).

Proof. This is a special case of [HY21, Lemma 6.1] in view of the fact that F_k^* is a $\mathbf{U}_p$-eigenform with eigenvalue 1 by (4.1.1), and the fact that the modular symbol $\xi_{F_k^*}\left(D\left(\begin{smallmatrix} p^n & b \\ 0 & 1\end{smallmatrix}\right)\right)\left(P\middle|\left(\begin{smallmatrix} p^n & b \\ 0 & 1\end{smallmatrix}\right)\right)$ converges by (3.2.6). ◻

Lemma 4.1.3. There is a family of measures $\mu(D)$ on $\mathbf{Q}_p^2 - \{(0,0)\}$ indexed by $D \in \mathrm{GL}_2(\mathbf{Q})(\mathrm{Div}^0(\Gamma\infty)\times\{1\})\,\mathrm{GL}_2(\widehat{\mathbf{Q}})$ such that

$$\int_{\mathbf{Z}_p\oplus\mathbf{Z}_p^\times} f(x/y)h(x,y)d\mu(D) = \int_{\mathbf{Z}_p} f(t)h(t,1)d\mu_{F_k^*}(D)$$

for every $h \in L_{k-2}(\mathbf{Q}_p)$ and continuous $f : \mathbf{Z}_p \to \mathbf{C}_p$, and

$$\int_{\mathbf{Z}_p^\times\oplus p\mathbf{Z}_p} g(x,y)d\mu(D) = \int_{\mathbf{Z}_p\oplus\mathbf{Z}_p^\times} g\left|\begin{pmatrix} 0 & 1 \\ -p & 0\end{pmatrix}\right.(x,y)d\mu\left(D\begin{pmatrix} 0 & 1 \\ -p & 0\end{pmatrix}_p\right)$$

doi:10.6342/NTU202201144

for every continuous $g : \mathbf{Z}_p^\times \oplus p\mathbf{Z}_p \to \mathbf{C}_p$, where the measure $d\mu_{F_k^*}(D)$ is defined in lemma (4.1.2).

Proof. This is [DD06, Theorem 4.2]. □

Lemma 4.1.4. Let $D = ((r) - (s), g_{\mathrm{f}}) \in \mathrm{GL}_2(\mathbf{Q})(\mathrm{Div}^0(\Gamma\infty) \times \{1\})\,\mathrm{GL}_2(\widehat{\mathbf{Q}})$ and $h \in L_{k-2}(\mathbf{Q}_p)$. Then

$$\int_{\mathbf{X}_p} h(x, y) d\mu((r) - (s), g_{\mathrm{f}}) = \int_r^s F_k^{**}(z, g_{\mathrm{f}}) h|g_p^{-1}(z, 1) dz.$$

Proof. Assume p is inert. By (4.1.3),

$$\begin{aligned}
&\int_{\mathbf{Z}_p \oplus \mathbf{Z}_p^\times \sqcup \mathbf{Z}_p^\times \oplus p\mathbf{Z}_p} h(x, y) d\mu(D) \\
&= \int_{\mathbf{Z}_p \oplus \mathbf{Z}_p^\times} h(x, y) d\mu(D) + \int_{\mathbf{Z}_p^\times \oplus p\mathbf{Z}_p} h(x, y) d\mu(D) \\
&= \int_{\mathbf{Z}_p} h(t, 1) d\mu_{F_k^*}(D) + \int_{\mathbf{Z}_p \oplus \mathbf{Z}_p^\times} h| \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} (x, y) d\mu \left(D \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_p \right) \\
&= \int_{\mathbf{Z}_p} h(t, 1) d\mu_{F_k^*}(D) + \int_{\mathbf{Z}_p} h| \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} (t, 1) d\mu \left(D \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_p \right)
\end{aligned}$$

By (4.1.2), this equals to

$$\begin{aligned}
&\xi_{F_k^*}(D)(h) + \xi_{F_k^*} \left(D \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_p \right) \left(h| \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \right) \\
&= \eta_{F_k^*}(D)(h|g_p^{-1}) + \eta_{F_k^*} \left(D \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_p \right) (h|g_p^{-1})
\end{aligned}$$

doi:10.6342/NTU202201144

$$= \int_r^s \left(F_k^*(z, g_{\mathrm{f}}) + F_k^*(z, g_{\mathrm{f}} \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_p \right) h|g_p^{-1}(z,1)dz$$

$$= \int_r^s F_k^{**}(z, g_{\mathrm{f}}) h|g_p^{-1}(z,1)dz,$$

where all of the terms involved are convergent by (3.2.6). Similarly, if p is split, then

$$\int_{\mathbf{Z}_p^\times \oplus \mathbf{Z}_p^\times} h(x,y)d\mu(D)$$

$$= \int_{\mathbf{Z}_p \oplus \mathbf{Z}_p^\times} \mathbb{I}_{\mathbf{Z}_p^\times}(\frac{x}{y})h(x,y)d\mu(D)$$

$$= \int_{\mathbf{Z}_p} \mathbb{I}_{\mathbf{Z}_p^\times}(t)h(t,1)d\mu_{F_k^*}(D)$$

$$= \int_{\mathbf{Z}_p^\times} h(t,1)d\mu_{F_k^*}(D)$$

$$= \int_{\mathbf{Z}_p} h(t,1)d\mu_{F_k^*}(D) - \int_{p\mathbf{Z}_p} h(t,1)d\mu_{F_k^*}(D)$$

$$= \xi_{F_k^*}(D)(h) - \xi_{F_k^*}\left(D \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p \right) \left(h| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \eta_{F_k^*}(D)(h|g_p^{-1}) - \eta_{F_k^*}\left(D \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p \right) (h|g_p^{-1})$$

$$= \int_r^s \left(F_k^*(z, g_{\mathrm{f}}) - F_k^*(z, g_{\mathrm{f}} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p \right) h|g_p^{-1}(z,1)dz$$

$$= \int_r^s F_k^{**}(z, g_{\mathrm{f}}) h|g_p^{-1}(z,1)dz,$$

as desired. □

doi:10.6342/NTU202201144

4.2 Construction of the p-adic L-functions

We prove the main theorem in this subsection. We write the distinguished period integral as a linear combination of p-adic modular symbols and construct the p-adic L-function for F. We also give the explicit interpolating formula.

Theorem 4.2.1. Let

$$c = -24 \sum_{d|q^2} dn_d \chi_{\overline{w}}(d^{-1}) |d^{-1}|_q^{1-\frac{k}{2}}$$

where q is split into $\overline{w}w$ in F. Then

$$c \prod_{v \neq p} L(1-\frac{k}{2}, \chi_v) = -\delta \prod_{v \text{ is split}} |2\theta|_v^{2-k} \operatorname{vol}(\widehat{\mathcal{O}}_F^\times)$$
$$\times \sum_{a \in \mathrm{Cl}^+(F)} \chi(a) \operatorname{N}(a)^{k-2} \int_{\mathbf{X}_p} Q(z,1)^{\frac{k-2}{2}} d\mu((\gamma_a^{-1}\infty) - (\Psi(\varepsilon)\gamma_a^{-1}\infty)), \Psi(a)\varsigma_{\mathrm{f}})$$

where

- $\delta = \overline{\theta} - \theta$,
- $F_+ \cap \widehat{\mathcal{O}}_F^\times = \varepsilon^{\mathbf{Z}}$,
- $Q(x,y) = \begin{cases} x^2 - \operatorname{Tr}(\theta)xy + \operatorname{N}(\theta)y^2 & \text{if } v \text{ is non-split,} \\ -xy/2 & \text{if } v \text{ is split,} \end{cases}$
- $\Psi(a)\varsigma_{\mathrm{f}} = \gamma_a u_a \in \mathrm{GL}_2(\mathbf{Q})^+ U_2(pq^2)$ for $a \in \mathrm{Cl}^+(F)$.

Define

$$\mathbf{P}(s,\chi) = -\delta \prod_{v \text{ is split}} \langle |2\theta|_v \rangle^{2-s} \operatorname{vol}(\widehat{\mathcal{O}}_F^\times)$$

doi:10.6342/NTU202201144

$$\times \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\langle \mathrm{N}(a)\rangle^{s-2} \int_{\mathbf{X}_p} \langle Q_\theta(z,1)\rangle^{\frac{s-2}{2}} d\mu((\gamma_a^{-1}\infty)-(\Psi(\varepsilon)\gamma_a^{-1}\infty)),\Psi(a)\varsigma_{\mathrm{f}}),$$

Then $\mathbf{P}(s,\chi)$ is a p-adic analytic function such that

$$\mathbf{P}(s,\chi) = cL_p(1-\frac{s}{2},\chi),$$

where $L_p(s,\chi)$ is the Kubota-Leopoldt p-adic L-functions.

Proof. By proposition (3.3.1) and lemma (4.1.4),

$$\begin{aligned}
&P_{\Phi,s,\chi}(\varsigma)\\
=&\mathrm{vol}(\widehat{\mathcal{O}}_F^\times)(2\sqrt{-1})^{-1}\delta(\sqrt{-1})^{\frac{k-2}{2}}\\
&\times \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}} \int_{\gamma_a\infty}^{\Psi(\varepsilon)\gamma_a\infty} F_k^{**}(z,\Psi(a)\varsigma_{\mathrm{f}})Q_\theta(z,1)^{\frac{k-2}{2}}dz\\
=&\mathrm{vol}(\widehat{\mathcal{O}}_F^\times)(2\sqrt{-1})^{-1}\delta(\sqrt{-1})^{\frac{k-2}{2}}\\
&\times \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{\frac{k-2}{2}} \int_{\mathbf{X}_p} Q_\theta|\gamma_a(z,1)^{\frac{k-2}{2}} d\mu((\gamma_a^{-1}\infty)-(\Psi(\varepsilon)\gamma_a^{-1}\infty)),\Psi(a)\varsigma_{\mathrm{f}})
\end{aligned}$$

where $Q_\theta|\gamma_a(x,y)=\mathrm{N}(a)Q(x,y)$. By proposition (2.1.4) and (2.2.5),

$$\begin{aligned}
&\prod_{v\neq p} L(1-\frac{k}{2},\chi_v)\left(-24\sum_{d|q^2} dn_d\chi_{\overline{w}}(d^{-1})|d^{-1}|_q^{1-\frac{k}{2}}\right)\\
=&\prod_{v \text{ is split}} |2\theta|_v^{2-k}2(\sqrt{-1})^{-\frac{k}{2}}(2\sqrt{-1})^{-1}\delta(\sqrt{-1})^{\frac{k-2}{2}}\,\mathrm{vol}(\widehat{\mathcal{O}}_F^\times)\\
&\times \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{k-2} \int_{\mathbf{X}_p} Q(z,1)^{\frac{k-2}{2}} d\mu((\gamma_a^{-1}\infty)-(\Psi(\varepsilon)\gamma_a^{-1}\infty)),\Psi(a)\varsigma_{\mathrm{f}})\\
=&-\delta\prod_{v \text{ is split}} |2\theta|_v^{2-k}\,\mathrm{vol}(\widehat{\mathcal{O}}_F^\times)\\
&\times \sum_{a\in \mathrm{Cl}^+(F)} \chi(a)\,\mathrm{N}(a)^{k-2} \int_{\mathbf{X}_p} Q(z,1)^{\frac{k-2}{2}} d\mu((\gamma_a^{-1}\infty)-(\Psi(\varepsilon)\gamma_a^{-1}\infty)),\Psi(a)\varsigma_{\mathrm{f}}),
\end{aligned}$$

doi:10.6342/NTU202201144

as desired. Assume $p-1$ divides $\frac{k-2}{2}$. Then

$$\begin{aligned}
&\prod_{v\neq p} L(1-\frac{k}{2},\chi_v)\left(-24\sum_{d|q^2} dn_d\chi_{\overline{w}}(d^{-1})\langle |d^{-1}|_q\rangle^{\frac{k-2}{2}}\right)\\
&=-\,\delta\prod_{v\text{ is split}}\langle|2\theta|_v\rangle^{2-k}\operatorname{vol}(\widehat{\mathcal{O}}_F^\times)\\
&\quad\times\sum_{a\in\mathrm{Cl}^+(F)}\chi(a)\langle\mathrm{N}(a)\rangle^{k-2}\int_{\mathbf{X}_p}\langle Q(z,1)\rangle^{\frac{k-2}{2}}d\mu((\gamma_a^{-1}\infty)-(\Psi(\varepsilon)\gamma_a^{-1}\infty)),\Psi(a)\varsigma_{\mathfrak{f}}),
\end{aligned}$$

which is $\mathbf{P}(k,\chi)$. □

References

[Bar78] Daniel Barsky. Fonctions zeta p-adiques d'une classe de rayon des corps de nombres totalement réels. In *Groupe d'Etude d'Analyse Ultramétrique (5e année: 1977/78)*, pages Exp. No. 16, 23. Secrétariat Math., Paris, 1978.

[BKL18] Alexander Beilinson, Guido Kings, and Andrey Levin. Topological polylogarithms and p-adic interpolation of L-values of totally real fields. *Math. Ann.*, 371(3-4):1449–1495, 2018.

[Bum97] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[Cas73] William Casselman. On some results of Atkin and Lehner. *Math. Ann.*, 201:301–314, 1973.

doi:10.6342/NTU202201144

[CD14] Pierre Charollois and Samit Dasgupta. Integral Eisenstein cocycles on $\mathbf{GL}_n$, I: Sczech's cocycle and p-adic L-functions of totally real fields. *Camb. J. Math.*, 2(1):49–90, 2014.

[CN79] Pierrette Cassou-Noguès. Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques. *Invent. Math.*, 51(1):29–59, 1979.

[DD06] Henri Darmon and Samit Dasgupta. Elliptic units for real quadratic fields. *Annals of mathematics*, pages 301–346, 2006.

[Dix03] A. C. Dixon. Summation of a certain Series. *Proc. Lond. Math. Soc.*, 35:284–289, 1903.

[DK] Samit Dasgupta and Mahesh Kakde. Brumer-stark units and hilbert's 12th problem. `https://arxiv.org/abs/2103.02516`.

[DR80] Pierre Deligne and Kenneth A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[Hsi12] Ming-Lun Hsieh. On the non-vanishing of Hecke L-values modulo p. *Amer. J. Math.*, 134(6):1503–1539, 2012.

[HY21] Ming-Lun Hsieh and Shunsuke Yamana. Restriction of eisenstein series and stark–heegner points. *Journal de Théorie des Nombres de Bordeaux*, 33(3):887–944, 2021.

[Kit94] Koji Kitagawa. On standard p-adic L-functions of families of elliptic cusp forms. In *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture*

doi:10.6342/NTU202201144

(Boston, MA, 1991), volume 165 of *Contemp. Math.*, pages 81–110. Amer. Math. Soc., Providence, RI, 1994.

[KL06] Andrew Knightly and Charles Li. *Traces of Hecke operators*, volume 133 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[Spi14] Michael Spiess. Shintani cocycles and the order of vanishing of p-adic Hecke L-series at $s = 0$. *Math. Ann.*, 359(1-2):239–265, 2014.

doi:10.6342/NTU202201144