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碩士論文

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Master Thesis

複合更新理論應用在散射邊界上

An Application of Compound Renewal Theory on a Diffuse Boundary

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本論文係蘇于瑞君（R07246010）在國立臺灣大學應用數學科學研究所完成之碩士學位論文，於民國 109 年 7 月 10 日承下列考試委員審查通過及口試及格，特此證明

口試委員：陳逸昆 ________________（簽名）

（指導教授）

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Abstract

A transport equation with a diffuse boundary condition is studied for the propagation of a gas near vacuum. Our work reaches large time asymptotic results. Also, some stochastic processes concerning the structure of the differential equation and the boundary condition were studied. The characteristic method leads to the explicit solution of the transport equation, which is a renewal function. The local limit theorem is our main tool studying this function. When turning to the specific renewal function in our problem, more structures were revealed and the space-time relationship is given by extra properties of the second rate function.

Keywords: *Diffuse Boundary Condition; Random Walks; Renewal Theory; Regenerative Processes; Compound Renewal Processes; Large Deviations; Local Limit Theorem; Second Rate Function*

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摘要

研究的是具有散射邊界條件的漂移方程，此模型描述在上半空間裡一團接近真空的氣體。我們的工作得到了當時間趨近無窮的漸近結果。此外，研究了一些與微分方程的結構和邊界條件有關的隨機過程。漂移方程的顯式解是一個更新函數。局部極限定理是我們研究此函數的主要工具。我們問題中它有更多結構，因此，這個漸進結果在不同的時空關係下由第二速率函數的額外屬性所決定。

關鍵字: 散射邊界條件;隨機漫步；更新理論；再生過程；複合更新過程；大偏差理論；局部極限定理；第二速率函數。

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1 Introduction

In the work of Liu and Yu (see [5]), a Boltzmann equation under a constant gravitational field on the upper half space is considered, for $(\mathbf{x}, t, \mathbf{v}) \in \mathbf{H}^+ \times \mathbb{R}^+ \times \mathbb{R}^3$,

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \mathbf{g} \cdot \nabla_{\mathbf{v}} f = Q(f) \tag{1.1}$$

with Maxwell diffuse boundary condition, for $\mathbf{x} \in \{z = 0\}$[1]

$$\begin{aligned} f(\mathbf{x}, t, \mathbf{v})|_{\mathbf{v}_3 > 0} &= \rho(\mathbf{x}, t)\sqrt{2\pi/\theta} M(\mathbf{v})|_{\mathbf{v}_3>0}, \\ \rho(\mathbf{x}, t) = \rho(x, y, t) :&= \int_{\mathbf{v}_3<0} -\mathbf{v}_3 f(\mathbf{x}, t, \mathbf{v}) d\mathbf{v}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{x} = (x, y, z) \in \mathbf{H}^+, &\qquad \mathbf{g} = (0, 0, g),\ g > 0, \\ \mathbf{H}^+ := \{\mathbf{x} \in \mathbb{R}^3 | z \geq 0\}, &\qquad \mathbb{R}^+ := \{t \geq 0\}, \\ M(\mathbf{v}) := \frac{e^{-|\mathbf{v}|^2/(2\theta)}}{(2\pi\theta)^{3/2}}. & \end{aligned}$$

Fix any time t, the quantity $f(\cdot, t, \cdot)$ is a probability density which gives the probability per unit phase-space volume if it's normalized.[2] This equation models the diffusion of gas near vacuum that being reflected off a flat surface with the Maxwellian distribution and being pulled back to the surface by the gravitational force. The quantity ρ is the flux that goes into the boundary $\{z = 0\}$. Under the boundary condition, this inward flux equals to the flux that goes outward from the boundary.

The following theorem gives an upper bound estimate of the gas density f showing that the propagation of the gas is similar to a two-dimensional heat flow in the central ($|\mathbf{x}| = o(t)$) part.

[1] We abuse the notation that when $\mathbf{x} \in \{z = 0\}$, $\mathbf{x}$ denotes either $(x, y, 0)$ or (x, y) according to the context.

[2] This is achieved when the initial data $f(\mathbf{x}, 0, \mathbf{v})$ is normalized by the fact that the solution f has a fixed total amount $||f||_t := \int f(\mathbf{x}, t, \mathbf{v}) d\mathbf{x} d\mathbf{v}$, which is constant for any $t \in \mathbb{R}^+$.

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Theorem 1.1 (LY[5]-B)**.** *There exists* $\epsilon_0 > 0$ *and* $C_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$, *if* $0 \le f(\boldsymbol{x}, 0, \boldsymbol{v}) \le \epsilon e^{-\sqrt{x^2+y^2}} e^{-gz/\theta} M(\boldsymbol{v})$, *then the solution satisfies*

$$0 \le f(\boldsymbol{x}, t, \boldsymbol{v}) \le C_0 \epsilon \begin{cases} (e^{-gz/\theta} M(\boldsymbol{v}))^{3/4} \\ \left(\frac{1}{1+t} e^{-\frac{x^2+y^2}{C_0(1+t)}} + e^{-\frac{\sqrt{x^2+y^2}+t}{C_0}}\right) \sqrt{e^{-gz/\theta} M(\boldsymbol{v})} \end{cases} \tag{1.2}$$

This theorem is obtained by first treating the collision term $Q(f)$ as a perturbation since the case of interest is the one near vacuum; thus, with $Q(f)$ omitted, a free transport equation

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \mathbf{g} \cdot \nabla_{\mathbf{v}} f = 0 \tag{1.3}$$

is studied first, which then leads to the following theorem. This theorem has a slightly stronger estimate than **Theorem 1.1**.

Theorem 1.2 (LY[5]-A)**.** *There exists* $C > 0$ *such that if* $|f(\boldsymbol{x}, 0, \boldsymbol{v})| \le e^{-\sqrt{x^2+y^2}} \left(e^{-gz/\theta} M(\boldsymbol{v})\right)^{\beta}$ *for a given* $\beta \in (0, 1)$, *then the solution* $\tilde{f}$ *of the transport equation satisfies*

$$|\tilde{f}(\boldsymbol{x}, t, \boldsymbol{v})| = \mathcal{O}(1) \Big(\frac{1}{1+t} e^{-\frac{x^2+y^2}{C(1+t)}} + e^{-\frac{\sqrt{x^2+y^2}+t}{C}} \Big) \left(e^{-gz/\theta} M(\boldsymbol{v})\right)^{\beta}.$$

Instead of upper bounds of f described in (1.2), our goal is to go further into its large time asymptotics. However, our current result is only on the transport part. We leave the non-linear collision part for future research.

In the case of a free transport equation, the behaviour of f in $\mathbf{H}^+ \times \mathbb{R}^+ \times \mathbb{R}^3$ is completely determined by the boundary flux ρ and its initial data; thus, the problem is reduced to study $\rho(\mathbf{x}, t)$ in $\mathbb{R}^2 \times \mathbb{R}^+$. (By characteristic method, the result for f can be easily recovered from ρ.) The above **Theorem 1.2** is obtained by estimates of upper bounds of $\rho(\mathbf{x}, t)$ in various regions of $(\mathbf{x}, t)$ (see [5, p.195]). Thus, asymptotics of f are reached when we have the corresponding asymptotics of ρ.

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Theorem 1.3 (Main Result)**.** *Given a radial space-time ratio* $\alpha = |\boldsymbol{x}|/t$*, suppose the initial data satisfies* $f^+(x,t) = \mathcal{O}(e^{-\lambda(\alpha)\cdot(\boldsymbol{x},t)})$.[3] *Then*

$$\rho(\boldsymbol{x},t) = ||f||\psi_1(D'(1,\alpha))\frac{C(\alpha)}{t}e^{-tD(1,\alpha)}(1+o(1)) \tag{1.4}$$

as $t \to \infty$*, where the remainder* $o(1)$ *is uniform for all sufficiently small compact set* K *such that* $\alpha \in K$*. The functions* C, D *are defined in* ***Section 2.2*** *and* λ *is an increasing function defined in* ***Section 3****.*

1.1 Method of Characteristics, the Renewal Measure

The transport equation can be solved explicitly by characteristic method. A characteristic curve $(\mathbf{x}(t), \mathbf{v}(t))$ satisfies $\frac{d}{dt}f(\mathbf{x}(t), t, \mathbf{v}(t)) = 0$; thus, given $t > t_0$,

$$\begin{aligned}\mathbf{v}(t) =& \mathbf{x}'(t) = \mathbf{v}(t_0) - \mathbf{g}(t-t_0),\\ \mathbf{x}(t) =& \mathbf{x}(t_0) + \mathbf{v}(t_0)\cdot(t-t_0) - \mathbf{g}\frac{(t-t_0)^2}{2}.\end{aligned}$$

Since f is invariant along a characteristic curve, for each point $(\mathbf{x}, \mathbf{v}) \in \mathbf{H}^+ \times \mathbb{R}^3$ the value of f can be shifted along a characteristic curve to its boundary of domain, which are either $\{t = 0\}$ and $\{z = 0\}$. In this way, separating the boundary flux ρ into two parts according to these two types of characteristic curves, we find that ρ satisfies a *renewal equation*. For $\mathbf{x} \in \{z = 0\}$,

[3] The initial flux f^+ is defined in (1.7); ψ_1 is the Laplace transform of f^+ being the only term concerning the initial data in (1.4).

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$$
\begin{aligned}
\rho(\mathbf{x},t) &= \int_{\mathbf{v}_3<0} -\mathbf{v}_3 f(\mathbf{x},t,\mathbf{v})d\mathbf{v} \\
&= \int_{\mathbf{v}_3<-\mathbf{g}t/2} -\mathbf{v}_3 f(\mathbf{x},t,\mathbf{v})d\mathbf{v} + \int_{-\mathbf{g}t/2<\mathbf{v}_3<0} -\mathbf{v}_3 f(\mathbf{x},t,\mathbf{v})d\mathbf{v} \\
&= \int_{\mathbf{v}_3<-\mathbf{g}t/2} -\mathbf{v}_3 f(x-\mathbf{v}_1 t, y-\mathbf{v}_2 t, -\mathbf{v}_3 t - \frac{g}{2}t^2, 0, \mathbf{v}+\mathbf{g}t)d\mathbf{v} \\
&\quad + \int_{-\mathbf{g}t/2<\mathbf{v}_3<0} -\mathbf{v}_3 f\big(\mathbf{x}+(2\mathbf{v}_1\mathbf{v}_3/g, 2\mathbf{v}_2\mathbf{v}_3/g, 0), t+2\mathbf{v}_3/g, \mathbf{v}-2(0,0,\mathbf{v}_3)\big)d\mathbf{v} \\
&= \int_{\mathbf{v}_3<-\mathbf{g}t/2} -\mathbf{v}_3 f(x-\mathbf{v}_1 t, y-\mathbf{v}_2 t, -\mathbf{v}_3 t - \frac{g}{2}t^2, 0, \mathbf{v}+\mathbf{g}t)d\mathbf{v} \\
&\quad + \int_{-\mathbf{g}t/2<\mathbf{v}_3<0} -\mathbf{v}_3 \rho\Big(x+\frac{2\mathbf{v}_1\mathbf{v}_3}{g}, y+\frac{2\mathbf{v}_2\mathbf{v}_3}{g}, t+\frac{2\mathbf{v}_3}{g}\Big)\sqrt{\frac{2\pi}{\theta}}M\Big(\mathbf{v}-2(0,0,\mathbf{v}_3)\Big)d\mathbf{v} \\
&= f^+(\mathbf{x},t) + Q(\mathbf{x},t) * \rho(\mathbf{x},t), \qquad (1.5)
\end{aligned}
$$

where the convolution "$*$" is taken on both $\mathbf{x}$ and t and [4]

$$
Q(\mathbf{x},t) = Q(x,y,t) := \begin{cases} \frac{g^2}{8\pi\theta^2 t}\exp\left(-\frac{x^2+y^2}{2\theta t^2} - \frac{g^2t^2}{8\theta}\right), & t>0 \\ 0 & t \le 0 \end{cases}, \qquad (1.6)
$$

$$
f^+(\mathbf{x},t) := \int_{\mathbf{v}_3<-\mathbf{g}t/2} -\mathbf{v}_3 f(x-\mathbf{v}_1 t, y-\mathbf{v}_2 t, -\mathbf{v}_3 t - \frac{g}{2}t^2, 0, \mathbf{v}+\mathbf{g}t)d\mathbf{v}. \qquad (1.7)
$$

If we interpret f^+ as the flux caused by particles that never been reflected, then moving forward in time along its characteristic curve, any particle with finite initial speed will eventually reach the "ground" $\{z=0\}$. Thus, if f is integrable,

$$
\begin{aligned}
||f^+|| := \int_0^\infty \int_{\mathbb{R}^2} f^+ d\mathbf{x}dt &= \int_{\mathbf{H}^+\times\mathbb{R}^3} f(\mathbf{x},0,\mathbf{v})d\mathbf{x}d\mathbf{v} \\
&= \int_{\mathbf{H}^+\times\mathbb{R}^3} f(\mathbf{x},t,\mathbf{v})d\mathbf{x}d\mathbf{v} =: ||f||.
\end{aligned}
$$

The renewal equation (1.5) can be solved uniquely (a proof can be found in e.g.

[4] The function $Q(\mathbf{x},t)$ defined in [5] has a typo.

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[6, p. 115]) with solution given by

$$\rho(\mathbf{x},t) = R * f^{+}(\mathbf{x},t), \tag{1.8}$$

$$R(\mathbf{x},t) := \sum_{n=0}^{\infty} Q^{n*}(\mathbf{x},t), \tag{1.9}$$

where Q^{0*} is defined to be the Dirac delta function. $R(\mathbf{x},t)$ is the *renewal function* associated with Q.

Being non-negative and $\int Q d\mathbf{x}dt = 1$, it can be treated as a probability density of a random vector ξ. Let ξ_i, $i = 1, 2, ...$ be a sequence of iid copies of ξ; by an elementary property of convolution, $S_n = \sum_{i=1}^{n} \xi_i$ is a random vector with the density Q^{n*} for each n. Then the renewal function $R(\mathbf{x},t)$ as sums of probability densities defines the *renewal measure*

$$H(B) := \sum_{n=0}^{\infty} P(S_n \in B) = \int_B R(\mathbf{x},t) d\mathbf{x}dt \tag{1.10}$$

for $B \subset \mathbb{R}^2 \times \mathbb{R}^+$, which is finite for every bounded measurable set B if (see [4, p.654]) either

1. The expectation $E\xi \neq 0$ exists.

2. $E\xi = 0$, the second moment matrix $E\xi^T\xi$ exists and has rank not less than 3.

In our case,

$$E\xi = \int (x,y,t)Q(x,y,t)dxdydt = (0, 0, \frac{\sqrt{2\pi\theta}}{g}) \neq 0.$$

In fact, since $Q(\mathbf{x},t) = 0$ for $t \leq 0$, the third component $\xi^{(3)} > 0$ almost surely and hence $S_n^{(3)} > 0$ almost surely for each n. This gives a further structure of the renewal measure H. To give this third component a special treatment, we name it τ and write $\xi = (\zeta, \tau)$.

As the random variable τ is positive, the finite sums of its iid copies form a

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sequence with strong order, which means it can be treated as an index and is referred as the renewal times. Some stochastic processes index by time t are defined later, which has a structure concerning these renewal times.

1.2 Numerical Computation

For small time t, the flux $\rho(\mathbf{x}, t)$ can be achieved numerically by computing the renewal function $R(\mathbf{x}, t) = \sum_{n=0}^{\infty} Q^{n*}(\mathbf{x}, t)$.[5] This subsection aims for gaining some intuitions by visualizing our problem.

For demonstration, the dimension of $\mathbf{x}$ is reduced to one (it's easier to plot the result) and coefficients are scaled and normalized. Let

$$Q(\mathbf{x}, t) = Q(x, t) := \frac{2}{\sqrt{\pi}} \exp\left(-\frac{x^2}{t^2} - t^2\right),$$
$$f_\tau(t) := \int_{-\infty}^{\infty} Q(x, t) dx = 2t \exp\left(-t^2\right).$$

The function Q chosen here leads to the same f_τ as the function Q defined by (1.6) up to a scaling.[6] It's an easy observation to see that

$$\int R(\mathbf{x}, t) d\mathbf{x} = \sum \int Q^{n*}(x, t) dx = \sum f_\tau^{n*}(t).$$

Under suitable initial data[7], the initial flux $f^+(\mathbf{x}, t)$ can be chosen to be the Dirac delta function, so that the flux $\rho(x, t) = \sum_{n=0}^{\infty} Q^{n*}(x, t)$ and the flow rate on $\mathbb{R}$ is $\int_{-\infty}^{\infty} \rho(x, t) dx = \sum_{n=0}^{\infty} f_\tau^{n*}(t)$. The density $\rho(x, t)$ and $Q^{n*}(x, t)$ are shown by contours in **Figure 1,2**, while **Figure 3** shows the flow rate $\int_{-\infty}^{\infty} \rho(x, t) dx$ and $f_\tau^{n*}(t), \quad n = 1, 2, 3, ...,$ for $x = 0$.

[5] "small" depends on one's computation power and required accuracy.

[6] Note that f_τ can be treated as a probability density, which defines a random variable τ concerning to processes defined in the next section.

[7] Choose the initial data $f(\mathbf{x}, 0, \mathbf{v}) = \delta_{\mathbf{x}=0; \mathbf{v}=(0,0,-1)}$.

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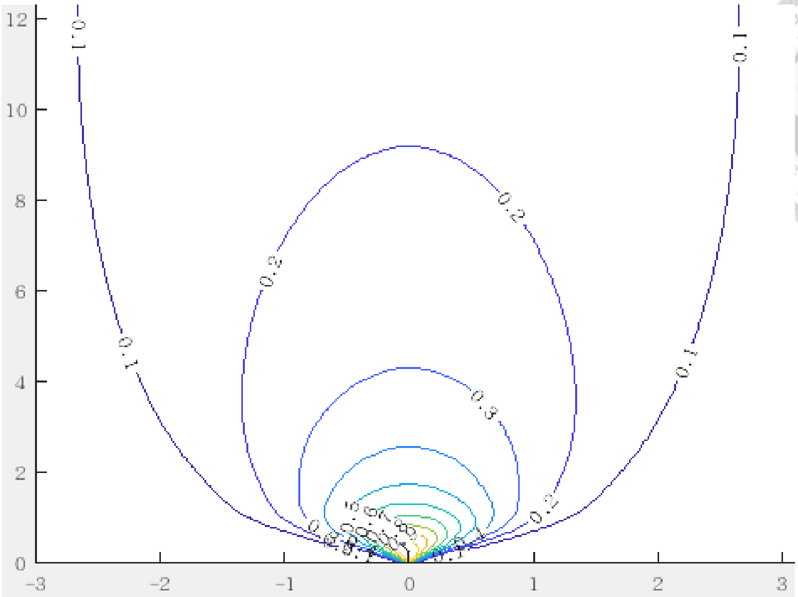

Figure 1: Contour of $\rho(x, t)$ for $0 < t < 12$ and $-3 < x < 3$. Started quite concentrated at the origin, it gradually spreads out as t grows.

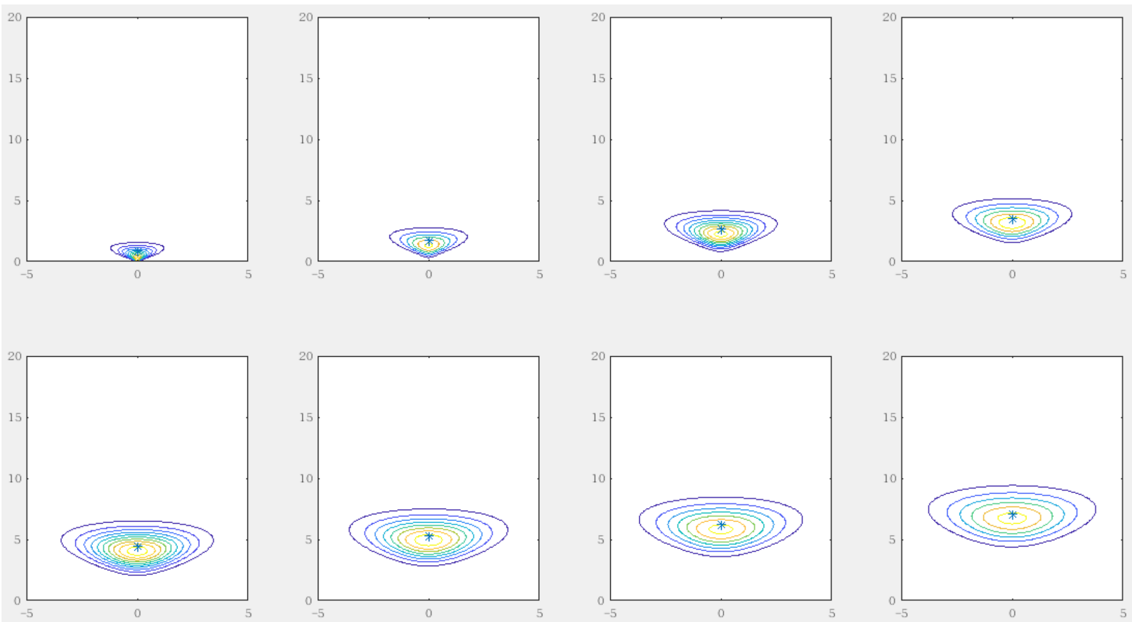

Figure 2: Contour of $Q^{n*}(x, t)$ for $n = 1, 2, ..., 8$, $0 < t < 20$ and $-5 < x < 5$. The center of $Q^{n*}(x, t)$ shifts with a unit pace as n grows; meanwhile, their shapes gradually deform to a normal distribution.

From **Figure 3**, one can see the flow rate $\int \rho(x, t)dx$ becomes almost a constant after oscillating a few periods (with the fixed period $E\tau$). The convergence of the

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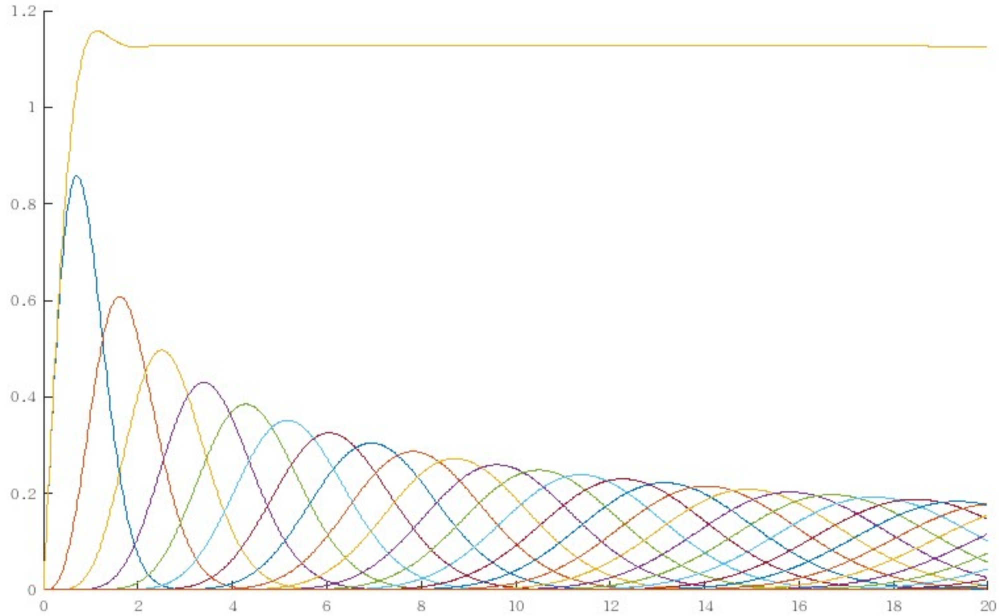

Figure 3: The flow rate $\int_{-\infty}^{\infty} \rho(x,t)dx$ and $f_\tau^{n*}(t)$, $n = 1, 2, 3, ...$, for $t < 20$. The flow rate $\int \rho(x,t)dx$ converges to a constant $1/E\tau = 2/\sqrt{\pi} \approx 1.128$. The peak point of $f_\tau^{n*}(t)$ is reached when $t = nE\tau$.

flow rate can be shown by an elementary renewal theory (see e.g. [6, Sec 2.2]) that

$$\lim_{t\to\infty} N(t)/t = 1/E\tau \quad \text{a.s.} \tag{1.11}$$

where $N(t)$ and τ are defined in **Definition 4.1**.

For small time t, the flux $\rho(x,t)$ is dominated by a finite number of $Q^{n*}(x,t)$ for n near $t/E\tau$ i.e. letting $m = \lfloor t/E\tau \rfloor$, $\rho(x,t)$ is well approximated by $\sum_{n=0}^{m+N} Q^{n*}(x,t)$ for some $N < \infty$ (with the error term of $\mathcal{O}(\exp(-N))$) estimated by exponential Chebyshev's inequality). This is why the numerical approximation is feasible. However, the required N(upon an accuracy) grows as t grows.[8] This is when the limit theorems kicks in.

[8]As can be seen that $Q^{n*}(x,t)$ spreads out as n grows so more effective overlaps of $Q^{n*}(x,t)$ on the t-large side of **Figure 3**.

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2 Local Limit Theorem

This section aims for the asymptotics of the renewal measure H defined in (1.10). If f^+ is the Dirac delta function, then $\rho(\mathbf{x}, t)$ is the density of H and it's regarded as the homogeneous case. For general f^+, the inhomogeneous renewal measure H_1 is defined as H besides that among the iid sequence of random variables ξ_i, ξ_1 is no longer being identical. Recall that $||f^+|| := \int f^+ d\mathbf{x}dt = ||f||$ and assign the probability density of ξ_1 with $f^+/||f||$, then $\rho(\mathbf{x}, t)/||f||$ becomes the density of H_1.[9]

Let us begin with a more general problem formulation. Set ξ be a nondegenerate random vector in $\mathbb{R}^d$, with distribution F, recall that $S_n = \sum_{i=1}^n \xi_i$, where $\xi_i,\ i = 1, 2, ...$ is a sequence of iid copies of ξ.

2.1 Local Limit Theorem for Sums of Independent Random Vectors

Before going to the asymptotics of H, we should first study the asymptotics of S_n. Let

$$\psi(\lambda) := Ee^{\lambda\cdot\xi},\ \lambda \in \mathbb{R}^d$$

be the Laplace transform of ξ and $\mathcal{A} := \{\psi < \infty\}$ be its effective domain. For $\lambda \in \mathcal{A}$, let F denotes the distribution of ξ, its *Cramér transform* is defined such that

$$F_\lambda(dy) = \frac{e^{\lambda\cdot y}}{\psi(\lambda)} F(dy). \tag{2.1}$$

[9]The normalization of f^+ can be omit i.e. ξ_1 need not be a random variable, doing so is just for an easier interpretation.

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Denoted by $\xi_{(\lambda)}$ is the random vector with this tilted, or re-weighted distribution and $S_{n(\lambda)}$ is defined accordingly. An easy computation gives[10]

$$E\xi_{(\lambda)} = (\ln \psi(\lambda))',\ V\xi_{(\lambda)} = (\ln \psi(\lambda))''.$$

The key point of the local limit theorem of S_n is choosing the right λ of the Cramér transform such that the center of $S_{n(\lambda)}$ shifts to the point of interest, so the distribution around this point can be approximated by the central limit theorem. Given $v \in \mathbb{R}^n$, by properties of the Laplace transform, we have

$$\frac{e^{\lambda \cdot v}}{\psi^n(\lambda)} P(S_n \in dv) = P(S_{n(\lambda)} \in dv),$$
$$P(S_n \in dv) = e^{-\lambda \cdot v + n \ln \psi(\lambda)} P(S_{n(\lambda)} \in dv).$$

We can replace $P(S_{n(\lambda)} \in dv)$ with $(2\pi n)^{-d/2} \det(V\xi_{(\lambda)})^{-1/2}(1+o(1))$ by the central limit theorem if λ is chosen such that $ES_{n(\lambda)} = v$. This is achieved when $(\ln \psi(\lambda))' = E\xi_{(\lambda)} = v/n$. Let $\alpha = v/n$ and assume such λ can be achieved and determined according to this value, i.e. $\lambda = \lambda(\alpha)$, we rewrite the above equation as

$$P(S_n \in dv) = \frac{e^{-n\left(\alpha \cdot \lambda(\alpha) - \ln \psi(\lambda(\alpha))\right)}}{\sqrt{(2\pi n)^d \det((\ln \psi(\lambda(\alpha))''))}}(1+o(1))dv \tag{2.2}$$

as $n \to \infty$.

The *deviation function* corresponding to ξ is defined as

$$\Lambda(\alpha) := \sup_{\lambda}\{\lambda \cdot \alpha - \ln \psi(\lambda)\}, \tag{2.3}$$

which is the conjugate of the convex function $\ln \psi(\lambda)$. Let $\mathcal{L}$ denotes the domain of analyticity of Λ.

---

10 Here $V\xi_{(\lambda)}$ denotes the covariance matrix $\mathrm{Cov}[\xi_{(\lambda)}]$.

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Theorem 2.1 (Local limit theorem for sums of independent random vectors [3]). *Given a compact set $K \subset \mathcal{L}$, for $\alpha = v/n \in K$, the equation $(\ln \psi(\lambda))' = \alpha$ can be solved uniquely so the function $\lambda(\alpha)$ described in (2.2) is well defined; furthermore, the function $\alpha \cdot \lambda(\alpha) - \ln \psi(\lambda(\alpha))$ coincides with $\Lambda(\alpha)$. Thus, rewrite (2.2),*

$$P(S_n \in dv) = \frac{e^{-n\Lambda(\alpha)}}{\sqrt{(2\pi n)^d \det((\ln \psi(\lambda(\alpha))'')}}(1 + o(1))dv \tag{2.4}$$

as $n \to \infty$, where the remainder $o(1)$ is uniform in K.

2.2 Local Limit Theorem for the Renewal Measure

The local limit theorem of the renewal measure has a similar form to (2.4) but with a different rate function. The *second rate function* (see [4])

$$D(\alpha) := \inf_{r>0} r\Lambda(\alpha/r),$$

in terms of convex analysis, is the *positively homogeneous convex function* generated by Λ. Let $\mathfrak{D}$ denotes its domain of analyticity. Let ψ_1 denotes the Laplace transform of ξ_1, $\mathcal{A}_1 := \{\psi_1 < \infty\}$ be its effective domain and $\mathcal{A}_K := \{D'(\alpha) \mid \alpha \in K\}$.

Theorem 2.2 (Integro-local limit theorem for the renewal measure [2][4]). *For $x \in \mathbb{R}^d$ and $T \in \mathbb{R}$, let $\alpha := x/T$. Suppose $\alpha \in K \subset \mathfrak{D}$ for a compact set K separated from the origin and $\mathcal{A}_K \subset \mathcal{A}_1$, then*

$$H(dx) = \frac{dx}{T^{(d-1)/2}} C(\alpha) e^{-TD(\alpha)}(1 + o(1)), \tag{2.5}$$

$$H_1(dx) = \psi_1(D'(\alpha))H(dx)(1 + o(1)) \tag{2.6}$$

as $T \to \infty$, where the remainder $o(1)$ is uniform in K.

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A brief outline of the proof is given here. For $x \in \mathbb{R}^d$,

$$H(dx) := \sum_{n=1}^{\infty} P(S_n \in dx).$$

Depending on the structure of the distribution of ξ, there're several ways showing that this series is approximately the partial sums

$$\sum_{\mathcal{N}_2} P(S_n \in dx),$$

where $\mathcal{N}_2 := \{c_1 T \le n \le c_2 T\}$ and c_1, c_2 are constants depending on α. Let $r := n/T$ and $L(r) := r\Lambda(\alpha/r)$, by **Theorem 2.1** and the Laplace method,[11]

$$\begin{aligned}
\sum_{\mathcal{N}_2} P(S_n \in dx) &= \sum_{\mathcal{N}_2} \frac{C_1(x/n)}{n^{d/2}} e^{-n\Lambda(x/n)}(1+o(1))dx \\
&= \sum_{r \in [c_1, c_2]} \frac{C_1(\alpha/r)}{(Tr)^{d/2}} e^{-Tr\Lambda(\alpha/r)}(1+o(1))dx \\
&= \Big(\int_{c_1}^{c_2} \frac{TC_1(\alpha/r)}{(Tr)^{d/2}} e^{-TL(r)} dr \Big)(1+o(1))dx \\
&= \frac{TC_1(\alpha/r_\alpha)}{(Tr_\alpha)^{d/2}} \sqrt{\frac{2\pi}{TL''(r_\alpha)}} e^{-TD(\alpha)}(1+o(1))dx.
\end{aligned}$$

For the inhomogeneous case ($\xi_1 \neq_d \xi$), we need the following lemma.[12]

Lemma 2.3. *Under the assumptions in* ***Theorem 2.2****, there exists* $c > 0$ *and* $C_0, T_0 < \infty$ *such that*

$$\big| H_1(dx) - E\Big(H\big(d(x-\xi_1)\big); |\xi_1| \le \ln^2 T \Big) \big| \le C_0 e^{-TD(\alpha) - c\ln^2 T} \tag{2.7}$$

for $T \ge T_0$.

[11] For details, see [2, p. 580].

[12] In fact, $H_1(B) = E\big(H(B-\xi_1)\big)$ so this lemma means if we put a restriction on ξ_1, then there is a corresponding error.

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For $|\xi_1| \leq \ln^2 T$,

$$-TD(\alpha - \xi_1/T) = -TD(\alpha) + D'(\alpha) \cdot \xi_1 + \mathcal{O}(\ln^4 T/T).$$

Thus,

$$\begin{aligned}
&E\Big(H\big(d(x-\xi_1)\big); |\xi_1| \leq \ln^2 T\Big) \\
=&E\Big(\frac{dx}{T^{(d-1)/2}} C(\alpha - \xi_1/T) e^{-TD(\alpha-\xi_1/T)} (1+o(1)); |\xi_1| \leq \ln^2 T\Big) \\
=&E\Big(\frac{dx}{T^{(d-1)/2}} C(\alpha) e^{-TD(\alpha)+D'(\alpha)\cdot\xi_1}\Big)(1+o(1)) \\
=&\psi_1(D'(\alpha))H(dx)(1+o(1))
\end{aligned}$$

as $T \to \infty$.

3 The Second Rate Function

The valid region of the asymptotic in **Theorem 2.2** depends on the second rate function D and the distribution of the initial step ξ_1. Recall that

$$Q(\mathbf{x}, t) = Q(x, y, t) := \begin{cases} \frac{g^2}{8\pi\theta^2 t} \exp\left(-\frac{x^2+y^2}{2\theta t^2} - \frac{g^2 t^2}{8\theta}\right), & t > 0 \\ 0, & t \leq 0 \end{cases},$$

which is now assigned as the density function of ξ, determines D. On the other hand, $f^+/||f||$ is assigned to be the density function of ξ_1.

For simplicity, we consider this nonparametrized version

$$Q(\mathbf{x}, t) = Q(x, y, t) := \begin{cases} C\frac{1}{t} \exp\left(-\frac{x^2+y^2}{t^2} - t^2\right), & t > 0 \\ 0, & t \leq 0 \end{cases},$$

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where C is the normalizing constant. First, the Laplace transform of ξ is computed.

$$\begin{aligned}\psi(s,u_1,u_2)=&\int_0^\infty\int_{\mathbb{R}^2}C\frac{1}{t}\exp\left(-\frac{x^2+y^2}{t^2}-t^2+st+u_1x+u_2y\right)dxdydt\\ =&C\int_0^\infty \pi t\exp\left(\frac{t^2(u_1^2+u_2^2)}{4}-t^2+st\right)dt.\end{aligned}$$

Note that ψ is analytic in its effective domain

$$\mathcal{A}:=\{\psi<\infty\}=\{(s,u_1,u_2)\in\mathbb{R}^3\mid u_1^2+u_2^2<4\}.$$

Write $\psi(s,\mu)=\psi(s,u_1,u_2)$ for $\mu=\sqrt{u_1^2+u_2^2}$. For $\mu\in(\mu^-,\mu^+):=(-2,2)$, define

$$\begin{aligned}A(\mu):=&-\sup\{s:\psi(s,\mu)\le 1\},\\ A^\infty(\mu):=&-\sup\{s:\psi(s,\mu)\le\infty\}=-\infty.\end{aligned}$$

Fix $\mu\in(\mu^-,\mu^+)$, $\psi(s,\mu)$ strictly increases as s grows so $\lambda=A(\mu)$ is the unique solution for the equation $\psi(-\lambda,\mu)=1$. By implicit function theorem, A is analytic in the region where $A(\mu)>A^\infty(\mu)$. This region happens to be (μ^-,μ^+). For $\alpha\in\mathbb{R}$, define

$$D_1(\alpha):=D(1,\alpha).$$

Some convex analysis arguments show that

$$D_1(\alpha)=\sup_\mu\{\mu\alpha-A(\mu)\},$$

which is analytic for $\alpha\in(A'(\mu^-+0),A'(\mu^+-0))$. Combining the facts

- The graph $(A(\mu),(\mu))$ is a contour of ψ.
- $\psi(s,\mu)$ is strictly increasing in s-component.
- The effective domain in μ-component of ψ is bounded.

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We conclude that $(A'(\mu^- + 0), A'(\mu^+ - 0)) = (-\infty, \infty)$ in our problem. Thus, for the homogeneous case, the local limit theorem for the renewal measure can be applied without restriction.

For the inhomogeneous case, the constraint on ξ_1 actually depends on the region of interest. Given $\alpha \in \mathbb{R}^+ \times \mathbb{R}^2$, it's sufficient that $D'(\alpha) \in (\mathcal{A}_1)$, which means, by selecting K a sufficiently small neighborhood of $D'(\alpha)$, let

$$\begin{aligned}\lambda_j^+ &:= \sup\{D'(\alpha)_j, \alpha \in K\}, \\ \lambda_j^- &:= \inf\{D'(\alpha)_j, \alpha \in K\}, \quad j = 1, 2, 3,\end{aligned}$$

then it's sufficient for ξ_1 to satisfy: there exists $r, C > 0$ such that

$$\begin{aligned}P((\xi_1)_j \geq \beta) &\leq Ce^{-\lambda_j^+ \beta}, \\ P((\xi_1)_j \leq -\beta) &\leq Ce^{-\lambda_j^- \beta}\end{aligned}$$

for $\beta \geq r$.

3.1 Normal and Moderate Large Deviations

When $\alpha = \mathbf{x}/t \to 0$, the asymptotic is said to be in the moderate large deviation region. In this case, $D_1(\alpha) \to 0$, $D_1'(\alpha) \to 0$ and $D_1''(\alpha) \to (\frac{E\zeta^2}{E\tau})^{-1} =: \Sigma^{-1}$; also, $\psi_1(D_1'(\alpha)) \to 1$ and $C(\alpha) \to (2\pi E\tau|\Sigma|)^{-1/2}$; thus, we have the following asymptotic

$$\frac{\rho(\mathbf{x}, t)}{||f||} \to ((2\pi)^2 E\tau|\Sigma|)^{-1/2} \frac{\exp\left(\frac{-\mathbf{x}^T\Sigma^{-1}\mathbf{x}}{2E\tau t}\right)}{t},$$

which is uniform in any compact set separated from the origin. Combining **Theorem 4.4**, we have

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Theorem 3.1 (Local limit theorem of the boundary flux in its central part [1, p.401]).

$$\frac{\rho(\boldsymbol{x},t)}{||f||} \to ((2\pi)^2 E\tau|\Sigma|)^{-1/2}\frac{\exp\left(\frac{-\boldsymbol{x}^T\Sigma^{-1}\boldsymbol{x}}{2E\tau t}\right)}{t} \tag{3.1}$$

as $t \to \infty$. This asymptotic is uniform when $x = o(t^{2/3})$.

Note that the initial data has no contribution to the asymptotic in this region.[13]

4 Stochastic Processes

Physical quantities satisfying conservation laws of the Boltzmann equation (1.1) or the free transport equation (1.3), which are of physical interest, can be treated as stochastic processes. What are conservation laws? For the transport equation, these physical quantities are required to be invariant along characteristic curves. For example, f, $(\mathbf{v}_1, \mathbf{v}_2)f$ and $(\frac{|\mathbf{v}|^2}{2} + gz)f$ give the distributions of the position, the horizontal momentum and the total energy of the gas in the phase-space respectively. Let us denote these processes by $Y_0(t)$, $Y_1(t)$ and $Y_2(t)$. Given $B \subset \mathbf{H}^+ \times \mathbb{R}^3$,

$$\begin{aligned}
P(Y_0(t) \in B) &= C_0 \int_B f(\mathbf{x},t,\mathbf{v})d\mathbf{x}d\mathbf{v},\\
P(Y_1(t) \in B) &= C_1 \int_B (\mathbf{v}_1,\mathbf{v}_2) f(\mathbf{x},t,\mathbf{v})d\mathbf{x}d\mathbf{v},\\
P(Y_2(t) \in B) &= C_2 \int_B \left(\frac{|\mathbf{v}|^2}{2} + gz\right) f(\mathbf{x},t,\mathbf{v})d\mathbf{x}d\mathbf{v},
\end{aligned}$$

where C_i are normalizing constants.

Similarly, $\int f d\mathbf{v}$, $\int (\mathbf{v}_1, \mathbf{v}_2) f d\mathbf{v}$ and $\int (\frac{|\mathbf{v}|^2}{2} + gz) f d\mathbf{v}$ give the distributions of the position, the horizontal momentum and the total energy of the gas in the upper-

[13] Though the tails of the initial f^+ are still required to be subject to exponential decay.

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space $\mathbf{H}^+$ respectively. Denote these processes by X_0, X_1 and X_2. For $U \subset \mathbf{H}^+$,

$$P(X_0(t) \in B) = C_0 \int_U \int_{\mathbb{R}^3} f(\mathbf{x}, t, \mathbf{v}) d\mathbf{v} d\mathbf{x},$$
$$P(X_1(t) \in B) = C_1 \int_U \int_{\mathbb{R}^3} (\mathbf{v}_1, \mathbf{v}_2) f(\mathbf{x}, t, \mathbf{v}) d\mathbf{v} d\mathbf{x},$$
$$P(X_2(t) \in B) = C_2 \int_U \int_{\mathbb{R}^3} (\frac{|\mathbf{v}|^2}{2} + gz) f(\mathbf{x}, t, \mathbf{v}) d\mathbf{v} d\mathbf{x},$$

The stochastic formulation of these quantities enable us to reach their limiting behaviours.

Additionally, these processes can be mapped onto the plane $\{z = 0\}$ and forms some two-dimensional processes. Among them, one of the simplest processes is the last hitting position $Z(t)$ of a single gas particle, which is obtained by mapping $Y_0(t)$ along its characteristic curve to $\{z = 0\}$. This process admits the following definition, being the main object studied in [1].

Definition 4.1 (Compound renewal processes [1])**.** *Consider a sequence of random vectors $\xi_n := (\tau_n, \zeta_n)$, which are iid to $\xi = (\tau, \zeta)$ for $n = 2, 3, ...$ and independent of ξ_1; $\tau > 0$ and $\tau_1 \geq 0$. For each n, ζ_n can depend on τ_n while being independent of τ_m for all $m \neq n$. Let $T_n := \sum_{j=1}^{n} \tau_j$, a* renewal process *is*

$$N(t) := \sum_{n=0}^{\infty} \mathbf{1}(T_n \leq t) \tag{4.1}$$

Let $Z(0) = 0$, then

$$Z(t) := \sum_{j=1}^{N(t)} \zeta_j \tag{4.2}$$

is a compound renewal process*.*

The compound renewal process can be associated with the renewal measure H_1 directly. Let us interpret $Q^{n*}(\mathbf{x}, t)$ as the probability density of a chicken walking n steps to move a displacement $\mathbf{x}$ while spending exactly t amounts of time (or

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reaching $(\mathbf{x}, t)$ from the origin $(0,0)$). Suppose further that with every step it takes, it lays an egg. If the chicken was at the origin at time $t = 0$, then $R(\mathbf{x}, t)$ becomes the increasing rate (with time) of the expected number of eggs per unit area at $\mathbf{x}$ and at time t. Writing probabilistically the chicken's behaviour, we get exactly the process defined in **Definition 4.1**.

As moments of f or sums of mappings of f along the characteristic curve, these processes are all belong to a class of stochastic process.

Definition 4.2 (Processes with regenerative increments [6, p. 193])**.** *Let* $\tau_n, \ n = 2, 3, ...$ *be a sequence of iid random variables with distributions identical to* τ *and* τ_1 *be another random variable independent to random variables* τ_n*;* $\tau > 0$ *and* $\tau_1 \geq 0$*.* $T_n := \sum_{j=1}^{n} \tau_j$ *is called a* renewal time*. An* increment *of a process* $G(t)$ *is given by*

$$g_n := \big(\tau_n, \{G(t + T_{n-1}) - G(T_{n-1}) : 0 \leq t < \tau_n\}\big).$$

A process $G(t)$ *is said to have* regenerative increments *if* g_n *are iid for* $n = 2, 3, ...$ *and* g_1 *is independent of increments* g_n *(not necessarily identical).*

These processes are all closely related to the renewal measure H_1. Their distributions can be written explicitly into the form as "a convolution of H_1 and a function F_0 plus another function F_1". Indeed,

$$\begin{aligned}
P(Z(t) \in U) - \mathbf{1}_{\{0 \in U\}} P(T_1 > t) &= P\Big(\bigcup_{n=1}^{\infty} \{Z_n \in U, \ T_n \leq t, \ T_{n+1} > t\}\Big) \\
&= \sum_{n=1}^{\infty} \big(P(Z_n \in U, \ T_n \leq t) - P(Z_n \in U, \ T_{n+1} \leq t)\big) \\
&= H_1([0,t] \times U) - \sum_{n=1}^{\infty} P(Z_n \in U, \ T_n + \tau \leq t) \\
&= H_1([0,t] \times U) - H_1([0,t] \times U) * P(\tau \leq t) \\
&= H_1([0,t] \times U) * P(\tau > t),
\end{aligned}$$

where $H_1([0,t] \times U) * P(\tau > t) := \int_0^t P(\tau > t - s) H_1(ds \times U)$. Likewise, by the

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characteristic method used in deriving (1.5), the distribution of $X_i(t)$, $Y_i(t)$ can also be written into this form.

With this connection to the renewal measure, the limiting behaviour of these physical quantities are revealed. Like the inhomogeneous case to the homogeneous case, an extra term arises in front of the renewal measure, which is the Laplace transform of the function F_0.

Theorem 4.3 (Integro-local limit theorems for the compound renewal process)**.** *Under the condition of* ***Theorem 2.2****,*

$$\begin{aligned} P(Z(t) \in dx) = \Big(\int_0^\infty e^{\lambda(\alpha) y} P(\tau > y) dy \Big) H_1(d(t,x))(1 + o(1)) \\ + \mathbf{1}(x = 0) P(T_1 > t) \end{aligned}$$

for $t \to \infty$*.*

There's a version of central limit theorem for a process with regenerative increment, which can be found in e.g. [6] (in this book, it's proved by using a functional central limit theorem)

Theorem 4.4 (Regenerative CLT (one dimensional) [6, Sec 2.13])**.** *In the context of* ***Definition 4.2****, suppose*

$$\begin{aligned} \mu &:= E\tau, \\ a &:= E[G(\tau)]/\mu, \\ M_n &:= \sup_{0 \le t < \tau_n} \{G(t + T_{n-1}) - G(T_{n-1})\}, \quad n \ge 1, \\ \sigma^2 &:= V[G(\tau) - a\tau] \end{aligned}$$

are finite. Then for the case $G(0) = 0$*,*

$$\frac{G(t) - at}{\sqrt{t}} \xrightarrow{d} N(0, \sigma^2/\mu), \quad \text{as } t \to \infty.$$

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When modified to its multi-dimensional version, this theorem matches our estimate of the boundary flux in its central part (see **Theorem 3.1**).

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