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有限自制力的表現定理

A Representation Theorem for “Self-Control with Limited Willpower”

謝松霖
Sung-Lin Hsieh

指導教授：梁孟玉博士
Advisor: Meng-Yu Liang, Ph.D.

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中文摘要

本文旨在建構一個表現定理，使個體可於數個集合中先做選擇，接著在此集合中選出單一物件。此定理考慮擁有「偏好承諾」此一特性的偏好關係，而這種偏好首先由 Gul and Pesendorfer (2001) 所提出。有別於文獻中假設個體擁有無限意志力抑或毫無意志力，本文的表現定理則允許個體擁有有限的意志力。我們與文獻的主要差異在於我們放寬了 Gul and Pesendorfer (2001) 文獻中的定理三之獨立公設並且保留其餘的公設。此外，在我們的模型中，若個體的意志力不足以抵抗誘惑，則會因意志力的限制而只能選取較低事前效用的選項；反之，成功抵抗誘惑的個體會保持與文獻中相同的行為。

關鍵字：偏好承諾、集合居中性質、有限意志力、獨立公設、混合空間

Abstract

This dissertation constructs a representation theorem for individual choice among sets of alternatives, from which the individual will later choose a single object. In particular, it concerns preference relations on sets of alternatives which satisfy "preference for commitment," first characterized by Gul and Pesendorfer (2001). Instead of assuming the individual either having unlimited willpower or none at all, we consider the same model as Gul and Pesendorfer but with limited willpower to resist temptation. We relax the Independence Axiom and keep the rest axioms of their Theorem 3 in our characterization. Agents with insufficient willpower to resist temptations are bound to choose an option with lower ex ante utility while the behaviors of agents who resist temptations remain unchanged.

Key words: preference for commitment, set betweenness, limited willpower, independence axiom, mixture space

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Chapter 1

Introduction

Imagine the following situation. A consumer loves to eat hamburger; however, she is on a diet. She knows that salad is more beneficial than hamburger for her. If she goes to a restaurant selling both salad and hamburger, she may still buy the salad in the end. Nevertheless, she would suffer the temptation from the hamburger. Therefore, if she can make a choice between a restaurant only with salad or the one selling both salad and hamburger, she may prefer the the former in order to avoid the temptation from the hamburger. When she goes to the restaurant with both products, there are two possible situations. First, she may be able to resist the temptation from the hamburger; that is, she exercises *self-control*. Second, she may give in to the temptation and eat the hamburger at last.

The main idea of the story above was introduced by Gul and Pesendorfer (2001) (henceforth GP) . Within several different menus with different goods, a decision maker first has to choose a menu, and then chooses a goods in that menu. In their model, the preference stated above is called *temptation preference with self-control* which can be represented by a function U of the form:

$$U(A) := \max_{x \in A}\{u(x) + v(x)\} - \max_{y \in A} v(y).$$

Here, A is a set of goods and both u and v are von Neumann-Morgenstern utility functions over lotteries. The function v measures the temptation. The function u measures the rank of singletons, or the utility without any temptation effect. The decision maker wants to maximize her utility but, at the same time, has to face the temptation. When making a choice, she has to care not only the benefit from the goods but also the temptation effect. In other words, she has to maximize $u+v$ instead of only u, the conventional utility function. For example, let x be the salad and y be the hamburger; we further assume $\{x\} \succ \{y\}$ in the following discussion. There are three possible relationship among $\{x\}$, $\{x, y\}$ and $\{y\}$ in the GP model. First, if $\{x\} \sim \{x, y\} \succ \{y\}$, it implies that the decision maker thinks the hamburger is not tempting since she feels indifferent between $\{x\}$ and $\{x, y\}$. In this case, $v(x) \geq v(y)$. Second, if $\{x\} \succ \{x, y\} \sim \{y\}$, it means that the decision maker gives in to the temptation due to $\{x, y\} \sim \{y\}$. Third, if $\{x\} \succ \{x, y\} \succ \{y\}$, it is a typical case of performing self-control because $\{x\} \succ \{x, y\}$ implies that she suffers the disutility from the temptation and $\{x, y\} \succ \{y\}$ indicates that x will be shown between x and y. In this model, the reason of giving in to the temptation is that the salad is not worthwhile enough to resist temptation. The salad is more beneficial than hamburger, which means $u(x) > u(y)$. However, when decision maker considers both benefit and temptation at the same time, y may be preferred to x. If y is preferred, it means that the benefit from x dose not exceed the disutility from resisting the temptation.

Another model introduced by Masatlioglu et al. (2011) (henceforth MNO) is a model with limited willpower. The utility function in this model is constructed on the space same as previous model. However, the preference of decision maker in this model can be represented by a function U of the form:

$$U(A) := \max_{x \in A} u(x)$$
$$s.t. \max_{y \in A} v(y) - v(x) \leq w.$$

Both u and v have the same meaning as in GP, and the constant w stands for the constant willpower. Take the example of salad and hamburger again. There are only two possible relationship among $\{x\}$, $\{x, y\}$ and $\{y\}$ in this model. The decision maker wants to maximize the utility, but is restricted by a willpower constraint, which means that the difference of temptation level between the most tempting goods and the selected one cannot be higher than her willpower. If $\{x\} \sim \{x, y\} \succ \{y\}$, the decision maker feels $\{x\}$ is indifferent with $\{x, y\}$ because x is more tempting than y or y is not too tempting. That is, $v(x) \geq v(y)$ or $v(y) - v(x) \leq w$. If $\{x\} \succ \{x, y\} \sim \{y\}$, it means that the y is too tempting to resist. Specifically, the decision maker gives in to the temptation because of insufficient willpower, or $v(y) - v(x) > w$.

Based on the discussion above, the decision maker gives in to the temptation for two possible reasons in the literature: resisting cost and limited willpower. On account of the reasonableness of both reasons, we construct a representation theorem taking both of them into consideration. In our model, the preference of the decision maker can be represented by a function of the form:

$$U(A) := \max_{x \in A}\{u(x) + v(x)\} - \max_{y \in A} v(y)$$

$$s.t. \max_{y \in A} v(y) - v(x) \leq w.$$

Since there are two possible reasons for the decision maker to give in to the temptation, then it requires a certain method to differentiate these two reasons. In other words, if the decision maker's preference is $\{x\} \succ \{x, y\} \sim \{y\}$, it is important to identify which reason causes it. In the salad and hamburger example, we provide the decision maker with a set consisting of salad and a lottery with a small chance to win the hamburger. If the reason is about the limited willpower, the decision maker would choose salad as long as the probability of winning hamburger is small enough. On the other hand, if the reason is due to the resisting cost, the decision maker will always choose the lottery because $u(y) + v(y) > u(x) + v(x)$. Also, from the linearity of u and v, the inequality still holds if y, the hamburger, is replaced by the lottery.

Except the GP model and the MNO model, there are some literature of representation theorem talking about self-control or willpower. In the topic of self-control, Dekel and Lipman (2009) provided a preference which measures the utility of a set with the n most beneficial and the m most tempting goods in the set instead of the most beneficial singleton and the most tempting singleton in the GP model. Noor (2011) mentioned the temptation effect from different menus, which means the decision maker suffers from the temptation not only when she makes a choice from a set of goods, but also when she makes the decision from different sets. In the topic of willpower, Benabou and Tirole (2002) focused on the self-confidence and the interaction of imperfect willpower and the imperfect memory. Ozdenoren et al. (2012) used the limited willpower to explain the time-inconsistency problem. These literature were talking about the extension and application of the concept of self-control or limited willpower. However, in our model, we focus on the combination of these two concepts and the axiomatic theorem. Note that Fudenberg and Levine (2012) tried to characterize the self-control with willpower in a dual-self model. Compared with our axiomatic theorem, their approach paid more attention to be an application of a dual-self model.

In chapter 2 , we introduce the construction of mixture space with sets of lotteries and give the formal definition of the method for differentiating the two possible reasons of giving in to the temptation. Following GP, we use the similar way to construct u and v in this chapter. In chapter 3, we define the willpower in a formal way and construct U. In the end, we provide a conclusion in chapter 4.

Chapter 2

Mixture space and linear utility function

In this dissertation, we focus on the two-period decision problem. First, the decision maker makes a choice among a list of menus. The menus are sets of consumption goods. In the second period, the decision maker can only choose a consumption goods from the menus which he chose a while ago. Following Kreps (1979), the consumption goods in our model can be a prize or a lottery.

Our notations mainly follows GP's notation. Let (Z, d) be a compact metric space, where Z is the of set finite many prizes. Let Δ be the set of all the lotteries generated by Z. Let $\mathscr{A}$ denote the family of all compact subset of Δ. We generate topology of $\mathscr{A}$ by the (*Hausdorff*) metric

$$d_h(A, B) = \max\{\max_A \min_B d_p(x, y), \max_B \min_A d_p(x, y)\}$$

where d_p is a metric that generates the weak topology. Define $h_\alpha(A, B) = \{\alpha x + (1-\alpha)y : x \in A, y \in B)\}$ for $A, B \in \mathscr{A}$, $\alpha \in [0, 1]$.

In this chapter, section 1 gives background knowledge of mixture space and its relation to a linear representation function U. Section 2 shows that, in general, $(\mathscr{A}, h_\alpha)$ is not a mixture pace. Section 3 gives Axioms on $\succeq$ and show a subset of $\mathscr{A}$ is a mixture space. Section 4 gives define to $u\,(x)$ and $v\,(x; a, b, \delta)$ from U for all $x \in \Delta$, and show Lemma 4 and 5 in Gul and Pesendorfer (2001) still hold in our model.

2.1 Background knowledge

The results in this Section are from Kreps (1988) (page 52 and 54).

Definition 1. *A mixture space is a set of objects* Π*, with typical elements* π, ρ, μ *and* v*, and a family of functions* $h_\alpha : \Pi \times \Pi \to \Pi$ *for* $\alpha \in [0, 1]$ *such that :*

(a) $h_1(\pi, \rho) = \pi$

(b) $h_\alpha(\pi, \rho) = h_{1-\alpha}(\rho, \pi)$*, and*

(c) $h_\alpha(h_\beta(\pi, \rho), \rho) = h_{\alpha\beta}(\pi, \rho)$*.*

Theorem 1. *Suppose* Π *is a mixture space and* $\succ$ *is a binary relation on* Π*. Then*

(a) $\succ$ *is a preference relation,*

(b) $\pi \succ \rho$ *and* $\alpha \in (0, 1]$ *implies* $h_\alpha(\pi, \mu) \succ h_\alpha(\rho, \mu)$ *for all* $\mu \in \Pi$*, and*

(c) $\pi \succ \rho \succ \mu$ *imply that there exist* $\alpha, \beta \in (0, 1)$ *such that* $h_\alpha(\pi, \mu) \succ \rho \succ h_\beta(\pi, \mu)$*,*

if and only if

(d) there exists a function $F : \Pi \to R$ *such that*

(di) $\pi \succ \rho$ *iff* $F(\pi) > F(\rho)$*, and*

(dii) $F(h_\alpha(\pi, \rho)) = \alpha F(\pi) + (1 - \alpha) F(\rho)$.

Moreover, if F *represents in the sense of (d),* F' *is another representation (in this sense) if and only if* $F' = \alpha F + \beta$ *for constants* $\alpha > 0$ *and* β*.*

2.2 Fail to be a mixture space

In Lemma 1 of Gul and Pesendorfer (2001), they said that $\mathscr{A}$ is a mixture and use the theorem above to construct a linear utility function over $\mathscr{A}$. However, we find out that $\mathscr{A}$ dose not completely fit the definition of mixture space.

Let us check Definition 1 (c):

$$h_\alpha(h_\beta(\pi, \rho), \rho) = h_{\alpha\beta}(\pi, \rho)$$

Let $\pi = \{a, b\}$, $\rho = \{x, y\}$, then

$$
\begin{aligned}
&h_\alpha\left(h_\beta\left(\pi,\rho\right),\rho\right)\\
=&h_\alpha\left(\beta\pi + \left(1-\beta\right)\rho\right),\rho)\\
=&h_\alpha(\{\beta p + \left(1-\beta\right) q : p \in \{a,b\}, q \in \{x,y\}\},\rho)\\
=&\{\alpha\left(\beta p + \left(1-\beta\right) q_1\right) + \left(1-\alpha\right) q_2 : p \in \{a,b\}, q_i \in \{x,y\}, i \in \{1,2\}\}
\end{aligned}
$$

and

$$h_{\alpha\beta}\left(\pi,\rho\right) = \{\alpha\beta p + \left(1-\alpha\beta\right) q : p \in \{a,b\}, q \in \{x,y\}\}$$

Suppose a, b, x, y represent four degenerate lotteries, $\delta_a, \delta_b, \delta_x, \delta_y$ which get different prize for sure. We can see $h_\alpha\left(h_\beta\left(\pi,\rho\right)\right)$ contains lotteries with prizes a, x, y and b, x, y, while $h_{\alpha\beta}\left(\pi,\rho\right)$ does not. Since we can not apply the theorem of mixture over $\mathscr{A}$, we try to find some proper subspace of $\mathscr{A}$ to apply this theorem.

2.3 GP's axioms and a mixture space

Since the main difference between our model and GP(2001) is on menus with no self-control, to get our representation function U over $\mathscr{A}$, we introduce some new notations for menus with no self-control. For $a \in \Delta$, define the set of tempting alternatives such that the decision maker (DM) has no self-control

$$T(a) := \{b \in \Delta : \{a\} \succ \{a,b\} \sim \{b\}\}.$$

When DM gives in a to some choice in $T\left(a\right)$,there are two possible situations : lack of willpower or not worth. Hence, we divide $T\left(a\right)$ into two sets: $L\left(a\right)$ and $L^c\left(a\right)$. If lack of willpower, then making b close enough to a DM will have self-control.

- $L(a) := \{b \in T(a) : \{a, h_\delta(b,a)\} \succ \{h_\delta(b,a)\}$ for some $\delta \in (0,1)\}$
- $L^c(a) := T(a) \setminus L(a)$

Note that if $L(a) = \phi$ for all $a \in \Delta$, we have the case same as having unlimited willpower. Suppose $L(a) \neq \phi$ for some $a \in \Delta$. This means that a decision maker cannot resist the craving for the hamburger and choose the salad, but he can resist the craving for a small chance to have a hamburger adding to the salad . Or b is a lottery when a and b present together, the decision maker can not resist the temptation of picking a lottery b. However, if we make a compound lottery from a and b, for sufficiently small chance of winning b, then this compound lottery becomes less tempting. Hence, DM is able to resist a small chance of winning the tempting lottery. Note that $L^c(a)$ captures the set of tempting alternatives that does not worth the effort to resist when it is presented with a in one menu. Hence, a small chance of wining it is still not worth the effort to resist it.

Definition 2. $\mathscr{L}(\succeq) = \{A \in \mathscr{A} : A = \{x, y\}$ *with* $\{x\} \succ \{y\}$ *and* $y \notin L(x)$*, or* $A = \{x\}\}$

Note that $\mathscr{L}(\succeq)$ collects all singleton sets and two-element sets excepts menus with no self-control due to lack of willpower. Let us have the following axioms as in Gul and Pesendorfer (2001) theorem 3.

Axiom 1 (Preference Relation)**.** $\succeq$ *is a complete and transitive binary relation.*

Axiom 2a (Upper Semi-Continuity)**.** *The sets* $\{B \in \mathscr{A} : B \succeq A\}$ *are closed.*

Axiom 2b (Lower von Neumann-Morgenstern Continuity)**.** $A \succ B \succ C$ *implies* $h_\alpha(A, C) \succ B$ *for some* $\alpha \in (0, 1)$.

Axiom 2c (Lower Singleton Continuity)**.** *The sets* $\{x : \{y\} \succeq \{x\}\}$ *are all closed.*

Axiom 3 (Conditional Independence)**.** *For* $A, B, C \in \mathscr{L}(\succeq)$*,* $A \succ B$ *and* $\alpha \in (0, 1)$ *implies* $h_\alpha(A, C) \succ h_\alpha(B, C)$.

Axiom 4 (Set Betweenness)**.** $A \succeq B$ *implies* $A \succeq A \cup B \succeq B$

Unlike theorem 1 in Gul and Pesendorfer (2001), we don't have general continuity axiom. When menus pass willpower constraint, a jump in utility is not avoidable. Hence, we adopt a relaxed version of continuity axiom as in their theorem 3. Moreover, the preference we would like to characterize does not satisfy general independence axiom. The reason is that the independence axiom fails when mixing menus that have no self-control due to lack of willpower. For example, if the reason of $A \succ B$ is that all the better choices in B are not available due to lack of willpower, then mixing with C may make those better choices in B become available, and hence it is possible to have $h_\alpha(B, C) \succ h_\alpha(A, C)$. The following lemma in GP(2001) shows that set betweenness implies that each finite set is indifferent to an appropriate two-element subset.

Lemma 1 (Gul and Pesendorfer (2001), lemma 2)**.** *Let U be a function that represents some $\succeq$ satisfying Axiom 4. If $A \in \mathscr{A}$ is a finite set, then* $U(A) = \max_{x \in A} \min_{y \in A} U(\{x, y\}) = \min_{y \in A} \max_{x \in A} U(\{x, y\})$. *Moreover, there is an x^*, y^* such that (x^*, y^*) solves the maxmin and (y^*, x^*) solves the minmax problem.*

For $a \in \Delta$, define the set of tempting alternatives such that the decision maker has costly self-control.

$$S(a) = \{b \in \Delta : \{a\} \succ \{a, b\} \succ \{b\}\}.$$

Use the above lemma we can show the following lemma holds without the requirement that U is a linear function.[1]

Lemma 2. *Let U be a function that represents some $\succeq$ satisfying Axioms 1, 2a-c, 3 and 4 and $A = h_\alpha(\{x, y\}, \{a, b\})$. If*

$$\{x, y\} \succ \{y\} \text{ and } \{a, b\} \succ \{b\}, \quad \text{then}$$

$$U(A) = \min_{z \in A} U(\{\alpha x + (1 - \alpha)a, z\})$$

and if

$$\{x\} \succ \{x, y\}, \{a\} \succ \{a, b\}, y \notin L(x) \text{ and } b \notin L(a), \quad \text{then}$$

$$U(A) = \max_{w \in A} U(\{w, \alpha y + (1 - \alpha)b\})$$

[1] We follow a similar argument of the proof of Lemma 3 in GP (2001, page 1422) without using their assumption that U is a linear function that represents $\succeq$. We use conditional Independence axiom directly.

Proof. By Lemma 1, there exists (w^*, z^*) such that $A \sim \{w^*, z^*\}$ and (w^*, z^*) solves the maxmin problem. First we show that $\{x, y\} \succ \{y\}$ and $\{a, b\} \succ \{b\}$ implies $w^* = \alpha x + (1-\alpha)a$. By Axiom 3, we have

$$A \succ \alpha\{y\} + (1-\alpha)\{a, b\},$$

$$A \succ \alpha\{x, y\} + (1-\alpha)\{b\}.$$

Suppose $w^* = \alpha x + (1-\alpha)b$. Then, since $A \sim \{w^*, z^*\}$ and it solves the maxmin problem, we have

$$A \preceq \{\alpha x + (1-\alpha)b, \alpha y + (1-\alpha)b\} = \alpha\{x, y\} + (1-\alpha)\{b\} \prec A,$$

which yields a contradiction. Similarly, if $w^* = \alpha y + (1-\alpha)a$, then

$$A \preceq \{\alpha y + (1-\alpha)b, \alpha y + (1-\alpha)a\} = \alpha\{y\} + (1-\alpha)\{a, b\} \prec A.$$

If $w^* = \alpha y + (1-\alpha)b$, then

$$A \preceq \{\alpha y + (1-\alpha)b, \alpha y + (1-\alpha)b\} = \{\alpha y + (1-\alpha)b\} \prec \alpha\{y\} + (1-\alpha)\{a, b\} \prec A.$$

Hence, $w^* = \alpha x + (1-\alpha)a$.

Suppose that we have $\{x\} \succ \{x, y\}$ and $\{a\} \succ \{a, b\}$ with $y \notin L(x)$ and $b \notin L(a)$. Then we can apply Axiom 3 and obtain

$$\alpha\{x\} + (1-\alpha)\{a, b\} \succ A,$$

$$\alpha\{x, y\} + (1-\alpha)\{a\} \succ A.$$

Then since $A \sim \{z^*, w^*\}$ and it solves the minmax problem, we can use a similar argument as above to show $z^* = \alpha y + (1-\alpha)b$. □

From the above lemma we can derive a mixture space in the following restricted domain. First, given preference $\succeq$, we define the following notation.

Definition 3. $\mathscr{M}(\succeq) = \{A \in \mathscr{A} : A = \{x, y\}$ *with* $y \in S(x)$, *or* $A = \{x\}\}$

Since $\mathscr{M}(\succeq)$ has at most two elements, and $h_\alpha(A, B)$ contains four elements when A and B are both two-element sets, from lemma 2 we select the equivalent two-element set when h_α has four elements. More specifically, for $A, B \in \mathscr{M}(\succeq)$ and $\alpha \in [0, 1]$, Let

$$\tilde{h}_\alpha(A, B)$$
$$= \begin{cases} \{\alpha a + (1-\alpha)x, \alpha b + (1-\alpha)y\} & \text{if } A = \{a, b\}, B = \{x, y\}, \text{ and } b \in S(a), y \in S(x), \\ h_\alpha(A, B) & \text{otherwise.} \end{cases} .$$

Now we are ready to show that $(\mathscr{M}(\succeq), \{\tilde{h}_\alpha\}_{\alpha \in [0,1]})$ is a mixture space.

Lemma 3. *If a preference $\succeq$ satisfies Axiom 3 and 4, then $(\mathscr{M}(\succeq), \{\tilde{h}_\alpha\}_{\alpha \in [0,1]})$ is a mixture space.*

Proof. First, we will show that $\tilde{h}_\alpha(A, B) \in \mathscr{M}(\succeq)$ for any $A, B \in \mathscr{M}(\succeq)$. From Lemma 2, it is known that $\tilde{h}_\alpha$ is either a singleton set or a two-element set. If $\tilde{h}_\alpha$ is a singleton set, then obviously it is in $\mathscr{M}(\succeq)$. If $\tilde{h}_\alpha$ has two elements, then it only takes one of the two possible forms, either $\tilde{h}_\alpha(\{a, b\}, \{x\})$, or $\tilde{h}_\alpha(\{a, b\}, \{x, y\})$ with $b \in S(a)$ and $y \in S(x)$. By Axiom 3, we have

$$\begin{aligned} \{\alpha a + (1-\alpha)x\} = \alpha\{a\} + (1-\alpha)\{x\} &\succ \{\alpha a + (1-\alpha)x, \alpha b + (1-\alpha)x)\} \\ &\succ \alpha\{b\} + (1-\alpha)\{x\} = \{\alpha b + (1-\alpha)x\} \end{aligned}$$

Hence, $\tilde{h}_\alpha(\{a, b\}, \{x\}) \in \mathscr{M}(\succeq)$. By Axiom 3, we also have

$$\begin{aligned} \{\alpha a + (1-\alpha)x\} \succ \alpha\{a, b\} + (1-\alpha)\{x\} &\succ \alpha\{a, b\} + (1-\alpha)\{x, y\} \\ &\succ \alpha\{a, b\} + (1-\alpha)\{y\} \succ \alpha\{b\} + (1-\alpha)\{y\} = \{\alpha b + (1-\alpha)y\} \end{aligned}$$

Hence, $\tilde{h}_\alpha(\{a, b\}, \{x, y\}) \in \mathscr{M}(\succeq)$ as well.

From the discussion in section 2, we know $h_\alpha(h_\beta(\pi,\rho),\rho) = h_{\alpha\beta}(\pi,\rho)$ when ρ is a singleton set. Since $\tilde{h}$ and h are the same function when ρ is a singleton set, all we need to show is $\tilde{h}_\alpha\left(\tilde{h}_\beta(\pi,\rho),\rho\right) = \tilde{h}_{\alpha\beta}(\pi,\rho)$ when ρ is a two-element set.

$$
\begin{aligned}
&\tilde{h}_\alpha\left(\tilde{h}_\beta(\{x,y\},\{a,b\}),\{a,b\}\right)\\
&= \tilde{h}_\alpha(\{\tilde{h}_\beta(x,a),\tilde{h}_\beta(y,b)\},\{a,b\})\\
&= \{\alpha\tilde{h}_\beta(x,a) + (1-\alpha)a, \alpha\tilde{h}_\beta(y,b) + (1-\alpha)b\}\\
&= \tilde{h}_{\alpha\beta}(\{x,y\},\{a,b\}).
\end{aligned}
$$

$\square$

Since Lemma 2 implies $\tilde{h}_\alpha(A,B) \sim h_\alpha(A,B)$ for all $A, B \in \mathscr{M}(\succeq)$, from the above lemma we apply Theorem 1 on $\mathscr{M}(\succeq)$ and obtain the following theorem.

Theorem 2. *A preference $\succeq$ satisfies Axiom 1, 2a-c, 3, 4 if and only if there is a linear function $U : \mathscr{M}(\succeq) \to R$ represent $\succeq$ for sets in $\mathscr{M}(\succeq)$. Moreover, U in the representation is unique up to a positive affine transformation. The restriction of U to singleton sets is continuous.*

2.4 U, u, v and their properties

From lemma 2, we have a unique (up to a positive affine transformation) linear function $U : \mathscr{M}(\succeq) \to \mathbb{R}$. Note that by Axiom 4, for any two-element set we have either $\{a\} \sim \{a,b\}$, $\{a,b\} \sim \{b\}$, or $\{a\} \succ \{a,b\} \succ \{b\}$. Hence, if there is a unique U defined over $\mathscr{M}(\succeq)$, then we have a unique U defined over all two-element sets. Using Lemma 1, then we obtain a unique U defined over $\mathscr{A}$.

Now we will move to define u, v over all points in Δ. First, we define the linear function $u : \Delta \to R$ as

$$u(x) := U(\{x\})$$

For any a, b, δ with $a, b \in \Delta$, $\delta \in (0,1)$, if (a,b,δ) such that $\{a,b\} \in \mathscr{M}(\succeq)$ and $\{a,(1-$

$\delta)b + \delta x\} \in \mathscr{M}(\succeq)$ for all $x \in \Delta$, we can define the function $v : \Delta \to R$.

$$v(x; a, b, \delta) \coloneqq \frac{U(\{a, b\}) - U(\{a, (1-\delta)b + \delta x)}{\delta}$$

For properties on v we adopt Lemma 4 of Gul and Pesendorfer (2001) but change the domain of U from $\mathscr{A}$ to $\mathscr{M}(\succeq)$. Note that the proof in GP(2001) is still valid here since all two-element sets using in their proof of Lemma 4 is in $\mathscr{M}(\succeq)$.

Lemma 4. *Let U be a linear function that represents some $\succeq$ over $\mathscr{M}(\succeq)$. Suppose that $\{a, (1-\delta)b + \delta x\} \in \mathscr{M}(\succeq)$ for all $z \in \Delta$. Then:*

(i) $\forall z$ *such that* $\{a, z\} \in \mathscr{M}(\succeq)$, $v(z; a, b, \delta) = U(\{a, b\}) - U(\{a, z\})$

(ii) $v(a; a, b, \delta) = U(\{a, b\}) - U(\{a\})$

(iii) $v(\alpha z + (1-\alpha) z'; a, b, \delta) = \alpha v(z; a, b, \delta) + (1-\alpha) v(z'; a, b, \delta)$.

(iv) $v(z; a, b, \delta) = v(z; a, b, \delta')$, $\forall \delta' \in (0, \delta)$

(v) Suppose that $\{a, (1-\delta)b + \delta x\} \in \mathscr{M}(\succeq)$ *for all* $z \in \Delta$.

Since $\mathscr{L}(\succeq)$ is not a mixture space, we don't have U is linear over $\mathscr{L}(\succeq)$. However, using an argument similar to the proof of Lemma 5.6 in Kreps (1988), we obtain a weaker version of linearity of U.

Lemma 5. *Let U be a function that represent some $\succeq$ satisfying Axiom 1,2a-c, 3 and 4. If $\{x, y\} \in \mathscr{L}(\succeq)$, then*

$$U(\alpha\{x, y\} + (1-\alpha) A) = \alpha U(\{x, y\}) + (1-\alpha) U(A) \textit{ for all } A \in \mathscr{M}(\succeq).$$

Proof. If $\{x, y\}$ satisfies $\{x\} \succ \{x, y\} \succ \{y\}$, the linearity is already proven in lemma 4. We have to deal with $\{x, y\}$ with $\{x\} \sim \{x, y\} \succ \{y\}$ or $\{x\} \succ \{x, y\} \sim \{y\}$. If $\{x, y\}$ satisfies $\{x\} \succ \{x, y\} \sim \{y\}$, we claim that $h_\alpha(\{x, y\}, A) \sim h_\alpha(\{y\}, A)$ for all $A \in \mathscr{M}(\succeq)$. By Axiom 3, $\{x\} \succ \{y\}$ implies $h_\alpha(\{x\}, A) \succ h_\alpha(\{y\}, A)$, and Axiom 4 further implies $h_\alpha(\{x\}, A) \succeq h_\alpha(\{x, y\}, A) \succeq h_\alpha(\{y\}, A)$. In this case, we only have to show that $h_\alpha(\{x, y\}, A) \succ h_\alpha(\{y\}, A)$ will lead to a contradiction. Let us take $A = \{a, b\}$ with $\{a\} \succ \{a, b\} \succ \{b\}$. Suppose that $h_\alpha(\{x, y\}, A) \succ h_\alpha(\{y\}, A)$. In this case, we have $\{x\} \succ \{x, y\} \sim \{y\}$. Since $\{x\} \succ \{y\}$, letting $\alpha, \beta \in (0, 1)$ and applying Axiom 3, we obtain

$$\beta\{x\} + (1-\beta)\{y\} \succ \beta\{y\} + (1-\beta)\{y\} = \{y\} \sim \{x, y\},$$

and

$$\alpha\{x'\} + (1-\alpha)A \succ \alpha\{x, y\} + (1-\alpha)A,$$

where $\{x'\} = \beta\{x\} + (1-\beta)\{y\}$.

Since $h_\alpha(\{x'\}, A) \succ h_\alpha(\{x, y\}, A) \succ h_\alpha(\{y\}, A)$, von Neumann-Morgenstern continuity implies there exists some $\gamma \in (0, 1)$ such that

$$\begin{aligned} & h_\alpha(\{x, y\}, A) \\ & \succ h_\gamma(h_\alpha(\{x'\}, A), h_\alpha(\{y\}, A)). \end{aligned}$$

We only deal with the case when $A = \{a, b\}$ with $\{a\} \succ \{a, b\} \succ \{b\}$ because when A is a singleton set, the argument is similar but easier. Since $h_\alpha(\{x'\}, \{a, b\})$ and $h_\alpha(\{y\}, \{a, b\})$ are both in $\mathscr{M}(\succeq)$, we uses Lemma 2 to obtain

$$\begin{aligned} h_\alpha(\{x, y\}, A) &\succ h_\gamma(h_\alpha(\{x'\}, \{a, b\}), h_\alpha(\{y\}, \{a, b\})) \\ &\sim \tilde{h}_\gamma(h_\alpha(\{x'\}, \{a, b\}), h_\alpha(\{y\}, \{a, b\})) \\ &\sim h_\alpha(h_\gamma(\{x'\}, \{y\}), \{a, b\}) \\ &\succ h_\alpha(\{x, y\}, \{a, b\}), \end{aligned}$$

which yields a contradiction. The last "$\succ$" uses the fact that $\{x'\} \succ \{y\}$, $h_\gamma(\{x'\}, \{y\}) \succ \{y\} \sim \{x, y\}$ and Axiom 4. Since $h_\alpha(\{x, y\}, A) \sim h_\alpha(\{y\}, A)$, we have $U(h_\alpha(\{x, y\}, A)) = U(h_\alpha(\{y\}, A))$. Using Lemma 2, we have $U(h_\alpha(\{y\}, A)) = \alpha U(\{y\}) + (1-\alpha) U(A) = \alpha U(\{x, y\}) + (1-\alpha) U(A)$ for all $A \in \mathscr{M}(\succeq)$

For $\{x, y\}$ with $\{x\} \sim \{x, y\} \succ \{y\}$, Axioms 3 and 4 implies $h_\alpha(\{x\}, \{a, b\}) \succeq h_\alpha(\{x, y\}, \{a, b\})$ in the previous discussion. By using the fact that $\{x\} \succ \{y\}$, and applying a similar argument, we can rule out the possibility that $h_\alpha(\{x\},\{a,b\}) \succ h_\alpha(\{x,y\},\{a,b\})$ and further obtain $U(h_\alpha(\{x,y\},A)) = U(h_\alpha(\{x\},A)) = \alpha U(\{x\}) + (1-\alpha)U(A) = \alpha U(\{x,y\}) + (1-\alpha)U(A)$ for all $A \in \mathscr{M}(\succeq)$ $\square$

The following lemma is our version of lemma 5 of Gul and Pesendorfer (2001).

Lemma 6. *Let U be a function that represents some $\succeq$ satisfying Axioms 1, 2a-c, 3 and 4. Suppose $L(a) = \phi$, $b \in S(a)$ and $\delta \in (0,1)$ satisfy $U(\{a\}) > U(\{a,(1-\delta)b+\delta z\}) > U(\{(1-\delta)b+\delta z\})$ for all $z \in \Delta$. Then*

$$U(\{a,y\}) = \max_{x\in\{a,y\}}\{u(x)+v(x;a,b,\delta)\} - \max_{z\in\{a,y\}} v(z;a,b,\delta),$$

Proof. For the case where $U(\{a\}) > U(\{a,y\}) > U(\{y\})$, by GP(2001), we know $v(y;a,b,\delta) \geq v(a;a,b,\delta)$ and $u(a)+v(a;a,b,\delta)-v(y;a,b,\delta) > u(y)+v(y;a,b,\delta)-v(y;a,b,\delta)$.

Let $A = (1-\delta)\{a,b\}+\delta\{a,y\}$. Since $\{a,b\} \in \mathscr{M}(\succeq)$, by Lemma 5 we have $U(A) = (1-\delta)U(\{a,b\})+\delta U(\{a,y\})$ for $\{a,y\} \in \mathscr{L}(\succeq)$. For the case where $U(\{a\}) = U(\{a,y\}) > U(\{y\})$, the first part of Lemma 2 establishes that $U(A) = \min_{z\in A} U(\{a,z\})$. Hence, we have $v(a;a,b,\delta) \geq v(y;a,b,\delta)$ for the by the same argument in GP(2001). For the case where $U(\{a\}) > U(\{a,y\}) = U(\{y\})$ and $y \in L^c(a)$, we will show $v(y;a,b,\delta) \geq v(a;a,b,\delta)+u(a)-u(y)$. From Gul and Pesendorfer (2001) this is equivalent to show that

$$U(\{a,(1-\delta)b+\delta y\}) \leq (1-\delta)U(\{a,b\})+\delta U(\{a,y\}) = U(A)$$

The above inequality holds because of the second part of Lemma 2 $U(A) = \max_{w\in A}(\{w,(1-\delta)b+\delta y\})$. $\square$

Chapter 3

Main theorem

The previous chapter focuses on sets without the constraint of willpower and obtains results similar to GP(2001). Since the main difference between our model and GP(2001) is on menus with no self-control due to lack of willpower, to get our representation function U over $\mathscr{A}$, we add one more axiom to regulate the willpower constraints across different menus. Let $\overline{A}$ be the closure of A.

Axiom 5. *For any $a, x \in \Delta$, if $b \in \overline{L(a)} \cap \overline{S(a)}$ and $y \in S(x)$, then $\frac{1}{2}\{a\} + \frac{1}{2}\{x, y\} \succeq \frac{1}{2}\{a, b\} + \frac{1}{2}\{x\}$.*

Axiom 5 requires that given the same chance of having to perform costly self-control, DM prefers facing a temptation that he does not have to exhaust his entire willpower.

From Lemma 3, we know that $S(a)$ is convex and $\delta a + (1-\delta) b \in S(a)$ if $b \in S(a)$. Since we are interested in the preferences that could reveal limited willpower and costly self-control, we add the following axiom.

Axiom 6 (Richness)**.** *There exists an a in the interior of Δ, such that $L(a) \neq \phi$ and $S(a)$ contains a non-empty open ball.*

First let us introduce a lemma before we show our main theorem.

Lemma 7. *Let U be a function that represents some $\succeq$ satisfying Axioms 1, 2a-c, and 3-6. Take a in the interior of Δ, such that $L(a) \neq \phi$ and $b \in S(a)$, $\delta \in (0, 1)$ satisfy*

$U(\{a\}) > U(\{a,(1-\delta)b+\delta z\}) > U(\{(1-\delta)b+\delta z\})$ for all $z \in \Delta$. Then

$$U(\{a,y\}) = \max_{x\in\{a,y\}}\{u(x)+v(x;a,b,\delta)\} - \max_{z\in\{a,y\}} v(z;a,b,\delta),$$
$$\textit{s.t.} \ \max_{z\in\{a,y\}} v(z;a,b,\delta) - v(x;a,b,\delta) \le w(a),$$

where $w(a) = \max_{z\in S(a)} v(z;a,b,\delta) - v(a;a,b,\delta)$.

Proof. For the case where $y \in S(a)$, we have $v(y;a,b,\delta) \le \max_{z\in S(a)} v(z;a,b,\delta)$. Hence, by Lemma 6, we have the desired result for all $\{a,y\} \in \mathscr{L}(\succeq)$. The remaining part of the proof shows that if $y \in L(a)$, then we must have $v(y;a,b,\delta) - v(a;a,b,\delta) \ge w(a)$, which is equivalent to show that $v(y;a,b,\delta) > \max_{z\in S(a)} v(z;a,b,\delta)$. Since $y \in L(a)$, $\exists\bar{\delta} \in (0,1)$ such that $\delta' y + (1-\delta')a \in S(a)$ if $\delta' \le \bar{\delta}$ and $\delta' y + (1-\delta')a \in L(a)$ if $\delta' > \bar{\delta}$. Let $\bar{y} = \bar{\delta}y + \left(1-\bar{\delta}\right)a$. Since $\bar{y} \in S(a)$, by Lemma 4 (iii), we have $\bar{\delta}v(y;a,b,\delta) + \left(1-\bar{\delta}\right)v(a;a,b,\delta) = v(\bar{y};a,b,\delta) \ge v(a;a,b,\delta)$. Hence, $v(y;a,b,\delta) \ge v(a;a,b,\delta)$. Moreover, $u(a) + v(a;a,b,\delta) > u(\bar{y}) + v(\bar{y};a,b,\delta) = \bar{\delta}(u(y)+v(y;a,b,\delta)) + \left(1-\bar{\delta}\right)(u(a)+v(a;a,b,\delta))$. Hence, $u(a)+v(a;a,b,\delta) > u(y)+v(y;a,b,\delta)$.

We claim that $\max_{z\in S(a)} v(z;a,b,\delta) = v(\bar{y};a,b,\delta)$. For any $b' \in S(a)$, from Axiom 5, we have $\frac{1}{2}\{a\} + \frac{1}{2}\{a,b'\} \succeq \frac{1}{2}\{a\} + \frac{1}{2}\{a,\bar{y}\}$. Since $\{a,b'\}$ and $\{a,\bar{y}\}$ are in $\mathscr{M}(\succeq)$, we have $\frac{1}{2}u(a) + \frac{1}{2}U(\{a,b'\}) \ge \frac{1}{2}u(a) + \frac{1}{2}U(\{a,\bar{y}\})$, $U(\{a,b'\}) = u(a) + v(a;a,b,\delta) - v(b';a,b,\delta)$ and $U(\{a,\bar{y}\}) = u(a) + v(a;a,b,\delta) - v(\bar{y};a,b,\delta)$. Hence, $v(b';a,b,\delta) \le v(\bar{y};a,b,\delta)$. $\square$

Now we are ready to prove our main theorem.

Theorem 3. *A preference $\succeq$ satisfies Axiom 1, 2a-c, 3-6 if and only if there are continuous linear function u, v and a constant w such that the function U defined as*

$$U(A) = \max_{x\in A}\{u(x)+v(x)\} - \max_{y\in A} v(y)$$
$$\textit{s.t.} \ \max_{y\in A} v(y) - v(x) \le w$$

for all $A \in \mathscr{A}$ and U present $\succeq$.

Proof. From Axiom 6, there exists x in the interior of Δ, such that $L(x) \neq \Phi$ and $y \in S(x)$, $\delta \in (0,1)$ satisfy $U(\{x\}) > U(\{x, (1-\delta)y + \delta z\}) > U(\{(1-\delta)y + \delta z\})$ for all $z \in \Delta$. By Lemma 7, we can let $u(z) := U(\{z\})$, $v(z) := v(z; x, y, \delta)$ for all $z \in \Delta$ and $w = \max_{z \in S(x)} v(z; x, y, \delta) - v(x; x, y, \delta)$. Now consider the set $A = \{a, b\}$, where a and b are in the relative interior of Δ. Assume without loss of generality, that $u(a) \geq u(b)$. Since a is in the interior of Δ, we can select α and a' such that $\{a, c\} = \alpha\{a'\} + (1-\alpha)\{x, y\}$. Hence, $u(\{a\}) > U(\{a, c\}) > u(c)$. Then for δ' sufficiently small we have

$$U(\{a\}) > U(\{a, (1-\delta')c + \delta' z\}) > U(\{(1-\delta')c + \delta' z\}) \text{ for all } z \in \Delta.$$

If $L(a) = \Phi$, then $\{a, b\} \in \mathscr{L}(\succeq)$. Hence, we can apply Lemma 6,

$$U(\{a, b\}) = \max_{z \in \{a,b\}} \{u(a) + v(a; a, c, \delta')\} - \max_{z \in \{a,b\}} v(z; a, c, \delta'),$$

Since the willpower constraint is relevant only when $b \in S(a)$, we need to show that $v(b; a, c, \delta') - v(a; a, c, \delta') \leq w$ for $b \in S(a)$. Since $b \in S(a)$, by Axiom 5, we have $\frac{1}{2}\{x\} + \frac{1}{2}\{a, b\} \succeq \frac{1}{2}\{a\} + \frac{1}{2}\{x, z\}$, for $z \in \overline{L(x)} \cap \overline{S(x)}$. This implies that $\frac{1}{2}u(x) + \frac{1}{2}U(\{a, b\}) \geq \frac{1}{2}u(a) + \frac{1}{2}U(\{x, z\})$, $U(\{a, b\}) = u(a) + v(a; a, b, \delta') - v(b'; a, b, \delta')$ and $U(\{x, z\}) = u(x) - w$. Hence, we have $v(b; a, c, \delta') - v(a; a, c, \delta') \leq w$. Let $\delta^* = \min\{\delta, \delta'\}$. By Lemma 4 (iv), $v(\cdot; x, y, \delta^*) = v(\cdot; x, y, \delta)$ and $v(\cdot; a, c, \delta^*) = v(\cdot; a, c, \delta')$. By Lemma 4 (v), for an appropriate constant k, $v(\cdot; x, y, \delta^*) = v(\cdot; a, c, \delta^*) + k$ and hence it follows that

$$U(\{a, b\}) = \max_{z \in \{a,b\}} \{u(z) + v(z)\} - \max_{z' \in \{a,b\}} v(z')$$

$$s.t. \max_{z' \in \{a,b\}} v(z') - v(z) \leq w$$

If $L(a) \neq \Phi$, we can apply Lemma 7,

$$U(\{a,b\}) = \max_{z\in\{a,b\}} \{u(a) + v(a;a,b,\delta')\} - \max_{z\in\{a,y\}} v(z';a,b,\delta'),$$
$$\text{s.t. } \max_{z'\in\{a,b\}} v(z';a,b,\delta) - v(z;a,b,\delta) \leq w(a),$$

where $w(a) = \max_{z\in S(a)} v(z;a,b,\delta') - v(a;a,b,\delta')$. Take $c \in \overline{L(a)} \cap \overline{S(a)}$ and $z \in \overline{L(x)} \cap \overline{S(x)}$. By Axiom 5, we have $w(a) = v(c;a,b,\delta') - v(a;a,b,\delta') = w$. Hence, follow the same argument, we have

$$U(\{a,b\}) = \max_{z\in\{a,b\}} \{u(z) + v(z)\} - \max_{z'\in\{a,b\}} v(z')$$
$$s.t. \max_{z'\in\{a,b\}} v(z') - v(z) \leq w$$

Now consider an arbitrary finite set A. We know that

$$\begin{aligned} U(A) &= \max_{a\in A} \min_{b\in A} U(\{a,b\}) \\ &= \max_{a\in A} \min_{b\in A} \begin{Bmatrix} \max_{z\in\{a,b\}}\{u(z)+v(z)\} - \max_{z'\in\{a,b\}} v(z') \\ s.t. \max_{z'\in\{a,b\}} v(z') - v(z) \leq w \end{Bmatrix} \\ &= \max_{a\in A} \min_{b\in A} \begin{Bmatrix} \max_{z\in\{a,b\}}\{u(z)+v(z)\} \\ s.t. \max_{z'\in\{a,b\}} v(z') - v(z) \leq w \end{Bmatrix} + \min_{b\in A}\{-v(b)\} \end{aligned}$$

Let $x \in \arg\max_{z'\in A} v(z')$. If $v(x) - v(a^*) > w$, then a^* does not solve the constraint maxminmax problem because for the pair $\{a^*, x\}$ we would choose x instead of a^*. Hence, a^* will not survive after the second requirement, i.e., $\min_{b\in A}$ when we take $b = x$. If $v(x) - v(a^*) \leq w$, then for any pair $\{a^*, z\}$ where $z \in A$, a^* satisfies the constraint. Hence, if $v(x) - v(z) \leq w$, then we choice a^* only when $u(a^*) + v(a^*) \geq u(z) + v(z)$. Hence, we have

$$U(A) = \max_{a\in A}\{u(a) + v(a)\} + \min_{b\in A}\{-v(b)\}$$
$$s.t. \max_{z'\in\{a,b\}} v(z') - v(z) \leq w$$

$\square$

Chapter 4

Conclusion

This dissertation is the first one to characterize both costly self-control and limited willpower with representation theorem. We provide a method to distinguish the two possible reasons. It separates the set $T(a)$ into two parts: $L(a)$ and $L^c(a)$, the reason of willpower and the reason of resisting cost respectively. The method we provide here is to construct a menu with a lottery and a less tempting prize. There is a possible alternative definition for $L(a)$ which can separate $T(a)$ into two identical partitions. Suppose $b \in T(a)$. If there exists a sequence of lotteries $\{x\}_{i=0}^{n}$ ($x_0 = a$, $x_n = b$), and $\{x_i, x_{i+1}\} \in \mathscr{M}(\succeq)$ for all i, then $b \in T(a)$. In this version of definition, no self-control caused by lack of willpower implies that we can find a series of lotteries like a bridge so that decision maker can perform self-control in every step of this bridge. This definition may be more nature for some readers. We are going to verify the equivalence of these two definitions.

Besides, we are going to compare preferences with different characteristic in our further study. For example, decision makers with same measurement of benefit and temptation, u and v, and with different level of willpower, the constant w, may have some relationship in the preference domain.

Furthermore, if possible, we want to find some axioms which are more likely to be realized in place of the current version. Our main purpose is finding a series of verifiable axioms corresponding to the model with costly self-control and limited willpower. Hence, any axiom which is easier to be verified would be much better.

Bibliography

Benabou, R. and J. Tirole (2002): “Self-Confidence and Personal Motivation,” *The Quarterly Journal of Economics*, 117, 871–915.

Dekel, E. and B. L. Lipman (2009): “Temptation-driven preferences,” *Review of Economic Studies*, 76, 937–971.

Fudenberg, D. and D. Levine (2012): “Timing and Self-Control,” *Econometrica*, 80, 1–42.

Gul, F. and W. Pesendorfer (2001): “Temptation and Self-Control,” *Econometrica*, 69, 1403–1435.

Kreps, D. (1979): “A Representation Theorem for Preference for Flexibility,” *Econometrica: Journal of the Econometric Society*, 47, 565–577.

——— (1988): *Notes on the theory of choice*, Westview Press.

Masatlioglu, Y., D. Nakajima, and E. Ozdenoren (2011): “Revealed Willpower,” .

Noor, J. (2011): “Temptation and Revealed Preference,” *Econometrica*, 79, 601–644.

Ozdenoren, E., S. W. Salant, and D. Silverman (2012): “Willpower and the Optimal Control of Visceral Urges,” *Journal of the European Economic Association*, 10, 342–368.