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廣義隨機矩陣特徵值之中央極限定理

Central Limit Theorem for Linear Eigenvalue Statistics of Generalized Wigner Matrices

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摘要

在本文中，我們考慮 n 維的實對稱 Wigner 矩陣 W，其中每個元 W_{jk} 均為獨立的隨機變數(但不需要是相同分佈)，而證明出此類矩陣之線性統計量所滿足之中央極限定理。

關鍵字: 隨機矩陣、特徵值、中央極限定理

Abstract

In this paper, we consider $n \times n$ real symmetric Wigner matrices W with independent (modulo symmetry condition), but not necessarily identically distributed, entries $\{W_{jk}\}_{j,k=1}^{n}$ and prove central limit theorem for linear eigenvalue statistics of such matrices.

Keywords: Random matrices, Eigenvalues, Central limit theorem

1. Introduction to generalized Wigner matrices and other generalities

1.1 Introduction

To study the asymptotic behavior of eigenvalues $\{\lambda_i\}_{i=1}^{n}$ of random matrices, we consider the linear eigenvalue statistics, defined via test function $\varphi : R \to C$ as $N_n[\varphi] = \sum_{i=1}^{n} \varphi(\lambda_i)$. In 2009, A.Lytova and L.Pastur proved central limit theorem for $N_n[\varphi]$ and calculated its variance explicitly under the condition that all entries in the matrices are independent and identically distributed.([1])

In 2010, László Erdös, Horng − Tzer Yau, and Jun Yin([2]) proposed the generalized Wigner matrices by assuming that the entries in the matrices are independent, but not necessarily identically distributed. They proved the semicircle law with sharp estimates. Here we prove central limit theorem of this model.

Following the idea of A.Lytova and L.Pastur, the proof is divided into two steps. First we prove central limit theorem for the random variable $N_n[\varphi]$ when the entries are Gaussian. Then we apply Taylor expansion to study the general case.

1.2 Preliminaries

In this section we introduce basic definitions about generalized Wigner matrices.

Definition 1.1 [Generalized Wigner Matrices]

For each $n \in \mathbb{N}$, let W be a $n \times n$ real, symmetric matrix where

(1) $\{W_{jk}(n)\}_{1\le j\le k\le n}^{n} = \{W_{jk}\}_{1\le j\le k\le n}^{n}$ are independent random variables

(2) $E[W_{jk}] = 0,\ E[W_{jk}^2(n)] = \sigma_{jk}^2(n) = \sigma_{jk}^2 < \infty$

(3) There is a constant C, independent of n, such that

(1.1) $$\max_{j,k}\{\sigma_{jk}^2\} \le \frac{C}{n}$$

(1.2) $$\sum_{j=1}^{n} \sigma_{jk}^2 = 1 \text{ for any fixed } k$$

(4) Denoting by $B = \{\sigma_{jk}^2\}$ the matrix of variances, there exist two positive constants δ_- and δ_+, independent of n, such that

(1.3) $$1 \text{ is a simple eigenvalue of } B \text{ and } \mathrm{Spec}(B) \subset [-1+\delta_-, 1-\delta_+] \cup \{1\}.$$

(5) The distributions of the matrix elements have a uniform subexponential decay in the sense that there exists a constant $\nu > 0$, independent of n, such that for any $x \geq 1$ and $1 \leq j, k \leq n$ we have

(1.4) $$P(|W_{jk}| \geq x\sigma_{jk}) \leq \nu^{-1} e^{-x^{\nu}}$$

Then W is called a generalized Wigner matrix.([2])

Through this paper we impose an additional assumption that

(1.5) $$\sup_{j,k} |(n\sigma_{jk}^2 - 1)| = o(1).$$

Definition 1.2 *[Linear Eigenvalue Statistics]*

With a test function $\varphi : R \to C$, and $\{\lambda_i\}_{i=1}^{n}$ the eigenvalues of a generalized Wigner matrix W, the linear eigenvalue statistic is defined as

(1.6) $$N_n[\varphi] = \sum_{i=1}^{n} \varphi(\lambda_i)$$

1.3 Lemmas

In this part we introduce some lemmas that will be helpful in proving our desired theorems.

Lemma 1.1 *[Duhamel Formula]*

Let M_1, M_2 be $n \times n$ matrices and $t \in \mathbb{R}$. Then we have

(1.7) $$e^{(M_1+M_2)t} = e^{M_1 t} + \int_0^t e^{M_1(t-s)} M_2 e^{(M_1+M_2)s} ds$$

proof. Define $A(t) = e^{-M_1 t} e^{(M_1+M_2)t}$

then $A'(t) = e^{-M_1 t}(-M_1)e^{(M_1+M_2)t} + e^{-M_1 t}(M_1 + M_2)e^{(M_1+M_2)t}$

$= e^{-M_1 t} M_2 e^{(M_1+M_2)t}$

Thus,

$A(t) - A(0) = \int_0^t A'(s)ds = \int_0^t e^{-M_1 s} M_2 e^{(M_1+M_2)s} ds$, or

$$A(t) = A(0) + \int_0^t e^{-M_1 s} M_2 e^{(M_1+M_2)s} ds$$

Multiply $e^{M_1 t}$ to both sides and we have the Duhamel formula :

$$e^{(M_1+M_2)t} = e^{M_1 t} + \int_0^t e^{M_1(t-s)} M_2 e^{(M_1+M_2)s} ds$$ ■

Lemma 1.2 [Properties of $U(t) = e^{itM}$]

Consider a real symmetric matrix $M = \{M_{jk}\}_{j,k=1}^{n}$ and set $U(t) = e^{itM}$, $t \in \mathbb{R}$.

Then U(t) is a symmetric unitary matrix possessing the properties

(1.8) (i) $U(t)U(s) = U(t+s)$ (ii) $\| U(t) \| = 1$

(iii) $\sum_{j=1}^{n} |U_{jk}(t)|^2 = 1$ (iv) $|U_{jk}(t)| \leq 1$

proof.

(i) Since $U(t) = e^{itM} = \sum_{k=0}^{\infty} \frac{(itM)^k}{k!}$, $U(s) = e^{isM} = \sum_{k=0}^{\infty} \frac{(isM)^k}{k!}$

Consider $U(t)U(s) = \sum_{k=0}^{\infty} a_k M^k$, then $a_k = \sum_{j=0}^{k} \frac{(it)^j}{j!} \cdot \frac{(is)^{k-j}}{(k-j)!}$

$$= \frac{1}{k!}\sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!}\, i^k \cdot t^j \cdot s^{k-j} = \frac{i^k}{k!}\sum_{j=0}^{k} \binom{k}{j} \cdot t^j \cdot s^{k-j} = \frac{i^k}{k!} \cdot (t+s)^k = \frac{[i(t+s)]^k}{k!}$$

Thus $U(t)U(s) = \sum_{k=0}^{\infty} \frac{[i(t+s)]^k}{k!} M^k = \sum_{k=0}^{\infty} \frac{[i(t+s)M]^k}{k!} = U(t+s)$.

(ii) Since M is real symmetric, M can be diagonalized as $M = CDC^{-1}$,

$\| U(t) \| = \| e^{itM} \| = \| e^{itCDC^{-1}} \|$

$= \| Ce^{itD}C^{-1} \| \quad \left(e^{itCDC^{-1}} = \sum_{k=0}^{\infty} \frac{(itCDC^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{itCD^kC^{-1}}{k!} = Ce^{itD}C^{-1}\right)$

$= \| e^{itD} \| = 1$

Since U(t)is unitary, it follows that (iii) $\sum_{j=1}^{n} \left|U_{jk}(t)\right|^2 = 1$

and (iv)$\left|U_{jk}(t)\right| \leq 1$ is obvious from (iii) ■

<u>Lemma 1.3</u> [Differential Formula]

The Duhamel formula(1.7) allows us to obtain the derivatives of U(t) with respect to the entries M_{jk} of M. (the symbol "*" stands for convolution of two functions.)

(1.9) $\quad \frac{\partial U_{ab}(t)}{\partial M_{jk}} = \frac{i}{1+\delta_{jk}}\left[\left(U_{aj} * U_{bk}\right)(t) + \left(U_{bj} * U_{ak}\right)(t)\right]$ where $\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$

It follows that for $r \in \mathbb{N}$, there is a constant c_r, independent of n, t such that

(1.10) $\quad \left|\frac{\partial^r U_{ab}(t)}{\partial {M_{jk}}^r}\right| \leq c_r |t|^r$

<u>proof.</u>

For (1.9),

let H^{jk} be a symmetric matrix with all entries zero except for

$$H^{jk}_{jk} = H^{jk}_{kj} = h$$

and $I_{jk} = \dfrac{H^{jk}}{h}$

By definition $\dfrac{\partial U_{ab}(t)}{\partial M_{jk}(t)} = \lim\limits_{h\to 0}\dfrac{\left[e^{it(M+H^{jk})}\right]_{ab} - \left[e^{itM}\right]_{ab}}{h}$

applying Duhamel's formula (1.6), with $M_1 = iM$ and $M_2 = iH^{jk}$, we get

$$\frac{\partial U_{ab}(t)}{\partial M_{jk}} = \lim_{h\to 0}\frac{\int_0^t \left[e^{iM(t-s)}(iH^{jk})e^{i(M+H^{jk})s}\right]_{ab} ds}{h}$$

$$= i \cdot \lim_{h\to 0}\int_0^t \left[e^{iM(t-s)} I_{jk} e^{i(M+H^{jk})s}\right]_{ab} ds$$

$$= i \cdot \int_0^t \left[e^{iM(t-s)} I_{jk} e^{iMs}\right]_{ab} ds \qquad \text{(by continuity of } e^{iMs}\text{)}$$

$$= i \cdot \int_0^t \left[U(t-s) I_{jk} U(s)\right]_{ab} ds$$

(a)If $j = k$:

$$I_{jk}U(s) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ U_{j1}(s) & U_{j2}(s) & \cdots & U_{jn}(s) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \rightarrow \text{jth row}$$

$$\Rightarrow \; U(t-s)I_{jk}U(s) = \begin{bmatrix} U_{1j}(t-s)U_{j1}(s) & U_{1j}(t-s)U_{j2}(s) & \cdots & U_{1j}(t-s)U_{jn}(s) \\ U_{2j}(t-s)U_{j1}(s) & U_{2j}(t-s)U_{j2}(s) & \cdots & U_{2j}(t-s)U_{jn}(s) \\ \vdots & \vdots & \ddots & \vdots \\ U_{nj}(t-s)U_{j1}(s) & U_{nj}(t-s)U_{j2}(s) & \cdots & U_{nj}(t-s)U_{jn}(s) \end{bmatrix}$$

In this case $\dfrac{\partial U_{ab}(t)}{\partial M_{jk}} = i \cdot \displaystyle\int_0^t \left[U(t-s)I_{jk}U(s)\right]_{ab} ds$

$$= i \cdot \int_0^t U_{aj}(t-s)U_{jb}(s)ds = i \cdot U_{aj} * U_{jb}(t) = i \cdot U_{aj} * U_{bj}(t)$$

(b)If $j \neq k$, without lose of generality say $j < k$:

$$
I_{jk}U(s) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ U_{k1}(s) & U_{k2}(s) & \cdots & U_{kn}(s) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ U_{j1}(s) & U_{j2}(s) & \cdots & U_{jn}(s) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \rightarrow \text{jth row} \\ \\ \\ \\ \rightarrow \text{kth row} \\ \\ \\ \\ \end{matrix}
$$

$\Rightarrow \quad U(t-s)I_{jk}U(s)$

$$
= \begin{bmatrix} U_{1j}(t-s)U_{k1}(s) + U_{1k}(t-s)U_{j1}(s) & U_{1j}(t-s)U_{k2}(s) + U_{1k}(t-s)U_{j2}(s) & \cdots & U_{1j}(t-s)U_{kn}(s) + U_{1k}(t-s)U_{jn}(s) \\ U_{2j}(t-s)U_{k1}(s) + U_{2k}(t-s)U_{j1}(s) & U_{2j}(t-s)U_{k2}(s) + U_{2k}(t-s)U_{j2}(s) & \cdots & U_{2j}(t-s)U_{kn}(s) + U_{2k}(t-s)U_{jn}(s) \\ \vdots & \vdots & \ddots & \vdots \\ U_{nj}(t-s)U_{k1}(s) + U_{nk}(t-s)U_{j1}(s) & U_{nj}(t-s)U_{k2}(s) + U_{nk}(t-s)U_{j2}(s) & \cdots & U_{nj}(t-s)U_{kn}(s) + U_{nk}(t-s)U_{jn}(s) \end{bmatrix}
$$

In this case $\dfrac{\partial U_{ab}(t)}{\partial M_{jk}} = i \cdot \displaystyle\int_0^t \left[U(t-s)I_{jk}U(s)\right]_{ab} ds$

$$
= i \cdot \int_0^t [U_{aj}(t-s)U_{bk}(s) + U_{ak}(t-s)U_{jb}(s)]ds
$$

$$
= i \cdot \left[\left(U_{aj} * U_{bk}\right)(t) + \left(U_{bj} * U_{ak}\right)(t) \right]
$$

From (a) and (b)we conclude that

$$
\frac{\partial U_{ab}(t)}{\partial M_{jk}} = \frac{i}{1+\delta_{jk}} \left[\left(U_{aj} * U_{bk}\right)(t) + \left(U_{bj} * U_{ak}\right)(t) \right] \quad \text{where } \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}
$$

Now we prove (1.10):

If r = 1,

Since $\left|U_{jk}(t)\right| \leq 1, \left|\dfrac{\partial U_{ab}(t)}{\partial M_{jk}}\right| \leq \left| \left(U_{aj} * U_{bk}\right)(t) + \left(U_{bj} * U_{ak}\right)(t) \right|$

$$
\leq \left| \int_0^t \left| \left(U_{aj}(t-s)U_{bk}(s) + U_{ak}(t-s)U_{jb}(s)\right) \right| ds \right|
$$

$$
\leq \left| \int_0^t (\left|U_{aj}(t-s)\right| \cdot |U_{bk}(s)| + \left|U_{aj}(t-s)\right| |U_{bk}(s)|) ds \right|
$$

$$\leq \left| \int_0^t 2\, ds \right| = 2|t|$$

For general $r \geq 2$, the differential formula (1.9) yields an r-dimensional integral.

Since $|U_{jk}(t)| \leq 1$, there is a constant c_r which depends on r that

$$\left| \frac{\partial^r U_{ab}(t)}{\partial {M_{jk}}^r} \right| \leq c_r |t|^r \qquad \blacksquare$$

Lemma 1.4 [Equalities and Inequalities about Gaussian random variables]

Let $\xi = \{\xi_r\}_{r=1}^p$ be independent Gaussian random variables of zero mean, and $\Phi : R^p \to C$ be a differentiable function with polynomially bounded partial derivatives $\Phi'_r, r = 1,2,\dots,p$.

Then we have

(1.11) (i) $E[\xi_r \Phi(\xi)] = E[\xi_r^2]\, E[\Phi'_r(\xi)],$ $\qquad r = 1,2,\dots,p.$

(1.12) (ii) $\mathrm{Var}[\Phi(\xi)] \leq \max_{1\leq r\leq p}\{\sigma_r^2\} \sum_{r=1}^p E[|\Phi'_r(\xi)|^2]$ (Poincaré inequality)

proof.

(i) $$E[\xi_r \Phi(\xi)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} x_r \Phi(x) \frac{1}{\sqrt{2\pi}^p \sigma_1 \sigma_2 \dots \sigma_p} e^{-\sum_{k=1}^p \frac{x_k^2}{2\sigma_k^2}} dx_1 dx_2 \dots dx_p$$

Applying integration by parts with respect to x_r, we get that this expected value is

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left\{ \left[-\sigma_r^2 \Phi(x) \frac{1}{\sqrt{2\pi}^p \sigma_1 \sigma_2 \dots \sigma_p} e^{-\sum_{k=1}^p \frac{x_k^2}{2\sigma_k^2}} \right]_{-\infty}^{\infty} \right.$$

$$\left. + \sigma_r^2 \int_{-\infty}^{\infty} \frac{\Phi'_r(x)}{\sqrt{2\pi}^p \sigma_1 \sigma_2 \dots \sigma_p} e^{-\sum_{k=1}^p \frac{x_k^2}{2\sigma_k^2}} dx_r \right\} dx_1 dx_2 \dots dx_{r-1} dx_{r+1} \dots dx_p$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left\{ \sigma_r^2 \int_{-\infty}^{\infty} \frac{\Phi'_r(x)}{\sqrt{2\pi}^p \sigma_1 \sigma_2 \dots \sigma_p} e^{-\sum_{k=1}^p \frac{x_k^2}{2\sigma_k^2}} dx_r \right\} dx_1 dx_2 \dots dx_{r-1} dx_{r+1} \dots dx_p$$

(since Φ has polynomially bounded partial derivatives.)

$= E[\xi_r^2]\, E[\Phi_r'(\xi)]$

(ii) Consider the Hilbert space $L^2(d\mu)$ where

$$d\mu = \frac{1}{\sqrt{2\pi}^p \sigma_1\sigma_2 \ldots \sigma_p} e^{-\sum_{k=1}^{p} \frac{x_k^2}{2\sigma_k^2}} dx_1 dx_2 \ldots dx_p$$

Observe that

$$< \nabla f, \nabla g >_\mu = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \nabla f \cdot \nabla g \, d\mu = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f \cdot (-\Delta g + \sum_{r=1}^{p} \frac{x_r}{\sigma_r^2} \frac{\partial g}{\partial x_r})\, d\mu$$

By defining $L = -\Delta + \sum_{r=1}^{p} \frac{x_r}{\sigma_r^2} \frac{\partial}{\partial x_r}$, we can rewrite the above equation as

$< \nabla f, \nabla g >_\mu = < f, Lg >_\mu = < Lf, g >_\mu$

Thus L is self-adjoint and positive.

Consider eigenvalues $\{\lambda_r\}$ of L, it is well known that $\lambda_0 = 0$ and $\lambda_r = \frac{1}{\sigma_r^2}$ $r = 1,2, \ldots, p$.

Let $e_0, e_1 \ldots, e_{p,\ldots}$ be orthonormal basis of L in $L^2(d\mu)$ with respect to eigenvalues $\lambda_0, \lambda_1 \ldots, \lambda_p, \ldots$

we write the decomposition of f:

$$f = < f, e_0 >_\mu + < f, e_1 >_\mu e_1 + < f, e_2 >_\mu e_2 + \ldots + < f, e_p >_\mu e_p + \ldots$$

Note that $< f, e_0 >_\mu = < f, 1 >_\mu = \int f d\mu = E[f]$, so

$$f - E[f] = < f, e_1 >_\mu e_1 + < f, e_2 >_\mu e_2 + \ldots + < f, e_p >_\mu e_p + \ldots$$

Since $\int E[f]\, e_r d\mu = 0$ for all $r = 1,2, \ldots, p, \ldots$ by writing $\Phi = f - E[f]$,

$$\Phi = < \Phi, e_1 >_\mu e_1 + < \Phi, e_2 >_\mu e_2 + \ldots + < \Phi, e_p >_\mu e_p + \ldots$$

Let $\lambda = \inf_{r \geq 1} \{\lambda_r\} = \inf_{r \geq 1} \left\{\frac{1}{\sigma_r^2}\right\} = \frac{1}{\sup_{r \geq 1} \{\sigma_r^2\}}$

$$L\Phi = <\Phi, e_1>_\mu \lambda_1 e_1 + <\Phi, e_2>_\mu \lambda_2 e_2 + \ldots + <\Phi, e_p>_\mu \lambda_p e_p + \ldots$$

and

$$<L\Phi, \Phi>_\mu = <\Phi, e_1>_\mu{}^2 \lambda_1 + <\Phi, e_2>_\mu{}^2 \lambda_2 + \ldots + <\Phi, e_p>_\mu{}^2 \lambda_p + \ldots$$

$$\geq \lambda\left(<\Phi, e_1>_\mu{}^2 + <\Phi, e_2>_\mu{}^2 + \ldots + <\Phi, e_p>_\mu{}^2 + \ldots\right) = \lambda <\Phi, \Phi>_\mu$$

Hence

$$\mathrm{Var}[\Phi(\xi)] = <\Phi, \Phi>_\mu \leq \frac{1}{\lambda} <L\Phi, \Phi>_\mu$$

$$= \sup_{r\geq 1}\{\sigma_r^2\} \|\nabla\Phi\|^2 = \sup_{r\geq 1}\{\sigma_r^2\} \sum_{l=1}^{p} E[|\Phi_r'(\xi)|^2] = \max_{r\geq 1}\{\sigma_r^2\} \sum_{l=1}^{p} E[|\Phi_r'(\xi)|^2] \quad \blacksquare$$

<u>Lemma 1.5</u> [An Integral Equation]([1])

Denote the correspondence between functions and their generalized Fourier transforms as $f \leftrightarrow \hat{f}$. If P, Q and R are locally Lipshitzian, satisfy $\sup_{t\geq 0} e^{-\delta t}|f(t)| < \infty$, and $1 + i\,\hat{Q}(z) \neq 0, \quad \mathrm{Im}\, Z < 0.$

then the equation

$$P(t) + \int_0^t Q(t-s)P(s)ds = R(t), \qquad t \geq 0,$$

has a unique locally Lipshitzian solution

$$P(t) = -i\int_0^t T(t-s)R(s)ds$$

where $T \leftrightarrow (1 + i\,\hat{Q})^{-1}$

In particular, if R(t) is differentiable, R(0) = 0, and $Q(t) = \int_0^t Q_1(s)ds$

then the equation

$$P(t) + \int_0^t ds \int_0^s Q_1(s-v)P(v)dv = R(t), \quad t \geq 0, \qquad (1.13)$$

has a unique locally Lipshitzian solution

$$P(t) = -\int_0^t T_1(t-s)R'(s)ds,$$

where $T_1 \leftrightarrow (z+\widehat{Q}_1)^{-1}$

provided by $z+\widehat{Q}_1 \neq 0$, $\operatorname{Im} Z < 0$.

Lemma 1.6 [Semicircle Law]([1][2])

Let W be a generalized Wigner matrix, and $N_n[\varphi]$ be the corresponding linear eigenvalue statistic(see (1.6)). Then we have for any bounded and continuous $\varphi : R\rightarrow C$ with probability 1 that

(1.14) $$\lim_{n\to\infty} n^{-1}N_n[\varphi] = \int_{-2}^{2} \varphi(\lambda)\cdot\frac{1}{2\pi}\sqrt{4-\lambda^2}d\lambda$$

2. Central Limit Theorem for linear eigenvalue statistics in the case of Gaussian entries

2.1 Bound of Variance

<u>*Theorem 2.1*</u> *[Bound of Var[N_n[φ]]]*

Let W be a generalized Wigner matrix as mentioned in definition 1.1, and φ be a bounded function with bounded derivative, then

$$\mathrm{Var}[\mathrm{N_n}[\varphi]\,] \le \frac{4C}{n}\cdot \mathrm{E}[\mathrm{Tr}\;\varphi'(W)(\varphi'(W))^*\,] \;\le\; 4C\cdot\left(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|\right)^2 \tag{2.1}$$

<u>proof.</u> By the spectral theorem for real symmetric matrices, we have (see [1])

$$\mathrm{N_n}[\varphi] = \sum_{i=1}^{n}\varphi(\lambda_i) = \mathrm{Tr}(\varphi(W)) \tag{2.2}$$

Thus, we can apply Lemma 1.3 to $\Phi(W) = \mathrm{Tr}(\varphi(W))$, viewing it as a differentiable function of the independent Gaussian random variables W_{jk} and obtain

$$\frac{\partial \mathrm{Tr}(\varphi(W))}{\partial W_{jk}} = \frac{2}{1+\delta_{jk}}\,\varphi'_{jk}(W) \tag{2.3}$$

where $\varphi'_{jk}(W)$ is the jk entry of $\varphi'(W)$.

(1.12) and (1.1) imply

$\mathrm{Var}[\mathrm{N_n}[\varphi]\,] = \mathrm{Var}[\mathrm{Tr}(\varphi(W))\,]$

$$\le \frac{C}{n}\sum_{1\le j,k\le n}\mathrm{E}\left[\left|\frac{2}{1+\delta_{jk}}\,\varphi'_{jk}(W)\right|^2\right]$$

$$\le \frac{4C}{n}\sum_{1\le j,k\le n}\mathrm{E}\left[\left|\,\varphi'_{jk}(W)\right|^2\right] = \frac{4C}{n}\cdot \mathrm{E}[\mathrm{Tr}\;\varphi'(W)(\varphi'(W))^*\,]$$

Note that for normal matrix A, we have $|\mathrm{Tr}(A)| \le n\|A\|$,

and for Hermitian matrix B with $\psi:\mathbb{R}\to\mathbb{C}$, $\|\psi(B)\| \le \sup_{\lambda\in\mathbb{R}}|\psi(\lambda)|$

Hence

$$\mathrm{E}[\mathrm{Tr}\;\varphi'(W)(\varphi'(W))^*] \le \left(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|\right)^2$$

and it follows that

$$\mathrm{Var}[N_n[\varphi]\,] \le \frac{4C}{n} \cdot \mathrm{E}[\mathrm{Tr}\,\varphi'(W)(\varphi'(W))^*\,] \le 4C \cdot \left(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|\right)^2 \qquad \blacksquare$$

2.2 Central Limit Theorem

<u>Theorem 2.2</u> [Central Limit Theorem in the case of Gaussian entries with specific φ]

Let $\varphi : R\to R$ be a bounded function with bounded derivative satisfying

$$(2.4) \qquad \hat{\varphi}(t) = \frac{1}{2\pi}\int e^{-it\lambda}\varphi(\lambda)d\lambda, \quad \int(1+|t|^2)|\hat{\varphi}(t)|\,dt < \infty$$

Suppose W is a generalized Wigner matrix with Gaussian entries, and let $N_n[\varphi]$ be the corresponding linear eigenvalue statistic of W. Then the random variable

$$N_n^0 = N_n[\varphi] - E[N_n[\varphi]]$$

converges in distribution to the Gaussian random variable N^0 with zero mean and variance

$$(2.5) \qquad \mathrm{Var}[N^0] = \frac{1}{2\pi^2}\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\varphi(\lambda)-\varphi(\mu)}{\lambda\text{-}\mu}\right)^2 \frac{4-\lambda\mu}{\sqrt{4-\mu^2}\sqrt{4-\lambda^2}}d\lambda\, d\mu$$

<u>proof.</u> Define

$$(2.6) \qquad Z_n(x) = E[e^{ixN_n^0}],$$

by the continuity theorem for characteristic functions, it’s sufficient to prove that for any $x \in \mathbb{R}$,

$$(2.7) \qquad \lim_{n\to\infty} Z_n(x) = Z(x),$$

where $Z(x) = e^{-\frac{x^2\mathrm{Var}[N^0]}{2}}$ is the characteristic function of normal random variable with zero mean and variance $\mathrm{Var}[N^0]$.

Consider the integral equation

$$Z(x) = 1 - \mathrm{Var}[N^0]\int_0^x yZ(y)dy$$

or

$$\log Z(x) = -\frac{1}{2}\mathrm{Var}[N^0]x^2 + c$$

Since Z(0) = 1, we have c = 0.

So

$$Z(x) = e^{-\frac{x^2 \mathrm{Var}[N^0]}{2}} \tag{2.8}$$

is the unique solution of

$$Z(x) = 1 - \mathrm{Var}[N^0]\int_0^x yZ(y)dy \tag{2.9}$$

It follows from (2.6) that,

$$Z_n'(x) = iE[N_n^0 e^{ixN_n^0}] \tag{2.10}$$

By theorem 2.1, Cauchy-Schwarz inequality and note that $E\left[e^{2ixN_n^0}\right] = E\left[e^{ix(2N_n^0)}\right] = 1$,

$$|Z_n'(x)| = \left|E\left[N_n^0 e^{ixN_n^0}\right]\right| \le E\left[\left|N_n^0 e^{ixN_n^0}\right|\right] \le \{ E[{N_n^0}^2]E[e^{2ixN_n^0}] \}^{\frac{1}{2}}$$

$$= \{\mathrm{Var}[N_n^0]\cdot 1\}^{\frac{1}{2}} = \{\mathrm{Var}[N_n[\varphi]]\}^{\frac{1}{2}}$$

$$\le \{4C\big(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|\big)^2\}^{\frac{1}{2}} = 2\sqrt{C}\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|$$

since φ is a bounded function with bounded derivative.

Now it's sufficient to prove that for any converging subsequences $\left\{Z_{n_j}\right\}$ and $\left\{Z'_{n_j}\right\}$,

$$\lim_{n_j\to\infty} Z_{n_j}(x) = Z(x), \qquad \lim_{n_j\to\infty} Z'_{n_j}(x) = -x\mathrm{Var}[N^0]Z(x). \tag{2.11}$$

The dominated convergence theorem then proves the result. Moreover, during the calculation of $\lim_{n_j\to\infty} Z'_{n_j}(x)$, we can write $\mathrm{Var}[N^0]$ explicitly.

To this end, we consider φ admits the Fourier transform (2.4), and recall the Fourier inversion formula

$$\varphi(\lambda) = \int e^{it\lambda}\hat{\varphi}(t)dt \tag{2.12}$$

Then for this certain class of functions φ , we have

$$N_n[\varphi] = \mathrm{Tr}\,\varphi(W) = \mathrm{Tr}\int e^{itW}\hat{\varphi}(t)dt = \int \mathrm{Tr}U(t)\cdot\hat{\varphi}(t)dt$$

$$\Rightarrow\ N_n^0 = \int \mathrm{Tr}U(t)\cdot\hat{\varphi}(t)dt - E\left[\int \mathrm{Tr}U(t)\cdot\hat{\varphi}(t)dt\right]$$

$$= \int (\mathrm{Tr}U(t) - E[\mathrm{Tr}U(t)])\cdot\hat{\varphi}(t)dt$$

$$= \int u_n{}^0(t)\cdot\hat{\varphi}(t)dt$$

by defining

(2.13) $$u_n(t) = \mathrm{Tr}U(t) \text{ and } u_n{}^0(t) = u_n(t) - E[u_n(t)]$$

Thus,

$$Z_n'(x) = iE\left[N_n^0 e^{ixN_n^0}\right] = iE\left[\int u_n{}^0(t)\cdot\hat{\varphi}(t)dt\cdot e^{ixN_n^0}\right]$$

$$= i\int E\left[u_n{}^0(t)\, e^{ixN_n^0}\right]\cdot\hat{\varphi}(t)dt := i\int Y_n(x,t)\cdot\hat{\varphi}(t)dt$$

by defining

(2.14) $$Y_n(x,t) = E\left[u_n{}^0(t)\, e^{ixN_n^0}\right].$$

Note that by (2.2) we have $u_n(t) = \mathrm{Tr}U(t) = \mathrm{Tr}\, e^{itW} = N_n[\,e^{it\lambda}]$, so it follows from Theorem 2.1 and $(e^{it\lambda})' = it\, e^{it\lambda}$ that

(2.15) $$\mathrm{Var}[u_n(t)] \le 4C\left(\sup_{\lambda\in\mathbb{R}}\left|it\, e^{it\lambda}\right|\right)^2 \le 4Ct^2$$

and similarly with $u_n'(t) = N_n[\,i\lambda e^{it\lambda}]$, $(\,i\lambda e^{it\lambda})' = ie^{it\lambda} - it\lambda e^{it\lambda} = ie^{it\lambda}(1 - t\lambda)$,

(2.16) $$\mathrm{Var}[u_n'(t)] \le \frac{4C}{n}\cdot E\left[\mathrm{Tr}\left(ie^{itW}(I - tW)\right)\left(ie^{itW}(I - tW)\right)^*\right]$$

$$= \frac{4C}{n}\cdot E[\mathrm{Tr}\,(I + t^2W^2)] = \frac{4C}{n}\cdot E\left[\sum_{k=1}^{n}(I + t^2W^2)_{kk}\right] = \frac{4C}{n}\sum_{k=1}^{n} E[(I + t^2W^2)_{kk}]$$

$$= \frac{4C}{n}\sum_{k=1}^{n} (1 + t^2 E[W_{kk}^2]) \le \frac{4C}{n}\sum_{k=1}^{n} (1 + \frac{Ct^2}{n})$$

$$= \frac{4C}{n} \cdot n \cdot \left(1 + \frac{Ct^2}{n}\right) = 4C + \frac{4C^2t^2}{n} \le 4C(1 + \frac{Ct^2}{n})$$

Now we are ready to prove that $\{Y_n(x,t)\} = \{E[u_n{}^0(t)\ e^{ixN_n^0}]\}_{n=1}^{\infty}$ is bounded, equicontinuous on any finite set of the plane $\mathbb{R}^2$ ={(x,t)}, and that its every uniformly converging on the set sub-sequence has the same limit. This proves the assertion of the theorem under condition (2.4). Indeed, let $\{Z_{n_r}\}_{r\ge1}$ be subsequence converging for $Z' \neq Z$. Consider the corresponding subsequence $\{Y_{n_r}\}_{r\ge1}$It contains a uniformly converging subsequence of $\{Z_{n_r}\}_{r\ge1}$ to converge to Z, a contradiction.

Since $|e^{ixN_n^0}| \le 1$, by Cauchy-Schwarz inequality and (2.15) we have the following bound:

$$(2.18) \qquad |Y_n(x,t)| = |E[u_n{}^0(t)\ e^{ixN_n^0}]| \le E[|u_n{}^0(t)|] \le (E[|u_n{}^0(t)|^2])^{\frac{1}{2}}$$

$$= Var[u_n(t)]^{\frac{1}{2}} \le (4Ct^2)^{\frac{1}{2}} = 2\sqrt{C}|t|$$

On the other hand, from (2.17)

$$(2.19) \qquad \left|\frac{\partial Y_n(x,t)}{\partial t}\right| = \left|\frac{\partial E[u_n{}^0(t)\ e^{ixN^0}]}{\partial t}\right| = \left|E\left[\frac{\partial u_n{}^0(t)\ e^{ixN^0}}{\partial t}\right]\right| \le E\left|\frac{\partial u_n{}^0(t)}{\partial t}\right|$$

$$\le Var[u_n'(t)]^{\frac{1}{2}} \le (4C(1 + \frac{Ct^2}{n}))^{\frac{1}{2}} = 2\sqrt{C(1 + \frac{Ct^2}{n})}$$

And again by the Cauchy-Schwarz inequality and (2.15),

$$(2.20) \qquad \left|\frac{\partial Y_n(x,t)}{\partial x}\right| = \left|\frac{\partial E[u_n{}^0(t)\ e^{ixN_n^0}]}{\partial x}\right| = |E[u_n{}^0(t) i\ N_n^0 e^{ixN_n^0}]| = |E[u_n{}^0(t)\ N_n^0 e^{ixN_n^0}]|$$

$$\le (Var[u_n{}^0(t)])^{\frac{1}{2}} \cdot (Var[N_n[\varphi]])^{\frac{1}{2}}$$

$$\leq \left(\frac{4Ct^2}{n}\right)^{\frac{1}{2}} \cdot \left(4C\left(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|\right)^2\right)^{\frac{1}{2}}$$

$$= \left(\frac{2\sqrt{C}|t|}{\sqrt{n}}\right) \cdot \left(2\sqrt{C}\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|\right) = 4C|t|\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|$$

So we conclude that $\{Y_n(x,t)\}_{n=1}^{\infty}$ is bounded, equicontinuous on any finite set of the plane $\mathbb{R}^2$ ={(x,t)}. Now we turn to prove every uniformly converging subsequence of $\{Y_n(x,t)\}_{n=1}^{\infty}$ has the same limit.

It follows from the Duhamel formula (1.7) with $M_1 = 0$ and $M_2 = iW$ that

$$e^{itW} = I + i\int_0^t We^{isW}\,ds \iff U(t) = I + i\int_0^t WU(s)\,ds$$

thus

$$(2.21) \qquad u_n(t) = \mathrm{Tr}U(t) = n + i\int_0^t \mathrm{Tr}\big(WU(s)\big)\,ds = n + i\int_0^t \sum_{j=1}^{n}\big(WU(s)\big)_{jj}\,ds$$

$$= n + i\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} W_{jk}U_{kj}(s)\,ds$$

Substitute in $Y_n(x,t)$:

$$(2.22) \qquad Y_n(x,t) = E\left[u_n{}^0(t)\,e^{ixN_n^0}\right] = E\left[(u_n(t) - E[u_n(t)])\,e^{ixN_n^0}\right]$$

$$= E\left[u_n(t)e^{ixN_n^0}\right] - E\left[E[u_n(t)]e^{ixN_n^0}\right]$$

$$= E\left[u_n(t)e^{ixN_n^0}\right] - E[u_n(t)]E\left[e^{ixN_n^0}\right]$$

$$= E\left[u_n(t)(e^{ixN_n^0} - E[e^{ixN_n^0}])\right]$$

$$= E\left[(n + i\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} W_{jk}U_{kj}(s)\,ds)(e^{ixN_n^0} - E[e^{ixN_n^0}])\right]$$

$$= nE[(e^{ixN_n^0} - E[e^{ixN_n^0}])] + iE\left[(\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} W_{jk}U_{kj}(s)\,ds)(e^{ixN_n^0} - E[e^{ixN_n^0}])\right]$$

$$= i\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} E\big[W_{jk}U_{kj}(s)(e^{ixN_n^0} - E[e^{ixN_n^0}])\big]\,ds$$

Now apply (1.11) by letting $\Phi(W_{jk}) = U_{jk}(s)(e^{ixN_n^0} - E[e^{ixN_n^0}])$,

(2.23) $$Y_n(x,t) = i\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} E\big[W_{jk}^2\big]E\left[\frac{\partial U_{jk}(s)(e^{ixN_n^0} - E[e^{ixN_n^0}])}{\partial W_{jk}}\right]ds$$

By using (2.3),

$$\frac{\partial e^{ixN_n^0}}{\partial W_{jk}} = \frac{2ix}{1+\delta_{jk}}\cdot \varphi_{jk}'(W)\ \cdot e^{ixN_n^0}$$

Under the assumption (2.4) and (2.12),

$$\varphi(W) = \int e^{itW}\hat{\varphi}(t)dt$$

(2.24) $$\varphi'(W) = i\int t\cdot e^{itW}\hat{\varphi}(t)dt = i\int tU(t)\hat{\varphi}(t)dt$$

So $$\frac{\partial e^{ixN_n^0}}{\partial W_{jk}} = \frac{2ix}{1+\delta_{jk}}\cdot e^{ixN_n^0}\cdot\left(i\int tU_{jk}(t)\hat{\varphi}(t)dt\right)$$

$$= \frac{-2x}{1+\delta_{jk}}\cdot e^{ixN_n^0}\cdot\int tU_{jk}(t)\hat{\varphi}(t)dt$$

This and (1.9) implies

$$\frac{\partial U_{jk}(s)(e^{ixN_n^0} - E[e^{ixN_n^0}])}{\partial W_{jk}}$$

$$= \frac{\partial U_{jk}(s)}{\partial W_{jk}}\cdot\left(e^{ixN_n^0} - E\big[e^{ixN_n^0}\big]\right) + U_{jk}(s)\cdot\frac{\partial(e^{ixN_n^0} - E[e^{ixN_n^0}])}{\partial W_{jk}}$$

$$= \frac{i}{1+\delta_{jk}}\left[(U_{jj} * U_{kk})(s) + (U_{kj} * U_{jk})(s)\ \right]\cdot\left(e^{ixN_n^0} - E\big[e^{ixN_n^0}\big]\right)$$

$$+ U_{jk}(s) \cdot \frac{-2x}{1+\delta_{jk}} \cdot e^{ixN_n^0} \cdot \int tU_{jk}(t)\hat{\varphi}(t)dt$$

$$= \frac{i}{1+\delta_{jk}}\left(U_{jj} * U_{kk}\right)(s) \cdot \left(e^{ixN_n^0} - E\left[e^{ixN_n^0}\right]\right) + \frac{i}{1+\delta_{jk}}\left(U_{kj} * U_{jk}\right)(s) \cdot \left(e^{ixN_n^0} - E\left[e^{ixN_n^0}\right]\right)$$

$$- \frac{2x}{1+\delta_{jk}} U_{jk}(s) \cdot e^{ixN_n^0} \cdot \int tU_{jk}(t)\hat{\varphi}(t)dt$$

Since

$$\left(U_{kj} * U_{jk}\right)(s) = \int_0^s U_{kj}(s-v)U_{kj}(v)dv = \int_0^s \left[e^{i(s-v)W}\right]_{kj} \left[e^{ivW}\right]_{jk} dv$$

$$= \int_0^s \left[e^{isW}\right]_{kk} dv = \int_0^s U_{kk}(s)\, dv = sU_{kk}(s)$$

we have

$$E\left[\frac{\partial U_{jk}(s)(e^{ixN_n^0} - E[e^{ixN_n^0}])}{\partial W_{jk}}\right]$$

$$= \frac{i}{1+\delta_{jk}} E\left[\left(U_{jj} * U_{kk}\right)(s) \cdot \left(e^{ixN_n^0} - E\left[e^{ixN_n^0}\right]\right)\right]$$

$$+ \frac{i}{1+\delta_{jk}} s \cdot E\left[U_{kk}(s) \cdot \left(e^{ixN_n^0} - E\left[e^{ixN_n^0}\right]\right)\right] - \frac{2x}{1+\delta_{jk}} E\left[U_{jk}(s) \cdot e^{ixN_n^0} \cdot \int tU_{jk}(t)\hat{\varphi}(t)dt\right]$$

$$= \frac{i}{1+\delta_{jk}} \int_0^s E\left[U_{jj}(s-v)U_{kk}(v)\right] \cdot \left(e^{ixN_n^0} - E\left[e^{ixN_n^0}\right]\right) dv$$

$$+ \frac{i}{1+\delta_{jk}} s \cdot E\left[U_{kk}(s) \cdot \left(e^{ixN_n^0} - E\left[e^{ixN_n^0}\right]\right)\right] - \frac{2x}{1+\delta_{jk}} E\left[U_{jk}(s) \cdot e^{ixN_n^0} \cdot \int tU_{jk}(t)\hat{\varphi}(t)dt\right]$$

Substitute these formula into (2.23), we get

$$Y_n(x,t) = E\left[u_n{}^0(t)\, e^{ixN_n^0}\right] = i\int_0^t \sum_{j=1}^n \sum_{k=1}^n E\left[W_{jk}^2\right] E\left[\frac{\partial U_{jk}(s)(e^{ixN_n^0} - E[e^{ixN_n^0}])}{\partial W_{jk}}\right] ds$$

$$= i\int_0^t \sum_{j=1}^n \sum_{k=1}^n E\left[W_{jk}^2\right] \left\{\frac{i}{1+\delta_{jk}} s \cdot E\left[U_{kk}(s) \cdot \left(e^{ixN_n^0} - E\left[e^{ixN_n^0}\right]\right)\right]\right\} ds$$

$$+\,i\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} E[W_{jk}^2]\left\{\frac{i}{1+\delta_{jk}}\int_0^s E[U_{jj}(s-v)U_{kk}(v)]\cdot\left(e^{ixN_n^0}-\ E[e^{ixN_n^0}]\right)dv\right\}ds$$

$$-\,i\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} E[W_{jk}^2]\left\{\frac{2x}{1+\delta_{jk}}E\left[U_{jk}(s)\cdot e^{ixN_n^0}\cdot\int tU_{jk}(t)\hat{\varphi}(t)dt\right]\right\}ds$$

$$=\ \ -\int_0^t s\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{1+\delta_{jk}}\right)\cdot\sigma_{jk}^2\cdot E\left[U_{kk}(s)\cdot\left(e^{ixN_n^0}-\ E[e^{ixN_n^0}]\right)\right]ds$$

$$-\int_0^t ds\int_0^s \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{1+\delta_{jk}}\right)\cdot\sigma_{jk}^2\cdot E\left[U_{jj}(s-v)U_{kk}(v)\cdot\left(e^{ixN_n^0}-\ E[e^{ixN_n^0}]\right)\right]dv$$

$$-\,2x\int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{1+\delta_{jk}}\right)\cdot\sigma_{jk}^2 E\left[e^{ixN_n^0}\cdot U_{jk}(s)\cdot\varphi_{jk}'(W)\right]ds$$

$$=F_n+\ G_n+\ H_n.$$

In view of (1.13), we can write solution about

$$P(t)+\ \int_0^t ds\int_0^s Q_1(s-v)P(v)dv=R(t),\quad t\geq 0,$$

So we claim that,

$$Y_n(x,t)+\ \int_0^t ds\int_0^s E[n^{-1}u_n(s-v)]Y_n(x,v)dv=-xZ_n(x)A_n(t)+\ r_n(t)$$

with $\lim_{n\to\infty} r_n(t)=0$ on any compact subset of $\{t\geq 0\}$.

To this end, first we consider

$$\lim_{n\to\infty}F_n=-\lim_{n\to\infty}\int_0^t s\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{1+\delta_{jk}}\right)\cdot\sigma_{jk}^2\cdot E\left[U_{kk}(s)\cdot\left(e^{ixN_n^0}-\ E[e^{ixN_n^0}]\right)\right]ds$$

$$=\ -\lim_{n\to\infty}\int_0^t s\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}\sigma_{jk}^2\cdot E\left[U_{kk}(s)\cdot\left(e^{ixN_n^0}-\ E[e^{ixN_n^0}]\right)\right]ds$$

$$= -\lim_{n\to\infty}\int_0^t s\cdot\sum_{k=1}^{n}(\sum_{j=1}^{n}\sigma_{jk}^2)E\left[U_{kk}(s)\cdot\left(e^{ixN_n^0}- E\left[e^{ixN_n^0}\right]\right)\right]ds$$

By (1.2) and $\sum_{k=1}^{n}U_{kk}(s) = \mathrm{Tr}\,U(s)$,

$$\lim_{n\to\infty}F_n = -\lim_{n\to\infty}\int_0^t s\cdot E\left[\mathrm{Tr}\,U(s)\cdot\left(e^{ixN_n^0}- E\left[e^{ixN_n^0}\right]\right)\right]ds$$

$$= -\lim_{n\to\infty}\int_0^t s\cdot E\left[u_n(s)\cdot\left(e^{ixN_n^0}- E\left[e^{ixN_n^0}\right]\right)\right]ds$$

Note that $\left|e^{ixN^0}- E\left[e^{ixN^0}\right]\right| \le 2$, so

$$\lim_{n\to\infty}|F_n| \le \lim_{n\to\infty} 2\int_0^t s\cdot E[u_n(s)]\,ds$$

Using (2.15) we get

$$\lim_{n\to\infty}|F_n| \le \lim_{n\to\infty} 2\int_0^t s\cdot \mathrm{Var}[u_n(s)]^{\frac{1}{2}}\,ds \le \lim_{n\to\infty}2\int_0^t s\cdot\left(\frac{4Cs^2}{n}\right)^{\frac{1}{2}}ds$$

$$= \lim_{n\to\infty}\frac{4\sqrt{C}}{\sqrt{n}}\int_0^t s^2\,ds = \lim_{n\to\infty}\frac{4\sqrt{C}}{\sqrt{n}}\cdot\frac{t^3}{3} = 0 \text{ on any compact subset of } \{t\ge 0\}.$$

On the other hand, we consider G_n by writing the following limit and use (1.4):

$$\lim_{n\to\infty}\left|-\int_0^t ds\int_0^s\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\sigma_{jk}^2-\frac{1}{n}\right)\cdot E\left[U_{jj}(s-v)U_{kk}(v)\cdot\left(e^{ixN_n^0}- E\left[e^{ixN_n^0}\right]\right)\right]dv\right|$$

$$= \lim_{n\to\infty}\left|\int_0^t ds\int_0^s\sum_{j=1}^{n}\sum_{k=1}^{n}E\left[U_{jj}(s-v)U_{kk}(v)\cdot\left(e^{ixN_n^0}- E\left[e^{ixN_n^0}\right]\right)\right]dv\right|$$

$$= \lim_{n\to\infty}\left|\int_0^t ds\int_0^s E\left[\mathrm{Tr}\,U(s-v)\cdot \mathrm{Tr}\,U(v)\cdot\left(e^{ixN_n^0}- E\left[e^{ixN_n^0}\right]\right)\right]dv\right|$$

$$= \lim_{n\to\infty}\left|\int_0^t ds\int_0^s E\left[u_n(s-v)\cdot u_n(v)\cdot\left(e^{ixN_n^0}- E\left[e^{ixN_n^0}\right]\right)\right]dv\right|$$

Use the same techniques we have

$$\lim_{n\to\infty}\left|\int_0^t ds\int_0^s E\left[u_n(s-v)\cdot u_n(v)\cdot\left(e^{ixN_n^0}-\ E\left[e^{ixN_n^0}\right]\right)\right]dv\right|$$

$$\leq\ \lim_{n\to\infty}2\int_0^t ds\int_0^s E[u_n(s-v)\cdot u_n(v)]dv=0.$$

So

$$\lim_{n\to\infty}(-G_n)=\ \lim_{n\to\infty}\int_0^t ds\int_0^s E[n^{-1}u_n(s-v)]\,E\left[u_n(v)\left(e^{ixN_n^0}-\ E\left[e^{ixN_n^0}\right]\right)\right]dv$$

$$=\ \lim_{n\to\infty}\int_0^t ds\int_0^s E[n^{-1}u_n(s-v)]\,E\left[(u_n(v)-E[u_n(t)])e^{ixN_n^0}\right]dv$$

$$=\ \lim_{n\to\infty}\int_0^t ds\int_0^s E[n^{-1}u_n(s-v)]\,E\left[{u_n}^0(v)e^{ixN_n^0}\right]dv$$

$$=\ \lim_{n\to\infty}\int_0^t ds\int_0^s E[n^{-1}u_n(s-v)]\,Y_n(x,v)dv.$$

Finally we consider

$$H_n=\ -2x\int_0^t\sum_{j=1}^n\sum_{k=1}^n\left(\frac{1}{1+\delta_{jk}}\right)\cdot\sigma_{jk}^2E\left[e^{ixN_n^0}\cdot U_{jk}(s)\cdot\varphi_{jk}'(W)\right]ds$$

recall that $Z_n(x)=\ E\left[e^{ixN^0}\right]$, hence with similar actions in the estimate of G_n,

and note that by (2.24), $i\int tU_{jk}(t)\widehat{\varphi}(t)dt=\ \varphi_{jk}'(W)$, so we have

$$\lim_{n\to\infty}H_n\ =\ \lim_{n\to\infty}[-xZ_n(x)]\cdot\frac{2}{n}\int_0^t\sum_{j=1}^n\sum_{k=1}^n E\left[U_{jk}(s)\cdot\varphi_{jk}'(W)\right]ds$$

$$=\ \lim_{n\to\infty}[-xZ_n(x)]\cdot\ 2\int_0^t E[n^{-1}\mathrm{Tr}\ U(s)\varphi'(W)]\,ds$$

Now, by (1.14) it's easy to see, if we write

$$(2.25) \qquad B(t) := \lim_{n\to\infty} E[n^{-1}u_n(s-v)] = \frac{1}{2\pi}\int_{-2}^{2} e^{i(s-v)\lambda}\sqrt{4-\lambda^2}d\lambda,$$

$$(2.26) \qquad A(t) := \lim_{n\to\infty} 2\int_0^t E[n^{-1}\mathrm{Tr}\, U(s)\varphi'(W)]\, ds = \frac{1}{\pi}\int_0^t ds\int_{-2}^{2} e^{is\lambda}\varphi'(\lambda)\sqrt{4-\lambda^2}d\lambda$$

and pass limit to $Y_n(x,t) = F_n + G_n + H_n$, we have that if

$Y(x,t)$ is a limit of every uniformly converging subsequence of $\{Y_n(x,t)\}$, then

$$(2.27) \qquad Y(x,t) + \int_0^t ds\int_0^s B(s-v)\, Y(x,v)dv = -xZ(x)\, A(t)$$

It follows from the spectral theorem that

$$\widehat{B} = f$$

where f is the Stieljes transform of the semicircle law (1.14). thus

$$T_1(t) = \frac{i}{2\pi}\int_L \frac{e^{izt}}{z+2f(z)}dz = -\frac{1}{\pi}\int_{-2}^{2}\frac{e^{it\lambda}}{\sqrt{4-\lambda^2}}d\lambda$$

Hence by (1.13), with

$$R(t) = -xZ(x)\, A(t) = -xZ(x)\cdot\frac{1}{\pi}\int_0^t ds\int_{-2}^{2} e^{is\lambda}\varphi'(\lambda)\sqrt{4-\lambda^2}d\lambda$$

$$\Leftrightarrow\ R'(t) = -xZ(x)A'(t) = -xZ(x)\cdot\frac{1}{\pi}\int_{-2}^{2} e^{is\lambda}\varphi'(\lambda)\sqrt{4-\lambda^2}d\lambda$$

We figure out that

$$Y(x,t) = -\int_0^t\left\{-\frac{1}{\pi}\int_{-2}^{2}\frac{e^{i(t-s)\mu}}{\sqrt{4-\mu^2}}d\mu\right\}\cdot\left\{-xZ(x)\cdot\frac{1}{\pi}\int_{-2}^{2} e^{is\lambda}\varphi'(\lambda)\sqrt{4-\lambda^2}d\lambda\right\}ds$$

$$= -\frac{xZ(x)}{\pi^2}\int_0^t\left\{\int_{-2}^{2}\frac{e^{i(t-s)\mu}}{\sqrt{4-\mu^2}}d\mu\right\}\cdot\left\{\int_{-2}^{2} e^{is\lambda}\varphi'(\lambda)\sqrt{4-\lambda^2}d\lambda\right\}ds$$

and

$$\int_0^t e^{i(t-s)\mu}\, e^{is\lambda}ds = e^{it\mu}\int_0^t e^{s(i\lambda\text{-}i\mu)}d\, s = e^{it\mu}\left[\frac{1}{i\lambda\text{-}i\mu}e^{s(i\lambda\text{-}i\mu)}\right]_0^t = \left(\frac{-i}{\lambda\text{-}\mu}\right)\left(e^{it\lambda}-e^{it\mu}\right)$$

So

$$Y(x,t) = \frac{ixZ(x)}{\pi^2}\int_{-2}^{2}\frac{1}{\sqrt{4-\mu^2}}d\mu\int_{-2}^{2}\frac{e^{it\lambda}-e^{it\mu}}{\lambda\text{-}\mu}\varphi'(\lambda)\sqrt{4-\lambda^2}d\lambda$$

Recall

$$Z_n'(x) = i\int Y_n(x,t)\cdot\hat{\varphi}(t)dt$$

Thus for the part of $Y(x,t)$ associated with t, write

$$\int(e^{it\lambda}-e^{it\mu})\cdot\hat{\varphi}(t)dt = \int e^{it\lambda}\,\hat{\varphi}(t)dt - \int e^{it\mu}\,\hat{\varphi}(t)dt = \varphi(\lambda)-\varphi(\mu)$$

Then

$$\lim_{n_r\to\infty} Z_{n_r}'(x) = \frac{-xZ(x)}{\pi^2}\int_{-2}^{2}\frac{1}{\sqrt{4-\mu^2}}d\mu\int_{-2}^{2}\frac{\varphi(\lambda)-\varphi(\mu)}{\lambda\text{-}\mu}\varphi'(\lambda)\sqrt{4-\lambda^2}d\lambda$$

Since

$$\varphi^{'}(\lambda)(\varphi(\lambda)-\varphi(\mu)) = \frac{1}{2}\frac{\partial}{\partial\lambda}(\varphi(\lambda)-\varphi(\mu))^2$$

We have the form

$$\lim_{n_r\to\infty} Z_{n_r}'(x) = \frac{-xZ(x)}{2\pi^2}\int_{-2}^{2}\frac{1}{\sqrt{4-\mu^2}}d\mu\int_{-2}^{2}\frac{\frac{\partial}{\partial\lambda}(\varphi(\lambda)-\varphi(\mu))^2}{\lambda\text{-}\mu}\sqrt{4-\lambda^2}d\lambda$$

performing integration by parts,

$$\int_{-2}^{2}\frac{\frac{\partial}{\partial\lambda}(\varphi(\lambda)-\varphi(\mu))^2}{\lambda\text{-}\mu}\sqrt{4-\lambda^2}d\lambda = \int_{-2}^{2}\left(\frac{\varphi(\lambda)-\varphi(\mu)}{\lambda\text{-}\mu}\right)^2\frac{4-\lambda\mu}{\sqrt{4-\lambda^2}}d\lambda$$

so

$$\lim_{n_r\to\infty} Z_{n_r}'(x) = \frac{-xZ(x)}{2\pi^2}\int_{-2}^{2}\frac{1}{\sqrt{4-\mu^2}}d\mu\int_{-2}^{2}\left(\frac{\varphi(\lambda)-\varphi(\mu)}{\lambda\text{-}\mu}\right)^2\frac{4-\lambda\mu}{\sqrt{4-\lambda^2}}d\lambda$$

$$= \frac{-xZ(x)}{2\pi^2}\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\varphi(\lambda)-\varphi(\mu)}{\lambda\text{-}\mu}\right)^2\frac{4-\lambda\mu}{\sqrt{4-\mu^2}\sqrt{4-\lambda^2}}d\lambda\,d\mu$$

In (2.11)

$$\lim_{n_r \to \infty} Z'_{n_r}(x) = -x\mathrm{Var}[N^0]Z(x)$$

Hence we completes the proof with

$$\mathrm{Var}[N^0] = \frac{1}{2\pi^2}\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\varphi(\lambda) - \varphi(\mu)}{\lambda\text{-}\mu}\right)^2 \frac{4 - \lambda\mu}{\sqrt{4 - \mu^2}\sqrt{4 - \lambda^2}}\, d\lambda\, d\mu$$

under condition (2.4). ■

Theorem 2.3 [Central Limit Theorem in the case of Gaussian entries with bounded φ *and* φ'*]*

Theorem 2.2 remains true when $\varphi : R \to C$ is a function with bounded derivative.

proof.

With bounded test function φ with bounded derivatives, there exists a sequence $\{\varphi_k\}$ satisfying (2.4) and for all A > 0,

(2.28) $$\sup_{\lambda\in\mathbb{R}}|\varphi_k'(\lambda)| \le \sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)| \;, \quad \lim_{k\to\infty}\sup_{|\lambda|\le A}|\varphi'(\lambda)-\varphi_k'(\lambda)| = 0$$

We have proved central limit theorem for any φ_k, and write $Z_n[\varphi], Z[\varphi]$ to represent the characteristic functions as (2.6) and (2.7) with respect to φ. Now

$$|Z_n[\varphi] - Z[\varphi]| \le |Z_n[\varphi] - Z_n[\varphi_k]| + |Z_n[\varphi_k] - Z[\varphi_k]| + |Z[\varphi_k] - Z[\varphi]|$$

where the second term $|Z_n[\varphi_k] - Z[\varphi_k]|$ vanishes as $n \to \infty$ from the above proof.

For the third term, applying mean value theorem

$$|Z[\varphi_k] - Z[\varphi]| = \left|e^{-\frac{Var[\varphi_k]x^2}{2}} - e^{-\frac{Var[\varphi]x^2}{2}}\right| \le \frac{x^2}{2}\,|Var[N^0[\varphi_k]] - Var[N^0[\varphi]]|$$

From continuity of the variance (2.5), it vanishes after the limit $k \to \infty$.

For the first term, using mean value theorem again, and from (2.6), (2.1)

$$|Z_n[\varphi] - Z_n[\varphi_k]| = \left|E[e^{ixN_n^0[\varphi]}] - E[e^{ixN_n^0[\varphi_k]}]\right| \le |x|\;E[N_n^0[\varphi] - N_n^0[\varphi_k]]$$

$$\le |x|\;Var^{\frac{1}{2}}[N_n^0[\varphi - \varphi_k]]$$

$$\le 2\sqrt{C}|x|\;E\left[\frac{1}{n}\mathrm{Tr}\;\varphi'(W)(\varphi'(W))^*\right]^{\frac{1}{2}}$$

$$= 2\sqrt{C}|x|\left(\int|\varphi'(\lambda) - \varphi_k'(\lambda)|^2 E[N_n(d\lambda)]\right)^{\frac{1}{2}}$$

Let A be a constant greater than 2, then the above integral can be divided into $[-A, A]$ and $\mathbb{R} \setminus [-A, A]$. The first part vanishes by (2.28), and the second one is zero since the support of semicircle law is $[-2,2]$.

Thus, we proved the central limit theorem for bounded functions with bounded derivatives. ■

3. Central Limit Theorem for linear eigenvalue statistics in general cases

3.1 Generalities

To prove the central limit theorem under the scheme of theorem 2.2 when the entries are not Gaussian, we need an analog of lemma 1.4. Recall that if a random variable X has a finite *p*th absolute moment $\mu_p,\ p \geq 1,$ then we have the expansions:

$$f(t) = E\left[e^{itX}\right] = \sum_{j=0}^{p} \frac{\mu_j}{j!}(it)^j + o(t^p)$$

and

$$L(t) = \log E\left[e^{itX}\right] = \sum_{j=0}^{p} \frac{\kappa_j}{j!}(it)^j + o(t^p), \quad t \to 0$$

where $\{\kappa_j\}$ are the cumulants of X, which can be expressed by $\mu_1, \mu_2, \ldots, \mu_j, \ldots$

In particular,

$$\kappa_1 = \mu_1, \qquad \kappa_2 = \mu_2 - {\mu_1}^2 = \mathrm{Var}[X], \qquad \kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2{\mu_1}^3, \ldots$$

And in general,

$$\kappa_j = \sum_{\lambda} c_\lambda \mu_\lambda$$

where the sum is over all additive partitions λ of the set $\{1,2,\ldots\ldots,j\}$, c_λ are known coefficients and $\mu_\lambda = \prod_{i\in\lambda} \mu_i$.

<u>*Lemma 3.1*</u> *[Taylor Expansion]*([1])

Let ξ be a random variable such that $E[|\xi|^{p+2}] < \infty$ for a certain nonnegative integer p. Then for any function $\Phi: \mathbb{R} \to \mathbb{C}$ of the class C^{p+1} with bounded derivatives $\Phi^{(r)}$, r = 1,2,…,p+1, we have

$$E[\xi\Phi(\xi)] = \sum_{r=0}^{p} \frac{\kappa_{r+1}}{r!} E[\Phi^{(r)}(\xi)] + \varepsilon_p \tag{3.1}$$

where the remainder term ε_p satisfy the bound

$$|\varepsilon_p| \leq \frac{(2p+3)^{p+1}}{(p+1)!} E[|\xi|^{p+2}] \sup_{t\in\mathbb{R}} |\Phi^{(p+1)}(t)| \tag{3.2}$$

<u>*Lemma 3.2*</u> *[Semicircle Law]*([1][2])

Let W be a generalized Wigner matrix satisfying the condition

$$w_3 = \sup_{n\in\mathbb{N}} \max_{1\leq j\leq k\leq n} E[|W_{jk}|^3] < \infty \tag{3.3}$$

and N_n be the normalized counting measure of its eigenvalues. Then

$$\lim_{n\to\infty} E[N_n] = \frac{1}{2\pi}\sqrt{4-\lambda^2}d\lambda \tag{3.4}$$

which is the semicircle law. The convergence here is understood as the weak convergence of measures.

3.2 Central Limit Theorem in general cases

Theorem 3.1 *[Central Limit Theorem in general cases]*

Let W be a generalized Wigner matrix satisfying the following conditions:

(3.5) $$w_5 = \sup_{n\in\mathbb{N}} \max_{1\le j,k\le n} E[|W_{jk}|^5] < \infty$$

(3.6) $$\mu_3 = E\left[(W_{jk})^3\right] < \infty, \quad \mu_4 = E\left[(W_{jk})^4\right] < \infty$$

(3.7) $$w_3 = \sup_{n\in\mathbb{N}} \max_{1\le j,k\le n} \mu_3 < \infty$$

(3.8) $$\sup_{n\in\mathbb{N}} \max_{1\le j,k\le n} |\kappa_{4,jk}| = 0$$

Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a test function whose Fourier transform $\hat{\varphi}$ satisfy the condition

(3.9) $$\int (1 + |t|^5)|\hat{\varphi}(t)|dt < \infty$$

Then the corresponding centered linear eigenvalue statistic N_n^0 converges in distribution to the Gaussian random variable of zero mean and $\mathrm{Var}[N^0]$ as in theorem 2.2.

proof. Let $\widehat{W}$ be a generalized Wigner matrix as theorem 2.2(with Gaussian entries) such that $E[W_{jk}^2] = E[\widehat{W}_{jk}^2]$ for all j, k and $\widehat{N}^0$ the corresponding linear eigenvalue statistic. In view of the previous theorem, it's sufficient to show

$$R_n(x) = E\left[e^{ixN_n^0}\right] - E\left[e^{ix\widehat{N}_n^0}\right] \to 0 \text{ as } n \to \infty$$

Denoting

(3.10) $$e_n(s,x) = e^{ix\mathrm{Tr}\varphi(W(s))^0}$$

where

(3.11) $$W(s) = s^{\frac{1}{2}}W + (1-s)^{\frac{1}{2}}\widehat{W} \qquad 0 \le s \le 1$$

Now $R_n(x) = \int_0^1 \frac{\partial}{\partial s} E[e_n(s,x)]\, ds$

$$(3.12) \qquad = \frac{ix}{2\sqrt{n}} \int_0^1 E[e_n^0(s,x) \mathrm{Tr}\varphi'(W(s))(s^{\frac{1}{2}}W + (1-s)^{\frac{1}{2}}\widehat{W})]\, ds$$

Note that by (2.12),

$$(3.13) \qquad \varphi'(W) = i \int \widehat{\varphi}(t) t U(t) dt$$

This and (3.12) imply

$$R_n(x) = -\frac{x}{2} \int_0^1 ds \int t\,\widehat{\varphi}(t)[A_n - B_n] dt$$

where

$$(3.14) \qquad A_n = \frac{1}{\sqrt{ns}} \sum_{j,k=1}^{n} E[W_{jk}\Phi_n]$$

$$(3.15) \qquad B_n = \frac{1}{\sqrt{n(1-s)}} \sum_{j,k=1}^{n} E[\widehat{W}_{jk}\Phi_n]$$

and

$$(3.16) \qquad \Phi_n = e_n^0(s,x) U_{jk}(s,t), \quad U(s,t) = e^{itW(s)}$$

Applying (3.1) with p = 3 to every term of (3.13) and (1.11) to every term of (3.14), we obtain

$$(3.17) \qquad A_n - B_n = T_2 + T_3 + \varepsilon_3$$

where

$$(3.18) \qquad T_r = \frac{s^{\frac{r-1}{2}}}{r!\, n^{\frac{r+1}{2}}} \sum_{j,k=1}^{n} \kappa_{r+1,jk} E[\frac{\partial^r}{\partial W_{jk}^r(s)} \Phi_n]$$

and by (3.2),

$$(3.19) \qquad |\varepsilon_3| \leq \frac{9^4 w_5}{4!\, n^{\frac{5}{2}}} \sum_{j,k=1}^{n} \sup_{M(s)} \left| \frac{\partial^4}{\partial W_{jk}^4(s)} \Phi_n \right|$$

Using (1.9), (1.10) and

$$(3.20) \qquad \frac{\partial e_n(x)}{\partial W_{jk}(s)} = \frac{-2}{1+\delta_{jk}} x e_n(x) \int t U_{jk}(t) \widehat{\varphi}(t) dt, \text{ where } e_n(x) = e^{ixN_n^0}$$

we have

$$(3.21)\qquad \left|\frac{\partial^r}{\partial W_{jk}^r(s)}\Phi_n\right| \le P_r(t,x) \quad 0 \le r \le 4$$

where $P_r(t,x)$ is a polynomial in $|t|, |x|$ of degree r independent of j, k and n. Hence,

$$(3.22)\qquad |\varepsilon_3| \le \frac{P_4(t,x)}{n^{\frac{1}{2}}}$$

By (3.18), (3.6) and (3.8),

$$(3.23)\qquad T_3 = \frac{s}{3!\,n^2}\sum_{j,k=1}^{n}\kappa_{4,jk}E\left[\frac{\partial^3}{\partial W_{jk}^3(s)}\Phi_n\right] \to 0 \text{ as } n \to \infty$$

Finally we consider T_2 by using (1.9), (3.20) and (3.6):

$$T_2 = \frac{s^{\frac{1}{2}}}{2!\,n^{\frac{3}{2}}}\sum_{j,k=1}^{n}\kappa_{3,jk}E\left[\frac{\partial^2}{\partial W_{jk}^2(s)}\Phi_n\right]$$

$$= -\frac{\sqrt{s}\mu_3}{n^{\frac{2}{3}}}\sum_{j,k=1}^{n}\left(\frac{1}{1+\delta_{jk}}\right)E\Big[e_n{}^0(s,x)\big\{\big(U_{jk}*U_{jk}*U_{jk}\big)(s,t)+3\big(U_{jk}*U_{jj}*U_{kk}\big)(s,t)\big\}$$

$$+\ 2xe_n(s,x)\big\{\big(U_{jk}*U_{jk}\big)(s,t)+\big(U_{jj}*U_{kk}\big)(s,t)\big\}\int\theta\widehat{\varphi}(\theta)U_{jk}(s,\theta)d\theta$$

$$-\ 2x^2e_n(s,x)U_{jk}(s,t)\left\{\int\theta\widehat{\varphi}(\theta)U_{jk}(s,\theta)d\theta\right\}^2$$

$$+\ ixe_n(s,x)U_{jk}(s,t)\int\theta\widehat{\varphi}(\theta)\big\{\big(U_{jk}*U_{jk}\big)(s,\theta)-\big(U_{jj}*U_{kk}\big)(s,\theta)\big\}d\theta\Big]$$

Consider the two types of sum in T_2 by writing

$$T_{21} = n^{-\frac{3}{2}}\sum_{j,k=1}^{n}U_{jk}(t_1)U_{jj}(t_2)U_{kk}(t_3)$$

$$T_{22} = n^{-\frac{3}{2}}\sum_{j,k=1}^{n}U_{jk}(t_1)U_{jk}(t_2)U_{jk}(t_3)$$

By lemma (1.2) and Schwarz inequality,

$$\sum_{j,k=1}^{n} \left|U_{jk}(t_1)U_{jk}(t_2)\right| \le \left(\sum_{j,k=1}^{n} \left|U_{jk}(t_1)\right|^2\right)^{\frac{1}{2}} \left(\sum_{j,k=1}^{n} \left|U_{jk}(t_2)\right|^2\right)^{\frac{1}{2}} = n$$

Hence $|T_{21}| \le n^{\frac{1}{2}}$ and $|T_{22}| \le n^{\frac{1}{2}}$, which imply

$$|T_2| \le \frac{P_2(t,x)}{n^{\frac{1}{2}}}$$

Together with (3.17), (3.22), (3.23) and (3.9) we complete the proof. ■

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