國立台灣大學理學院數學所

博士論文

Department of Mathematics

College of Science

National Taiwan University

Doctoral Dissertation

質量重分配及其在自旋-1
玻色-愛因斯坦凝聚基態上之應用

Mass Redistribution and Its Applications to the Ground States of Spin-1 Bose-Einstein Condensates

林立人

Liren Lin

指導教授：陳宜良 博士

Advisor: I-Liang Chern Ph.D.

中華民國 102 年 7 月

July, 2013

國立臺灣大學博士學位論文
口試委員會審定書

質量重分配及其在自旋-1
玻色-愛因斯坦凝聚基態上之應用

Mass Redistribution and Its Applications to the Ground States of Spin-1 Bose-Einstein Condensates

本論文係林立人（學號 D97221009）在國立臺灣大學數學研究所完成之博士學位論文，於民國 102 年 6 月 26 日承下列考試委員審查通過及口試及格，特此證明

口試委員：陳宜良（簽名）
（指導教授）

陳俊全

林太家

方永富

陳建隆

系主任、所長 王振男（簽名）

誌　謝

國立臺灣大學

感謝我的指導教授陳宜良老師。這篇論文的產生就從他將自旋玻色-愛因斯坦凝聚介紹給我開始。先是一個小問題(雖然一開始也不覺得小)，然後發現越來越多有趣的事情。回想起來，我能夠知道這個問題眞是非常幸運。從大三、大四、碩士一直到博士班的頭幾年，不停地換領域，找不到一個可以下定決心的問題，我可以說是沒有一樣東西是專精的。代數會一些(差不多都忘了)，幾何會一些。一直到爲了考資格考才終於認眞學了實分析。本來想做應用數學也好，到頭來卻又覺得自己還是擁有一個純數學家的靈魂。混到了博三，偏微分方程的近代理論只知道一些最皮毛的事。索伯列夫空間，弱解等概念還說得出來，其他的基本上是零。就在這情形下，老師告訴我自旋玻色-愛因斯坦凝聚的某個計算結果。我於是沒有任何武器，直接面對這個有三個分量，看起來相當可怕的系統。也因爲這個系統看起來太複雜了，什麼都不會的我，只能從一些最天眞的想法去做。而就因爲這樣，才讓我發現到問題背後原來有一個非常簡單的道理。在這之中，還是由於老師堅信某些“幸運”的背後必定有些重要的事情在其中，讓我發現質量重分配這個方法。否則，我的工作可能就只是一些幸運的猜測，然後在一堆莫名的計算中驗證了所要證明的結果。

除此之外，在博士期間，常常聽老師分享他對各式各樣數學的看法，包含跟我的研究有關與無關的，拓展了我的視野。這些想法雖然很多離我很遠，有些確實給了一些可以實行的想法。很多想法在短時間之內也還沒辦法去弄清楚，似乎讓未來有做不完的事。

還有非常重要的，由於老師交遊廣闊，也讓我認識了許多學者。包括新加坡國立大學的包維柱教授，他與他的合作者們的數值模擬工作在我的研究上給了我許多想法。還有新竹教育大學的陳人豪教授，因爲老師的邀請特別來到台大一起討論。像我這麼不擅與不認識的人往來的人，這些機會對我來說都是非常難得的。

摘要

自旋-1玻色-愛因斯坦凝聚是一類特殊的，含有三個分量函數的系統。通常以 $\Psi = (\psi_1, \psi_0, \psi_{-1})$ 表示。它的行爲由一個能量泛函 $E[\Psi]$ 及兩個限制條件所描述。這兩個限制分別是原子數守恆與磁化量守恆，也就是說 $\int |\Psi|^2$ 及 $\int (|\psi_1|^2 - |\psi_{-1}|^2)$ 是兩個固定的數。而所謂的基態即指在這兩個條件之下，使能量 E 達到最小的狀態 Ψ。要解釋這篇論文所討論的問題，我們還得指出，根據能量 E 的表達式裡的某個參數的正負號，自旋-1玻色愛因斯坦凝聚被分成順磁性與反磁性兩類。這兩類系統表現出來的行爲有本質上的不同。這篇論文裡的工作，其動機來自於兩個現象，恰好一個屬於順磁性，另一個反磁性。

1. 任何順磁性系統中的基態，必定滿足下列形式

$$\Psi = (\gamma_1\psi, \gamma_0\psi, \gamma_{-1}\psi),$$

其中 γ_j 皆爲常數，而 ψ 爲函數。這個形式稱作單模近似。由其名稱即可知，這原本只被視爲一種簡化的假設。然而在後來的研究中卻發現，對順磁性系統的基態來說，此形式是完全正確，而非只是近似。

2. 考慮外加一個均勻磁場的情形。若將磁場的強度由零慢慢增加，當強度超過某個特定的數值時，反磁性系統的基態會經歷一個從 $\psi_0 \equiv 0$ 到 $\psi_0 \neq 0$ 的分歧。

雖然這兩個現象很早就已經在數值模擬中被發現，但在我們的研究之前，還沒有一個眞正嚴格的數學證明。這篇論文包含我們在 [16, 17] 這兩篇論文裡的工作，它們分別給出了上面兩個現象的嚴格證明。比起兩篇原本的論文，在本文中我們盡可能把所有的細節都交待清楚。我們的證明方法主要是使用了下面這個原理：質量密度(也就是 $|\psi_1|^2$, $|\psi_0|^2$ 及 $|\psi_{-1}|^2$)的重分配將必定導致動能的下降。這個原理可視爲某個廣爲人知的梯度的凸性不等式的簡單推廣。我們將會說明這個原理如何給出解決上面問題的一個統一的想法。

關鍵詞: 自旋，旋量，玻色-愛因斯坦凝聚系統，薛丁格系統，單模近似，質量重分配，分歧

Abstract

Spin-1 Bose-Einstein condensate (BEC) is a special three-component system, written as $\Psi = (\psi_1, \psi_0, \psi_{-1})$. Its behavior is described by an energy functional $E[\Psi]$ with two constraints: the conservation of the number of atoms and the conservation of total magnetization. That is $\int |\Psi|^2$ and $\int (|\psi_1|^2 - |\psi_{-1}|^2)$ are fixed numbers. And a ground state is a minimizer of E under the constraints. To explain the problems considered in this thesis, we remark that according to the sign of a specific parameter in the energy E, spin-1 BECs are classified into two groups: ferromagnetic ones and antiferromagnetic ones. They exhibit rather different behaviors. The works in this thesis are motivated by the following two phenomena.

1. Any ground state of a ferromagnetic system is of the form

$$\Psi = (\gamma_1 \psi, \gamma_0 \psi, \gamma_{-1} \psi),$$

where γ_j are constants and ψ a function. This is called single-mode approximation. According to the name, this form was originally only used as a simplified assumption, while from later studies it is found to be exactly the case for ferromagnetic ground states.

2. When an external magnetic field is applied, the ground state of an antiferromagnetic system undergoes a bifurcation from $\psi_0 \equiv 0$ to $\psi_0 \neq 0$ as the strength of the magnetic field surpasses a critical value.

Although these phenomena have been well-known from numerical simulations for quite a long time, there were no rigorous mathematical justifications before our investigations. In this thesis, our works [16, 17] on their proofs are given, with more details. The proofs rely on a principle which says that a redistribution of the mass densities (i.e. $|\psi_1|^2$, $|\psi_0|^2$ and $|\psi_{-1}|^2$) will decrease the kinetic energy. This principle can be regarded as a simple generalization of a well-known convexity inequality for gradients. We will show how this principle can give a rather unified approach toward our problems.

Keywords: spin-1, spinor, BEC system, Schrödinger system, single-mode approximation, mass redistribution, bifurcation

Contents

國立臺灣大學 · 敦品勵學 · 愛國愛人

Chapter 1

Introduction

When a Bose-Einstein condensate (BEC) of dilute atomic gas is confined by an optical trap, all its hyperfine spin states can be active simultaneously. In this situation, a spin-f BEC is described by a $(2f+1)$-component order parameter

$$\Psi = (\psi_f, \psi_{f-1}, ..., \psi_{-f}),$$

where the components ψ_j are complex-valued functions in the mean-field theory. Since the first realization of such spinor BECs in 1997 [19] (spin-1 BEC of ^{23}Na), their rich structures have drawn great interest and a lot of researches.

This thesis focuses on some facts exhibited by the ground states of spin-1 BEC which have been well-known from numerical simulations for a long time. The aim is to provide rigorous mathematical justifications of them, based on a principle which says that a redistribution of the mass densities between different components will decrease the kinetic energy. Before further discussion, we shall first introduce the mathematical model.

1.1 Mathematical model of spin-1 BEC

A spin-1 BEC, as mentioned above, is described by a three-component vector function $\Psi = (\psi_1, \psi_0, \psi_{-1})$, where each ψ_j is a complex-valued function on $\mathbb{R}^3$. We leave the specification of the suitable function space for Ψ to §2.1, although it should be very clear from the energy functional given below. Also note that we consider Ψ as being independent of time since we will only be interested in ground states. For the dynamical law, see e.g. [15].

The energy of the system is

$$E[\Psi] = E_{kin}[\Psi] + E_{pot}[\Psi] + E_0[\Psi] + E_1[\Psi] + E_{Zee}[\Psi],$$

where[1]

$$\begin{aligned} E_{kin}[\Psi] &= \int \sum_j |\nabla \psi_j|^2 \\ E_{pot}[\Psi] &= \int V(x)|\Psi|^2 \\ E_0[\Psi] &= \int \beta_0 |\Psi|^4 \\ E_1[\Psi] &= \int \beta_1 |\Psi^* F \Psi|^2 \\ E_{Zee}[\Psi] &= \int \left[p(|\psi_1|^2 - |\psi_{-1}|^2) + q(|\psi_1|^2 + |\psi_{-1}|^2)\right]. \end{aligned}$$

$V(x)$ is a real-valued function, and β_0, β_1, p, q are real constants. In the definition of $E_1[\Psi]$, Ψ is regarded as a column vector and Ψ^* is its conjugate transpose. F stands for the triple (F_x, F_y, F_z) of 3×3 matrices given by

$$F_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; F_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus

$$\Psi^* F \Psi = (\Psi^* F_x \Psi, \Psi^* F_y \Psi, \Psi^* F_z \Psi).$$

The notation $|\Psi|$ denotes the Euclidean length $(\sum_j |\psi_j|^2)^{1/2}$, and similarly for $|\nabla \psi_j|$ and $|\Psi^* F \Psi|$.

Physically, V represents a state-independent trap potential, the terms with coefficients β_0 and β_1 describe the interactions between the atoms, p and q are the linear

---

[1] **Remark on notation.** When the domain of an integration is not specified, it's understood to be $\mathbb{R}^3$. Also, the dummy variable x as well as the differential dx in integrals are almost never written explicitly. Nevertheless, we shall sometimes retain the variable x for the trap potential V. This convention seems better in some places.

and quadratic Zeeman effects induced by an external uniform magnetic field, and the components of F are called the spin-1 Pauli matrices.

Besides the energy, the system is described to have the following two conserved quantities:

$$\text{Number of atoms} \quad \mathcal{N}[\Psi] = \int |\Psi|^2,$$
$$\text{Total magnetization} \quad \mathcal{M}[\Psi] = \int \big(|\psi_1|^2 - |\psi_{-1}|^2\big).$$

And a ground state is a minimizer of E under fixed $\mathcal{N}$ and $\mathcal{M}$. By normalization, we can assume $\mathcal{N}[\Psi] = 1$. And $\mathcal{M}[\Psi] = M$ for some constant M. Note that $|\mathcal{M}[\Psi]| \leq \mathcal{N}[\Psi]$ for every state Ψ, so we must have $|M| \leq 1$. Due to the symmetry of the roles of ψ_1 and ψ_{-1}, we will only consider $0 \leq M \leq 1$. The general assumptions on the parameters of E are the following:

(A1) $V \in L^\infty_{loc}(\mathbb{R}^3)$, and $V(x)$ tends to infinity as $|x|$ tends to infinity. Precisely[2]

$$\lim_{R\to\infty} \big(\inf_{|x|\geq R} V(x)\big) = \infty.$$

Note that in particular V is bounded from below.

(A2) $\beta_0 > |\beta_1| > 0$.

(A3) $q \geq 0$.

(A4) $V \geq 0$ and $p = 0$.

We give some remarks for these assumptions.

1. (A2) indicates a repulsive nature of the system. (A1) will then guarantee that $V(x)$ traps the system essentially in a localized region, which will be crucial in some places, including the proof of the existence result.

2. I'm not sure whether (A2) must be true in principle, but it holds for real systems as far as I know. (For example spin-1 BEC of ^{23}Na and ^{87}Rb.) Mathematically,

---

[2] We'll write $\inf$ (and the like) also for $\operatorname{ess\,inf}$.

the fact $|\beta_1| < \beta_0$ is also helpful in the proof of existence when $\beta_1 < 0$. For $\beta_1 > 0$, the assumption $\beta_1 < \beta_0$ is in fact not used in this thesis. By the way, the case $\beta_0 = \beta_1 = 0$ or only $\beta_1 = 0$ can also be studied mathematically. We'll however not consider them since they exhibit no further difficulty but only result in some degenerate situations not of much interest.

3. According to the sign of β_1, spin-1 BECs are classified into two groups: *ferromagnetic* ones for $\beta_1 < 0$, and *antiferromagnetic* ones for $\beta_1 > 0$. The typical examples are respectively ^{23}Na and ^{87}Rb. They show very different behaviors from each other.
4. Physically, the values of p and q can be tuned by modifying the applied magnetic field. It's also possible to make q negative, but we do not consider this case in this thesis.
5. Due to the conservations of $\mathcal{N}$ and $\mathcal{M}$, ground states are not changed by shifting the values of V and p by any constants. Hence (A4) causes no loss of generality for our purposes.

Note. The model of spin-1 BEC appeared very soon after the first realization. The earliest papers being [9], [18] and [12]. As to the understanding of the model, I however most benefited from [22] and various papers by Dr. Weizhu Bao and his collaborators, for example [3], [2] and [15]. Besides them, the review article [11] is also a reference I consulted from time to time. Our expression of the energy functional is mostly similar to that given in [22], [15] and [11].

1.2 The motivations

The whole study is motivated by two phenomena, pertaining to ferromagnetic and antiferromagnetic systems respectively.

1. For a ferromagnetic system, when there is no external magnetic field (i.e. $q = 0$), its ground state Ψ obeys the single-mode approximation (SMA)

$$\Psi = (\gamma_1\psi, \gamma_0\psi, \gamma_{-1}\psi),$$

   where each γ_j is a constant, and ψ is a function independent of j.

2. For an antiferromagnetic system, as q increases from zero, its ground state Ψ undergoes a bifurcation from $\psi_0 \equiv 0$ to $\psi_0 \neq 0$ at a critical $q_c > 0$.

The SMA, as the name indicates, was originally only regarded as an approximation, which was used to simplify the study of spin-1 BEC. As later investigations showed, it turns out to be exactly the case but not only an approximation for ferromagnetic systems (and is in general not suited for antiferromagnetic systems).

These phenomena have been known from numerical simulations for a long time. For clear declarations and discussions of them, see respectively [21] and [22, 15]. The bifurcation phenomenon was recently also observed in experiments [10]. Nevertheless, there seems to be no sound mathematical reasonings for the validity of these facts before. In theoretical discussions on the bifurcation, like in [22], the researchers usually assume Ψ is a constant vector, which is of course not a satisfactory demonstration.

We will later first consider $q = 0$. Due to the SMA, a ferromagnetic ground state can be characterized as a one-component system. On the other hand, the antiferromagnetic ground state has only two components since $\psi_0 \equiv 0$. They will be referred to as simplified characterizations in this thesis. Their proofs first appeared in our paper [16]. It's interesting that, by using the mentioned redistribution method, they can be proved in almost the same way.

On the other hand, the bifurcation phenomenon can also be deduced by using mass redistribution, while a lot more technical details are involved. The most difficult part is to prove that we do have $\psi_0 \equiv 0$ for some q strictly larger than zero. The proof first appeared in [17], where there are also many relevant discussions on the redistribution method. I am recently also preparing a simplified version, which go straight to the verification of the bifurcation phenomenon.

Note. The outline of the thesis is very clear from the contents. Moreover, I'll use a few words in the beginning of every chapter or section to indicate what we are going to do.

Chapter 2

Preliminaries

In this chapter we give some preliminaries which are essential for the discussions in the rest of this thesis. In Section 2.1, we introduce a reduction which says that we can simply consider $(|\psi_1|, |\psi_0|, |\psi_{-1}|)$ for our purposes. Many notations are also given in the same section. In Section 2.2, the fundamental facts such as the existence of ground state, the Euler-Lagrange system and its direct corollaries are given. In Section 2.3, the idea of mass redistribution is introduced.

2.1 A reduction of the model

We shall write H^1 for $H^1(\mathbb{R}^3, \mathbb{C})$, and similarly for other function spaces. Let

$$\mathbb{B} = \left\{ (\psi_1, \psi_0, \psi_{-1}) \;\middle|\; \psi_j \in H^1 \cap L^2_V \cap L^4 \text{ for each } j \right\},$$

where L^2_V is the V-weighted L^2 space. That is, a measurable function f belongs to L^2_V if $\|f\|^2_{L^2_V} := \int V(x)|f|^2$ is finite. Note that L^2_V is nothing but L^2 space with respect to the (σ-finite) measure $V(x)dx$. By endowing $\mathbb{B}$ with the norm

$$\|\Psi\| = \sum_j \left(\|\psi_j\|_{H^1} + \|\psi_j\|_{L^2_V} + \|\psi_j\|_{L^4} \right), \tag{2.1}$$

$\mathbb{B}$ is a Banach space. Obviously, $\mathbb{B}$ is the appropriate space for our variational model. Precisely, ground states are minimizers of the following problem:

$$\min E \quad \text{over} \quad \{ \Psi \in \mathbb{B} \,|\, \mathcal{N}[\Psi] = 1, \ \mathcal{M}[\Psi] = M \}.$$

For our purposes, we can reduce this model on $\mathbb{B}$ to a model on $\mathbb{B}_+$, where

$$\mathbb{B}_+ = \{(u_1, u_0, u_{-1}) \in \mathbb{B} \mid u_j \geq 0 \text{ for each } j\}\,.$$

We give the reduction in the following.

Given $\Psi = (\psi_1, \psi_0, \psi_{-1}) \in \mathbb{B}$, we have

$$E_{kin}[\Psi] = \sum_j |\nabla \psi_j|^2 \geq \sum_j |\nabla |\psi_j||^2$$

by the convexity inequality for gradients (Section 8.1). Moreover, let

$$\psi_j = |\psi_j| e^{i\theta_j},$$

then it's easy to check that

$$E_1[\Psi] = \int \beta_1 \Big\{ 2|\psi_0|^2 \Big[|\psi_1|^2 + |\psi_{-1}|^2 + 2|\psi_1||\psi_{-1}| \cos(\theta_1 - 2\theta_0 + \theta_{-1}) \Big] + (|\psi_1|^2 - |\psi_{-1}|^2)^2 \Big\}.$$

Hence

$$E_1[\Psi] \geq \int \beta_1 \Big\{ 2|\psi_0|^2 (|\psi_1| - \operatorname{sgn}(\beta_1)|\psi_{-1}|)^2 + (|\psi_1|^2 - |\psi_{-1}|^2)^2 \Big\},$$

where

$$\operatorname{sgn}(\beta_1) = \begin{cases} 1 & \text{if } \beta_1 > 0 \\ -1 & \text{if } \beta_1 < 0. \end{cases}$$

And the equality holds if

$$\cos(\theta_1 - 2\theta_0 + \theta_{-1}) \equiv -\operatorname{sgn}(\beta_1). \tag{2.2}$$

For other parts of the energy, we obviously have

$$E_{pot}[\Psi] = E_{pot}[(|\psi_1|, |\psi_0|, |\psi_{-1}|)],$$

$$E_0[\Psi] = E_0[(|\psi_1|, |\psi_0|, |\psi_{-1}|)],$$

$$E_{Zee}[\Psi] = E_{Zee}[(|\psi_1|, |\psi_0|, |\psi_{-1}|)].$$

We thus obtain

$$E[\Psi] \geq \mathcal{E}[(|\psi_1|, |\psi_0|, |\psi_{-1}|)],$$

where (remember that we have assumed $p = 0$)

$$\begin{aligned}\mathcal{E}[\mathbf{u}] := \int \Bigg\{ \sum_j |\nabla u_j|^2 + V(x)|\mathbf{u}|^2 + \beta_0|\mathbf{u}|^4 \\ + \beta_1\Big[2u_0^2(u_1 - \operatorname{sgn}(\beta_1)u_{-1})^2 + (u_1^2 - u_{-1}^2)^2\Big] + q(u_1^2 + u_{-1}^2)\Bigg\}\end{aligned}$$

for $\mathbf{u} = (u_1, u_0, u_{-1}) \in \mathbb{B}_+$. Also, note that the conservations of $\mathcal{N}$ and $\mathcal{M}$ are actually constraints on $|\psi_j|$ and have nothing to do with the phases. These observations lead us to replace the original variational problem by the following one:

$$\min_{\mathbf{u}\in\mathbb{A}} \mathcal{E}[\mathbf{u}], \tag{2.3}$$

where the admissible class

$$\mathbb{A} = \{\mathbf{u} \in \mathbb{B}_+ \,|\, \mathcal{N}[\mathbf{u}] = 1,\ \mathcal{M}[\mathbf{u}] = M\}.$$

The validity of using this reduced model is provided by Corollary 8.5, Corollary 8.6 and Corollary 8.8, of which we can say the last one is the only not totally trivial assertion. We give careful examinations of them for the sake of being completely rigorous. For later discussions, one can indeed just forget the original model and focus on (2.3).

We introduce some more notations to conclude this section. Define

$$E_g = \min_{\mathbf{u}\in\mathbb{A}} \mathcal{E}[\mathbf{u}],$$

and

$$\mathbb{G} = \{\mathbf{u} \in \mathbb{A} \,|\, \mathcal{E}[\mathbf{u}] = E_g\}.$$

Thus E_g is the ground-state energy, and $\mathbb{G}$ is the set of minimizers of (2.3), which are exactly the objects to study. Since many assertions and discussions in this thesis involve different values of M and q, in later parts of this thesis we will write $\mathbb{A}_M$ (the admissible class has nothing to do with q), $\mathbb{G}_{M,q}$ and $E_g(M, q)$ to specify their values explicitly.

Similar to E, we will use $\mathcal{E}_{kin}$, $\mathcal{E}_{pot}$, $\mathcal{E}_0$, $\mathcal{E}_1$, and $\mathcal{E}_{Zee}$ to denote the five parts of $\mathcal{E}$. Moreover, we will use $H(\mathbf{u})$ to denote the integrand of $\mathcal{E}[\mathbf{u}]$, i.e.

$$\mathcal{E}[\mathbf{u}] = \int H(\mathbf{u}).$$

H_{kin}, H_{pot}, etc. are similarly defined for the corresponding parts.

2.2 Fundamental properties

In some aspects our three-component system can be regarded as a generalization of the one-component system studied in [14]. The fundamental properties about the one-component model hold and can be proved similarly for our model. (The uniqueness is however a remarkable exception. See Remark 2.2 below. Detailed discussions are given in §7.1.) We summarize them in the following.

Theorem 2.1. $\mathbb{G} \neq \emptyset$. $\mathbf{u} \in \mathbb{G}$ *is at least of class* C^1*, and satisfies the Euler-Lagrange system*

$$\begin{cases} (\mu+\lambda)u_1 = \mathcal{L}u_1 + 2\beta_1\left[u_0^2(u_1 - \mathrm{sgn}(\beta_1)u_{-1}) + u_1(u_1^2 - u_{-1}^2)\right] + qu_1 & \text{(2.4a)} \\ \mu u_0 = \mathcal{L}u_0 + 2\beta_1 u_0(u_1 - \mathrm{s}(\beta_1)u_{-1})^2 & \text{(2.4b)} \\ (\mu-\lambda)u_{-1} = \mathcal{L}u_{-1} + 2\beta_1\left[u_0^2(u_{-1} - \mathrm{sgn}(\beta_1)u_1) + u_{-1}(u_{-1}^2 - u_1^2)\right] + qu_{-1} & \text{(2.4c)} \end{cases}$$

in the sense of distribution, where $\mathcal{L} = -\Delta + V + 2\beta_0|\mathbf{u}|^2$*, and* λ *and* μ *are the Lagrange multipliers induced by the constraints* $\mathcal{N}[\mathbf{u}] = 1$ *and* $\mathcal{M}[\mathbf{u}] = M$ *respectively. Moreover, for each* u_j*, either* $u_j \equiv 0$ *or* $u_j > 0$ *on all of* $\mathbb{R}^3$.

The existence result can be proved by the standard direct method in the calculus of variations, in which one tries to show that a minimizing sequence in $\mathbb{A}$ has a subsequence which weakly converges to an element in $\mathbb{G}$. The only difference from a typical situation is that here the system is on the whole space but not a bounded domain. As a result, we do not have compact embedding $H^1 \hookrightarrow L^2$ to guarantee that the weak limit is still in $\mathbb{A}$. Instead, we should use the assumption (A1), which implies that, in some sense, most part of $\mathbf{u}$ is really contained in bounded domains, on which compact embedding applies. A precise argument can be given almost the same as in Lemma A.2 of [14]. (See also [1, 6].) Nevertheless, besides the conclusion of existence, some observations from its proof will also be needed later. We give them in Proposition 2.2 below. For convenience we give the proof in Section 8.3. The most important point is that we actually have strong convergence but not only weak convergence for the extracted subsequence of the minimizing sequence. This holds for our model since the norm of $\mathbb{B}$ is bounded by a constant multiple of the energy functional.

Proposition 2.2. *Let $\{\mathbf{u}^n\}$ be a sequence in $\mathbb{B}_+$. Suppose*

$$\mathcal{N}[\mathbf{u}^n] \to 1,\ \mathcal{M}[\mathbf{u}^n] \to M,$$

and $\mathcal{E}[\mathbf{u}^n]$ is uniformly bounded in n, then $\{\mathbf{u}^n\}$ has a subsequence $\{\mathbf{u}^{n(k)}\}_{k=1}^{\infty}$ converging weakly to some $\mathbf{u}^\infty \in \mathbb{A}$, which satisfies

$$\mathcal{E}[\mathbf{u}^\infty] \le \liminf_{k\to\infty} \mathcal{E}[\mathbf{u}^{n(k)}].$$

If we assume further that $\mathcal{E}[\mathbf{u}^n] \to E_g$, then $\mathbf{u}^\infty \in \mathbb{G}$, and $\mathbf{u}^{n(k)} \to \mathbf{u}^\infty$ in the norm of $\mathbb{B}$.

The Euler-Lagrange system (2.4) is called a time-independent Gross-Pitaevskii system (GP system). We remark that (2.4) is indeed valid not only in the sense of distribution, but also when tested by elements in $\mathbb{B}$. In fact, $\mathcal{E}$, $\mathcal{N}$ and $\mathcal{M}$ are continuously (Fréchet) differentiable as functionals from $\mathbb{B}$ into $\mathbb{R}$, and (2.4), after multiplied by 2, is exactly

$$\mu\mathcal{N}'[\mathbf{u}] + \lambda\mathcal{M}'[\mathbf{u}] = \mathcal{E}'[\mathbf{u}].$$

We omit the verification of this fact. Once (2.4) is obtained, that $\mathbf{u} \in \mathbb{G}$ is continuously differentiable follows standard regularity theorem (see e.g. [13], 10.2). And the strict positivity of a nonvanishing component can be obtained by the strong maximum principle. We give the proof of this last assertion also in Section 8.3. We shall usually use this fact tacitly to avoid repeatedly referring to Theorem 2.1.

Corollary 2.3. *Let* $\mathbf{u} \in \mathbb{G}$. *If* $0 < M < 1$, *then* $u_j \neq 0$ *for* $j = 1, -1$.

Proof. Since $\int (u_1^2 - u_{-1}^2) = M > 0$, $u_1 \neq 0$, and hence $u_1 > 0$. To prove $u_{-1} \neq 0$, assume otherwise, then (2.2c) gives $u_0^2 u_1 = 0$, and so $u_0 = 0$. Thus among the three components only $u_1 \neq 0$, which implies $M = 1$ from the constraint $\mathcal{N}[\mathbf{u}] = 1$, contradicting to our assumption. □

The two-component ground state

Since we will investigate whether $u_0 \equiv 0$ for $\mathbf{u} \in \mathbb{G}$, it will be convenient to introduce the two-component admissible class

$$\mathbb{A}^{two} = \{\mathbf{u} \in \mathbb{A} \,|\, u_0 \equiv 0\}\,.$$

Note that for $\mathbf{u} \in \mathbb{A}^{two}$ the constraints are equivalent to

$$\int u_1^2 = \frac{1+M}{2} \quad \text{and} \quad \int u_{-1}^2 = \frac{1-M}{2}.$$

Due to the following uniqueness result, there is no need to introduce the corresponding class of minimizers $\mathbb{G}^{two}$.

Theorem 2.4. *There exists exactly one element* $\mathbf{z} = (z_1, 0, z_{-1}) \in \mathbb{A}^{two}$ *which minimizes the energy* $\mathcal{E}$ *over* $\mathbb{A}^{two}$. *Moreover,* $\mathbf{z}$ *is independent of the value of* $q \in [0, \infty)$.

Proof. The existence of $\mathbf{z}$ can be proved as for the general three-component case. On the other hand, the fact that $\mathbf{z}$ is independent of q follows the simple observation that $\mathcal{E}_{Zee}$ equals the constant q over $\mathbb{A}^{two}$, and hence plays no role in the minimization. We prove the uniqueness of $\mathbf{z}$ in the following.

Given $\mathbf{u}, \mathbf{v} \in \mathbb{A}^{two}$. Let $\mathbf{w} \in \mathbb{B}_+$ be defined by $w_j^2 = (u_j^2 + v_j^2)/2$ for $j = 1, 0, -1$, then $\mathbf{w}$ is also in $\mathbb{A}^{two}$. Let $D = (\mathcal{E}[\mathbf{u}] + \mathcal{E}[\mathbf{v}])/2 - \mathcal{E}[\mathbf{w}]$, then $D = D_{kin} + D_n + D_s$, where

$$D_{kin} = \frac{\mathcal{E}_{kin}[\mathbf{u}] + \mathcal{E}_{kin}[\mathbf{v}]}{2} - \mathcal{E}_{kin}[\mathbf{w}] = \int \sum_{j=1,-1} \left(\frac{|\nabla u_j|^2 + |\nabla v_j|^2}{2} - |\nabla w_j|^2 \right),$$

which is nonnegative by the convexity inequality for gradients. Also,

$$D_n = \frac{\mathcal{E}_n[\mathbf{u}] + \mathcal{E}_n[\mathbf{v}]}{2} - \mathcal{E}_n[\mathbf{w}] = \frac{\beta_0}{4} \int \left(|\mathbf{u}|^2 - |\mathbf{v}|^2\right)^2 \geq 0,$$

and

$$D_s = \frac{\mathcal{E}_s[\mathbf{u}] + \mathcal{E}_s[\mathbf{v}]}{2} - \mathcal{E}_s[\mathbf{w}] = \frac{\beta_1}{4} \int \left(u_1^2 - u_{-1}^2 - v_1^2 + v_{-1}^2\right)^2 \geq 0,$$

as are easily checked. Now assume, moreover, $\mathbf{u}$ and $\mathbf{v}$ both minimize $\mathcal{E}$ over $\mathbb{A}^{two}$, then we must have $D_{kin} = D_n = D_s = 0$. Otherwise we get the contradiction $\mathcal{E}[\mathbf{w}] < (\mathcal{E}[\mathbf{u}] + \mathcal{E}[\mathbf{v}])/2$. From $D_n = 0$ and $D_s = 0$ we then conclude that $\mathbf{u} = \mathbf{v}$. This proves the uniqueness of $\mathbf{z}$. $\square$

Remark 2.1. Let $\mathbf{u} \in \mathbb{G}$. The assertion $u_0 \equiv 0$ is obviously equivalent to $\mathbf{u} = \mathbf{z}$. We will show in Section 3.2 that $\mathbf{z} \in \mathbb{G}$ when $q = 0$. As a corollary, the assertions in Theorem 2.1 for elements in $\mathbb{G}$ also apply to $\mathbf{z}$.

Remark 2.2. The convexity argument used to prove the uniqueness of $\mathbf{z}$ is standard. The idea however fails for general $\mathbb{G}$, due to the term $H_1(\mathbf{u})$. Although the uniqueness will not be needed essentially, the lack of it still causes troubles in some of our presentations. See Section 7.1 for more discussions on the uniqueness problem.

2.3 Mass redistribution

Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ be an n-tuple of real-valued function in $H^1(\mathbb{R}^d)$, and let $g = |\mathbf{f}|$. The convexity inequality for gradients (Section 8.1) says

$$|\nabla g|^2 \leq \sum_k |\nabla f_k|^2.$$

This fact has a simple while interesting generalization, when $f_1^2, \ldots, f_n^2$ do not sum to a single g^2, but are distributed into multiple parts. To be precise, we give the following definition.

Definition 2.1. Let $\mathbf{f}$ be as above, and let $\mathbf{g} = (g_1, g_2, \ldots, g_m)$ be an m-tuple of nonnegative functions. We say $\mathbf{g}$ is a mass redistribution of $\mathbf{f}$ if

$$
\begin{aligned}
g_1^2 &= a_{11}f_1^2 + a_{12}f_2^2 + \cdots + a_{1n}f_n^2 \\
g_2^2 &= a_{21}f_1^2 + a_{22}f_2^2 + \cdots + a_{2n}f_n^2 \\
&\vdots \\
g_m^2 &= a_{m1}f_1^2 + a_{m2}f_2^2 + \cdots + a_{mn}f_n^2,
\end{aligned}
$$

where $a_{\ell k}$ ($\ell = 1, \ldots, m$; $k = 1, \ldots, n$) are nonnegative constants satisfying

$$\sum_{\ell=1}^{m} a_{\ell k} = 1 \quad \text{for each} \quad k = 1, \ldots, n.$$

That is, the coefficients of every column sum to 1.

Note that $g = |\mathbf{f}|$ is the only mass redistribution of $\mathbf{f}$ for $m = 1$. For general m, we have the following result.

Proposition 2.5. *Let* $\mathbf{g}$ *be a mass redistribution of* $\mathbf{f}$ *as in Definition 2.1, then we have*

(1) $|\mathbf{g}| = |\mathbf{f}|$, and

(2) $\sum_{\ell=1}^{m} |\nabla g_\ell|^2 \leq \sum_{k=1}^{n} |\nabla f_k|^2$.

Proof. The first assertion follows directly from the definition of mass redistribution. For the second assertion, apply the convexity inequality for gradients to

$$g_\ell = \left(\left(\sqrt{a_{\ell 1}} f_1\right)^2 + \left(\sqrt{a_{\ell 2}} f_2\right)^2 + \cdots + \left(\sqrt{a_{\ell n}} f_n\right)^2 \right)^{1/2},$$

and we get

$$|\nabla g_\ell|^2 \leq a_{\ell 1} |\nabla f_1|^2 + a_{\ell 2} |\nabla f_2|^2 + \cdots + a_{\ell n} a_{\ell n} |\nabla f_n|^2.$$

The assertion is then obtained by summing over $\ell = 1, 2, \ldots, m$. □

Remark 2.3. It should be clear why we use the word "redistribution". On the other hand, we will consider mass redistributions of $\mathbf{u} \in \mathbb{A}$. The adjective "mass" is added since the square of u_j represents the mass density of the j-th component. Indeed, we might as well use the term "square redistribution". For convenience, however, we shall later only say redistribution.

Let's write $\mathbb{A}_M$ and $\mathbb{G}_{M,q}$ here. Redistribution provides a simple and concrete way to variate an element in $\mathbb{A}_M$ into another element, in the same space or in another $\mathbb{A}_{M'}$. Indeed, if $\mathbf{v}$ is a redistribution of some $\mathbf{u} \in \mathbb{A}_M$, then (1) of Proposition 2.5 implies $\mathcal{N}[\mathbf{v}] = 1$, and one needs only to take care of the value of $\mathcal{M}[\mathbf{v}]$. Also, it's easy to compare $\mathcal{E}[\mathbf{v}]$ with $\mathcal{E}[\mathbf{u}]$. Precisely, again from (1) we have

$$\mathcal{E}_{pot}[\mathbf{v}] = \mathcal{E}_{pot}[\mathbf{u}] \quad \text{and} \quad \mathcal{E}_0[\mathbf{v}] = \mathcal{E}_0[\mathbf{u}], \tag{2.5}$$

and from (2) we have

$$\mathcal{E}_{kin}[\mathbf{v}] \leq \mathcal{E}_{kin}[\mathbf{u}]. \tag{2.6}$$

As will be seen, these features make it easy to deduce some facts by using redistribution, which might otherwise be harder to obtain or need more elaboration.

The true merit of redistribution (in my opinion and for our purpose) exhibits in Chapter 5, when we use it to obtain simple inequalities satisfied by ground states. To be precise, let $\mathbf{u} \in \mathbb{G}_{M,q}$ for some M, q. Then for any redistribution $\mathbf{v}$ of $\mathbf{u}$ in the same class $\mathbb{A}_M$, the fact $\mathcal{E}[\mathbf{u}] \leq \mathcal{E}[\mathbf{v}]$ together with (2.5) imply

$$\mathcal{E}_{kin}[\mathbf{u}] + \mathcal{E}_1[\mathbf{u}] + \mathcal{E}_{Zee}[\mathbf{u}] \leq \mathcal{E}_{kin}[\mathbf{v}] + \mathcal{E}_1[\mathbf{v}] + \mathcal{E}_{Zee}[\mathbf{v}].$$

And (2.6) further implies

$$\mathcal{E}_1[\mathbf{u}] + \mathcal{E}_{Zee}[\mathbf{u}] \leq \mathcal{E}_1[\mathbf{v}] + \mathcal{E}_{Zee}[\mathbf{v}]. \tag{2.7}$$

Inequality (2.7) is particularly simple in that it involves only algebraic expressions of $\mathbf{u}$ ($\mathbf{v}$ is practically also expressed in terms of $\mathbf{u}$). This inequality, with suitable constructions of $\mathbf{v}$, will be sufficient for our proof of the bifurcation phenomenon.

The "best" way to gain sharper inequalities from redistribution will be the topic of Section 6

Chapter 3

The Simplified Characterizations

In this chapter we assume $q = 0$, i.e. no external magnetic field. Thus

$$\mathcal{E} = \mathcal{E}_{kin} + \mathcal{E}_{pot} + \mathcal{E}_0 + \mathcal{E}_1 \text{ and } H = H_{kin} + H_{pot} + H_0 + H_1.$$

In Section 3.1, we prove the SMA. And in Section 3.2, we consider the phenomenon $u_0 \equiv 0$ for $\mathbf{u} \in \mathbb{G}$. A direct consequence of these results is that we can characterize elements in $\mathbb{G}$ (and hence ground states) by systems with fewer (one or two) components.

It's interesting that, though these two phenomena look quite different, they can be proved in essentially the same way. To explain the idea, let $\mathcal{P}$ denote the property (SMA or $u_0 \equiv 0$) to be justified. We will prove that, for every $\mathbf{u} \in \mathbb{A}$, there corresponds a redistribution $\widetilde{\mathbf{u}}$ of $\mathbf{u}$ which also lies in $\mathbb{A}$, such that

(a) $\widetilde{\mathbf{u}}$ has the property $\mathcal{P}$, and

(b) $\mathcal{E}[\widetilde{\mathbf{u}}] \leq \mathcal{E}[\mathbf{u}]$.

From (b), we have $\mathcal{E}[\widetilde{\mathbf{u}}] = \mathcal{E}[\mathbf{u}]$ provided $\mathbf{u} \in \mathbb{G}$, from which we shall prove $\mathbf{u}$ is exactly $\widetilde{\mathbf{u}}$, and hence $\mathbf{u}$ has the property $\mathcal{P}$.

3.1 The single-mode approximation

In this section we assume $\beta_1 < 0$. Let

$$\mathbb{A}_1 = \{\mathbf{u} \in \mathbb{A} \,|\, \mathbf{u} = (\gamma_1 f, \gamma_0 f, \gamma_{-1} f) \text{ for some constants } \gamma_j \text{ and some function } f\}.$$

The goal is to prove $\mathbb{G} \subset \mathbb{A}_1$.

Now given any $\mathbf{u} \in \mathbb{A}$. It's easy to see that a redistribution of $\mathbf{u}$ in $\mathbb{A}_1$ can be expressed as $\boldsymbol{\gamma}|\mathbf{u}|$, where $\boldsymbol{\gamma} = (\gamma_1, \gamma_0, \gamma_{-1})$ is any triple of nonnegative constants satisfying

$$\begin{cases} \gamma_1^2 + \gamma_0^2 + \gamma_{-1}^2 = 1 \\ \gamma_1^2 - \gamma_{-1}^2 = M. \end{cases} \tag{3.1}$$

Let Γ denote the set containing all such $\boldsymbol{\gamma}$:

$$\Gamma := \left\{ (\gamma_1, \gamma_0, \gamma_{-1}) \in \mathbb{R}^3 \,\middle|\, \gamma_j \geq 0 \text{ for each } j, \ \boldsymbol{\gamma} \text{ satisfies } (3.1) \right\}. \tag{3.2}$$

Then for each $\boldsymbol{\gamma} \in \Gamma$, since $\boldsymbol{\gamma}|\mathbf{u}|$ is a redistribution of $\mathbf{u}$, we have

$$H_{pot}(\boldsymbol{\gamma}|\mathbf{u}|) \equiv H_{pot}(\mathbf{u}) \text{ and } H_0(\boldsymbol{\gamma}|\mathbf{u}|) \equiv H_0(\mathbf{u}). \tag{3.3}$$

Also,

$$H_{kin}(\boldsymbol{\gamma}|\mathbf{u}|) = |\nabla|\mathbf{u}||^2 \leq \sum_j |\nabla u_j|^2 = H_{kin}(\mathbf{u}). \tag{3.4}$$

On the other hand,

$$H_1(\boldsymbol{\gamma}|\mathbf{u}|) = \beta_1 P(\boldsymbol{\gamma})|\mathbf{u}|^4,$$

where

$$P(\boldsymbol{\gamma}) = 2\gamma_0^2(\gamma_1 + \gamma_{-1})^2 + M^2.$$

Since $\beta_1 < 0$, to make $\mathcal{E}[\boldsymbol{\gamma}|\mathbf{u}|] \leq \mathcal{E}[\mathbf{u}]$ as possible as we can, we compute the maximum of $P(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma} \in \Gamma$. It's easy to check that there is a unique $\boldsymbol{\gamma}^\star \in \Gamma$ such that

$$\max_{\boldsymbol{\gamma} \in \Gamma} P(\boldsymbol{\gamma}) = P(\boldsymbol{\gamma}^\star) = 1.$$

Indeed the maximizer $\boldsymbol{\gamma}^\star = (\gamma_1^\star, \gamma_0^\star, \gamma_{-1}^\star)$ is given by

$$\gamma_1^\star = \frac{1}{2}(1 + M), \ \gamma_0^\star = \sqrt{\frac{1}{2}(1 - M^2)}, \ \gamma_{-1}^\star = \frac{1}{2}(1 - M).$$

We can now state our main theorem in this section.

Theorem 3.1. *Assume* $q = 0$ *and* $\beta_1 < 0$. *If* $\mathbf{u} \in \mathbb{G}$, *then* $\mathbf{u} = \boldsymbol{\gamma}^\star |\mathbf{u}|$.

Proof. By direct calculation we have

$$\begin{aligned} H_1(\mathbf{u}) - H_1(\boldsymbol{\gamma}^\star|\mathbf{u}|) &= -\beta_1 \Big\{ |\mathbf{u}|^4 - \big[2u_0^2(u_1 + u_{-1})^2 + (u_1^2 - u_{-1}^2)^2 \big] \Big\} \\ &= -\beta_1 (u_0^2 - 2u_1 u_{-1})^2 \geq 0. \end{aligned} \tag{3.5}$$

By (3.3), (3.4) and (3.5), we have $H(\mathbf{u}) \geq H(\boldsymbol{\gamma}^\star|\mathbf{u}|)$ for every $\mathbf{u} \in \mathbb{A}$. And hence $\mathbf{u} \in \mathbb{G}$ implies $\mathcal{E}[\mathbf{u}] = \mathcal{E}[\boldsymbol{\gamma}^\star|\mathbf{u}|]$, and the equality holds if and only if the inequalities in (3.4) and (3.5) are equalities. That is

$$\sum_j |\nabla u_j|^2 - |\nabla |\mathbf{u}||^2 = 0, \tag{3.6}$$

and

$$u_0^2 - 2u_1 u_{-1} = 0. \tag{3.7}$$

From (8.1), the equality (3.6) holds iff

$$u_j \nabla u_k - u_k \nabla u_j = 0 \quad \text{for} \quad j \neq k. \tag{3.8}$$

Since $\mathbf{u}$ is not identically zero (by the assumption $\mathcal{N}[\mathbf{u}] = 1$), at least one component of $\mathbf{u}$ is strictly positive everywhere. Assume $u_1 > 0$. Then from (3.8) we have

$$\nabla \left(\frac{u_0}{u_1} \right) = \nabla \left(\frac{u_{-1}}{u_1} \right) = 0,$$

which implies u_0 and u_{-1} are both constant multiples of u_1. This shows $\mathbf{u} \in \mathbb{A}_1$. That $\mathbf{u}$ must be $\boldsymbol{\gamma}^\star|\mathbf{u}|$ then follows either by (3.7) or by the fact that $\boldsymbol{\gamma}^\star$ is the unique maximizer of P over Γ. The case $u_0 > 0$ or $u_{-1} > 0$ can be proved similarly. □

Remark 3.1. Since $|\mathbf{u}|$ is bounded away from zero, we can also conclude from (3.6) and Corollary 8.3 that $\mathbf{u} \in \mathbb{A}_1$.

The above theorem implies that searching for ground states of a ferromagnetic spin-1 BEC can be reduced to an one-component minimization problem. Precisely, let

$$\mathbb{A}^s = \{ |\mathbf{u}| \mid \mathbf{u} \in \mathbb{A} \} = \left\{ f \in H^1 \cap L^4 \cap L_V^2 \,\middle|\, f \geq 0 \text{ and } \textstyle\int f^2 = 1 \right\}, \tag{3.9}$$

and define $\mathcal{E}^s : \mathbb{A}^s \to \mathbb{R}$ by

$$\mathcal{E}^s[f] = \int \Big\{ |\nabla f|^2 + V f^2 + (\beta_0 + \beta_1) f^4 \Big\}.$$

Then $\mathcal{E}[\boldsymbol{\gamma}^\star f] = \mathcal{E}^s[f]$ for $f \in \mathbb{A}^s$. Also let

$$\mathbb{G}^s = \Big\{ f \in \mathbb{A}^s \,\Big|\, \mathcal{E}^s[f] = \min_{g \in \mathbb{A}^s} \mathcal{E}^s[g] \Big\}.$$

Then if $\mathbf{u} \in \mathbb{G}$, by Theorem 3.1 we have for every $f \in \mathbb{A}^s$

$$\mathcal{E}^s[|\mathbf{u}|] = \mathcal{E}[\boldsymbol{\gamma}^\star |\mathbf{u}|] \le \mathcal{E}[\boldsymbol{\gamma}^\star f] = \mathcal{E}^s[f].$$

Thus $|\mathbf{u}| \in \mathbb{G}^s$. Conversely if $f \in \mathbb{G}^s$, then for every $\mathbf{u} \in \mathbb{A}$ we have

$$\mathcal{E}[\boldsymbol{\gamma}^\star f] = \mathcal{E}^s[f] \le \mathcal{E}^s[|\mathbf{u}|] = \mathcal{E}[\boldsymbol{\gamma}^\star |\mathbf{u}|] \le \mathcal{E}[\mathbf{u}].$$

Hence $\boldsymbol{\gamma}^\star f \in \mathbb{G}$. We thus obtain the following one-component characterization of $\mathbb{G}$.

Corollary 3.2. $\mathbb{G} = \{\boldsymbol{\gamma}^\star f \mid f \in \mathbb{G}^s\}$.

3.2 The vanishing of u_0

We assume $\beta_1 > 0$ in this section. Recall the definition of $\mathbb{A}^{two}$ in Section 2.2 We want to show that $\mathbb{G} \subset \mathbb{A}^{two}$. Now, similarly, for every $\mathbf{u} \in \mathbb{A}$ we want to find an appropriate redistribution $\widetilde{\mathbf{u}} = (\widetilde{u}_1, 0, \widetilde{u}_{-1}) \in \mathbb{A}^{two}$ so that $\mathcal{E}[\widetilde{\mathbf{u}}] \le \mathcal{E}[\mathbf{u}]$. This time, however, the assumption $\widetilde{\mathbf{u}} \in \mathbb{A}^{two}$ alone doesn't give us an obvious candidate of $\widetilde{\mathbf{u}}$. In view that such $\widetilde{\mathbf{u}}$ satisfies $|\widetilde{\mathbf{u}}| = |\mathbf{u}|$ and hence $\mathcal{N}[\widetilde{\mathbf{u}}] = 1$, as a guess, we try just imposing the additional assumption that $\widetilde{\mathbf{u}}$ also satisfies

$$\widetilde{u}_1^2 - \widetilde{u}_{-1}^2 = u_1^2 - u_{-1}^2,$$

to make $\mathcal{M}[\widetilde{\mathbf{u}}] = M$ automatically. This results in only one possibility, that is

$$\widetilde{u}_j = \sqrt{u_j^2 + \frac{u_0^2}{2}} \quad \text{for} \quad j = 1, -1. \tag{3.10}$$

It's fortunate that it works, and we obtain our main theorem of this section as follows.

Theorem 3.3. *Assume* $\beta_1 > 0$ *and* $M > 0$*, then* $\mathbf{u} \in \mathbb{G}$ *implies* $u_0 \equiv 0$*.*

Proof. For $\mathbf{u} \in \mathbb{A}$, define $\widetilde{\mathbf{u}} \in \mathbb{A}^{two}$ by (3.10). Again since $\widetilde{\mathbf{u}}$ is a redistribution of $\mathbf{u}$, we have

$$H_{pot}(\widetilde{\mathbf{u}}) = H_{pot}(\widetilde{\mathbf{u}}) \text{ and } H_0(\widetilde{\mathbf{u}}) = H_0(\widetilde{\mathbf{u}}),$$

and

$$H_{kin}(\widetilde{\mathbf{u}}) \leq H_{kin}(\mathbf{u}).$$

Also obviously

$$H_1(\mathbf{u}) - H_1(\widetilde{\mathbf{u}}) = 2\beta_1 u_0^2 (u_1 - u_{-1})^2 \geq 0.$$

Thus for $\mathbf{u} \in \mathbb{G}$ we have $\mathcal{E}[\mathbf{u}] = \mathcal{E}[\widetilde{\mathbf{u}}]$, and

$$u_0^2 (u_1 - u_{-1})^2 \equiv 0.$$

From this equality, we have either $u_0 \equiv 0$ or $u_1 \equiv u_{-1}$. However, since we assume $M > 0$, we cannot have $u_1 \equiv u_{-1}$, and hence $u_0 \equiv 0$. ☐

Remark 3.2. From Theorem 3.3 and Theorem 2.4, $\mathbf{z}$ is then the unique element in $\mathbb{G}$ when $0 < M \leq 1$ and $q = 0$. From Theorem 3.4 below, $\mathbf{z}$ is also an element in $\mathbb{G}$ when $M = q = 0$, but is not the unique one.

3.3 Some degenerate situations

The requirement $M > 0$ in Theorem 3.3 is necessary. In fact, for $M = 0$, ground states are not unique, and $u_0 \equiv 0$ corresponds to only one possible state. Moreover, the SMA is again valid. Precisely, consider the following variational problem:

$$\min_{f \in \mathbb{A}^s} \int \Big\{ |\nabla f|^2 + V f^2 + \beta_0 f^4 \Big\}, \tag{3.11}$$

where $\mathbb{A}^s$ is defined by (3.9). We have the following characterization.

Theorem 3.4. *Assume* $\beta_1 > 0$ *and* $M = 0$*, then*

$$\mathbb{G} = \left\{ \big(t, \sqrt{1-2t^2}, t\big) f \;\middle|\; 0 \le t \le 1/\sqrt{2},\ f \text{ is a minimizer of } (3.11) \right\}.$$

Proof. Since $M = 0$, $\boldsymbol{\gamma} \in \Gamma$ (defined by (3.2)) implies

$$\boldsymbol{\gamma} = \big(t, \sqrt{1-2t^2}, t\big) \quad \text{for some} \quad t \in \big[0, 1/\sqrt{2}\big].$$

Now it's easy to see that for any $\mathbf{u} \in \mathbb{A}$ and $\boldsymbol{\gamma} \in \Gamma$ we have

$$H(\boldsymbol{\gamma}|\mathbf{u}|) = |\nabla|\mathbf{u}||^2 + V|\mathbf{u}|^2 + \beta_0|\mathbf{u}|^4,$$

which is independent of $\boldsymbol{\gamma}$. Obviously, $H(\boldsymbol{\gamma}|\mathbf{u}|) \le H(\mathbf{u})$. It remains to show that $\mathbf{u} \in \mathbb{G}$ (and hence $\mathcal{E}[\boldsymbol{\gamma}|\mathbf{u}|] = \mathcal{E}[\mathbf{u}]$) implies $\mathbf{u} = \boldsymbol{\gamma}|\mathbf{u}|$ for some $\boldsymbol{\gamma} \in \Gamma$. The proof is almost the same as before and we omit it. □

In contrast to the above result, the following corollary of Theorem 3.3 shows that SMA is almost never the case when $M > 0$.

Corollary 3.5. *Assume* $\beta_1 > 0$ *and* $0 < M < 1$*, then* $\mathbf{u} \in \mathbb{G} \cap \mathbb{A}_1$ *implies* u_1 *and* u_{-1} *are constants. Moreover, such* $\mathbf{u}$ *exists only if* V *is a constant.*

Proof. By Theorem 3.3, the Euler-Lagrange system (2.4) is reduced to the following two-component system:

$$\begin{cases} (\mu+\lambda)u_1 = \mathcal{L}u_1 + 2\beta_1 u_1(u_1^2 - u_{-1}^2) \\ (\mu-\lambda)u_{-1} = \mathcal{L}u_{-1} + 2\beta_1 u_{-1}(u_{-1}^2 - u_1^2), \end{cases} \tag{3.12}$$

where $\mathcal{L} = -\Delta + V + 2\beta_0(u_1^2 + u_{-1}^2)$.

Since $0 < M < 1$, for $j = 1, -1$, $u_j > 0$. So $\mathbf{u} \in \mathbb{A}_1$ implies $u_{-1} = \kappa u_1$ for some constant $0 < \kappa < 1$. The system (3.12) then gives the following two equations for u_1:

$$(\mu+\lambda)u_1 = -\Delta u_1 + Vu_1 + 2\beta_0(1+\kappa^2)u_1^3 + 2\beta_1(1-\kappa^2)u_1^3, \tag{A}$$

$$(\mu-\lambda)u_1 = -\Delta u_1 + Vu_1 + 2\beta_0(1+\kappa^2)u_1^3 + 2\beta_1(\kappa^2-1)u_1^3. \tag{B}$$

Now

$$\frac{1}{2}((A)-(B)) \implies \lambda u_1 = 2\beta_1(1-\kappa^2)u_1^3.$$

Since $u_1 > 0$, we get

$$u_1 = \sqrt{\frac{\lambda}{2\beta_1(1-\kappa^2)}}\,.$$

In particular u_1 and $u_{-1} = \kappa u_1$ are constants. Hence $\Delta u_1 = 0$. Then,

$$\frac{1}{2}((A)+(B)) \implies \mu u_1 = Vu_1 + 2\beta_0(1+\kappa^2)u_1^3,$$

from which we get

$$V = \mu - 2\beta_0(1+\kappa^2)u_1^2.$$

And hence V is also a constant. $\square$

Chapter 4

Some Further Properties

Now that we have proved the two simplified characterizations in the situation without external magnetic field, in the remaining of this thesis (except Chapter 8) we shall, more or less, focus on the bifurcation phenomenon. For convenience, we will thus only consider $\beta_1 > 0$, despite the fact that some assertions hold also for $\beta_1 < 0$. Also, we will, of course, not always assume $q = 0$.

In this chapter, we use the notations $\mathbb{A}_M$, $\mathbb{G}_{M,q}$ and $E_g(M, q)$ to specify the values of M and q. We will establish some more results for elements in $\mathbb{G}_{M,q}$. Most of the results are directly relevant to the proof of the bifurcation phenomenon. Some of them however are just given for completeness or serving as illustrations of using the redistribution technique.

4.1 Continuity and monotonicity of $E_g(M, q)$

In this section we prove that $E_g(M, q)$ is continuous and increasing in each variable. Since the two variables are of quite different natures, we treat them separately.

4.1.1 E_g as a function of M

In this subsection we fix a $q \in [0, \infty)$ and consider $E_g(\cdot, q)$. The proof of continuity will rely on the monotonicity, and hence we prove the latter first. For this we need the following lemma.

Lemma 4.1. *$\mathcal{E}$ is bounded on* $\cup_{0 \leq M \leq 1} \mathbb{G}_{M,q}$.

Proof. The assertion is equivalent to say that we can choose for every $M \in [0,1]$ an $\mathbf{f}^M \in \mathbb{A}_M$, such that $\mathcal{E}[\mathbf{f}^M]$ is uniformly bounded in M. This is easy to do. For example, let f be any nonnegative function in $H^1 \cap L_V^2 \cap L^4$ such that $\int f^2 = 1$. Then for each $M \in [0,1]$, let $\mathbf{f}^M = ((\frac{1+M}{2})^{1/2} f, 0, (\frac{1-M}{2})^{1/2} f)$. We have $\mathbf{f}^M \in \mathbb{A}_M$ and

$$\mathcal{E}[\mathbf{f}^M] = \int \Big\{ |\nabla f|^2 + V f^2 + \beta_0 f^4 + \beta_1 M^2 f^4 + q \Big\},$$

which is bounded above by the finite number $\mathcal{E}[\mathbf{f}^1]$. □

Proposition 4.2. *$E_g(\cdot, q)$ is strictly increasing on $[0,1]$.*

Proof. Let $\mathbf{u} \in \mathbb{G}_{M,q}$. We first consider $0 < M \leq 1$. For small $\delta \geq 0$, let $\mathbf{u}(\delta)$ be the redistribution of $\mathbf{u}$ defined by

$$\begin{cases} u_1(\delta)^2 = (1-\delta) u_1^2 \\ u_0(\delta)^2 = u_0^2 + \delta u_1^2 + \delta u_{-1}^2 \\ u_{-1}(\delta)^2 = (1-\delta) u_{-1}^2 \,. \end{cases}$$

Then $\mathbf{u}(\delta) \in \mathbb{A}_{(1-2\delta)M}$. Since $\mathbf{u}(\delta)$ is a redistribution of $\mathbf{u}$, $\mathcal{E}_{kin}[\mathbf{u}(\delta)] \leq \mathcal{E}_{kin}[\mathbf{u}]$. One can also check by direct computation that

$$\mathcal{E}_{Zee}[\mathbf{u}] - \mathcal{E}_{Zee}[\mathbf{u}(\delta)] = q\delta \int (u_1^2 + u_{-1}^2) \geq 0,$$

and

$$\mathcal{E}_1[\mathbf{u}] - \mathcal{E}_1[\mathbf{u}(\delta)] = \beta_1 \delta \int (u_1 - u_{-1})^2 \big[2u_0^2 + 4u_1 u_{-1} + \delta (u_1 - u_{-1})^2 \big] \geq 0. \quad (4.1)$$

Moreover, if $\delta > 0$, strict inequality holds in (4.1). To see this, for $0 < M < 1$, note that $u_1 u_{-1} > 0$ (Corollary 2.3) and that $(u_1 - u_{-1})^2$ can not be identically zero (otherwise $M = 0$). While for $M = 1$, only $u_1 > 0$, and the positivity of (4.1) is obvious. Thus we obtain

$$E_g((1-2\delta)M, q) \leq \mathcal{E}[\mathbf{u}(\delta)] < \mathcal{E}[\mathbf{u}] = E_g(M, q)$$

for each small $\delta > 0$, which shows $E_g(\cdot, q)$ is strictly increasing on $(0, 1]$.

It remains to show that $E_g(\cdot, q)$ is strictly increasing at 0. Let $\{M_n\}$ be a sequence in $(0, 1)$ such that $M_n \to 0^+$. And let $\mathbf{u}^n \in \mathbb{G}_{M_n,q}$ for each n. By Lemma 4.1, $\mathcal{E}[\mathbf{u}^n]$ is uniformly bounded, and hence Lemma 2.2 implies there is a subsequence $\{\mathbf{u}^{n(k)}\}$ of $\{\mathbf{u}^n\}$ such that $\mathbf{u}^{n(k)} \rightharpoonup \mathbf{u}^\infty$ weakly in $\mathbb{B}$ for some $\mathbf{u}^\infty \in \mathbb{A}_0$. Moreover,

$$E_g(0, q) \leq \mathcal{E}[\mathbf{u}^\infty] \leq \liminf_{k\to\infty} \mathcal{E}[\mathbf{u}^{n(k)}] = \liminf_{k\to\infty} E_g(M_{n(k)}, q) = \inf_{0<M\leq 1} E_g(M, q).$$

The last equality is due to the just proved monotonicity of $E_g(\cdot, q)$ on $(0, 1]$. Thus $E_g(0, q) \leq E_g(M, q)$ for every $M \in (0, 1]$. To see why strict inequality must hold, assume $E_g(0, q) = E_g(M, q)$ for some $M > 0$. Then since $E_g(\cdot, q)$ is strictly increasing on $(0, 1]$, we have $E_g(M/2, q) < E_g(0, q)$, a contradiction. $\square$

Proposition 4.3. $E_g(\cdot, q)$ *is continuous on* $[0, 1]$.

Proof. The ideas of proving the left continuity and the right continuity are different. We first prove the right continuity. Let $\mathbf{u} \in \mathbb{G}_{M,q}$ for some $0 \leq M < 1$. For small $\delta \geq 0$, let $\mathbf{u}(\delta)$ be the redistribution of $\mathbf{u}$ defined by

$$\begin{cases} u_1(\delta)^2 = u_1^2 + \delta u_0^2 + \delta u_{-1}^2 \\ u_0(\delta)^2 = (1-\delta)u_0^2 \\ u_{-1}(\delta)^2 = (1-\delta)u_{-1}^2 \,. \end{cases}$$

Let's use M_δ to denote $\mathcal{M}[\mathbf{u}(\delta)]$. Then $M_\delta = M + \delta \int (u_0^2 + 2u_{-1}^2)$. Since $M < 1$, u_0 and u_{-1} cannot both vanish, and hence $M_\delta > M$ for $\delta > 0$. Obviously $M_\delta \to M^+$ as $\delta \to 0^+$. Now since $E_g(\cdot, q)$ is strictly increasing, we have

$$0 < E_g(M_\delta, q) - E_g(M, q) \tag{4.2}$$

for $\delta > 0$. On the other hand, since $\mathbf{u} \in \mathbb{G}_{M,q}$ while $\mathbf{u}(\delta)$ need not lie in $\mathbb{G}_{M_\delta,q}$, we have $E_g(M_\delta, q) - E_g(M, q) \leq \mathcal{E}[\mathbf{u}(\delta)] - \mathcal{E}[\mathbf{u}]$. Thus

$$E_g(M_\delta, q) - E_g(M, q) \leq (\mathcal{E}_1[\mathbf{u}(\delta)] - \mathcal{E}_1[\mathbf{u}]) + (\mathcal{E}_{Zee}[\mathbf{u}(\delta)] - \mathcal{E}_{Zee}[\mathbf{u}]) \tag{4.3}$$

from (2.5) and (2.6). It's easy to check that the right-hand side of (4.3) tends to zero as $\delta \to 0^+$, and hence we obtain

$$\limsup_{\delta\to 0^+}(E_g(M_\delta, q) - E_g(M, q)) \le 0.$$

This together with (4.2) imply the right continuity of $E_g(\cdot, q)$ on $[0, 1)$.

For the left-continuity on $(0, 1]$, we prove by contradiction. Let $M \in (0, 1]$. Assume there is a sequence $\{M_n\}$ in $(0, 1)$ such that $M_n \to M^-$, and $E_g(M_n, q)$ doesn't converge to $E_g(M, q)$. By choosing a suitable subsequence, we can assume without loss of generality that the sequence $\{M_n\}$ itself satisfies

$$E_g(M, q) - E_g(M_n, q) > \varepsilon \quad \text{for each } n, \text{ for some } \varepsilon > 0.$$

Now for each n choose one $\mathbf{u}^n \in \mathbb{G}_{M_n,q}$. Lemma 2.2 implies that there is a subsequence $\{\mathbf{u}^{n(k)}\}_{k=1}^\infty$ such that $\mathbf{u}^{n(k)} \to \mathbf{u}^\infty$ for some $\mathbf{u}^\infty \in \mathbb{A}_M$. Moreover, we have

$$E_g(M, q) \le \mathcal{E}[\mathbf{u}^\infty] \le \liminf_{k\to\infty} \mathcal{E}[\mathbf{u}^{n(k)}] = \liminf_{k\to\infty} E_g(M_{n(k)}, q) \le E_g(M, q) - \varepsilon,$$

a contradiction. ☐

Proposition 4.3 implies the following approximation result.

Corollary 4.4. *For any* $M \in [0, 1]$*, we can find a sequence* $\mathbf{u}^n \in \mathbb{G}_{M_n,q}$ *such that* $M_n \in [0, 1]$*,* $M_n \to M$*,* $M_n \neq M$ *for each* n*, and* $\mathbf{u}^n \to \mathbf{u}^\infty$ *in the norm of* $\mathbb{B}$ *for some* $\mathbf{u}^\infty \in \mathbb{G}_{M,q}$.

Proof. Let $\{M_n\}$ be a sequence in $[0, 1]$ such that $M_n \to M$ and $M_n \neq M$ for each n. Then let $\mathbf{u}^n \in \mathbb{G}_{M_n,q}$ for each n. By definition we have $\mathcal{N}[\mathbf{u}^n] = 1$ and $\mathcal{M}[\mathbf{u}^n] \to M$. Since $\mathcal{E}[\mathbf{u}^n] = E_g(M_n, q)$, by continuity of $E_g(\cdot, q)$ we also have $\mathcal{E}[\mathbf{u}^n] \to E_g(M, q)$. Thus by Lemma 2.2, $\{\mathbf{u}^n\}$ has a subsequence $\{\mathbf{u}^{n(k)}\}_{k=1}^\infty$ such that $\mathbf{u}^{n(k)} \to \mathbf{u}^\infty$ strongly in $\mathbb{B}$ for some $\mathbf{u}^\infty \in \mathbb{G}_{M,q}$. The sequence $\mathbf{u}^{n(k)} \in \mathbb{G}_{M_{n(k)},q}$ thus satisfies the assertion to be proved. ☐

Remark 4.1. Suppose we have uniqueness of element in $\mathbb{G}_{M,q}$, and let $\mathbf{u}^M$ be the unique element in $\mathbb{G}_{M,q}$. Then Corollary 4.4 simply says the map $M \mapsto \mathbf{u}^M$ is continuous from $[0,1]$ into $\mathbb{B}$. In particular, let's here write $\mathbf{z}^M$ for $\mathbf{z}$ to specify the dependence on M explicitly. Then we have the corollary that $M \mapsto \mathbf{z}^M$ is continuous from $[0,1]$ into $\mathbb{B}$. (We'll use this fact in §5.2.2.) To be rigorous, this is true since $\mathbf{z}^M$ is the unique element in $\mathbb{G}_{M,0}$ for $0 < M \leq 1$, and as $M \to 0^+$, the limit of $\mathbf{z}^M$ in $\mathbb{B}$, which should lie in $\mathbb{G}_{0,0}$ by Corollary 4.4, must be $\mathbf{z}^0$. Of course, we might as well just prove the analogue of Corollary 4.4 for the "two-component world", and the continuity of $M \mapsto \mathbf{z}^M$ follows Theorem 2.4 directly.

4.1.2 E_g as a function of q

Now we consider the function $E_g(M,\cdot)$ for fixed $M \in [0,1]$. For $\mathbf{u} \in \mathbb{B}_+$, let's here write $\mathcal{E}[\mathbf{u},q]$ instead of $\mathcal{E}[\mathbf{u}]$ to indicate the value of q. The proofs of monotonicity and continuity of $E_g(M,\cdot)$ are much easier than those of $E_g(\cdot,q)$ above, and the proof of continuity doesn't rely on the monotonicity. We put the assertions in a single proposition.

Proposition 4.5. *For fixed $M \in [0,1]$, $E_g(M,\cdot)$ is an increasing and continuous function on $[0,\infty)$. Moreover, it's strictly increasing if $M > 0$.*

Proof. Let $q_1 > q_2 \geq 0$ and $\mathbf{u} \in \mathbb{G}_{M,q_1}$. We have

$$\begin{aligned} E_g(M,q_1) - E_g(M,q_2) &\geq \mathcal{E}[\mathbf{u},q_1] - \mathcal{E}[\mathbf{u},q_2] \\ &= (q_1 - q_2)\int (u_1^2 + u_{-1}^2) \geq 0, \end{aligned} \tag{4.4}$$

which implies $E_g(M,\cdot)$ is an increasing function on $[0,\infty)$. If $M > 0$, we have $u_1 > 0$, and hence the last inequality in (4.4) is strict, which proves the strict monotonicity .

We next prove the continuity. Given any $q_1, q_2 \geq 0$, let $\mathbf{u}^k = (u_1^k, u_0^k, u_{-1}^k) \in \mathbb{G}_{M,q_k}$ for $k = 1,2$. Since $\mathcal{E}[\mathbf{u}^1,q_1] = E_g(M,q_1)$ and $\mathcal{E}[\mathbf{u}^1,q_2] \geq E_g(M,q_2)$, we have

$$\begin{aligned} (q_1 - q_2)\int \left((u_1^1)^2 + (u_{-1}^1)^2\right) &= \mathcal{E}[\mathbf{u}^1,q_1] - \mathcal{E}[\mathbf{u}^1,q_2] \\ &\leq E_g(M,q_1) - E_g(M,q_2). \end{aligned} \tag{4.5}$$

Similarly,

$$\begin{aligned} E_g(M,q_1) - E_g(M,q_2) &\leq \mathcal{E}[\mathbf{u}^2, q_1] - \mathcal{E}[\mathbf{u}^2, q_2] \\ &= (q_1 - q_2) \int \left((u_1^2)^2 + (u_{-1}^2)^2\right). \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), and the fact $\int \left((u_1^k)^2 + (u_{-1}^k)^2\right) \leq \mathcal{N}[\mathbf{u}^k] = 1$ for $k = 1, 2$, we find

$$|E_g(M,q_1) - E_g(M,q_2)| \leq |q_1 - q_2|,$$

and hence $E_g(M,\cdot)$ is continuous. $\square$

Remark 4.2. $E_g(0,\cdot)$ is not strictly increasing. Indeed, by Proposition (4.7) below, for $q > 0$, $\mathbf{u} \in \mathbb{G}_{0,q}$ satisfies $u_1 = u_{-1} = 0$. Such one-component ground state, as the two-component $\mathbf{z}$, is unique and independent of q. This is easily obtained by imitating the proof of Theorem 2.4). Thus $E_g(0,\cdot)$ is a constant function on $(0,\infty)$, and hence on $[0,\infty)$ by continuity.

With the continuity of $E_g(M,\cdot)$, we can show the following analogue of Corollary 4.4. The proof is the same as that of Corollary 4.4 by changing the roles of M and q, and hence we omit it.

Corollary 4.6. *For any $q \in [0,\infty)$, there is a sequence $\mathbf{u}^n \in \mathbb{G}_{M,q_n}$ such that $q_n \in [0,\infty)$, $q_n \to q$, $q_n \neq q$ for each n, and $\mathbf{u}^n \to \mathbf{u}^\infty$ in the norm of $\mathbb{B}$ for some $\mathbf{u}^\infty \in \mathbb{G}_{M,q}$.*

4.2 u_{-1} is no larger than u_1

The goal in this section is indicated by the title. The relevant assertions are Proposition 4.7 (and Remark 4.3 following it), Proposition 4.8, and Proposition 4.11.

Proposition 4.7. *Suppose $q > 0$ and $\mathbf{u} \in \mathbb{G}_{0,q}$ (i.e. $M = 0$). We have $u_1 = u_{-1} = 0$.*

Proof. Let $\mathbf{v} = (v_1, v_0, v_{-1})$ be the element in $\mathbb{A}_0$ defined by

$$\begin{cases} v_1^2 = v_{-1}^2 = (u_1^2 + u_{-1}^2)/2 \\ v_0^2 = u_0^2\,. \end{cases}$$

Then $\mathcal{E}[\mathbf{u}] - \mathcal{E}[\mathbf{v}] = (\mathcal{E}_{kin}[\mathbf{u}] - \mathcal{E}_{kin}[\mathbf{v}]) + \mathcal{E}_1[\mathbf{u}]$. Since $\mathbf{v}$ is a redistribution of $\mathbf{u}$, $\mathcal{E}_{kin}[\mathbf{u}] - \mathcal{E}_{kin}[\mathbf{v}] \geq 0$. Also, $\mathcal{E}_1[\mathbf{u}] \geq 0$, and hence $\mathcal{E}[\mathbf{u}] - \mathcal{E}[\mathbf{v}] \geq 0$. Nevertheless, $\mathbf{u} \in \mathbb{G}_{0,q}$, so we must have $\mathcal{E}[\mathbf{u}] - \mathcal{E}[\mathbf{v}] = 0$. Thus actually $\mathcal{E}_{kin}[\mathbf{u}] - \mathcal{E}_{kin}[\mathbf{v}] = \mathcal{E}_1[\mathbf{u}] = 0$. In particular the term $(u_1^2 - u_{-1}^2)^2$ in $H_1(\mathbf{u})$ is zero, which implies $u_1 = u_{-1}$. To see why they must vanish, note that now we have

$$\begin{aligned}\mathcal{E}[\mathbf{u}] &= \int \Big\{ \sum_j |\nabla u_j|^2 + V(x)|\mathbf{u}|^2 + \beta_0 |\mathbf{u}|^4 + q(u_1^2 + u_{-1}^2) \Big\} \\ &\geq \int \Big\{ |\nabla |\mathbf{u}||^2 + V(x)|\mathbf{u}|^2 + \beta_0 |\mathbf{u}|^4 \Big\} = \mathcal{E}[(0, |\mathbf{u}|, 0)].\end{aligned} \tag{4.7}$$

Again since $\mathbf{u} \in \mathbb{G}_{0,q}$ and $(0, |\mathbf{u}|, 0) \in \mathbb{A}_0$, we must have $\mathcal{E}[\mathbf{u}] = \mathcal{E}[(0, |\mathbf{u}|, 0)]$. Thus the inequality in (4.7) is equality, which implies $\mathbf{u} = (0, |\mathbf{u}|, 0)$ since $\sum_j |\nabla u_j|^2 \geq |\nabla |\mathbf{u}||^2$ and $q > 0$. □

Remark 4.3. From Theorem 3.4, for $M = q = 0$, we also have $u_1 = u_{-1}$, while $u_1 = u_{-1} = 0$ corresponds to only one possibility. This together with Proposition 4.7 provide satisfactory descriptions of the degenerate situation $M = 0$. Also, on the other extreme $M = 1$, only $u_1 > 0$. Therefore, there is no need to consider the bifurcation phenomenon for $M = 0, 1$.

Proposition 4.8. *For every* $0 \leq M \leq 1$ *and* $q \geq 0$, $\mathbf{u} \in \mathbb{G}_{M,q}$ *satisfies* $u_{-1} \leq u_1$.

Proof. Let $\mathbf{v}$ be defined by $v_1 = \max(u_1, u_{-1})$, $v_{-1} = \min(u_1, u_{-1})$, and $v_0 = u_0$. Then we have $\mathcal{E}[\mathbf{v}] = \mathcal{E}[\mathbf{u}]$. To check this equality, for the kinetic part $\mathcal{E}_{kin}$ one can use the formula

$$v_j = \frac{1}{2}\left(u_j + u_{-j} + j|u_j - u_{-j}|\right)$$

for $j = 1, -1$. Then direct computation gives

$$|\nabla v_1|^2 + |\nabla v_{-1}|^2 = \frac{1}{2}\Big\{ |\nabla u_1|^2 + |\nabla u_{-1}|^2 + 2\nabla u_1 \cdot \nabla u_{-1} + 2\big|\nabla |u_1 - u_{-1}|\big|^2 \Big\}.$$

And $|\nabla v_1|^2 + |\nabla v_{-1}|^2 = |\nabla u_1|^2 + |\nabla u_{-1}|^2$ is obtained by applying the fact

$$|\nabla |f||^2 = |\nabla f|^2 \text{ a.e. for every } f \text{ of class } H^1.$$

The equalities of the other parts are obvious. Thus, we have

$$E_g(\mathcal{M}[\mathbf{v}], q) \le \mathcal{E}[\mathbf{v}] = \mathcal{E}[\mathbf{u}] = E_g(M, q).$$

Since $E_g(\cdot, q)$ is strictly increasing, we thus obtain

$$\mathcal{M}[\mathbf{v}] \le M. \tag{4.8}$$

On the other hand, it's also obvious by definition that

$$v_1^2 - v_{-1}^2 \ge u_1^2 - u_{-1}^2. \tag{4.9}$$

(4.8) and (4.9) imply $v_1^2 - v_{-1}^2 = u_1^2 - u_{-1}^2$, that is $v_1^2 - u_1^2 = v_{-1}^2 - u_{-1}^2$, of which the left-hand side is nonnegative while the right-hand side is nonpositive by definition of $\mathbf{v}$. Thus we really have $v_1 = u_1$ and $v_{-1} = u_{-1}$, which means $u_{-1} \le u_1$. □

Proposition 4.8 can be used to improve itself. Precisely, we shall prove that strict inequality $u_{-1} < u_1$ holds when $M > 0$, by using the strong maximum principle. In doing so, the knowledge of the non-strict inequality itself is needed.

Lemma 4.9. *Let* $\mathbf{f} \in L^1(\mathbb{R}^3, \mathbb{R}^3)$ *be such that the distributional divergence* $\nabla \cdot \mathbf{f} \in L^1(\mathbb{R}^3)$. *Then* $\int \nabla \cdot \mathbf{f} = 0$.

Proof. For $R > 0$, let $\varphi_R : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\varphi_R(x) = \begin{cases} 1, & |x| < R \\ R + 1 - |x|, & R \le |x| < R + 1 \\ 0, & R + 1 \le |x|. \end{cases}$$

Then it's obvious that

$$\lim_{R\to\infty} \int (\nabla \cdot \mathbf{f})\, \varphi_R = \int \nabla \cdot \mathbf{f}.$$

On the other hand,

$$\int (\nabla \cdot \mathbf{f})\, \varphi_R = -\int_{R \le |x| < R+1} \mathbf{f}(x) \cdot \mathbf{n}(x),$$

where $\mathbf{n}(x) = x/|x|$. Thus $\int (\nabla \cdot \mathbf{f})\, \varphi_R \to 0$ as $R \to \infty$, which proves the assertion.

□

Corollary 4.10. *Let* $\mathbf{u} \in \mathbb{G}_{M,q}$*. If* $0 < M < 1$*, the Lagrange multiplier* λ *in the GP system (2.4) is positive.*

Proof. (2.4a) multiplied by u_{-1} minus (2.4c) multiplied by u_1 gives

$$2\lambda u_1 u_{-1} = \nabla \cdot (-u_{-1}\nabla u_1 + u_1 \nabla u_{-1}) + 2\beta_1 (u_1^2 - u_{-1}^2)(u_0^2 + 2u_1 u_{-1}).$$

By Lemma 4.9, $\int \nabla \cdot (-u_{-1}\nabla u_1 + u_1 \nabla u_{-1}) = 0$, and hence

$$\lambda \int u_1 u_{-1} = \beta_1 \int (u_1^2 - u_{-1}^2)(u_0^2 + 2u_1 u_{-1}). \tag{4.10}$$

Now $u_1 u_{-1} > 0$ by Corollary 2.3, and hence $\int u_1 u_{-1} > 0$. On the other hand, by Proposition 4.8 we have $u_1^2 - u_{-1}^2 \geq 0$, which cannot be identically zero since $M > 0$. Thus we also have $\int (u_1^2 - u_{-1}^2)(u_0^2 + 2u_1 u_{-1}) > 0$, and (4.10) implies $\lambda > 0$. ☐

Proposition 4.11. *For* $0 < M \leq 1$ *and* $q \geq 0$*,* $\mathbf{u} \in \mathbb{G}_{M,q}$ *satisfies* $u_{-1} < u_1$*.*

Proof. If $M = 1$, we have $u_1 > 0 \equiv u_{-1}$. For $0 < M < 1$, let $w = u_1 - u_{-1}$. Then (2.4a) minus (2.4c) gives

$$\Delta w + Qw = -\lambda(u_1 + u_{-1}) - \mu w, \tag{4.11}$$

where

$$Q = -V - 2\beta_0 |\mathbf{u}|^2 - 2\beta_1 \left[2u_0^2 + (u_1 + u_{-1})^2\right] - q.$$

Since $\lambda > 0$ and $w \geq 0$, by subtracting $|\mu| w$ from both sides of (4.11), we obtain $\Delta w + \widetilde{Q} w \leq 0$, where $\widetilde{Q} = Q - |\mu|$ is locally bounded. By Corollary 8.11, either $w > 0$ everywhere or $w \equiv 0$. But $w \equiv 0$ means $u_1 = u_{-1}$, contradicting to the assumption $M > 0$. Thus $w > 0$, which is what we want to show. ☐

Remark 4.4. The subtraction of $|\mu| w$ in the proof above is indeed not necessary since we also have $\mu > 0$ for $0 < M < 1$. This is easy to obtain by using (2.4b) when $u_0 > 0$, and by using (2.4c) when $u_0 \equiv 0$. We omit the details.

Recall the definition of **z** from Theorem 2.4. Since $\mathbf{z} \in \mathbb{G}_{M,0}$ (for any $0 \leq M \leq 1$), we have the following corollary.

Corollary 4.12. *For* $0 < M \leq 1$, $z_{-1} < z_1$.

Remark 4.5. Although $\mathbf{z}$ is independent of q, it's dependent on M. To be precise we shall sometimes write $\mathbf{z}^M = (z_1^M, 0, z_{-1}^M)$ to specify this dependence. For notational simplicity, we will however not do so when such explicitness is not really necessary.

4.3 Exponential decay of ground states

In this section we prove the exponential decay of ground states with the aid of Proposition 4.8, The approach of using the fundamental solution of Helmholtz equation is exactly taken from [14], Lemma A.5.

Proposition 4.13. *Let* $\mathbf{u} \in \mathbb{G}_{M,q}$, *for arbitrary* $0 \leq M \leq 1$ *and* $q \geq 0$. *For any* $a > 0$, *there exist constants* $U_j(a)$ $(j = 1, 0, -1)$ *such that* $u_j(x) \leq U_j(a)e^{-a|x|}$.

Proof. (2.4b) can be arranged as $(-\Delta + a^2)u_0 = Q_0 u_0$, where

$$Q_0 = a^2 + \mu - V - 2\beta_0|\mathbf{u}|^2 - 2\beta_1(u_1 - u_{-1})^2. \tag{4.12}$$

Thus

$$u_0(x) = (Y_a * (Q_0 u_0))(x) = \int Y_a(x-y)Q_0(y)u_0(y)dy,$$

where $Y_a(x) = e^{-a|x|}/(4\pi|x|)$ is the fundamental solution of the operator $-\Delta + a^2$. (Y_a is also referred to as the Yukawa potential. See [13], 6.23.) By the assumption (A1), $Q_0 < 0$ outside a bounded set, say $B(R_0)$, the open ball centered at the origin with radius R_0. Thus we obtain

$$u_0(x) \leq \int_{|y|<R_0} Y_a(x-y)Q_0(y)u_0(y)dy = e^{-a|x|}\int_{|y|<R_0} \frac{e^{a(|x|-|x-y|)}}{4\pi|x-y|}Q_0(y)u_0(y)dy.$$

Thus $u_0(x) \leq U_0(a)e^{-a|x|}$, where (see also Lemma 4.14 below)

$$U_0(a) = \sup_{x\in\mathbb{R}^3}\int_{|y|<R_0} \frac{e^{a(|x|-|x-y|)}}{4\pi|x-y|}Q_0(y)u_0(y)dy < \infty. \tag{4.13}$$

For u_j, $j = 1, -1$, we similarly have

$$(-\Delta + a^2)u_j = Q_j u_j - 2\beta_1 u_0^2(u_j - u_{-j})$$

from (2.4a) and (2.4c), where

$$Q_j = a^2 + \mu + j\lambda - V - 2\beta_0|\mathbf{u}|^2 - 2\beta_1(u_j^2 - u_{-j}^2) - q.$$

Now since $u_{-1} \leq u_1$, Q_1 is also negative outside $B(R_1)$ for some radius R_1, and

$$\begin{aligned} u_1(x) &= \int Y_a(x-y)\big[Q_1(y)u_1(y) - 2\beta_1 u_0(y)^2(u_1(y) - u_{-1}(y))\big]dy \\ &\leq \int Y_a(x-y)Q_1(y)u_1(y)dy \\ &\leq \int_{|y|<R_1} Y_a(x-y)Q_1(y)u_1(y)dy. \end{aligned}$$

As above we conclude that $u_1(x) \leq U_1(a)e^{-a|x|}$, where $U_1(a)$ is given by (4.13) with all the indices 0 replaced by 1. In contrast, the fact $u_{-1} \leq u_1$ makes it difficult to apply the same argument to u_{-1}. Nevertheless, also since $u_{-1} \leq u_1$, at least we can choose $U_{-1}(a) = U_1(a)$. □

For our next result, we give the following estimate of $U_j(a)$.

Lemma 4.14. *For $j = 1$ and 0,*

$$U_j(a) \leq \frac{e^{aR_j}}{4\pi} \sup_{x\in\mathbb{R}^3} \left(\int_{|y|<R_j} \frac{Q_j(y)^2}{|x-y|^2} dy \right)^{1/2}.$$

Proof. Since $|x| - |x-y| \leq |y|$, we have for $j = 1, 0$

$$\begin{aligned} \int_{|y|<R_j} \frac{e^{a(|x|-|x-y|)}}{4\pi|x-y|} Q_j(y)u_j(y)dy &\leq \int_{|y|<R_j} \frac{e^{aR_j}}{4\pi|x-y|} Q_j(y)u_j(y)dy \\ &= \frac{e^{aR_j}}{4\pi} \int_{|y|<R_j} \frac{Q_j(y)}{|x-y|} u_j(y)dy \\ &\leq \frac{e^{aR_j}}{4\pi} \left(\int_{|y|<R_j} \frac{Q_j(y)^2}{|x-y|^2} dy \right)^{1/2}, \end{aligned}$$

where the last inequality is obtained by Hölder's inequality and the fact

$$\int (u_j)^2 \leq \int |\mathbf{u}|^2 = 1.$$

We thus obtain the assertion of the lemma. □

The assertion of exponential decay is indeed far stronger than what we need. On the other hand, we will consider sequences $\{\mathbf{u}^n\}$ of ground states corresponding to different values of q, and hence different Lagrange multipliers, where estimates independent of n are required. We give what we really need in the following.

Corollary 4.15. *Given a sequence* $\mathbf{u}^n = (u_1^n, u_0^n, u_{-1}^n) \in \mathbb{G}_{M,q_n}$. *Let* μ_n *and* λ_n *be the Lagrange multipliers corresponding to* $\mathbf{u}^n$. *If the sequences* $\{\mu_n\}$ *and* $\{\lambda_n\}$ *are both bounded, then for any* $\varepsilon > 0$*, there is* $r_j > 0$ *(*$j = 1, 0, -1$*) independent of* n *such that* $u_j(x) \leq \varepsilon$ *for* $|x| \geq r_j$.

Proof. The assertion is easily seen by repeating the proof of Proposition 4.13 for $\mathbf{u}^n$. It suffices to give $U_j^n(a)$ (the analogue of $U_j(a)$ for $\mathbf{u}^n$) an upper bound independent of n. Take $U_0^n(a)$ for example. By assumption, there is $c > 0$ such that $\mu_n < c$ for every n. From (4.12) we have

$$\begin{aligned} Q_0^n &= a^2 + \mu_n - V - 2\beta_0 |\mathbf{u}^n|^2 - 2\beta_1 (u_1^n - u_{-1}^n)^2 \\ &\leq a^2 + \mu_n - V \\ &< a^2 + c - V. \end{aligned}$$

Hence we can find R_0 independent of n so that $Q_0^n < 0$ outside $B(R_0)$. Then, by Lemma 4.14, we have

$$\begin{aligned} U_0^n(a) &\leq \frac{e^{aR_0}}{4\pi} \sup_{x\in\mathbb{R}^3} \left(\int_{|y|<R_0} \frac{Q_0^n(y)^2}{|x-y|^2} dy \right)^{\frac{1}{2}} \\ &\leq \frac{e^{aR_0}}{4\pi} \sup_{x\in\mathbb{R}^3} \left(\int_{|y|<R_0} \frac{(a^2 + c - V(y))^2}{|x-y|^2} dy \right)^{\frac{1}{2}}, \end{aligned}$$

which is independent of n. $U_1^n(a)$ can be estimated similarly, and again for $U_{-1}^n(a)$ we use the fact $U_{-1}^n(a) \leq U_1^n(a)$. □

Chapter 5

The Bifurcation Phenomenon

We begin our proof of the bifurcation phenomenon. According to Remark 4.3, it suffices to consider $0 < M < 1$.

Our main theorem is the following.

Theorem 5.1. *For fixed* $0 < M < 1$*, there is a* $q_c > 0$ *such that for* $q > q_c$*,* $\mathbf{u} \in \mathbb{G}_{M,q}$ *satisfies* $u_0 > 0$*, while for* $0 \leq q < q_c$*,* $\mathbf{z}$ *is the unique element in* $\mathbb{G}_{M,q}$*.*

Remark 5.1. We do not know what happens at the critical q_c. Since E_g is continuous with respect to q, $\mathbf{z}$ is of course an element in $\mathbb{G}_{M,q_c}$. However, since we do not prove the uniqueness of ground state, we are not sure if it's possible that there are other three-component ground states at q_c. We give more detailed discussions in §7.1.

The proof idea of Theorem 5.1 is to use (2.7) to derive some conditions on the situation *to be excluded*. More precisely, to prove that $\mathbf{u} \in \mathbb{G}_{M,q}$ cannot have some property, we assume the opposite, then exploit the fact that any redistribution $\mathbf{v} \in \mathbb{A}_M$ of $\mathbf{u}$ (in particular those not having the property) satisfy (2.7). How this idea works will be clear in the proof.

We regard M as a fixed number in $(0, 1)$ in the following. The proof is divided into three claims.

Claim 1. *For* q *large enough,* $\mathbf{z}$ *is not an element in* $\mathbb{G}_{M,q}$*.*

Proof. Assume $\mathbf{z} \in \mathbb{G}_{M,q}$ for some M, q. Since $\mathbf{z}$ is independent of q, it's quite easy to prove the claim by (2.7). For example, consider $\mathbf{v}$ to be the redistribution of $\mathbf{z}$ defined by

$$\begin{cases} v_1^2 = (1-\sigma) z_1^2 \\ v_0^2 = \sigma z_1^2 + z_{-1}^2 \\ v_{-1}^2 = 0\,, \end{cases} \tag{5.1}$$

where $\sigma = (1-M)/(1+M)$, which is just the constant making $\mathbf{v} \in \mathbb{A}_M$. Then (2.7) implies

$$\mathcal{E}_{Zee}[\mathbf{z}] - \mathcal{E}_{Zee}[\mathbf{v}] \leq \mathcal{E}_1[\mathbf{v}] - \mathcal{E}_1[\mathbf{z}]. \tag{5.2}$$

It's easy to check that the left-hand side of (5.2) equals $(1-M)q$. In contrast, the right-hand side, no matter what it is, is independent of q. Thus, since $M < 1$, (5.2) gives an upper bound of q. That is there is an upper bound of q for $\mathbf{z}$ to be in $\mathbb{G}_{M,q}$. □

Since three-component elements in $\mathbb{G}_{M,q}$ in general depend on q, it's far more difficult to prove that there is a positive lower bound of q for the existence of $\mathbf{u} \in \mathbb{G}_{M,q}$ with $u_0 > 0$. We leave this to the last claim. We shall first give the observation that the two-component regime and the three-component regime are really separated by a specific q_c. That is, we exclude by the next claim the possibility that two-component and three-component ground states will alternately be the case (in any range on the q-axis).

Claim 2. *Assume for some q there exists $\mathbf{u} \in \mathbb{G}_{M,q}$ with $u_0 > 0$, then for every $q' > q$, $\mathbf{z} \notin \mathbb{G}_{M,q'}$.*

Proof. Let's here write $\mathcal{E}[\mathbf{u}, q]$ instead of $\mathcal{E}[\mathbf{u}]$ to specify the value of q. Since $\mathbf{u} \in \mathbb{G}_{M,q}$, $\mathcal{E}[\mathbf{u}, q] \leq \mathcal{E}[\mathbf{z}, q]$. Thus, by the assumption $u_0 > 0$, for $q' > q$ we have

$$\begin{aligned} \mathcal{E}[\mathbf{u}, q'] &= \mathcal{E}[\mathbf{u}, q] + (q'-q) \int \left(u_1^2 + u_{-1}^2 \right) \\ &< \mathcal{E}[\mathbf{z}, q] + (q'-q) \int \left(z_1^2 + z_{-1}^2 \right) = \mathcal{E}[\mathbf{z}, q']. \end{aligned}$$

Hence $\mathbf{z} \notin \mathbb{G}_{M,q'}$. □

Now define

$$q_c = \inf\left\{q \;\middle|\; \mathbf{z} \notin \mathbb{G}_{M,q'} \text{ for } q' > q\right\}.$$

From Claim 1, $q_c < \infty$. By definition of q_c, for any $q > q_c$ and $\mathbf{v} \in \mathbb{G}_{M,q}$, we have $v_0 > 0$. Moreover, Claim 2 implies that for any $0 \leq q < q_c$, $\mathbf{z}$ is the unique element in $\mathbb{G}_{M,q}$. To complete the proof of Theorem 5.1, it remains to show $q_c > 0$. That is we have to prove the following assertion.

Claim 3. *There exists $q > 0$ such that $\mathbf{z} \in \mathbb{G}_{M,q}$.*

Since the proof of Claim 3 requires much more effort, we give it in a separate section.

5.1 Proof of Claim 3

Let $\mathbf{u} \in \mathbb{G}_{M,q}$. To give a restriction on the presence of u_0, we consider the redistribution $\mathbf{v} \in \mathbb{A}_M$ of $\mathbf{u}$ defined by

$$\begin{cases} v_1^2 = u_1^2 + \dfrac{1}{2}u_0^2 \\ v_0^2 = 0 \\ v_{-1}^2 = u_{-1}^2 + \dfrac{1}{2}u_0^2 \,. \end{cases} \tag{5.3}$$

Then (2.7) implies

$$q \int u_0^2 \geq 2\beta_1 \int u_0^2 (u_1 - u_{-1})^2. \tag{5.4}$$

From (5.4), it's easy to see $u_0 \equiv 0$ if $q = 0$ (we can not have $u_1 - u_{-1} \equiv 0$ since $M > 0$). This is exactly the argument used in the proof of Theorem 3.3. For $q > 0$, however, no matter how small it is, whether $u_0 \equiv 0$ is not so obvious. We shall prove that, for q small enough, there does exist a positive constant c independent of q, such that the right-hand side of (5.4) is no less than $c \int u_0^2$, and hence obtain a lower bound of q for $u_0 > 0$. This is made possible by the assertions of the following lemma.

Lemma 5.2. *Given $q \in [0, \infty)$. Let $\mathbf{u}^n \in \mathbb{G}_{M,q_n}$ and $\mathbf{u}^\infty \in \mathbb{G}_{M,q}$ be as claimed in Corollary 4.6, then the following assertions hold.*

(a) *There exists a large enough R such that*

$$\frac{1}{2}\int (u_0^n)^2 \le \int_{B(R)} (u_0^n)^2 \quad \text{for all } n, \tag{5.5}$$

where $B(R) = \{x \in \mathbb{R}^3 \,|\, |x| < R\}$.

(b) $\mathbf{u}^n \to \mathbf{u}^\infty$ *uniformly.*

We first prove Claim 3 by this lemma.

Proof of Claim 3. Let $\mathbf{u}^n = (u_1^n, u_0^n, u_{-1}^n) \in \mathbb{G}_{M,q_n}$ be as claimed in Corollary 4.6 for $q = 0$. Then $\mathbf{u}^n \to \mathbf{z}$ in $\mathbb{B}$ since $\mathbf{z}$ is the unique element in $\mathbb{G}_{M,0}$ for $0 < M < 1$. For this sequence, let R be the corresponding radius asserted in (a) of Lemma 5.2, and let $k = \inf_{B(R)}(z_1 - z_{-1})$. Note that $k > 0$ by Corollary 4.12. Now by (b) of Lemma 5.2, $\mathbf{u}^n \to \mathbf{z}$ uniformly, and hence $(u_1^n - u_{-1}^n) \ge k/2$ on $B(R)$ for n large enough. From this fact and (5.5) we obtain

$$\begin{aligned} \int (u_0^n)^2(u_1^n - u_{-1}^n)^2 &\ge \int_{B(R)} (u_0^n)^2(u_1^n - u_{-1}^n)^2 \\ &\ge \frac{k^2}{4}\int_{B(R)} (u_0^n)^2 \\ &\ge \frac{k^2}{8}\int (u_0^n)^2 \end{aligned} \tag{5.6}$$

for n large enough. On the other hand, for any n, (5.4) implies

$$q_n \int (u_0^n)^2 \ge 2\beta_1 \int (u_0^n)^2(u_1^n - u_{-1}^n)^2. \tag{5.7}$$

Since $q_n \to 0$, (5.6) and (5.7) imply $u_0^n \equiv 0$ for n large enough, which completes the proof. □

Now we prove Lemma 5.2. The proofs of both assertions need the following observation.

Lemma 5.3. *Given* $q \in [0,\infty)$*. Let* $\mathbf{u}^n \in \mathbb{G}_{M,q_n}$ *and* $\mathbf{u}^\infty \in \mathbb{G}_{M,q}$ *be as claimed in Corollary 4.6, then the Lagrange multipliers* μ_n, λ_n *corresponding to* $\mathbf{u}^n$ *converge respectively to those corresponding to* $\mathbf{u}^\infty$*, denoted by* $\mu_\infty, \lambda_\infty$*.*

Proof. Multiply (2.4a) by u_1 and multiply (2.4c) by u_{-1}, and take integration, we obtain

$$(\mu + j\lambda)\int u_j^2 = F_j(\mathbf{u},q) \quad \text{for} \quad j = 1, -1, \tag{5.8}$$

where

$$\begin{aligned} F_j(\mathbf{u},q) = \int \Big\{ &|\nabla u_j|^2 + V u_j^2 + 2\beta_0 |\mathbf{u}|^2 u_j^2 \\ &+ 2\beta_1 \big[u_0^2 u_j (u_j - u_{-j}) + u_j^2 (u_j^2 - u_{-j}^2)\big] + q u_j^2 \Big\}. \end{aligned}$$

If $\int u_1^2$ and $\int u_{-1}^2$ are positive, we can solve (5.8) for μ and λ, and obtain

$$\begin{aligned} \mu &= \big[F_1(\mathbf{u},q)/(\textstyle\int u_1^2) + F_{-1}(\mathbf{u},q)/\textstyle\int u_{-1}^2\big]/2 \\ \lambda &= \big[F_1(\mathbf{u},q)/(\textstyle\int u_1^2) - F_{-1}(\mathbf{u},q)/\textstyle\int u_{-1}^2\big]/2. \end{aligned} \tag{5.9}$$

Now since we consider M as being fixed in $(0,1)$, $\int (u_j^n)^2$ and $\int (u_j^\infty)^2$ ($j = 1, -1$) are bounded away from zero. Thus (5.9) applies for μ_n, λ_n and $\mu_\infty, \lambda_\infty$, and it's easy to see that $\mu_n \to \mu_\infty$ and $\lambda_n \to \lambda_\infty$ follow the fact $\mathbf{u}^n \to \mathbf{u}^\infty$ in $\mathbb{B}$. $\square$

Proof of Lemma 5.2 (a). From the above lemma, $\mu_n \to \mu_\infty$, and in particular $\{\mu_n\}$ is a bounded sequence, say $\mu_n \le C$ for some constant $C > 0$. Multiply (2.4b) for $\mathbf{u}^n$ by u_0^n, and take integration, we obtain

$$\mu_n \int (u_0^n)^2 = \int \Big\{ |\nabla u_0^n|^2 + V(x)(u_0^n)^2 + 2\beta_0 |\mathbf{u}^n|^2 (u_0^n)^2 + 2\beta_1 (u_0^n)^2 (u_1^n - u_{-1}^n)^2 \Big\},$$

which implies

$$\int V(x)(u_0^n)^2 \le \mu_n \int (u_0^n)^2 \le C \int (u_0^n)^2. \tag{5.10}$$

On the other hand, by the assumption (A1), there exists $R > 0$ such that $V(x) \ge 2C$ for $|x| > R$, and hence

$$\int V(x)(u_0^n)^2 \ge \int_{B(R)^c} V(x)(u_0^n)^2 \ge 2C \int_{B(R)^c} (u_0^n)^2. \tag{5.11}$$

From (5.10) and (5.11), we obtain $\int (u_0^n)^2 \geq 2 \int_{B(R)^c} (u_0^n)^2$, which is easily checked to be equivalent to (5.5). $\square$

Proof of Lemma 5.2 (b). The idea is that, if the GP system (2.4) for $\mathbf{u}^n$ tends to that for $\mathbf{u}^\infty$ in a suitable sense, then uniform convergence can be obtained by the global boundedness result for elliptic operators. We take (2.4a) for example.

Let $v_1^n = u_1^n - u_1^\infty$. Subtract (2.4a) for $\mathbf{u}^\infty$ from (2.4a) for $\mathbf{u}^n$, we obtain

$$\Delta v_1^n - V(x) v_1^n = P_n - P_\infty + S_n - S_\infty, \tag{5.12}$$

where

$$\begin{aligned} P_n &= -(\mu_n + \lambda_n - q_n) u_1^n, \\ S_n &= 2\beta_0 |\mathbf{u}^n|^2 u_1^n + 2\beta_1 \big[(u_0^n)^2 (u_1^n - u_{-1}^n) + u_1^n \big((u_1^n)^2 - (u_{-1}^n)^2 \big) \big], \end{aligned}$$

and P_∞ and S_∞ are given by the same expressions with n replaced by ∞ (q_∞ is understood to be q). Apply global boundedness theorem for elliptic operators (see e.g. [8], Theorem 8.16) to (5.12), we obtain for every $r > 0$

$$\sup_{B(r)} |v_1^n| \leq \sup_{\partial B(r)} |v_1^n| + C \| P_n - P_\infty + S_n - S_\infty \|_{L^2}, \tag{5.13}$$

where $C > 0$ depends only on the radius r and $\sup_{B(r)} V$. Now since $q_n \to q$, $\mu_n \to \mu_\infty$, $\lambda_n \to \lambda_\infty$ (by Lemma 5.3), and $\mathbf{u}^n \to \mathbf{u}^\infty$ in $\mathbb{B}$, we see $P_n - P_\infty \to 0$ in L^2. Also, $S_n - S_\infty \to 0$ in L^2 since H^1 is continuously embedded in L^6. On the other hand, $\mu_n \to \mu_\infty$ and $\lambda_n \to \lambda_\infty$ also implies $\mu_n, \lambda_n, \mu_\infty$ and λ_∞ all lie in a bounded set. By Corollary 4.15, given $\varepsilon > 0$, we can find r_1 such that each u_1^n as well as u_1^∞ are bounded above by ε outside $B(r_1)$. In particular, we have

$$\sup_{|x| \geq r_1} |v_1^n(x)| \leq 2\varepsilon \quad \text{for all } n. \tag{5.14}$$

Let $r = r_1$ in (5.13), and let $n \to \infty$, we obtain

$$\limsup_{n \to \infty} \Big(\sup_{x \in B(r_1)} |v_1^n(x)| \Big) \leq 2\varepsilon. \tag{5.15}$$

From (5.14) and (5.15) we have

$$\sup_{x\in\mathbb{R}^3} |v_1^n(x)| \le 3\varepsilon \quad \text{for } n \text{ large enough.}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $v_1^n \to 0$ uniformly on $\mathbb{R}^3$. Similarly v_0^n and v_{-1}^n converge to zero uniformly, which complete the proof. $\square$

5.2 What remains

We have completed the proof of our main theorem. Some remarks are however worth mentioning.

5.2.1 Estimates of q_c from the proof

Although our statement of Theorem 5.1 is a qualitative one, our proof does provide some quantitative information. For example, as the proof of Claim 1 says, (5.2) gives an upper bound of q for $\mathbf{z} \in \mathbb{G}_{M,q}$, which is hence an upper bound of q_c. Similarly, (5.4) provides a lower bound of q_c. In view of the fact that $\mathbf{z}$ can be obtained by minimizing $\mathcal{E}$ over the two-component class $\mathbb{A}_M^{two}$ (instead of the much larger $\mathbb{A}_M$), and the fact that $\mathbf{z}$ is independent of q, the upper bound, which is expressed in terms of $\mathbf{z}$, is particularly useful. Since we will need this upper bound in §5.2.2 below, we now compute it out.

Let $\mathbf{v}$ be as in the proof of Claim 1. The right-hand side of (5.2) is

$$\begin{aligned}
&\mathcal{E}_1[\mathbf{v}] - \mathcal{E}_1[\mathbf{z}] \\
&\quad = \beta_1 \int \Big\{ \Big[2v_0^2(v_1 - v_{-1})^2 + (v_1^2 - v_{-1}^2)^2 \Big] - \Big[(z_1^2 - z_{-1}^2)^2 \Big] \Big\} \\
&\quad = \beta_1 \int \Big\{ \Big[2(\sigma z_1^2 + z_{-1}^2)(1-\sigma) z_1^2 + (1-\sigma)^2 z_1^4 \Big] - \Big[(z_1^2 - z_{-1}^2)^2 \Big] \Big\} \\
&\quad = \beta_1 \int \Big\{ \Big[2\sigma(1-\sigma) + (1-\sigma)^2 - 1 \Big] z_1^4 + \Big[2(1-\sigma) + 2 \Big] z_1^2 z_{-1}^2 - z_{-1}^4 \Big\} \\
&\quad = \beta_1 \int \Big\{ -\sigma^2 z_1^4 + (4 - 2\sigma) z_1^2 z_{-1}^2 - z_{-1}^4 \Big\},
\end{aligned}$$

where $\sigma = (1-M)/(1+M)$. Thus (5.2) gives

$$q_c \le \frac{\beta_1}{1-M} \int \Big\{ -\sigma^2 z_1^4 + (4-2\sigma) z_1^2 z_{-1}^2 - z_{-1}^4 \Big\}. \tag{5.16}$$

As to such quantitative consideration, it is then of interest to find as sharp inequalities as possible from redistribution. However, for a $\mathbf{u} \in \mathbb{G}_{M,q}$, it's not quite clear which redistribution $\mathbf{v} \in \mathbb{A}_M$ is the best one in that (2.7) gives the sharpest inequality. It turns out that, instead of choosing a specific redistribution in (2.7), a better way to derive sharp inequalities might be by using "redistributional perturbations" $\mathbf{u}(\delta) = (u_1(\delta), u_0(\delta), u_{-1}(\delta))$. We'll consider this in Chapter 6.

5.2.2 The boundedness of q_c with respect to M

Let's here write $q_c(M)$ to specify its dependence on M. We are interested in the behavior of the curve $q_c(M)$ in the (M, q)-plane. Numerical simulations show that $q_c(M)$ is continuous and increasing in of M, with

$$\lim_{M \to 0^+} q_c(M) = 0. \tag{5.17}$$

We recommend [**?**], Figure 5 for a clear diagram of the curve. Note that from Proposition 4.7, (5.17) is quite natural. Unfortunately, it seems not easy to prove the continuity and the monotonicity from our method. Nevertheless, one fact that is not quite clear numerically can be settled. That is, as $M \to 1^-$, whether $q_c(M)$ tends to infinity or some finite number. The following theorem says that it's the latter that is the case.

Theorem 5.4. *$q_c(M)$ is uniformly bounded for $0 < M < 1$.*

Proof. Let's write $\mathbf{z}^M$ for $\mathbf{z}$. From (5.16) we obtain

$$\begin{aligned} q_c(M) &\leq \frac{\beta_1}{1-M} \int \left\{ -\sigma^2 \left(z_1^M\right)^4 + (4-2\sigma)\left(z_1^M\right)^2 \left(z_{-1}^M\right)^2 - \left(z_{-1}^M\right)^4 \right\} \\ &\leq \frac{\beta_1(4-2\sigma)}{1-M} \int \left(z_1^M\right)^2 \left(z_{-1}^M\right)^2 \\ &\leq \frac{\beta_1(4-2\sigma)}{1-M} \|z_1^M\|_{L^\infty} \int \left(z_{-1}^M\right)^2 . \end{aligned}$$

Since $\int \left(z_{-1}^M\right)^2 = (1-M)/2$, we get

$$q_c(M) \leq \beta_1(2-\sigma)\|z_1^M\|_{L^\infty}.$$

Now $\sigma = (1 - M)/(1 + M)$ is bounded for $0 < M < 1$, and it remains to show the boundedness of $\|z_1^M\|_{L^\infty}$. From Remark 4.1, $M \mapsto \mathbf{z}^M$ is continuous from $[0, 1]$ into $\mathbb{B}$. With this result, we can in fact prove that $M \mapsto \mathbf{z}^M$ is also continuous from $[0, 1]$ into L^∞, in the same spirit as the proof of the assertion (b) of Lemma 5.2. This completes the proof. □

Remark 5.2. It might be surprising that, by the same argument, we have trouble to prove that $M \mapsto z_{-1}^M$ is also continuous from $[0, 1]$ into L^∞. Indeed, the problem only occurs at $M = 1$, where z_{-1}^M is equal to zero. See §7.2 for discussion of such problem.

Chapter 6

Redistributional Perturbation in a Fixed Admissible Class

Let $\mathbf{u} \in \mathbb{G}_{M,q}$. We have seen it's sometimes useful to construct a "redistributional perturbation" $\mathbf{u}(\delta)$ of $\mathbf{u}$, where $\delta \geq 0$ is a small parameter, and $\mathbf{u}(0) = \mathbf{u}$. In previous examples (namely proofs of Proposition 4.2 and Proposition 4.3), the $\mathbf{u}(\delta)$ are so constructed to be in different $\mathbb{A}_{M'}$, in order to compare ground states with different magnetizations. In this chapter we consider similar constructions lying in a fixed admissible class. Thus $\mathcal{E}[\mathbf{u}] \leq \mathcal{E}[\mathbf{u}(\delta)]$. By letting $I(\delta) = \mathcal{E}[\mathbf{u}(\delta)]$, $\delta = 0$ is then an endpoint minimum of I. Hence

$$I'(0^+) \geq 0, \tag{6.1}$$

where $I'(0^+)$ is the right derivative of I at 0. It turns out that the existence of such derivative needs some verification. We will give two examples, Proposition 6.1 and Proposition 6.2, as more delicate treatments of (5.1) and (5.3) respectively. We remark that δ will always denote a nonnegative parameter, which is small enough so that all involved expressions make sense.

6.1 Inequality from redistributional perturbation

For convenience we give some notations and remarks first.

1. As above, whenever a construction of $\mathbf{u}(\delta)$ is considered, we write $I(\delta)$ for $\mathcal{E}[\mathbf{u}(\delta)]$. Similarly $I_{kin}(\delta)$, $I_1(\delta)$ and $I_{Zee}(\delta)$ stand for the corresponding parts. Note that $I_{pot}(\delta)$

and $I_0(\delta)$ are constant functions, and hence (6.1) says

$$I'_{kin}(0^+) + I'_1(0^+) + I'_{Zee}(0^+) \geq 0. \tag{6.2}$$

As $I'_{Zee}(0^+)$, if exists, must be nonpositive since $\mathbf{u}(\delta)$ are redistributions of $\mathbf{u}$, we have

$$I'_{kin}(0^+) + I'_1(0^+) \geq 0. \tag{6.3}$$

Using (6.3) for a construction of $\mathbf{u}(\delta)$ is much of the same spirit as using (2.7) for a specific choice of $\mathbf{v}$, which has the advantage of involving only algebraic expressions of $\mathbf{u}$. Nevertheless, in this chapter we aim to gain as complete information from redistribution as possible, and hence we will use the full inequality (6.2).

2. When a construction of $\mathbf{u}(\delta)$ is considered, we write

$$D(\mathbf{u}(\delta)) = \frac{H(\mathbf{u}(\delta)) - H(\mathbf{u})}{\delta}$$

for small $\delta > 0$. $D_{kin}(\mathbf{u}(\delta))$, $D_1(\mathbf{u}(\delta))$ and $D_{Zee}(\mathbf{u}(\delta))$ are similarly defined. Thus

$$I'(0^+) = \lim_{\delta \to 0^+} \int D(\mathbf{u}(\delta)) = \int \frac{\partial}{\partial \delta} H(\mathbf{u}(\delta)) \bigg|_{\delta = 0^+}$$

if differentiation under the integral sign is valid.

3. For $\mathbf{u} \in \mathbb{G}_{M,q}$, we write (as in Section 8.1)

$$S(u_i, u_j) = |u_i \nabla u_j - u_j \nabla u_i|^2.$$

When computing $D_{kin}(\mathbf{u}(\delta))$, we will use the following fact:

- Whenever $\sum_j a_j u_j^2 > 0$ for some nonnegative constants a_j ($j = 1, 0, -1$), we have

$$\sum_j a_j |\nabla u_j|^2 - \left| \nabla \sqrt{\sum_j a_j u_j^2} \right|^2 = \frac{\sum_{k<\ell} a_k a_\ell S(u_k, u_\ell)}{\sum_j a_j u_j^2}.$$

This formula is just (8.1) with $\mathbf{f} = (\sqrt{a_1} u_1, \sqrt{a_0} u_0, \sqrt{a_{-1}} u_{-1})$.

We now give our examples of (6.1).

Proposition 6.1. *For $0 < M < 1$ and $0 \leq q \leq q_c(M)$ (so that $\mathbf{z} \in \mathbb{G}_{M,q}$), we have*

$$4\beta_1 \int z_1 z_{-1}(z_1 - z_{-1})(\tau z_{-1} - z_1) \geq q(1+M) + \int \frac{\tau S(z_1, z_{-1})}{(z_1)^2 + \tau(z_{-1})^2}, \tag{6.4}$$

where $\tau = (1+M)/(1-M)$.

Proof. Consider the redistribution $\mathbf{u}(\delta)$ of $\mathbf{z}$ defined by

$$\begin{cases} u_1(\delta)^2 = (1-\delta)z_1^2 \\ u_0(\delta)^2 = \delta z_1^2 + \tau\delta z_{-1}^2 \\ u_{-1}(\delta)^2 = (1-\tau\delta)z_{-1}^2 \, . \end{cases} \tag{6.5}$$

It's easy to check $\mathbf{u}(\delta) \in \mathbb{A}_M$ for each small $\delta > 0$. We compute $I'(0^+)$ as follows. First,

$$D_{kin}(\mathbf{u}(\delta)) = \frac{1}{\delta}\Big\{ |\nabla u_0(\delta)|^2 - (\delta|\nabla z_1|^2 + \tau\delta|\nabla z_{-1}|^2) \Big\} = -\frac{\tau S(z_1, z_{-1})}{z_1^2 + \tau z_{-1}^2},$$

which is independent of δ, and hence

$$I'_{kin}(0^+) = -\int \frac{\tau S(z_1, z_{-1})}{z_1^2 + \tau z_{-1}^2}.$$

Second,

$$\begin{aligned} H_1(\mathbf{u}(\delta)) = \beta_1 \Big\{ & 2\delta(z_1^2 + \tau z_{-1}^2)\Big(\sqrt{1-\delta}\,z_1 - \sqrt{1-\tau\delta}\,z_{-1}\Big)^2 \\ & + \Big[(1-\delta)z_1^2 - (1-\tau\delta)z_{-1}^2\Big]^2 \Big\}. \end{aligned}$$

It's not hard to see that $\frac{\partial}{\partial\delta}H_1(\mathbf{u}(\delta))$ is a homogeneous polynomial of $\mathbf{z}$ with degree 4, and for $\delta \geq 0$ in a fixed small neighborhood of 0, we have

$$\left| \frac{\partial}{\partial\delta} H_1(\mathbf{u}(\delta)) \right| \leq C|\mathbf{z}|^4 \in L^1$$

for some constant C independent of δ. Thus it's valid to differentiate $I_1(\delta)$ under the integral sign, which gives

$$\begin{aligned} I_1'(0^+) &= 2\beta_1 \int \Big\{ (z_1^2 + \tau z_{-1}^2)(z_1 - z_{-1})^2 + (z_1^2 - z_{-1}^2)(-z_1^2 + \tau z_{-1}^2) \Big\} \\ &= 4\beta_1 \int z_1 z_{-1}(z_1 - z_{-1})(\tau z_{-1} - z_1). \end{aligned}$$

Finally,

$$H_{Zee}(\mathbf{u}(\delta)) = q\big[(1-\delta)z_1^2 + (1-\tau\delta)z_{-1}^2\big],$$

and we obviously have

$$I'_{Zee}(0^+) = q\left(-\int z_1^2 - \tau\int z_{-1}^2\right) = -q(1+M).$$

(6.4) now follows $I'(0^+) = I'_{kin}(0^+) + I'_1(0^+) + I'_{Zee}(0^+) \geq 0.$ ☐

Proposition 6.2. *For* $0 < M < 1$ *and* $q \geq 0$*, every* $\mathbf{u} \in \mathbb{G}_{M,q}$ *satisfies*

$$q\int u_0^2 \geq \beta_1 \int u_0^2(u_1 - u_{-1})^2\left(2 + \frac{u_0^2}{u_1 u_{-1}}\right) + \frac{1}{2}\int \sum_{j=1,-1} \frac{S(u_j,u_0)}{u_j^2}. \tag{6.6}$$

Proof. Let $\mathbf{u}(\delta)$ be defined by

$$\begin{cases} u_1(\delta)^2 = u_1^2 + \delta u_0^2 \\ u_0(\delta)^2 = (1-2\delta)u_0^2 \\ u_{-1}(\delta)^2 = u_{-1}^2 + \delta u_0^2 \,. \end{cases} \tag{6.7}$$

It's easy to see $\mathbf{u}(\delta) \in \mathbb{A}_M$ for each small $\delta > 0$. Now

$$H_{Zee}(\mathbf{u}(\delta)) = q\left(u_1^2 + u_{-1}^2 + 2\delta u_0^2\right),$$

and it's also obvious that

$$I'_{Zee}(0^+) = 2q\int u_0^2.$$

On the other hand,

$$\begin{aligned} D_{kin}(\mathbf{u}(\delta)) &= \sum_{j=1,-1} \frac{\left|\nabla\sqrt{u_j^2+\delta u_0^2}\right|^2 - \left(|\nabla u_j|^2 + \delta|\nabla u_0|^2\right)}{\delta} \\ &= -\sum_{j=1,-1} \frac{S(u_j,u_0)}{u_j^2+\delta u_0^2}, \end{aligned} \tag{6.8}$$

and it's not clear if $|D_{kin}(\mathbf{u}(\delta))|$, for small $\delta \geq 0$, is bounded by an L^1 function independent of δ. Hence the operation

$$I'_{kin}(0^+) = \int \lim_{\delta\to 0^+} D_{kin}(\mathbf{u}(\delta)) = -\int \sum_{j=1,-1} \frac{S(u_j,u_0)}{u_j^2}$$

is not valid immediately. Similar problem occurs with $I_1'(0^+)$. Precisely,

$$H_1(\mathbf{u}(\delta)) = \beta_1\Big[2(1-2\delta)u_0^2\,(u_1(\delta)-u_{-1}(\delta))^2 + \big(u_1^2 - u_{-1}^2\big)^2\Big].$$

By using the fact $(\partial u_j/\partial\delta)(\delta) = u_0^2/(2u_j(\delta))$ for $j = 1, -1$, we have

$$\begin{aligned}
&\frac{\partial}{\partial\delta}H_1(\mathbf{u}(\delta))\\
&\quad = \beta_1\bigg[-4u_0^2\,(u_1(\delta)-u_{-1}(\delta))^2\\
&\qquad + 2(1-2\delta)u_0^2\cdot 2\,(u_1(\delta)-u_{-1}(\delta))\left(\frac{u_0^2}{2u_1(\delta)} - \frac{u_0^2}{2u_{-1}(\delta)}\right)\bigg]\\
&\quad = -2\beta_1\left[2u_0^2\,(u_1(\delta)-u_{-1}(\delta))^2 + (1-2\delta)\frac{u_0^4\,(u_1(\delta)-u_{-1}(\delta))^2}{u_1(\delta)u_{-1}(\delta)}\right]\\
&\quad = -2\beta_1 u_0^2\,(u_1(\delta)-u_{-1}(\delta))^2\left[2 + \frac{(1-2\delta)u_0^2}{u_1(\delta)u_{-1}(\delta)}\right],
\end{aligned} \tag{6.9}$$

and we are not sure if $|\frac{\partial}{\partial\delta}H_1(\mathbf{u}(\delta))|$, for small $\delta \geq 0$, can be bounded by an L^1 function independent of δ. This prevents us from computing $I_1'(0^+)$ by differentiation under the integral. To be rigorous, we avoid these problems as follows.

Since $\int D(\mathbf{u}(\delta)) \geq 0$ for $\delta > 0$,

$$\int D_{Zee}(\mathbf{u}(\delta)) \geq -\int D_1(\mathbf{u}(\delta)) - \int D_{kin}(\mathbf{u}(\delta)). \tag{6.10}$$

Now $D_{kin}(\mathbf{u}(\delta)) \leq 0$ since $\mathbf{u}(\delta)$ is a redistribution of $\mathbf{u}$. Also, from the result of (6.9), $\frac{\partial}{\partial\delta}H_1(\mathbf{u}(\delta)) \leq 0$ for $\delta > 0$, and hence we also have $D_1(\mathbf{u}(\delta)) \leq 0$ for small $\delta > 0$. Thus, after taking limit inferior as $\delta \to 0^+$, we can apply Fatou's lemma to the right-hand side of (6.10), and we obtain

$$2q\int u_0^2 \geq \int -\frac{\partial}{\partial\delta}H_1(\mathbf{u}(\delta))\bigg|_{\delta=0^+} + \int -\frac{\partial}{\partial\delta}H_{kin}(\mathbf{u}(\delta))\bigg|_{\delta=0^+}. \tag{6.11}$$

From (6.8) and (6.9), we see (6.11), after divided by 2, gives (6.6). □

Remark 6.1. Now that the terms of the right-hand side of (6.11) are finite, we have

$$\frac{S(u_j,u_0)}{u_j^2} \in L^1 \;\; (j = 1, -1) \quad \text{and} \quad \frac{u_0^4(u_1-u_{-1})^2}{u_1u_{-1}} \in L^1.$$

We can obviously use them to find suitable L^1 bounds of $|D_{kin}(\mathbf{u}(\delta))|$ and $|\frac{\partial}{\partial\delta}H_1(\mathbf{u}(\delta))|$ independent of δ. Hence $I_{kin}'(0^+)$ and $I_1'(0^+)$ can really be obtained by differentiation under the integrals. One might suspect that such operations of taking differentiation should be valid for all similar constructions of $\mathbf{u}(\delta)$. This is probably true. However, there are cases of which the validity are still open. See §7.3 for for an example and discussions.

6.1.1 Comparison with previous results

(6.5) and (6.7) can be regarded respectively as perturbation versions of (5.1) and (5.3). For the relation between (6.5) and (5.1), note that $\sigma = (1 - M)/(1 + M) = 1/\tau$, and we claim that we can in fact replace (6.5) by the following to get the same inequality (6.4):

$$\begin{cases} u_1(\delta)^2 = (1 - \sigma\delta)z_1^2 \\ u_0(\delta)^2 = \sigma\delta z_1^2 + \delta z_{-1}^2 \\ u_{-1}(\delta)^2 = (1 - \delta)z_{-1}^2 \,. \end{cases}$$

We omit the easy verification of the claim. Note however that this should be quite natural, since the true point of this construction is to share parts of z_1 and z_{-1} to the middle component. The ratio of the amounts shared is totally determined by the constraint $\mathcal{M} = M$ and is not a controllable parameter.

We have promised in §5.2.1 that it might be better to use redistributional perturbations to get inequalities. Let's now examine if this claim is true. For Proposition 6.2, the inequality (6.6) is obviously sharper than (5.4), since the right-hand side of (5.4) is only a part of the right-hand side of (6.6). We will see in the next section that (6.6) is indeed the sharpest possible inequality, in that it's really an equality.

As for Proposition 6.1, things are not so obvious. Whether (6.4) is sharper than (5.16) cannot be answered from their appearances. (Of course, we have to compare them after omitting (or adding) the contributions of the kinetic part for both of them.) To see this, let's write (5.16) as $q_c \le U_1[\mathbf{z}]$. And similarly (6.4) gives $q_c \le U_2[\mathbf{z}]$, where

$$U_2[\mathbf{z}] = \frac{4\beta_1}{1+M} \int z_1 z_{-1}(z_1 - z_{-1})(\tau z_{-1} - z_1)$$
$$= \frac{4\beta_1}{1-M} \int z_1 z_{-1}(z_1 - z_{-1})(z_{-1} - \sigma z_1).$$

For (6.4) to be better than (5.16), we must have $U_2[\mathbf{z}] \leq U_1[\mathbf{z}]$. Direct calculation gives

$$U_2[\mathbf{z}] - U_1[\mathbf{z}] = \frac{\beta_1}{1-M} \int \left\{ \sigma^2 z_1^4 - 4\sigma z_1^3 z_{-1} + 6\sigma z_1^2 z_{-1}^2 - 4 z_1 z_{-1}^3 + z_{-1}^4 \right\},$$

of which the integrand, as a polynomial, is not identically positive or negative. It's interesting, however, that from some numerical simulations, the integrand is really a negative function when we take $\mathbf{z}$ to be the two-component ground state, and hence the inequality from redistributional perturbation wins again. We don't know how to prove this fact rigorously, but there is an intuitive reason. To see this, we claim that, although we have shown in Corollary 3.5 that $\mathbf{z}$ in general doesn't obey the SMA, numerical results show that they are not far from SMA. If we are willing to take the assumption $z_{-1} = \kappa z_1$, where it's easy to check that κ must be $\sigma^{1/2}$, then

$$U_2[\mathbf{z}] - U_1[\mathbf{z}] = \frac{\beta_1}{(1-M)} \int \left\{ \sigma^2 - 4\sigma^{3/2} + 6\sigma^2 - 4\sigma^{3/2} + \sigma^2 \right\} z_1^4$$
$$= \frac{\beta_1}{1-M} \int 8\sigma^{3/2}(\sigma^{1/2} - 1) z_1^4.$$

Thus the integrand is negative.

6.2 From the viewpoint of the GP system

There is another point of view on what we did above, which leads us to find (6.6) is really an equality but not merely an inequality. We discuss it in the following.

At any rate, a redistributional perturbation $\mathbf{u}(\delta)$ is a kind of perturbation, and it's natural to ask whether the results above could also be obtained from the GP system (2.4), which, by its derivation, consists of information from general perturbations. The only obstruction is that (2.4) is obtained from smooth perturbation, while $\mathbf{u}(\delta)$ is kind of singular at $\delta = 0$. Indeed, using chain rule formally we have

$$I'(0^+) = \frac{d}{d\delta} \mathcal{E}[\mathbf{u}(\delta)] \bigg|_{\delta = 0^+} = \mathcal{E}'[\mathbf{u}](\mathbf{u}'(0^+)),$$

and one expects (6.1) might be a consequence of testing (2.4) by $\mathbf{u}'(0^+)$. To carry out this idea rigorously, however, we have to take care of the problem that $\mathbf{u}'(0^+)$ may not be good enough (precisely in $\mathbb{B}$) so that $\mathcal{E}'[\mathbf{u}](\mathbf{u}'(0^+))$ makes sense. It turns out that we can follow the idea for $\mathbf{u}(\delta)$ defined by (6.7), and find that the equality holds in (6.6). While in our argument the inequality itself plays a critical role. For $\mathbf{u}(\delta)$ defined by (6.5), the same idea doesn't work directly. We'll discuss the problem in §6.2.2

6.2.1 Validity of the equality of (6.6)

We first give a computational result.

Lemma 6.3. *Assume $f, g \in C^1$ and $f, g > 0$, then*

$$g^2\left(\frac{\Delta f}{f} - \frac{\Delta g}{g}\right) = -\nabla \cdot \left(fg\nabla\left(\frac{g}{f}\right)\right) + \left|f\nabla\left(\frac{g}{f}\right)\right|^2 \tag{6.12}$$

in the sense of distribution.

Proof. Assume $f, g \in C^2$ and $f, g > 0$, then

$$\begin{aligned}
\frac{\Delta f}{f} - \frac{\Delta g}{g} = \frac{g\Delta f - f\Delta g}{fg} &= -\frac{1}{fg}\nabla \cdot (f\nabla g - g\nabla f) \\
&= -\frac{1}{g^2}\frac{g}{f}\nabla \cdot \left(f^2\nabla\left(\frac{g}{f}\right)\right) \\
&= -\frac{1}{g^2}\left\{\nabla \cdot \left(\frac{g}{f} \cdot f^2\nabla\left(\frac{g}{f}\right)\right) - f^2\nabla\left(\frac{g}{f}\right) \cdot \nabla\left(\frac{g}{f}\right)\right\} \\
&= -\frac{1}{g^2}\left\{\nabla \cdot \left(fg\nabla\left(\frac{g}{f}\right)\right) - \left|f\nabla\left(\frac{g}{f}\right)\right|^2\right\}.
\end{aligned}$$

Thus (6.12) holds. It's then very natural to expect that (6.12) also holds in the sense of distribution when f, g are only of class C^1. That is

$$\begin{aligned}
&\int \left\{ -\nabla f \cdot \nabla\left(\frac{g^2}{f}\varphi\right) + \nabla g \cdot \nabla(g\varphi)\right\} \\
&\qquad = \int \left\{fg\nabla\left(\frac{g}{f}\right) \cdot \nabla\varphi + \left|f\nabla\left(\frac{g}{f}\right)\right|^2 \varphi\right\}
\end{aligned} \tag{6.13}$$

for every smooth function φ with compact support. Rigorous justification can be done by the following observations:

1. We can arrange (6.13) into the form

$$\int \Big\{ a(f,g,\nabla f,\nabla g)\varphi + \mathbf{A}(f,g,\nabla f,\nabla g)\cdot\nabla\varphi \Big\} = 0,$$

   where a is a scalar function and $\mathbf{A}$ is a vector function.

2. Given $f, g \in C^1$, $f, g > 0$. By mollifying f and g, we get smooth functions f_ε and g_ε such that $f_\varepsilon \to f$, $\nabla f_\varepsilon \to \nabla f$, $g_\varepsilon \to g$, and $\nabla g_\varepsilon \to \nabla g$ uniformly on any compact set.

We omit the routine details. ☐

Theorem 6.4. *The inequality* (6.6) *is an equality.*

Proof. If $u_0 \equiv 0$, the assertion is trivial. So assume $u_0 > 0$. The discussion before Lemma 6.3 suggests we test the GP system by

$$\mathbf{u}'(0^+) = \left(u_0^2/(2u_1), -u_0, u_0^2/(2u_{-1})\right).$$

That is computing

$$\text{(2.4a)} \times \left(\frac{u_0^2}{2u_1}\right) + \text{(2.4b)} \times (-u_0) + \text{(2.4c)} \times \left(\frac{u_0^2}{2u_{-1}}\right).$$

After some rearrangement, the result is

$$qu_0^2 = \beta_1 u_0^2 (u_1 - u_{-1})^2 \left(2 + \frac{u_0^2}{u_1 u_{-1}}\right) + \frac{1}{2}\sum_{j=1,-1} u_0^2 \left(\frac{\Delta u_j}{u_j} - \frac{\Delta u_0}{u_0}\right). \tag{6.14}$$

By Lemma 6.3, for $j = 1, -1$ we have

$$u_0^2 \left(\frac{\Delta u_j}{u_j} - \frac{\Delta u_0}{u_0}\right) = -\nabla \cdot \left(u_0 u_j \nabla\left(\frac{u_0}{u_j}\right)\right) + \left|u_j \nabla \left(\frac{u_0}{u_j}\right)\right|^2. \tag{6.15}$$

Note that

$$\left|u_j \nabla\left(\frac{u_0}{u_j}\right)\right|^2 = \frac{S(u_j, u_0)}{u_j^2}, \tag{6.16}$$

which lies in L^1 by (6.6). Also by (6.6) we have $u_0^2(u_1 - u_{-1})^2 \left(2 + u_0^2/(u_1 u_{-1})\right) \in L^1$, and it remains to show

$$\int \nabla \cdot \left(u_0 u_j \nabla \left(\frac{u_0}{u_j}\right)\right) = 0.$$

This is true by Lemma 4.9. To see why $u_0 u_j \nabla(u_0/u_j) \in L^1(\mathbb{R}^3, \mathbb{R}^3)$, note that $u_0 \in L^2$, and $u_j \nabla(u_0/u_j) \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ by (6.16). ☐

Remark 6.2. To eliminate the unwanted term $\nabla \cdot (u_0 u_j \nabla(u_0/u_j))$, in the proof above we use the inequality (6.6) to guarantee its integrability. It looks somewhat pedantic, but seems unavoidable. Similar problems happen when we try to prove equalities from other constructions of $\mathbf{u}(\delta)$. Thus the inequalities obtained from redistribution are not direct consequences of the GP system. This declaration however may be overthrown if we can prove some comparison results of the decaying rates of the three components. See §7.3 for discussion.

6.2.2 Discussions on (6.4)

We'd like to prove the same thing for (6.4). However, note that since $\mathbf{z}$ is independent of q, it's impossible that (6.4) be an equality for varied q. Indeed, following the above idea, we get a trouble at the very beginning: With $\mathbf{u}(\delta)$ defined by (6.5), we have

$$\mathbf{u}'(\delta) = \frac{1}{2}\left(\frac{-z_1}{\sqrt{1-\delta}}, \frac{z_1^2 + \tau z_{-1}^2}{\sqrt{\delta z_1^2 + \tau \delta z_{-1}^2}}, \frac{-\tau z_{-1}}{\sqrt{1-\tau\delta}} \right).$$

Thus

$$\mathbf{u}'(0^+) = (-z_1/2, +\infty, -\tau z_{-1}/2),$$

which suggests we multiply (2.4b) for $\mathbf{z}$, i.e. the trivial equation $0 = 0$, by infinity. And if we ignore this part and just using (2.4a) and (2.4c), the result is really far from (6.4).

This problem can be avoided if there is a sequence $\mathbf{u}^n \in \mathbb{G}_{M_n,q_n}$ such that $u_0^n > 0$, $M_n \to M \in (0,1)$, $q_n \to q_c(M)$, and $\mathbf{u}^n \to \mathbf{z} \in \mathbb{G}_{M,q_c(M)}$ in $\mathbb{B}$. Note that the existence of such sequence is not proved. Using Corollary 4.4 and Corollary 4.6, it's only guaranteed that there exists such $\mathbf{u}^n$ that converges to some $\mathbf{u}^\infty \in \mathbb{G}_{M,q_c(M)}$, and, as pointed out in Remark 5.1, we have no reason to say $\mathbf{u}^\infty = \mathbf{z}$. Since anyway such sequence may really exist, we still illustrate how we can use it to avoid the above problem in the following.

The idea is to consider the same construction as (6.5) for $\mathbf{u} \in \mathbb{G}_{M,q}$ with $u_0 > 0$.

Precisely, for such $\mathbf{u}$ we consider $\mathbf{u}(\delta)$ defined by

$$\begin{cases} u_1(\delta)^2 = (1-\delta)u_1^2 \\ u_0(\delta)^2 = u_0^2 + \delta u_1^2 + \tau_{\mathbf{u}}\delta u_{-1}^2 \\ u_{-1}(\delta)^2 = (1-\tau_{\mathbf{u}}\delta)u_{-1}^2\,, \end{cases}$$

where $\tau_{\mathbf{u}} = (\int u_1^2)/(\int u_{-1}^2)$, the constant making $\mathbf{u}(\delta) \in \mathbb{A}_M$. We claim without proof that, by using this redistributional perturbation, and following the idea of proving Theorem 6.4, we have the following result.

Theorem 6.5. *Let* $M \in (0,1)$ *and* $q > q_c(M)$. $\mathbf{u} \in \mathbb{G}_{M,q}$ *satisfies*

$$\begin{aligned} 4\beta_1 \int u_1 u_{-1}(u_1 - u_{-1})(\tau_{\mathbf{u}} u_{-1} - u_1) + 2\beta_1 \int u_0^2(u_1 - u_{-1})(\tau_{\mathbf{u}} u_{-1} - u_1) \\ = q \int (u_1^2 + \tau_{\mathbf{u}} u_{-1}^2) + \int \frac{S(u_0, u_1) + \tau_{\mathbf{u}} S(u_0, u_{-1})}{u_0^2}. \end{aligned}$$

Now assume there exists $\mathbf{u}^n \in \mathbb{G}_{M,q_n}$, where $M \in (0,1)$ and $q_n \to q_c(M)^+$, such that $\mathbf{u}^n \to \mathbf{z}$. Define $\tau_n = \tau_{\mathbf{u}^n}$. Note that $\tau_n \to \tau = (1+M)/(1-M)$. By applying Theorem 6.5 to $\mathbf{u}^n$, and letting $n \to \infty$, we obtain

$$\begin{aligned} 4\beta_1 \int z_1 z_{-1}(z_1 - z_{-1})(\tau z_{-1} - z_1) \\ = (1+M)q_c + \lim_{n\to\infty} \int \frac{S(u_0^n, u_1^n) + \tau_n S(u_0^n, u_{-1}^n)}{(u_0^n)^2}. \end{aligned} \tag{6.17}$$

This is the equality corresponding to (6.4). To see how (6.17) implies (6.4), we have the following result.

Lemma 6.6. *For general positive functions* $v_1, v_0, v_{-1} \in C^1$*, we have the following identity:*

$$\frac{S(v_0, v_1) + S(v_0, v_{-1})}{v_0^2} = (v_1^2 + v_{-1}^2)\left|\frac{\nabla v_0}{v_0} - \frac{v_1 \nabla v_1 + v_{-1}\nabla v_{-1}}{v_1^2 + v_{-1}^2}\right|^2 + \frac{S(v_1, v_{-1})}{v_1^2 + v_{-1}^2}.$$

Proof. Let $\mathbf{f} = \nabla v_0 / v_0$. We have

$$\begin{aligned} &\frac{S(v_0, v_1) + S(v_0, v_{-1})}{v_0^2} \\ &= |\nabla v_1 - v_1\mathbf{f}|^2 + |\nabla v_{-1} - v_{-1}\mathbf{f}|^2 \\ &= (v_1^2 + v_{-1}^2)|\mathbf{f}|^2 - 2(v_1\nabla v_1 + v_{-1}\nabla v_{-1}) \cdot \mathbf{f} + |\nabla v_1|^2 + |\nabla v_{-1}|^2 \\ &= (v_1^2 + v_{-1}^2)\left|\mathbf{f} - \frac{v_1 \nabla v_1 + v_{-1}\nabla v_{-1}}{v_1^2 + v_{-1}^2}\right|^2 + \frac{S(v_1, v_{-1})}{v_1^2 + v_{-1}^2}. \end{aligned}$$

Thus, by letting $v_1 = u_1^n$, $v_0 = u_0^n$, and $v_{-1} = \sqrt{\tau_n} u_{-1}^n$, we obtain

$$\frac{S(u_0^n, u_1^n) + \tau_n S(u_0^n, u_{-1}^n)}{(u_0^n)^2} \geq \frac{\tau_n S(u_1^n, u_{-1}^n)}{(u_1^n)^2 + \tau_n (u_{-1}^n)^2}. \tag{6.18}$$

Since $\mathbf{u}^n \to \mathbf{z}$ in $\mathbb{B}$, there is a subsequence $\mathbf{u}^{n(k)}$ of $\mathbf{u}^n$ such that

$$\mathbf{u}^{n(k)} \to \mathbf{z} \quad \text{and} \quad \nabla \mathbf{u}^{n(k)} \to \nabla \mathbf{z} \quad \text{almost everywhere.}$$

$\mathbf{u}^{n(k)} \to \mathbf{z}$ and $\nabla \mathbf{u}^{n(k)} \to \nabla \mathbf{z}$ *almost everywhere*. Applying Fatou's lemma to (6.18), we finally obtain

$$\lim_{k \to \infty} \int \frac{S(u_0^{n(k)}, u_1^{n(k)}) + \tau_{n(k)} S(u_0^{n(k)}, u_{-1}^{n(k)})}{(u_0^{n(k)})^2} \geq \int \frac{\tau S(z_1, z_{-1})}{z_1^2 + \tau z_{-1}^2}. \tag{6.19}$$

And hence (6.17) implies (6.4).

Open Problem. It's very interesting to know if the equality of (6.19) holds, which is equivalent to the equality of (6.4) at $q = q_c$. Were this true, (6.4) doesn't only provide an upper bound of q_c, but a characterization. From Lemma 6.6, the gap is provided by the limiting behavior of

$$(u_1^2 + \tau_{\mathbf{u}} u_{-1}^2) \left| \frac{\nabla u_0}{u_0} - \frac{u_1 \nabla u_1 + \tau_{\mathbf{u}} u_{-1} \nabla u_{-1}}{u_1^2 + \tau_{\mathbf{u}} u_{-1}^2} \right|^2,$$

as $\mathbf{u} \to \mathbf{z}$.

Chapter 7

Discussions of some Open Problems

We discuss some open problems arising from this study. They are categorized into three sections.

7.1 Uniqueness

Uniqueness is a standard and prominent problem to be settled in variational problems. Even in this thesis, although it's not essential for our main considerations, the lack of it causes troubles in some places. Examples are given by Remark 4.1 and Remark 5.1, which also haunt the discussion in §6.2.2.

We have mentioned in Remark 2.2 that our energy functional $\mathcal{E}$ doesn't have the suitable convexity property due to the term H_1. Let's also consider only $\beta_1 > 0$ here. Then, more precisely, it's the term $2\beta_1 u_0^2(u_1 - u_{-1})^2$ appearing in $H_1(\mathbf{u})$ that causes problem. As to remedy this difficulty, there are two natural ideas:

(a) Although $\mathcal{E}$ is not convex on $\mathbb{B}$, it might be convex on a fixed $\mathbb{A}_M$, which is sufficient to prove uniqueness.

(b) In this paper there is no assumption on the magnitude of β_1, while for real spin-1 BECs it's very small compared to β_0, and hence $\mathcal{E}_1$ contributes to a rather insignificant amount of the whole energy. If we are willing to take this fact into consideration, maybe the convexity of other parts will outweigh the nonconvexity of $\mathcal{E}_1$.

Unfortunately, these ideas do not work since there are $\mathbf{u}, \mathbf{v} \in \mathbb{A}_M$ such that

$$\frac{\mathcal{E}[\mathbf{u}] + \mathcal{E}[\mathbf{v}]}{2} - \mathcal{E}[\mathbf{w}] = \frac{\mathcal{E}_1[\mathbf{u}] + \mathcal{E}_1[\mathbf{v}]}{2} - \mathcal{E}_1[\mathbf{w}] < 0,$$

where $\mathbf{w} \in \mathbb{A}_M$, as in the proof of Theorem 2.4, is defined by $w_j^2 = (u_j^2 + v_j^2)/2$ for each j. We give an example below.

Let f, g, h be any three nonnegative functions in $H^1 \cap L_V^2 \cap L^4$ such that

(1) f, g and h are supported on disjoint sets,

(2) $\int (f^2 + g^2 + h^2) = 1$, and

(3) $\int g^2 = \int h^2 > 0$ and $\int (f^2 - g^2) = M$.

Then let $\mathbf{u} = (f, g, h)$ and $\mathbf{v} = (f, h, g)$. We have $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{A}_M$. It's easy to see

$$\frac{\mathcal{E}[\mathbf{u}] + \mathcal{E}[\mathbf{v}]}{2} - \mathcal{E}[\mathbf{w}] = \frac{\mathcal{E}_1[\mathbf{u}] + \mathcal{E}_1[\mathbf{v}]}{2} - \mathcal{E}_1[\mathbf{w}].$$

To check that it is negative, note that

$$\begin{aligned} &\frac{H_1(\mathbf{u}) + H_1(\mathbf{v})}{2} - H_1(\mathbf{w}) \\ &\quad = \beta_1 \bigg\{ \frac{2g^2(f-h)^2 + (f^2 - h^2)^2}{2} + \frac{2h^2(f-g)^2 + (f^2 - g^2)^2}{2} \\ &\qquad\qquad - (g^2 + h^2)\bigg(f - \sqrt{\frac{g^2 + h^2}{2}}\bigg)^2 - \bigg(f^2 - \frac{g^2 + h^2}{2}\bigg)^2 \bigg\}. \end{aligned}$$

Let $\Omega_f = \text{supp}(f)$, $\Omega_g = \text{supp}(g)$ and $\Omega_h = \text{supp}(h)$, then we have

$$\begin{aligned} \int_{\Omega_f} \bigg\{ \frac{H_1(\mathbf{u}) + H_1(\mathbf{v})}{2} - H_1(\mathbf{w}) \bigg\} &= \beta_1 \int_{\Omega_f} \bigg\{ \frac{f^4}{2} + \frac{f^4}{2} - 0 - f^4 \bigg\} = 0, \\ \int_{\Omega_g} \bigg\{ \frac{H_1(\mathbf{u}) + H_1(\mathbf{v})}{2} - H_1(\mathbf{w}) \bigg\} &= \beta_1 \int_{\Omega_g} \bigg\{ 0 + \frac{g^4}{2} - \frac{g^4}{2} - \frac{g^4}{4} \bigg\} = -\frac{\beta_1}{4} \int g^4, \\ \int_{\Omega_h} \bigg\{ \frac{H_1(\mathbf{u}) + H_1(\mathbf{v})}{2} - H_1(\mathbf{w}) \bigg\} &= \beta_1 \int_{\Omega_h} \bigg\{ \frac{h^4}{2} + 0 - \frac{h^4}{2} - \frac{g^4}{4} \bigg\} = -\frac{\beta_1}{4} \int h^4. \end{aligned}$$

Thus, no matter how small β_1 is, $\mathcal{E}$ doesn't have the desired convexity property on $\mathbb{A}_M$. Of course, the $\mathbf{u}$ and $\mathbf{v}$ above are far from ground states, especially due to the assumption that the supports of their components are disjoint. We can go on to suspect $\mathcal{E}$ might satisfy the convexity property when $\mathbf{u}, \mathbf{v} \in \mathbb{A}_M$ are "similar to" ground states. Anyway, uniqueness for our model, if holds, can not be easily obtained from the standard method.

On the other hand, it's also not quite clear whether uniqueness holds from numerical simulations. The trickiest part lies on the bifurcation point $q_c(M)$. To have a better understanding of the problem, remember that the "nonuniqueness" point $(M, q) = (0, 0)$ connects two boundary regimes which sharply contrast each other:

> For $0 < M \leq 1$ and $q = 0$, $\mathbf{u} \in \mathbb{G}_{M,0}$ has $u_0 \equiv 0$, while for $M = 0$ and $q > 0$, u_0 is the only nonvanishing component of $\mathbf{u} \in \mathbb{G}_{0,q}$ (Proposition 4.7).

It's observed in numerical simulations that such sharp contrast also occurs at $q_c(M)$ for $0 < M < 1$, and it's not easy to tell whether u_0 shrinks to zero rapidly as $q \to q_c(M)^+$, or indeed there are both two-component and three-component ground states at $q_c(M)$. In [15], the latter (nonuniqueness) is claimed to be the case. However, in other simulations by using numerical continuation method (Not published private discussions. See [7] for related study.), it looks possible to track the changes of ground state from three-component profiles to the two-component one as $q \to q_c(M)^+$, and hence ground state is unique (for $(M, q) \neq (0, 0)$).

7.2 Uniform convergence at boundary regimes

We have stated the bifurcation phenomenon in terms of varying q and fixed M. This choice is physically natural as the value of q can be tuned by modifying the applied magnetic field. From a mathematical point of view, we might as well consider the bifurcation with respect to other parameters. Somewhat unexpectedly at first sight, there are two difficulties to imitate the proof of Theorem 5.1 if we consider M as the varying parameter. The first one is that we lack an analogue of Claim 2 in Chapter 5. That is, we do not know how to prove that if for some $M \in (0, 1)$ there is $\mathbf{u} \in \mathbb{G}_{M,q}$ with $u_0 > 0$, then every $\mathbf{v} \in \mathbb{G}_{M',q}$ with $0 \leq M' < M$ must have $v_0 > 0$, or equivalently $\mathbf{z}^M \in \mathbb{G}_{M,q}$ implies $\mathbf{z}^{M'} \in \mathbb{G}_{M',q}$ for $1 \geq M' > M$. Thus, we can't prove that there exists a number $M_c(q)$ which definitely separates the two-component regime and the three-component one.

The second problem, which is more fundamental, is that we are not sure whether the Lagrange multipliers will converge as M tends to 1^- or 0^+. Note that in either case $\int u_{-1}^2 \to 0$ for $\mathbf{u} \in \mathbb{G}_{M,q}$, and we can not use the formula (5.9) directly. As a consequence, we can't obtain uniform convergence when $M \to 1^{-1}$ or 0^+ as in Lemma 5.2. Despite of this, we remark that in either situation it's known that the component which is not tending to zero does converge uniformly. For example, let $M_n \to 1^-$ and $\mathbf{u}^n \in \mathbb{G}_{M_n,q}$ converges in $\mathbb{B}$ to the unique element in $\mathbb{G}_{1,q}$, which we denote here also by $\mathbf{u}^\infty = (u_1^\infty, 0, 0)$, then we have $u_1^n \to u_1^\infty$ uniformly. This is because we can still prove $\mu_n + \lambda_n$ converges by using (2.4a), and (2.4a) for $\mathbf{u}^n$ tends to (2.4a) for $\mathbf{u}^\infty$. What really left open is whether u_0^n and u_{-1}^n converge to zero uniformly. This lack of uniform convergence (of u_{-1}^n precisely) then prevents us from imitating the proof of Claim 3 in Chapter 5 to conclude that $u_0^n = 0$ for large n. Similarly, when $M \to 0^+$, we only know u_0 converges uniformly but not for u_1 and u_{-1}. (Of course, this is sufficient to conclude that $u_0 > 0$ when M is close to zero.) As we have mentioned in the remark after Theorem 5.4, such problem also occurs for $\mathbf{z}^M$ when $M \to 1^-$, where z_{-1}^M converges to zero in $\mathbb{B}$, and we don't know if it converges uniformly.

7.3 Comparison of the decaying rates

We are not sure whether $I'(0^+)$ exists for some constructions of redistributional perturbation $\mathbf{u}(\delta)$. An example is given by

$$\begin{cases} u_1(\delta)^2 = (1-\delta)u_1^2 + \tau_{\mathbf{u}}\delta u_{-1}^2 \\ u_0(\delta)^2 = u_0^2 \\ u_{-1}(\delta)^2 = \delta u_1^2 + (1-\tau_{\mathbf{u}}\delta)u_{-1}^2 \,, \end{cases}$$

where $\mathbf{u} \in \mathbb{G}_{M,q}$ is such that $u_j > 0$ for each j, and $\tau_{\mathbf{u}} = (\int u_1^2)/(\int u_{-1}^2)$ is the constant making $\mathbf{u}(\delta) \in \mathbb{A}_M$. Since $u_0(\delta) = u_0$ for each small $\delta \geq 0$, to compute

I' it suffices to compute I'_{kin} and I'_1, of which we do not know the existence of both. Indeed, if we differentiate them formally under the integral signs, we obtain

$$\begin{aligned}
&\int \left\{ \frac{\tau_{\mathbf{u}} S(u_1, u_{-1})}{u_1^2} + \frac{S(u_{-1}, u_1)}{u_{-1}^2} \right\} \\
&\qquad \le 2\beta_1 \int (u_1^2 - u_{-1}^2)(\tau_{\mathbf{u}} u_{-1}^2 - u_1^2) \left(\frac{u_0^2}{u_1 u_{-1}} + 2 \right),
\end{aligned} \tag{7.1}$$

which might be an equation saying $\infty \le \infty$. Note that for the left-hand side of (7.1), we know

$$\frac{S(u_1, u_{-1})}{u_1^2} = \left| \nabla u_{-1} - \frac{u_{-1}}{u_1} \nabla u_1 \right|^2 \in L^1$$

since $u_{-1} \le u_1$, and it's $S(u_1, u_{-1})/u_{-1}^2$ that causes trouble. The problem here is very similar to that mentioned in Remark 6.2. Roughly speaking, they are all due to the fact that we do not have a comparison of the decaying rates of different components. To be precise, we remark directly that some numerical results show that

$$u_0(x) < u_{-1}(x) < u_1(x) \quad \text{for } |x| \text{ large.} \tag{7.2}$$

In fact it looks like

$$u_0(x) = o(u_{-1}(x)) \quad \text{and} \quad u_{-1}(x) = o(u_1(x)) \quad \text{as} \quad |x| \to \infty.$$

If (7.2) can be proved, then the right-hand side of (7.1) is finite, and we can justify the differentiation by Fatou's lemma as in the proof of Proposition 6.2. Also, one can see that all the integrability to be justified in the proof of Theorem 6.4 are obvious, and Theorem 6.4 can be obtained from the GP system (2.4) without using Proposition 6.2.

Chapter 8

Appendices

This chapter contains results which are not the main focuses of the thesis, while are used or at least relevant.

We give some remarks on the notation.

- By a *domain* in $\mathbb{R}^d$ we mean a connected open subset of $\mathbb{R}^d$.
- We use $A \subset\subset B$ to denote the fact that $\bar{A}$ is a compact subset of B, where $\bar{A}$ is the closure of A.

8.1 Convexity inequality for gradients

In the following, let Ω be a domain in $\mathbb{R}^d$. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, where each component is a real-valued function in $H^1(\Omega)$.

Theorem 8.1. $|\mathbf{f}| \in H^1(\Omega)$, *and*

$$\nabla|\mathbf{f}| = \begin{cases} \dfrac{f_1\nabla f_1 + \ldots + f_n\nabla f_n}{|\mathbf{f}|} & \text{on where} \quad |\mathbf{f}| > 0 \\ 0 & \text{on where} \quad |\mathbf{f}| = 0. \end{cases}$$

We omit the proof of this theorem, which is a direct generalization of Theorem 6.17 of [13].

Let $S(f_k, f_\ell)$ denote $|f_k\nabla f_\ell - f_\ell\nabla f_k|^2$. From Theorem 8.1, it's easy to check that

$$\sum_k |\nabla f_k|^2 - |\nabla|\mathbf{f}||^2 = \begin{cases} \dfrac{\sum_{k<\ell} S(f_k, f_\ell)}{|\mathbf{f}|^2} & \text{on where} \quad |\mathbf{f}| > 0 \\ 0 & \text{on where} \quad |\mathbf{f}| = 0. \end{cases} \tag{8.1}$$

Hence

$$|\nabla|\mathbf{f}||^2 \leq \sum_k |\nabla f_k|^2 . \tag{8.2}$$

And the condition of equality given in §7.8 of [13] can also be generalized as follows.

Theorem 8.2. *If $f_n > 0$ and is locally bounded away from zero, i.e. $\inf_K f_n > 0$ for every $K \subset\subset \Omega$, then equality of (8.2) holds a.e. iff there are constants $c_1, c_2, \ldots, c_{n-1}$ such that $f_k = c_k f_n$ a.e. for $k = 1, 2, \ldots, n-1$.*

For uses in this thesis, we give a simple generalization.

Corollary 8.3. *If $|\mathbf{f}|$ is locally bounded away from zero, then equality of (8.2) holds a.e. iff there are constants $c_1, c_2, \ldots, c_n$ such that $f_k = c_k|\mathbf{f}|$ for each k.*

Proof. Let $\mathbf{g} = (g_1, \ldots, g_{n+1}) = (f_1, f_2, \ldots, f_n, |\mathbf{f}|)$. Then $|\mathbf{g}| = \sqrt{2}|\mathbf{f}|$, and it's easy to see that

$$|\nabla|\mathbf{g}||^2 = \sum_{k=1}^{n+1} |\nabla g_k|^2 \quad \text{iff} \quad |\nabla|\mathbf{f}||^2 = \sum_{k=1}^{n} |\nabla f_k|^2 .$$

Thus, by applying Theorem 8.2 to g, we obtain the assertion of the corollary. ☐

8.2 Equivalence of the u-model and the Ψ-model

Define

$$\mathbf{u}^\Psi = (|\psi_1|, |\psi_0|, |\psi_{-1}|) \quad \text{for } \Psi \in \mathbb{B},$$

and, for a fixed triple $(\theta_1, \theta_0, \theta_{-1})$ of real constants satisfying (2.2),

$$\Psi^{\mathbf{u}} = (u_1 e^{i\theta_1}, u_0 e^{i\theta_0}, u_{-1} e^{i\theta_{-1}}) \quad \text{for } \mathbf{u} \in \mathbb{B}_+.$$

We have the following observation.

Lemma 8.4. *If Ψ is a ground state, then $\mathbf{u}^\Psi \in \mathbb{G}$. Conversely, if $\mathbf{u} \in \mathbb{G}$, then $\Psi^{\mathbf{u}}$ is a ground state.*

Proof. We have known from Section 2.1 that $E[\Psi] \geq \mathcal{E}[\mathbf{u}^{\Psi}]$ for every $\Psi \in \mathbb{B}$, and $\mathcal{E}[\mathbf{u}] = E[\Psi^{\mathbf{u}}]$ for every $\mathbf{u} \in \mathbb{B}_{+}$. Thus, for the first statement, given any $\mathbf{v} \in \mathbb{A}$, we have

$$\mathcal{E}[\mathbf{v}] = E[\Psi^{\mathbf{v}}] \geq E[\Psi] \geq \mathcal{E}[\mathbf{u}^{\Psi}].$$

Hence $\mathbf{u}^{\Psi} \in \mathbb{G}$. Similarly, for the second statement, for any $\Phi \in \mathbb{B}$ satisfying the constraints $\mathcal{N}[\Phi] = 1$ and $\mathcal{M}[\Phi] = M$,

$$E[\Phi] \geq \mathcal{E}[\mathbf{u}^{\Phi}] \geq \mathcal{E}[\mathbf{u}] = E[\Psi^{\mathbf{u}}].$$

And hence $\Psi^{\mathbf{u}}$ is a ground state. ☐

Corollary 8.5. *The assertion "Every ground state Ψ has $\psi_0 \equiv 0$" is equivalent to "Every $\mathbf{u} \in \mathbb{G}$ has $u_0 \equiv 0$".*

Proof. Assume the first assertion. Then for any $\mathbf{u} \in \mathbb{G}$, the fact $\Psi^{\mathbf{u}}$ is a ground state implies $u_0 e^{i\theta_0} \equiv 0$. Hence $u_0 \equiv 0$. Conversely, assume the second assertion. Then for any ground state Ψ, the fact $\mathbf{u}^{\Psi} \in \mathbb{G}$ implies $|\psi_0| \equiv 0$, that is $\psi_0 \equiv 0$. ☐

Therefore, using the original Ψ-model is equivalent to using the $\mathbf{u}$-model for studying the bifurcation phenomenon. For the SMA, one direction of the implications is still obvious.

Corollary 8.6. *The assertion "Every ground state obeys the SMA" implies "Every element in $\mathbb{G}$ obeys the SMA".*

Proof. Assume the first statement. Let $\mathbf{u} \in \mathbb{G}$, then

$$\Psi^{\mathbf{u}} = (\gamma_1 \psi, \gamma_0 \psi, \gamma_{-1} \psi)$$

for some constants γ_j and some function $\psi \in H^1(\mathbb{R}^3) \cap L^2_V(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$. That is

$$\mathbf{u} = (e^{-i\theta_1} \gamma_1 \psi, e^{-i\theta_0} \gamma_0 \psi, e^{-i\theta_{-1}} \gamma_{-1} \psi),$$

the form of the SMA. ☐

For the converse, we need the following result. (See relevant discussions after Corollary 8.8.)

Lemma 8.7. *Let Ψ be a ground state, then*

$$\Psi = (|\psi_1|e^{i\eta_1}, |\psi_0|e^{i\eta_0}, |\psi_{-1}|e^{i\eta_{-1}})$$

for some real constants η_1, η_0 and η_{-1}.

Proof. Since Ψ is a ground state, the fact $E[\Psi] \geq \mathcal{E}[\mathbf{u}^\Psi] = E[\Psi^{\mathbf{u}^\Psi}]$ implies

$$E[\Psi] = \mathcal{E}[\mathbf{u}^\Psi]. \tag{8.3}$$

From the derivation of the reduction in Section 2.1, the validity of (8.3) implies

$$|\nabla\psi_j| = |\nabla|\psi_j|| \quad \text{for each } j. \tag{8.4}$$

Now since $\mathbf{u}^\Psi \in \mathbb{G}$, for each j, $|\psi_j| \equiv 0$ or $|\psi_j| > 0$ everywhere. For some fixed j, if $|\psi_j| \equiv 0$, we can choose η_j to be any real constant. On the other hand, if $|\psi_j| > 0$, the fact that $|\psi_j| \in C^1$ implies $|\psi_j|$ is locally bounded away from zero. Thus, applying Corollary 8.3 for $\mathbf{f} = (\mathrm{Re}(\psi_j), \mathrm{Im}(\psi_j))$, we find the equality (8.4) is equivalent to

$$\psi_j = a|\psi_j| + ib|\psi_j| = |\psi_j|(a + ib)$$

for some real constants a and b. Obviously $|a + ib| = 1$, and hence $a + ib = e^{i\eta_j}$ for some constant η_j. ☐

Corollary 8.8. *The assertion "Every element in $\mathbb{G}$ obeys the SMA" implies "Every ground state obeys the SMA".*

Proof. Assume the first assertion. Then let Ψ be a ground state, we have

$$\mathbf{u}^\Psi = (|\psi_1|, |\psi_0|, |\psi_{-1}|) = (\gamma_1 u, \gamma_0 u, \gamma_{-1} u),$$

for some constants γ_j and some function $u \in H^1 \cap L^2_V \cap L^4$. By Lemma 8.7, we then have

$$\Psi = (e^{i\eta_1}\gamma_1 u, e^{i\eta_0}\gamma_0 u, e^{i\eta_{-1}}\gamma_{-1} u)$$

for some real constants η_j. Hence Ψ obeys the SMA. ☐

Discussion 1

One might think, naturally, that the constants η_j in Lemma 8.7 should satisfy (2.2), i.e.

$$\cos(\eta_1 - 2\eta_0 + \eta_{-1}) = -\mathrm{sgn}(\beta_1), \tag{8.5}$$

as is indicated by the reduction. This is not exactly true. Indeed, from the reduction in Section 2.1, The equality (8.3) holds iff (8.4) holds and

$$|\psi_0|^2|\psi_1||\psi_{-1}|\cos(\eta_1 - 2\eta_0 + \eta_{-1}) = -\mathrm{sgn}(\beta_1)|\psi_0|^2|\psi_1||\psi_{-1}|.$$

Hence (8.5) is required only when $\psi_j \neq 0$ for each j. While if one of the ψ_j vanishes, its phase plays no role and in principle can be arbitrary. And the phase(s) corresponding to nonvanishing component(s) can be arbitrary real constant(s).

Discussion 2

There is a more intuitive, and frequently adopted, way to see why the phases of the components of a ground state should be constants. That is by directly differentiating the polar form of ψ_j. To see this, we again let $\psi_j = |\psi_j|e^{i\eta_j}$, of which η_j is not yet known to be a constant. Then we have

$$\nabla\psi_j = e^{i\eta_j}\nabla|\psi_j| + i|\psi_j|e^{i\eta_j}\nabla\eta_j, \tag{8.6}$$

from which

$$|\nabla\psi_j|^2 = |\nabla|\psi_j||^2 + |\psi_j|^2|\nabla\eta_j|^2.$$

Hence $|\nabla\psi_j|^2 \geq |\nabla|\psi_j||^2$, and the equality holds if $|\psi_j|^2|\nabla\eta_j|^2 \equiv 0$. From this result, we obtain the same conclusion that η_j is a constant for nonvanishing ψ_j.

The differentiation (8.6) is however somewhat formal. There is no problem for $|\psi_j|$, which lies in H^1 as long as ψ_j does. Nevertheless, the differentiability of η_j is not automatically ensured, even if we have known that $\psi_j \in C^1$. For example, ix ($x \in \mathbb{R}$) is a smooth function, while its phase $\theta(x)$ satisfies

$$e^{i\theta(x)} = \begin{cases} i & \text{if } x > 0 \\ -i & \text{if } x < 0. \end{cases}$$

And hence $\theta(x)$ must have a jump discontinuity at $x = 0$. This problem is resolved again by the fact that $|\psi_j|$ is either identically zero or positive everywhere. The proof however relies on a nontrivial result asserting the possibility of "lifting" a S^1-valued function without losing regularity. Precisely, the validity of (8.6) for ground state Ψ is given by the following fact.

Lemma 8.9. *Let $\Omega \subset \mathbb{R}^d$ be a (smooth) bounded domain which is simply connected. Let $f \in H^1(\Omega, \mathbb{C})$ be such that $|f|$ is bounded away from zero, then $f = |f|e^{i\theta}$ for some $\theta \in H^1(\Omega, \mathbb{R})$.*

Proof. Since $|f|$ is bounded away from zero, $f/|f| \in H^1(\Omega, S^1)$. Thus $f/|f| = e^{i\theta}$ for some $\theta \in H^1(\Omega, \mathbb{R})$ [4]. ☐

Remark 8.1. The θ in the above proof is called a lifting of $f/|f|$. One can consult [5] for more about assertions on the regularity of lifting.

8.3 Complements to Section 2.2

The main results in this section are Proposition 8.12 and Proposition 8.14. They contain respectively the following assertions given in Section 2.2:

- the strict positivity of nonvanishing components of elements in $\mathbb{G}$, and
- Proposition 2.2.

8.3.1 Positivity of nonvanishing components

We first give a special case of Theorem 8.19 (the strong maximum principle) of [8].

Proposition 8.10. *Let Ω be a domain in $\mathbb{R}^d$. Suppose $v \in H^1(\Omega, \mathbb{R})$ satisfies (in the sense of distribution)*

$$\Delta v + d(x)v \geq 0 \text{ in } \Omega$$

for some measurable function $d(x)$ which is bounded and nonpositive. Then, if for some ball $B \subset\subset \Omega$ we have

$$\sup_B v = \sup_\Omega v \geq 0,$$

the function v must be constant in Ω.

Corollary 8.11. *Let Ω be a domain in $\mathbb{R}^d$. Suppose $u : \Omega \to [0, \infty)$ is of class C^1. If u satisfies*

$$\Delta u + d(x)u \leq 0 \text{ in } \Omega$$

for some $d(x) \in L^\infty_{loc}(\Omega)$, then either $u \equiv 0$ or $u > 0$ on Ω.

Proof. Assume $u(x_0) = 0$ for some $x_0 \in \Omega$. We want to prove $u \equiv 0$. Let Ω_1 be a subdomain of Ω such that $x_0 \in \Omega_1 \subset\subset \Omega$. Since $d(x) \in L^\infty_{loc}(\Omega)$, $d(x)$ is bounded on Ω_1. Let c be a positive constant such that $\widetilde{d}(x) := d(x) - c \leq 0$ on Ω_1. Then $v := -u$ satisfies

$$\Delta v + \widetilde{d}(x)v = -\big(\Delta u + d(x)u\big) + cu \geq 0 \text{ in } \Omega_1.$$

Thus we can apply Proposition 8.10 to v, and $v \equiv 0$ (i.e. $u \equiv 0$) follows the fact

$$\sup_B v = \sup_{\Omega_1} v = v(x_0) = 0,$$

for arbitrary ball B satisfying $x_0 \in B \subset\subset \Omega_1$. $\square$

Proposition 8.12. *Let $\mathbf{u} \in \mathbb{G}$. Then for each u_j, either $u_j \equiv 0$ or $u_j > 0$ everywhere.*

Proof. (2.4b) can be arranged as

$$\Delta u_0 + d_0(x)u_0 = 0,$$

where

$$d_0(x) = \mu - V(x) - 2\beta_0|\mathbf{u}|^2 - 2\beta_1\left(u_1 - \operatorname{sgn}(\beta_1)u_{-1}\right)^2.$$

Similarly, from (2.4a) and (2.4c) we have, for $j = 1, -1$,

$$\Delta u_j + d_j(x) u_j = -2\beta_1 \mathrm{sgn}(\beta_1) u_0^2 u_{-j} \leq 0,$$

where

$$d_j(x) = \mu + j\lambda - V - 2\beta_0 |\mathbf{u}|^2 - 2\beta_1 \left(u_0^2 + u_j^2 - u_{-j}^2 \right) - q.$$

By the fact that $\mathbf{u} \in C^1$ and the assumption (A1), $d_j(x)$ is locally bounded for each $j = 1, 0, -1$. The assertion of the proposition thus follows Corollary 8.11 ☐

8.3.2 Proof of Proposition 2.2

We next prove Proposition 2.2. Note that we write H^1 for $H^1(\mathbb{R}^3, \mathbb{C})$, and similarly for L_V^2 and L^4. A sequence $\{(u_1^n, u_0^n, u_{-1}^n)\}$ in $(H^1)^3$ is said to be (weakly) convergent in H^1 if it is (weakly) convergent in $(H^1)^3 = H^1 \oplus H^1 \oplus H^1 = H^1(\mathbb{R}^3, \mathbb{R}^3)$, which is equivalent to say $\{u_j^n\}$ is (weakly) convergent in H^1 for each $j = 1, 0, -1$. The same convention applies to (weak) convergence in L_V^2 and in L^4.

We'll use without proof the following facts.

1. $\mathbb{B}$ is a reflexive Banach space, in which weak convergence is equivalent to weak convergence in H^1, in L_V^2, and in L^4 separately.
2. Since $\mathbb{B}_+$ is a convex and closed subset of $\mathbb{B}$, $\mathbb{B}_+$ is a weakly closed subset of $\mathbb{B}$ (Mazur's theorem).
3. H^1, L_V^2, and L^4 are uniformly convex.

Remark 8.2. For our purpose, we can in fact "define" weak convergence in $\mathbb{B}$ to be weak convergence in H^1, in L_V^2 and in L^4, without knowing that this definition is really equivalent to weak convergence in the Banach space $\mathbb{B}$. Some arguments should then be modified, for example the reason $\mathbb{B}_+$ is weakly closed in $\mathbb{B}$. This, though works, is of course very unsatisfactory.

Remark 8.3. Although these facts are well-known, some of them usually do not appear in standard courses. Indeed, I myself got the answer of the first claim from *mathoverflow.net* (Thanks Dr. William B. Johnson), and the uniform convexity of general Sobolev spaces (but not only Lebesgue spaces) was found on a page of *math.stackexchange.com* (asked by Tomás and answered by martini).

We'll also need the following observation.

Lemma 8.13. *For* $\beta_1 < 0$ *(and* $|\beta_1| < \beta_0$ *by the assumption (A2)), we have for every* $\mathbf{u} \in \mathbb{A}$

$$\mathcal{E}_0[\mathbf{u}] + \mathcal{E}_1[\mathbf{u}] = (\beta_0 + \beta_1) \int |\mathbf{u}|^4 - \beta_1 \int \left(u_0^2 - 2u_1 u_{-1}\right)^2 .$$

In particular,

$$\mathcal{E}_0[\mathbf{u}] + \mathcal{E}_1[\mathbf{u}] \geq (\beta_0 + \beta_1) \int |\mathbf{u}|^4.$$

Proof. The assertion is a direct consequence of the following identity:

$$|\mathbf{u}|^4 - \left[2u_0^2(u_1 + u_{-1})^2 + (u_1^2 - u_{-1}^2)^2\right] = \left(u_0^2 - 2u_1 u_{-1}\right)^2 . \tag{8.7}$$

□

Remark 8.4. Identity (8.7) is also used in the proof of Theorem 3.1 (equation 3.5).

For convenience we restate Proposition 2.2 below.

Proposition 8.14. *Let* $\{\mathbf{u}^n\}$ *be a sequence in* $\mathbb{B}_+$*. Suppose* $\mathcal{N}[\mathbf{u}^n] \to 1$, $\mathcal{M}[\mathbf{u}^n] \to M$*, and* $\mathcal{E}[\mathbf{u}^n]$ *is uniformly bounded in* n*, then* $\{\mathbf{u}^n\}$ *has a subsequence* $\{\mathbf{u}^{n(k)}\}_{k=1}^{\infty}$ *converging weakly to some* $\mathbf{u}^\infty \in \mathbb{A}$*, which satisfies* $\mathcal{E}[\mathbf{u}^\infty] \leq \liminf_{k\to\infty} \mathcal{E}[\mathbf{u}^{n(k)}]$*. If we assume further that* $\mathcal{E}[\mathbf{u}^n] \to E_g$*, then* $\mathbf{u}^\infty \in \mathbb{G}$*, and* $\mathbf{u}^{n(k)} \to \mathbf{u}^\infty$ *in the norm of* $\mathbb{B}$*.*

Proof. We first remark that with the norm defined by (2.1), $\mathbb{B}$ is a reflexive Banach space, in which weak convergence is equivalent to weak convergence in H^1, in L^2_V, and in L^4 separately. We omit the verifications of these standard facts. Moreover, since

$\mathbb{B}_+$ is a convex and closed subset of $\mathbb{B}$, $\mathbb{B}_+$ is a weakly closed subset of $\mathbb{B}$ (Mazur's theorem).

Note that the uniform boundedness of $\mathcal{E}[\mathbf{u}^n]$ implies $\{\mathbf{u}^n\}$ is a bounded sequence in $\mathbb{B}$. This is obvious if $\beta_1 > 0$, and is also true for $\beta_1 < 0$ by Lemma 8.13. Thus, by the reflexivity of $\mathbb{B}$, $\{\mathbf{u}^n\}$ has a weakly convergent subsequence $\{\mathbf{u}^{n(k)}\}_{k=1}^{\infty}$ in $\mathbb{B}$, of which we denote the weak limit by $\mathbf{u}^\infty$. We have $\mathbf{u}^\infty \in \mathbb{B}_+$ since $\mathbb{B}_+$ is weakly closed in $\mathbb{B}$.

To prove $\mathbf{u}^\infty \in \mathbb{A}$, we shall prove

$$\int \left(u_j^\infty\right)^2 = \lim_{k\to\infty} \int \left(u_j^{n(k)}\right)^2. \tag{8.8}$$

First, by the weak lower semi-continuity of a norm, we have

$$\int \left(u_j^\infty\right)^2 \le \liminf_{k\to\infty} \int \left(u_j^{n(k)}\right)^2. \tag{8.9}$$

On the other hand, to give a suitable estimate of $\limsup_k \int (u_j^{n(k)})^2$, we exploit the facts that (i) (A1) implies $(u_j^{n(k)})^2$ is very small outside a large enough bounded set, and that (ii) on any bounded set, $u_j^{n(k)} \to u_j^\infty$ in L^2 by compact embedding $H^1 \hookrightarrow L^2$. As to (i), note that since $\{\mathbf{u}^n\}$ is a bounded sequence, we have in particular $\int V|\mathbf{u}^{n(k)}|^2 \le C$ for some $C > 0$ independent of k. By the assumption (A1), for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $V(x) \ge C/\varepsilon$ for $|x| \ge R_\varepsilon$. Thus we have

$$C \ge \int V|\mathbf{u}^{n(k)}|^2 \ge \int_{B(R_\varepsilon)^c} V|\mathbf{u}^{n(k)}|^2 \ge \frac{C}{\varepsilon} \int_{B(R_\varepsilon)^c} |\mathbf{u}^{n(k)}|^2,$$

and hence $\int_{B(R_\varepsilon)^c} |\mathbf{u}^{n(k)}|^2 \le \varepsilon$ for each k. In particular

$$\int_{B(R_\varepsilon)^c} \left(u_j^{n(k)}\right)^2 \le \varepsilon \quad \text{for each } k \in \mathbb{N} \text{ and } j = 1, 0, -1.$$

From this fact and the strong convergence mentioned in (ii), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \int \left(u_j^{n(k)}\right)^2 &= \limsup_{k\to\infty} \left(\int_{B(R_\varepsilon)^c} \left(u_j^{n(k)}\right)^2 + \int_{B(R_\varepsilon)} \left(u_j^{n(k)}\right)^2 \right) \\ &\le \varepsilon + \int_{B(R_\varepsilon)} \left(u_j^\infty\right)^2 \\ &\le \varepsilon + \int \left(u_j^\infty\right)^2. \end{aligned} \tag{8.10}$$

Since $\varepsilon > 0$ is arbitrary, (8.9) and (8.10) implies (8.8), and hence $\mathbf{u}^\infty \in \mathbb{A}$.

Next, the assertion $\mathcal{E}[\mathbf{u}^\infty] \leq \liminf_k \mathcal{E}[\mathbf{u}^{n(k)}]$ follows a general weak lower semi-continuity theorem. See e.g. Theorem 1.6 of [20]. Indeed, by that theorem we have

$$\int \left|\nabla u_j^\infty\right|^2 \leq \liminf_{k\to\infty} \int \left|\nabla u_j^{n(k)}\right|^2,$$
$$\int V(x)\left(u_j^\infty\right)^2 \leq \liminf_{k\to\infty} \int V(x)\left(u_j^{n(k)}\right)^2,$$

and

$$\int f(u_1^\infty, u_0^\infty, u_{-1}^\infty) \leq \liminf_{k\to\infty} \int f(u_1^{n(k)}, u_0^{n(k)}, u_{-1}^{n(k)}) \tag{8.11}$$

for every continuous function $f : \mathbb{R}^3 \to [0, \infty)$. As a consequence, we have

$$\begin{aligned} \mathcal{E}_{kin}[\mathbf{u}^\infty] &\leq \liminf_{k\to\infty} \mathcal{E}_{kin}[\mathbf{u}^{n(k)}] \\ \mathcal{E}_{pot}[\mathbf{u}^\infty] &\leq \liminf_{k\to\infty} \mathcal{E}_{pot}[\mathbf{u}^{n(k)}] \\ \mathcal{E}_0[\mathbf{u}^\infty] + \mathcal{E}_1[\mathbf{u}^\infty] &\leq \liminf_{k\to\infty} \left(\mathcal{E}_0[\mathbf{u}^{n(k)}] + \mathcal{E}_1[\mathbf{u}^{n(k)}]\right) \\ \mathcal{E}_{Zee}[\mathbf{u}^\infty] &\leq \liminf_{k\to\infty} \mathcal{E}_{Zee}[\mathbf{u}^{n(k)}]. \end{aligned} \tag{8.12}$$

Note carefully that (8.11) requires that f be nonnegative, and hence the assertion of the weak lower semi-continuity of $\mathcal{E}_0 + \mathcal{E}_1$ in (8.12) also uses Lemma 8.13. It is then clear that the limit inferiors in (8.12) must all be limits provided $\mathcal{E}[\mathbf{u}^n]$ tends to the ground-state energy E_g. Otherwise we get the contradiction

$$E_g = \lim_{k\to\infty} \mathcal{E}[\mathbf{u}^{n(k)}] > \mathcal{E}[\mathbf{u}^\infty].$$

Now $\mathcal{E}_{kin}[\mathbf{u}^{n(k)}] \to \mathcal{E}_{kin}[\mathbf{u}^\infty]$ and (8.8) imply

$$\left\|u_j^{n(k)}\right\|_{H^1} \to \left\|u_j^\infty\right\|_{H^1}. \tag{8.13}$$

Since H^1 is uniformly convex, (8.13) together with the fact $\mathbf{u}^{n(k)} \rightharpoonup \mathbf{u}^\infty$ weakly in H^1 imply $\mathbf{u}^{n(k)} \to \mathbf{u}^\infty$ strongly in H^1. Similarly we can prove $\mathbf{u}^{n(k)} \to \mathbf{u}^\infty$ in L_V^2 and in L^4, and hence $\mathbf{u}^{n(k)} \to \mathbf{u}^\infty$ in the norm of $\mathbb{B}$. $\square$

Bibliography

[1] Weizhu Bao and Yongyong Cai. Ground states of two-component Bose-Einstein condensates with an internal atomic Josephson junction. *EAJAM*, 1(1):49–81, 2011.

[2] Weizhu Bao and Fong Yin Lim. Computing ground states of spin-1 Bose-Einstein condensates by the normalized gradient flow. *SIAM J. Sci. Comput.*, 30(4):1925–1948, 2008.

[3] Weizhu Bao and Hanquan Wang. A mass and magnetization conservative and energy-diminishing numerical method for computing ground state of spin-1 Bose-Einstein condensates. *SIAM J. Numer. Anal.*, 45(5):2177–2200, 2007.

[4] Fabrice Bethuel and Xiaomin Zheng. Density of smooth functions between two manifolds in sobolev spaces. *Journal of Functional Analysis*, 80(1):60 – 75, 1988.

[5] Jean Bourgain, Haim Brezis, and Petru Mironescu. Lifting in Sobolev spaces. *J. Anal. Math.*, 80:37–86, 2000.

[6] Daomin Cao, I-Liang Chern, and Jun-Cheng Wei. On ground state of spinor Bose-Einstein condensates. *NoDEA Nonlinear Differential Equations Appl.*, 18(4):427–445, 2011.

[7] R-H Chen, I-L Chern, and W-C Wang. Exploring ground states and excited states of spin-1 Bose-Einstein condensates by continuation methods. *Journal of Computational Phyics*, 2011.

[8] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[9] Tin-Lun Ho. Spinor Bose Condensates in Optical Traps. *Phys. Rev. Lett.*, 81(4):742–745, Jul 1998.

[10] David Jacob, Lingxuan Shao, Vincent Corre, Tilman Zibold, Luigi De Sarlo, Emmanuel Mimoun, Jean Dalibard, and Fabrice Gerbier. Phase diagram of spin-1 antiferromagnetic Bose-Einstein condensates. *Phys. Rev. A*, 86:061601, Dec 2012.

[11] Yuki Kawaguchi and Masahito Ueda. Spinor Bose-Einstein condensates. *Physics Reports*, 520(5):253 – 381, 2012.

[12] C. K. Law, H. Pu, and N. P. Bigelow. Quantum spins mixing in spinor Bose-Einstein condensates. *Phys. Rev. Lett.*, 81(24):5257–5261, Dec 1998.

[13] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[14] Elliott H. Lieb, Robert Seiringer, and Jakob Yngvason. Bosons in a trap: A rigorous derivation of the gross-pitaevskii energy functional. *Phys. Rev. A*, 61:043602, 2000.

[15] Fong Yin Lim and Weizhu Bao. Numerical methods for computing the ground state of spin-1 Bose-Einstein condensates in a uniform magnetic field. *Phys. Rev. E*, 78(6):066704, Dec 2008.

[16] L. Lin and I. Chern. Proofs of some simplified characterizations of the ground states of spin-1 Bose-Einstein condensates. *ArXiv e-prints*, February 2011.

[17] L. Lin and I. Chern. Bifurcation between 2-component and 3-component ground states of spin-1 Bose-Einstein condensates in uniform magnetic fields. *ArXiv e-prints*, February 2013.

[18] Tetsuo Ohmi and Kazushige Machida. Bose-Einstein condensation with internal degrees of freedom in alkali atom gases. *Journal of the Physical Society of Japan*, 67(6):1822–1825, 1998.

[19] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle. Optical confinement of a Bose-Einstein condensate. *Phys. Rev. Lett.*, 80(10):2027–2030, Mar 1998.

[20] Michael Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, fourth edition, 2008.

[21] S. Yi, Ö. E. Müstecaplıoğlu, C. P. Sun, and L. You. Single-mode approximation in a spinor-1 atomic condensate. *Phys. Rev. A*, 66(1):011601, Jul 2002.

[22] Wenxian Zhang, Su Yi, and Li You. Mean field ground state of a spin-1 condensate in a magnetic field. *New Journal of Physics*, 5(1):77, 2003.