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穩態中的不穩定狀態
以偏微分方程為出發之研究與探討

A Survey of Instability of Steady States

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致　謝

在這兩年半的學期過程中，不免會遇到學業上的問題，與一般的研究生不同，我先經歷了教師實習後才進入研究所，多了半年的間隔讓我一開始不太適應，感謝身旁總是有同學的幫助，一起討論問題鑽研數學與程式語言，謝謝王柏川、吳志強、郭彥祥、林士閔，以及學長的教導，非常感謝王藹農教授每當我遇到問題時，耐心地、細心地教導我，直到完全確定我清楚了解才會安心的離去，也感謝我的父母、孫鈺婷在研究所個過程中給我支持與鼓勵，還有我的大學同學洪崇峰、陳昱達、黃俊淵，讓我在寫論文之餘也不會與教育脫節，兩年半來要感謝的人太多了，期許自己能夠在未來發揮所學，把這兩年半來了學習貢獻給學校與社會，我想這就是最好的回饋方式吧!

Contents

中文摘要

在這篇論文中，我們主要討論拋物方程與雙曲方程穩態解的情形。我們會為讀者準備充分的先備知識，由淺入深地從基本的定義開始介紹，最終會接到我們的主題 -由橢圓方程 $L\varphi = f(x, \varphi)$,$x \in \mathbb{R}^n$ 出發，其中 φ 是時間獨立的解，在某些假定的條件之下，我們將可以由穩態的解中導出不穩定狀態的結果，為了完成我們的工作，我們主要的參考文獻為 Manoussos Grillakis, Jalal Shatah, 以及 Walter Strauss 合力完成的 [2] 與 Paschalis Karageorgis, Walter A Strauss 共同完成的 [3]，基本知識的準備我們主要參考 Lawrence C. Evans 的著作 [1] 與 Walter A. Strauss 的著作 [4]。

關鍵字：非線性熱方程、非線性波方程、穩態、時間獨立

Abstract

We consider the steady states solutions of parabolic and hyperbolic equations such as $\partial_t u - \Delta u = f(x,u)$ and $\partial_{tt} u - \Delta u = f(x,u)$. Steady state which means a system that has numbers of properties that are unchanged in time. For instance, property p of the steady state system has zero partial derivative with respect to time : $\frac{\partial p}{\partial t} = 0$.

In this thesis we will give a proof about the instability results about the solutions of a general elliptic equation of the form $L\varphi = f(x,\varphi)$,$x \in \mathbb{R}^n$,where L is a linear,second-order elliptic differential operator whose coefficients are smooth and bounded. φ is the time-independent solution of $Lu = f(x,u)$,$x \in \mathbb{R}^n$. **To complete our work, we mainly consult paper[2] and [3].Also for some basic preliminaries we consult text books[1] and [4].**

Keywords: Nonlinear heat equation; Nonlinear wave equation; Steady states; Instability

Chapter 1

Introduction

In this thesis we consider the parabolic equation

$$\partial_t u + Lu = f(x,u), x\ \mathbb{R}^n \tag{1.1}$$

and the more complicated hyperbolic equation

$$\partial_t^2 u + a\partial_t u + Lu = f(x,u), x \in \mathbb{R}^n \tag{1.2}$$

while the L is the linear second-order elliptic differential operator whose coefficients are smooth and bounded. a is arbitrary real number and possibly zero. $f(x,u)$ is the nonlinear term. To studying many topics in PDE it's quite important to understanding the physical properties. When we studying the heat equation(also known as diffusion equation), we know that it describes the distribution of heat in a region. When studying the wave equation, it related to the models for vibrating string(n=1), membrane(n=2) and elastic solid(n=3). Now we focus on time-independent solutions, which known as the steady states in this thesis. Our main work is to provide sufficient conditions such that we can draw nonlinear instability form linearized instability.

Before we come to the main instability result we have the following assumptions about the time-independent solution φ and the nonlinear term $f(x,u)$. The assumptions are introduce in two parts, first part is meant to conclude that the φ is a nonlinearly unstable solution of both (1.1) and (1.2). The second part is meant to improve the conclusion of first part that the instability occurs by blow up.

First Part:

(A1)The equation $L\varphi = f(x,\varphi)$ has a C^2 solution φ.

(A2)The adjoint linearized operator $L^* - f_u(x,\varphi)$ has a negative eigenvalue $-\sigma$ and a corresponding eigenfunction $\chi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ which is non-negative.

(A3)The nonlinear term $f(x,s)$ is convex in s and is C^1.

(A4)Both $f(x,\varphi)$ and $f_u(x,\varphi)$ are bounded.

Second Part:

(A5)The product $\varphi\chi$ is integrable, where χ is the eigenfunction mentinoed in (A2).

(A6)There exist $C_0 > 0$ and $p > 1$ such that $f(x,s) \geq C_0|s|^p$ for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}$

In our assumptions, (A2) is to ensure that the presence of an negative eigenvalue. Besides, the time independent solution φ is not necessarily to be bounded. In Chapter3 we are going to prove the instability by using the first part of the assumptions and the instability by blow up by using the both parts of the assumptions. There is a special case that when $f(u) = |u|^p$ for some $p > 1$ and φ is bounded. We can get the instability by blow up only us assumptions (A1) and (A2).

1.1 Literature Review

The paper [2] talks about that with arbitrary nonlinearities and more general class of bounded states, but the assumptions they give are more restrictive.

The paper [3] is our mainly consult. With the assumptions (A1)-(A6) they talk about the instability in a harder way. We are going to redesign and rearrange the statements to make the reader more readily to appreciate the results. Besides, we use [1] and [4] to give some basics for the readers.

Chapter 2

Background

2.1 Some Inequalities

2.1.1 Convex Function

Definition 1. A function $F : \mathbb{R}^n \to \mathbb{R}$ is convex provided that

$$f(\tau x + (1-\tau)y) \leq \tau f(x) + (1-\tau)f(y)$$

for all $x, y \in \mathbb{R}^n$ and for $0 \leq \tau \leq 1$

If a function f is a convex function then f has the following properties

* f is concave if $-f$ is convex

* f is strictly convex if f satisfies $f(\tau x+(1-\tau)y) < \tau f(x)+(1-\tau)f(y)$, for all $x, y \in \mathbb{R}^n$ and $0 \leq \tau \leq 1$

* (Supporting hyperplanes)Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Then for each $x \in \mathbb{R}^n$ there exists $r \in \mathbb{R}^n$ such that the inequality $f(y) \geq f(x) + r \cdot (y - x)$ holds for all $y \in \mathbb{R}$.

2.1.2 Elementary Inequalities

In this section we will introduce some basic inequality that might be used in our thesis

I.Cauchy's Inequality

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \quad a, b \in \mathbb{R} \tag{2.1}$$

Proof. $0 \leq (a-b)^2 = a^2 - 2ab + b^2$ □

II.Young's inequality Let $1 < p, q < \infty$,also $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad a, b > 0 \tag{2.2}$$

Proof. Consider the mapping $x \mapsto e^x$ is convex, by what we've mentioned above we have

$$ab = e^{loga+logb} = e^{\frac{1}{p}loga^p + \frac{1}{q}logb^q} \leq \frac{1}{p}e^{loga^p} + \frac{1}{q}e^{logb^q} = \frac{a^p}{p} + \frac{b^q}{q}$$

□

III.H$\ddot{o}$lder's inequality Assume that $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(U)$,$v \in L^q(U)$, we have

$$\int_U |uv| dx \le ||u||_{L^p(U)} ||v||_{L^q(U)}$$

Proof. By homogeneity we may assume that $||u||_{L^p} = ||v||_{L^q} = 1$. Then we apply Young's inequality, for $1 < p, q < \infty$ we have that

$$\int_U |uv| dx \le \frac{1}{p} \int_U |u|^p dx + \frac{1}{q} \int_U |v|^q dx = 1 = ||u||_{L^p} ||v||_{L^q}$$

□

The previous three inequalities will be use in our thesis, the readers can get acquaintance with these inequalities for the further reading. For more details, readers can check reference [1] APPENDIX B.

2.2 Integration by Parts

One of our assumption for the main instability result contains the convex function, hence we have a brief talk about it in the previous section. Also integration by parts has been used in the proof of the main instability result. The following is the integration by parts formula, which will be wildly used in our thesis

For $u = u(x)$, $v = v(x)$

$$\int u dv = uv - \int v du$$

Chapter 3

Instability Result

In [3] has some discussion about the instability of steady states. We will use an easier way to talk about the main instability results with the solution of a general elliptic equation of the form

$$L\varphi = f(x, \varphi),\ x \in \mathbb{R}^n$$

For both L and φ are what we've defined in Chapter 1. L is a linear, second-order elliptic differential operator whose coefficients are smooth and bounded. φ is the time-independent solution. Besides the assumptions $(A1) - (A6)$ mentioned in Chapter 1 will be used in our proof, we'll not repeat them here.

First, we deal with the parabolic case which is much easier than the hyperbolic case. The reader can get acquaintance with the proof technique in parabolic case, after that we will give two lemmas for hyperbolic case.

Finally we'll come to the most complicated part the hyperbolic case.

3.1 Parabolic Equation

First we focus on the equation

$$\partial_t u + Lu = f(x,u),\ \ u(x,0) = \varphi(x) + \psi_0(x)$$

The steady state φ will be an exact solution when the perturbation $\psi_0 \equiv 0$. Now we consider the case that the perturbation ψ_0 is small.

Theorem 1. (Parabolic Equation)

For the first part of the assumptions (A1-A4) we let $\psi_0 \in L^\infty(\mathbb{R}^n)$ be continuous with

$$\int_{\mathbb{R}^n} \chi(x)\psi_0(x)dx > 0 \tag{3.1}$$

Let $0 < T \leq \infty$ and let u be a solution of

$$\partial_t u + Lu = f(x,u),\ \ u(x,0) = \varphi(x) + \psi_0(x) \tag{3.2}$$

on [0,T) such that $u - \varphi$ is continuous and bounded.

(a)If $T = \infty$ the the norm $||u - \varphi||_{L^\infty(\mathbb{R}^n)}$ must grow exponentially.

(b)Add assumptions (A5),(A6) then we can have that $T < \infty$

Proof. (a)First consider the function

$$G(t) = \int_{\mathbb{R}^n} \chi(x) \cdot w(x,t)dx, \quad w(x,t) = u(x,t) - \varphi(x) \tag{3.3}$$

We have

$$|G(t)| \leq ||\chi||_{L^1(\mathbb{R}^n)} \cdot ||w||_{L^\infty(\mathbb{R}^n)}$$

It's quite simple for us to find that the $L^\infty - norm$ has to either grow exponentially or blow up whenever $G(t)$ does.

Let us first focus on part (a) and assume that $w = u - \varphi$ is continuous on [0,∞). By distribution we have that w is a solution of

$$\partial_t w + Lw = f(x, w + \varphi) - f(x, \varphi) \ , \quad w(x, 0) = \psi_0(x)$$

Then in view of the convexity assumption (A3) we have

$$\partial_t w + [L - \frac{f(x, w + \varphi) - f(x, \varphi)}{(w + \varphi - \varphi)}]w \geq 0 \tag{3.4}$$

which gives that

$$\partial_t w + [L - f_u(x, \varphi)]w \geq 0$$

Also in assumption (A2) we note that $\chi(x) \geq 0$ ($\chi(x)$ is non-negative), we may multiply the inequality by $\chi(x)\theta(t)$, where $\theta(t)$ is an arbitrary non-negative test function. Then we obtain the following inequality

$$\partial_t w\chi(x)\theta(t) + [L - f_u(x, \varphi)]w\chi(x)\theta(t) \geq 0$$

Then we integral both parts to get the following

$$\int_0^t \int_{\mathbb{R}^n} \chi(x) \cdot \partial_t w \cdot \theta(\tau) dx d\tau + \int_0^t \int_{\mathbb{R}^n} [L^* - f_u(x, \varphi)]\chi(x) \cdot w \cdot \theta(\tau) dx d\tau \geq 0 \tag{3.5}$$

To simplify the first part of the integral we note that

$$G'(t) = \int_{\mathbb{R}^n} \chi(x)\partial_t w(x,t)dx$$

Which is a continuous function by what we've defined at (3.3), for $\partial_t w$ is continuous with values in $L^2(\mathbb{R}^n)$.To simplify the second part of the integral, by assumption (A2) we have

$$[L^* - f_u(x,\varphi)]\chi = -\sigma^2\chi$$

Then the equation (3.5) can be reduce to

$$\int_0^t G'(\tau)\cdot\theta'(\tau)d\tau - \sigma^2\int_0^t G(\tau)\cdot\theta(\tau)d\tau \geq 0$$

For all non-negative test function $\theta(t)$,eigenfunction χ, equivalently we have

$$G'(t) - \sigma^2 G(t) \geq 0 \tag{3.6}$$

Since $G(0) > 0$ by the assumption, the exponential growth of $G(t)$ follows.

(b)Next we add assumptions (A5) and (A6) for part (b). Suppose that $T = \infty$. As we have just shown $G(t)$ must grow exponentially fast. Using our assumption (A4) Both $f(x,\varphi), f_u(x,\varphi)$ are bounded and (A6) There exists $C_0 > 0$ and $p > 1$ such that $f(x,s) \geq C_0|s|^p$ for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}$, we have the following

$$\partial_t w + [L - f_u(x,\varphi)]w = f(x, w+\varphi) - f(x,\varphi) - f_u(x,\varphi)w \geq$$

$$C_0|w+\varphi|^p - C_1 - C_1|w|$$

Then we multiply both sides by the non-negative eigenfunction χ and integrating over space, we can have

$$G'(t) - \sigma^2 G(t) \geq C_0 \int_{\mathbb{R}^n} \chi |w + \varphi|^p dx - C_1 \int_{\mathbb{R}^n} \chi dx - C_1 \int_{\mathbb{R}^n} \chi |w| dx \quad (3.7)$$

Then by triangular inequality we have the last integral is at most

$$\int \chi |w| dx = \int \chi |(w + \varphi) - \varphi| dx \leq \int \chi |w + \varphi| dx + \int \chi |\varphi| dx$$

We can therefore reduce equation (3.7) to be

$$G'(t) \geq \sigma^2 G(t) + C_0 \int_{\mathbb{R}^n} \chi |x + \varphi|^p dx - C_1 \int_{\mathbb{R}^n} \chi dx - C_1 \left(\int_{\mathbb{R}^n} \chi |w + \varphi| dx + \int_{\mathbb{R}^n} \chi |\varphi| dx \right)$$

Since $\chi \in L^1$ by (A2) and $\chi\varphi \in L^1$ by (A5), we then deduce that

$$C_1 \int_{\mathbb{R}^n} \chi dx + C_1 \int_{\mathbb{R}^n} \chi |\varphi| dx \leq C_2$$

$$G'(t) \geq \sigma^2 G(t) + C_0 \int_{\mathbb{R}^n} \chi |x + \varphi|^p dx - C_1 \int_{\mathbb{R}^n} \chi |w + \varphi| dx - C_2$$

Since $G(t)$ grows exponentially fast we have

$$G'(t) \geq C_0 \int_{\mathbb{R}^n} \chi |w + \varphi|^p dx - C_1 \int_{\mathbb{R}^n} \chi |w + \varphi| dx \equiv C_0 A(t) - C_1 B(t) \quad (3.8)$$

by distribution and for all large enough t.

For $B(t)$ grows exponentially fast by the triangular inequality we have

$$G(t) \leq \int_{\mathbb{R}^n} \chi |w| dx \leq \int_{\mathbb{R}^n} \chi |w + \varphi| dx + \int_{\mathbb{R}^n} \chi |\varphi| dx = B(t) + C_3 \quad (3.9)$$

For $A(t)$ grows exponentially fast by the Hölder's inequality we have

$$B(t) = \int_{\mathbb{R}^n} \chi|w+\varphi|dx \le \left(\int_{\mathbb{R}^n} \chi dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} \chi|w+\varphi|^p dx\right)^{\frac{1}{p}} = C_4 A(t)^{\frac{1}{p}} \tag{3.10}$$

This inequality gives that $A(t)$ grows faster than $B(t)$ i.e.

$$A(t) \ge C_4^{-p} B(t)^p \tag{3.11}$$

Hence the equation $A(t)$ dominate $B(t)$. Then combining (3.8), (3.9) and (3.11) we have the inequality

$$G'(t) \ge C_5 A(t) \ge C_6 B(t)^p \ge C_7 G(t)^p \tag{3.12}$$

Since $G(t)$ is positive, we can deduce that $T < \infty$ as needed. □

3.2 Hyperbolic Equation

To complete the part(a) of Hyperbolic case, in [3] we have the following lemma.

Lemma 1. Let $a \in \mathbb{R}$ and $b > 0$. Suppose $y(t)$ is a C^1 function such that

$$y'' + ay' - by \ge 0$$

on some interval $[0, T)$ in the sense of distributions. If

$$\frac{a + \sqrt{a^2 + 4b}}{2} \cdot y(0) + y'(0) > 0 \tag{3.13}$$

the both $y(t)$ and $y'(t)$ must grow exponentially on $[0, T)$

Proof. Let λ_1, λ_2 be the roots of the characteristic equation $\lambda^2 + a\lambda - b = 0$ and set $z = y' - \lambda_1 y$ i.e. $z' = y'' - \lambda_1 y'$. Besides, the roots $\lambda_1 < 0 < \lambda_2$. Then

$$y'' + ay' - by \geq 0 \rightarrow y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y \geq 0$$

$$(y'' - \lambda_1 y') + \lambda_2(y' - \lambda_1 y) = z' - \lambda_2 z \geq 0$$

Using test function $\theta(t)$ we obtain the another test function $e^{-\lambda_2 t}\theta(t)$, use integration by parts it follows that

$$-\int_0^t z(\tau)e^{-\lambda_2\tau}\theta'(\tau)d\tau \geq 0$$

Choosing $\theta(t)$ to be an approximation of the characteristic function on the interval $(0, t)$, we can easily deduce that $z(t)e^{-\lambda_2 t} - z(0) \geq 0$ i.e. $z(t) \geq e^{\lambda_2 t}z(0)$. Thus we have $y' - \lambda_1 y \geq e^{\lambda_2 t}z(0)$ (for we have $z = y' - \lambda_1 y$), which implies the following inequality

$$y(t) \geq e^{\lambda_1 t}y(0) + \tfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \cdot z(0)$$

Then by what we've let at the beginning that $\lambda_1 < 0 < \lambda_2$, the exponential growth of y the follows, provided that

$$z(0) = y'(0) - \lambda_1 y(0) = y'(0) + \frac{a + \sqrt{a^2 + 4b}}{2} \cdot y(0) \quad \textit{is positive.}$$

Then in the view of our assumption (3.13), the exponential growth of y thus follows.

Finally for the exponential growth of $y'(t)$. Similarly we assume $w = y' - \lambda_2 y$ which satisfies

$$y'' + ay' - by = (y'' - \lambda_1 y) + \lambda_2(y' - \lambda_1 y) = w' - \lambda_1 w \geq 0$$

As what we've done above, we have $w(t) \geq e^{\lambda_1 t} w(0)$. Since $y(t)$ grows exponentially by above, then the equation

$$y'(t) \geq \lambda_2 y(t) + e^{\lambda_1 t} w(0)$$

force $y'(t)$ to grow exponentially fast as well because $\lambda_1 < 0 < \lambda_2$ the we complete the proof.

The result will be very useful in the proof of Theorem 2.-Hyperbolic Equation. The reader can think more carefully about the Lemma before the further reading. □

To complete part(b) of Hyperbolic case,we need the following lemma. In [5] has a special case which is similar to the case of the following lemma. We give a little modification of the property in [5] to complete our proof.

Lemma 2. Let $a \in \mathbb{R}$, $b > 0$ and $p > 0$. Suppose $y(t)$ is a non-negative C^1 function such that

$$y(T_1) > 0, \;\; y'(T_1) > 0, \;\; y''(t) + ay'(t) \geq by(t)^p$$

on some interval $[T_1, T_2)$ in the sense of distributions. Then $T_2 < \infty$

Proof. We'll start in an easier way. To claim that $y''(t) + ay'(t) \geq by(t)^p$. First we start form the special case $y \in C^2$ and a=1. Now we consider the in equality $F''(t) + F'(t) \geq C_0(t+K)^A|F(t)|^{1+r}$, and $t > 0$ with $C_0 > 0$ such that $F(0) > 0$ and $F'(0) > 0$. We are going to claim that $F(t)$ blows up in finite time. Now we take the auxiliary initial value problem

$$Y'(t) = v(t+K)^A[Y(t)]^{1+r/2}, F(0) > 0 \tag{3.14}$$

Where $v > 0$ is a small number to be chosen later. Since

$$Y(t) = \Big(Y(0)]^{-r/2} - \tfrac{vr}{2(A+1)}[(t+K)^{A+1} - K^{A+1}]\Big)^{-2/r}$$

and A>-1, the solution $Y(t)$ of the above problem blows up at finite time T_0 and satisfies $Y(t) > Y(0) > 0$ for $0 \leq t < T_0$. Then we compute the second derivative

$$Y''(t) = v(1+r/2)(t+K)^A[Y(t)]^{r/2}Y'(t) + vA(t+K)^{A-1}[Y(t)]^{1+r/2}$$

$$\geq v^2(1+r/2)(t+K)^{2A}[Y(T)]^{1+r} \tag{3.15}$$

where we have that $A \geq 0$ and Y satisfies (3.14). Now we add (3.14) and (3.15) also observing that $2A \geq A$ (For we have $A \geq 0$) and $[Y(t)]^{1+r/2} < [Y(0)]^{-r/2}[Y(t)]^{1+r}$, then we have

$$Y''(t) + Y'(t) \geq v^2(1+r/2)(t+K)^{2A}[Y(t)]^{1+r} + v(t+K)^A[Y(t)]^{1+r/2}$$

$$\geq B(t+K)^A[Y(t)]^{1+r}$$

Where $B = v^2(1 + r/2) + vA[Y(0)]^{-r/2}$

Further, we choose v so small such that

$$B = v^2(a + r/2) + vA[Y(0)]^{-r/2} < C_0$$

$$Y'(0) = vK^A[Y(0)]^{1+r/2} < F'(0)$$

Then we have the following inequality

$$Y''(t) + Y'(t) \geq C_0(t + K)^A[Y(t)]^{1+r} \tag{3.16}$$

and the initial condition $Y(0) \leq F(0)$ and $Y'(0) < F'(0)$. Now we show that $F(t) \geq Y(t)$ for $0 \leq t < T_0$, so we have F(t) also blows up at in finite time. From $F'(0) > Y'(0)$ we have $F'(t) > Y'(t)$ for t small enough the we set

$$t_0\text{=sup}\{t \in [0, T_0)|F'(\tau) > Y'(\tau) \text{ for } 0 \leq \tau < t\}$$

Suppose $t_0 < T_0$, where T_0 is the blow up time for $Y(t)$. Thus we have $F'(t) > Y'(t)$ for $t \in [0, t_0)$ and $F'(t_0) = Y'(t_0)$. Since $F'(t) - Y'(t) > 0$, the function $F(t) - Y(t)$ is strictly increasing in the interval $0 \leq t < t_0$. In particular $F(t) - Y(t) > F(0) - Y(0) = 0$ for such t. Note that $F(t_0) > Y(t_0)$, because if $F(t_0) = Y(t_0)$ then the function $F(t) - Y(t)$ would have zeros at 0 and t_0, so the derivative will vanish between 0 and t_0. Therefore, $F(t_0) > Y(t_0)$ and $F'(t_0) = Y'(t_0)$.

On the other hand, by (3.14) to (3.16) we have the following

$$[F''(t) - Y''(t)] + [F'(t) - Y'(t)] \geq C(t+K)^{2A}\{[F(t)]^{1+r} - [Y(t)]^{1+r} \geq 0$$

For $0 \leq t \leq t_0$. We rewrite the above inequality in the form

$$\frac{d}{dt} e^t [F'(t) - Y'(t)] \geq 0$$

Then we integral the above inequality over $[0, t_0]$ to obtain

$$e^{t_0}[F'(t_0) - Y'(t_0)] \geq F'(0) - Y'(0)$$

Which gives that $F'(t_0) - Y'(t_0) > 0$. We come to a contradiction, thus we have $t_0 \geq T_0$. Then the proof of the special case is complete.

After the case $a = 1$, The case $a > 0$ and $a \leq 0$ is similar and much easier. If y is merely C^1, we can apply the same test function of the previous Lemma to complete our proof. The reader can have more details in [5]

□

By the preceding lemmas we will come to the hardest part of this thesis-the hyperbolic equation. For hyperbolic case we focus on the following equation.

$$\partial_t^2 u + a\partial_t u + Lu = f(x,u), \;\; u(x,0) = \varphi(x) + \psi_0(x), \;\; \partial_t u(x,0) = \psi_1(x). \quad (3.17)$$

As what we've mentioned in 3.1 the steady state φ will be the exact solution when the perturbations $\psi_0 \equiv \psi_1 \equiv 0$, also we concerned with the case that the perturbation (ψ_0, ψ_1) is small.

Theorem 2. (Hyperbolic Equation)

Let $a \in \mathbb{R}$. As what we've done in section 3.1, we consider the first part of the assumptions at the beginning and let $(\psi_0, \psi_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be such that

$$\frac{a + \sqrt{a^2 + 4\sigma^2}}{2} \int_{\mathbb{R}^n} \chi(x)\psi_0(x)dx + \int_{\mathbb{R}^n} \chi(x)\psi_1(x)dx > 0 \tag{3.18}$$

Let $0 < T \leq \infty$ and let u be a solution of (3.13) on $[0, T)$ such that $u - \varphi$ is continuous in t with values in the energy space and $f(x, u)$ is locally integrable.

(a)If $T = \infty$, then the energy norm

$$||u(t) - \varphi||_e \equiv ||u(\bullet, t) - \varphi(\bullet)||_{H^1(\mathbb{R}^n)} + ||\partial_t u(\bullet, t)||_{L^2(\mathbb{R}^n)} \tag{3.19}$$

must grow exponentially.

(b)Add assumptions (A5),(A6). Then we can have that $T < \infty$

Proof. (a)As what we've done in Theorem1. consider the function

$$G(t) = \int_{\mathbb{R}^n} \chi(x) \cdot w(x, t)dx, \quad w(x, t) = u(x, t) - \varphi(x)$$

By the assumption(A2) we have the following

$$|G(t)| \leq ||\chi||_{L^2(\mathbb{R}^n)} \cdot ||w(t)||_{L^2(\mathbb{R}^n)} \leq C||w(t)||_e$$

For we know that $\chi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ i.e. $||\chi||_{L^2(\mathbb{R}^n)}$ is finite. By the inequality we can easily have that the $G(t)$ is well-defined and bounded as long as the energy norm of w is bounded; also the energy norm grows exponentially provided that $G(t)$ does and the energy norm goes infinite whenever $G(t)$ does.

Fist we focus on part (a). Assume that $w = u - \varphi$ is continuous on $[0, \infty)$ with values in energy space. Then by distribution we have that w is a solution of the following equation

$$\partial_t^2 + a\partial_t w + Lw = f(x, w + \varphi) - f(x, \varphi)$$

Then with the assumption (A3) the convexity of $f(x, s)$ we have

$$\partial_t^2 + a\partial_t w + [L - F_u(x, \varphi)]w \geq 0$$

As what we've done in Parabolic case. We consider an arbitrary non-negative test function $\theta(t)$, and by assumption (A2) that $\chi(x)$ is also non-negative. Multiply the inequality by $\chi(x)\theta(t)$ we obtain

$$\partial_t^2 w\chi(x)\theta(t) + a\partial_t w\chi(x)\theta(t) + [L - f_u(x, \varphi)]w \geq 0$$

Be careful that for the Hyperbolic case we should deal with the second order partial derivative of t i.e. the ∂_t^2 part. Now we integral both parts

$$\int_0^t \int_{\mathbb{R}^n} \partial_t^2 w \cdot \chi(x)\theta(\tau) dx d\tau + \int_0^t \int_{\mathbb{R}^n} \chi(x) \cdot a\partial_t w \cdot \theta(\tau) dx d\tau \\ + \int_0^t \int_{\mathbb{R}^n} [L^* - f_u(x, \varphi)]\chi(x) \cdot w \cdot \theta(\tau) dx d\tau \geq 0$$

Use integration by parts we can simplify the first integral to get

$$
\begin{aligned}
-\int_0^t \int_{\mathbb{R}^n} \chi(x) \cdot \partial_t w \cdot \theta'(\tau) dx d\tau + \int_0^t \int_{\mathbb{R}^n} \chi(x) \cdot a \partial_t w \cdot \theta(\tau) dx d\tau \\
+ \int_0^t \int_{\mathbb{R}^n} [L^* - f_u(x,\varphi)] \chi(x) \cdot w \cdot \theta(\tau) dx d\tau \geq 0
\end{aligned} \tag{3.20}
$$

Now the first and the second integral has ∂_t element, to simplify them we use the same technique as what we've done at Parabolic case. Note that

$$G'(t) = \int_{\mathbb{R}^n} \chi(x) \cdot \partial_t w(x,t) dx$$

$G'(t)$ is a continuous function by our definition of $G(t)$. For we know that $\partial_t w$ is continuous with values in $L^2(\mathbb{R}^n)$. To simplify the third part of the integral, by the assumption (A2) we have

$$[L^* - f_u(x,\varphi)]\chi = -\sigma^2 \chi$$

Then the equation (3.17) can be reduced to

$$-\int_0^t G'(t) \cdot \theta'(\tau) d\tau + a \int_0^t G'(t) \cdot \theta(\tau) d\tau - \sigma^2 \int_0^t G(\tau) \cdot \theta(\tau) d\tau \geq 0$$

$\theta(t)$ is the non-negative test function and apply integration by parts we have

$$G''(t) + aG'(t) - \sigma^2 G(t) \geq 0$$

Then by our assumption (3.15) note that

$$\frac{a + \sqrt{a^2 + 4\sigma^2}}{2} \cdot G(0) + G'(0) > 0$$

Now we apply the Lemma1. Both $G(t)$ and $G'(t)$ must grow exponentially fast. Then we finish the part (a).

Finally we turn to the part (b). Now we add assumptions (A5) and (A6). Suppose that $T = \infty$. For we've known that both $G(t)$ and $G'(T)$ must grow exponentially fast by the proof of part (a). Now consider the assumptions (A4) and (A6) we have the following

$$\partial_t^2 w + a\partial_t w + [L - f_u(x,\varphi)]w = f(x, w+\varphi) - f(x,\varphi) - f_u(x,\varphi)w \geq$$
$$C_0|w+\varphi|^p - C_1 - C_1|w|$$

Then we multiply both sides by the non-negative eigenfunction χ and integrating over space to obtain the following equation

$$G''(t) + aG'(t) - \sigma^2 G(t) \geq \int_{\mathbb{R}^n} \chi|w+\varphi|^p dx - C_1 \int_{\mathbb{R}^n} \chi dx - C_1 \int_{\mathbb{R}^n} \chi|w|dx \tag{3.21}$$

What's more difficult than the Parabolic case (3.7), we should deal with the second derivative of $G(t)$. As what we have done in Theorem1 we have

$$G''(t) + aG'(t) \geq \sigma^2 G(t) + C_0 \int_{\mathbb{R}^n} \chi|w+\varphi|^p dx - C_1 \int_{\mathbb{R}^n} \chi|x+\varphi|dx - C_2$$

By using the triangular inequality and the assumptions (A2)$\chi \in L^1$, (A5)The product $\chi\varphi$ is integrable i.e. $\chi\varphi \in L^1$.

Since $G(t)$ grows exponentially fast, the previous inequality gives that

$$G''(t) + aG'(t) \geq C_0 \int_{\mathbb{R}^n} \chi|w+\varphi|^p dx - C_1 \int_{\mathbb{R}^n} \chi|w+\varphi|dx \equiv C_0 A(t) - C_1 B(t) \tag{3.22}$$

by distribution and for large enough t.

For $B(t)$, consider the triangular inequality

$$G(t) \leq \int_{\mathbb{R}^n} \chi |w| dx \leq \int_{\mathbb{R}^n} \chi |w + \varphi| dx + \int_{\mathbb{R}^n} \chi |\varphi| dx = B(t) + C_3 \tag{3.23}$$

Hence we have that $B(t)$ grows exponentially fast.

$A(t)$ grows exponentially fast by Hölder's inequality we have

$$B(t) = \int_{\mathbb{R}^n} \chi |w + \varphi| dx \leq \left(\int_{\mathbb{R}^n} \chi dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} \chi |w + \varphi|^p dx \right)^{\frac{1}{p}} = C_4 A(t)^{\frac{1}{p}} \tag{3.24}$$

This inequality gives that $A(t)$ grows faster than $B(t)$ i.e.

$$A(t) \geq C_4^{-p} B(t)^p \tag{3.25}$$

Hence the value of $A(t)$ dominate $B(t)$. Then we combining the equation (3.19),(3.20) and (3.22) by distribution we have the following

$$G''(t) + aG'(t) \geq C_5 A(t) \geq C_6 B(t)^p \geq C_7 G(t)^p \tag{3.26}$$

Besides both $G(t)$ and $G'(t)$ are positive by above. Now we apply Lemma2 for we have $G''(t) + aG'(t) \geq C_7 G(t)^p$ the we have the contradiction that $T < \infty$. Hence the proof of part (b) is completed. □

After reading these Theorems and Lemmas readers can get a closer look about instability of steady states in an easier way. For some applications we will give some examples in the next chapter.

Chapter 4

The Special Case $f(u) = |u|^p$

In chapter 3 of [3] gives an example of convexity nonlinearity with potential term. Now we are going to introduce a special case for the convexity nonlinearity. For the original problem is

$$-\Delta\varphi + V(x)\cdot\varphi = f(\varphi) \quad x \in \mathbb{R}^n \tag{4.1}$$

Now we suppose that the non-linear term $f(\varphi) = |\varphi|^p$ for $p > 1$ and φ is bounded. Under these condition we only need assumptions (A1) and (A2) the we can still get the conclusion that the instability by blow up. In [3] only talks about this fact with few words. We are going to show this fact in the following proof. To claim the statement holds true, we are going to check that whether the assumptions (A3)-(A6) holds for $f(\varphi) = |\varphi|^p$ for $p > 1$ and φ is bounded

Proof. In this proof we are going to check that under the special case $f(u) = |u|^p$ and $p > 1$, φ the assumptions (A3)-(A6) still holds.

(A3)The non-linear term $f(x,s)$ is convex in s and is C^1:

For $f(x,u) = f(u)$ in our case, $f(u) = |u|^p$ by the convex function properties given in Chpater 2 we can easily have that $f(u) = |u|^p$ is convex in u. Also it's clear that $f(u)$ is C^1. Hence we have the assumption (A3) holds.

(A4)Both $f(x,\varphi)$ and $f_u(x,\varphi)$ are bounded :

For $f(\varphi) = |\varphi|^p$ and φ is bounded we can easily get the assumption (A4) holds.

(A5)The product $\varphi\chi$ is integrable: (χ is the eigenfunction in (A2))

For φ is bounded we have the product $\varphi\chi$ must integrable.

(A6)There exist $C_0 > 0$ and $p > 1$ such that $f(x,s) \geq C_0|s|^p$ for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}$:

It's quite simple that for $f(u) = |u|^p$ we can find $C_0 > 0$ and $p > 1$ such that $f(s) \geq C_0|s|^p$.

For we have the assumptions (A1)-(A6) all holds in this special case $f(u) = |u|^p$. Therefore we can improve the example in [3] that instability by blow up by using the Theorem 1 and Theorem 2 in Chapter 3.

□

References

[1] Lawrence Evans. Partial differential equations. 1998.

[2] Manoussos Grillakis, Jalal Shatah, and Walter Strauss. Stability theory of solitary waves in the presence of symmetry, i. *Journal of Functional Analysis*, 74(1):160–197, 1987.

[3] Paschalis Karageorgis and Walter A Strauss. Instability of steady states for nonlinear wave and heat equations. *Journal of Differential Equations*, 241(1):184–205, 2007.

[4] Walter A Strauss. Partial differential equations. an introduction. *New York*, 1992.

[5] Grozdena Todorova and Borislav Yordanov. Critical exponent for a nonlinear wave equation with damping. *Journal of Differential Equations*, 174(2):464–489, 2001.