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使用二元樹評價亞式一籃子選擇權

Asian Basket Option Pricing by a Simple Binomial Tree

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本論文係詹益齊君（學號 R01922016）在國立臺灣大學資訊工程學系完成之碩士學位論文，於民國 103 年 6 月 23 日承下列考試委員員審查通過及口試及格，特此證明

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摘要

亞式一籃子選擇權同時具備亞式選擇權跟一籃子選擇權的特性，故難以找到選擇權價格的封閉解。在這篇論文中，我們使用平移對數常態分配 (shifted lognormal)以及負平移對數常態分配(negative shifted lognormal)搭配動差擬合(moment matching)找出三個參數(shape, scale and shift)來近似一籃子資產的價格。之後，我們利用這三個參數觀察到的性質建構一個可以近似一籃子資產價值的二元樹。最後搭配 Hull-White methodology 找出美式跟歐式的亞式一籃子選擇權的價格。數值實驗的結果顯示我們的方法所找出來的歐式選擇權價格與蒙地卡羅方法找出來的價格十分接近，但是美式選擇權價格與最小平方蒙地卡羅法找出來的價格相比，我們的方法明顯地高估。

關鍵詞：亞式一籃子選擇權、平移對數常態分佈、動差擬合、封閉解、Hull-White 法

Abstract

Asian basket option is hard to price. This thesis presents a new approach to price European-style and American-style Asian basket options. First, we use approximation and moment-matching techniques to find the random variable following the shifted lognormal distribution to approximate the basket value. Second, we use the random variable to build a binomial tree and combine it with the Hull-White methodology for pricing path-dependent options to price Asian basket options. Finally, we compare our numerical results with Monte Carlo simulation for European-style Asian basket options and with the least-squares Monte Carlo for American-style ones. They show that the European-style Asian basket option prices obtained by our approach are accurate and the American-style ones are overpriced by our approach.

Key Words: Asian basket option, shifted lognormal distribution, moment matching, closed-form solution, Hull-White methodology.

Contents

Introduction

Option is a contract between two counterparties. In this contract, the option buyer has the right (but not obligation) to buy or sell the underlying assets. On the other hand, the seller has to sell or buy depending on the buyer's action. Various types of options have been designed for various purposes.

Among the types of options, the Asian option is a very popular financial derivative with two alternative average methods (arithmetic and geometric) and two different monitoring types (continuous and discrete). Continuously monitored Asian option and geometric averaging Asian option are rare in the market; therefore, only Asian options with discrete monitoring and arithmetic averaging are considered in this thesis. Basket option is an option whose payoff depends on a basket of assets. Examples include currency basket options and equity index options. The underlying assets may number more than ten, and the weight of each asset may be positive or negative. In particular, when there are only two assets with different weight signs, the basket option becomes a spread option because the payoff depends on the price difference between the two assets. Finally, an Asian basket option is an option with an Asian exercise feature and the underlying asset being a basket of different assets. Asian basket options are attractive for two reasons. First, the option price is lower than the basket option because of the averaging property which decreases the variance of the option. Second, the Asian basket option is an appropriate hedging tool because the payoff depends on averaging price in the basket and across time. Asian basket options are important for energy companies as they often hold a portfolio of different energy commodities with positive or negative weights. Therefore, they have incentive to hold Asian basket options for hedging purposes (Krekel, Kock, Korn and Man, 2006).

Asian options are hard to price for the following reason: Although the lognormal distribution captures the distribution of individual stock prices, the sum of lognormally distributed random variables is no longer lognormal. Hence, it is difficult to find a closed-form solution for Asian options. Various kinds of methods are published for pricing European and American style Asian option including approximation, Monte Carlo simulation, the PDE methods and the modified tree methods. Hull and White (1993) propose an efficient procedure to value European and American path-dependent options such as Asian options based on the modified tree technique. Klassen (2001) refines the Hull-White methodology by using Richardson extrapolation to improve convergence and save time. Only the Hull-White methodology will be used in this thesis although other methods can clearly be applied to improve upon it.

Like the Asian option, the basket option has the same problem that the sum of lognormal distributions is not lognormal. The Taylor expansion proposed by Ju (2002) performs well in most cases except inhomogeneous volatilities according to Krekel, Kock, Korn and Man (2006). The conditional expectation technique by Beisser (1999) slightly underprices the option, thus serving as a lower bound of the true price. Monte Carlo simulation is attractive but time-consuming. Milevsky and Posner (1998) apply the reciprocal gamma distribution to approximate basket value and obtain a closed-form solution for the option price. But this approach is only applicable for positive weights. Borovkova (2007) proposes a new closed-form approach for pricing basket options. This approach uses the shifted lognormal distribution to approximate the basket value and finds the parameters in the shifted lognormal distribution by moments matching. It works quite well: It obtains a price within the 95% confidence interval of the price obtained by Monte Carlo simulation. Besides, it can handle more than ten assets in the basket and with both positive and negative weights. Borokova

(2012) extends the shifted lognormal approximation of Borovkova (2007) to build a binomial tree to price American-style basket options.

There is no closed-form solution for American-style Asian basket options. Borovkova's (2007) approximate closed-form solution for European-style Asian basket options cannot be applied to American-style Asian basket options. Monte-Carlo simulation is applicable but computation intensive.

This thesis offers a straightforward approach to price both American-style and European-style Asian basket options by using the binomial tree from Borokova (2012) and the Hull-White methodology. Furthermore, this approach can also be used for other types of basket options in the future. The European-style price we obtain is close to the price from Monte Carlo simulation if the bucket size is large enough.

The rest of this thesis organizes as follow. Section 2 presents the assumptions and procedures to build the binomial tree. Section 3 describes how we apply the Hull-White methodology to the tree in section 2. Section 4 firstly compares the numerical results from our approach with the results by Monte Carlo simulation for European-style Asian basket options and proceed to compare the option prices by our approach with those by the least-squares Monte Carlo for American-style Asian basket options.

1. The Model

1.1 Generalized lognormal approach

The basket option is hard to price because the sum of lognormally distributed random variables is no longer lognormal. Levy (1992) and Milevsky (1998) use lognormal and reciprocal gamma distribution, respectively, to approximate the basket value for a closed-form solution of option price. However, they can only be used for the basket option with positive weights and may be biased with large volatilities according to Fan (2009). Borovkova (2007) introduces a new method of generalized lognormal distribution to approximate the basket value, obtaining the closed-form solution for basket options. The so-called GLN (Generalized LogNormal) approach can not only obtain prices close to those from Monte-Carlo simulation but also spend less time than Monte-Carlo simulation.

Let N represent the total number of assets in the basket and $(a_i)_{i=1}^{N}$ are the weights of the assets, which may be positive or negative. We assume that the underlying assets are commodity futures, which means that they follow zero-drift geometric Brownian motions. Therefore, if the futures prices at time t are $\left(F_i(t)\right)_{i=1}^{N}$, $F_i(t)$ follows

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i dW_i(t), \quad i = 1,\ 2,\ ......,\ N \tag{1}$$

where σ_i is the volatility of future i. $W_i(t)$ is the Brownian motion driving futures i, and $W_i(t)$ and $W_j(t)$ have correlation $\rho_{i,j}$ (i.e., $dW_i(t)dW_j(t) = \rho_{i,j}dt$). The basket value is

$$B(t) = \sum_{i=1}^{N} a_i F_i(t) \tag{2}$$

1.2 Finding parameters in GLN and binomial tree

Three are four different types in generalized lognormal distribution family: regular lognormal, shifted lognormal, negative lognormal and negative shifted lognormal. In this thesis, we only use the negative shifted lognormal and the shifted lognormal because the regular lognormal is a special case of the shifted lognormal with zero shifted parameter (same with the negative lognormal and the negative shifted lognormal). Furthermore, we use the skewness of the basket value to decide between using the negative shifted lognormal and the shifted lognormal. For example, if the skewness is positive, we use the shifted lognormal distribution to approximate the basket value, and if the skewness is negative, we use the shifted negative lognormal.

If Y is a random variable which follows the shifted lognormal distribution, we can use three parameters: m (scale), s (shape) and τ (shift) to characterize it as follows (Fan, 2009):

$$X \sim \text{lognorm}\left(m, s^2\right)$$
$$f_X(x) = \frac{1}{\sqrt{2\pi} sx} e^{-\frac{(\ln x - m)^2}{2s^2}},\ x > 0$$
$$Y = X - \tau$$

On the other hand, if Y follows the negative shifted lognormal distribution, it can be characterized as follows (Borovkova, 2007):

$$-X \sim \text{lognorm}\left(m, s^2\right)$$
$$f_X(x) = \frac{-1}{\sqrt{2\pi} sx} e^{-\frac{(\ln(-x) - m)^2}{2s^2}},\ x < 0$$
$$Y = X - \tau$$

In order to find these three parameters, we apply the moment-matching techniques to match the first three moments (Borovkova, 2007). The first three moments of the basket value are

$$B_1(t) = E\left(B(t)\right) = \sum_{i=1}^{N} a_i F_i(0) \tag{3}$$

$$B_2(t) = E\left(B(t)^2\right) = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j F_i(0) F_j(0) \exp\left(\rho_{i,j}\sigma_i\sigma_j t\right) \tag{4}$$

$$\begin{aligned} B_3(t) &= E\left(B(t)^3\right) \\ &= \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j a_k F_i(0) F_j(0) F_k(0) \exp\left[\left(\rho_{i,j}\sigma_i\sigma_j + \rho_{j,k}\sigma_j\sigma_k + \rho_{i,k}\sigma_i\sigma_k\right)t\right] \end{aligned} \tag{5}$$

where t is the time from 0. On the other hand, the first three moments of the shifted lognormal distribution are

$$M_1(t) = \tau(t) + \exp\left(m(t) + \frac{1}{2}s(t)^2\right) \tag{6}$$

$$M_2(t) = \tau(t)^2 + 2\tau(t)\exp\left(m(t) + \frac{1}{2}s(t)^2\right) + \exp\left(2m(t) + 2s(t)^2\right) \tag{7}$$

$$\begin{aligned} M_3(t) = {} & \tau(t)^3 + 3\tau(t)^2 \exp\left(m(t) + \frac{1}{2}s(t)^2\right) + 3\tau(t)\exp\left(2m(t) + 2s(t)^2\right) \\ & + \exp\left(3m(t) + \frac{9}{2}s(t)^2\right) \end{aligned} \tag{8}$$

according to Borovkova (2007). If we use the negative shifted lognormal distribution, we only need to change $M_1(t)$ and $M_3(t)$ to $-M_1(t)$ and $-M_3(t)$.

Matching the first three moments of the basket value and the shifted lognormal distribution by solving the following nonlinear system:

$$\begin{aligned} M_1(t) &= B_1(t) \\ M_2(t) &= B_2(t) \\ M_3(t) &= B_3(t) \end{aligned}$$

we can obtain $m(t)$, $s(t)$ and $\tau(t)$ numerically. Chang, Chen, and Wu (2012) offer a faster way to solve the nonlinear system, which is faster and more accurate than the MATLAB “lsqnonlin” function or other MATLAB functions.

To figure out how $m(t)$, $s(t)$ and $\tau(t)$ behave, we use the following basket:

$F_0 = [100;120]$, $\sigma = [0.2;0.3]$, $a = [-1;1]$, $\rho_{1,2} = 0.9$ and $T = 1$ year as an example. If there are 250 days in a year, we then calculate $m(t)$, $s^2(t)$ and $\tau(t)$ for each day, plotting the three parameters as t changes. We find that $s^2(t)$ and $m(t)$ are near linear functions of time t (see Fig. 1 and Fig. 2), whereas $\tau(t)$ changes little as t changes, which is consistent with Borovkova (2012). Therefore, $\tau(t)$ can be replaced by the mean value of $\tau(t)$ (see Fig. 3).

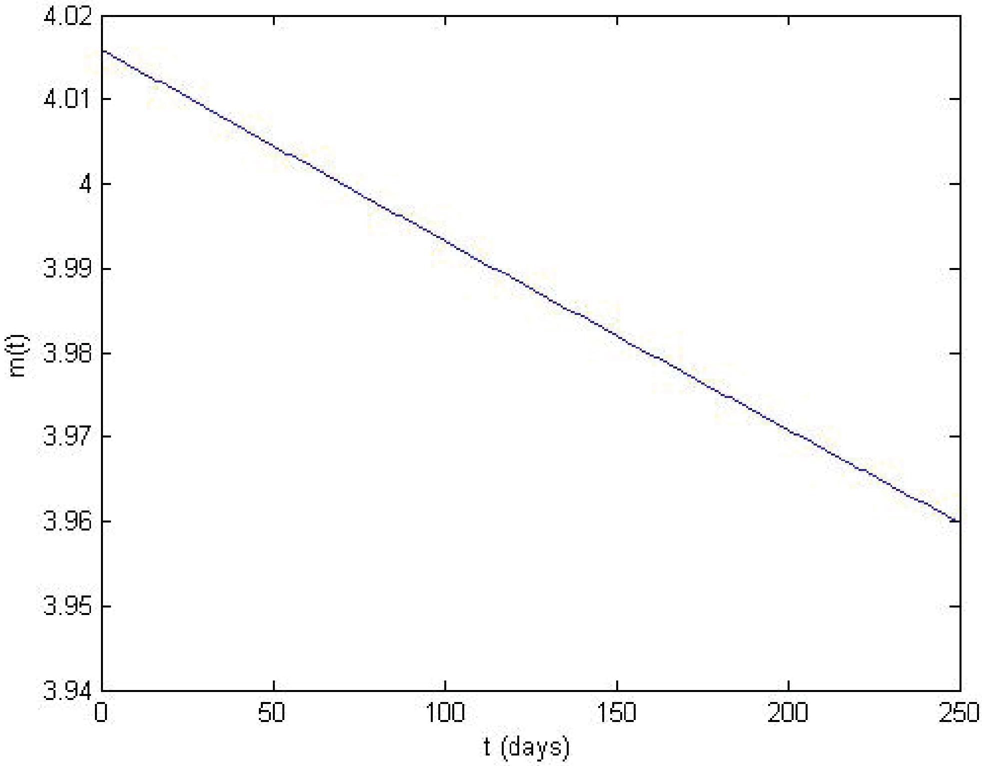

Fig. 1: **Parameter $m(t)$ of the Shifted Lognormal Distribution as a Function of Time t.** It shows that the parameter $m(t)$ is a near linear function of time t.

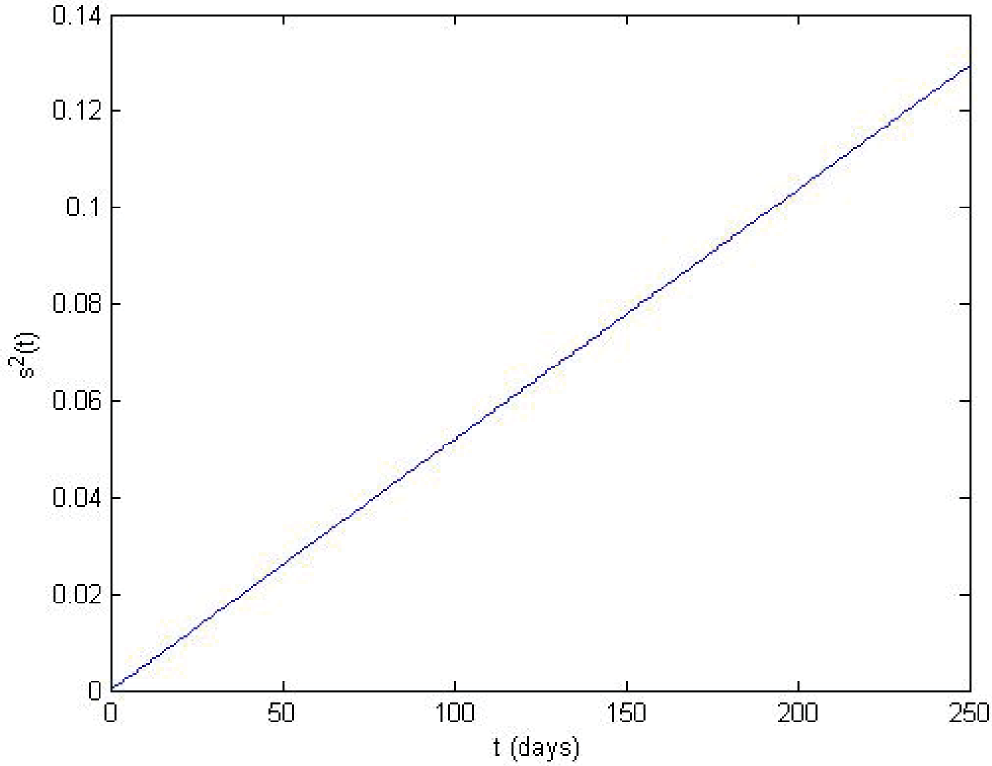

Fig. 2: **Parameter $s^2(t)$ of the Shifted Lognormal Distribution as a Function of Time t.** It shows that the parameter $s^2(t)$ is also a near linear function of time t.

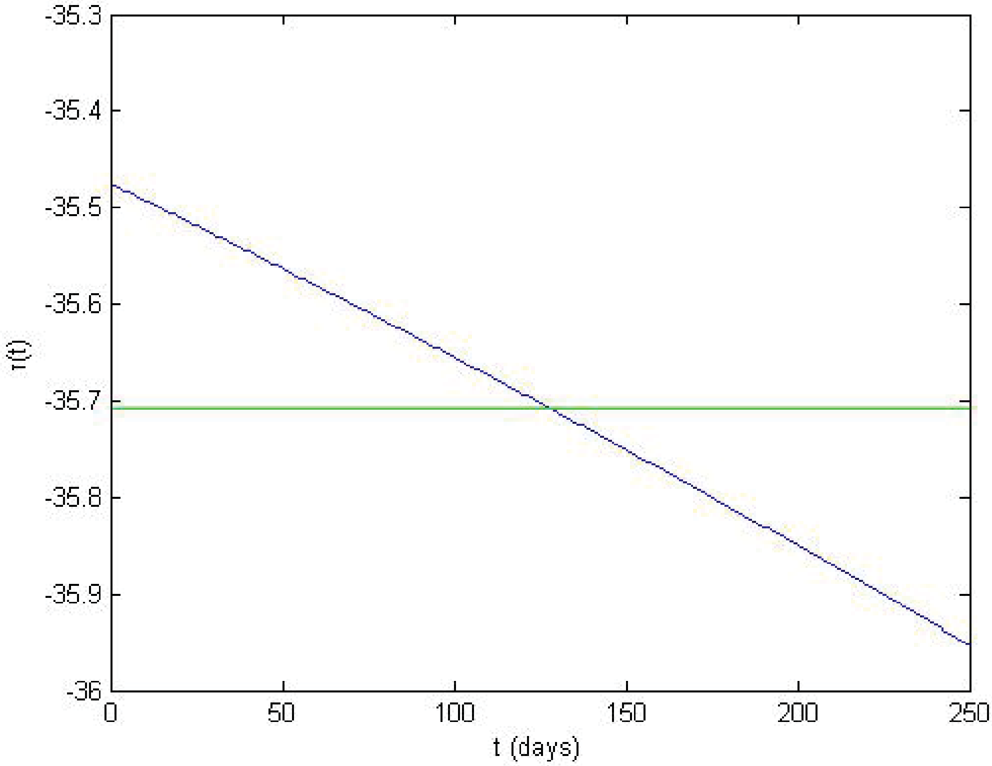

Fig.3: **Parameter $\tau(t)$ of the Shifted Lognormal Distribution as a Function of Time t.** The blue line is the parameter $\tau(t)$ as a function of time t and the pink line is the mean value of $\tau(t)$ for t over $[0,250]$. It shows that $\tau(t)$ changes little as t changes; therefore, we can replace $\tau(t)$ with the mean values τ.

We conclude that $m(t)$ is a near linear function of time t with an intercept and $s^2(t)$ is also a near linear function of time t. We then use this feature to build a binomial tree with a constant mean and a constant volatility to approximate the basket value through time in terms of GLN distribution as follows (Borovkova, 2012). Let T be the expiration date and τ be the mean value of $\tau(t)$ for t over $[0,T]$. First, we define

$$B^*(t) = B(t) - \tau \text{ or } B^*(t) = -B(t) - \tau\,, \tag{9}$$

depending on what generalized lognormal distribution we choose. Furthermore, we use $B^{\text{appr}}(t)$, which follows the following stochastic process, to approximate $B^*(t)$:

$$\frac{dB^{\text{appr}}(t)}{B^{\text{appr}}(t)} = \mu^* dt + \sigma^* dW(t) \tag{10}$$

where $W(t)$ follows Brownian motion and both μ^* and σ^* are constant, which are found by matching mean and variance of $B^{\text{appr}}(t)$ and $B^*(t)$. Note that

$$\text{Var}\left(\log B^{\text{appr}}(t)\right) = \sigma^{*2} t = s^2(t)$$

$$E\left(\log B^{\text{appr}}(t)\right) = \log B^{\text{appr}}(0) + \left(\mu^* - \frac{1}{2}\sigma^{*2}\right) t = m(t)$$

Hence,

$$\sigma^* = \sqrt{\frac{s^2(t)}{t}} \tag{11}$$

$$\mu^* = \frac{m(t) - \log B^{\text{appr}}(0)}{t} + \frac{1}{2}\sigma^{*2} \tag{12}$$

Finally, we use μ^* and σ^* to build our binomial tree to approximate the basket price in order to price Asian basket options.

1.3 Building binomial tree

Having obtained μ^* and σ^* from moment-matching and solving nonlinear systems, we use them to calculate q (the probability to go up), $1-q$ (the probability to go down), u (the factor when the price go up) and d (the factor when the price go down) as follows (Borovkova, 2012):

$$u = \exp\left(\left(\mu^* - \frac{1}{2}\sigma^{*2}\right)\Delta t + \sigma^* \sqrt{\Delta t}\right) \tag{13}$$

$$d = \exp\left(\left(\mu^* - \frac{1}{2}\sigma^{*2}\right)\Delta t - \sigma^* \sqrt{\Delta t}\right) \tag{14}$$

$$q = \frac{1}{2} \tag{15}$$

We then translate the binomial tree for $B^{\text{appr}}(t)$ into $B(t)$ by $B(t) = B^{\text{appr}}(t) + \tau$ if we use the shifted lognormal and $B(t) = -B^{\text{appr}}(t) - \tau$ if we use the negative shifted lognormal. By this means, we successfully build a simple binomial tree which approximates the basket value $B(t)$ (Borovkova, 2012).

2. The Hull-White Methodology

2.1 Finding the running averages

When we want to price Asian options based on a tree model without approximation, we need to record every possible average price at each node. However, the number of average prices grows exponentially, making it infeasible. Hull and White (1993) propose a methodology based on the tree model to price path-dependent options such as American lookback options and Asian options. In the following paragraphs, we will show how the methodology works and how to apply it to our binomial tree in the previous section.

In the Hull-White methodology, we firstly calculate the maximum and the minimum averaging prices at each node by the formula of geometric sequence (Lyuu, 2002). For each node $N(j,i)$, the maximum of running average $A_{\max}$ is

$$S_0\left(1+u+u^2+\cdots+u^{j-i}+u^{j-i}d+\cdots+u^{j-i}d^i\right)=S_0\frac{1-u^{j-i+1}}{1-u}+S_0u^{j-i}d\frac{1-d^i}{1-d} \qquad (15)$$

where S_0 is the underlying asset price at $t=0$ (now). The minimum running average $A_{\min}$ can be derived as

$$S_0\left(1+d+d^2+\cdots+d^i+d^iu+\cdots+d^iu^{j-i}\right)=S_0\frac{1-u^{j-i+1}}{1-u}+S_0u^{j-i}d\frac{1-d^i}{1-d} \qquad (16)$$

We then equally divide the maximum and the minimum running average into $k+1$ parts as follows

$$A_m(j,i)=\left(\frac{k-m}{k}\right)A_{\min}(j,i)+\left(\frac{m}{k}\right)A_{\max}(j,i), \quad m=0,1,\ldots,k \qquad (17)$$

where k is chosen by us and may affect the precision of the option price. Following this, we use backward induction to obtain the option price at each node and decide to early exercise if the value of exercising immediately exceeds the value of holding the option for American style option.

2.2 Backward induction

First, we calculate the payoffs of each node at the expiration day for all $k+1$ running averages $\overline{A}$ with exercise price X by $\max\left(\overline{A}-X,0\right)$ for call options and $\max\left(X-\overline{A},0\right)$ for put options. Assume the current node is $N\left(j,i\right)$ and the running average is a. If the node goes up to node $N\left(j+1,i\right)$, the running average would be

$$A_u = \frac{\left(j+1\right)a + S_0 u^{j+1-i} d^i}{j+2} \tag{18}$$

However, A_u may not be one of the running averages at $N\left(j+1,i\right)$. Hence, we need to use interpolation between two running averages that bracket it:

$$A_l\left(j+1,i\right) \le A_u \le A_{l+1}\left(j+1,i\right) \tag{20}$$

where l is an integer from 0 to k. By linear interpolation, A_u can be expressed as

$$A_u = xA_l\left(j+1,i\right) + \left(1-x\right)A_{l+1}\left(j+1,i\right),\ 0 \le \mathrm{x} \le 1 \tag{21}$$

Finally, we obtain the option price when the node $N\left(j,i\right)$ goes up to $N\left(j+1,i\right)$ as

$$C_u = xC_l\left(j+1,i\right) + \left(1-x\right)C_{l+1}\left(j+1,i\right) \tag{22}$$

where $C_l\left(t,s\right)$ denotes the option value at node $N\left(t,s\right)$ with running average $A_l\left(t,s\right)$. Similarly, the option price when the node $N\left(j,i\right)$ goes down to $N\left(j+1,i+1\right)$ is

$$C_d = yC_l\left(j+1,i+1\right) + \left(1-y\right)C_{l+1}\left(j+1,i+1\right) \tag{23}$$

where y can be obtained by the process similar to x. Once we obtain both C_u and C_d, we can obtain the option price of $N\left(j,i\right)$ by

$$C = \left[pC_u + \left(1-p\right)C_d\right]e^{-r\Delta t} \tag{24}$$

where Δt is the duration of a time step and p is the probability to go up. In chapter 1, we build a binomial tree with parameters: q, $1-q$, u and d. The Hull-White methodology can be applied to the binomial tree, helping us obtain the prices of European-style and American-style Asian basket options. We turn to the numerical results in the next chapter.

3. Experiment results

For European-style Asian basket options, we apply our approach to options with different basket compositions, and then compare the option prices obtained by our tree approach with those obtained by Monte Carlo simulation. However, for American-style Asian basket options, there are few methods except least-squares Monte Carlo for pricing the American-style Asian basket options. Therefore, there are no publicly available benchmarks.

Consider the following baskets for the Asian basket call and put options:

Basket 1: $F_0 = [50;50]$; $\sigma = [0.3;0.2]$; $\rho_{1,2} = 0.6$; $a = [0.3;0.7]$; $X = 50$.

Basket 2: $F_0 = [100;120]$; $\sigma = [0.2;0.3]$; $\rho_{1,2} = 0.9$; $a = [-1;1]$; $X = 20$.

Basket 3: $F_0 = [150;100]$; $\sigma = [0.3;0.2]$; $\rho_{1,2} = 0.7$; $a = [-1;1]$; $X = -50$.

Basket 4: $F_0 = [95;90;105]$; $\sigma = [0.2;0.3;0.25]$; $\rho_{1,2} = \rho_{1,3} = 0.9$; $\rho_{2,3} = 0.8$; $a = [1;-0.8;-0.5]$; $X = -30$.

Basket 5: $F_0 = [100;90;95]$; $\sigma = [0.25;0.3;0.2]$; $\rho_{1,2} = \rho_{1,3} = 0.9$; $\rho_{2,3} = 0.8$; $a = [0.6;0.8;-1]$; $X = 40$.

We assume the risk-free interest rate is 5% per annum and the time to expiry (T) is 1 year. Furthermore, the number of time steps for the tree is 100 and the bucket size k is 300. For Monte Carlo simulation, the number of simulation runs is 100000.

Basket	Approx. distribution	τ	μ^*	σ^*
Basket 1	Shifted	1.0668	0	0.2115
Basket 2	Shifted	−35.7071	0	0.3602
Basket 3	Neg. shifted	−59.0260	0	0.3141
Basket 4	Neg. shifted	−31.9634	0	0.3149
Basket 5	Shifted	−30.1925	0	0.3133

Table 1: Parameters of GLN obtained by moment matching.

Table 1 shows the parameters of GLN obtained by moment matching. τ may be positive or negative, μ^* must be zero and σ^* may be higher or lower than the volatilities of the assets in the basket.

The European-style Asian basket call and put option prices obtained by our tree approach are given in column 3 and column 5 in Table 2. Column 4 and column 6 in Table 2 are the corresponding option prices obtained by Monte Carlo simulation; and the standard errors of the option prices are given in parenthesis after the option prices by Monte Carlo simulation.

Basket	Approx. distribution	European-style call option		European-style put option	
		Our approach	Monte Carlo simulation	Our approach	Monte Carlo simulation
Basket 1	Shifted	2.2666	2.2532 (0.0119)	2.2666	2.2581 (0.0101)
Basket 2	Shifted	4.3862	4.3509 (0.0247)	4.3862	4.3470 (0.0184)
Basket 3	Neg. shifted	7.4898	7.4286 (0.0321)	7.4898	7.4369 (0.0416)
Basket 4	Neg. shifted	4.4959	4.4615 (0.0187)	4.0203	3.9989 (0.0229)
Basket 5	Shifted	3.4354	3.4105 (0.0223)	6.2891	6.2490 (0.0229)

Table 2: European-style Asian basket option prices

Table 2 shows that the prices obtained by our tree approach are within the 95% confidence intervals of those obtained by Monte Carlo simulation for both positive and negative skewness, with both positive and negative weights. Our approach is attractive for two reasons. First, the time complexity of Hull-White methodology is $O(kn^2)$. Therefore, when n is small, the CPU time of our approach is less than Monte Carlo simulation with 100000 runs. Second, Monte Carlo simulation can only generate the paths of the baskets with a positive definite correlation matrix. In contrast, our approach can price not only options with a positive definite correlation matrix for the assets but also options with a non-positive definite correlation matrix, which cannot be processed by Monte Carlo simulation without further adjustments.

For American-style Asian basket options, Monte Carlo simulation is not appropriate because it cannot handle early exercise. Therefore, we compare our approach with the least-squares Monte Carlo (LSMC) method (Longstaff and Schwartz, 2001). There are two methods for LSMC. First, we approximate the basket by GLN, then use a single asset to represent the whole basket. After that, we use LSMC on the single asset to obtain the option prices (we call it LSMC-1). Or, we can run LSMC directly on the basket paths generated by the correlated geometric Brownian motion to obtain the option prices (given in LSMC-2). LSMC-2 should serve as the benchmark theoretically.

Basket	Approx. distribution	American-style call option			American-style put option		
		Our approach	LSMC-1	LSMC-2	Our approach	LSMC-1	LSMC-2
Basket 1	Shifted	2.6131	2.2744 (0.0114)	2.2430 (0.0114)	2.5467	2.2509 (0.0095)	2.2534 (0.096)
Basket 2	Shifted	5.1051	4.3942 (0.0237)	4.3544 (0.0234)	4.8865	4.3828 (0.0176)	4.3451 (0.0175)
Basket 3	Neg. shifted	8.3658	7.4634 (0.0305)	7.4025 (0.0304)	8.6916	7.4396 (0.0393)	7.4224 (0.0393)
Basket 4	Neg. shifted	5.0434	4.4641 (0.0177)	4.4398 (0.0177)	4.6396	3.9722 (0.0217)	4.0109 (0.0219)
Basket 5	Shifted	3.8724	3.3966 (0.0213)	3.4003 (0.0211)	7.1946	6.2962 (0.0219)	6.2412 (0.0218)

Table 3: American-style Asian basket option prices

Table 3 shows the prices of American-style Asian basket options with the same baskets as before. It implies GLN approximation is suitable to handle the early exercise feature because the option prices obtained by LSMC-1 are close to those obtained by LSMC-2. We find that the American-style option prices obtained by LSMC-1 and LSMC-2 in Table 3 are also close to the European-style ones by Monte Carlo simulation in Table 2 mainly because less than 3% paths in LSMC-2 and no path in LSMC-1 are early exercised. We observe that the option prices obtained by our tree approach are too high compared to both LSMC-1 and LSMC-2. In fact, nearly 5% of the states are early exercised in the tree, and they are mainly located in the middle of the tree. But, we recall, no path is early exercised in LSMC-1 and less than 3% paths are early exercised in LSMC-2. Therefore, we conjecture that it is the Hull-White methodology of our tree approach that results in overpricing, not the approximation by GLN.

We use the Asian basket call option for Basket 2 to analyze the convergence of the tree with the Hull-White methodology. When we choose a larger bucket size k per node to obtain more precise option prices, the option prices would converge to the prices obtained by Monte Carlo simulation for European-style Asian basket options (see Fig.4). However, when we do the same comparison for *American-style* Asian basket options with LSMC-2, the option prices obtained by our tree approach would overprice no matter how large bucket size we choose (see Fig. 5).

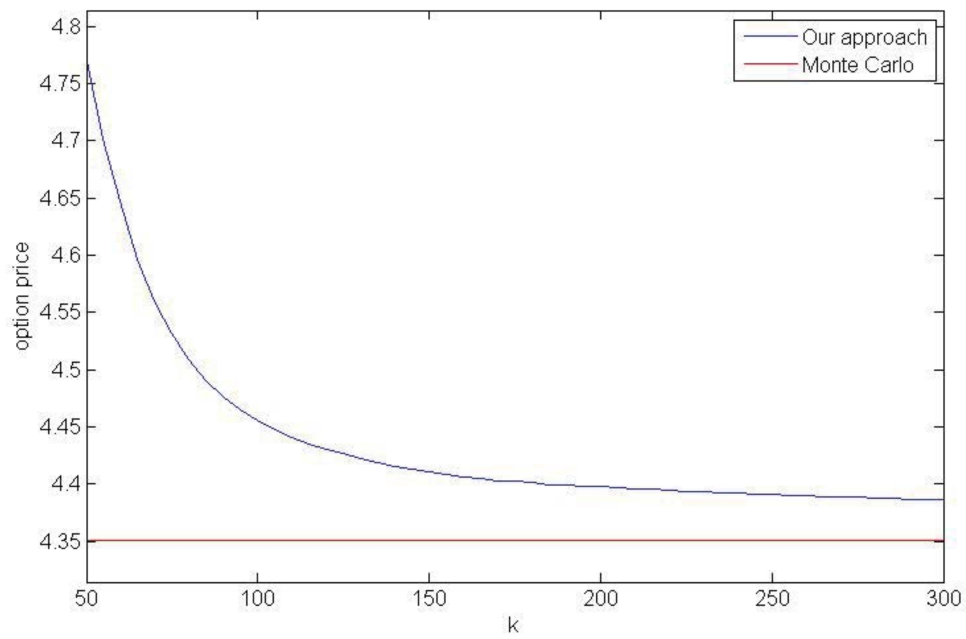

Fig. 4: **Convergence of European-style Asian Basket Option Price with k.**

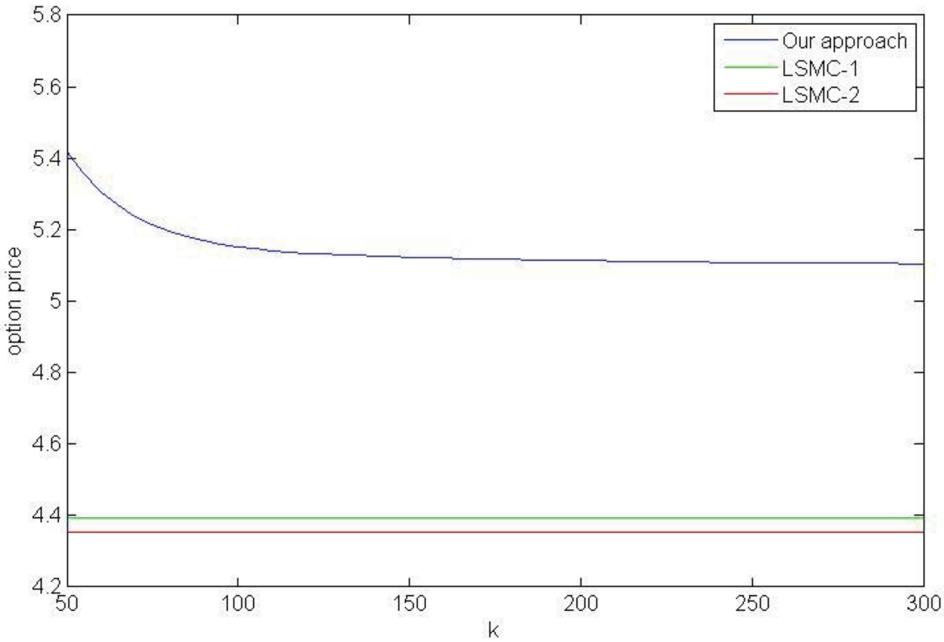

Fig. 5: **Convergence of American-style Asian Basket Option Price with k.**

4. Conclusion

Asian basket options are hard to obtain a closed-form solution because the sum of lognormally distributed random variables is no longer lognormal. In this thesis we presents the Generalized lognormal Hull-White approach for pricing both European-style and American-style Asian basket options based on approximating the distribution of the average by the generalized lognormal distribution, moment matching techniques, building a binomial tree by the parameters of the generalized lognormal distribution and the Hull-White methodology. For European-style Asian basket options, by comparing the option prices obtained by our tree approach with those by Monte Carlo simulation, our approach works well for baskets with positive and negative weights and consisting of more than two assets. For American-style ones, we compare the option prices obtained by our approach with those by the least-squares Monte Carlo. The approximation by our approach still leads to accurate prices, but the tree with the Hull-White methodology overprices the option. Although our tree approach overprices the option, the prices obtained by the least-squares Monte Carlo on the single asset approximated by the generalized lognormal are still accurate. Therefore, we believe that the approximation by the generalized lognormal can be implemented with other appropriate approaches for a single asset in order to price other types of basket options.

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