國立臺灣大學理學院數學系

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master Thesis

正特徵數中的極小模型理論

Minimal Model Program in Positive Characteristic

陳延安

Yen-An Chen

指導教授：陳榮凱 博士

Advisor：Jungkai Alfred Chen, Ph.D.

中華民國 103 年 6 月

June, 2014

國立臺灣大學碩士學位論文
口試委員會審定書

正特徵數中的極小模型理論
Minimal Model Program in Positive Characteristic

本論文係陳延安君 (R01221007) 在國立臺灣大學數學系完成之碩士學位論文，於民國 103 年 6 月 26 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

________________________ (簽名)

(指導教授)

________________　________________

________________　________________

________________　________________

________________　________________

系主任、所長 ________________________ (簽名)

誌謝

從大學部到碩士班，在臺大也有六年的時光。回想最初選擇代數幾何，主要是因為當初在學完大學部代數後，偶然看了 Hartshorne 的《代數幾何》，對於代數結構與幾何結構的相互呼應感到驚嘆，使我不由自主的想更認識它。

首先，我想要感謝我的指導教授陳榮凱老師，他常常在百忙之中抽空來指導我，對於我的各種問題更是不厭其煩的為我解答。尤其是在決定論文方向時，給了我一盞明燈，使我找到我想要走的道路。

再來，我要感謝 Caucher Birkar 、于靖老師、林惠雯老師以及齊震宇老師，對於作研究、求學與生涯規畫給的寶貴建議與鼓勵。我還要感謝我的學長陳正傑、陳星谷、林金毅以及鄭偉哲，在我報告後與我討論、給我建議和分享不同的看法。

最後我要感謝我的家人，他們總是默默的支持我，讓我可以專心的做研究，完成學業。

陳延安

2014.6.30

中文摘要

本文分為兩個部分。第一個部分介紹極小模型理論是如何運作的，尤其是在正特徵數的時候。第二個部分，Mori 在特徵數零時對三維 terminal 奇點的分類中，我確定了大部分在正特徵數時，仍然會是 terminal 奇點。

關鍵字：極小模型理論、正特徵數、terminal 奇點。

Abstract

In this thesis, there are two parts. The first part is to introduce what the minimal model program (MMP) is and how it works, especially in positive characteristic. Also, I illustrate some differences between characteristic zero and positive characteristic. In the second part, I verify that the classification Mori gave in characteristic zero for the terminal singularities in dimension 3 is mostly true in *positive characteristic*.

Key words : Minimal Model Program, Positive Characteristic, Terminal Singularity.

Contents

List of Figures

Chapter 1

What is MMP ?

MMP stands for minimal model program. Roughly speaking, it is a “method” to understand varieties further.

1.1 Classification

In all fields in mathematics, we usually want to understand objects fully by classification. If the number of all objects we are interested in is finite, then classification means to know which object is isomorphic to others. But if the number is infinite, then classification has different meaning from the finite case. This is because the human beings can only live for a finite time. Thus, we cannot check it case by case. So classification means to describe some properties that use finitely many objects.

In algebraic geometry, we are interested in varieties. So the first natural problem we may ask is

Problem 1.1.1. *Classify varieties up to isomorphism.*

But this problem is unrealistic even for curves because the parameter space has very high dimension as the genus is high. So we relax us to ask

Problem 1.1.2. *Classify projective varieties up to birational isomorphism.*

It is then natural to study birational geometry. The purpose of MMP is to find a good representative inside a birational equivalence class, which plays the pivotal role in the recent development of algebraic geometry.

1.2 What is a minimal model ?

As we said above, we want to find some "nice" objects which could describe all objects. More precisely, in our case, we want to find a good representative in any fixed birational isomorphism classes. We call it a minimal model.

For example, there exists a unique smooth projective curve in any fixed birational isomorphism classes. (For a proof, see [Har77, chapter 1, section 6].) So the smooth projective curves are what we want for minimal models. However, this is a special feature for curves. For the case of surfaces or 3-folds, we will discuss later.

1.3 Asymptotic Riemann-Roch Theorem

I give a version of Riemann-Roch Theorem which will be used later.

Theorem 1.3.1. *Let X be a normal projective variety of dimension n and D, E be two divisors. Then we have*

$$P(m) = \chi(mD + E) = \frac{D^n}{n!}m^n - \frac{D^{n-1}.(K_X - 2E)}{2(n-1)!}m^{n-1} + (\text{ lower order terms }).$$

Proof. We prove by induction on n. For $n = 1$, we have

$$P(m) = \deg(mD + E) - g + 1 = (\deg D)m - \frac{1}{2}\deg(K_X - 2E).$$

Now for $n > 1$, we choose a general ample divisor H such that $H + D$ is ample. Choose a general element $G \in |H + D|$. Note that H and G are normal projective varieties. Then we consider the following two exact sequences.

$$0 \longrightarrow mD + E \longrightarrow mD + E + H \longrightarrow (mD + E + H)|_H \longrightarrow 0$$

$$0 \longrightarrow (m-1)D + E \longrightarrow mD + E + H \longrightarrow (mD + E + H)|_G \longrightarrow 0$$

Thus, we have

$$\begin{aligned}
P(m) - P(m-1) &= \chi((mD+E+H)|_G) - \chi((mD+E+H)|_H) \\
&= \left(\frac{G.D^{n-1}}{(n-1)!} - \frac{H.D^{n-1}}{(n-1)!}\right) m^{n-1} + (\text{ lower order terms }) \\
&= \frac{D^n}{(n-1)!} m^{n-1} + (\text{ lower order terms }).
\end{aligned}$$

The second term is

$$\begin{aligned}
& -\left(\frac{G.D^{n-2}.(K_G - 2(E+H)|_G)}{2(n-2)!} - \frac{H.D^{n-2}.(K_H - 2(E+H)|_H)}{2(n-2)!}\right) m^{n-2} \\
= & -\left(\frac{D^{n-2}.((K_X+G).G - (K_X+H).H) - 2D^{n-1}.(E+H)}{2(n-2)!}\right) m^{n-2} \\
= & -\left(\frac{D^{n-1}.(K_X - 2E) + D^n}{2(n-2)!}\right) m^{n-2}
\end{aligned}$$

Note that if $P(m) = a_n m^n - a_{n-1} m^{n-1} + \ldots$, then

$$P(m) - P(m-1) = n a_n m^{n-1} - \left(\frac{n(n-1)}{2} a_n + (n-1) a_{n-1}\right) m^{n-2} + \ldots.$$

Thus, we have

$$a_n = \frac{D^n}{n!} \text{ and } a_{n-1} = \frac{D^{n-1}.(K_X - 2E)}{2(n-1)!}.$$

Chapter 2

MMP in characteristic zero

In this chapter, k is always an algebraically closed field of *characteristic zero*. I give an introduction on how MMP works for surfaces and 3-folds over k.

2.1 Surface case

Given that X is a projective variety over k. In 1964, Hironaka proved the following theorem in [Hir64].

Theorem 2.1.1 (Hironaka). *For a normal variety, it is possible to resolve singularities of varieties over k by blowing up finitely many times along nonsingular subvarieties.*

Therefore, given a surface X, we may replace X by its birational model. Unlike curves, this smooth surface X is not unique since we may blow up any point further. Moreover, blowing up a point will produce an exceptional curve C with $C^2 = -1$. Note also that this curve C is isomorphic to $\mathbb{P}^1$. We call such curve a (-1)-curve. Thus, our ideal minimal models will be smooth varieties without any (-1)-curves.

Using Castelnuovo's contraction theorem (for a proof, see [Har77, Chapter 5, Theorem 5.7]),

Theorem 2.1.2 (Castelnuovo). *Given a (-1)-curve C on a smooth surface X, there exists a smooth surface X_0 such that X is the blowup of X_0 along some points $P \in X_0$ with exceptional curve C.*

we get a smooth surface without (-1)-curves, such surfaces are called relative minimal model of original X.

Remark 2.1.3. A smooth surface is not necessary to have finitely many (-1)-curves. For example, the blowup of $\mathbb{P}^2$ at 9 points in general position has infinitely many (-1)-curves. (For more detail, see [Har77, Chapter 5, exercise 4.15].)

Proposition 2.1.4. *For every surface X, there exists a relative minimal model.*

Proof. This proof is essentially due to [Har77].

By Castelnuovo's theorem, we have a sequence $X = X_0 \to X_1 \to X_2 \to \cdots$. It suffices to show that it must stop. Put E_i' be the exceptional curve of $X_i \to X_{i+1}$ and E_i be the proper transform of E_i' to X. Note that we have $E_i^2 = -1$ for all i and $E_i.E_j = 0$ for $i < j$. Put $e_i = c(E_i)$ be the cohomology class of E_i in $H^1(X, \Omega_X)$. Then we have $\langle e_i, e_i \rangle = -1$ for all i and $\langle e_i, e_j \rangle = 0$ for $i < j$. Thus, $\{e_i\}_{i \geq 0}$ is linearly independent in $H^1(X, \Omega_X)$. Since $H^1(X, \Omega_X)$ is finite dimensional, then $\{e_i\}_{i \geq 0}$ is a finite set. Hence, it must stop. □

Remark 2.1.5. X is a relative minimal model if and only if every birational morphism $X \to X'$ to a smooth surface X' is actually an isomorphism. (For detail, [Har77, Chapter 5, section 5].)

Remark 2.1.6. The relative minimal model is not unique for rational or ruled surfaces. For example, we consider the blowup of $\mathbb{P}^2$ at two points. We may either blow down two exceptional curves to get $\mathbb{P}^2$ or blow down the proper transform of the line connecting two points to $\mathbb{P}^1 \times \mathbb{P}^1$.

In order to generalize to higher dimensional case, Mori gives another criterion for minimal models.

Definition 2.1.7. A smooth projective variety X is minimal if K_X is *nef*. That is, $K_X.C \geq 0$ for all curves C in X.

Remark 2.1.8. If a surface X contains a (-1)-curve C, then by adjunction formula, $(K_X + C).C = 2g(C) - 2 = -2$. Thus, $K_X.C = -1 < 0$. Hence, a minimal model is a relative minimal model.

Remark 2.1.9. Note that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are relative minimal models but not minimal models.

Then what happen to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$? Although they have no (-1)-curves, they still have many curves C that make K_X not nef, that is, $K_X.C < 0$.

Then we can ask whether we have a "Castelnuovo's theorem" for such curves ? Mori said that we have an "extremal contraction" $X \to X'$ if K_X is not nef. (See Theorem 2.2.1.) In other words, we can contract some special curves that make K_X not nef. But it is not necessary that $\dim X' = \dim X$. In general, we have $\dim X' \leq \dim X$ and, for $\dim X' < \dim X$, we call $X \to X'$ a *Mori fiber space*.(Mfs)

Conjecture 2.1.10. *If $\kappa(X) = -\infty$, then X is birational to a Mfs. If $\kappa(X) \geq 0$, then X is birational to a minimal model Y.*

Remark 2.1.11. It is still open in dimension ≥ 5.

Remark 2.1.12 (Enriques-Kodaira classification of surfaces). If $\kappa(X) = -\infty$, then X is rational or a $\mathbb{P}^1$-bundle.

If $\kappa(X) = 0$, then X is birational to a K3 surface, an Enriques surface or an étale quotient of an abelian surface.

If $\kappa(X) = 1$, then X is birational to an elliptic surface.

If $\kappa(X) = 2$, then X is of general type.

In the end of this section, I recall some definitions which are used later. For detail, see, for example, [KM08].

Definition 2.1.13. A *pair* (X, B) consists of a normal variety X and a $\mathbb{Q}$-divisor $B = \sum b_i B_i$ such that $K_X + B$ is $\mathbb{Q}$-Cartier.

Definition 2.1.14. X is $\mathbb{Q}$-factorial if every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.

Definition 2.1.15 (discrepancy)**.** Let $f : Y \to X$ is a birational morphism of normal varieties and K_X is $\mathbb{Q}$-Cartier. Let E be an irreducible exceptional divisor, $e \in E$ is a general point of E and $\{y_i\}$ is a local coordinate at $e \in Y$ such that $E = (y_1 = 0)$. Then near e, we have

$$f_*(\text{ local generator of } \mathcal{O}_X(mK_X)) = y_1^{m \cdot a(E,X)}(\text{unit})(dy_1 \wedge \ldots \wedge dy_n)^{\otimes m}$$

where $m \in \mathbb{Z}$ such that mK_X is Cartier. Note that $a(E, X)$ is independent of m and is called the *discrepancy* of E with respect to X.

Remark 2.1.16. If f is a proper morphism and K_Y is Cartier, for example smooth, then

$$mK_Y \sim f^*(mK_X) + \sum(m \cdot a(E_i, X))E_i$$

Using numerical equivalence, we may divide by m and get

$$K_Y \equiv f^*K_X + a(E_i, X)E_i$$

Definition 2.1.17.

$$K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum_{E_i:\text{ exceptional}} a(E_i, X, \Delta)E_i$$

Definition 2.1.18 (discrepancy)**.**

$$\text{discrep }(X, \Delta) := \inf_E \{a(E, X, \Delta) : E \text{ is an exceptional divisor over } X.\}$$

Definition 2.1.19. Let (X, Δ) be a pair where X is a normal variety and $\Delta = \sum a_i D_i$. Assume $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then we say that (X, Δ) is

$terminal$		> 0
$canonical$		≥ 0
klt	if discrep(X, Δ)	> -1 and $\lfloor \Delta \rfloor \leq 0$
plt		> -1
lc		≥ -1

2.2 Higher dimensional case

In this section, we will mainly discuss 3-folds. Given a projective variety X of dimension ≥ 3, we may assume that X is nonsingular by Hironaka's resolution theorem. Then we ask whether K_X is nef or not. If K_X is nef, then we get a minimal model. If K_X is not nef, then we have an extremal contraction $X \to X_1$. More precisely,

Theorem 2.2.1. *If K_X is not nef, then there exists an extremal ray R of $\overline{NE}(X)$ with $K_X.R < 0$ and a morphism $f : X \to X_1$ such that*

1. *f is not an isomorphism.*

2. *A curve C in X is contracted to a point if and only if $[C] \in R$.*

3. *X_1 is a normal projective variety and f has connected fibers.*

Such f is called an extremal contraction with respect to R.

This theorem is based on the following theorems. (For a proof, see [KM08].)

Theorem 2.2.2 (Vanishing theorem)**.** *Let $\sum d_i D_i$ be a $\mathbb{Q}$-divisor and L be a line bundle. Assume that $D = L + \sum d_i D_i$ is nef and big and $\sum D_i$ is simple normal crossing. Then $H^i(X, K_X + \lceil D \rceil) = 0$ for $i > 0$.*

Theorem 2.2.3 (Nonvanishing theorem)**.** *Let D be a nef Cartier divisor and D' a $\mathbb{Q}$-divisor. Suppose $aD + D' - K_X$ is $\mathbb{Q}$-Cartier, nef and big for some $a > 0$ and $(X, -D')$ is klt. Then $H^0(X, mD + \lceil D' \rceil) \neq 0$ for all $m \gg 0$.*

Theorem 2.2.4 (Base point free theorem)**.** *Let (X, Δ) be a klt pair and Δ be effective. Let D be a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$. Then $|bD|$ has no basepoints for $b \gg 0$.*

Theorem 2.2.5 (Rationality theorem)**.** *Let (X, Δ) be a klt pair and Δ be effective such that $K_X + \Delta$ is not nef. Let $a(X) > 0$ be an integer such that $a(X)(K_X + \Delta)$ is Cartier. Let H be a nef and big Cartier divisor. We define*

$$r = r(H) = \min\{t \in \mathbb{R} \mid H + t(K_X + \Delta) \text{ is nef}\}.$$

Then $r \in \mathbb{Q}$ and is of the form $\frac{u}{v}$ where $0 < v \leq a(X)(\dim X + 1)$.

Corollary 2.2.6. *Then there exists an extremal ray R such that $R.(K_X + \Delta) < 0$ and $R.(H + r(K_X + \Delta)) = 0$.*

Theorem 2.2.7 (Cone theorem)**.** *Let (X, Δ) be a klt pair and Δ be effective. Then*

1. *There are (countably many) rational curves C_j in X such that*

$$0 < -(K_X + \Delta).C_j \leq 2 \dim X$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

2. *Given any $\epsilon > 0$ and ample $\mathbb{Q}$-divisor H, then*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta+\epsilon H)\geq 0} + \sum_{\textit{finite}} \mathbb{R}_{\geq 0}[C_j].$$

3. *For any $F \subset \overline{NE}(X)$ a $(K_X + \Delta)$-negative extremal face, there exists a unique morphism $f_F : X \to Y$ with Y being projective such that $(f_F)_*(\mathcal{O}_X) = \mathcal{O}_Y$. Moreover, for any curve C in X, $f_F(C)$ is a point if and only if $[C] \in F$.*

4. *Let Y and f_F as above. For any line bundle $\mathcal{L}$ on X such that $\mathcal{L}.C = 0$ for all curves C with $[C] \in F$, then there exists a line bundle $\mathcal{L}_Y$ on Y such that $\mathcal{L} = (f_F)^*\mathcal{L}_Y$.*

Remark 2.2.8. Combining (3) in cone theorem with the corollary to rationality theorem, we have the extremal contraction we want.

Here comes a problem, is X_1 still smooth ? The answer is *No*. There is even an example of a smooth projective 3-fold X such that K_X is not nef and, for any extremal contraction $f : X \to X'$, X' is a singular 3-fold. For an example, see [Mat02, Example 3-1-3].

Thus, we should go out the category of smooth varieties. In other word, we should allow some mild singularities. Hence, we consider a category $\mathcal{C}$ which consists of normal projective $\mathbb{Q}$-factorial varieties with only terminal singularities.

Note that the notion of K_X being nef still make sense since X is $\mathbb{Q}$-factorial.

Next, extremal contraction holds in $\mathcal{C}$. More precisely, for any object X in $\mathcal{C}$, the corresponding X_1 in Theorem 2.2.1 is an object in $\mathcal{C}$.

Let $f : X \to Y$ be a birational extremal contraction. If codimension of exceptional set of f is 1 (resp. ≥ 2), then we call such f a *divisorial contraction* (resp. *small contraction*).

However, for a small contraction $f : X \to Y$, K_Y is not $\mathbb{Q}$-Cartier. Indeed, if K_Y is $\mathbb{Q}$-Cartier, then $K_X = f^*K_Y$ since f is small. For any curve C contracted by f, $0 = f^*K_Y.C = K_X.C < 0$, a contradiction.

So we go out the category $\mathcal{C}$ again. This time, the singularities of Y may too wild to control. Nevertheless, Mori gives a solution to this problem. It is "*flip*", a new operation that make us stay in the category $\mathcal{C}$.

Definition 2.2.9 (flip)**.** Let $f : X \to Y$ be a small contraction of an extremal ray with respect to K_X and $X \in \mathcal{C}$. More explicitly,

1. f is a birational morphism onto Y and $X \in \mathcal{C}$.
2. f is small.
3. $-K_X$ is f-ample.
4. The relative Picard number $\rho(X/Y) = 1$.

The morphism $f^+ : X^+ \to Y$ is called a *flip* of f if

1. f^+ is a birational morphism onto Y and $X^+ \in \mathcal{C}$.
2. f^+ is small.
3. K_{X^+} is f^+-ample.
4. The relative Picard number $\rho(X^+/Y) = 1$.

Example 2.2.10. Consider $Z = V(xy - zt) \subset \mathbb{A}^4$. It has only one singularity $P = (0, 0, 0, 0)$. Blowing up P, we get a birational morphism $Y \to Z$ from a smooth variety Y. This morphism contracts one divisor $E = V(xy - zt) \subset \mathbb{P}^3$. Note that $E \cong \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

by Segre embedding. Then we can contract E in two different directions given by the two projections on $\mathbb{P}^1$.

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ X & \overset{\phi}{\dashrightarrow} & X^+ \\ \searrow & & \swarrow \\ & Z & \end{array}$$

Note that $K_X \sim_Z 0$ and $K_{X^+} \sim_Z 0$. Then we choose an ample divisor B^+ on X^+ and put $B = \phi^{-1}(B^+)$. Then $K_X + B$ is negative over Z but $K_{X^+} + B^+$ is positive over Z.

Proposition 2.2.11. *Let $f : X \to Y$ be the small extremal contraction with respect to K_X from a normal projective $\mathbb{Q}$-factorial variety X with only terminal singularities. Then a flip $f^+ : X^+ \to Y$ exists if and only if the canonical ring $R := \oplus_{m\geq 0} f_* \mathcal{O}_X(mK_X)$ is a finitely generated $\mathcal{O}_Y$-algebra. Moreover, in the latter case, the flip has the description $f^+ : \operatorname{Proj} R \to Y$. Hence, it is unique.*

If flips exist, then it not only makes us come back to the category $\mathcal{C}$ but also makes X closer to being nef.

Remark 2.2.12. By [BCHM10], flips always exist.

In 1986, Shokurov proved that the termination of flips in dimension 3, see [Sho86]. In 1987, Kawamata, Matsuda and Matsuki proved for dimension 4, see [KMM87]. For dimension ≥ 5, it is still open.

To sum up, we have the following theorem and flowchart.

Theorem 2.2.13. *Let (X, Δ) be a projective normal $\mathbb{Q}$-factorial varieties of dimension 3 with only terminal singularities where Δ is an effective $\mathbb{Q}$-divisor. Then there exists a sequence of birational morphisms*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{s-1}} (X_s, \Delta_s)$$

with the following properties.

1. *Each (X_i, Δ_i) is a projective normal $\mathbb{Q}$-factorial varieties with only terminal singularities of dimension 3.*

2. *For each i, the relative Picard number $\rho(X_i/X_{i+1}) = 1$ and $(K_{X_i} + \Delta_i).C_i < 0$ for all ϕ_i-contracted curves C_i.*

3. *X_s satisfies one of the following conditions.*

   (a) (X_s, Δ_s) is a minimal model, that is, $K_{X_s} + \Delta_s$ is nef.

   (b) There is a surjective morphism $\phi : X_s \to Y$ to a projective normal variety Y with $\dim Y < 3$ such that $\phi : X_s \to Y$ is a Mori fiber space.

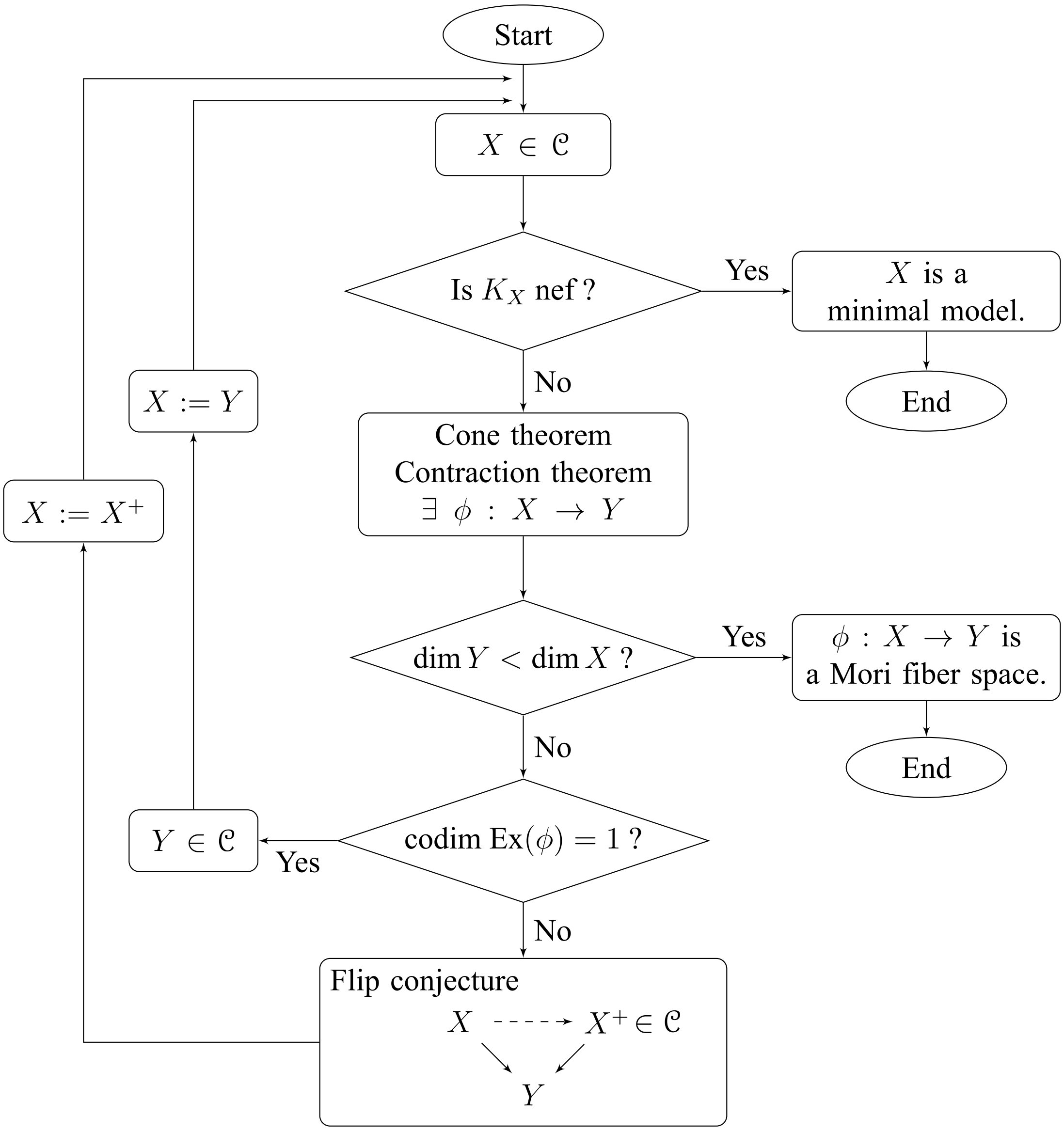


Figure 2.1: Flowchart for MMP in dimension 3 or higher.

Chapter 3

MMP in positive characteristic

In this chapter, k is an algebraically closed field and, if I do not mention, of *positive characteristic* p.

3.1 Differences with characteristic zero

In characteristic zero, there are two tools that are used frequently, resolution of singularities and Kodaira vanishing.

Abhyankar proved [Abh56] that resolution of singularities hold for surfaces in all characteristic. He then proved [Abh66] that this also holds for 3-folds in characteristic > 5. In 2008 and 2009, Cossart and Piltant proved [CP08], [CP09] for 3-folds in any characteristic. For dimension ≥ 4, it is still open.

For Kodaira vanishing, it does not hold in general in positive characteristic. (even in dimension 2, for an example, see [Ray78])

Besides these, there are some geometric difference.

Theorem 3.1.1 (Generic smoothness)**.** *Suppose char*$(k) = 0$*. Let* $f : X \to Y$ *be a morphism with* X *being smooth. Then there exists an open subset* U *in* Y *such that the fibers of* f *over any points in* U *are smooth.*

Example 3.1.2 (Counterexample to generic smoothness). Consider $X = (y^2 + x^p + t) \subseteq \mathbb{A}^3_{x,y,t}$ and $Y = \mathbb{A}^1_t$ over an algebraically closed field k with positive characteristic p. Put

$f : X \to Y$ be the projection. Then the fibers of f over $c \in Y$ is $(g = y^2 + x^p + c = 0) \subseteq \mathbb{A}^2$. Note that

$$\frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial g}{\partial y} = 2y = 0 \text{ if } y = 0$$

Then, $(x_c, 0)$ is a singular point of the fiber where $x_c^p + c = 0$. Hence, all fibers are singular. This gives a counterexample to generic smoothness in positive characteristic.

Next, I give some properties and phenomenon only occur in positive characteristic.

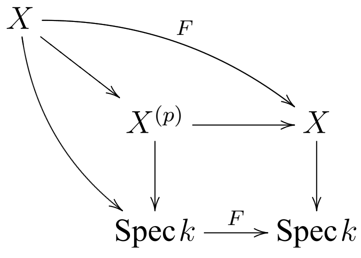
Example 3.1.3. Suppose $\text{char}(k) = p > 0$. Let $X = Y = \mathbb{A}^1$. Consider a ring homomorphism $k[t] \to k[t]$ sends t to t^p. Then it corresponds to $f : X \to Y$ be given by $a \mapsto a^p$. For any $a \in Y$, the fiber over a is given by $(t - b)^p = 0$ where $b^p = a$. That is, we have non-reduced fibers.

As we see above, in positive characteristic, the Frobenius map plays the most important role.

Definition 3.1.4. Let X be a scheme over k with $\text{char}(k) > 0$. The (absolute) Frobenius morphism $F : X \to X$ is an identity on points and p-th power on its sections.

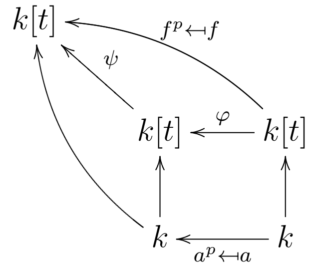
Example 3.1.5 ($X = \operatorname{Spec} A$). The corresponding ring homomorphism is $f : A \to A$ which sends a to a^p.

Then we have a diagram



where $X^{(p)} = \operatorname{Spec} k \times_{\operatorname{Spec} k} X$. The morphism $X \to X^{(p)}$ is called the geometric Frobenius.

Example 3.1.6 ($X = \mathbb{A}^1$). The corresponding diagram is



where $\varphi : \sum a_i t^i \mapsto \sum a_i^p t^i$ and $\psi : t \mapsto t^p$.

The following proposition tells us that the geometry over $\overline{\mathbb{F}}_p$ is different from other fields.

Proposition 3.1.7. *Let k be an algebraically closed field of arbitrary characteristic.*

1. *If $k \neq \overline{\mathbb{F}}_p$, then all nonzero abelian varieties over k have infinite rank.*

2. *If $k = \overline{\mathbb{F}}_p$, then all group schemes of finte type are torsion groups.*

Proof. See [FJ74, Theorem 10.1] and [Tan12, Fact 2.4].

Corollary 3.1.8. *Given X be a projective variety over $\overline{\mathbb{F}}_p$ and D be a Cartier divisor. If $D \equiv 0$, then D is a torsion in Pic X.*

Proof. See [Kee99, Lemma 2.16].

3.2 F-Singularities

In this section, I introduce how to define singularities via the Frobenius morphism. Surprisingly, they have something to do with singularities defined by discrepancies. (See Theorem 3.2.8.)

Let (X, B) be a pair with an effective $\mathbb{Q}$-divisor B.

Definition 3.2.1. (X, B) is sharply F-pure at $x \in X$ if the map

$$(\mathcal{O}_X)_x \to (F^e_* \mathcal{O}_X(\lceil (p^e - 1)B \rceil))_x$$

splits for all $e \in \mathbb{N}$.

Definition 3.2.2. (X, B) is strongly F-regular at $x \in X$ if, for all effective divisors E, there exists an integer $e > 0$ such that $(\mathcal{O}_X)_x \to (F^e_* \mathcal{O}_X(\lceil (p^e - 1)B \rceil + E))_x$ splits.

Example 3.2.3 ($X = \mathbb{A}^1$). The map $\mathcal{O}_X \to F_* \mathcal{O}_X$ corresponds to a ring homomorphism $k[t] \to k[t]$ which sends f to f^p. Note that we have a diagram

$$\begin{array}{ccc} k[t] & \xrightarrow[f \mapsto f^p]{F} & k[t] \\ & {\scriptstyle \sum a_i t^i \mapsto \sum a_i^p t^i}\ \searrow^{\alpha} \quad {}^{\beta}\nearrow\ {\scriptstyle t \mapsto t^p} & \\ & k[t] & \end{array}$$

Note that α is a bijection and β splits. Then F splits and $\mathbb{A}^1$ is sharply F-pure.

Example 3.2.4. If X is smooth, then we want to show that X is sharply F-pure. Fix a closed point $x \in X$. Put $R = (\mathcal{O}_X)_x$ and it is a regular local ring of dimension $= \dim X$. We have an exact sequence $0 \to R \to F^e_* R \to N \to 0$ where N is the cokernel of $R \to F^e_* R$. Note that $F^e_* R = (F^e_* \mathcal{O}_X)_x$ and $R \to F^e_* R$ splits if N is a free R-module.

Let $t \in R$ be an element of system of parameters at x. Then $R/\langle t \rangle$ is a regular local ring of dimension $= \dim X - 1$. Consider the following diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \langle t \rangle & \to & F^e_* \langle t \rangle & \to & \langle t \rangle N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R & \to & F^e_* R & \to & N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R/\langle t \rangle & \to & F^e_* R/\langle t \rangle & \to & N/\langle t \rangle N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Note that N is finitely generated since R is a regular local ring and $F^e_* R$ is free. So if $N/\langle t \rangle N$ is free, then N is a free R-module by Nakayama's lemma. Thus, by induction on $\dim X$, we only need to show for $\dim X = 0$, which is automatically true since R is a field.

Before proving an important theorem 3.2.8 in this section, I show some properties for F-singularities.

Proposition 3.2.5. *If (X, B) is strongly F-regular, then it is sharply F-pure.*

Proof. Fix any $e \in \mathbb{N}$, we may choose $m \in \mathbb{N}$ such that all coefficients of $m\lceil (p^e - 1)B \rceil$ are greater than p^e. By strongly F-regularity of (X, B), there exists $e' \in \mathbb{N}$ such that

$$\mathcal{O}_X \to F_*^{e'}\mathcal{O}_X(\lceil (p^{e'} - 1)B \rceil + m\lceil (p^e - 1)B \rceil) \tag{3.1}$$

splits.

Claim. $e' \geq e$.

Indeed, if $e' < e$, then the morphism 3.1 above factors through $\mathcal{O}_X \to \mathcal{O}_X(S)$ for any component S of $\lceil (p^e - 1)B \rceil$. Thus, $\mathcal{O}_X \to \mathcal{O}_X(S)$ splits, a contradiction.

Note that the morphism 3.1 factors through $\mathcal{O}_X \to F_*^e\mathcal{O}_X(\lceil (p^e - 1)B \rceil)$ since $e' \geq e$. Hence it splits. □

Proposition 3.2.6. *(X, B) is strongly F-regular if and only if, for all effective divisors E, there exists $q' = p^{e'}$ such that the map $\mathcal{O}_X \to F_*^e\mathcal{O}_X(\lceil p^e B \rceil + E)$ splits for all $q = p^e \geq q'$.*

Proof (due to [HW02]). The part 'if' holds since the map

$$\mathcal{O}_X \to F_*^e\mathcal{O}_X(\lceil p^e B \rceil + E)$$

factors through $F_*^e\mathcal{O}_X(\lceil (p^e - 1)B \rceil + E)$.

To show 'only if', let E be any effective divisor and choose an effective divisor D such that $\lceil qB \rceil \leq \lceil (q-1)B \rceil + D$ for all $q = p^e$. Since (X, B) is strongly F-regular, there exists $q' = p^{e'}$ such that

$$\mathcal{O}_X \to F_*^{e'}\mathcal{O}_X(\lceil (q' - 1)B \rceil + E + D + \lceil B \rceil)$$

splits. Note that this map factors through $F_*^{e'}\mathcal{O}_X(\lceil q'B \rceil + E + \lceil B \rceil)$, then the map $\mathcal{O}_X \to F_*^{e'}\mathcal{O}_X(\lceil q'B \rceil + E + \lceil B \rceil)$ splits. On the other hand, by Proposition 3.2.5, (X, B) is sharply

F-pure, hence X is sharply F-pure. Then the map $F^{e'}_* \mathcal{O}_X \hookrightarrow F^{e+e'}_* \mathcal{O}_X$ splits for all $e \in \mathbb{N}$. Applying $\otimes_{F^{e'}_* \mathcal{O}_X} F^{e'}_* \mathcal{O}_X(\lceil q'B \rceil)$, we have $F^{e'}_* \mathcal{O}_X(\lceil q'B \rceil) \hookrightarrow F^{e+e'}_* \mathcal{O}_X(q\lceil q'B \rceil)$ also split. Thus, the map

$$\mathcal{O}_X \to F^{e+e'}_* \mathcal{O}_X(q\lceil q'B \rceil + E + \lceil B \rceil)$$

splits for all $e \in \mathbb{N}$. Note that this map factors through $F^{e+e'}_* \mathcal{O}_X(\lceil qq'B \rceil + E)$. Hence, the map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil p^e B \rceil + E)$ splits for all $q = p^e \geq q'$. $\square$

Proposition 3.2.7. *Suppose (X, B) is sharply F-pure (resp. strongly F-regular) with X being normal and B being an effective $\mathbb{Q}$-Cartier divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then the coefficients of B ≤ 1 (resp. < 1).*

Proof (due to [HW02]). Suppose there exists a component S of B with coefficient > 1 (resp. ≥ 1). Then there is a $q = p^e$ such that the coefficient of $(q-1)B$ (resp. qB) in S is at least q. So we have $\lceil (q-1)B \rceil \geq qS$ (resp. $\lceil qB \rceil \geq qS$). Therefore, the map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (q-1)B \rceil) \text{ (resp. } \mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil qB \rceil))$$

factors through $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(S) \hookrightarrow F^e_* \mathcal{O}_X(qS)$, which does not split. Thus, by definition (resp. Proposition 3.2.6), (X, B) is not sharply F-pure (resp. not strongly F-regular), a contradiction. $\square$

In fact, sharply F-purity and strongly F-regularity are analogous to log canonical and Kawamata log terminal. Moreover, they share the similar properties. One of the most important theorem is as follows [HW02].

Theorem 3.2.8. *If (X, B) is sharply F-pure (resp. strongly F-regular), then (X, B) is log canonical (resp. klt).*

Proof. We may assume that X is affine, say $\operatorname{Spec} R$. Let $f : \widetilde{X} \to X$ be a proper birational morphism with $\widetilde{X}$ being normal. Write $K_{\widetilde{X}} + \widetilde{B} = f^*(K_X + B) + \sum a_j E_j$.

Step 0 We have the following duality :

Let X be a normal variety over k with char$(k) = p > 0$ and D be a divisor. Then for all $e \in \mathbb{N}$, we have an $F^e_*\mathcal{O}_X$-isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \cong F^e_*\mathcal{O}_X((1-p^e)K_X - D)$$

Indeed, by duality of finite morphisms, we have

$$\mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \cong F^e_*\mathrm{Hom}(\mathcal{O}_X(D), \omega_{X/X})$$

where $\omega_{X/X}$ is the relative dualizing sheaf of F^e. Note that

$$\omega_{X/X} = \mathcal{O}_X(K_X - p^e K_X).$$

So $\mathrm{Hom}(\mathcal{O}_X(D), \omega_{X/X}) = \mathcal{O}_X(-D + (1-p^e)K_X)$.

Step 1 Fix $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \hookrightarrow L = k(\widetilde{X})$. Then we can choose a nonzero $b \in R$ such that $b \cdot H^0(X, \omega_X^{-i}) \subseteq H^0(\widetilde{X}, \omega_{\widetilde{X}}^{-i})$ for $i = 0, 1, \ldots, r-1$ where r will be fixed later.

Step 2 Assume that (X, B) is sharply F-pure. Let $\phi : F^e_*\mathcal{O}_X(\lceil (p^e-1)B \rceil) \to \mathcal{O}_X$ be the splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e-1)B \rceil)$. Then it induces a splitting $\widetilde{\phi} : F^e_*L \to L$ of $L \hookrightarrow F^e_*L$. Put $Y = \widetilde{X} \backslash Z$ where $Z = \mathrm{Supp}\ ((\widetilde{\phi}(F^e_*\mathcal{O}_{\widetilde{X}}) + \mathcal{O}_{\widetilde{X}})/\mathcal{O}_{\widetilde{X}})$. Since $F^e_*\mathcal{O}_{\widetilde{X}}$ is a coherent $\mathcal{O}_{\widetilde{X}}$-module, Y is an open subset of $\widetilde{X}$ and is the maximal open subset of $\widetilde{X}$ where $\widetilde{\phi}$ induces an $\mathcal{O}_Y$-linear map $f^*(\phi) : F^e_*\mathcal{O}_Y \to \mathcal{O}_Y$.

Claim. *$f^*(\phi)$ gives an F-splitting of Y.*

Proof. Let U be an open subset of $\widetilde{X}$ where f is an isomorphism.
Note that $\mathrm{codim}(X \backslash U, X) \geq 2$ and U is dense in Y. Also, $\mathcal{O}_Y \hookrightarrow F^e_*\mathcal{O}_Y \to \mathcal{O}_Y$ determines an element of $H^0(Y, \mathcal{O}_Y) \subseteq H^0(U, \mathcal{O}_U) = R$ which is the identity on U. Thus it is an identity on Y.

Step 3 Apply duality on $\widetilde{X}$ and (X, B), we get

$$\begin{cases} \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(F_*^e\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}) \cong F_*^e\mathcal{O}_{\widetilde{X}}((1-p^e)K_{\widetilde{X}}) \\ \mathrm{Hom}_{\mathcal{O}_X}(F_*^e\mathcal{O}_X(\lceil (p^e-1)B\rceil), \mathcal{O}_X) \cong F_*^e\mathcal{O}_X((1-p^e)K_X - \lceil (p^e-1)B\rceil) \end{cases}$$

So $\phi \in F_*^e\mathcal{O}_X((1-p^e)K_X - \lceil (p^e-1)B\rceil)$. Note that

$$f_*(rf^*(K_X+B)) = f_*(r(K_{\widetilde{X}} + \widetilde{B}) - r\sum a_jE_j) = r(K_X+B).$$

Then $H^0(\widetilde{X}, rf^*(K_X+B)) = H^0(X, r(K_X+B))$ is principally generated as an R-module. Here, we require that r is divided by the index of K_X+B. Write $p^e-1 = mr+i$ with $0 \le i < r$. Then

$$\begin{aligned} \mathcal{O}_X((1-p^e)K_X &- \lceil (p^e-1)B\rceil) \subseteq \mathcal{O}_X(-mr(K_X+B) - iK_X) \\ &\subseteq b^{-1}H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mrf^*(K_X+B) - iK_{\widetilde{X}})) \\ &\subseteq b^{-1}H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}((1-p^e)K_{\widetilde{X}} + \sum mra_jE_j)) \end{aligned}$$

Step 4 Suppose $a_j < -1$, then we choose $r \in \mathbb{N}$ satisfying the requirements above and $a_j \le -1 - \frac{1}{r}$. Put ξ be the generic point of E_j. Note that

$$\begin{aligned} -\nu - mra_j - q &\ge -\nu + m(r+1) - mr - i + 1 \\ &= m - \nu - i + 1 \to \infty \text{ as } q \to \infty. \end{aligned}$$

where $\nu = v_{E_j}(b)$ and $m = \lfloor \frac{q-1}{r} \rfloor$. So we can take q sufficiently large such that $-\nu - mra_j \ge q$. Then at ξ, $f^*(\phi) \in F_*^e(\pi^q\mathcal{O}_{\widetilde{X}}((1-p^e)K_{\widetilde{X}})) = \pi\mathrm{Hom}(F_*^e\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}))$ where π is a regular parameter. Thus, $\xi \in Y$ and $f^*(\phi)(F_*^e\mathcal{O}_{Y,\xi}) \subseteq \pi\mathcal{O}_{Y,\xi}$, which contradicts to that $f^*(\phi)$ gives a splitting of Y.

Step 5 Suppose (X, B) is strongly F-regular. We have known that all $a_j \ge -1$ and $\lfloor B \rfloor = 0$ by Proposition 3.2.7. Now we suppose that $a_j = -1$ for some j. We consider a Cartier divisor E defined by $c \in R$. Then by assumption, we have a splitting $\psi : F^e_*\mathcal{O}_X(\lceil (p^e-1)B\rceil + E) \to \mathcal{O}_X$ of $\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)B\rceil + E)$. Note that ψ also gives a splitting of $\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)B\rceil)$. Moreover,

$$\begin{aligned}\psi &\in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(\lceil (p^e-1)B\rceil + E), \mathcal{O}_X) \\ &= F^e_*\mathcal{O}_X((1-p^e)K_X - \lceil (p^e-1)B\rceil - E) \\ &\subseteq F^e_*(b^{-1}H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}((1-p^e)K_{\widetilde{X}} + \sum mra_jE_j - E))).\end{aligned}$$

We can choose c such that $t = v_{E_j}(c) \geq r + \nu$. Then $t - \nu + mr \geq (m+1)r \geq q$. Thus, at ξ, we have $f^*(\phi) \in F^e_*(\pi^q\mathcal{O}_{\widetilde{X}}((1-p^e)K_{\widetilde{X}})) = \pi\mathrm{Hom}(F^e_*\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}))$ where π is a regular parameter. This leads to a contradiction as above.

□

Remark 3.2.9 ([HW02]). The converse statement to the part of sharply F purity does not hold.

3.3 MMP for surfaces

Before proving that MMP works for surfaces, I recall some theorems.

Theorem 3.3.1 ([CTX13]). *Given a pair (X, B) which is strongly F-regular with K_X+B being not nef and an ample divisor A. Let t be the smallest number such that $L = K_X + B + tA$ is nef. Then $t \in \mathbb{Q}$.*

Remark 3.3.2. It is analogous to rationality theorem in characteristic zero.

Theorem 3.3.3 (Cone Theorem, [Tan12]). *Let X be a projective normal surface and B be an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Given any ample $\mathbb{Q}$-Cartier divisor H, we have*

1. $\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$.

2. $\overline{NE}(X) = \overline{NE}(X)_{K_X+B+H\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$.

Remark 3.3.4. This theorem is proved by Bend-and-Break technique.

Theorem 3.3.5 (Keel's semi-ample theorem, [Kee99]). *Let X be projective scheme over k and L be a nef $\mathbb{Q}$-divisor. Define $\mathbb{E}(L)$ be the union of integral subschemes V of* dim > 0 *and $L|_V$ is not big, then L is semi-ample $\Leftrightarrow$ $L|_{\mathbb{E}(L)}$ is semi-ample.*

Proposition 3.3.6 ([Fuj84]). *Let X be a projective normal surface of arbitrary characteristic and L be a nef line bundle. If $\kappa(L) = 1$, then L is semi-ample.*

Proof. Note that we may assume that X is nonsingular. Consider the Iitaka fibration $\varphi = \varphi_{|mL|} : X \dashrightarrow C$ where C is a curve. Then we have $mL = E + \varphi^*\mathcal{O}_C(1)$ where E is an effective divisor. Put $F = \varphi^*\mathcal{O}_C(1)$. We have

$$0 \leq E.F = (E + F).F = mL.F \leq mL.(E + F) = m^2L^2 = 0.$$

Then $E.F = L.F = E.L = 0$. Write $E = \sum r_iE_i$ be the prime decomposition. Thus, $\varphi_*(E_i)$ is a point and E_i is the multiple of $\varphi^*(\varphi_*(E_i))$. Hence, E_i is semi-ample. □

Now we are ready to prove.

Theorem 3.3.7 (Contraction Theorem, [Tan12]). *Let $(X, B = \sum b_jB_j)$ be a projective normal surface satisfying one of the following condition.*

(QF) *X is $\mathbb{Q}$-factorial and $0 \leq b_j \leq 1$ for all j.*

(FP) *$k = \overline{\mathbb{F}}_p$ and $b_j \geq 0$ for all j.*

(LC) *(X, B) is log canonical.*

Let $R := \mathbb{R}_{\geq 0}[C]$ be a $(K_X + B)$-negative extremal ray. Then there exists a morphism cont$_R : X \to X'$ to a projective variety X' with the following properties :

1. *$(\text{cont}_R)_*(\mathcal{O}_X) = \mathcal{O}_{X'}$.*

2. *For any curve C' on X, $[C'] \in R$ if and only if cont$_R(C')$ is a point.*

3. *$\rho(X') = \rho(X) - 1$.*

Proof. I skecth the proof for (QF). For detail, see [Tan12].

Claim. If $C^2 > 0$, then $\rho(X) = 1$.

Indeed, given any curve C', by Kodaira's lemma, we have $mC \sim nC' + E$ where $m, n \in \mathbb{N}$ and E is an effective divisor. Then since $\mathbb{R}_{\geq 0}[C]$ is an extremal ray, $[C'] = q[C]$ for some $q \in \mathbb{Q}$.

Hence, for $C^2 > 0$, $X' = \{\text{a point}\}$ is what we want.

Claim. If $C^2 = 0$, then C is semi-ample.

Indeed, we have $K_X.C < 0$ from $(K_X + B).C < 0$ and $C^2 = 0$. Consider a resolution $f : Y \to X$, we have $(f^*C)^2 = 0$ and $K_Y.f^*C = K_X.C < 0$. By asymptotic Riemann-Roch theorem 1.3.1, we have $\chi(Y, f^*C) \geq 1$. Since $h^2(Y, nf^*C) = h^0(Y, K_Y - nf^*C) = 0$ for $n \gg 0$, we have $\kappa(Y, f^*C) \geq 1$. Then $\kappa(X, C) \geq 1$. Since $C^2 = 0$, $\kappa(X, C) = 1$. By Proposition 3.3.6, C is semi-ample.

Hence, $\varphi_{|mC|}$ is the contraction we want.

Claim. If $C^2 < 0$, then C is isomorphic to $\mathbb{P}^1$.

Indeed, we have $C.(K_X + C) < 0$ from $C.(K_X + B) < 0$ and $C^2 < 0$. Put r be any positive integer such that $r(K_X + C)$ is Cartier. Then $h^0(C, \omega_X(C)^{[r]}|_C) = 0$ implies $h^1(C, \mathcal{O}_C) = 0$ by [Tan12, Lemma 5.2].

Put A be any ample divisor and $q \in \mathbb{Q}_{\geq 0}$ such that $(A + qC).C = 0$. Then $G = A + qC$ is nef and big. Note that, for any curve C', $G.C' = 0$ iff $C' = C$. Thus $G|_C \equiv 0$. Since $C = \mathbb{P}^1$, $G|_C$ is free, hence semi-ample. By Keel's Theorem 3.3.5, G is semi-ample.

Remark 3.3.8. In [Tan12], Tanaka shows that

1. When dim $X' = 2$, X' is still $\mathbb{Q}$-factorial.

2. A normal projective surface over $\overline{\mathbb{F}}_p$ is $\mathbb{Q}$-factorial.

3. A curve C with $C^2 < 0$ on a normal projective surface over $\overline{\mathbb{F}}_p$ can be contracted. This is not true in general, for an example, see [Har77, Chapter 5, Example 5.7.3].

4. Using (2) and (3), we get the contraction theorem for (FP).

5. For the case (LC), the proof is similar to (QF). Basically, he uses Bertini's theorem to deal with the part where he uses $\mathbb{Q}$-factoriality in the case (QF).

In summary, we get the minimal model program.

Theorem 3.3.9. *Let (X, B) be a normal projective surface satisfying one of* (QF)*,* (FP) *and* (LC)*. Then there exists a sequence of birational morphisms*

$$(X, B) = (X_0, B_0) \xrightarrow{\phi_0} (X_1, B_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{s-1}} (X_s, B_s)$$

with the following properties.

1. *Each X_j is a normal projective surface.*

2. *If (X_j, B_j) satisfies* (QF) *(resp.* (FP) *resp.* (LC)*), then so is (X_{j+1}, B_{j+1}).*

3. *Each* $\mathrm{Ex}(\phi_j) = C_j$ *is a proper irreducible curve such that $(K_{X_j} + B_j).C_j < 0$.*

4. *(X_s, B_s) is a minimal model or a Mori fiber space.*

3.4 MMP for 3-folds

Given that (X, B) with $\dim X = 3$. Note that MMP for 3-folds over characteristic zero heavily relies on Kodaira vanishing theorem which is not true for positive characteristic. Moreover, from the surface case in positive characteristic, we know that we need many efforts to achieve our goal, that is, MMP for 3-folds. So far, Hacon and Xu prove [HX13] the case when $k = \overline{\mathbb{F}}_p$ for $p > 5$, X is smooth and $B = 0$.

If we accept a more general MMP, then in [HX13], Hacon and Xu show that

Theorem 3.4.1. *Let (X, B) be a $\mathbb{Q}$-factorial projective three dimensional canonical pair over an algebraically closed field k of characteristic $p > 5$. Assume that all coefficients of B are in the standard set $\{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ and $N_\sigma(K_X + B) \wedge B = 0$. If $K_X + B$ is pseudo-effective, then*

1. *There exists a minimal model X_{min} of (X, B).*

2. *If* $k = \overline{\mathbb{F}}_p$*, then* X_{min} *can be obtained by running the usual* $(K_X + B)$*-MMP.*

Remark 3.4.2. The minimal model in (1) for $k \neq \overline{\mathbb{F}}_p$ is not obtained by running the usual MMP. The process they give is similar to usual MMP. But their process may go out of the category of schemes. More explicitly, Keel shows that the extremal contraction exists in the category of algebraic spaces. In order to come back to the category of schemes, they introduced the *generalized flip* which bring us back to the category of schemes. This notion is similar to the usual flip. Also, they show the existence and the termination of the generalized flip.

Chapter 4

Resolution for terminal singularities

In characteristic zero, we have known that the category which consists of varieties with terminal singularities is the smallest category that MMP could work in higher dimension. Moreover, the development of MMP in dimension 3 heavily depends on the understanding of terminal singularities.

In the beginning, I will introduce some results in characteristic zero for classification of terminal singularities in 3-folds. These are due to Mori, Kollár and Shepherd-Barron.

For the rest of this chapter, I try to verify or determine whether the classification for the terminal 3-fold singularities in characteristic zero is still terminal or not in positive characteristic.

More explicitly, I show that

Proposition 4.0.1. *Let k be of positive characteristic. We have*

1. *Every quotient singularity of type $\frac{1}{r}(a, r-a, 1)$ where $(r, a) = 1$ is terminal.*

2. *Every isolated singularity of type cA, cA/r, $cAx/2$ or cD is terminal.*

The methods I use are weighted blowups and (algebraic) change of variables. And the choice of weighted blowups is essentially due to [Che13].

4.1 Characteristic zero

In this section, k is an algebraically closed field of characteristic zero. We have known that

Theorem 4.1.1 ([Rei83], [Rei87]). *Any terminal singularities in 3-folds are cyclic quotient of isolated cDV singularities (may be nonsingular).*

Recall that we have known

Theorem 4.1.2. *Up to isomorphism, the possible du Val singularities are as follows.*

1. $A_n : x^2 + y^2 + z^{n+1} = 0.$

2. $D_n : x^2 + y^2 z + z^{n-1} = 0$ *for* $n \geq 4$.

3. $E_6 : x^2 + y^3 + z^4 = 0.$

4. $E_7 : x^2 + y^3 + yz^3 = 0.$

5. $E_8 : x^2 + y^3 + z^5 = 0.$

Mori gives a list for necessary condition for being cDV singularities with quotients.

Theorem 4.1.3 ([Mor85], [Rei87]). *Let $P \in X = (Q \in Y)/\mu_r$ be a terminal hyperquotient singularities where $r > 1$ and $Q \in Y$ is singular. Then $P \in X$ belongs to one of the following families*

1. $cA/r : (xy + f(z,u) = 0) \subseteq \mathbb{A}^4/\frac{1}{r}(a, r-a, 1, 0)$

2. $cAx/2 : (x^2 + y^2 + f(z,u) = 0) \subseteq \mathbb{A}^4/\frac{1}{2}(1,0,1,0)$

3. $cAx/4 : (x^2 + y^2 + f(z,u) = 0) \subseteq \mathbb{A}^4/\frac{1}{4}(1,3,1,2)$

4. $cD/2 : (\varphi = 0) \subseteq \mathbb{A}^4/\frac{1}{2}(1,1,0,1)$ *where φ is one of the following*

$$\begin{cases} x^2 + yzu + y^{2a} + u^{2b} + z^c, \text{ where } a \geq b \geq 2, c \geq 3 \\ x^2 + y^2 z + \lambda y u^{2\ell+1} + f(z, u^2) \end{cases}$$

5. $cD/3 : (\varphi = 0) \subseteq \mathbb{A}^4/\frac{1}{3}(0,2,1,1)$ *where φ is one of the following*

$$\begin{cases} x^2 + y^3 + zu(z+u) \\ x^2 + y^3 + zu^2 + yg(z,u) + h(z,u), g \in m^4, h \in m^6 \\ x^2 + y^3 + z^3 + yg(z,u) + h(z,u), g \in m^4, h \in m^6 \end{cases}$$

6. $cE/2 : (x^2 + y^3 + yg(z,u) + h(z,u) = 0) \subseteq \mathbb{A}^4/\frac{1}{2}(1,0,1,1)$

Then Kollár and Shepherd-Barron proved that

Theorem 4.1.4 ([KSB88]). *Every isolated singularity in the list above is terminal.*

4.2 Cyclic quotient singularities in positive characteristic

From now on, k is always an algebraically closed field of *positive characteristic* p.

For this type of singularities, I use the language of toric varieties. The method is the same as in characteristic zero. (For detail in characteristic zero, see [Ful93].)

4.2.1 Toric varieties

$N = \mathbb{Z}^n$ is the lattice of rank n. $M = N^\vee$ is the dual lattice of N.

Fix any strongly convex rational polyhedral cone σ in $N_\mathbb{R}$, we have dual cone

$$\sigma^\vee = \{u \in M_\mathbb{R} \mid \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\} \subset M_\mathbb{R}$$

Let $S_\sigma = \sigma^\vee \cap M$ and $A_\sigma = k[S_\sigma]$.

Then we can construct the correspond toric variety $U_\sigma = \operatorname{Spec} A_\sigma$.

Definition 4.2.1. σ is said to be *regular* if its minimal generators form a part of a $\mathbb{Z}$-basis of N.

Lemma 4.2.2 (Gordon's lemma). *S_σ is a finitely generated semigroup.*

The proof is characteristic-free. For detail, see [Ful93, page 12].

Lemma 4.2.3. *If σ is regular, then the corresponding U_σ is nonsingular.*

Proof. Since σ is regular, we have minimal generators $u_1, u_2, \ldots, u_n$ of σ.

Then $\chi^{u_1}, \chi^{u_2}, \ldots, \chi^{u_n}$ are algebraic independent. Indeed, if $\prod(\chi^{u_i})^{a_i} = \prod(\chi^{u_i})^{b_i}$ where $a_i, b_j \in \mathbb{N}\cup\{0\}$, then $\sum a_i u_i = \sum b_i u_i$. Since $u_1, u_2, \ldots, u_n$ form a basis, we have $a_j = b_j$ for all $j = 1, \ldots, n$.

Thus, $U_\sigma = \operatorname{Spec} k[\chi^{u_1}, \chi^{u_2}, \ldots, \chi^{u_n}] \cong \mathbb{A}^n$. □

4.2.2 Cyclic quotients

Let ξ_r be the r-th roots of unity. Unlike in characteristic zero, ξ_r may be 1. For example, $r = p$.

Given $\mathbb{A}^n$ with coordinates $x_1, \ldots, x_n$. Consider an action of $\mu_r = (\mathbb{Z}/r\mathbb{Z}, +)$ on $\mathbb{A}^n$ by $\bar{1} \cdot (x_1, \ldots, x_n) = (\xi_r^{a_1} x_1, \ldots, \xi_r^{a_n} x_n)$. Then we can get a quotient $\mathbb{A}^n/\mu_r = \operatorname{Spec} k[x_1, \ldots, x_n]/I$ where I is an ideal generated by the relations.

Example 4.2.4. Consider $\mu_3 = \frac{1}{3}(2, 1)$. Then

$$k[x, y]^{\mu_3} = k[x^3, y^3, xy] = k[x_1, x_2, x_3]/(x_1 x_2 - x_3^3)$$

Although $\mathbb{A}^n/\mu_p$ is $\mathbb{A}^n$, we still denote $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ by $\mathbb{A}^n/\mu_p$ where I is an ideal generated by the relations for viewing ξ_r just a symbol with $\xi_r^r = 1$. This is reasonable since the language of toric varieties cares about the relation but not an actual action.

Proposition 4.2.5. *$\mathbb{A}^n/\mu_r$ is a toric variety.*

Proof. For detail, see [Ful93, page 35].

In fact, every fan can be refined to a regular fan by adding finitely many rays. Moreover, adding a ray corresponds to a weighted blowup.

To sum up, we may resolve such singularities by finitely many weighted blowups. Then computing the discrepancy, we have the following result.

Proposition 4.2.6. *$\mathbb{A}^3/\mu_r$ is a terminal singularity if and only if $\mu_r = \frac{1}{r}(a, r-a, 1)$.*

For detail, see [Rei87].

4.3 cA_n type singularities

From now on, I show the second part of Proposition 4.0.1.

Put $\mathrm{char}(k) = p > 0$. I assume that the singularities are isolated[1].

For the case of characteristic zero, cA_n type singularities are of the form

$$X = (xy + z^{n+1} + tg(x, y, z, t) = 0) \subset \mathbb{A}^4.$$

Note that, by an easy change of variables, it is isomorphic to $\widetilde{X} = (x^2 + y^2 + z^{n+1} + tg(x, y, z, t) = 0) \subset \mathbb{A}^4$ except for $p = 2$.

For $p = 2$, $x^2 + y^2 + z^{n+1} = 0$ is not a du Val singularity. More explicitly, it is not normal since the singular locus is $x = y, z = 0$, which is one dimensional.

Proposition 4.3.1. *Any* isolated cA_n *type singularities of the form*

$$(xy + z^{n+1} + tg(x, y, z, t) = 0)$$

with $p > 0$ *are terminal.*

For some technical reasons, I consider

$$X = (f = xy + z^{n+1}h(z) + tg(x, y, z, t) = 0) \subset \mathbb{A}^4$$

where $h(0) = 1$.

Write $g(x, y, z, t) = g_0 + g_1 + g_2 + g_{>2}$ where g_j is the homogeneous part of g with weight j in x, y, z. More precisely, write $g_0 = g_0(t)$, $g_1 = g^x(t)x + g^y(t)y + g^z(t)z$, and $g_2 = \sum_{1 \le i \le j \le 3} g^{x_i x_j}(t) x_i x_j$.

For $T(t) = \sum a_i t^i$, we define

$$\mathrm{ord}_t\, T(t) = \min\{i \mid a_i \neq 0\},\ \mathrm{LC}(T) = a_{\mathrm{ord}_t T} \text{ and } \mathrm{C}_j(T) = a_j.$$

[1]Here I do not give an explicit definition for isolated singularities in positive characteristic. What I use is that the point is not isolated if the singular locus through it has dimension greater than 1.

Put $m = \min\{\mathrm{ord}_t\, g^x, \mathrm{ord}_t\, g^y\}$ and $\ell = \mathrm{ord}_t\, g^z$.

By using following change of variables

$$\begin{cases} x \to x - tg^y \\ y \to y - tg^x \\ z \to z \\ t \to t \end{cases}$$

we may assume that $m \geq 1$.

Before proving the proposition, I give a flowchart for my proof.

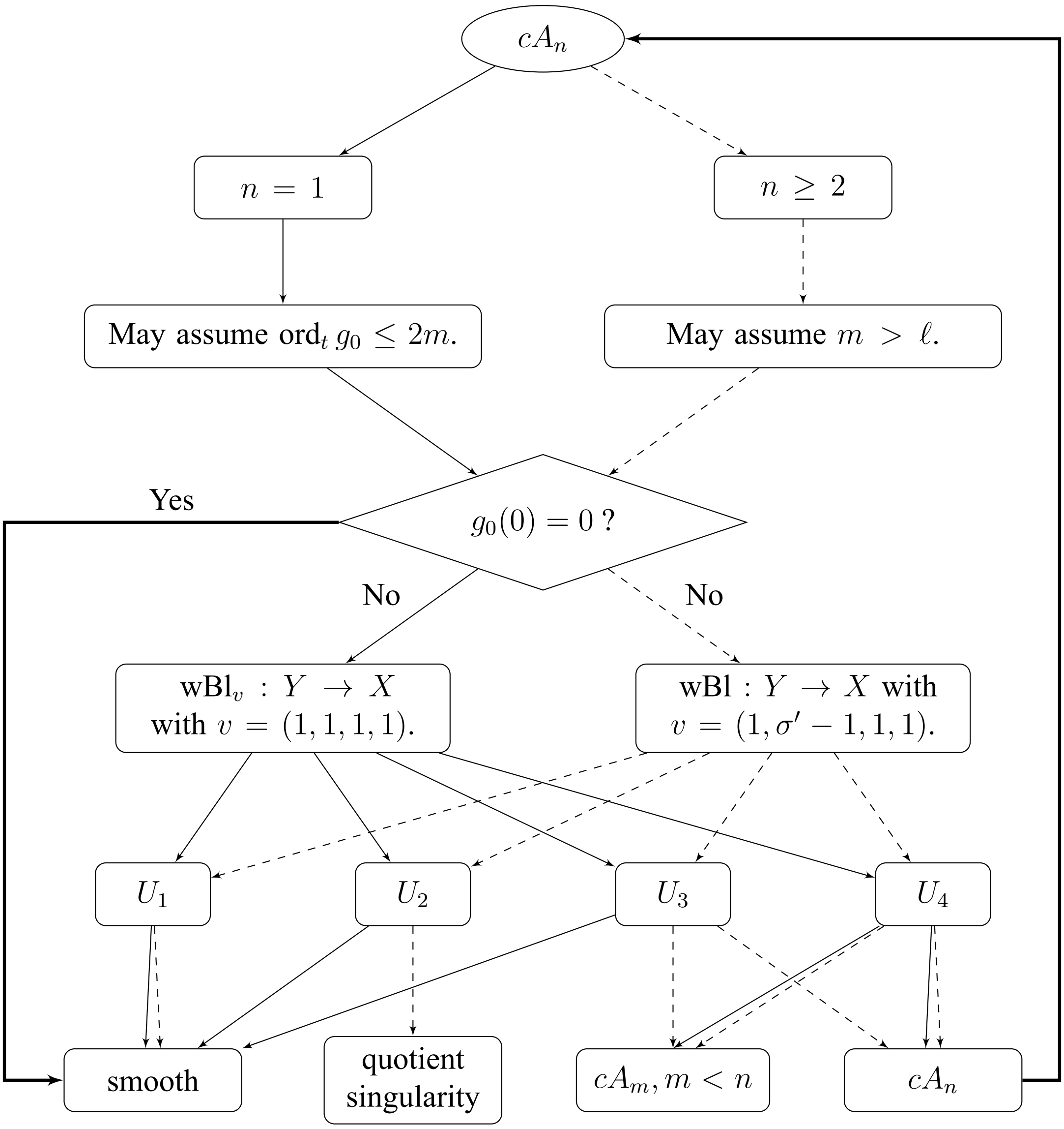


Figure 4.1: Flowchart for cA_n.

4.3.1 $n = 1$

Claim. *We may assume that* $\operatorname{ord}_t g_0 \leq 2m$.

Proof. If $\operatorname{ord}_t g_0 \geq 2m+1$, then we consider the change of variables

$$\begin{cases} x \to x - tg^y \\ y \to y - tg^x \\ z \to z - ct^{m+1} \\ t \to t \end{cases}$$

Then we have $xy + z^2 h(z) + t(\widetilde{g}_0 + \widetilde{g}_1 + \widetilde{g}_2 + \widetilde{g}_{>2}) = 0$ where

$$\begin{aligned} \widetilde{g}_0 = g_0 \; & - \; tg^x g^y + c^2 t^{2m+1} h(-ct^{m+1}) - ct^{m+1} g^z \\ & + \; t^2 g^{x^2} (g^y)^2 + t^2 g^{y^2} (g^x)^2 + c^2 t^{2m+2} g^{z^2} \\ & + \; t^2 g^{xy} g^x g^y + ct^{m+2} g^{xz} g^y + ct^{m+2} g^{yz} g^x \\ & + \; (\text{terms with order } \geq 3m+3) \end{aligned}$$

$$\begin{aligned} \widetilde{g}^x &= -2tg^{x^2} g^y - tg^{xy} g^x - ct^{m+1} g^{xz} + (\text{terms with order } \geq 2m+2) \\ \widetilde{g}^y &= -2tg^{y^2} g^x - tg^{xy} g^y - ct^{m+1} g^{yz} + (\text{terms with order } \geq 2m+2) \\ \widetilde{g}^z &= g^z - 2ct^m h(-t^{m+1}) + c^2 t^{2m+1} h'(-t^{m+1}) \\ & \quad -2ct^{m+1} g^{z^2} - tg^{xz} g^y - tg^{yz} g^x + (\text{terms with order } \geq 2m+2). \end{aligned}$$

Without loss of generality, we may assume that $m = \operatorname{ord}_t g^x \leq \operatorname{ord}_t g^y$.

Note that $\widetilde{m} \geq m+1$.

- If $\operatorname{ord}_t g^z < m$, then by choosing $c \neq 0$, we have

$$\operatorname{ord}_t \widetilde{g}_0 < 2m+1 \leq 2(\widetilde{m}-1)+1 < 2\widetilde{m}.$$

- If $\operatorname{ord}_t g^z \geq m$, then by choosing c such that

$$\mathrm{C}_{2m+1}(g_0) - \mathrm{LC}(g^x)\,\mathrm{C}_m(g^y) + c^2 - c\,\mathrm{C}_m(g^z) \neq 0$$

we have $\operatorname{ord}_t \widetilde{g}_0 = 2m+1 \leq 2(\widetilde{m}-1)+1 < 2\widetilde{m}$.

□

- If $g_0(0) \neq 0$, then it is smooth.
- If not, then we consider $\mathrm{wBl}_v : Y \to X$ with weight $v = (1,1,1,1)$. Y is smooth on[2] $U_1 \cup U_2 \cup U_3$. Also, the discrepancy ≥ 1. On U_4, we have

$$xy + z^2\widetilde{h}(z,t) + \frac{g_0(t)}{t} + g^x x + g^y y + g^z z + tH(x,y,z,t) = 0 \text{ for some } H.$$

On[3] $U_4 \cap E$, we have $xy + z^2 h(0) + g'(0) + g^z(0)z = 0$.

  - If $g'(0) \neq 0$.
    * If $h(0)z^2 + g^z(0)z + g'(0)$ is not a perfect square, then Y is smooth.
    * If it is a perfect square, then, after change of variables, Y has only cA_1 type singularities. The termination will be proved later.
  - If not.
    * If $\operatorname{ord}_t g_0 \leq 2\operatorname{ord}_t g^z$, then induction[4] on $\operatorname{ord}_t g_0$.
    * If $\operatorname{ord}_t g_0 > 2\operatorname{ord}_t g^z$, then after taking some blowups, we may assume that $g^z(0) \neq 0$ and $g_0(0) = 0$. In this case, by taking a blowup further, we have $xy + z^2 h(0) + g^z(0)z = 0$ on $E \cap U_4$. Clearly, $\widetilde{Y}$ is smooth.

4.3.2 $n \geq 2$

Claim. *May assume that $m > \ell$*

[2] From now on, $\{U_i\}_{i=1}^4$ is always an open affine covering of Y and Q_i is the origin on U_i.
[3] From now on, E is always the exceptional divisor of wBl : $Y \to X$.
[4] Note that we have $\operatorname{ord}_t \widetilde{g}_0 \leq 2\widetilde{m}$.

Proof. If $m \leq \ell$

Consider the change of variables

$$\begin{cases} x \to x - tg^y \\ y \to y - tg^x \\ z \to z - ctg^z \\ t \to t \end{cases}$$

Then we have $xy + z^{n+1}h(z) + t(\widetilde{g}_0 + \widetilde{g}_1 + \widetilde{g}_2 + \widetilde{g}_{>2}) = 0$ where

$$\begin{aligned} \widetilde{g}_0 = g_0 \quad &+ \quad -tg^x g^y + c^{n+1}t^n (g^z)^n h(-ctg^z) \\ &+ \quad t^2 g^{x^2} (g^y)^2 + t^2 g^{y^2} (g^x)^2 + c^2 t^2 g^{z^2} (g^z)^2 \\ &+ \quad t^2 g^{xy} g^x g^y + ct^2 g^{xz} g^y g^z + ct^2 g^{yz} g^x g^z \\ &+ \quad \text{(terms with order } \geq 3m+3) \end{aligned}$$

$$\begin{aligned} \widetilde{g}^x &= -2tg^{x^2} g^y - tg^{xy} g^x - ctg^{xz} g^z + \text{(terms with order } \geq 2m+2) \\ \widetilde{g}^y &= -2tg^{y^2} g^x - tg^{xy} g^y - ctg^{yz} g^z + \text{(terms with order } \geq 2m+2) \\ \widetilde{g}^z &= g^z + (n+1)t^{n-1}(-cg^z)^n h(-ctg^z) + t^n(-cg^z)^{n+1} h'(-ctg^z) \\ & \quad -2ctg^{z^2} g^z - tg^{xz} g^y - tg^{yz} g^x + \text{(terms with order } \geq 2m+2) \end{aligned}$$

We may choose $c \neq 1$ such that $\text{ord}_t\, \widetilde{g}^z \leq \ell$. Note that $\widetilde{m} \geq m+1$. Thus, after finitely many times, we are done. □

Define $\sigma' = \min\{i_1 + i_3 + i_4 + 1 \mid x^{i_1} z^{i_3} t^{i_4} \in g\}$ and $\sigma = \min\{\sigma', n+1\}$. We may assume that all monomials $x^{i_1} z^{i_3} t^{i_4}$ with $i_1 + i_3 + i_4 + 1 = \sigma'$ have $i_1 = 0$.

- If $\sigma' = 1$, then X is smooth.
- If $2 \leq \sigma' < n+1$, then may assume that $t^{\sigma'-1} \in g$. By taking $\text{wBl}_v : Y \to X$ with weight $v = (1, \sigma'-1, 1, 1)$, Y is smooth on U_1. On U_2, the only singularity is at Q_2 which is a terminal quotient singularity of index $\sigma' - 1$. Also, the discrepancy ≥ 1.

On $U_3 \cap E$, we have $xy+g(t)=0$ where $\deg_t\ g(t) < n+1$. Write $g(t)=\prod(t-\alpha_i)^{r_i}$ where $0 < \sum r_i \leq n$. Then Y has only cA_m type singularities with $m < n$. Thus, by induction on n, we are done.

On $U_4 \cap E$, we have $xy + g(z) = 0$ where $\deg_z\ g(z) < n+1$. Similar to U_3, we are done.

- If $\sigma' \geq n+1$, then we take $\text{wBl}_v : Y \to X$ with weight $v = (1, n, 1, 1)$. Write $z^{n+1} + \sum z^i t^j = \prod(z - \alpha_i t)^{r_i}$ where $z^i t^j \in tg(x, y, z, t)$ and $i + j = n + 1 = \sum r_i$. It is similar as above, Y is smooth on U_1 and has a terminal quotient singularity at Q_2. Note that all remaining singularities are covered by U_4.

  On $U_4 \cap E$, we have $xy + \prod(z - \alpha_i)^{r_i} = 0$.

  – If not all α_i are the same, then we are done.

  – If all $\alpha_i = \alpha$

    * if $\alpha = 0$, then σ' will decrease and we back to the first step.
    * If $\alpha \neq 0$, then after doing a change of variables (this may increase σ'), we back to the first step.

Then we must to show that it is impossible that, in each step, we always have $\sigma' \geq n+1$ and same α_i. In other words, the process will terminate.

4.3.3 Termination

Now I assume that we have infinite steps.

Remark 4.3.2. For any given N, we may assume that $\text{ord}_t\, g^{x^{i_1}y^{i_2}z^{i_3}} \geq N$ for $i_1 \geq 1$ or $i_2 \geq 1$. Since I use finitely many identities and relations to get a contradiction, I could pick an N which is large enough to have the following arguments.

Note that, after finitely many steps, the original $g_0, g^z, \ldots, g^{z^n}$ will disappear.

Let $S = \{j | g^{z^j} z^j \in g \text{ and } j \geq n+1\}$.

- If $S = \emptyset$, then the singular locus of $\widetilde{X} = (xy + z^{n+1} = 0)$ passing through the origin has dimension ≥ 1, a contradiction.

- If $|S| \geq 1$. Write $g^{z^j} = \sum_{\kappa=1} t^{N_\kappa^j}$ for $j \in S$. Doing $z \to z + \alpha_0 t$, we have[5]

$$xy + z^{n+1} + \sum_{j \in S} \sum_{\kappa=1} \sum_{i_0=0}^{j} \xi_{i_0,\kappa}^j \binom{j}{i_0} \alpha_0^{j-i_0} t^{N_\kappa + j - i_0} z^{i_0} \text{ where } \xi_{i,\kappa}^j \in k.$$

  Taking a blowup, we get

$$xy + z^{n+1} + \sum_{j \in S} \sum_{\kappa=1} \sum_{i_0=0}^{j} \xi_{i_0,\kappa}^j \binom{j}{i_0} \alpha_0^{j-i_0} t^{N_\kappa + j - n - 1} z^{i_0}.$$

  Then doing a change of variables $z \to z + \alpha_1 t$ and taking a blow up, we get

$$xy + z^{n+1} + \sum_{j \in S} \sum_{\kappa=1} \sum_{i_1=0}^{j} \sum_{i_0=i_1}^{j} \xi_{i_0,\kappa}^j \binom{j}{i_0} \binom{i_0}{i_1} \alpha_0^{j-i_0} \alpha_1^{i_0-i_1} t^{N_\kappa + j - 2(n+1) + i_0} z^{i_1}.$$

  By induction, we have

$$xy + z^{n+1} + \sum_{j \in S} \sum_{\kappa=1} \sum_{S_\ell^j} \xi_{i_0,\kappa}^j \binom{j, i_0, \ldots, i_{\ell-1}}{i_0, i_1, \ldots, i_\ell} \mathbf{a}^{\mathbf{i}} t^{N_\kappa + j - (\ell+1)(n+1) + I_0^{\ell-1}} z^{i_\ell}$$

  where[6]

$$S_\ell^j = \{(i_\ell, \ldots, i_0) \in \mathbb{Z}^{\ell+1} \mid 0 \leq i_\ell \leq i_{\ell-1} \leq \cdots \leq i_0 \leq j\}.$$

$$\binom{j, i_0, \ldots, i_{\ell-1}}{i_0, i_1, \ldots, i_\ell} = \binom{j}{i_0} \binom{i_0}{i_1} \cdots \binom{i_{\ell-1}}{i_\ell}.$$

$$\mathbf{a}^{\mathbf{i}} = \alpha_0^{j-i_0} \alpha_1^{i_0-i_1} \cdots \alpha_\ell^{i_{\ell-1}-i_\ell}.$$

$$I_0^{\ell-1} = i_0 + i_1 + \cdots + i_{\ell-1}.$$

  We may assume that $\xi_{i,\kappa} \neq 0$ for the largest j, κ and for all i.

  For $i_\ell = 0$, we have

$$N_\kappa^j + j - (\ell+1)(n+1) = n + 1. \tag{4.1}$$

[5] The expression is not complete. I only write down the terms that I want to keep track of.

[6] We make a convention that $0^0 = 1$.

Then $N_\kappa^j + j = (\ell+2)(n+1)$.

Next, for $i_0 = i_1 = \cdots = i_\ell = j_0$ and $i_{\ell+1} = \cdots = i_{\ell'} = 0$ where j_0 will be chosen later, we have

$$N_\kappa^j + j - (\ell'+1)(n+1) + j_0\ell = n+1.$$

Then

$$\ell' = \frac{n+1+j_0}{n+1}\ell.$$

Again, for $i_0 = i_1 = \cdots = i'_\ell = j_0$ and $i_{\ell'+1} = \cdots = i_{\ell''} = 0$, we have

$$N_\kappa^j + j - (\ell''+1)(n+1) + j_0\ell' = n+1.$$

Then

$$\frac{\ell''-\ell'}{\ell'-\ell} = \frac{j_0}{n+1}.$$

Inductively, we get

$$\frac{\ell^{(r)}-\ell^{(r-1)}}{\ell^{(r-1)}-\ell^{(r-2)}} = \frac{j_0}{n+1}$$

Then

$$\begin{aligned} \ell^{(r)} - \ell^{(r-1)} &= \left(\frac{j_0}{n+1}\right)^r \ell \\ \ell^{(r)} &= \ell\left(1 + \frac{j_0}{n+1} + \cdots + \left(\frac{j_0}{n+1}\right)^r\right) \end{aligned}$$

– If $n+1 \nmid j$, then we choose $j_0 = j$. Thus, we have $\ell^{(r)} \notin \mathbb{N}$ for some $r \in \mathbb{N}$, a contradiction.

– If $n+1 \mid j$.

  * If $p \nmid j$, then we choose $j_0 = 1$, we get a contradiction as above.

  * If $p \mid j$ and $p \nmid n+1$, then we write $j = p^m u$ where $p \nmid u$. Choose $j_0 = p^m$. Note that $p \nmid \binom{j}{p^m}$ and $\frac{p^m}{n+1} \notin \mathbb{Z}$. Thus, we get a contradiction as above.

  * If $p \mid j$ and $p \mid n+1$, then N_κ^j must be divisible by p because of the equa-

tion 4.1. Then we consider the second maximal $N^j_{\kappa'}$. If the corresponding $\xi_{0,\kappa'} = 0$, then $N^j_\kappa + j - a(n+1) = N^j_{\kappa'} + j - (n+1)$. Then $p \mid N_{\kappa'}$. If $\xi_{0,\kappa'} \neq 0$, then we also have $N^j_{\kappa'} + j - (\ell+1)(n+1) = n+1$. Thus, we still have $p \mid N^j_{\kappa'}$.

Next, I consider the second maximal element j_2 in S if exists. Note that $p \mid j_2$ since $p \mid n+1$. Otherwise, it will produce $t^b z$ for some b and it cannot be eliminated by "z^{n+1}", contradiction to infinite steps. Then $p \mid N^{j_2}_\kappa$. Inductively, we get

$$\widetilde{X} = (xy + (G(x,y,z,t))^p + H(x,y,z,t) = 0)$$

where every term in $H(x,y,z,t)$ must be divisible by x or y and degree in each variable is not divisible by p. If all terms in $H(x,y,z,t)$ can be divided by x^2, xy or y^2, then the singular locus through of $\widetilde{X}$ has dimesion ≥ 1, a contraction. But if we have the term only divided by x, then we can do a change of coordinate $x \to x + at^r$ and $z \to z + bt^r$ for some suitable a, b and a large r to make $tg_0(t)$ not the p-th power of some polynomials. Then we are done.

4.4 cA/r type singularities

Put char$(k) = p > 0$ and $r \geq 2$. I assume that the singularities are isolated. For characteristic zero, cA/r type singularities are of the form

$$X = (\varphi = xy + z^{n+1} + tg(x,y,z,t) = 0) \subset \mathbb{A}^4/\frac{1}{r}(s, r-s, 1, r).$$

Proposition 4.4.1. *Any* isolated *cA/r type singularities of the form above with $p > 0$ are terminal.*

I consider a more general form $X = (\varphi = xy + z^{n+1}h(z) + tg(x,y,z,t) = 0) \subset \mathbb{A}^4/\frac{1}{r}(s, r-s, 1, r)$ where $h(0) = 1$ and $(r,s) = 1$. Note that $r \mid n+1$. Write $f =$

$z^{n+1} + tg(x, y, z, t)$.

Put

$$\begin{aligned} \kappa^{\#} &= \min\{m \mid t^m \in f\} \\ \kappa &= \min\{\frac{s}{r}i_1 + \frac{1}{r}i_3 + i_4 \mid x^{i_1}z^{i_3}t^{i_4} \in f\} \\ \kappa^{*} &= \min\{\frac{s}{r}i_1 + \frac{1}{r}i_3 + i_4 \mid x^{i_1}z^{i_3}t^{i_4} \in f, i_1 \geq 1\} \end{aligned}$$

Note that $1 \leq \kappa \leq \kappa^*$, $\kappa \leq \kappa^{\#} < \infty$.

Claim. *We may assume that $\kappa \neq \kappa^*$.*

Proof. For any monomial $cx^{i_1}z^{i_3}t^{i_4} \in f$ with $\frac{s}{r}i_1 + \frac{1}{r}i_3 + i_4 = \kappa$ and $i_1 \geq 1$. Consider a change of variables $y \to y - cx^{i_1-1}z^{i_3}t^{i_4}$, then we can eliminate this monomial and it may increase κ and κ^*, but not $\kappa^{\#}$. It is because such monomial should be xt^{i_4} which is not fixed by $\frac{1}{r}(s, r-s, 1, r)$. Thus, after finitely many times, we are done. □

- If $\kappa^{\#} = \kappa = 1$, then it is a terminal quotient singularity.

- If $\kappa^{\#} + \kappa > 2$, then we take $\mathrm{wBl}_v : Y \to X$ with weight $v = \frac{1}{r}(s, \kappa r - s, 1, r)$. On U_1 (resp. U_2), the only singularity is the origin and it is a terminal quotient singularity with index s (resp. $\kappa r - s$).[7]

  On U_3, we have $\tilde{\varphi} = xy + z^{-\kappa}f(xz^{s/r}, yz^{(\kappa r - s)/r}, z^{1/r}, zt)$. On $U_3 \cap E$, we have $xy + \overline{f}(t) = 0$ where $\overline{f}(t)$ is not a zero polynomial. Then there are only finitely many singularities. And the singularities are at worst of cA type.

  On $U_4 \cap E$, we have $xy + \overline{f}(z) = 0$ where $\overline{f}(z)$ is not a zero polynomial. Except for the origin, the possible singularities are of cA type. For the origin, it is of cA/r type singularity with $\tilde{\kappa}^{\#} = \kappa^{\#} - \kappa$. Note that the discrepancy > 0. Hence, by induction on $\kappa^{\#}$, we are done.

[7] It is nonsingular if $s = 1$ (resp. $\kappa r - s = 1$).

4.5 $cAx/2$ and cD type singularities

Before proving, I give flowcharts for my proof. See Figure 4.2 and 4.3. In Figure 4.3, the dashed line means that some cases will go to $cAx/2$ type with $\tau = \mu^b + 1$. Note that we do induction on both $cAx/2$ type and cD type at the same time.

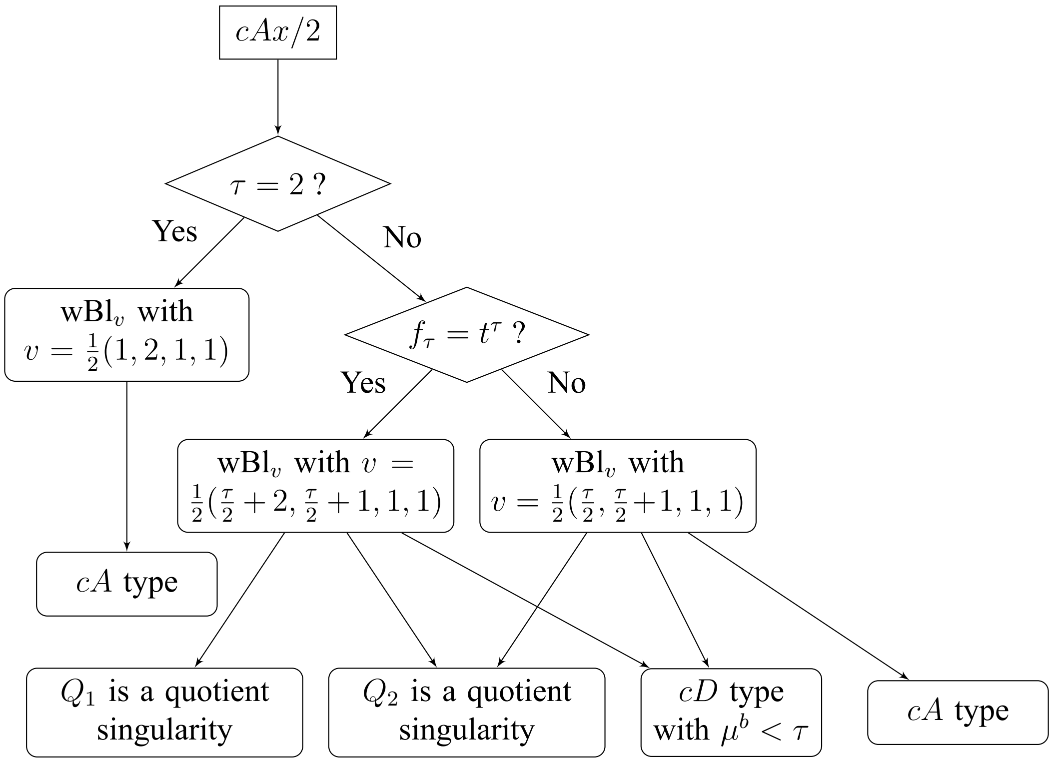


Figure 4.2: Flowchart for $cAx/2$.

4.5.1 Settings

Put $\mathrm{char}(k) = p > 0$. I assume that the singularities are isolated. For $cAx/2$ type singularity, it is, in characteristic zero, of the form

$$(\varphi = x^2 + y^2 + z^{n+1} + tg(x, y, z, t) = 0) \subset \mathbb{A}^4/\frac{1}{2}(1, 0, 1, 1).$$

I consider a more general form

$$(\varphi = x^2 + y^2 + z^{n+1}h(z) + tg(x, y, z, t) = 0) \subset \mathbb{A}^4/\frac{1}{2}(1, 0, 1, 1)$$

where $h(0) = 0$ if and only if $h \equiv 0$. For $p = 2$, $(x^2 + y^2 + z^{n+1}h(z) = 0)$ is not normal, hence it is not a du Val singularity. Note that we may assume that $x \notin g$.

Define $\tau = \min\{i + j \mid z^i t^j \in \varphi\}$. Note that $2 \mid \tau$.

Now we may assume that $\deg_t g^{xz^i} > n + 1$ and $\deg_t g^{yz^j} > n + 1$ for $i \in \mathbb{N}$ (resp. $i \in \mathbb{Z}_{\geq 0}$) and $j \in \mathbb{Z}_{\geq 0}$ (resp. $j \in \mathbb{N}$) if $\frac{\tau}{2}$ is odd (resp. even).

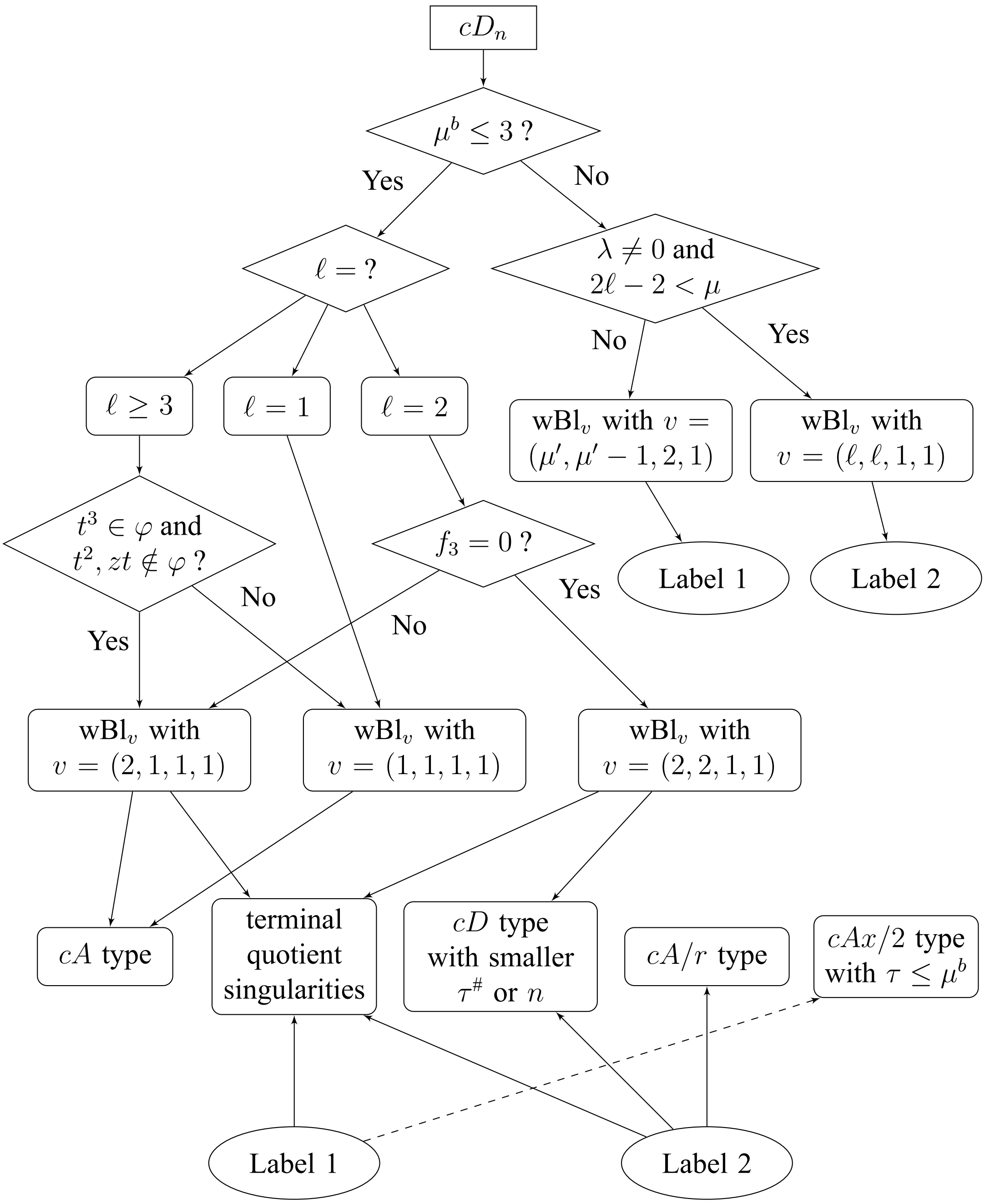


Figure 4.3: Flowchart for cD_n.

For cD_n type singularity with $n \geq 4$, it is, in characteristic zero, of the form

$$X = (\varphi = x^2 + y^2z + z^{n-1} + tg(x, y, z, t) = 0) \subset \mathbb{A}^4.$$

Now I consider a more general form

$$X = (\varphi = x^2 + y^2z + z^{n-1}h(z) + tg(x, y, z, t) = 0) \subset \mathbb{A}^4$$

where $h(z) \equiv 0$ if $h(0) = 0$. I redefine g and write

$$\varphi = x^2 + y^2z + z^{n-1}h(z) + yt^\ell + tg(x, y, z, t)$$

where yt^ℓ is the monomial with smallest ℓ among such form in the original $tg(x, y, z, t)$. (ℓ may be infinite.)

For $p = 2$, we note that $(x^2 + y^2z + z^{n-1}h(z) = 0)$ is not normal since the singular locus has a component of codimension 1. Thus it is not a du Val singularity.

Put

$$\begin{aligned} \mu &= \min\{2i + j \mid z^it^j \in \varphi\} \\ \mu^b &= \min\{\mu, 2\ell - 2\} \\ \tau^\# &= \min\{i + j \mid z^it^j \in \varphi \text{ and } i = 0, 1\}. \end{aligned}$$

Now we may assume that $\deg_t g^{xz^i} > 2n - 2$ and $\deg_t g^{yz^{i+1}} > 2n - 2$ for all integer i with $0 \leq i \leq n - 1$. Moreover, we may assume that $f_\mu = \sum_{2i+j=\mu} z^it^j = t^\mu$.

4.5.2 The base case

Proposition 4.5.1. *Any* isolated $cAx/2$ *type singularities of the form above with* $p > 0$ *and* $\tau = 2$ *are terminal.*

Proof. We have two cases.

- $x^2 + y^2 + z^2 + tg = 0$. In this case, we may assume further that $z \notin g$ and $g_0(t) \not\equiv 0$.

Then taking $\text{wBl}_v : Y \to X$ with weight $v = \frac{1}{2}(1,2,1,1)$, we get that it is smooth on $U_1 \cup U_2 \cup U_3$. On U_4, it is of cA type singularity.

- $x^2 + y^2 + z^{n+1} + tg = 0$ with $n > 1$. From assumption, at least one of t or z is in g. Taking $\text{wBl}_v : Y \to X$ with weight $v = \frac{1}{2}(1,2,1,1)$, we get that it is smooth on $U_1 \cup U_2 \cup U_4$. On U_3, it is smooth or of cA type singularity.

Proposition 4.5.2. *Any* isolated *cD_n type singularities of the form above with $p > 0$ and $\mu^b \leq 3$ are terminal.*

We may assume that $g_0(0) = 0$.

To prove the proposition, I divide it into three cases with respect to ℓ.

- $\ell = 1$. Take $\text{wBl}_v : Y \to X$ with weight $v = (1,1,1,1)$. Y is smooth on $U_1 \cup U_4$.

  On $U_2 \cap E$, we have $x^2 + t + t(g_0'(0)t + g^z(0)z) = 0$.

  - If $g^z(0) \neq 0$, then only possible singularity is $(0,0,-1/g^z(0),0)$. And then we do a change of variables, it's not hard to see that it is at worst of cA_1 type.
  - If $g^z(0) = 0$, then it is smooth.

  On $U_3 \cap E$, we have $x^2 + yt + t(g_0'(0)t + g^z(0)) = 0$. The only possible singularity is $(0, -g^z(0), 0, 0)$ and it is at worst of cA_1 type.

- $\ell \geq 3$. So we have $\mu \leq 3$ from the assumption $\mu^b \leq 3$. Thus, at least one of t, t^2, z, zt, t^3 is in φ.

  - If $t \in \varphi$, then it is smooth.
  - From the form of φ, we have $z \notin \varphi$.
  - If $zt \in \varphi$, then blow up with weight $(1,1,1,1)$. It is smooth on U_1, U_3 and U_4. On $U_2 \cap E$, the only possible singularity is the origin. And it is at worst of cA_1 type.

– If $zt \notin \varphi$ and $t^2 \in \varphi$, then we take $\mathrm{wBl}_v : Y \to X$ with weight $v = (1,1,1,1)$. It is smooth on $U_1 \cup U_4$. On U_2 and U_3, there are isolated singularities which are at worst of cA_n-like singularities.[8]

– If $t^3 \in \varphi$ and $t^2, zt \notin \varphi$, then we take $\mathrm{wBl}_v : Y \to X$ with weight $v = (2,1,1,1)$. Write

$$\sum_{i+j=3} c_{ij} z^i t^j = \prod (\alpha_i t - \beta_i z)^{r_i}$$

where $\sum r_i = 3$. By $y \to y + \beta t$ for some suitable β, we may assume that $r_i \leq 2$ for all i.

On $U_1 \cap E$, we have $y^2 z + \prod(\alpha_i t - \beta_i z)^{r_i} + cyzt = 0$. For $z = t = 0$, we have $y = 0$ and $\frac{\partial \widetilde{\varphi}}{\partial x}|_{y=z=t=0} = 1$. Then the origin is a terminal quotient singularity. Other singularities are on $U_3 \cup U_4$.

On $U_2 \cap E$, we have $z + \prod(\alpha_i t - \beta_i z)^{r_i} + czt = 0$. For $t = 0$, then it is smooth.

On $U_4 \cap E$, we have $y^2 z + \prod(\alpha_i - \beta_i z)^{r_i} + cyz = 0$. Since $t^3 \in \varphi$, we have $z \neq 0$. That is, the singularities are on U_3.

On $U_3 \cap E$, we have $y^2 + \prod(\alpha_i t - \beta_i)^{r_i} + cyt = 0$. We could show that it has only finitely many singularities. Each is of the form

$$z^2 + y^2(ay + b) + cyz + dxt + x^2 t + yt\widetilde{g} = 0.$$

where a and d are not zero. Taking $\mathrm{wBl}_v : Z \to Y$ with weight $v = (1,1,1,1)$, we could show that the singularities of Z are at worst of cA_1 type.

- $\ell = 2$. I divide it into two cases. As above, we write

$$f_3 = \sum_{i+j=3} c_{ij} z^i t^j = \prod (\alpha_i t - \beta_i z)^{r_i}$$

where $\sum r_i = 3$.

– $f_3 \neq 0$. We may assume that f_3 is not cubic.

[8]Here, the form is difference from the origin cA_n type singularity. But the method for cA_n can be applied to this case. Moreover, it has at worst of cA_n type singularities after taking a weighted blowup.

We consider $\mathrm{wBl}_v : Y \to X$ with weight $v = (2, 1, 1, 1)$. Note that the origin on U_1 is a terminal quotient singularity.

On $U_2 \cap E$, we have $z + t^2 + f_3(z, t) + czt = 0$. The origin is smooth. Other singularities are on $U_3 \cup U_4$.

On $U_4 \cap E$, we have $y^2z + y + f_3(z, 1) + cyz = 0$. Then we have $2yz + 1 + cz = 0$. This tells us that the singularities are on U_3.

On $U_3 \cap E$, we have $y^2 + yt^2 + f_3(1, t) + cyt = 0$. We could show that it has only finitely many singularities. Each is of the form

$$dxz + x^2z + y^2 + yt^2 + t^2(at + b) + c'yt + yt\tilde{g} = 0.$$

where d is not zero. Taking $\mathrm{wBl}_v : Z \to Y$ with weight $v = (1, 1, 1, 1)$, we could show that the singularities of Z are at worst of cA_1 type.

– $f_3 = 0$. Note that $xt^2 \notin \varphi$. Write $f_4 = \sum_{i+j=4} z^i t^j$. We consider $\mathrm{wBl}_v : Y \to X$ with weight $v = (2, 2, 1, 1)$.

  On $U_1 \cap E$, we have $1 + yt^2 + f_4(z, t) = 0$. Then the only singularity is the origin which is a terminal quotient singularity.

  On $U_4 \cap E$, we have $x^2 + y + f_4(z, 1) = 0$. Then it is smooth.

  On $U_3 \cap E$, we have $x^2 + yt^2 + f_4(1, t) = 0$. The only possible singularity is the origin and it is of the form $x^2 + y^2z + yt^2 + t\tilde{g} = 0$. By induction on $\tau^{\#}$, we are done.

  On $U_2 \cap E$, we have $x^2 + t^2 + f_4(z, t) = 0$. The only possible singularity is the origin and it is of the form $(x^2 + yz + t^2 + f_4(z, t) + yt\tilde{g} = 0) \subset \mathbb{A}^4/\frac{1}{2}(0, 1, 1, 1)$. Taking $\mathrm{wBl}_v : Z \to Y$ with weight $v = (1, 1, 1, 1)$, we get only terminal quotient singularities.

4.5.3 Inductive step

Proposition 4.5.3. *Given any* isolated *$cAx/2$ type singularities P with $p > 0$ and $\tau_0 := \tau(P) \geq 4$. Suppose that cD type and $cAx/2$ type isolated singularities with $\mu^b < \tau_0$ and*

$\tau < \tau_0$ *are terminal. Then* P *is terminal.*

Proof. Write $f_{\tau_0} = \sum_{i+j=\tau_0} c_{ij} z^i t^j = \prod(\alpha_j z + \beta_j t)^{r_j}$. I divide it into two cases.

- If $f_{\tau_0} = t^{\tau_0}$. Without loss of generality, we study the case that $\tau_0/2$ is odd. Doing a change of variables, we have $x^2 + 2xt^{\tau_0/2} + y^2 + f_{\geq \tau_0+2} = 0$. We consider $\mathrm{wBl}_v : Y \to X$ with weight $v = \frac{1}{2}(\tau_0/2+2, \tau_0/2+1, 1, 1)$.

  On $U_4 \cap E$, we have $2x + y^2 + f_{\tau_0+2}(z, 1) = 0$. Then Y is smooth.

  On $U_3 \cap E$, we have $2xt^{\tau_0/2} + y^2 + f_{\tau_0+2}(1, t) = 0$. Then $y = 0, t = 0$. And $x = 0$ from $\frac{\partial \widetilde{\varphi}}{\partial z}$. Thus, Q_3 is the only singularity which is of the form

  $$y^2 z + 2yt^{\tau_0/2} + x^2 + z^{(n+1-\tau_0)/2}\widetilde{h}(z) + f_{\tau_0+2}(1, t) + t\widetilde{g} = 0.$$

  It is of cD type with $\mu^b \leq 2(\tau_0/2) - 2 < \tau_0$.

  On $U_2 \cap E$, we have $1 + 2xt^{\tau_0/2} + f_{\tau_0+2} = 0$. We may consider $z = t = 0$, then it is smooth except for Q_2 which is a terminal quotient singularity.

  On $U_1 \cap E$, we have $y^2 + 2t^{\tau_0/2} + f_{\tau_0+2} = 0$. Note that we only need to consider the origin, which is a terminal quotient singularity.

- For f_{τ_0} is not t^{τ_0}.

  We consider $\mathrm{wBl}_v : Y \to X$ with weight $v = \frac{1}{2}(\tau_0/2, \tau_0/2+1, 1, 1)$.

  On $U_1 \cap E$, we have $1 + f_{\tau_0}(z, t) = 0$.

  Clearly, the only possible singularities are on U_3 and U_4. (In fact, it is smooth since the singularities on $U_3 \cup U_4$ have $x = 0$.)

  On $U_2 \cap E$, we have $x^2 + f_{\tau_0}(z, t) = 0$. We have $x = 0$. And the origin is a terminal quotient singularity. Others singularities are on $U_3 \cup U_4$.

  On $U_3 \cap E$, we have $x^2 + f_{\tau_0}(1, t) = 0$. Clearly, we have finitely many singularities and $x = 0$. We make f_{τ_0+2} have t^{τ_0+2} in advance. I divide it into two cases.

  - If $y \neq 0$, then the singularities are of the form $x^2 + z^r\xi(z) + cyz + t\widetilde{g} = 0$ where

$r \geq 2$, $y \in \widetilde{g}$ and $x \notin \widetilde{g}$. Taking $\mathrm{wBl}_v : Z \to Y$ with weight $v = (1,1,1,1)$, we get that Z is smooth on[9] $V_1 \cup V_2$. and has cA_1 type singularities on $V_3 \cup V_4$.

- If $y = 0$, then the only possible singularity is the origin since $t^{\tau_0+2} \in f_{\tau_0+2}$. And it is of the form $x^2 + y^2 z + \widetilde{h}(z) + t^r \xi(t) + t\widetilde{g} = 0$ where $r \geq 2$ and $\xi(0) \neq 0$. Note that it is of cD type with $\mu^b \leq r < \tau_0$ since f_{τ_0} is not t^{τ_0}.

On $U_4 \cap E$, we have $x^2 + f_{\tau_0}(z,1) = 0$. The origin is smooth since $t^{\tau_0+2} \in f_{\tau_0+2}$. The remaining is $y \neq 0$ and $x = z = 0$. It is of the form $x^2 + z^r \xi(z) + t\widetilde{g} = 0$ where $y \in \widetilde{g}$ and $x \notin \widetilde{g}$. Taking $\mathrm{wBl}_v : Z \to Y$ with weight $v = (1,1,1,1)$, we get that Z is smooth on $V_1 \cup V_4$ and has at worst of cA_1 type singularities on $V_2 \cup V_3$.

Proposition 4.5.4. *Given any* isolated *cD type singularities P with $p > 0$ and $\mu_0 := \mu^b \geq 4$. Suppose that cD type and $cAx/2$ type isolated singularities with $\mu^b < \mu_0$ and $\tau \leq \mu_0$ are terminal. Then P is terminal.*

Proof. Define $\mu' = \lfloor \frac{\mu_0}{2} \rfloor$. We consider $\mathrm{wBl}_v : Y \to X$ with weight $v = (\mu', \mu' - 1, 2, 1)$. I divide it into several cases.

- $\ell = \infty$ and $2\mu' = \mu_0$.

  Y is smooth except for Q_1, Q_2 and Q_4. For Q_1 and Q_2, they are terminal quotient singularities. For Q_3, it is of the form

$$(x^2 + y^2 + z^{n-1-\mu_0} h(z) + t^{\mu_0} + t\widetilde{g} = 0) \subset \mathbb{A}^4 / \frac{1}{2}(1,0,1,1).$$

  Note that it is of $cAx/2$ type singularity with $\tau \leq \mu_0$.

- $\ell = \infty$ and $2\mu' + 1 = \mu_0$.

  This case is similar to above. There is one difference. For Q_3, the form is

$$(x^2 + y^2 + z^{n-\mu_0} h(z) + zt^{\mu} + t\widetilde{g} = 0) \subset \mathbb{A}^4 / \frac{1}{2}(1,0,1,1).$$

[9] From now on, $\{V_i\}_{i=1}^4$ is always an open affine covering of Z.

It is of $cAx/2$ type singularity with $\tau \leq \mu_0 + 1$. If $\tau \leq \mu_0$, then we are done. If $\tau = \mu_0 + 1$, then we still apply the previous proposition. Note that there is only one possibility to come back to cD type singularities with $\widetilde{\mu} = \mu_0$. And in this case, it has smaller n. Thus, by induction on n, we are done.

- $\lambda \neq 0$ and $2\ell - 2 = \mu := \mu(P)$.

  On $U_2 \cap E$, we have $x^2 + z + t^\ell + t^\mu = 0$. Then we get that the only singularity is Q_2, which is a terminal quotient singularity.

  On $U_1 \cap E$, we have $1 + y^2 z + yt^\ell + t^\mu = 0$. Now we only need to consider $y = 0$. Then $t = 0$ from $\frac{\partial \widetilde{\varphi}}{\partial y}|_{x=0}$. So it is smooth except for Q_1 which is a terminal quotient singularity.

  On $U_3 \cap E$, we have $x^2 + y^2 + yt^\ell + t^\mu = 0$. The only possible singularity is Q_3. It is of the form $(x^2 + y^2 + z^{2n-2-\mu} h(z) + t\widetilde{g} = 0) \subset \mathbb{A}^4/\frac{1}{2}(1, 0, 1, 1)$ which is of $cAx/2$ type singularity with $\tau \leq \mu = \mu_0$.

  It is clear that U_4 is smooth.

- $\lambda \neq 0$ and $2\ell - 2 > \mu$.

  It is the same as the case $\ell = \infty$.

- $\lambda \neq 0$ and $2\ell - 2 < \mu$.

  In this case, we consider $\mathrm{wBl}_v : Y \to X$ with weight $v = (\ell, \ell, 1, 1)$ instead.

  On $U_4 \cap E$, we have $x^2 + y + \delta^\mu_{2\ell} = 0$, then U_4 is smooth.

  On $U_1 \cap E$, we have $1 + yt^\ell + \delta^\mu_{2\ell} t^\mu = 0$. Then U_1 is smooth except for Q_1 which is a terminal quotient singularity.

  On $U_2 \cap E$, we have $x^2 + t^\ell + \delta^\mu_{2\ell} t^\mu = 0$. Now we only need to consider $x = t = 0$. From $\frac{\partial \widetilde{\varphi}}{\partial y}|_{y=0}$, we get that $z = 0$. Then the only possible singularity is Q_2. It is of the form $(xy + z^2 + t\widetilde{g} = 0) \subset \mathbb{A}^4/\frac{1}{\ell}(1, -1, \ell, 1)$. Use the method for cA/r, we could resolve this singularity and get that it is a terminal singularity.

On U_3, we only need to consider the origin. It is of the form

$$x^2 + y^2 z + z^{n-1-2\ell} h(z) + y t^\ell + t\widetilde{g} = 0.$$

Then by induction on n, we are done.

Bibliography

[Abh56] S. S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic* $p \neq 0$, Ann. of Math. (2) **63** (1956), 491–526.

[Abh66] ———, *Resolution of singularities of embedded algebraic surfaces*, Pure and Applied Mathematics, Vol. 24, Academic Press, New York-London, 1966.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. M^{c}Kernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[Che13] J. A. Chen, *Explicit resolution of three dimensional terminal singularities*, arXiv preprint arXiv:1310.6445 (2013).

[CP08] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*, J. Algebra **320** (2008), no. 3, 1051–1082.

[CP09] ———, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra **321** (2009), no. 7, 1836–1976.

[CTX13] P. Cascini, H. Tanaka, and C. Xu, *On base point freeness in positive characteristic*, arXiv preprint arXiv:1305.3502 (2013).

[FJ74] G. Frey and M. Jarden, *Approximation theory and the rank of abelian varieties over large algebraic fields*, Proc. London Math. Soc. (3) **28** (1974), 112–128.

[Fuj84] T. Fujita, *Fractionally logarithmic canonical rings of algebraic surfaces*, Journal of the Faculty of Science, The University of Tokyo, Sect. 1 A, Mathematics **30** (1984), no. 3, 685–696.

[Ful93] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326.

[HW02] N. Hara and K. Watanabe, *F-regular and F-pure rings vs. log terminal and log canonical singularities*, J. Algebraic Geom. **11** (2002), no. 2, 363–392.

[HX13] C. D. Hacon and C. Xu, *On the three dimensional minimal model program in positive characteristic*, arXiv preprint arXiv:1302.0298 (2013).

[Kee99] S. Keel, *Basepoint freeness for nef and big line bundles in positive characteristic*, Ann. of Math. (2) **149** (1999), no. 1, 253–286.

[KM08] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, vol. 134, Cambridge University Press, 2008.

[KMM87] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem*, Advanced Studies in Pure Math. 10(1987) **10** (1987), 283–360.

[KSB88] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338.

[Mat02] K. Matsuki, *Introduction to the Mori program*, Springer, 2002.

[Mor85] S. Mori, *On* 3*-dimensional terminal singularities*, Nagoya Math. J. **98** (1985), 43–66.

[Ray78] M. Raynaud, *Contre-exemple au "vanishing theorem" en caractéristique* $p > 0$, C. P. Ramanujam—a tribute, Tata Inst. Fund. Res. Studies in Math., vol. 8, Springer, Berlin-New York, 1978, pp. 273–278.

[Rei83] M. Reid, *Minimal models of canonical* 3*-folds*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 131–180.

[Rei87] ———, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414.

[Sho86] V. V. Shokurov, *The nonvanishing theorem*, Mathematics of the USSR-Izvestiya **26** (1986), no. 3, 591–604.

[Tan12] H. Tanaka, *Minimal models and abundance for positive characteristic log surfaces*, arXiv preprint arXiv:1201.5699 (2012).