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垂向拉起沉沒長方形塊之解析研究

Analytical solutions to the vertical lifting of a submerged rectangular block

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Analytical solutions to the vertical lifting of a submerged rectangular block

本論文係張雲君（學號 r01521303）在國立臺灣大學土木工程學系碩士班完成之碩士學位論文，於民國 103 年 7 月 21 日承下列考試委員審查通過及口試及格，特此證明

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ABSTRACT

Lift-up problem is a process during which an object immersed in fluid is initially extricated from a bed (Fig. 2-1). In ocean and offshore engineering, it has many applications such as removing hydraulic structures, salvaging sunken ships, and operating submarines. Based on field experience (Liu (1969)), the lifting force is considerably large and the operation time is extremely long. It is a process that slowly increases the gap between the object and the bed until a turning point when the object is abruptly unfastened. This is called breakout phenomenon. Because of it and the lifting force, it is helpful to ocean and offshore engineering if the mechanics of the problem can be understood.

This study applies Stokes flow to the gap, investigating a lift-up problem with a rigid impermeable bed. Also, this study applies Stokes flow to the gap and Song and Huang's (2000) laminar poroelasticity theory to the porous medium, analyzing problems with a rigid porous bed and a hard poroelastic bed, respectively. Stokes flow can react to the horizontal and vertical velocities, Song and Huang's (2000) laminar poroelasticity theory can respond to the viscous effect of pore flow, and the complete

boundary conditions can react to the continuity of velocity and stress.

The first stage of this study provides the exact solution to the two-dimensional lift-up problem with a rigid impermeable bed. The exact solution reveals the tiny error of the pressure in adhesion approximation (Acheson (1990)), and verifies that the tiny error does not influence the kinematics of the flow and the dynamic force acting on the object.

The second stage of this study proposes a more general solution to the two-dimensional lift-up problem with a rigid porous bed. The solution has been verified by Mei's (1985) experiments, and reveals many mechanics of the problem. The dynamic force acting on the object and the breakout phenomenon are displayed, and other mechanics are demonstrated. In fact, they are the mechanics that cannot be presented by Mei's (1985) approach, and suggest that adhesion approximation (Acheson (1990)), Darcy's law, and Beavers and Joseph's (1967) partial-slip flow might not be suitable to problems with porous beds.

This study finally offers the solution to the two-dimensional lift-up problem with a hard poroelastic bed. The fluid and solid parts in the porous medium can be decoupled. The mechanics of the fluid is exactly the same as those in the problem with a rigid porous bed. Deformation lines, deformations and effective stresses of the solid in a hard poroelastic bed are revealed. The porous medium can be influenced to the depth of L, half the length of the object, and the solid effective stresses are only influenced by the pressure distribution in the porous medium.

Keyword: Lift-up problem, Breakout phenomenon, Stokes flow, Brinkman equation, Porous medium, Laminar Poroelasticity

中文摘要

拉起問題為將沉沒於水中且靜置於底床之物體，由底床垂直拉起之問題。此問題於海岸或水利工程中有許多應用，如水工構造物之移除、海底沈船打撈、潛水艇操作等。實際經驗中，拉起之力量甚大，且耗時良久，直到某個轉捩點此力量才會突然鬆懈，而能將物體順利拉起，此特徵也因此被稱為突破現象。若能認識突破現象之物理機制，對於拉起物體之過程必然有益。

本研究利用史托氏流(Stokes flow) 描述縫隙中之流體，研究由不透水、不變形底床拉起物體之問題。此外，本研究還利用史托氏流描述縫隙中之流體，利用 Song and Huang (2000) 提出之層流多孔彈性理論於多孔介質，分別研究由透水、不變形底床拉起物體之問題，以及透水、變形底床拉起物體之問題。史托氏流完整呈現水平及垂直方向之動量方程式，Song and Huang 之層流多孔彈性理論能反應出孔隙流之黏滯性，且其理論之邊界條件能保有流體之速度及應力連續。

本研究第一階段求得二維、於不透水底床拉起物體之精確解。此解揭示了黏附近似之壓力誤差，也驗證流場之流線與速度不受其大刀闊斧之近似所影響，是不錯的近似方法。

本研究第二階段求得二維、於透水但不變形底床拉起物體之解，並利用 Mei (1985)研究中之實驗驗證其合理性。此解呈現出許多於透水但不變形底床拉起物體之物理機制，首要是拉起力量之改變以及突破現象，即研究拉起問題之初衷，此外還有如流體之流線、壓力、剪力、渦度、黏滯性應等，完整呈現出此問題之物理機制。此外，這些物理機制顯示出 Mei 所採用之黏附近似、達西定律、以及

滑移邊界條件(Beavers and Joseph (1967))很可能不適用於有孔隙之拉起問題。

本研究最終求得二維、於透水可變形底床拉起物體之解。求解過程中，多孔介質之流體與固體之動量方程可拆開分別求解，孔隙流體之解和由透水但不變形底床拉起物體之解相同，也因此有同樣的建議。孔隙固體之解可呈現孔隙固體之變形函數、變形量、有效應力等物理機制，揭示拉起過程中孔隙固體受影響之深度可達拉起物體長度之半，以及孔隙固體有效應力只和孔隙壓力之分布有關。

關鍵字：垂直拉起、突破現象、史托氏流、布林克曼流、多孔介質、層流多孔彈性理論

NOTATION

$\vec{u}$	fluid velocity vector
u	horizontal fluid velocity
v	vertical fluid velocity
p	fluid dynamic pressure
F	fluid dynamic force acting on the object
μ	fluid viscosity
ψ	fluid stream function
V	object velocity
h	gap width
L	half the gap length
U	horizontal velocity scale
x, y	physical coordinates
x^*, y^*	dimensionless coordinates
u^*	dimensionless horizontal fluid velocity
v^*	dimensionless vertical fluid velocity

p^* dimensionless fluid dynamic pressure

t^* dimensionless time

Re Reynolds number

ε ratio of the gap width to half the gap length

p_0 external pressure

(1) in the gap

(2) in the porous medium

$\vec{u}^{(1)}$ fluid velocity vectors in the gap

$u^{(1)}$ horizontal fluid velocity in the gap

$v^{(1)}$ vertical fluid velocity in the gap

$p^{(1)}$ fluid dynamic pressure in the gap

$\psi^{(1)}$ fluid stream function in the gap

$\vec{u}^{(2)}$ fluid velocity vector in the porous medium

$u^{(2)}$ horizontal fluid velocity in the porous medium

$v^{(2)}$ vertical fluid velocity in the porous medium

$p^{(2)}$ dynamics dynamic pressure in the porous medium

n_0 porosity

k_p specific permeability

$\psi^{(2)}$ fluid stream function in the porous medium

D_{50}	medium grain size
ρ	fluid density
ρ_s	solid density
$\vec{d}^{(2)}$	solid deformation vector
$d_1^{(2)}$	horizontal solid deformation
$d_2^{(2)}$	vertical solid deformation
$\dot{\vec{d}}^{(2)}$	solid velocity vector
$\dot{d}_1^{(2)}$	horizontal solid velocity
$\dot{d}_2^{(2)}$	vertical solid velocity
$\underline{\underline{\sigma}}^{(2)}$	fluid stress tensor in the porous medium
$\underline{\underline{\sigma_s}}^{(2)}$	solid stress tensor in the porous medium
$F(k)$	frequency correction factor
$\underline{\underline{\tau}}^{(2)}$	fluid shear stress tensor in the porous medium
$\underline{\underline{\tau_s}}^{(2)}$	solid effective stress tensor in the porous medium
$\underline{\underline{I}}$	identity matrix
μ'	second fluid viscosity
G	Lamb’s constant
λ	Lamb’s constant
$\psi_s^{(2)}$	solid deformation function in the porous medium

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$h/L = 10^{-1.7}$.

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Chapter 1

INTRODUCTION

Lift-up problem is a process during which an object immersed in fluid is initially extricated from a bed (Fig. 2-1). In ocean and offshore engineering, it has many applications such as removing hydraulic structures, salvaging sunken ships, and operating submarines. Based on field experience (Liu (1969)), the lifting force is considerably large and the operation time is extremely long. It is a process that slowly increases the gap between the object and the bed until a turning point when the object is abruptly unfastened. This is called breakout phenomenon. Because of it and the lifting force, it is helpful to ocean and offshore engineering if the mechanics of the problem can be understood.

Adhesion approximation, under the basis of dimensional analysis, has been extensively applid to lift-up problems. It significantly simplifies the Navier-Stokes equations so that the problems can be easily analyzed. Landau and Lifshitz (1959) first applied it to a disc pushed toward a rigid impermeable plane, whereas Acheson (1990) applied it to a disc pulled away from the plane. Batchelor (1967) applied it to two spheres separated from each other.

Mei, Yeung, and Liu (1985) (Mei) proposed an approach that treats the seabed as a rigid porous bed, and did experiments. In Mei's approach, the flow in the gap is described by adhesion approximation, the pore flow obeys Darcy's law, and the slip-flow boundary condition proposed by Beavers and Joseph (1967) is applied to the permeable interface.

Biot (1941, 1962) developed a poroelasticity theory for saturated porous medium in which the soil solid is described by Hooke's law, and the pore flow by Darcy's law. Foda (1982) employed Biot's (1941, 1962) theory to investigate the breakout phenomenon from a poroelastic seabed. In Foda's model, the flow in the gap is described by adhesion approximation, the porous medium obeys Biot's (1941, 1962) theory.

However, although horizontal velocity dominates the gap flow, there may be a small region in which the vertical velocity is greater than horizontal velocity (e.g. the center of the gap). Adhesion approximation in the vertical momentum equation might thus lead to some error to the solution.

Mei's (1985) approach might not be suitable to the problem for three reasons. First, when the object is initially lifted up from the bed, the width of the gap may be less than the dimension of the interstices of the porous medium. The flow in the gap may mainly come from the pore of bed rather than from the gap periphery. Hence the flow in the gap might not be described by adhesion approximation.

Second, when the width of the gap is greater than the dimension of the interstices of the porous medium, the flow in the gap mainly comes from the gap periphery. The horizontal velocity dominates the gap flow, so viscous effect becomes essential. Both of Darcy’s law and Beavers and Joseph’s (1967) slip–flow boundary condition, however, could not respond to the viscous effect in the porous medium. Actually, Beavers and Joseph (1967) argued that there is a boundary layer below the bed interface in which the pore flow does not satisfy Darcy’s law; the slip-flow boundary condition is just an empirical boundary condition to connect the upper and the pore flow.

Third, Beavers and Joseph (1967) proposed the boundary condition under the condition that only horizontal velocity exists in the upper flow. In this problem, however, vertical velocity also exists in the gap; the situation is so different that the boundary condition might not be appropriate.

Foda’s (1982) work might not be suitable to the problem for three reasons. The first and second reasons to Mei (1985) are also valid here. Third, one of the boundary conditions might not be appropriate. Foda (1982) applied the continuity of fluid pressure to the boundary condition. However, Biot (1941, 1962) suggested the boundary conditions in the case where both the upper and pore flow are inviscid. Song and Huang (2000) suggested that if the viscous effect of fluid is essential, the continuity of stress should be applied; and lift-up problem is indeed such a case.

After the famous Biot's theory (1941, 1962), poroelastic theory has been further extended. On a basis of the first law of thermodynamics, Song and Huang (2000) proposed a more general theory for laminar poroelastic media flow that takes viscosity of fluid into account. In their theory, the governing equation becomes the Biot equation (1941, 1962) if fluid viscous effect is dropped and mass coupling effect for flow is considered; it reduces to the Brinkman equation (Brinkman (1947, 1948, and 1949)) if rigid solid skeleton is assumed and inertial term for fluid is neglected; and it further degenerates to Darcy's law if fluid viscosity is also neglected. Also, referring to Deresiewicz and Skalak (1963), Song and Huang (2000) proposed a complete set of boundary conditions (continuity of velocity and stress) for adjacent porous interfaces.

The above backgrounds motivate this master thesis. In this study, Chapter 2 examines the lift-up problem with a rigid impermeable bed. Chapter 3 investigates the problem with a rigid porous bed. Chapter 4 discovers the problem with a hard poroelastic bed. In each chapter, more general solutions to the two-dimensional lift-up problems will be provided, and the discrepancy due to the different analysis approaches will be examined. Finally, Chapter 5 concludes the comparison between the former and present researches in each topic, and summarizes the contributions by our approach.

Chapter 2

A TWO-DIMENSIONAL LIFT-UP PROBLEM WITH A RIGID IMPERMEABLE BED

2.1 Introduction

Lift-up problem is a process during which an object immersed in fluid is initially extricated from a bed (Fig. 2-1). In ocean and offshore engineering, it has many applications such as removing hydraulic structures, salvaging sunken ships, and operating submarines. Based on field experience (Liu (1969)), the lifting force is considerably large and the operation time is extremely long. It is a process that slowly increases the gap between the object and the bed until a turning point when the object is abruptly unfastened. This is called breakout phenomenon. Because of it and the lifting force, it is helpful to ocean and offshore engineering if the mechanics of the problem can be understood.

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However, although horizontal velocity dominates the gap flow, there may be a small region in which the vertical velocity is greater than horizontal velocity (e.g. the center of the gap). The approximation in the y-momentum might thus lead to some error to the solution. On the other hand, when an object is initially lifted up, the process is very slow. The flow in the gap may be described as Stokes flow, a governing equation with more complete x- and y-momentum equations.

This study analytically reinvestigates the lift-up problem with a rigid impermeable bed. Instead of adhesion approximation, Stokes flow is applied to the gap. A more general solution to the two-dimensional lift-up problem with a rigid impermeable bed will be provided, and the discrepancy due to the different analysis approaches will be examined.

2.2 The exact solution

Assuming the bottom of a rigid object and the top of a rigid impermeable bed to be flat, we study a problem in which the object is uniformly lifted up from the bed without tilt (Fig 2-1).

2.2.1 Governing equations

In the gap, the fluid is assumed to be incompressible. Since lift-up problem is a very slow process, the flow in the gap can be treated as Stokes flow:

$$\nabla \cdot \vec{u} = 0, \tag{2.2.1}$$

$$0 = -\nabla p + \mu \Delta \vec{u}, \tag{2.2.2}$$

where $\vec{u} = (u, v)$ is the fluid velocity, p the fluid dynamic pressure, and μ the fluid viscosity. Note that the fluid pressure can be separated into dynamic and static pressure because the fluid surface is assumed to be flat and the fluid density is constant. The dynamic pressure, which causes the extremely large suction force on the object, is our concern, while the static pressure, which yields buoyancy on the object and is much less than the suction force, need not to be discussed here. Second, we focus on the object initially lifted up, so $h \ll L$. Instead of adhesion approximation, we apply Stokes flow. Third, since the inertial term has been dropped, this problem becomes a boundary value problem. By (2.2.1), we introduce a stream function ψ such that

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}. \tag{2.2.3}$$

Taking the curl of Eq. (2.2.2) and using Eq. (2.2.3) yield

$$\Delta \Delta \psi = 0. \tag{2.2.4}$$

Thus, the governing equation becomes biharmonic equation, a fourth-order partial differential equation.

2.2.2 Boundary conditions

Because the problem is symmetric about the y-axis, we give the symmetric conditions

$$u=0\,,\quad v_x=0\,,\qquad \text{on}\ \ x=0\,,\ \ 0\le y\le h\,. \tag{2.2.5}$$

At the top and the bottom of the gap, no-slip condition gives

$$u=0\,,\quad v=0\,,\qquad \text{on}\ \ y=0\,,\ \ 0\le x\le L\,, \tag{2.2.6}$$

$$u=0\,,\quad v=V\,,\qquad \text{on}\ \ y=h\,,\ \ 0\le x\le L\,. \tag{2.2.7}$$

Because the gap is very small, at the entrance of the gap, we may assume the flow is parallel. Hence

$$u_x=0\,,\ \ v=0\,,\qquad \text{on}\ \ x=L\,,\ \ 0\le y\le h\,. \tag{2.2.8}$$

Since the surrounding fluid has no boundary, the hole has relatively small length scale. We therefore assume the fluid outside the gap has a negligible effect and give the external pressure

$$p=0,\qquad \text{on}\ \ x=L\,,\ \ 0\le y\le h\,. \tag{2.2.9}$$

Note that (2.2.8), (2.2.9) are quite common as long as $h \ll L$; we can ignore the "end effect" at the far filed. (e.g. Karman's rotating disc in Frank White (2006)).

2.2.3 Solutions

Employing the method of separation of variables (refer to Selvadurai (2000) and Wang (2003)), we find the solutions

$$\psi(x,y)=\sum_{n=0}^{\infty}(A_n e^{\lambda_n y}+B_n e^{-\lambda_n y}+C_n y e^{\lambda_n y}+D_n y e^{-\lambda_n y})\sin\lambda_n x\,, \tag{2.2.10}$$

$$p=\mu\sum_{n=0}^{\infty}\left(2\lambda_n C_n e^{\lambda_n y}+2\lambda_n D_n e^{-\lambda_n y}\right)\cos\lambda_n x\,,\quad \lambda_n=(n+\frac{1}{2})\frac{\pi}{L}\,, \tag{2.2.11}$$

The dynamic force applied on the bottom of the object is

$$F=2\int_0^L p(x,h)dx=4\mu\sum_{0}^{\infty}\left(C_n e^{\lambda_n h}+D_n e^{-\lambda_n h}\right)\sin\lambda_n L\,. \tag{2.2.12}$$

The coefficients are presented in Appendix A1. Besides the solutions, we also seek the terms that will be used in the later discussion.

$$\frac{\partial p}{\partial y}=\mu\sum_{n=0}^{\infty}\left(2\lambda_n^2 C_n e^{\lambda_n y}-2\lambda_n^2 D_n e^{-\lambda_n y}\right)\cos\lambda_n x\,, \tag{2.2.13}$$

$$\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})=\mu\sum_{n=0}^{\infty}\left(2\lambda_n^2 C_n e^{\lambda_n y}-2\lambda_n^2 D_n e^{-\lambda_n y}\right)\cos\lambda_n x\,. \tag{2.2.14}$$

Note that despite the discontinuity of $v^{(1)}$ in (2.2.7) and (2.2.8), the infinite series solution indeed converges because of the characteristics of Fourier series.

2.3 Adhesion approximation

Adhesion approximation has been extensively used. It significantly simplifies the Navier-Stokes equations so that many problems can be easily analyzed; notably, Mei (1985) and Foda (1982) applied it to the lifting of a large object from a seabed.

2.3.1 Order of magnitude analysis and leading order equations

Referring to Panton (2005) and Acheson (1990), this section revisits the order of magnitude analysis that derives the leading order equations in the two-dimensional problem. Denote the velocity scale for the y-direction by V, the width scale by h, and the length scale L. Since both terms in the continuity Eq. (2.2.1) have the same order, the velocity scale for the x-direction is

$$U = V\frac{L}{h}. \tag{2.3.1}$$

Define non-dimensional variables as

$$x^* = \frac{x}{L},\ y^* = \frac{y}{h},\ u^* = \frac{u}{V\frac{L}{h}},\ v^* = \frac{v}{V},$$

$$p^* = \frac{p-p_0}{\frac{\mu V}{h}\left(\frac{L}{h}\right)^2},\ t^* = \frac{t}{h/V},\ \mathrm{Re} = \frac{\rho V h}{\mu},\ \varepsilon = \frac{h}{L}, \tag{2.3.2}$$

where p_0 is the external pressure. Substituting them into the x-momentum equation produces

$$\mathrm{Re}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \varepsilon^2\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{2.3.3}$$

and into the y-momentum equation produces

$$\mathrm{Re}\,\varepsilon^2\left(\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} + \varepsilon^4\frac{\partial^2 v^*}{\partial x^{*2}} + \varepsilon^2\frac{\partial^2 v^*}{\partial y^{*2}}. \tag{2.3.4}$$

Under the conditions $\mathrm{Re} \ll 1$ and $\varepsilon \ll 1$, the leading order equations of motion reduce to

$$0=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}, \tag{2.3.5}$$

$$0=\frac{\partial p}{\partial y}. \tag{2.3.6}$$

2.3.2 Boundary conditions

The boundary conditions for adhesion approximation are

$$u=0, \quad v=0, \qquad \text{on } y=0,\ 0\le x\le L \tag{2.3.7}$$

$$u=0, \quad v=V, \qquad \text{on } y=h,\ 0\le x\le L \tag{2.3.8}$$

$$p=0, \qquad \text{on } x=L,\ 0\le y\le h \tag{2.3.9}$$

2.3.3 Solutions to adhesion approximation

Referring to Landau and Lifshitz (1959) and Acheson (1990), this section solves the adhesion approximation problem. Eq. (2.3.6) indicates that p is only a function of x. Then integrating Eq. (2.3.5) twice with respect to y and applying the boundary conditions $u=0$ at $y=0$ and at $y=h$ gives

$$u=\frac{1}{2\mu}\frac{\partial p}{\partial x}y(y-h). \tag{2.3.10}$$

Substituting (2.3.10) for u, integrating Eq. (2.2.1) with respect to y, and then applying the boundary conditions $v=0$ at $y=0$ and $v=V$ at $y=h$, we obtain

$$\frac{d}{dx}\left(\frac{\partial p}{\partial x}\right)=\frac{12\mu V}{h^3}. \tag{2.3.11}$$

Integrating Eq. (2.3.11) with respect to x and applying the boundary conditions

(2.3.9), we obtain

$$p = \frac{6\mu V}{h^3}\left(x^2 - L^2\right). \tag{2.3.12}$$

Hence the dynamics force exerted on the object is

$$F = 2\int_0^L p(x,h)dx = -\frac{8\mu V L^3}{h^3}. \tag{2.3.13}$$

The negative sign means downward direction. Note that both the exact solution and adhesion approximation show that the suction force goes to infinite when h is approaching to zero.

In addition, taking (2.3.12) into (2.3.10) gives

$$u = \frac{6V}{h^3} xy(y-h). \tag{2.3.14}$$

Taking (2.3.14) into Eq. (2.2.1), integrating Eq. (2.2.1) with respect to y, and applying the boundary condition $v = 0$ at $y = 0$ give

$$v = -\frac{6V}{h^3}\left(\frac{y^3}{3} - \frac{y^2}{2}h\right). \tag{2.3.15}$$

Then the viscous term is

$$\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = -\mu\frac{6V}{h^3}(2y-h). \tag{2.3.16}$$

2.4 Discussion

The main difference between the exact solution and adhesion approximation is the y-momentum equation, (2.2.2) and (2.3.6). Exact solution retains the viscous terms, whereas adhesion approximation ignores it. On the basis of the order of magnitude analysis, adhesion approximation shows that the order of $\partial p/\partial y$ is greater than that of the viscous term, thereby giving the key equation $\partial p/\partial y = 0$ so that the solution can be easily obtained (the first procedure deriving (2.3.10)). However, the viscous term (2.3.16) in adhesion approximation, which should be less than $\partial p/\partial y = 0$, turns out to be greater than it. This result contradicts the order of magnitude analysis, so the analysis may not be true. On the other hand, the $\partial p/\partial y$ (2.2.13) and viscous term (2.2.14) in the exact solution displays their ratio is exactly unit. Stokes flow reveals not only the exact solution but also the correct order.

We assign parameters in Mei's (1985) experiments to examine the discrepancy between the different approaches: $V = 10^{-5} m/s$, $L = 7.86cm$, and $h/L = 10^{-2}$. Fig. 2-2 shows the dimensionless pressure contour at the center of the gap, revealing the main discrepancy between adhesion approximation and the exact solution. $\partial p/\partial y$ is surely zero by the adhesion approximation, whereas the exact one is not. On the other hand, Fig. 2-3 shows the streamline in the entire gap and at the center of it. Fig. 2-4 shows the dimensionless velocity contour at the center of the gap. The kinematics by the adhesion approximation almost coincides with the exact one. In addition, Fig. 2-5 shows the percentage error of the dynamic force acting on the object by adhesion approximation, from $h/L = 10^{-3}$ to $h/L = 10^{-1.7}$. The percentage error is at most 0.025%, displaying that the dynamic force by adhesion approximation is almost the same as the exact one, and it is not influenced by the tiny error of the pressure. In short, the tiny error of the pressure in adhesion approximation does not influence the kinematics and the dynamic force acting on the object.

2.5 Conclusions

This study reinvestigates the lift-up problem with a rigid impermeable bed that has been studied by Landau and Lifshitz (1959) and Acheson (1990). They applied adhesion approximation to the problem, which significantly simplifies the Navier-Stokes equations so that the solution can be easily obtained. However, the inconsistent approximation in the y-momentum may lead to some error to the solution. For this situation, this study applies Stokes flow to the gap. The x- and y-momentum equations are more compete.

This study provides the exact solution to the two-dimensional lift-up problem with a rigid impermeable bed, a boundary value problem with a fourth order partial differential equation. The exact solution reveals the tiny error of the pressure in adhesion approximation, and verifies that the tiny error does not influence the kinematics of the flow and the dynamic force acting on the object.

This chapter investigates the lift-up problem with a rigid impermeable bed. In fact, it will be closer to reality if a rigid but porous bed is considered; therefore, the next chapter studies the lift-up problem with a rigid but porous bed.

Chapter 3

A TWO-DIMENSIONAL LIFT-UP PROBLEM WITH A RIGID POROUS BED

3.1 Introduction

Lift-up problem is a process during which an object immersed in fluid is initially extricated from a bed (Fig. 3-1). In ocean and offshore engineering, it has many applications such as removing hydraulic structures, salvaging sunken ships, and operating submarines. Based on field experience (Liu (1969)), the lifting force is considerably large and the operation time is extremely long. It is a process that slowly increases the gap between the object and the bed until a turning point when the object is abruptly unfastened. This is called breakout phenomenon. Because of it and the lifting force, it is helpful to ocean and offshore engineering if the mechanics of the problem can be understood.

Lift-up problem has been theoretically studied. Biot (1941, 1962) developed a poroelasticity theory for saturated porous medium in which the soil solid is described by Hooke's law, and the pore flow by Darcy's law. Foda (1982) employed Biot's

(1941, 1962) theory to investigate the breakout phenomenon from a poro-elastic seabed. Mei, Yeung, and Liu (1985) (Mei) proposed an approach that treats the seabed as rigid porous bed, reinvestigate the problem of Foda (1982), and did experiments. In Mei's approach, the flow in the gap is described by adhesion approximation (Chap. 2), the pore flow obeys Darcy's law, and the slip-flow boundary condition proposed by Beavers and Joseph (1967) is applied to the permeable interface.

Mei's (1985) approach, however, might not be suitable to the problem for three reasons. First, when the object is initially lifted up from the bed, the width of the gap may be less than the dimension of the interstices of the porous medium. The flow in the gap may mainly come from the pore of bed rather than from the gap periphery. Hence the flow in the gap might not be described by adhesion approximation.

Second, when the width of the gap is greater than the dimension of the interstices of the porous medium, the flow in the gap mainly comes from the gap periphery. The horizontal velocity dominates the gap flow, so viscous effect becomes essential. Both of Darcy's law and Beavers and Joseph's (1967) slip–flow boundary condition, however, could not respond to the viscous effect in the porous medium. Actually, Beavers and Joseph (1967) argued that there is a boundary layer below the bed interface in which the pore flow does not satisfy Darcy's law; the slip-flow boundary condition is just an empirical boundary condition to connect the upper and the pore flow.

Third, Beavers and Joseph (1967) proposed the boundary condition under the condition that only horizontal velocity exists in the upper flow. In this problem, however, vertical velocity also exists in the gap; the situation is so different that the boundary condition might not be appropriate.

After the famous Biot's theory (1941, 1962), poroelastic theory has been further extended. On a basis of the first law of thermodynamics, Song and Huang (2000) proposed a more general theory for laminar poro-elastic media flow that takes viscosity of fluid into account. In their theory, the governing equation becomes the Biot's equation (1941, 1962) if fluid viscous effect is dropped and mass coupling effect for flow is considered; it reduces to the Brinkman equation (Brinkman (1948, 1948, and 1949)) if rigid solid skeleton is assumed and inertial term for fluid is neglected; and it further degenerates to Darcy's law if fluid viscosity is also neglected. Also, referring to Deresiewicz and Skalak (1963), Song and Huang (2000) proposed a complete set of boundary conditions (continuity of velocity and stress) for adjacent porous interfaces.

Song and Huang's (2000) laminar poroelastic theory has been applied to some investigated problems with rigid porous medium. First, Huang and Chiang (1997) applied the theory to reinvestigate laminar channel flow passing over porous bed. In their work, the boundary layer can be clearly shown and the parameter α that is empirically determined for the slip-flow BC in Beavers and Joseph (1967) could be exactly presented. Second, Hsu, Huang, and Hsieh (2004) applied the theory to reinvestigate the low Reynolds number uniform flow past a porous spherical shell. The continuities of tangential velocity and shear stress at the boundaries of the shell, which cannot be satisfied in Jone (1973), are successfully being preserved.

This study analytically reinvestigates the lift-up problem with a rigid porous bed. Instead of adhesion approximation introduced in Chap. 2, Stokes flow, with more complete momentum equations, is employed to the gap. Instead of Darcy's law and the slip-flow BC by Beavers and Joseph (1967), Song and Huang's (2000) laminar poroelastic theory is employed for the porous medium (Fig 3-1). A more general solution to the two-dimensional lift-up problem with a rigid porous bed, verified by Mei's experiments, will be provided. The mechanics of the problem will be revealed. And the discrepancy between Mei's approach and our approach will be examined.

3.2 Mathematical formulation

Assuming the bottom of a rigid object and the top of a rigid porous bed to be flat, we study a problem in which the object is initially lifted up from the bed without tilt (Fig. 3-1).

3.2.1 Governing equations

In the gap, the fluid is assumed to be incompressible. Since lift-up problem is a very slow process, the flow in the gap can be treated as Stokes flow:

$$\nabla \cdot \vec{u}^{(1)} = 0, \tag{3.2.1}$$

$$-\nabla p^{(1)} + \mu \Delta \vec{u}^{(1)} = 0, \tag{3.2.2}$$

where $\vec{u}^{(1)} = (u^{(1)}, v^{(1)})$ is the fluid velocity, $p^{(1)}$ the fluid dynamic pressure, μ the fluid viscosity and superscript (1) denotes the gap region. Note first that the fluid pressure can be separated into dynamic and static pressure because the fluid surface is assumed to be flat and the fluid density is constant. The dynamic pressure, which causes the extremely large suction force on the object, is our concern, while the static pressure, which yields buoyancy on the object and is much less than the suction force, need not to be discussed here. Second, we focus on the object initially lifted up, so $h \ll L$. Mei employed the adhesion approximation to describe the gap flow, whereas we apply Stokes flow to it. Third, since the inertial term has been dropped, this problem becomes a boundary value problem. By (3.2.1), we introduce a stream function $\psi^{(1)}$ such that

$$u^{(1)} = -\frac{\partial \psi^{(1)}}{\partial y}, \quad v^{(1)} = \frac{\partial \psi^{(1)}}{\partial x}. \tag{3.2.3}$$

Taking the curl of Eq. (3.2.2) and using the stream function yield

$$\Delta\Delta\psi^{(1)} = 0\,. \tag{3.2.4}$$

In the porous bed, the fluid is assumed to be incompressible, and the solid is assumed to be rigid. Then the porous medium flow is governed by the Brinkman equations (Brinkman (1947, 1948, and 1949), and Song and Huang (2000)):

$$\nabla\cdot\vec{u}^{(2)} = 0\,, \tag{3.2.5}$$

$$-\nabla p^{(2)} + \mu\Delta\vec{u}^{(2)} - \mu b\vec{u}^{(2)} = 0\,, \tag{3.2.6}$$

where $\vec{u}^{(2)} = (u^{(2)}, v^{(2)})$ is the true fluid velocity, $p^{(2)}$ the dynamic pressure in porous medium, $b = n_0 / k_p$, n_0 the porosity, k_p the specific permeability (Scheidegger (1974)), and superscript (2) denotes the porous media region. In the following, we call the second term in (3.2.6) the Brinkman term, the third term in (3.2.6) Darcy's term. Note that instead of (3.2.6), Mei (1985) applied (3.2.6) without the Brinkman term (viscous term), which is Darcy's formula. By (3.2.5), we introduce a stream function $\psi^{(2)}$ such that

$$u^{(2)} = -\frac{\partial\psi^{(2)}}{\partial y}\,, \quad v^{(2)} = \frac{\partial\psi^{(2)}}{\partial x}\,. \tag{3.2.7}$$

Taking the curl of Eq. (3.2.6) and using the stream function yield

$$\Delta\Delta\psi^{(2)} - b\Delta\psi^{(2)} = 0\,. \tag{3.2.8}$$

Thus, the governing equations become two fourth order partial differential equations, (3.2.4) and (3.2.8).

3.2.2 Boundary conditions

1. On the boundary of the gap

The symmetric conditions about the y-axis give

$$u^{(1)}=0,\quad v_x^{(1)}=0,\qquad \text{on } x=0,\ 0\le y\le h. \tag{3.2.9}$$

The no-slip condition on the top of the gap gives

$$u^{(1)}=0,\quad v^{(1)}=V,\qquad \text{on } y=h,\ 0\le x\le L. \tag{3.2.10}$$

Since the gap is very small, at the periphery of the gap, we may assume the flow is parallel. Hence

$$u_x^{(1)}=0,\ v^{(1)}=0,\qquad \text{on } x=L,\ 0\le y\le h. \tag{3.2.11}$$

Because the surrounding fluid has no boundary, the hole has relatively small length scale. We therefore assume the fluid outside the gap has a negligible effect and give

$$p^{(1)}=0,\qquad \text{on } x=L,\ 0\le y\le h. \tag{3.2.12}$$

2. On the boundary of the porous medium

The symmetric conditions about the y-axis give

$$u^{(2)}=0,\quad v_x^{(2)}=0,\qquad \text{on } x=0,\ -\infty\le y\le 0. \tag{3.2.13}$$

We assume that no disturbance exist in the bottom of the porous medium. Thus

$$u^{(2)}=0,\quad v^{(2)}=0,\qquad \text{on } y=-\infty,\ 0\le x\le L. \tag{3.2.14}$$

Conditions below the periphery of the gap are given as follow, which is explained below

$$u_x^{(2)} = 0\,,\;\; v^{(2)} = 0\,, \qquad \text{on}\;\; x = L\,,\;\; -\infty \le y \le 0\,. \tag{3.2.15}$$

Because of the velocity boundary conditions (3.2.15), we give the pressure condition

$$p^{(2)} = 0\,, \qquad \text{on}\;\; x = L\,,\;\; -\infty \le y \le 0\,. \tag{3.2.16}$$

Note that (3.2.11), (3.2.12) are quite common as long as $h \ll L$; we can ignore the "end effect" at the far filed. (e.g. Karman's rotating disc in Frank White (2006)). Also note that we give (3.2.15) based on (3.2.11). First, it is obviously impossible to choose $v < 0$. Second, if there is any point with $v > 0$ below the periphery of the gap, $\partial v/\partial y < 0$. Then by continuity $\partial u/\partial x > 0$. It means that the flow accelerate toward $x = 0$, contradicting to the physics in this problem. Therefore, by this inference, we think (3.2.15) is reasonable.

3. On the bed surface

The continuity of normal flow velocity gives

$$v^{(1)} = n_0 v^{(2)}\,, \qquad \text{on}\;\; y = 0,\;\; 0 \le x \le L\,. \tag{3.2.17}$$

The continuity of tangential flow velocity gives

$$u^{(1)} = n_0 u^{(2)}\,, \qquad \text{on}\;\; y = 0,\;\; 0 \le x \le L\,. \tag{3.2.18}$$

The continuity of normal fluid stress gives

$$-p^{(1)} + 2\mu \frac{\partial v^{(1)}}{\partial y} = -p^{(2)} + 2\mu \frac{\partial v^{(2)}}{\partial y}\,, \qquad \text{on}\;\; y = 0,\;\; 0 \le x \le L\,. \tag{3.2.19}$$

The continuity of tangential fluid stress gives

$$\frac{\partial u^{(1)}}{\partial y}+\frac{\partial v^{(1)}}{\partial x}=\frac{\partial u^{(2)}}{\partial y}+\frac{\partial v^{(2)}}{\partial x}, \qquad \text{on } \ y=0, \ \ 0\le x\le L. \tag{3.2.20}$$

Song and Huang (2000) proposed eight physical quantities that should be continuous on the permeable interface. In this study, only the above four are needed: normal flow velocity, tangential flow velocity, normal fluid stress, and tangential fluid stress. Note that these boundary conditions are more complete than Beavers and Joseph's (1967) partial-slip condition, and they have been verified by Huang and Chiang (1997) and Hsu, Huang, and Hsieh (2004).

3.2.3 Solutions

By the method of separation of variables (Selvadurai (2000) and Wang (2003)), we find the solutions

$$\psi^{(1)}(x,y)=\sum_{n=0}^{\infty}(A_n e^{\lambda_n y}+B_n e^{-\lambda_n y}+C_n y e^{\lambda_n y}+D_n y e^{-\lambda_n y})\sin\lambda_n x, \tag{3.2.21}$$

$$p^{(1)}=\mu\sum_{n=0}^{\infty}\left(2\lambda_n C_n e^{\lambda_n y}+2\lambda_n D_n e^{-\lambda_n y}\right)\cos\lambda_n x, \tag{3.2.22}$$

$$\psi^{(2)}(x,y)=\sum_{n=0}^{\infty}(E_n e^{\lambda_n y}+F_n e^{-\lambda_n y}+G_n e^{(\lambda_n{}^2+b)^{1/2}y}+H_n e^{-(\lambda_n{}^2+b)^{1/2}y})\sin\lambda_n x, \tag{3.2.23}$$

$$p^{(2)}=-b\mu\sum_{n=0}^{\infty}\left(E_n e^{\lambda_n y}\right)\cos\lambda_n x, \qquad \lambda_n=(n+\frac{1}{2})\frac{\pi}{L}. \tag{3.2.24}$$

The dynamic force applied on the bottom of the object is

$$F=2\int_0^L p^{(1)}(x,h)dx=4\mu\sum_{0}^{\infty}\left(C_n e^{\lambda_n h}+D_n e^{-\lambda_n h}\right)\sin\lambda_n L. \tag{3.2.25}$$

The coefficients are presented in Appendix A2. Note that despite the discontinuity of $v^{(1)}$ in (3.2.10) and (3.2.11), the infinite series solution indeed converges because of the characteristics of Fourier series.

3.3 Discussion

3.3.1 Verification of the solutions

Mei (1985) proposed an approach and did experiments to investigate the lift-up problem. Their approach might not be suitable to the problem, which will be illustrated in the following, but the experiments are worth comparing. In order to compare with Mei's (1985) experiments, we choose the following parameters: half the object length $L = 7.86cm$, medium grain size $D_{50} = 0.245mm$, porosity $n_0 = 0.41$ and thus permeability $k_p = 6.6\times10^{-11}m^2$ by Carman-Kozeny formula (Kaviany (1991)) ($L = 7.86cm$, $D_{50} = 0.245mm$, and $k_p = 6.5\times10^{-11}m^2$ in Mei's (1985) experiments). The boundary condition (3.2.10) is assigned as $V = 10^{-5}\, m/s$ for Figs. 3-3 to 3-10, to satisfy $\mathrm{Re} = \rho Vh/\mu \ll 1$ and Mei's experiments. Force will be transformed as the unit uplift, N/m^2, in order to compared with Mei's experiments.

Some difference should be clarified. Mei's (1985) experiments give a constant external force to the object to estimate the lift-up time, whereas our approach assigns a velocity to the object (BC (3.2.10)) to estimate the lift-up force. That is, we have to estimate the corresponding velocity for a given force, and then calculate the lift-up time. To be honest, this circuitousness is the trade-off in our approach, but the physically based theories can reveal the mechanics in the porous medium.

Fig. 3-2 shows the force change with respect to the ratio of h/L under different constant velocities. Observe first that in each case, there are two turning points, $\log h/L = -2.7$ and -1.7. In this study, we call them the first turning point and the breakout point (explain below), respectively. Second, before the first turning point, the force remains constant as the objected is lifted-up. After that, the force decreases rapidly until the breakout point, at which the force reduces to zero. Third, even though different constant velocities are given, they correspond to almost the same second turning points, $h = 0.16cm$. This point may indicate the breakout phenomenon because the force for a initially given velocity reduces to zero and because it falls within the range of the breakout height in Mei's (1985) experiments, $0.15 - 0.5cm$. Thus we call it the breakout point.

The unit uplift $3 - 15N/m^2$ in Fig. 3-2 is the magnitude proceeded in Mei's experiments. Estimating the corresponding velocity for a given constant force in each interval, and then calculating the breakout time, we obtain that constant unit uplift $3N/m^2$ corresponds to about $100s$, and $15N/m^2$ corresponds to about $20s$. These quantities are consistent with those in Mei's experiments (Fig. 5 in Mei (1985)) and therefore verify our solution.

Fig. 3-2 can account for some descriptions in Mei's experiments. First, the constant velocities before the first turning point accounts for the description, "The chart recorder was started for $10-15s$ to attain constant speed, then the plate was released." Second, $\log h/L=-1.7$ and $\log h/L=-1.2$ corresponds to the range of breakout height in Mei's experiment, $0.15-0.5cm$. The gap width needs less than $3s$ to vary in this interval, accounting for the difficult to define and measure a definite breakout height in experiments. Third, the velocity changing rapidly in the above interval accounts for the description, "When breakout is approached, the chart trace rises sharply." These observations explain some phenomena in Mei's experiments, and show the compatibility of our solution and Mei's experiments.

3.3.2 Mechanics in the lift-up problem

Note that because of Beavers and Joseph's (1967) slip-flow boundary condition, no mechanics in the porous medium can be shown by Mei's (1985) approach. In the following, our solution reveals not only the significant phenomenon in this problem, but also the unsuitability of Mei's approach.

1. Dynamic force acting on the object

Fig. 3-3 shows the forces change with respect to height, with constant velocity $V=10^{-5}m/s$. The solid lines are for rigid porous beds; the dashed line is for a rigid impermeable bed. Observe that the force for the rigid impermeable bed goes to infinity as the gap is very small, but the force for the rigid porous beds remain constant before the first turning point. In addition, for the rigid porous bed, four essential concepts should be pointed out. First, when the object is initially lifted up, the force for fine sand bed (small permeability) is about 10 times that for medium sand bed (large permeability). Second, the force for fine sand bed decreases earlier than that for medium sand bed. Third, although the force for fine sand bed decreases earlier, it is still greater than medium sand bed until breakout is approaching. Forth, although different soil beds correspond to different forces and first turning points, their breakout heights are almost the same.

2. Streamline

Fig. 3-4 shows the streamlines in the gap and the porous medium. Fig. 3-4a reveals that before the first turning point, the gap flow mainly comes from the porous bed; only little comes from the periphery ($x = L$). It should be pointed out that at this stage, the width of the gap is much less than the dimension of the interstices of the porous medium (estimated by sand diameter D_{50} and porosity n_0). Hence it makes sense that the gap flow mainly comes from the porous bed.

Fig. 3-4b shows that at the first turning point, the gap flow gradually comes from the periphery. In fact, as $h/L = 10^{-2.7}$, the width of the gap is almost the same as the dimension of the interstices of the porous medium. That is, when the width of the gap and the dimension of the interstices are of the same order, the gap flow gradually comes from the periphery, and the dynamic force acting on the object start to decrease (Fig. 3-2).

Fig. 3-4c shows that when the breakout occurs, the gap flow almost comes from the periphery of the gap. Fig. 3-5 shows percentage of the horizontal flux from the gap periphery. About 98% flow in the gap comes from the periphery as the breakout occurs.

Note that, before the first turning point, the boundary condition (3.2.11) is still being satisfied, but this can be visualized only when we plot the region near the gap periphery. Also note that Mei (1985) concerned with a thin boundary layer below $y = 0$, thereby applying Beavers and Joseph's (1967) slip-flow boundary condition, a boundary condition connecting the horizontal velocity in the interface. However, Figs. 3-4a and 4b show that at the two stages, the boundary layer may not exist. Thus the slip-flow boundary condition might not be suitable to the two stages. Furthermore, the vertical velocity dominates the flow near the interface. Hence adhesion approximation might not be appropriate to the two stages either.

3. Pressure

Fig. 3-6 shows the dimensionless pressure contour in the porous medium and the gap. It shows that although before the first turning point, although the flow mainly comes from the porous medium, $\partial p / \partial y$ is still very small. The reason is that the velocities in the gap are almost the same as the object velocity, so $\partial p / \partial y$ is still very small as the one in adhesion approximation.

4. Shear stress and Vorticity

Fig. 3-7 shows the dimensionless shear stress contour. Fig. 3-8 shows the dimensionless vorticity contour. And Fig. 3-9 shows the ratio of the Brinkman term to the sum of the Brinkman and Darcy's terms. Observe in Figs. 3-7a and 3-8a that before the first turning point, the shear stress and vorticity are very small; in Fig. 3-9a that Darcy's term dominates the pore flow, whereas the Brinkman term (viscous term) is not significant.

Figs. 3-7b and 3-8b show that at the first turning point, the shear stress and vorticity become much greater; Fig. 3-9b shows that the Brinkman term becomes more significant, indicating that Darcy formula applied by Mei (1985) is not sufficient. Note that Figs. 3-4 and 3-5 display that at the first turning point, the gap flow gradually comes from the periphery of it ($x = L$); therefore, the viscous effect by horizontal flow gradually influences the pore flow. Also note that these figures demonstrate the boundary layer thickness, $(k_p/n_0)^{1/2}$, which can be obtained by the order of magnitude analysis.

Fig. 3-9c shows the ratio of the viscous term when the breakout occurs, displaying the boundary layer mentioned by Bevearse and Joseph (1967). However, although the Brinkman term becomes the most significant at this stage, it has not yet dominated the pore flow (at most 50%). Also, observe in Figs. 3-7c and 3-8c that the shear stress and vorticity is very small. The reason is that little flow comes into the gap from the porous medium (Figs. 3-4 and 3-5), thereby causing the small values of the shear stress and vorticity. Fig. 3-10 shows the circulation around the porous medium boundary. The circulation is still increasing at the first turning point, $h/L = 10^{-2.7}$, and starts to decrease nearly at $h/L = 10^{-2.4}$, at which the percentage of the horizontal flux from the gap periphery is about 20% (Fig. 3-5).

Note that from the first turning point, the viscous effect gradually influences the pore flow. This actually cannot be described by Darcy's law and Beavers and Joseph's slip-flow boundary condition.

3.4 Conclusions

This study reinvestigates the lift-up problem with a rigid porous bed that has been studied by Mei (1985). Mei studied the breakout phenomenon of the problem by assuming adhesion approximation to the gap flow and Darcy's law to the porous medium, and applying Beavers and Joseph's (1967) partial-slip boundary condition to the bed interface. This approach, however, might not be appropriate if initially the gap width is less than the dimension of the interstices of the porous medium, and if later the viscous effect of pore flow gradually becomes essential. For this situation, this study applies Stokes flow to the gap, and Song and Huang's (2000) laminar poroelasticity theory to the porous mdeium, modifying the governing equation and boundary conditions in Mei's (1985) approach. Stokes flow can react to the horizontal and vertical velocities, the Brinkman equations can respond to the viscous effect of pore flow, and the complete boundary conditions can react to the continuity of velocity and stress.

This study proposes a more general solution to the two-dimensional lift-up problem with a rigid porous bed, a boundary value problem with two fourth order partial differential equations. The solution has been verified by Mei's experiments, and reveals many mechanics of the problem. The dynamic force acting on the object and the breakout phenomenon are displayed, the initial motivation to investigate the lift-up problem. Also, other mechanics are demonstrated: streamlines, pressure, shear stress, voticity, and viscous effect. In fact, these are the mechanics that cannot be presented by Mei's (1985) approach, and suggests that adhesion approximation, Darcy's law, and Beavers and Joseph's (1967) slip-flow boundary condition might not be suitable to any lift-up problem with porous beds.

This chapter investigates the lift-up problem with a rigid porous bed. In fact, the study can be further extended if the deformation of bed is considered; therefore, the next chapter studies the lift-up problem with a hard poroelastic bed.

Chapter 4

A TWO-DIMENSIONAL LIFT-UP RPOBLEM WITH A HARD POROELASTIC BED

4.1 Introduction

Lift-up problem is a process during which an object immersed in fluid is initially extricated from a bed (Fig. 4-1). In ocean and offshore engineering, it has many applications such as removing hydraulic structures, salvaging sunken ships, and operating submarines. Based on field experience (Liu (1969)), the lifting force is considerably large and the operation time is extremely long. It is a process that slowly increases the gap between the object and the bed until a turning point when the object is abruptly unfastened. This is called breakout phenomenon. Because of it and the lifting force, it is helpful to ocean and offshore engineering if the mechanics of the problem can be understood.

Lift-up problem has been theoretically studied. Biot (1941, 1962) developed a poroelasticity theory for saturated porous medium in which the soil solid is described by Hooke's law, and the pore flow by Darcy's law. Foda (1982) employed Biot's theory to investigate the breakout phenomenon from a poroelastic seabed. Mei, Yeung, and Liu (1985) (Mei) proposed an approach that treats the seabed as rigid porous bed, reinvestigate the problem of Foda (1982), and did experiment. In Foda's model, the flow in the gap is described by adhesion approximation (Chap. 2), the porous medium obeys Biot's (1941, 1962) theory. Also, Foda applied the boundary layer theory proposed by Mei and Foda (1981) to approximate the porous medium.

Actually, the case investigated by Foda (1982) is the problem with a soft poroelastic bed. Chen, Huang, and Song (1997) discovered that the boundary layers examined by Simons (1977) and Mei and Foda (1981) appear only in soft poroelastic bed, in which the wavelength of the second longitudinal wave in the porous bed is much shorter than that of water wave.

Foda's (1982) approach, however, might not be suitable to the problem for three reasons. First, when the object is initially lifted up from the bed, the width of the gap may be less than the dimension of the interstices of the porous medium. The flow in the gap may mainly come from the pore of bed rather than from the gap periphery. Hence the flow in the gap might not be described by adhesion approximation.

Second, when the width of the gap is greater than the dimension of the interstices of the porous medium, the flow in the gap mainly comes from the gap periphery. The horizontal velocity dominates the gap flow, so viscous effect becomes essential. However, Darcy's law, incorporated in Biot's theory (1941, 1962) and thus in Foda's (1982) model, could not respond to the viscous effect in the porous medium. Actually, Beavers and Joseph (1967) argued that there is a boundary layer below the bed interface in which the pore flow does not satisfy Darcy's law.

Third, one of the boundary conditions might not be appropriate. Foda (1982) applied the continuity of fluid pressure to the boundary condition. However, Biot (1962) suggested the boundary conditions in the case where both the upper and pore flow are inviscid. Song and Huang (2000) suggested that if the viscous effect of fluid is essential, the continuity of stress should be applied; and the problem Foda (1982) investigated is indeed such a case.

After the famous Biot's theory (1941, 1962), poroelastic theory has been further extended. On a basis of the first law of thermodynamics, Song and Huang (2000) proposed a more general theory for laminar poro-elastic media flow that takes viscosity of fluid into account. In their theory, the governing equation becomes the Biot equation (1941, 1962) if fluid viscous effect is dropped and mass coupling effect for flow is considered; it reduces to the Brinkman equation (Brinkman (1947, 1948, and 1949)) if rigid solid skeleton is assumed and inertial term for fluid is neglected; and it further degenerates to Darcy's law if fluid viscosity is also neglected. Also, referring to Deresiewicz and Skalak (1963), Song and Huang (2000) proposed a complete set of boundary conditions (continuity of velocity and stress) for adjacent porous interfaces.

Song and Huang (2000) applied their theory to a problem in which viscous water wave passes over a poroelastic bed, giving the limiting solution to a rigid porous bed for the experiment conducted by Liu, Davis, and Downing (1996). Later, Hsieh, Dai and Huang (2003) applied Song and Huang's (2000) laminar theory to investigate the interaction of oscillatory laminar water waves and a steady non-uniform current passing over a hard poroelastic bed, revealing the importance of fluid viscosity versus the pressure gradient.

This study analytically investigates a lift-up problem with a hard poroelastic bed (simply differentiated by Lame's constants G and λ), the simplest case in problems with different kinds of poroelastic beds, in order to offer a preliminarily correct solution to the complicated problems. Instead of adhesion approximation, Stokes flow, with more complete momentum equations, is employed to the gap. Instead of Biot's (1941, 1962) poroelastic theory and the boundary conditions that Foda (1982) applied, Song and Huang's (2000) laminar poroelastic theory is employed for the porous medium (Fig 4-1). The solution to the two-dimensional lift-up problem with a hard poroelastic bed will be provided, and the mechanics in the problem will be revealed.

4.2 Mathematical formulation

Assuming the bottom of a rigid object and the top of a hard poroelastic bed to be flat, we study a problem in which the object is initially lifted up from the bed without tilt (Fig. 4-1).

4.2.1 Governing equations

In the gap, the fluid is assumed to be incompressible. Since lift-up problem is a very slow process, the flow in the gap can be treated as Stokes flow:

$$\nabla \cdot \vec{u}^{(1)} = 0, \tag{4.2.1}$$

$$-\nabla p^{(1)} + \mu \Delta \vec{u}^{(1)} = 0, \tag{4.2.2}$$

where $\vec{u}^{(1)} = (u^{(1)}, v^{(1)})$ is the fluid velocity, $p^{(1)}$ the fluid dynamic pressure, μ the fluid viscosity and superscript (1) denotes the gap region. Note first that the fluid pressure can be separated into dynamic and static pressure because the fluid surface is assumed to be flat and the fluid density is constant. The dynamic pressure, which causes the extremely large suction force on the object, is our concern, while the static pressure, which yields buoyancy on the object and is much less than the suction force, need not to be discussed here. Second, we focus on the object initially lifted up, so

$h \ll L$. Foda employed the adhesion approximation to describe the gap flow, whereas we apply Stokes flow to it. Third, since the inertial term has been dropped, this problem becomes a boundary value problem. By (4.2.1), we introduce a stream function $\psi^{(1)}$ such that

$$u^{(1)} = -\frac{\partial \psi^{(1)}}{\partial y}, \quad v^{(1)} = \frac{\partial \psi^{(1)}}{\partial x}. \tag{4.2.3}$$

Taking the curl of Eq. (2.2) and using the stream function yield

$$\Delta\Delta\psi^{(1)} = 0. \tag{4.2.4}$$

For the porous medium, Song and Huang's (2000) laminar poroelastic theory gives the continuity equations of fluid and solid

$$\frac{\partial}{\partial t}(n_0\rho) + \nabla\cdot(n_0\rho\vec{u}^{(2)}) = 0, \tag{4.2.5}$$

$$\frac{\partial}{\partial t}[(1-n_0)\rho_s] + \nabla\cdot[(1-n_0)\rho_s\vec{d}^{(2)}] = 0, \tag{4.2.6}$$

and the momentum equations of fluid and solid

$$\nabla\underline{\underline{\sigma}}^{(2)} = n_0\rho\dot{\vec{u}}^{(2)} + F(k)\frac{\mu n_0^2}{k_p}(\vec{u}^{(2)} - \dot{\vec{d}}^{(2)}), \tag{4.2.7}$$

$$\nabla\underline{\underline{\sigma_s}}^{(2)} = (1-n_0)\rho_s\ddot{\vec{d}}^{(2)} - F(k)\frac{\mu n_0^2}{k_p}(\vec{u}^{(2)} - \dot{\vec{d}}^{(2)}), \tag{4.2.8}$$

where $\vec{u}^{(2)} = (u^{(2)}, v^{(2)})$, $\dot{\vec{d}}^{(2)} = (\dot{d}_1^{(2)}, \dot{d}_2^{(2)})$ are the fluid and solid velocity vectors, ρ, ρ_s the fluid and solid densities, $\underline{\underline{\sigma}}^{(2)}$, $\underline{\underline{\sigma_s}}^{(2)}$ the fluid and solid stress tensors, n_0 the porosity, k_p the specific permeability, $F(k)$ frequency correction factor, which can be set as unity for low frequency, and superscript (2) denotes the porous media region. In (4.2.7) and (4.2.8), stress tensors are

$$\underline{\underline{\sigma}}^{(2)} = n_0 \underline{\underline{\tau}}^{(2)} - n_0 p^{(2)} \underline{\underline{I}}\,, \tag{4.2.9}$$

$$\underline{\underline{\sigma_s}}^{(2)} = (1-n_0)\underline{\underline{\tau_s}}^{(2)} - (1-n_0) p^{(2)} \underline{\underline{I}}\,, \tag{4.2.10}$$

, where $\underline{\underline{\tau}}^{(2)}$, $\underline{\underline{\tau_s}}^{(2)}$ are the shear stress of the fluid and the effective stress of the solid skeleton, $p^{(2)}$ the dynamic pressure in porous medium, and $\underline{\underline{I}}$ the identity matrix. The shear stress of the fluid and the effective stress of the solid are

$$\underline{\underline{\tau}}^{(2)} = \mu[\nabla \vec{u}^{(2)} + (\nabla \vec{u}^{(2)})^T] + \mu'(\nabla \cdot \dot{\vec{d}}^{(2)})\underline{\underline{I}}\,, \tag{4.2.11}$$

$$\underline{\underline{\tau_s}}^{(2)} = G[\nabla \vec{d}^{(2)} + (\nabla \vec{d}^{(2)})^T] + \lambda(\nabla \cdot \vec{d}^{(2)})\underline{\underline{I}}\,, \tag{4.2.12}$$

where μ' is the second fluid viscosity, G, λ the Lamb's constants of elasticity.

In the porous medium, the pore flow is assumed to be incompressible. Grain compressibility is usually negligible; that is, $d\rho_s / dt = 0$ (Mei and Foda (1981), Song and Huang (2000)). Then the continuity equations of fluid and solid can be written as

$$\nabla \cdot \vec{u}^{(2)} = 0\,, \tag{4.2.13}$$

$$\nabla \cdot \dot{\vec{d}}^{(2)} = 0\,. \tag{4.2.14}$$

Due to the hard bed, we may assume that the solid velocity is very small. Time is merely a parameter even to the solid deformation. Thus the continuity equation of solid can be further simplified as

$$\nabla \cdot \vec{d}^{(2)} = 0\,, \tag{4.2.15}$$

Since in the gap the inertial of the fluid have been dropped, the inertial terms in (4.2.7) and (4.2.8) have to be neglected. The momentum equations of fluid and solid can be simplified as

$$-n_0\nabla p^{(2)} + n_0\mu\Delta\vec{u}^{(2)} = \frac{\mu n_0^2}{k_p}\vec{u}^{(2)}, \tag{4.2.16}$$

$$-(1-n_0)\nabla p^{(2)} + (1-n_0)G\Delta\vec{d}^{(2)} = -\frac{\mu n_0^2}{k_p}\vec{u}^{(2)}. \tag{4.2.17}$$

By (4.2.13) and (4.2.15), we introduce a stream function $\psi^{(2)}$ for pore flow and a deformation function $\psi_s^{(2)}$ for solid such that

$$u^{(2)} = -\frac{\partial\psi^{(2)}}{\partial y}, \quad v^{(2)} = \frac{\partial\psi^{(2)}}{\partial x}, \tag{4.2.18}$$

$$d_1^{(2)} = -\frac{\partial\psi_s^{(2)}}{\partial y}, \quad d_2^{(2)} = \frac{\partial\psi_s^{(2)}}{\partial x}. \tag{4.2.19}$$

Taking the curl of Eq. (4.2.16) and using the steam function (4.2.18) yield

$$\Delta\Delta\psi^{(2)} - \frac{n_0}{k_p}\Delta\psi^{(2)} = 0. \tag{4.2.20}$$

Taking the curl of Eq. (4.2.17) and using the stream function (4.2.18) and the deformation function (4.2.19) yield

$$(1-n_0)\Delta\Delta\psi_s^{(2)} = -\frac{\mu n_0^2}{k_p}\Delta\psi^{(2)}. \tag{4.2.21}$$

Thus, the governing equations become three fourth-order partial differential equations, (4.2.4), (4.2.20) and (4.2.21). Note that whereas Foda (1982) analyzed the problem with a soft poroelastic bed, this study investigates the problem with a hard poroelastic bed, the simplest case in problems with different kinds of poroelastic beds.

4.2.2 Boundary conditions

1. On the boundary of the gap

The symmetric conditions about the y-axis give

$$u^{(1)} = 0, \quad v_x^{(1)} = 0, \qquad \text{on} \ \ x = 0, \ \ 0 \le y \le h. \tag{4.2.22}$$

The no-slip condition on the top of the gap gives

$$u^{(1)} = 0, \quad v^{(1)} = V, \qquad \text{on} \ \ y = h, \ \ 0 \le x \le L. \tag{4.2.23}$$

Since the gap is very small, at the periphery of the gap, we may assume the flow is parallel. Hence

$$u_x^{(1)} = 0, \quad v^{(1)} = 0, \qquad \text{on} \ \ x = L, \ \ 0 \le y \le h. \tag{4.2.24}$$

Because the surrounding fluid has no boundary, the hole has relatively small length scale. We therefore assume the fluid outside the gap has a negligible effect and give

$$p^{(1)} = 0, \qquad \text{on} \ \ x = L, \ \ 0 \le y \le h. \tag{4.2.25}$$

2. On the boundary of the porous medium

Fluid:

The symmetric conditions about the y-axis give

$$u^{(2)} = 0, \quad v_x^{(2)} = 0, \qquad \text{on} \ \ x = 0, \ \ -\infty \le y \le 0. \tag{4.2.26}$$

We assume that no disturbance exist in the bottom of the porous medium. Thus

$$u^{(2)} = 0, \quad v^{(2)} = 0, \qquad \text{on} \ \ y = -\infty, \ \ 0 \le x \le L. \tag{4.2.27}$$

Conditions below the periphery of the gap are given as follow, which is explained below

$$u_x^{(2)} = 0\,,\ \ v^{(2)} = 0\,, \qquad \text{on}\ \ x = L\,,\ \ -\infty \le y \le 0\,. \tag{4.2.28}$$

Because of the velocity boundary conditions (2.28), we give the pressure condition

$$p^{(2)} = 0\,, \qquad \text{on}\ \ x = L\,,\ \ -\infty \le y \le 0\,. \tag{4.2.29}$$

Note that (4.2.24), (4.2.25) are quite common as long as $h \ll L$; we can ignore the "end effect" at the far filed. (e.g. Karman's rotating disc in Frank White (2006)). Also note that we give (4.2.28) based on (4.2.24). First, it is obviously impossible to choose $v < 0$. Second, if there is any point with $v > 0$ below the periphery of the gap, $\partial v/\partial y < 0$. Then by continuity $\partial u/\partial x > 0$. It means that the flow accelerate toward $x = 0$, contradicting to the physics in this problem. Therefore, by this inference, we think (4.2.28) is reasonable.

Solid:

The symmetric conditions about the y-axis give

$$d_1^{(2)} = 0\,,\quad (d_2^{(2)})_x = 0\,,\quad \text{on}\ \ x = 0\,,\ \ -\infty \le y \le 0\,. \tag{4.2.30}$$

We assume that no disturbance exist in the bottom of the porous medium. Thus

$$d_1^{(2)} = 0\,,\quad d_2^{(2)} = 0\,, \qquad \text{on}\ \ y = -\infty\,,\ \ 0 \le x \le L\,. \tag{4.2.31}$$

Because of the flow condition (4.2.28), we give

$$(d_1^{(2)})_x = 0\,,\ \ d_2^{(2)} = 0\,,\quad \text{on}\ \ x = L\,,\ \ -\infty \le y \le 0\,. \tag{4.2.32}$$

3. On the bed surface

The continuity of normal flow velocity gives

$$v^{(1)} = n_0 v^{(2)}, \quad \text{on} \quad y=0, \quad 0 \le x \le L. \tag{4.2.33}$$

The continuity of tangential flow velocity gives

$$u^{(1)} = n_0 u^{(2)}, \quad \text{on} \quad y=0, \quad 0 \le x \le L. \tag{4.2.34}$$

The continuity of normal fluid stress gives

$$-p^{(1)} + 2\mu \frac{\partial v^{(1)}}{\partial y} = -p^{(2)} + 2\mu \frac{\partial v^{(2)}}{\partial y}, \quad \text{on} \quad y=0, \quad 0 \le x \le L. \tag{4.2.35}$$

The continuity of tangential fluid stress gives

$$\frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} = \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x}, \quad \text{on} \quad y=0, \quad 0 \le x \le L. \tag{4.2.36}$$

The continuity of normal total stress gives

$$-p^{(1)} + 2\mu \frac{\partial v^{(1)}}{\partial y} = -p^{(2)} + 2G \frac{\partial d_2^{(2)}}{\partial y}, \quad \text{on} \quad y=0, \quad 0 \le x \le L. \tag{4.2.37}$$

The continuity of tangential total stress gives

$$\mu\left(\frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x}\right) = G\left(\frac{\partial d_1^{(2)}}{\partial y} + \frac{\partial d_2^{(2)}}{\partial x}\right), \quad \text{on} \quad y=0, \quad 0 \le x \le L. \tag{4.2.38}$$

Song and Huang (2000) proposed eight physical quantities that should be continuous on the permeable interface. In this case, only the above six are needed: normal flow velocity, tangential flow velocity, normal fluid stress, tangential fluid stress, normal total stress, and tangential total stress. Note that instead of the continuity of fluid pressure employed by Foda (1982), we apply the continuity of fluid stress (4.2.35) and (4.2.36) as the boundary conditions, and they have been verified by Song and Huang (2000), and Hsieh, Dai, and Huang (2003).

4.2.3 Solutions

Observe the governing equation (4.2.20) and boundary conditions (4.2.22)-(4.2.29) and (4.2.33)- (4.2.36); these are actually the governing equation and boundary conditions to the problem with a rigid porous bed (Chapter 2). That is, in this study, the fluid and solid parts in the porous medium are decoupled; the fluid part (Eq. 4.2.20) can be first solved, and then the solid part (Eq. 4.2.21). The solution to the fluid part is the same as Chapter 2, they are

$$\psi^{(1)}(x,y)=\sum_{n=0}^{\infty}(A_n e^{\lambda_n y}+B_n e^{-\lambda_n y}+C_n y e^{\lambda_n y}+D_n y e^{-\lambda_n y})\sin\lambda_n x\,, \tag{4.2.39}$$

$$p^{(1)}=\mu\sum_{n=0}^{\infty}\left(2\lambda_n C_n e^{\lambda_n y}+2\lambda_n D_n e^{-\lambda_n y}\right)\cos\lambda_n x\,, \tag{4.2.40}$$

$$\psi^{(2)}(x,y)=\sum_{n=0}^{\infty}(E_n e^{\lambda_n y}+F_n e^{-\lambda_n y}+G_n e^{(\lambda_n{}^2+b)^{1/2}y}+H_n e^{-(\lambda_n{}^2+b)^{1/2}y})\sin\lambda_n x\,, \tag{4.2.41}$$

$$p^{(2)}=-b\mu\sum_{n=0}^{\infty}\left(E_n e^{\lambda_n y}\right)\cos\lambda_n x\,, \qquad \lambda_n=(n+\frac{1}{2})\frac{\pi}{L}. \tag{4.2.42}$$

Then Taking the solution (4.2.41) into Eq. (4.2.21), solving the governing equation (4.2.21) with the boundary conditions (4.2.30)-(4.2.32), and (4.2.37)-(4.2.38), we find the solutions

$$\psi_s^{(2)}(x,y)=\sum_{n=0}^{\infty}(I_n e^{\lambda_n y}+K_n y e^{\lambda_n y}-\frac{\mu n_0}{(1-n_0)G}G_n e^{(\lambda_n^2+b)^{1/2}y})\sin\lambda_n x\,. \tag{4.2.43}$$

The coefficients are presented in Appendix A3. Note that despite the discontinuity of $v^{(1)}$ in (4.2.23) and (4.2.24), the infinite series solution indeed converges because of the characteristics of Fourier series.

4.3 Discussion

Foda (1982) investigated a problem with a poroelastic bed. In his model, the flow in the gap is described by adhesion approximation, the porous medium obeys Biot's (1941, 1962) theory. Also, Foda applied the boundary layer theory proposed by Mei and Foda (1981) to approximate the porous medium. According to the analysis by Chen, Huang, and Song (1997), the case investigated by Foda (1982) is the problem with a soft poroelastic bed.

Foda's [5] approach, however, might not be suitable to the problem for three reasons. First, when the object is initially lifted up from the bed, the width of the gap may be less than the dimension of the interstices of the porous medium. The flow in the gap may mainly come from the pore of bed rather than from the gap periphery. Hence the flow in the gap might not be described by adhesion approximation.

Second, when the width of the gap is greater than the dimension of the interstices of the porous medium, the flow in the gap mainly comes from the gap periphery. The horizontal velocity dominates the gap flow, so viscous effect becomes essential. However, Darcy's law, incorporated in Biot's theory (1941, 1962) and thus in Foda's (1982) model, could not respond to the viscous effect in the porous medium. Actually, Beavers and Joseph (1967) argued that there is a boundary layer below the bed interface in which the pore flow does not satisfy Darcy's law.

Third, one of the boundary conditions might not be appropriate. Foda (1982) applied the continuity of fluid pressure to the boundary condition. However, Biot (1962) suggested the boundary conditions in the case where both the upper and pore flow are inviscid. Song and Huang (2000) suggested that if the viscous effect of fluid is essential, the continuity of stress should be applied; and lift-up problem is indeed such a case.

Whereas Foda (1982) analyzed the problem with a soft poroelastic bed, this study investigates the problem with a hard poroelastic bed, the simplest case in problems with different kinds of poroelastic beds, in order to offer a preliminarily correct solution to the problems.

This chapter assigns Lame's constant $G = 5 \times 10^8 \, N/m^2$ for sand bed material (Song and Huang (2000)). The other parameters are totally the same as those in Chap. 3: half the object length $L = 7.86cm$, medium grain size $D_{50} = 0.245mm$, porosity $n_0 = 0.41$, and permeability $k_p = 6.6 \times 10^{-11} m^2$, object velocity $V = 10^{-5} \, m/s$.

In this case, the fluid and solid parts in the porous medium can be decoupled.

The mechanics of the fluid is exactly the same as those in the problem with rigid porous bed, which has been studied in Chap. 3.

Fig. 4-2 shows the solid deformation lines. Before the first turning point, the solid is most influenced, and at the breakout point, the solid is less influenced. Fig. 4-3 shows the corresponding solid deformations. The horizontal deformation d_1 mainly occurs below the periphery of the gap, $x = L$, especially at the depth of L, half the length of the object. The vertical deformation d_2 mainly occurs on the bed interface, especially at the center of the gap, $x = 0$. It is shown that the porous medium can be influenced to the depth of half the object length L.

Actually, the solid behavior in the lift-up problem with a hard poroelastic bed is influenced by the pressure patterns. Since the Brinmkan term take at most 50% as the breakout occurs, and it is confined in the thin boundary layer with thickness (Chap. 3), we may write the fluid momentum equation (4.2.16) as

$$n_0 \nabla p^{(2)} + \frac{\mu n_0^2}{k_p} \vec{u}^{(2)} \approx 0, \tag{4.3.1}$$

for the pressure distribution in the porous medium during the whole lift-up process. Taking (4.3.1) into the solid momentum equation (4.2.17) gives

$$-\nabla p^{(2)} + (1 - n_0) G \Delta \vec{d}^{(2)} \approx 0. \tag{4.3.2}$$

This equation accounts for that the solid effective stresses are only influenced by the pressure distribution in the porous medium.

Figs. 4-4, 4-5, and 4-6 show the contours of the solid effective stresses $(\tau_s)_{11}^{(2)}$, $(\tau_s)_{22}^{(2)}$, and $(\tau_s)_{12}^{(2)}$, respectively. Since the pressure gradient is balanced by the solid effective stresses, the stress distribution can be explained by the pressure distribution. Fig. 3-6 shows that the maximum $p_x^{(2)}$, pressure gradient with respect to x in the porous medium, occurs near the gap periphery ($x = L$) and the bed interface. By (4.3.2), the $(\tau_s)_{11}^{(2)}$ varies most rapidly in the x-direction and the $(\tau_s)_{12}^{(2)}$ varies most rapidly in the y-direction in that region. On the other hand, the minimum $p_x^{(2)}$ occurs below the center of the gap, so the $(\tau_s)_{11}^{(2)}$ varies most slowly in the x-direction and $(\tau_s)_{12}^{(2)}$ varies very slowly in the y-direction in that region.

Fig. 3-6 also shows that the maximum $p_y^{(2)}$, pressure gradient with respect to y in the porous medium, occurs just below the center of the gap, so the $(\tau_s)_{22}^{(2)}$ varies most rapidly in the y-direction and the $(\tau_s)_{12}^{(2)}$ varies most rapidly in the x-direction in that region. On the other hand, the minimum $p_y^{(2)}$ occurs below the gap periphery, so the $(\tau_s)_{22}^{(2)}$ varies most slowly in the y-direction and $(\tau_s)_{12}^{(2)}$ varies most slowly in the x-direction in that region.

4.4 Conclusions

Foda (1982) studied the problem with a soft poroelastic bed by assuming adhesion approximation to the gap flow and Biot's (1941, 1962) theory to the porous medium. However, Chapter 3 suggests that adhesion approximation and Darcy's law, which is included in Biot's (1941, 1962) theory, might not be suitable to any lift-up problem with porous beds. In addition, one of the boundary conditions applied by Foda (1982) might not be appropriate.

Instead of soft poroelastic bed, this study investigates the problem with a hard poroelastic bed, in order to offer a preliminarily correct solution to problems with different kinds of poroelastic beds. This study applies Stokes flow to the gap, and Song and Huang's (2000) laminar poroelasticity theory to the porous medium. Stokes flow can react to the horizontal and vertical velocities, Song and Huang's (2000) laminar poroelasticity theory can respond to the viscous effect of pore flow, and the complete boundary conditions can react to the continuity of velocity and stress.

This study offers the solution to the two-dimensional lift-up problem with a hard poroelastic bed, a boundary value problem with three fourth order partial differential equations. The fluid and solid parts in the porous medium can be decoupled. The mechanics of the fluid is exactly the same as those in the problem with a rigid porous bed. Deformation lines, deformations and effective stresses of the solid in the hard poroelastic bed are revealed. The porous medium can be influenced to the depth of L, half the length of the object, and the solid effective stresses are only influenced by the pressure distribution in the porous medium.

This chapter investigates the lift-up problem with a hard poroelastic bed. In fact, the study will be complete if the problem with a soft poroelastic bed can be analyzed. However, it is a very difficult problem: The inertial term in the porous medium may appear, and the porosity may change with respect to time. To be honest, this problem is far beyond what I can deal with now, and I just hope this topic can be finished in the future.

Chapter 5

CONCLUDING REMARKS

This study reinvestigates the lift-up problem with a rigid impermeable bed that has been studied by Landau and Lifshitz (1959) and Acheson (1990); the lift-up problem with a rigid porous bed that has been studied by Mei, Yeung, and Liu (1985) (Mei); and the lift-up problem with a hard poroelastic bed.

Landau and Lifshitz (1959) and Acheson (1990) applied adhesion approximation to the problem with a rigid impermeable bed, which significantly simplifies the Navier-Stokes equations so that the solution can be easily obtained. But the inconsistent approximation in the y-momentum may lead to some error to the solution.

Mei (1985) studied the problem with a rigid porous bed by assuming adhesion approximation to the gap flow and Darcy's law to the porous medium, and applying Beavers and Joseph's (1967) partial-slip boundary condition to the bed interface. But Mei's approach might not be suitable if initially the gap width is less than the dimension of the interstices of the porous medium, and if later the viscous effect of pore flow gradually becomes essential.

Foda (1982) studied the problem with a soft poroelastic bed by assuming adhesion approximation to the gap flow and Biot's (1941, 1962) theory to the porous medium. But Foda's approach might not be suitable if initially the gap width is less than the dimension of the interstices of the porous medium, and if later the viscous effect of pore flow gradually becomes essential. In addition, one of the boundary conditions might not be appropriate.

This study applies Stokes flow to the gap, investigating the lift-up problem with a rigid impermeable bed. Also, this study applies Stokes flow to the gap and Song and Huang's (2000) laminar poroelasticity theory to the porous medium, analyzing the lift-up problems with a rigid porous bed and a hard poroelastic bed, respectively. Stokes flow can react to the horizontal and vertical velocities, Song and Huang's (2000) laminar poroelasticity theory can respond to the viscous effect of pore flow, and the complete boundary conditions can react to the continuity of velocity and stress.

The first stage of this study provides the exact solution to the two-dimensional lift-up problem with a rigid impermeable bed, a boundary value problem with a fourth order partial differential equation. The exact solution reveals the tiny error of the pressure in adhesion approximation, and verifies that the tiny error does not influence the kinematics of the flow and the dynamic force acting on the object.

The second stage of this study proposes a more general solution to the two-dimensional lift-up problem with a rigid porous bed, a boundary value problem with two fourth order partial differential equations. The solution has been verified by Mei's experiments, and reveals many mechanics of the problem. The dynamic force acting on the object and the breakout phenomenon are displayed, and other mechanics are demonstrated. They are the mechanics that cannot be presented by Mei's (1985) approach, and suggests that adhesion approximation, Darcy's law, and Beavers and Joseph's (1967) partial-slip flow might not be suitable to the problem with porous bed.

This study finally offers the solution to the two-dimensional lift-up problem with a hard poroelastic bed, a boundary value problem with three fourth order partial differential equations. The fluid and solid parts in the porous medium can be decoupled. The mechanics of the fluid is exactly the same as those in a problem with rigid porous bed. Deformation lines, deformations and effective stresses of the solid in hard poroelastic bed are revealed. The porous medium can be influenced to the depth of L, half the length of the object, and the solid effective stresses are only influenced by the pressure distribution in the porous medium.

FIGURES

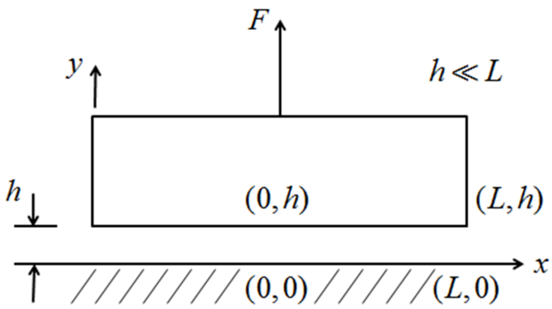


Figure 2- 1: Definition sketch. Lifting an object off a rigid impermeable bed.

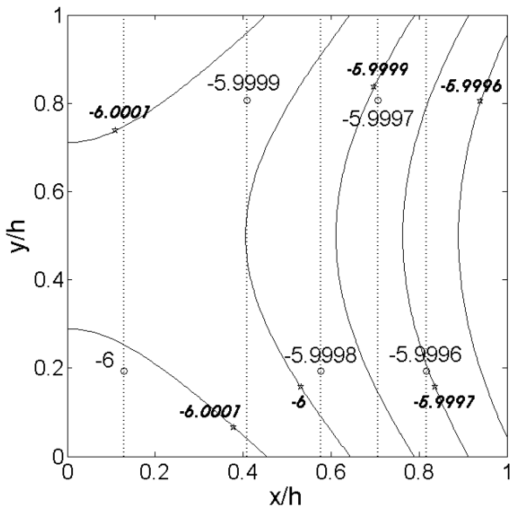
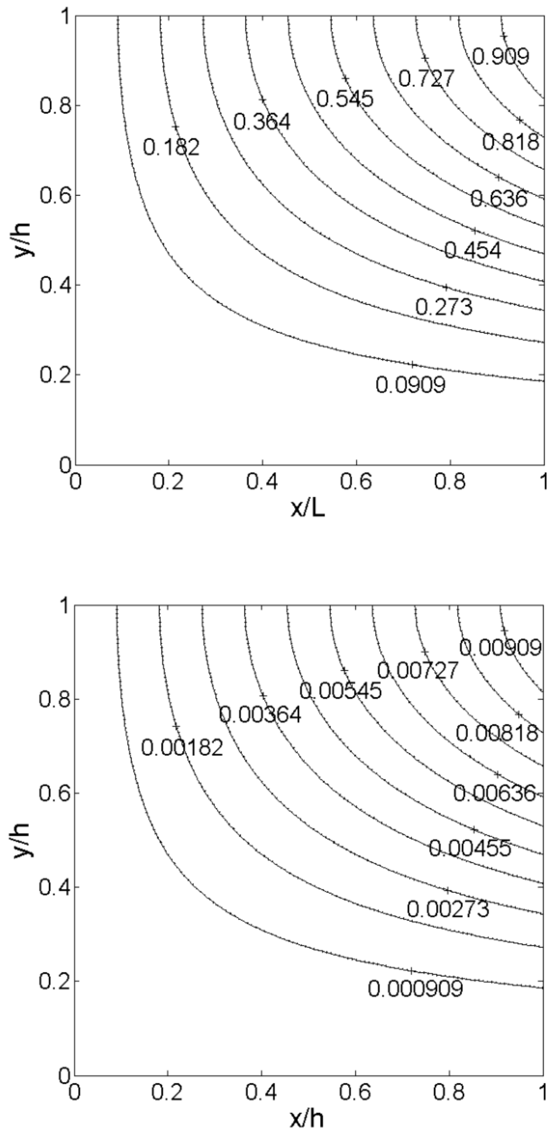


Figure 2- 2: Dimensionless pressure contour ($p\Big/\frac{\mu V L^2}{h^3}$) at the center of the gap. Solid lines are by the exact solution, and dashed lines by adhesion approximation.



(a)

(b)

Figure 2- 3: Dimensionless streamlines (ψ / VL). Solid lines are by the exact solution, and dashed lines by adhesion approximation. (a) In the entire gap. (b) At the center of the gap.

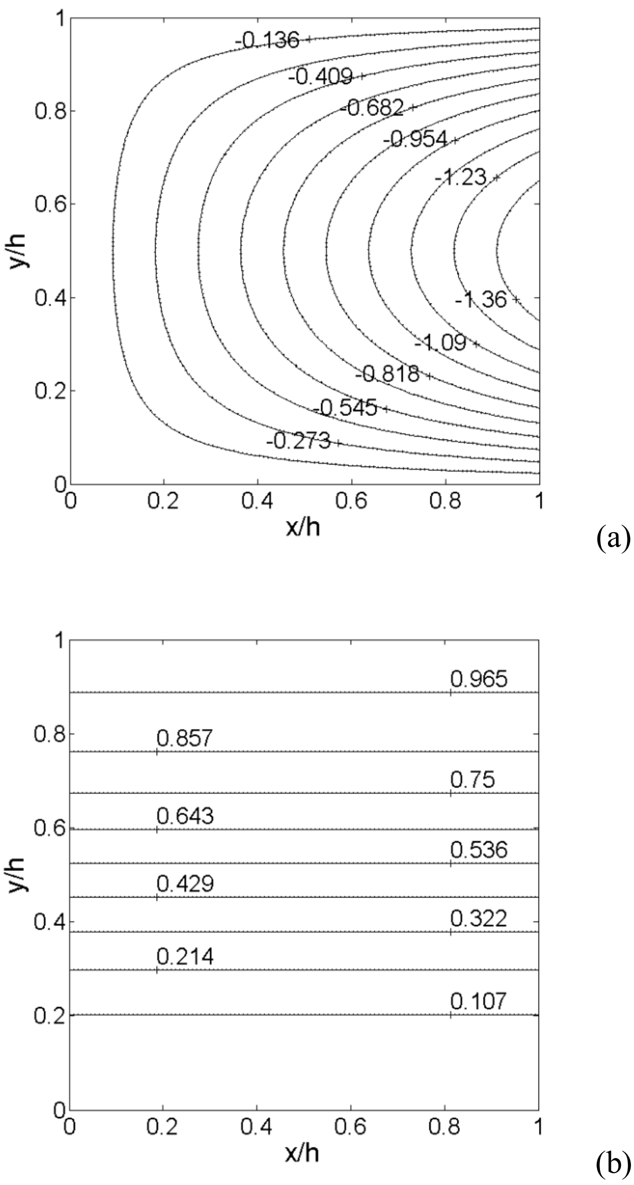


Figure 2- 4: Dimensionless velocity contour at the center of the gap. Solid lines are by the exact solution, and dashed lines by adhesion approximation.

(a) u/V . (b) v/V .

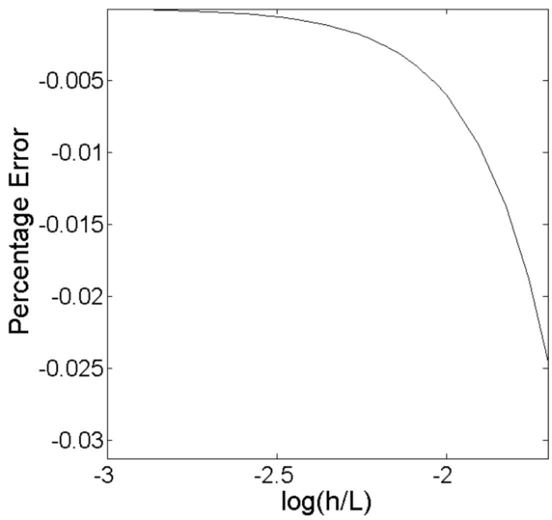


Figure 2- 5: Percentage Error of the dynamic force acting on the object by adhesion approximation.

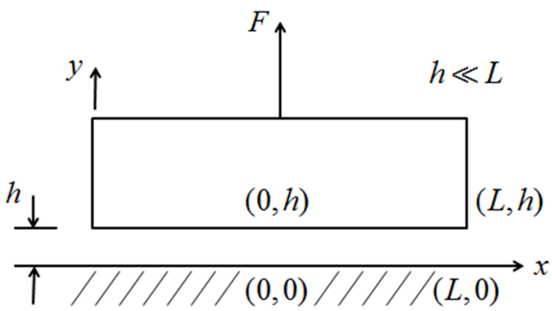


Figure 3- 1: Definition sketch. Lifting an object off a rigid porous bed.

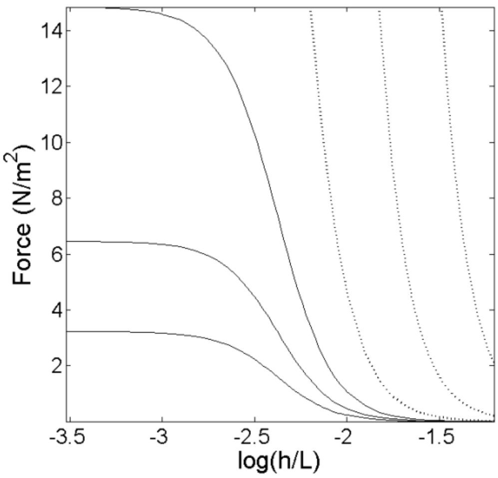


Figure 3- 2: Force per unit area under a constant velocity. Solid lines from top to bottom correspond to $V = 2.3\times10^{-5},\ 10^{-5},\ 5\times10^{-6}\ m/s$; dashed lines from left to right correspond to $V = 10^{-4},\ 10^{-3},\ 10^{-2}\ m/s$.

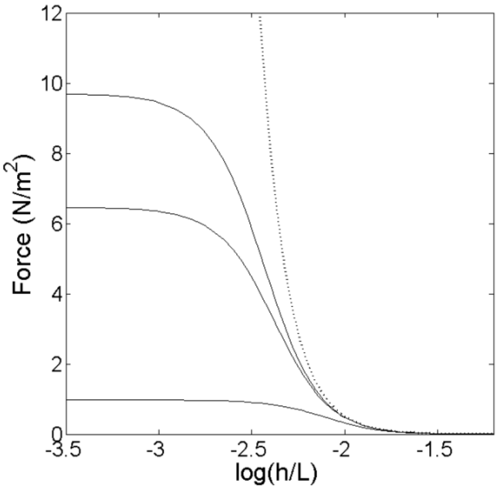


Figure 3- 3: Force change with respect to different height. Solid lines are for rigid porous beds, and from top to bottom correspond to $D_{50} = 0.2,\ 0.245,\ 0.63mm$, which are the sizes of fine sand, the grain in Mei's experiments, and medium sand, respectively (Mei (1985), BSI (2009)). The dashed line is for a rigid impermeable bed.

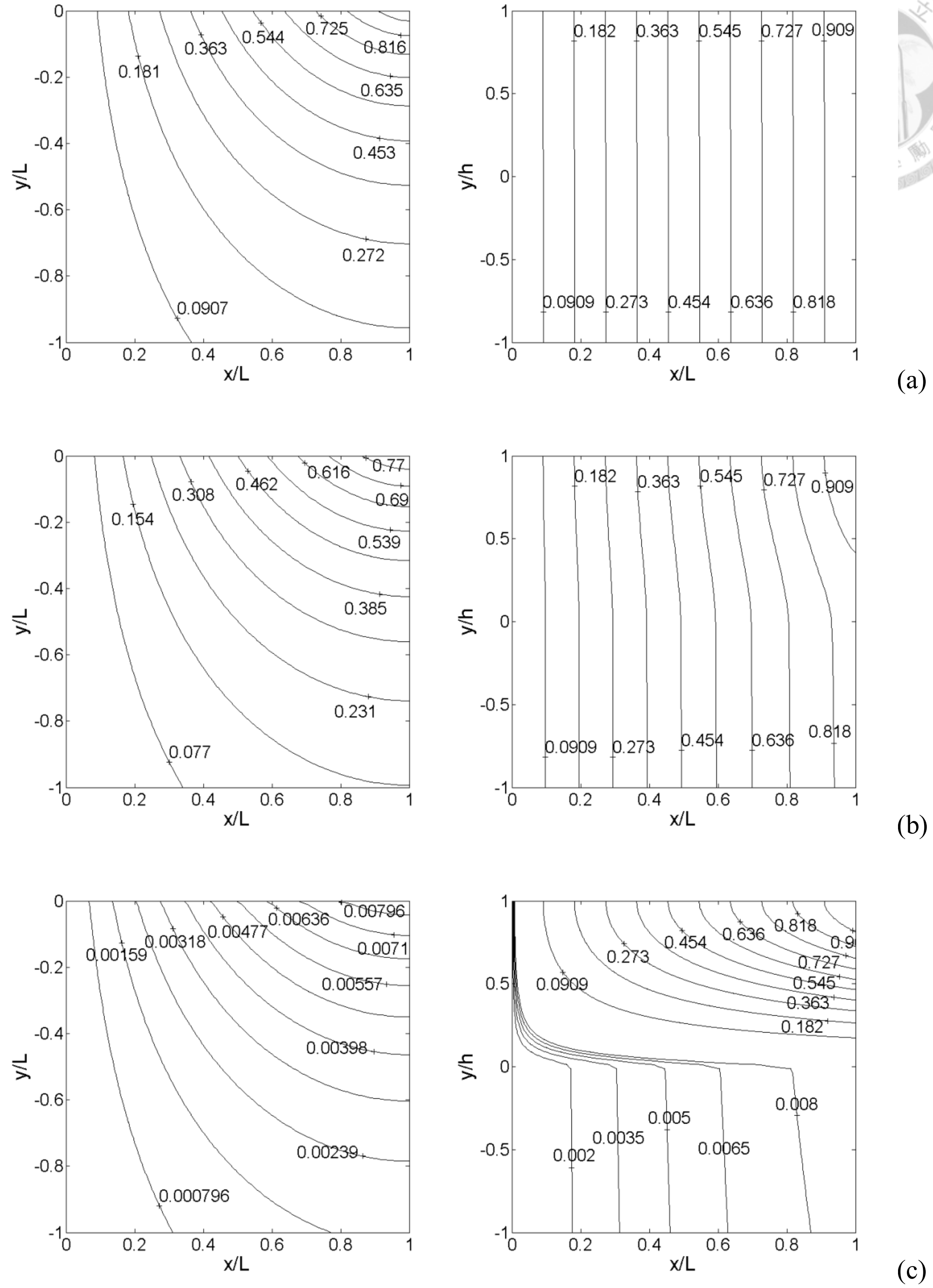


Figure 3- 4: Dimensionless streamlines (ψ / VL). Left column is in the porous medium, right column near the interface. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.

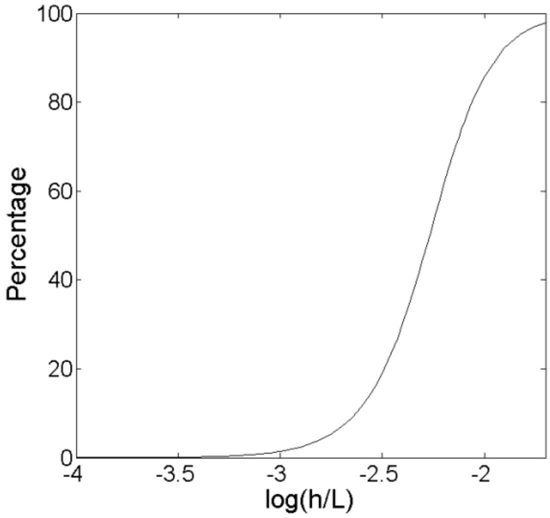


Figure 3- 5: Percentage of the horizontal flux from the gap periphery.

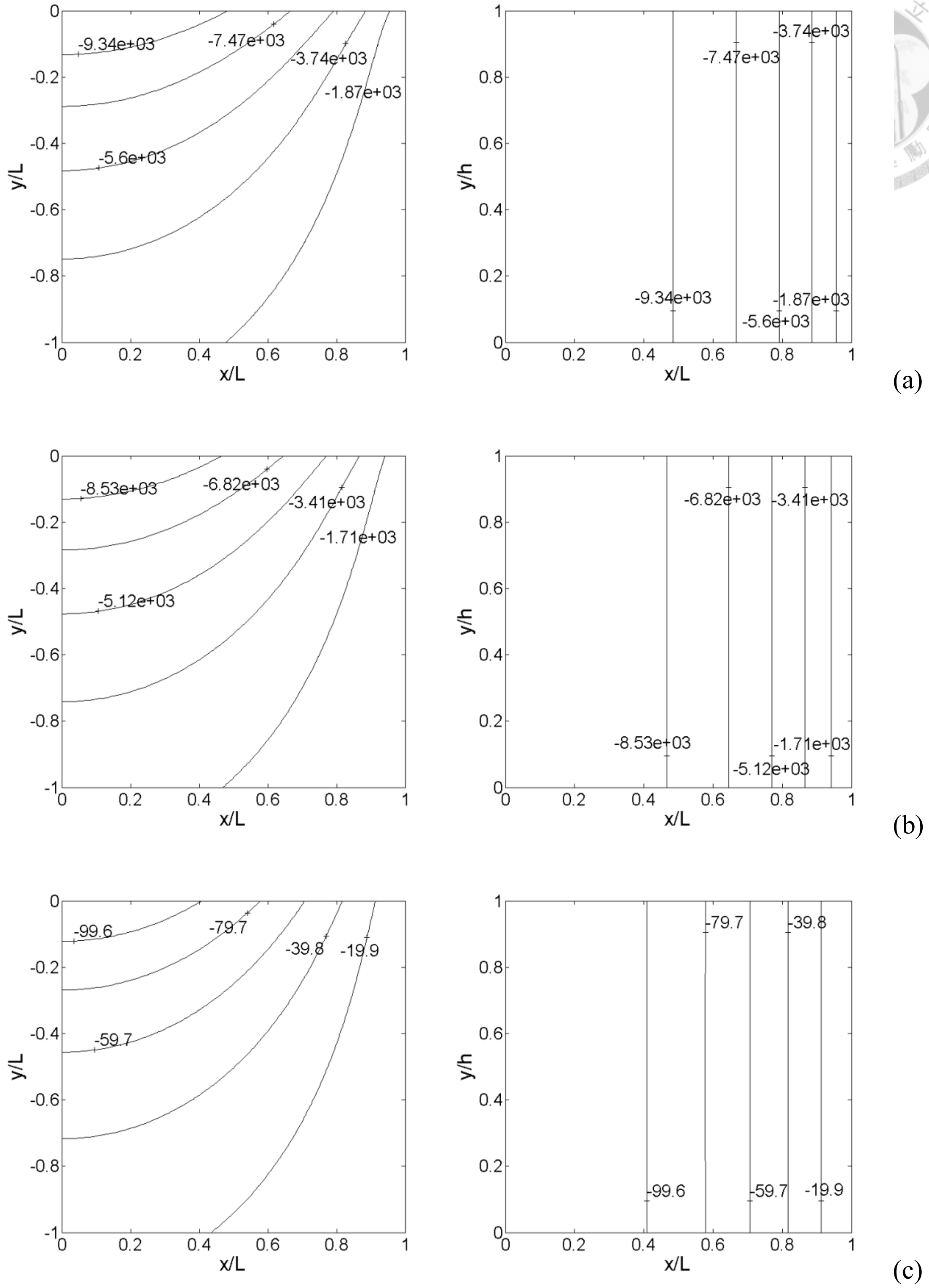
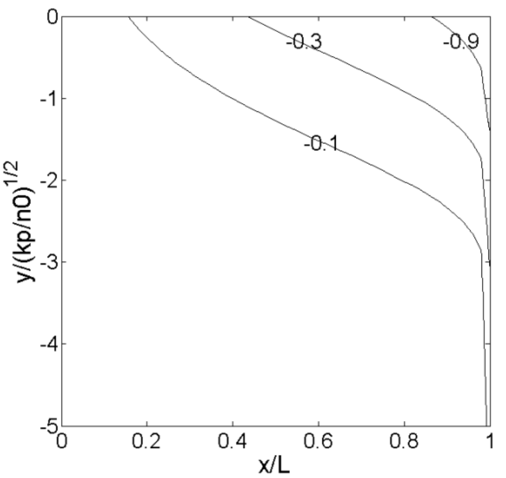
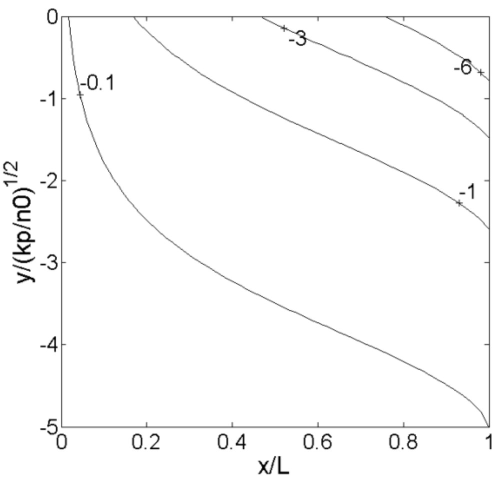
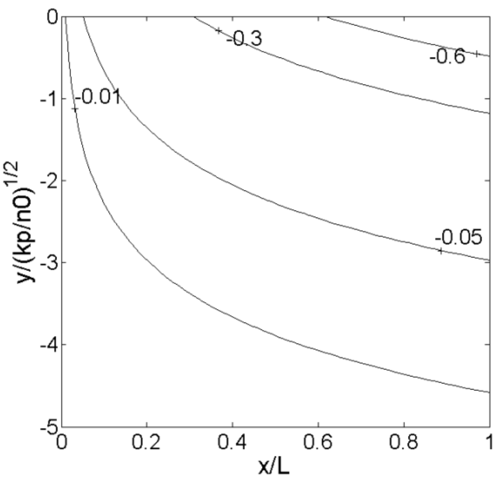


Figure 3- 6: Dimensionless pressure contour ($p / \frac{\mu V}{\sqrt{k_p / n_0}}$). Left column is in the porous medium, right column in the gap. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.



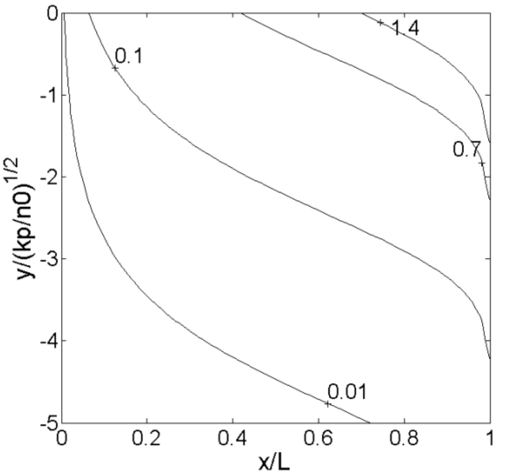
(a)



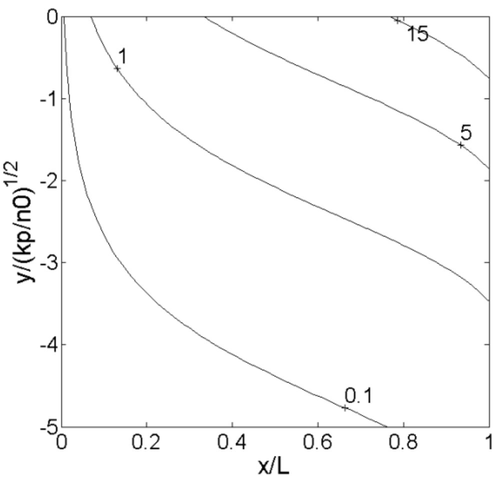


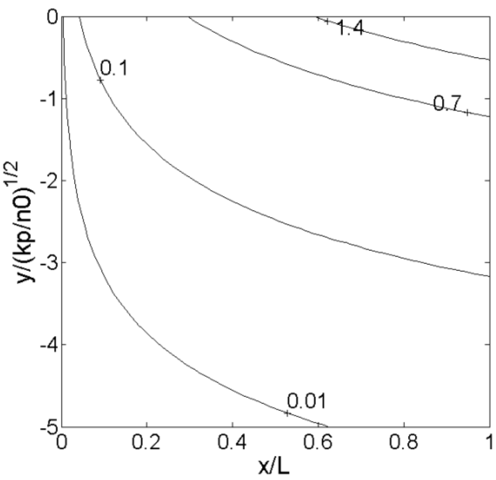
(b) (c)

Figure 3- 7: Dimensionless shear stress ($\tau_{12}^{(2)} / \frac{\mu V}{\sqrt{k_p / n_0}}$) in the porous medium. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.



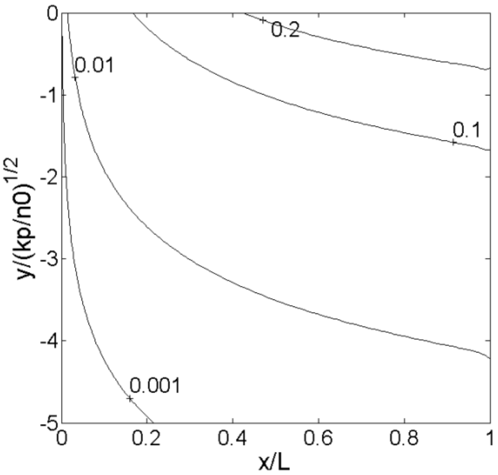
(a)



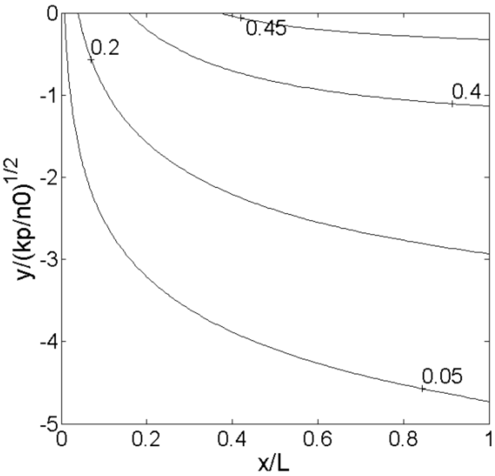


(b) (c)

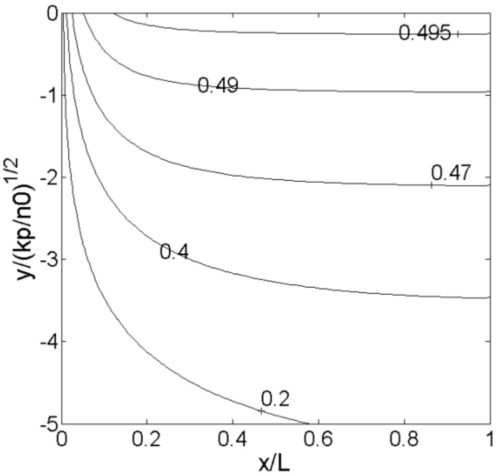
Figure 3- 8: Dimensionless vorticity contour ($\omega^{(2)} / \frac{V}{\sqrt{k_p/n_0}}$) in the porous medium. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.



(a)



(b)



(c)

Figure 3- 9: Ratio of the Brinkman term to the sum of the Brinkman and Darcy terms. (a) Before the first turning point. $h/L=10^{-3.5}$. (b) At the first turning point. $h/L=10^{-2.7}$. (c) At the breakout point. $h/L=10^{-1.7}$.

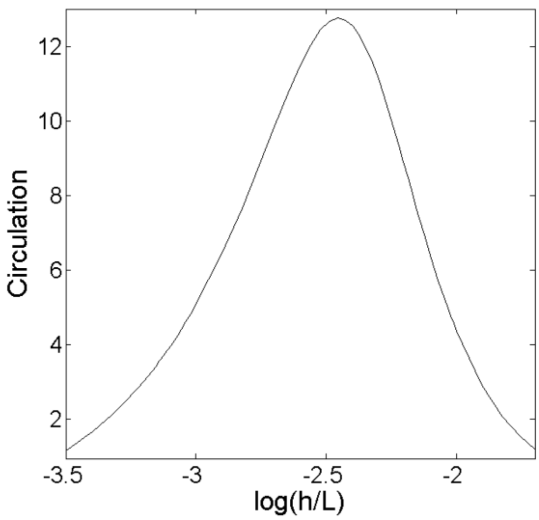


Figure 3- 10: Dimensionless circulation (Γ/VL)

around the boundary of the porous medium

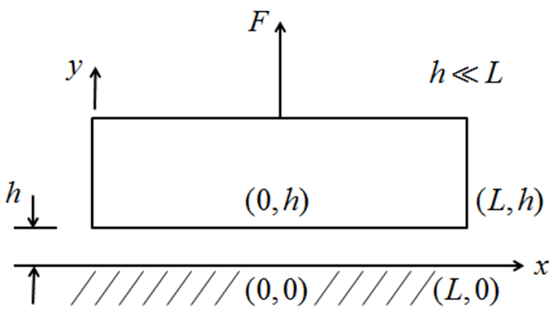
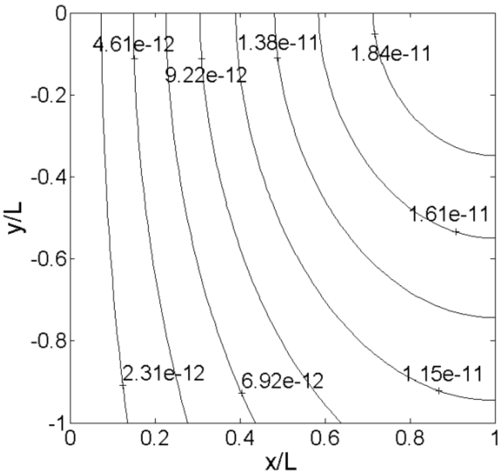
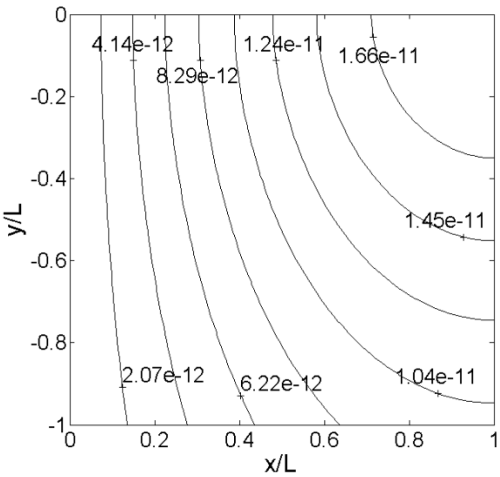


Figure 4- 1: Definition sketch. Lifting an object off a hard poroelastic bed.



(a)



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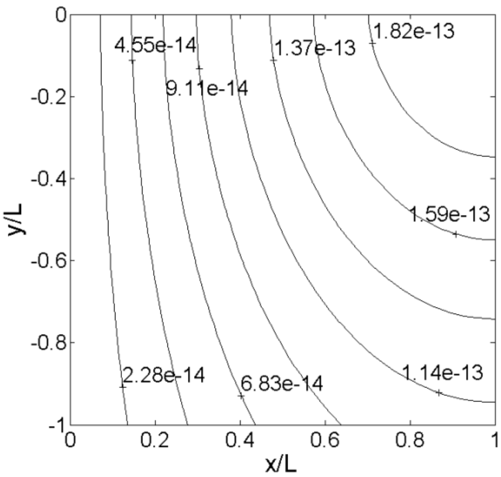


(c)

Figure 4- 2: Solid deformation lines. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.

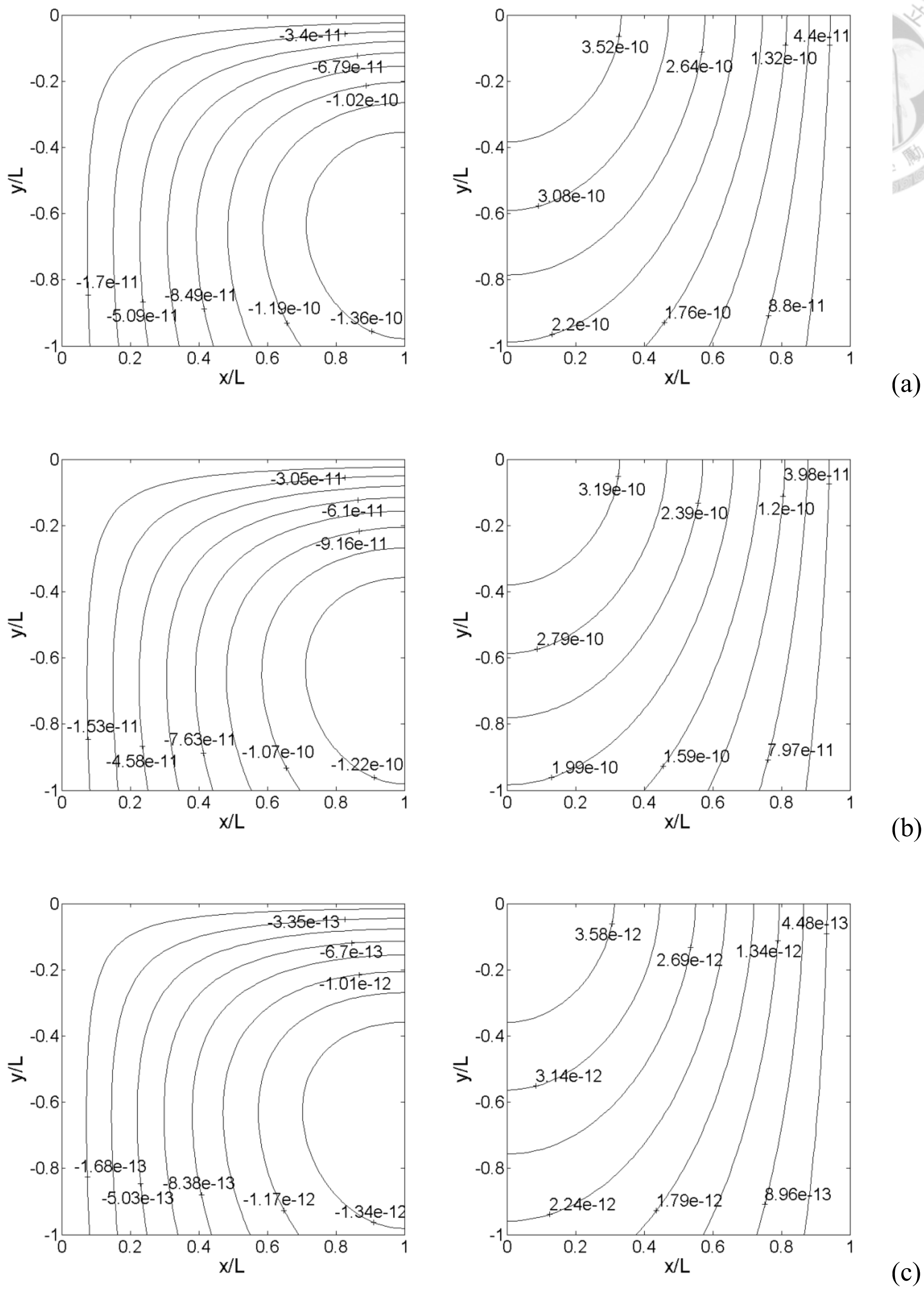


Figure 4- 3: Solid deformation contour. Left column is for d_1, right column for d_2. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.

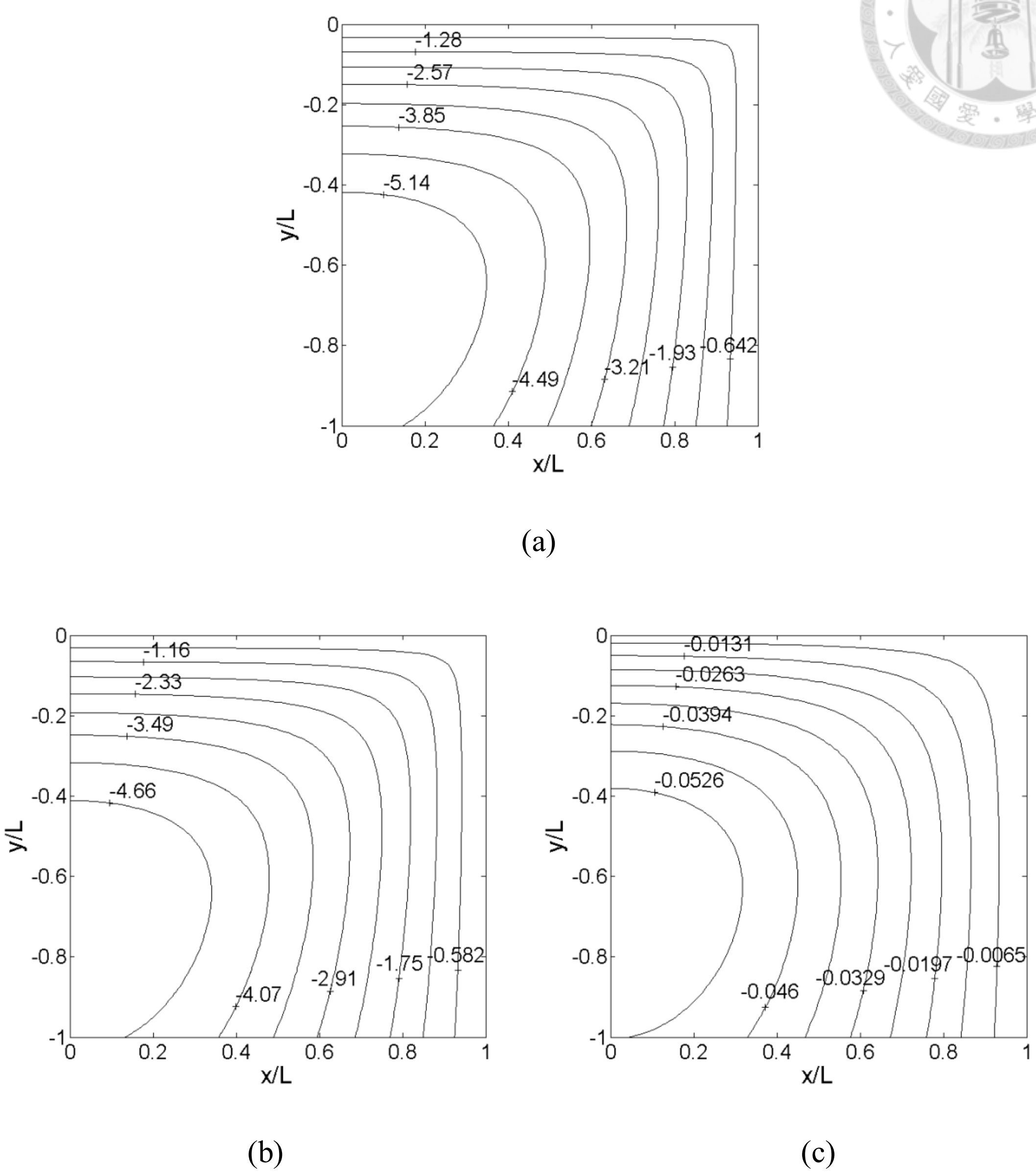
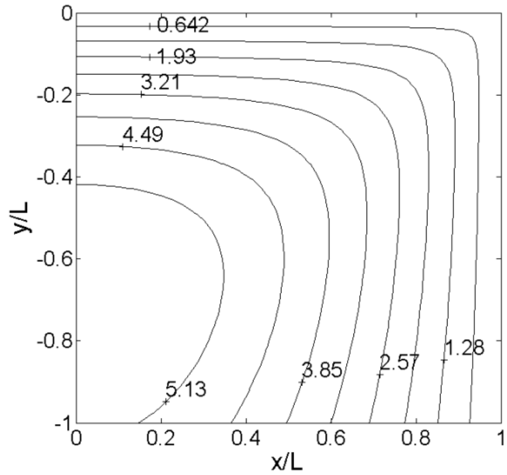
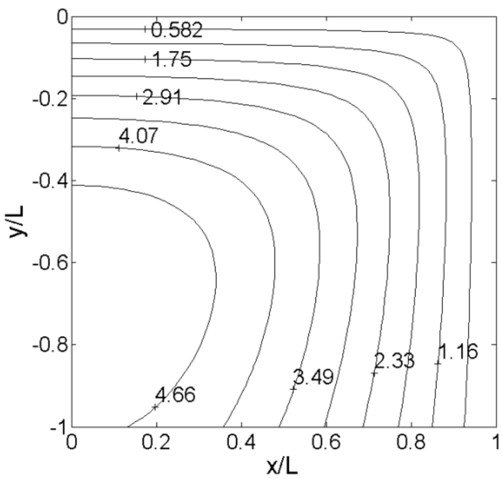


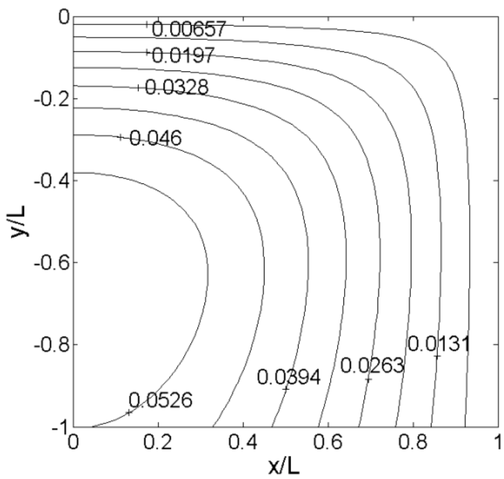
Figure 4- 4: Solid effective stress $(\tau_s)_{11}^{(2)}$. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.



(a)

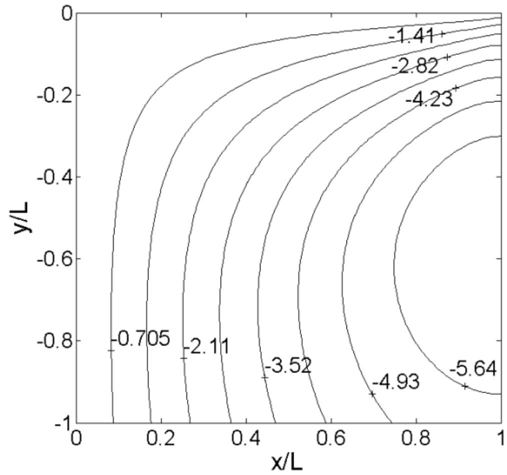


(b)

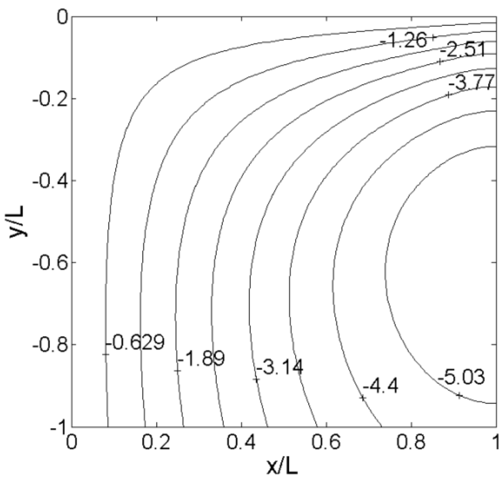


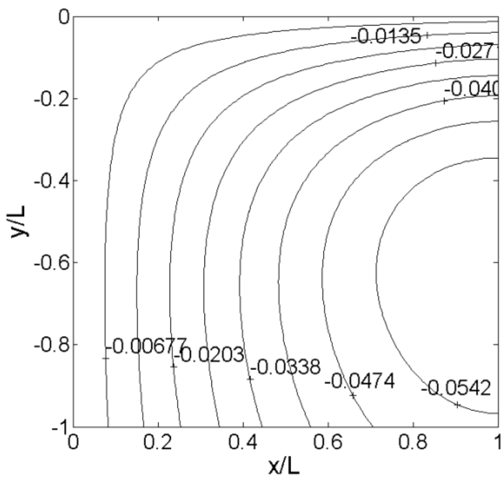
(c)

Figure 4- 5: Solid effective stress $(\tau_s)_{22}^{(2)}$. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.



(a)





(b) (c)

Figure 4- 6: Solid effective stress $(\tau_s)_{12}^{(2)}$. (a) Before the first turning point. $h/L = 10^{-3.5}$. (b) At the first turning point. $h/L = 10^{-2.7}$. (c) At the breakout point. $h/L = 10^{-1.7}$.

TABLES

Parameter	Notation	Value	Unit
object velocity	V	10^{-5}	m/s
half object length	L	0.0786	m
gap width	h	0.0786×10^{-2}	m
fluid viscosity	μ	10^{-3}	$N\cdot s/m^2$
fluid density	ρ	1000	kg/m^3

Table 1: Parameters in Chapter 2.

Parameter	Notation	Value	Unit
object velocity	V	10^{-5}	m/s
half object length	L	0.0786	m
gap width	h	$0.0786\times10^{-3.5}$ ~ $0.0786\times10^{-1.7}$	m
fluid viscosity	μ	10^{-3}	$N\cdot s/m^2$
fluid density	ρ	1000	kg/m^3
medium grain size	D_{50}	0.245	mm
permeability	k_p	6.6×10^{-11}	m^2
porosity	n_0	0.41	

Table 2: Parameters in Chapter 3.

Parameter	Notation	Value	Unit
object velocity	V	10^{-5}	m/s
half object length	L	0.0786	m
gap width	h	$0.0786\times10^{-3.5}\sim$ $0.0786\times10^{-1.7}$	m
fluid viscosity	μ	10^{-3}	$N\cdot s/m^2$
fluid density	ρ	1000	kg/m^3
medium grain size	D_{50}	0.245	mm
permeability	k_p	6.6×10^{-11}	m^2
porosity	n_0	0.41	
Lame's constant	G	5×10^8	N/m^2

Table 3: Parameters in Chapter 4.

APPENDICES

A1 Coefficients to the solutions in Chapter 2

$$A_n = -B_n,$$

$$B_n = \frac{1}{2\lambda_n}\left(C_n + D_n\right),$$

$$C_n = \frac{1}{he^{\lambda_n h}}\left(\frac{g_n}{2} - \frac{e^{-\lambda_n h} - e^{\lambda_n h}}{2\lambda}D_n\right),$$

$$D_n = \frac{2\lambda_n he^{\lambda h}}{e^{2\lambda h} + e^{-2\lambda h} - 2 - 4\lambda^2 h^2}\left[\frac{g_n\left(e^{-\lambda h} - e^{\lambda h}\right)}{2he^{\lambda h}} - \lambda_n g_n\right],$$

$$g_n = \frac{2V\sin(\lambda_n L)}{\lambda^2 L}.$$

A2 Coefficients to the solutions in Chapter 3

$A_n = n_0 E_n + n_0 G_n - B_n$,

$B_n = \dfrac{1}{2\lambda_n}(C_n + D_n + \tilde{j} G_n)$,

$C_n = D_n + \dfrac{1}{2\lambda_n}(\tilde{c} E_n + \tilde{d} G_n)$,

$D_n = \dfrac{1}{4\mu\lambda_n}[(\tilde{a} - \mu\tilde{c}) E_n - (\tilde{b} - \mu\tilde{d}) G_n]$,

$E_n = \dfrac{1}{\tilde{n}}(-\lambda_n g_n - \tilde{o} G_n)$,

$G_n = \dfrac{1}{\tilde{r}}(\lambda_n g_n \dfrac{\tilde{p}}{\tilde{n}} + g_n)$,

$g_n = \dfrac{2V \sin(\lambda_n L)}{\lambda^2 L}$,

$\tilde{a} = -\mu b + 2\mu\lambda_n^2 (n_0 - 1)$,

$\tilde{b} = 2\mu\lambda_n (\lambda_n^2 + b)^{1/2} (n_0 - 1)$,

$\tilde{c} = 2\lambda_n^2 - 2 n_0 \lambda_n$,

$\tilde{d} = 2\lambda_n^2 (1 - n_0) + b$,

$\tilde{f} = [n_0 (\lambda_n^2 + b)^{1/2} - n_0 \lambda_n] e^{-\lambda_n h}$,

$\tilde{i} = (e^{\lambda_n h} - e^{-\lambda_n h}) / 2\lambda_n$,

$\tilde{j} = -n_0 (\lambda_n^2 + b)^{1/2} + n_0 \lambda_n$,

$\tilde{k} = (h e^{\lambda_n h} - \tilde{i}) / 2\lambda_n$,

$\tilde{l} = (2\lambda_n \tilde{i} + \tilde{e}) / 4\mu\lambda_n$,

$\tilde{m} = (2\lambda_n \tilde{k} + h e^{-\lambda_n h} - \tilde{i}) / 4\mu\lambda_n$,

$\tilde{n} = \tilde{c} \cdot \tilde{i} + \tilde{l} (\tilde{a} - \mu\tilde{c})$,

$$\tilde{o} = \tilde{d} \cdot \tilde{i} + \tilde{f} + \tilde{l}(\tilde{b} - \mu\tilde{d}),$$

$$\tilde{p} = \tilde{c} \cdot \tilde{k} + n_0 e^{\lambda_n h} + \tilde{m}(\tilde{a} - \mu\tilde{c}),$$

$$\tilde{q} = \tilde{d} \cdot \tilde{k} + n_0 e^{\lambda_n h} - \tilde{i} \cdot \tilde{j} + \tilde{m}(\tilde{b} - \mu\tilde{d}),$$

$$\tilde{r} = \tilde{q} - \tilde{p} \cdot \tilde{o} / \tilde{n}.$$

A3 Coefficients to the solutions in Chapter 4

$A_n = n_0 E_n + n_0 G_n - B_n$,

$B_n = \dfrac{1}{2\lambda_n}(C_n + D_n + \tilde{j} G_n)$,

$C_n = D_n + \dfrac{1}{2\lambda_n}(\tilde{c} E_n + \tilde{d} G_n)$,

$D_n = \dfrac{1}{4\mu\lambda_n}[(\tilde{a} - \mu\tilde{c})E_n - (\tilde{b} - \mu\tilde{d})G_n]$,

$E_n = \dfrac{1}{\tilde{n}}(-\lambda_n g_n - \tilde{o} G_n)$,

$G_n = \dfrac{1}{\tilde{r}}(\lambda_n g_n \dfrac{\tilde{p}}{\tilde{n}} + g_n)$,

$I_n = \dfrac{1}{2G\lambda_n^2}[\mu(2\lambda_n^2 A_n + 2\lambda_n^2 B_n + 2\lambda_n C_n - 2\lambda_n D_n) + \dfrac{\mu n_0 (2\lambda_n^2 + \mathrm{b})}{1 - n_0} G_n - 2G\lambda_n K_n]$,

$K_n = \dfrac{1}{2(1 - n_0)G\lambda_n}[4\mu\lambda_n^2 B_n + 2\mu\lambda_n C_n - 2\mu\lambda_n D_n + \dfrac{\mu n_0}{1 - n_0}(2\lambda_n^2 + \mathrm{b} - 2\lambda_n(\lambda_n^2 + \mathrm{b})^{1/2})G_n]$,

$g_n = \dfrac{2V\sin(\lambda_n L)}{\lambda^2 L}$,

$\tilde{a} = -\mu b + 2\mu\lambda_n^2(n_0 - 1)$,

$\tilde{b} = 2\mu\lambda_n(\lambda_n^2 + b)^{1/2}(n_0 - 1)$,

$\tilde{c} = 2\lambda_n^2 - 2n_0\lambda_n$,

$\tilde{d} = 2\lambda_n^2(1 - n_0) + b$,

$\tilde{f} = [n_0(\lambda_n^2 + b)^{1/2} - n_0\lambda_n]e^{-\lambda_n h}$,

$\tilde{i} = (e^{\lambda_n h} - e^{-\lambda_n h})/2\lambda_n$,

$\tilde{j} = -n_0(\lambda_n^2 + b)^{1/2} + n_0\lambda_n$,

$\tilde{k} = (he^{\lambda_n h} - \tilde{i})/2\lambda_n$,

$\tilde{l} = (2\lambda_n\tilde{i} + \tilde{e})/4\mu\lambda_n$,

$$\tilde{m} = (2\lambda_n \tilde{k} + he^{-\lambda_n h} - \tilde{i}) / 4\mu\lambda_n \, ,$$

$$\tilde{n} = \tilde{c} \cdot \tilde{i} + \tilde{l}\,(\tilde{a} - \mu\tilde{c}) \, ,$$

$$\tilde{o} = \tilde{d} \cdot \tilde{i} + \tilde{f} + \tilde{l}\,(\tilde{b} - \mu\tilde{d}) \, ,$$

$$\tilde{p} = \tilde{c} \cdot \tilde{k} + n_0 e^{\lambda_n h} + \tilde{m}(\tilde{a} - \mu\tilde{c}) \, ,$$

$$\tilde{q} = \tilde{d} \cdot \tilde{k} + n_0 e^{\lambda_n h} - \tilde{i} \cdot \tilde{j} + \tilde{m}(\tilde{b} - \mu\tilde{d}) \, ,$$

$$\tilde{r} = \tilde{q} - \tilde{p} \cdot \tilde{o} / \tilde{n} \, .$$

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