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三維卡拉比-丘空間奇異點及模空間連結性研究

The Connectedness Problem of Calabi-Yau Moduli Spaces

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本論文係王賜聖（學號 D98221004）在國立臺灣大學數學系完成之博士學位論文，於民國 104 年 06 月 03 日承下列考試委員審查通過及口試及格，特此證明

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摘 要

本文探討在雙有理映射及形變理論的操作下，給出判別三維卡拉比–丘簇的奇異點是否為節點（即米爾諾數等於一）的條件。同時也對於 P.S. Green 和 T. Hübsch 教授的結果：在乘積射影空間裡的三維完全交集卡拉比—丘流形皆可由錐過渡變換連接，提供一個詳細的證明。

關鍵字— 卡拉比–丘；錐過渡變換

Abstract

We develop criteria for a Calabi–Yau 3-fold to be a conifold, i.e. to admit only ODPs as singularities, in the context of extremal transitions. There are birational contraction and smoothing involved in the process, and we give such a criterion in each aspect.

More precisely, given a small projective resolution $\pi : \widehat{X} \to X$ of Calabi–Yau 3-fold X, we show that (1) If the fiber over a singular point $P \in X$ is irreducible then P is a cA_1 singular point, and an ODP if and only if there is a normal surface which is smooth in a neighborhood of the fiber. (2) If the natural closed immersion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X)$ is an isomorphism then X has only ODPs as singularities.

There are topological constraints associated to a smoothing $\widetilde{X}$ of X. It is well known that $e(\widehat{X}) - e(\widetilde{X}) = 2\,|\mathrm{Sing}(X)|$ if and only if X is a conifold. Based on this and a Bertini-type theorem for degeneracy loci of vector bundle morphisms, we supply a detailed proof of the result by P.S. Green and T. Hübsch that all complete intersection Calabi–Yau 3-folds in product of projective spaces are connected through *projective* conifold transitions (known as the standard web).

Keywords— Calabi–Yau threefold; conifold transition; small contraction; determinantal contraction; standard web.

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Contents

Chapter 1

Introduction

Calabi–Yau conifolds, i.e. Calabi–Yau 3-folds with only ordinary double points (ODPs), arise naturally in algebraic geometry and string theory, where a Calabi–Yau 3-fold X is a projective Gorenstein 3-fold with at worst terminal singularities such that $K_X \sim 0$ and $H^1(\mathscr{O}_X) = 0$. M. Reid [43] had proposed to study the moduli spaces of smooth Calabi–Yau 3-folds through conifold transitions. One major question asked there is if all the moduli spaces are indeed connected through conifold transitions. This is usually referred as the *Reid's fantasy*. While non-projective conifold transitions are also considered in the literature, in this thesis we stick on the projective ones.

A special yet fundamental question arising from the connectedness problem is the following: Suppose that there is a small (extremal) transition between smooth Calabi–Yau 3-folds $\widehat{X}$ and $\widetilde{X}$. By this we mean there is a projective small contraction $\widehat{X} \xrightarrow{\pi} X$ from $\widehat{X}$ to a Calabi–Yau 3-fold X, with general terminal singularities, that X is smoothable to $\widetilde{X}$ in a flat family. It is called a conifold transition if X has such only ODPs as its singularities.

Question 1. Is it true $\widehat{X}$ can in fact be connected to $\widetilde{X}$ through a sequence of conifold transitions (through a different X of course)?

Let $\widehat{X} \to X$ be a small projective resolution of a Calabi–Yau 3-fold X. We already know that the singularities of X are terminal singularities of index 1, i.e., isolated cDV singularities. For such singularities, the dual graph of the exceptional curves and their normal bundles had been studied systematically in [27, 33, 42]. The analysis of such singularities is based on the lemma of M. Reid [42, (1.1), (1.14)], namely a general hyperplane section through the singularity has a Du Val surface singularity at it, and the inverse image E of the surface gives a partial resolution of the Du Val singularity. Notice that the normal surface E contains the exceptional curves and is a *relatively trivial* divisor. Applying a theory of rational Gorenstein surfaces to the surface E, D.R. Morrison gave a description of the dual graphs for such singularities [33, (5.5)].

Our first goal is to study such singularities in the case of Calabi–Yau 3-folds. Instead of a relatively trivial divisor we will consider a *relatively antiample* divisor. For example, a 3-fold X with only cA_1-singularities which contains a smooth surface D passing through all singularities admits a small projective resolution by blowing up X along D. Let E be the inverse image of D. It is simple to show that for any irreducible exceptional curve C the intersection number $E.C$ is the negative number $\deg_C(\mathscr{O}_E(-1)|_C)$. Thus the smooth divisor E is relatively antiample.

The first result is the following.

Theorem 1.1 (= Theorem 4.1). *Let $\pi : \widehat{X} \to X$ be a small projective resolution of a normal variety X of dimension 3. Suppose that $K_{\widehat{X}}$ is π-trivial and that there is an irreducible normal surface D in $\widehat{X}$ such that $-D$ is π-ample. If the fiber over a singular point $P \in X$ is irreducible, then the analytic type of the singular point P is*

$$x^2 + y^2 + z^2 + w^{2m} = 0 \text{ for some } m \in \mathbb{N}.$$

Furthermore, the singular point P is an ODP if and only if the surface D is smooth in a neighborhood of the fiber.

Such small projective morphism π is called a flopping contraction (cf. Definition 2.17). Remark that, for the ODP criterion in Theorem 1.1, the "if" implication is well known and frequently used in the literature (cf. Lemma 2.23). We prove that the existence of such normal surface D, which is smooth in a neighborhood of the fiber, is also a necessary condition.

The case with cA_1-singularities different from ODPs do occur (cf. Example 2.21 and Example 4.6).

Note that there are no small contractions with $-K$ being ample by Mori's classification of K-negative extremal rays on smooth projective 3-folds.

We briefly explain the idea of the proof. Notice that $\pi(D)$ is non $\mathbb{Q}$-Cartier. We will prove that it is a smooth Weil divisor by using a theory of normal Gorenstein surfaces (cf. Proposition 2.27), due to H. Laufer and F. Sakai. Applying a result of D.R. Morrison and a simple observation (cf. Lemma 2.23 and Corollary 2.32), we can show that the normal bundle of the irreducible curve $\pi^{-1}(P)$ in $\widehat{X}$ is either of type $\mathscr{O}_{\mathbb{P}^1}(-1) \bigoplus \mathscr{O}_{\mathbb{P}^1}(-1)$ or of type $\mathscr{O}_{\mathbb{P}^1} \bigoplus \mathscr{O}_{\mathbb{P}^1}(-2)$ and Theorem 1.1 follows.

Corollary 1.2 (= Corollary 4.2). *With notation as in Theorem 1.1, the scheme theoretical fiber structure on $\pi^{-1}(P)$ is reduced, that is, $\pi^{-1}m_P \cdot \mathscr{O}_{\widehat{X}} = \mathscr{I}$, where $\mathscr{I}$ is the ideal sheaf of the fiber with the reduced structure.*

There is an application of Theorem 1.1 to 3-dimensional pl flipping contractions (cf. Definition 2.18).

Corollary 1.3 (= Corollary 4.3). *Let $\pi : Y \to X$ be a pl flipping contraction for a 3-dimensional plt pair $(Y, S + B)$. If the fiber C over a singular point $P \in X$ is irreducible, Y is smooth in a neighborhood of C and $K_Y.C = 0$, then*

(1) The scheme theoretical fiber structure on $\pi^{-1}(P)$ is reduced,

(2) The analytic type of the singular point P is

$$x^2 + y^2 + z^2 + w^{2m} = 0 \text{ for some } m \in \mathbb{N}.$$

Furthermore, the singular point P is an ODP if and only if the surface S is smooth in a neighborhood of the fiber C.

In the case of Calabi–Yau 3-folds which admit small projective resolutions, the characterizations of singularities induced by irreducible fibers are:

Theorem 1.4 (= Theorem 4.4). *Let $\pi : \widehat{X} \to X$ be a small projective resolution of a Calabi–Yau 3-fold X. Then*

(1) Given a singular point $P \in X$, then the following are equivalent:

(a) The fiber over the singular point P is irreducible;

(b) The scheme theoretical fiber $\pi^{-1}(P)$ is integral;

(c) The analytic type of the singular point P is

$$x^2 + y^2 + z^2 + w^{2m} = 0 \text{ for some } m \in \mathbb{N};$$

More generally, the same conclusion holds if X is a projective Gorenstein terminal 3-fold.

(2) The singularities of X are of type cA_1 if and only if there is a smooth Weil divisor S containing the singular locus such that $\mathrm{Bl}_S X$ is $\mathbb{Q}$-factorial. In

this case, π is isomorphic to the blowing up of X along a smooth Weil divisor.

Furthermore, if S is as in part (2), the singularities of X are ODPs if and only if the normal surface $\pi^{-1}(S)$ is smooth.

We remark that $\mathrm{Bl}_S X$ is in fact smooth in this case (cf. Remark 4.5). In the literature, to construct a small projective resolution π of a conifold, one usually assumes that there is a such Weil divisor S passing through all of the ODPs. Conversely, our result proves the existence of such smooth Weil divisors.

There is an example of Calabi–Yau 3-folds with cA_1-singularities different from ODPs which admits a small projective resolution (cf. Example 4.6). The projectivity assumption in Theorem 1.4 plays a key technical role (cf. Example 2.22).

Note that, for an isolated cDV singularity which admits a small resolution with an irreducible exceptional set, it was classified in [20, Main Theorem] into three types: cA_1, cD_4 and cE_n. The key difficulty in the proof of Theorem 1.4 is thus to show that such small contractions contract its irreducible fibers to cA_1 singularities. We can prove that there are such normal surfaces (as in Theorem 1.1) by a Bertini-type theorem for normality (c.f Proposition 2.39). Nakamaye's theorem on augmented base loci (cf. Proposition 2.35) plays an essential role in the proof.

In the second part of the thesis, we discuss how to relate projective small transitions to conifold transitions (cf. Section 4.2). There is a simple ODPs criterion (involved topological constraints) for small transitions.

Proposition 1.5 (= Proposition 4.8)**.** *Let $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ be a small transition. Then the difference of the topological Euler numbers $e(\widehat{X}) - e(\widetilde{X})$ equals the number $2\,|\mathrm{Sing}(X)|$ if and only if the singularities of X are ODPs.*

We will also introduce the primitive small transitions (cf. Definition 4.9) and prove the following result:

Theorem 1.6 (= Theorem 4.10)**.** *Let $\pi : \widehat{X} \to X$ be a small resolution of a Calabi–Yau 3-fold X. If the natural closed immersion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X)$ of Kuranishi spaces is an isomorphism then the singularities of X are ODPs. Moreover, the number of ODPs is equal to the relative Picard number $\rho(\widehat{X}/X)$.*

Theorem 1.6 is a generalization of the case of relative Picard number one which have been studied in [15, (5.1)]. Using the deformation properties of X and $\widehat{X}$ and the minimal model theory, we will prove it by induction on the relative Picard number.

Corollary 1.7 (= Corollary 4.12)**.** *Let $\pi : \widehat{X} \to X$ be a small resolution of a Calabi–Yau 3-fold X. Suppose that, for any Calabi–Yau 3-fold $\widehat{X}'$ which is birationally equivalent to $\widehat{X}$ and any factorization $\widehat{X}' \to X' \to X$ with $X' \neq \widehat{X}'$, the Calabi–Yau 3-fold X' is not smoothable. Then the singularities of X are ODPs. Moreover, the number of ODPs is equal to the relative Picard number $\rho(\widehat{X}/X)$.*

Remark that $\widehat{X}$ and $\widehat{X}'$ are connected by a sequence of flops [21, 23].

In final chapter, we review the result of P.S. Green and T. Hübsch and the construction of determinantal contractions between complete intersection Calabi–Yau (CICY) configuration matrices. The fibers of determinantal contractions are irreducible (cf. Remark 5.15) and thus these contractions provide a lot of examples of small projective resolution of Calabi–Yau 3-folds with the exceptional set being a disjoint union of irreducible rational curves.

Theorem 1.8. *(= Theorem 5.18) Any two (parameter spaces of) complete intersection Calabi–Yau 3-folds in products of projective spaces are connected by a finite sequence of conifold transitions.*

In [17, §3 p.435], the authors deferred the proof of the existence of (projective) conifold transitions to a forthcoming paper, which unfortunately has not yet been available.

By analysing the normal bundle of (irreducible) 1-dimensional fibers of determinantal contractions, we have inferred that all singularities are cA_1-singularities (cf. Theorem 1.4). To prove the connectedness theorem of moduli spaces of CICY 3-folds as discovered by P.S. Green and T. Hübsch, we have to show that the singularities appeared in determinantal contractions are ODPs. It turns out that we need only use the more topological criterion Proposition 1.5 rather than the geometric criterion Theorem 1.4 to deduce Theorem 1.8.

Following the outline sketched in [3, 17], we will prove the existence of determinantal contractions between CICY configuration matrices by using another Bertini-type theorem for degeneracy loci of morphisms of vector bundles (cf. Proposition 2.36). According to a description of degeneracy loci and explicit computations (cf. Proposition 5.4 and Corollary 5.16), Theorem 1.8 then follows from Proposition 1.5.

We hope that Theorem 1.4 and Theorem 1.6 will be useful for further studies on primitive small transitions.

The contents of this paper are organized as follows.

Chapter 1 briefly describes some motivations and the results of this thesis.

Chapter 2 contains some definitions and materials needed for our proof of main results. We recall the definitions of singularities and their small (and crepant) resolutions in Section 2.1, 2.2. In Section 2.3, the theory of normal Gorenstein surfaces, due to H. Laufer, D.R. Morrison and F. Sakai, are reviewed. Section 2.4, 2.5 give Nakamaye's theorem for augmented base locus and Bertini-type theorems for vector bundle and the normality on smooth varieties. In Section 2.6, we review the mixed Hodge Structures on varieties with normal crossings following the point of view of Griffiths and Schmid and prove a Lefschetz-type theorem for *reducible* ample hyperplane sections.

In Chapter 3, we establish the main result, the characterizations of cA_1 singularities in the case of Calabi–Yau 3-folds which admit small projective resolutions, it also contains further discussions on conifold transitions and their relations to the primitive small transitions.

In Chapter 4, we review the definitions and basic results on CICY configuration matrices and the formal correspondence between such matrices, and give a proof of the connectedness of parameter spaces of Calabi–Yau complete intersections.

Chapter 2

Preliminaries

2.1 Singularities

Definition 2.1. Let (X, Δ) be a pair, that is, X is a normal variety and $\Delta = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X, where D_i are distinct, irreducible and $0 \leqslant d_i \leqslant 1$, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a birational morphism from a normal variety Y. Then we can write

$$K_Y \equiv f^*(K_X + \Delta) + \sum a(E, X, \Delta)E,$$

where the sum runs over all the distinct prime divisors $E \subseteq Y$, and $a(E, X, \Delta)$ is a rational number. We define

$$\operatorname{discrep}(X, \Delta) := \inf\{a(E, X, \Delta) \mid E \text{ is exceptional over } X\}.$$

We say that (X, Δ) is

$$\left.\begin{array}{c}\text{terminal}\\ \text{canonical}\\ \text{klt}\\ \text{plt}\end{array}\right\} \text{if } \operatorname{discrp}(X, \Delta) \left\{\begin{array}{l} > 0\\ \geqslant 0\\ > -1 \text{ and } \lfloor \Delta \rfloor = 0\\ > -1\end{array}\right. .$$

Here klt is an abbreviation for *Kawamata log terminal* and plt for *purely log terminal*. If $\Delta = 0$ then the notations klt and plt coincide and in this case we say that X has *log terminal* singularities.

The rational number $a(E, X, \Delta)$ is called the discrepancy of E with respect to (X, Δ). Note that it only depends on the valuation of the function field $K(X)$, corresponding to the discrete valuation ring $\mathscr{O}_{E,Y} \subseteq K(X)$.

The *index* of a singularity is the smallest r for which rK_X is Cartier in a neighbourhood of the singularity. If $r = 1$ and it is Cohen-Macaulay, then it is called Gorenstein.

Example 2.2 (The surface case). Let S be a surface, let $\lambda : \widetilde{S} \to S$ be a minimal resolution and $\{E_i\}$ the family of all exceptional divisors. Since the intersection matrix $(E_i.E_j)$ is negative definite, the surface S is terminal if and only if it is smooth.

In the case of canonical singularities, we have $K_{\widetilde{S}} = \lambda^* K_S$. Namely, the canonical surface singularities are the Du Val singularities [26, (4.20)], which are analytically isomorphic to isolated hypersurface singularities defined by one of the equations

$$\begin{aligned}
&A_n : x^2 + y^2 + z^{n+1} = 0 \quad (n \geqslant 1);\\
&D_n : x^2 + y^2 z + z^{n-1} = 0 \quad (n \geqslant 4);\\
&E_6 : x^2 + y^3 + z^4 = 0;\\
&E_7 : x^2 + y^3 + yz^3 = 0;\\
&E_8 : x^2 + y^3 + z^5 = 0.
\end{aligned} \tag{2.1.1}$$

For log terminal surface singularities (or more general surface singularities) see [26, Section 4.1].

Remark 2.3. Let (X, Δ) be an n-dimensional plt pair. Then X has rational singularities, i.e., for any resolution $f : Y \to X$ we have $R^i f_* \mathscr{O}_Y = 0$ for $i > 0$ [26, (5.22)]. For the case of surfaces, it is a rational Gorenstein singularity if and only if it is a Du Val singularity. In general, we call a Gorenstein point $p \in X$ elliptic if $R^{n-1} f_* \mathscr{O}_Y$ is a 1-dimensional vector space at p.

We recall the following proposition due to M. Reid [41]. The statements are taken from [26, (5.30), (5.35)].

Proposition 2.4 ([41, 26]). *Let (X, p) be an index* 1 *canonical* 3*-fold singularity.*

(1) If $p \in H \subseteq X$ is a general hypersurface section, then (H, p) is either a Du Val or an elliptic singularity.
(2) The following are equivalent:
   (a) The general hypersurface section $p \in H \subseteq X$ is elliptic.

(b) *If $f : Y \to X$ is any resolution of singularities then there is a divisor $E \subseteq f^{-1}(p)$ such that $a(E, X) = 0$.*

Definition 2.5. A point $p \in X$ is called a compound Du Val singularity (or cDV singularity) if a general hypersurface section $p \in H \subseteq X$ is a Du Val surface singularity, that is, it is analytically equivalent to a three-dimensional hypersurface singularity given by an equation of the form

$$f(x, y, z) + tg(x, y, z, t) = 0$$

where f is the equation of a Du Val singularity (as in (2.1.1)), and g is an arbitrary polynomial. We will say that $p \in X$ is a cA_n, cD_n, cE_6, cE_7 or cE_8 singularity to specify the general hypersurface section through p.

Example 2.6. A singularity $p \in X$ is a cA_1 if and only if it is analytically equivalent to $x^2 + y^2 + z^2 + w^n = 0$ for some $n \in \mathbb{N}$, denoted it by $A_1(n - 1)$. If $n = 2$, it is called an ordinary double point or ODP for short.

According to Example 2.2 and Proposition 2.4 (1), it follows that an index 1 terminal 3-fold singularity is isolated (in general, a terminal variety is smooth in codimension two [26, (5.18)]). Indeed, we have a characterization of such singularities, which is a consequence of Proposition 2.2.

Corollary 2.7. *[42, (1.1)] Let (X, p) be a 3-fold singularity. Then (X, p) is terminal of index* 1 *if and only if it is an isolated cDV singularity.*

The following proposition will be used in the proof of Theorem 4.4.

Proposition 2.8. *Let X be a projective Gorenstein* 3*-fold with only terminal singularities. Then for a general hypersurface section $H \supseteq \mathrm{Sing}(X)$, the surface singularity (H, p) is a Du Val singularity for all $p \in \mathrm{Sing}(X)$.*

Proof. The idea is to find a general hypersurface section though all singular points of X in the sense of Definition 2.5 in [41].

Set

$$\mathscr{I}_1 := \bigoplus\nolimits_{p \in \mathrm{Sing}(X)} m_p$$

and

$$\mathscr{I}_2 := \bigoplus\nolimits_{p \in \mathrm{Sing}(X)} m_p / m_p^2,$$

where m_p is the maximal ideal of $\mathscr{O}_{X,p}$. There is a canonical surjective morphism from $\mathscr{I}_1$ to $\mathscr{I}_2$ and let $\mathscr{K}$ be the kernel of this morphism.

Choose a sufficiently ample divisor A on X such that $A \otimes \mathscr{I}_1$ and $A \otimes \mathscr{I}_2$ are generated by global sections and $H^1(X, A \otimes \mathscr{K}) = 0$, Then the composition map

$$H^0(X, A \otimes \mathscr{I}_1) \otimes \mathscr{O}_X \twoheadrightarrow H^0(X, A \otimes \mathscr{I}_2) \otimes \mathscr{O}_X \twoheadrightarrow A \otimes \mathscr{I}_2 \tag{2.1.2}$$

is surjective.

Let $A \otimes \mathscr{I}_1$ be generated by global sections $s_1, \cdots, s_n$. For each $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n$, a global section $\Sigma_{i=1}^n \alpha_i s_i$ defines a hypersurface section $H_\alpha \subseteq X$ containing $\mathrm{Sing}(X)$. By the Bertini theorem, a general hypersurface section $H \supseteq \mathrm{Sing}(X)$ is irreducible and smooth away from the singularities of X, i.e., there is a Zariski open set U_0 in $\mathbb{C}^n$ such that, for $\alpha \in U_0$, H_α is irreducible and $\mathrm{Sing}(H_\alpha) \subseteq \mathrm{Sing}(X)$. Note that H_α is a normal surface since X is Cohen–Macaulay (by Serre's criterion).

For each $p \in \mathrm{Sing}(X)$, let $V_p \subseteq m_p$ be a vector space which is the image of the subspace generated by $s_1, \cdots, s_n$ under the natural map

$$H^0(X, A \otimes \mathscr{I}_1) \otimes \mathscr{O}_X \to A \otimes \mathscr{I}_1 \to m_p.$$

By the surjectivity of the morphism (2.1.2), the vector space $V_p \subseteq m_p$ is mapped onto m_p/m_p^2. Hence an element in V_p gives a hypersurface section through the point p (in the sense of [41, (2.5)]). From Proposition 2.4 (2) and that $p \in X$ is a terminal singularity, a general hypersurface section through the singularity p is Du Val. That is, there is a Zariski open set $U_p \subseteq \mathbb{C}^n$ such that (H_α, p) is a Du Val singularity for $\alpha \in U_p$. Let U be the intersection of all U_p and U_0. Then for $\alpha \in U$, H_α is a Du Val surface with $\mathrm{Sing}(H) = \mathrm{Sing}(X)$. This completes the proof. □

2.2 Small Birational Morphisms

Let $\pi : Y \to X$ be a birational morphism. The π-exceptional set $\mathrm{Exc}(\pi)$ is the set of points in Y where π is not a local isomorphism.

Proposition 2.9. *[6, (1.40)] Let $\pi : Y \to X$ be a birational morphism of a normal variety X. If X is $\mathbb{Q}$-factorial, then every irreducible component of* $\mathrm{Exc}(\pi)$ *has codimension one in Y*

Proof. Let $y \in \mathrm{Exc}(\pi)$ and $x = \pi(y)$. If we can find a codimension one component of $\mathrm{Exc}(\pi)$ through y then the proposition follows immediately.

Identify the function fields $K(X)$ and $K(Y)$ via the isomorphism π^*, so that $\mathscr{O}_{X,x}$ is a proper subring of $\mathscr{O}_{Y,y}$. Pick an element $t \in m_{Y,y} \backslash \mathscr{O}_{X,x}$, and write $\operatorname{div}(t) = D_1 - D_2$ where D_i's are effective divisors without common components.

Since X is $\mathbb{Q}$-factorial, there exists a $m \in \mathbb{N}$ such that mD_i's are Cartier divisors, hence define elements u and v of $\mathscr{O}_{X,x}$ such that $t^m = uv^{-1}$ in $K(Y)$. Obviously, $u = t^m v$ belongs to $m_{Y,y} \cap \mathscr{O}_{X,x} = m_{X,x}$. We claim that v is also in $m_{X,x}$. If otherwise, then $t^m = uv^{-1}$ would be in $\mathscr{O}_{X,x}$. Therefore $t \in \mathscr{O}_{X,x}$, since $\mathscr{O}_{X,x}$ is integral closed, a contradiction.

The equations $u = v = 0$ define a subscheme Z containing x, which has codimension two in some neighborhood of x (it is the intersection of the codimension one subschemes mD_1 and mD_2). On the other hand, $\pi^{-1}(Z)$ is defined by $t^m v = v = 0$, hence by the sole equation $v = 0$. It has codimension one in Y and thus is contained in $\operatorname{Exc}(\pi)$. □

Definition 2.10. A birational morphism is called small if the exceptional set has codimension at least two.

Small projective resolutions play important roles in this thesis. By Proposition 2.9, if we are interested in small projective birational morphism $f : Y \to X$ then X is forced to be non $\mathbb{Q}$-factorial. In general, there is small $\mathbb{Q}$-factorializations for projective terminal 3-folds.

Theorem 2.11. *[21, (4.5)] Let X be a projective 3-fold with terminal singularities. Then there is a small projective birational morphism $f : Y \to X$ such that Y is $\mathbb{Q}$-factorial with at most terminal singularities. The morphism f is said to be a (small) $\mathbb{Q}$-factorialization of X.*

Another way to construct small projective morphisms is:

Lemma 2.12. *Let X be a Cohen–Macaulay variety of dimension n with only isolated singularities. Suppose that the embedding dimension of X at each singularity is $n+1$ and there is a Gorenstein prime divisor D which is not $\mathbb{Q}$-Cartier and contains a subset S of $\operatorname{Sing}(X)$.*

Let $\pi : \mathrm{Bl}_D X \to X$ be the blowing-up of X with center D. Then $\pi^{-1}(s) \cong \mathbb{P}^1_{k(s)}$ for all $s \in S$. In particular, the fiber over s is irreducible.

Proof. Pick a singular point $s \in S$. Since the embedding dimension at s is $n+1$, there is a $(n+1)$-dimensional regular affine scheme M, an neighborhood U containing only the singularity s, and a closed immersion $U \hookrightarrow M$.

Note that π is an isomorphism over $U \backslash \{s\}$. Let $\mathscr{I}$ be the defining ideal of $D \cap M$ in M. Since the codimension of $\mathscr{I}_s$ in $\mathscr{O}_{M,s}$ is 2 and $\mathscr{O}_{D,s}$ is Gorenstein, the ideal $\mathscr{I}_s$ is generated by a regular sequence f_1, f_2 by [8, (21.20)].

Let $\widetilde{M} = \mathrm{Bl}_{D \cap M} M$. Consider the diagram

$$\begin{tikzcd} \iota^{-1}(\widetilde{M}) \arrow[r, hook] \arrow[d] & \widetilde{M} \arrow[d] \\ \operatorname{Spec} \mathscr{O}_{M,s} \arrow[r, hook, "\iota"] & M. \end{tikzcd}$$

Then $\iota^{-1}(\widetilde{M})$ is the blowing-up of Spec $\mathscr{O}_{M,s}$ along $\mathscr{I}_s$ and

$$\iota^{-1}(\widetilde{M}) \cong \operatorname{Proj} \mathscr{O}_{M,s}[X, Y]/(f_2 X - f_1 Y)$$

(see e.g. [9, IV-25]). Hence the fiber over s is isomorphic to $\mathbb{P}^1_{k(s)}$.

According to the commutative diagram

$$\begin{tikzcd} Bl_{D \cap U} U \arrow[r, hook] \arrow[d, "\pi"] & \widetilde{M} \arrow[d] \\ U \arrow[r, hook] & M, \end{tikzcd}$$

it follows that $\pi^{-1}(s)$ is isomorphic to $\mathbb{P}^1_{k(s)}$. This completes the proof. $\square$

A *crepant* contraction is a proper morphism $\pi : Y \to X$ of normal verities with connected fibers such that K_Y is a pullback of a Catier divisor for X. Any small resolution $\pi : Y \to X$ of a Gorenstein 3-fold X is crepant ,i.e., $K_Y = \pi^* K_X$, because the exceptional set contains no divisors.

Remark 2.13. If X admits a crepant resolution then all discrepancies are zero and hence it has canonical singularities. If X admits a small resolution π then, by definition, X has terminal singularities.

Proposition 2.14. *[1, (5.4)] Let $\pi : Y \to Z$ be a crepant contraction from a smooth projective variety Y to an affine normal variety Z. Assume that the fiber $F = \pi^{-1}(z)_{red}$, with reduced structure, is locally complete intersection with the conormal bundle $N^\vee_{F/Y} = \mathscr{I}_F / \mathscr{I}_F^2$. Suppose moreover that the blow up $\beta : \widehat{Y} \to Y$ of Y along F has log terminal singularities. By $\widehat{F}$ we denote the exceptional divisor of the blow-up. Then the following conditions are equivalent:*

(1) $N^\vee_{F/Y}$ *is generated by global sections on* F*;*
(2) $\mathscr{O}_{\widehat{Y}}(-\widehat{F})$ *is generated by global sections at any point of* $\widehat{F}$*;*
(3) $\pi^{-1}m_z \cdot \mathscr{O}_Y = \mathscr{I}_F$ *or, equivalently, the scheme theoretic fiber structure of* F *is reduced and contains no embedded components.*

Proof. First we note that $\pi_*\mathscr{I}_F = m_z \subseteq \mathscr{O}_Z$ and the scheme theoretic structure on $\pi^{-1}(P)$ is defined by the ideal sheaf which is the image of the evaluation $\pi^{-1}\pi_*\mathscr{I}_F \to \mathscr{I}_F$ [1, (5.1)].

Claims (2) and (3) are equivalent because

$$\beta^{-1}\mathscr{I}_F = \mathscr{O}_{\widehat{Y}}(-\widehat{F}) \text{ and } \beta_*\mathscr{O}_{\widehat{Y}}(-\widehat{F}) = \mathscr{I}_F.$$

The implication (2) $\Rightarrow$ (1) is obvious. To prove the converse implication, we consider a short exact sequence

$$0 \to \mathscr{O}_{\widehat{Y}}(-2\widehat{F}) \to \mathscr{O}_{\widehat{Y}}(-\widehat{F}) \to \mathscr{O}_{\widehat{F}}(-\widehat{F}) \to 0.$$

Since, by assumption, $-2\widehat{F} - K_{\widehat{Y}} = -(\dim Y - \dim F + 1)\widehat{F} - \beta^*K_Y$ is $(\pi \circ \beta)$-big and nef and $\widehat{Y}$ has log terminal singularities it follows that $H^1(\widehat{Y}, \mathscr{O}_{\widehat{Y}}(-2\widehat{F})) = 0$ and global sections of $\mathscr{O}_{\widehat{F}}(-\widehat{F})$ extends to $\widehat{Y}$. Hence any section of $N^\vee_{F/Y}$ extends to a function in $H^0(Y, \mathscr{O}_Y)$ vanishing along F, that is the natural map

$$\begin{aligned} \pi^! : m_z &\to H^0(F, N^\vee_{F/Y}) = H^0(\widehat{F}, \mathscr{O}_{\widehat{F}}(-\widehat{F})) \\ f &\longmapsto (y \mapsto [f \circ \pi] \in (\mathscr{I}_F/\mathscr{I}_F^2)_y) \end{aligned}$$

is surjective. □

We will use the shorthand $(a, b) \in \mathbb{Z}^2$ for the vector bundle of rank two $\mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b)$ on $\mathbb{P}^1$.

Example 2.15. Suppose that X is a 3-fold with only k ODPs. Let $\widetilde{X}$ denote the blowup of X in all its singular points; $\varphi : \widetilde{X} \to X$ is a smooth projective 3-fold with k exceptional divisors $E_1, \cdots, E_k$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. There are two projections $\varphi_i^\pm : E_i \to \mathbb{P}^1$, rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, corresponding to a choice of $\mathbb{P}^1$ factor. Then the fibres of every $\varphi_i^\pm$ can be blown-down to yield a 3-dimensional Moishezon manifold Z, i.e. a compact complex manifold with three algebraically independent meromorphic functions. Thus we obtain 2^k Moishezon small resolutions Z of the X in which each ODP has been replaced by a smooth rational curve with normal bundle $(-1, -1)$.

Suppose that X is a terminal 3-fold which admits a small resolution $\pi : \widehat{X} \to X$. Let $\widetilde{X}$ denote the blowup of $\widehat{X}$ in all its exceptional curves over ODPs. Then we obtain a Moishezon small partial resolution $f : Z \to X$ factoring $\widetilde{X} \to X$ [12, 35].

We recall the definition of flipping contractions.

Definition 2.16. Let X be a normal variety and D a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. A $(K+D)$-*flipping contraction* is a projective birational morphism $f : X \to Z$ to a normal variety Z such that f is small and $-(K_X + D)$ is f-ample.

A $(K+D)$-*flip* of f is a small projective birational morphism $f^+ : X^+ \to Z$ such that $(K_{X^+} + D^+)$ is $\mathbb{Q}$-Cartier and f^+-ample, where D^+ is the proper transform of D in X^+.

Note that if $\rho(X/Z) = 1$ then the $(K+D)$-flip does not depend on the choice of D [26, (6.5)]. In this case we call $f^+ : X^+ \to Z$ the *flip* of f.

Definition 2.17. Let X be a normal variety. A flopping contraction is a projective birational morphism $f : X \to Z$ to a normal variety Z such that f is small and K_X is (numerically) f-trivial.

If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $-(K_X + D)$ is f-ample, then the $(K+D)$-flip of f is also called the *D-flop*.

Definition 2.18. Let $(X, \Delta = S + B)$ be a plt pair. The notation means that $\lfloor \Delta \rfloor = S$ is a prime Weil divisor and B a $\mathbb{Q}$-divisor having no component in common with S. A *pl (prelimiting) contraction* is a $(K+S+B)$-flipping contraction $f : X \to Z$ such that $-S$ is f-ample, X is $\mathbb{Q}$-factorial and $\rho(X/Z) = 1$.

Remark 2.19. This definition is more restrictive than the Shokurov's definition of pl flipping contraction. We adopt the above Definition, which follows BCHM, in this thesis.

Finally, let (X, x) be the germ of an isolated Gorenstein 3-fold singularity with a small resolution $\pi : \widehat{X} \to X$. It is a terminal singularity (cf. Remark 2.13).

The following proposition describes the structure of $\pi^{-1}(x)$.

Proposition 2.20. *[33, (5.5)] Denote the exceptional set $\pi^{-1}(x) = \bigcup C_i$ with reduced structure. Then the $C_i \cong \mathbb{P}^1$ meet transversally. Moreover, the normal bundle of C_i in $\widehat{X}$ is either $(-1,-1)$, $(0,-2)$ or $(1,-3)$.*

We remark that three branches are allowed to meet at a point, with the singularity of $\pi^{-1}(x)$ analytically of the form $\operatorname{Spec}\mathbb{C}[[x,y,z]]/(xy,xz,yz)$.

Example 2.21. Suppose that $\pi : \widehat{X} \to X$ is irreducible, i.e. the exceptional curve C is irreducible. The singularity is an ODP if and only if the normal bundle is $(-1,-1)$.

We recall a simple example for the case $(0,-2)$: Let $(X,0) \subseteq (\mathbb{C}^4,0)$ be a cA_1-singularity which is defined by $xy = u(v^2-u)$. It admits a small resolution $\pi : \widehat{X} \to X$ which is given by the blowing up of the plane S defined by $x = u = 0$. We remark that the inverse image of S is a normal surface with an isolated singularity $0 \in \pi^{-1}(S)$.

In general, the normal bundle $N_{C/\widehat{X}}$ is $(0,-2)$ if and only if it is a cA_1 singularities

$$x^2+y^2+z^2+w^{2m}=0$$

for some $m \in \mathbb{N}$ [27, p. 273] or [42, Section 5].

Example 2.22. By the method of Laufer, we can construct an example for the case $(1,-3)$ (cf. [27, Example 2.3.], [40, Example 10.]). Consider the hypersurface singularity $(X,0) \subseteq (\mathbb{C}^4,0)$

$$x^2+z^3+(y^2z-w^2)y=0.$$

It is easy to see that its general hypersurface section is the Du Val singularity D_4 (cf. Example 2.1.1), i.e., $(X,0)$ is a cD_4-singularity. The blow up $Y \to X$ of the ideal I generated by the 2×2 minors of

$$\begin{pmatrix} y & z & w & x \\ -w & -x & yz & z^2 \end{pmatrix}$$

resolve the singularity of X, so that Y is smooth, and the exceptional locus of the blow up is a $\mathbb{P}^1$ with normal bundle $(1,-3)$.

The following lemma is a criterion for ODPs.

Lemma 2.23. *Let $\pi : Y \to X$ be a small resolution of a 3-fold X and let C be an irreducible exceptional curve. Suppose that there is a smooth surface S in Y such that $S \supseteq C$ and $\pi(C) \in \pi(S)$ is a smooth surface point. Then the normal bundle of C in Y is $(-1,-1)$.*

Proof. Consider the normal bundle exact sequence

$$0 \to N_{C/S} \to N_{C/Y} \to N_{S/Y}|_C \to 0. \tag{2.2.1}$$

Since S is a smooth surface with $\pi(C) \in \pi(S)$ being a smooth point, the normal bundle $N_{C/S}$ is $\mathscr{O}_{\mathbb{P}^1}(-1)$. By Lemma 2.20 and the short sequence (2.2.1), we get

$$\deg N_{S/Y}|_C = \deg N_{C/Y} + 1 = -1,$$

where the $\deg N_{C/Y} \in \mathbb{Z}$ is the degree of its determinant line bundle. According to the fact $\mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(-1), \mathscr{O}_{\mathbb{P}^1}(-1)) = 0$, it follows that the exact sequence (2.2.1) is split and thus the normal bundle $N_{C/Y}$ is $(-1,-1)$. □

The following technical lemma will be used in the sequel.

Lemma 2.24. *Let $C = \bigcup C_i$ be a curve in a smooth 3-fold Y such that the irreducible components C_i meet in a finite set of points. Then there exsits an injection $\bigoplus_i H^2_{C_i}(Y, \Omega^2_Y) \hookrightarrow H^2_C(Y, \Omega^2_Y)$. Moreover, it is an isomorphism if C_i are mutually disjoint.*

Proof. By induction on the number of components of C, we may assume that $C = C_1 \cup C_2$. From the Mayer–Vietoris sequence, we get

$$H^2_{C_1 \cap C_2}(\Omega^2_Y) \to H^2_{C_1}(\Omega^2_Y) \oplus H^2_{C_2}(\Omega^2_Y) \to H^2_C(\Omega^2_Y).$$

Since Ω^2_Y is locally free and $\mathrm{depth}_{C_1 \cap C_2} \mathscr{O}_Y = 3$, we have the result. □

2.3 Normal Gorenstein Surfaces

This section is developed to prove Corollary 2.32, which plays an important role in the proof of our main result. We first recall a theory of normal Gorenstein surfaces [44, 45, 46].

Let S be a normal surface and let π be a resolution of S. The *inverse image* π^*C of a $\mathbb{Q}$-Weil divisor C is defined to be $\widetilde{C} + \sum \alpha_i E_i$ where $\widetilde{C}$ is the proper transform of C, those curves E_i are contracted by π and $\alpha_i \in \mathbb{Q}$ are determined by the relations: $(\widetilde{C} + \sum \alpha_i E_i).E_j = 0$ for all j (cf. [26, (4.1)], [45]). Even if C is integral, π^*C is in general a $\mathbb{Q}$-divisor. For two $\mathbb{Q}$-Weil divisor C and C', the intersection number $C.C' \in \mathbb{Q}$ is defined by $\pi^*C.\pi^*C'$.

Definition 2.25. An irreducible curve C on S is called an exceptional curve of the first kind if $K_S.C < 0$ and $C^2 < 0$.

Remark 2.26. From [46, (1.1)], the proper transform of an exceptional curve of the first kind on S by the minimal resolution of S is a (-1)-curve, where a $(-k)$-curve on a smooth surface means a smooth rational curve with self-intersection $-k$.

Proposition 2.27. *[31, 33, 46] Let S be a normal Gorenstein surface, $\lambda : \widetilde{S} \to S$ the minimal resolution and C an exceptional curve of the first kind on S.*

(1) The inverse image $\lambda^{-1}(C)$ consists of a chain of (-2)-curves and (-1)-curve $\widetilde{C}$ with the following dual graph Γ_n $(n \geqslant 1)$:

$$\overbrace{\underset{C_1}{\bullet} - \underset{C_2}{\bullet} - \cdots - \underset{C_{n-1}}{\bullet}}^{n-1} - \overset{\widetilde{C}}{\circ} \tag{2.3.1}$$

where $\bullet$ denotes a (-2)-curve C_i, $C^2 = -1/n$ and $\widetilde{C}$ is the proper transform of C.

(2) Let T be a normal surface and let $\pi : S \to T$ be a proper birational morphism. Suppose that the irreducible curve C is the fiber over a point $P \in T$. Then P is a smooth point of T.

We remark that the statement (1) was known in [46, Example 1.2] and (2) in [33, (1.2)], [31, (0.1)]. For the convenience of the reader, we supply a proof here.

Proof. (1) First we note that C meets only Du Val singularities of S. Indeed, there is a unique effective integral divisor Δ supported in $\lambda^{-1}(\mathrm{Sing}(S))$ such that $\lambda^* K_S = K_{\widetilde{S}} + \Delta$. Given $P \in C \cap \mathrm{Sing}(S)$, let Δ_P be the component of Δ which is supported in $\lambda^{-1}(P)$. Since Δ is integral, if $\Delta_P > 0$, we have

$$K_S.C = K_{\widetilde{S}}.\widetilde{C} + \Delta.\widetilde{C} = -1 + \Delta.\widetilde{C} \geqslant 0,$$

a contradiction. Then $\Delta_P = 0$ and thus the assertion holds by [44, p.1235].

From Remak 2.26, the proper transform $\widetilde{C}$ is a (-1)-curve. If $\widetilde{C}$ meets an irreducible component E of $\mathrm{Exc}(\lambda)$, then $(\widetilde{C} + E)^2 = 2(\widetilde{C}.E) - 3$ (since $E^2 = -2$). The negative definiteness of $\lambda^{-1}(C)$ and $\widetilde{C}.E > 0$ imply that $\widetilde{C}.E = 1$.

Suppose that $\widetilde{C}$ meets two irreducible components E_1 and E_2 of $\mathrm{Exc}(\lambda)$. Then $(2\widetilde{C} + E_1 + E_2)^2 = 2E_1.E_2 \geqslant 0$ which would contradict the negative definiteness of $\lambda^{-1}(C)$. Hence $\widetilde{C}$ meets a unique curve E_1 transversely.

Assume that $E_1 + \cdots + E_k$ is an A_k-configuration inside $\lambda^{-1}(C)$, and that $\widetilde{C}$ meets E_1. If E_k meets two exceptional curves E_{k+1} and E_{k+2}, then

$$\left(2\widetilde{C} + 2\left(\sum_{i=1}^{k} E_i\right) + E_{k+1} + E_{k+2}\right)^2 = 0$$

which once again would contradict the negative definiteness of $\lambda^{-1}(C)$.

(2) Suppose that the dual graph of C is of type Γ_n. From $C = \pi^{-1}(P)$, the fiber $(\pi \circ \lambda)^{-1}(P)$ consists of $\widetilde{C}$ and (-2)-curves E_i. Since $\widetilde{C}$ is (-1)-curve, we get the map $\pi_1 : \widetilde{S} \to S_1$ by blowing down $\widetilde{C}$. Then S_1 is smooth along $\cup \pi_1(E_i)$, and there is a birational map $S_1 \to T$ taking $\cup \pi_1(E_i)$ to the point P. If $n > 1$, then there is exactly one i for which $\widetilde{C}$ meets E_i. Hence

$$K_{S_1}.\pi_1(E_i) = K_{\widetilde{S}}.E_i - \widetilde{C}.E_i = -1.$$

that is, $\pi_1(E_i)$ is a (-1)-curve in S_1. Similarly, $\pi_1(E_j)^2 = -2$ for $j \neq i$. We can then blow down $\pi_1(E_i)$ and repeat the argument. Continue in this way until there is nothing left to blow down, at which point there is a birational map $\nu : S_n \to T$ with S_n smooth and with $\nu^{-1}(P)$ having dimension zero. Then, by Zariski's main theorem, ν is an isomorphism and thus P is a smooth point of T. $\square$

We also need a result of D.R. Morrison, which computes the normal bundle of a rational curve in a smooth 3-fold.

Theorem 2.28. *[33, (2.1)] Let S be a surface with Du Val singularities $P_1, \cdots, P_r$, and let $\lambda : \widetilde{S} \to S$ be the minimal resolution. Let C be an irreducible curve which has multiplicity one in the fundamental cycle, and let*

$$\Gamma = \overline{(\cup_i \lambda^{-1}(P_i)) \setminus C}.$$

Then there is an exact sequence of $\mathscr{O}_{\widetilde{S}\setminus\Gamma}$-modules

$$0 \to \mathscr{O}_B \to \lambda^*(\Omega_S)|_{\widetilde{S}\setminus\Gamma} \to \Omega_{\widetilde{S}}|_{\widetilde{S}\setminus\Gamma} \to \mathscr{O}_B \to 0$$

where $B = m(C \setminus (\Gamma \cap C))$, and m is given by Table 2.1.

Table 2.1:

$\lambda(C)$	C	m
A_n	C_{k+1}	$\min(k+1, n-k)$
D_n	C_1	2
	C_{n-1}, C_n	[n / 2]
E_6	C_1, C_6	2
E_7	C_7	3

Definition 2.29. Let C be a smooth rational curve on a surface S with Du Val singularities. The conormal sheaf $N^{\vee}_{C/S}$ of C in S is defined by the kernel of the morphism $\Omega_S|_C \twoheadrightarrow \Omega_C$.

Notice that if S is smooth in a neighborhood of C, then this agrees with the standard definition of conormal invertible sheaf of C. However, if S is singular at a point of C, the conormal sheaf will not in general be invertible.

Proposition 2.30. *[33, (3.1)] Let C be a smooth Weil divisor on a surface S with Du Val singularities, and let $P_1, \cdots, P_l$ be the singular points of S contained in C. Let $\lambda : \widetilde{S} \to S$ be the minimal resolution, and let $\widetilde{C}$ be the proper transform of C; we identify $\widetilde{C}$ and C via λ. Then there is a zero-dimensional subscheme $Z = \sum m_i P_i \subseteq C$ such that*

$$0 \to \mathscr{O}_Z \to N^{\vee}_{C/S} \to N^{\vee}_{C/\widetilde{S}} \to \mathscr{O}_Z \to 0$$

is an exact sequence of $\mathscr{O}_C$-modules. Moreover, if C_i is the component of $\lambda^{-1}(P_i)$ meeting C, then the coefficient m_i is given by Table 2.1

Let C be a smooth rational curve in a smooth 3-fold Y. Since $N_{C/Y}$ is a rank two vector bundle on $C \cong \mathbb{P}^1$, we write $N_{C/Y} = (a, b)$ with $a \geqslant b$. Suppose that there is a surface S with Du Val singularities such that $Y \supseteq S \supseteq C$ and $C \cap \mathrm{Sing}(S) \neq \varnothing$. By Proposition 2.30, we have the exact sequence

$$0 \to \mathscr{O}_Z \to N^{\vee}_{C/S} \to N^{\vee}_{C/\widetilde{S}} \to \mathscr{O}_Z \to 0 \tag{2.3.2}$$

Proposition 2.31. *Under the above condition, if $K_Y.C = 0$ and $N^{\vee}_{C/\widetilde{S}} = \mathscr{O}_C(1)$, then $\deg Z = m \geqslant 1$ and $N_{C/Y} = (m-1, -m-1)$.*

Proof. First note that $a+b=-K_Y.C-2=-2$. Since $C \subseteq S \subseteq Y$, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N^\vee_{C/Y} \arrow[r] \arrow[d] & \Omega_Y|_C \arrow[r] \arrow[d, two heads] & \Omega_C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N^\vee_{C/S} \arrow[r] & \Omega_S|_C \arrow[r] & \Omega_C \arrow[r] & 0
\end{tikzcd}$$

so that $N^\vee_{C/Y} \to N^\vee_{C/S}$ is surjective. Combining this with (2.3.2), we get a surjection

$$N^\vee_{C/Y} \to \ker(N^\vee_{C/\widetilde{S}} \to \mathscr{O}_Z) = \mathscr{I}_Z(1). \tag{2.3.3}$$

By assumption,

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{O}_C}(N^\vee_{C/Y}, N^\vee_{C/\widetilde{S}}) &= \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_C(-a) \oplus \mathscr{O}_C(-b), \mathscr{O}_C(1)) \\
&= \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_C, \mathscr{O}_C(a+1) \oplus \mathscr{O}_C(b+1)).
\end{aligned}$$

Hence a morphism $\alpha : N^\vee_{C/Y} \to N^\vee_{C/\widetilde{S}}$ corresponds to a global section of $\mathscr{O}_C(a+1) \oplus \mathscr{O}_C(-a-1)$, and $\mathrm{Im}(\alpha) = \mathscr{I}_Z(1)$ if the corresponding section vanishes exactly along Z, that is, if its zero locus has degree m.

Now, under the convention $a \geqslant b$, $\mathscr{O}_C(a+1) \oplus \mathscr{O}_C(-a-1)$ has non-trivial global sections if and only if $a \geqslant 0$, and the non-trivial section of $\mathscr{O}_C(a+1) \oplus \mathscr{O}_C(-a-1)$ vanish along a scheme of degree $a+1$. By (2.3.3), we get $m = a+1$, as desired. □

The following corollary will be used in the proof of Theorem 4.1.

Corollary 2.32. *Let C, S and Y be as in Proposition 2.31. If C is an exceptional curve of the first kind on S, then the normal bundle $N_{C/Y}$ is $(0,-2)$.*

Proof. Recall that C contains at least one singular points of S. By Proposition 2.27, the dual graph of C is of type Γ_n with $n > 1$, and thus all (-2)-curves are contracted to an A_{n-1} surface singularity $P \in C$. Then $Z = mP$ for some $m \in \mathbb{Z}_{\geqslant 0}$. From Proposition 2.30, the degree m of Z is $\min(1, n-1) = 1$. Hence the normal bundle $N_{C/Y}$ is $(0,-2)$. □

2.4 Augmented Base Locus

In this section, we discuss a theorem of Nakamaye describing the augmented base locus of a big and nef divisor, which will be used in the proof of Theorem 4.4.

Let Y be a smooth projective variety and D a divisor on Y. The stable base locus of D is defined by

$$\mathbf{B}(D) = \bigcap_{m \geqslant 1} \mathrm{Bs}(|mD|).$$

Equivalently, $\mathbf{B}(D) = \mathrm{Bs}(|mD|)$ for sufficiently large and divisible m. The augmented base locus of D is a Zariski closed set

$$\mathbf{B}_{+}(D) = \mathbf{B}(D - \varepsilon A)$$

for any ample $\mathbb{Q}$-divisor A and $0 < \varepsilon \ll 1$. It is well-defined and depend only on the numerical class of D.

Suppose further that D is big, i.e. $D^{\dim X} > 0$. Then the restriction of $\mathscr{O}_Y(D)$ to any subvariety V of Y is nef, but it may not be big, that is, it may happen that $D^{\dim V}.V = 0$.

Definition 2.33. Given a big and nef divisor D on Y, the null locus $\mathrm{Null}(D)$ of D is defined to be the union of all subvarieties $V \subseteq Y$ with dimension $d > 0$ such that $D^d.V = 0$.

Observe that $\mathrm{Null}(D)$ is a proper subset of Y since $D^{\dim Y} > 0$. In fact, it is a subvariety of Y.

Proposition 2.34 ([34])**.** *Let D be a big and nef divisor on Y. Then* $\mathrm{Null}(D)$ *is a Zariski closed subset of Y.*

Proof. Suppose that $\{V_i\}_{i\in I}$ is a collection of subvarieties of Y having the property that $D^{\dim V_i}.V_i = 0$ for each $i \in I$ and that the Zariski closure of this union is an irreducible subvariety V. We will show that then V is contained in the null locus $\mathrm{Null}(D)$.

Suppose to the contrary that $D^{\dim V}.V > 0$, so that $D|_V$ is big. Then given an ample divisor A there is an integer $m \gg 0$ such that $\mathscr{O}_V(mD - A)$ has a non-vanishing section. Let $W \subsetneq V$ be the zero locus of this section. Then any subvariety $V' \nsubseteq W$, the restriction $\mathscr{O}_{V'}(mD - A)$ of this bundle to V' also has a non-vanishing section. It follows that $\mathscr{O}_{V'}(D)$ is big. Therefore all the V_i in the collection must lie in W, contradicting the fact that V is the Zariski closure of their union. □

Obviously, we have $\mathrm{Null}(D) \subseteq \mathbf{B}_+(D)$. Indeed, for such subvariety $V \subseteq Y$, the diviosr $D|_V$ is not big. Then $\mathscr{O}_V(mD - \varepsilon mA)$ cannot have non-vanishing section for any $m \in \mathbb{N}$ and thus $V \subseteq \mathbf{B}(D - \varepsilon A) = \mathbf{B}_+(D)$.

Nakamaye's theorem states that such null subvarieties account for every irreducible components of $\mathbf{B}_+(D)$:

Proposition 2.35 ([34]). *If Y is a smooth projective variety of dimension $\geqslant 2$ and D a big and nef divisor on Y, then $\mathbf{B}_+(D) = \mathrm{Null}(D)$.*

2.5 Bertini-type theorems

The aim of the section is to introduce Bertini-type theorem, which is used to construct suitable general sections in the sequel.

Let $\sigma : \mathscr{E} \to \mathscr{F}$ be a morphism of vector bundles of ranks m and n on a variety M. Note that there is a bijection between morphisms $\mathscr{E} \to \mathscr{F}$ and global sections of $\mathscr{E}^\vee \otimes \mathscr{F}$.

For $k \leqslant \min(m, n)$, we define the k-th degeneracy locus of σ by

$$D_k(\sigma) = \{x \in M \mid \mathrm{rank}(\sigma_x) \leqslant k\}.$$

Its ideal is locally generated by $(k+1)$-minors of a matrix for σ. We can show that the codimension of $D_k(\sigma)$ in M is less than or equal to $(m-k)(n-k)$ [39, (2.7)], which is called its expected codimension. Notice that the 0-th degeneracy locus of σ is the zero scheme $Z(\sigma)$.

Now we state a Bertini-type theorem for vector bundles. The statement is taken from [39, (2.8)]. For the reader's convenience, we review its proof.

Theorem 2.36. *Let $\mathscr{E}$ and $\mathscr{F}$ be vector bundles of ranks m and n on a smooth variety M and let $\mathscr{E}^\vee \otimes \mathscr{F}$ be generated by global sections. If $\sigma : \mathscr{E} \to \mathscr{F}$ is a general morphism, then one of the following holds:*

(1) $D_k(\sigma)$ is empty;

(2) $D_k(\sigma)$ has expected codimension $(m-k)(n-k)$ and the singular locus of $D_k(\sigma)$ is $D_{k-1}(\sigma)$.

Here the "general" means that there is a Zariski open set in the vector space $H^0(\mathscr{E}^\vee \otimes \mathscr{F})$ such that every global section σ in the open set satisfies (1) or (2).

Proof. Let $\mathbf{V}$ be the variety $\mathbf{Spec}(Sym((\mathscr{E}^\vee \otimes \mathscr{F})^\vee))$, with projection morphism $\pi : \mathbf{V} \to M$. Note that there is a subvariety $\Sigma_k \subseteq \mathbf{V}$ of codimension $(m-k)(n-k)$ whose fibers over M are isomorphic to the determinantal variety $Y_k := \{A \in M_n(\mathbb{C}) \mid \operatorname{rank}(A) \leqslant k\}$.

Since $\mathscr{E}^\vee \otimes \mathscr{F}$ is globally generated, the evaluation map

$$\mathscr{O}_M \otimes H^0(\mathscr{E}^\vee \otimes \mathscr{F}) \to \mathscr{E}^\vee \otimes \mathscr{F}$$

is surjective. It induces the projection

$$p : M \times H^0(\mathscr{E}^\vee \otimes \mathscr{F}) \to \mathbf{V}$$

which is regular everywhere. Then p^{-1} preserves codimension and singular locus, i.e. the subvariety $Z := p^{-1}(\Sigma_k)$ has codimension $(m-k)(n-k)$ and $\operatorname{Sing}(Z) = p^{-1}(\operatorname{Sing}(\Sigma_k))$.

Let $q : Z \to H^0(\mathscr{E}^\vee \otimes \mathscr{F})$ be the projection morphism. Notice that the fiber $q^{-1}(\sigma)$ is isomorphic to $D_k(\sigma)$ and

$$q^{-1}_{Z\setminus \operatorname{Sing} Z}(\sigma) \simeq D_k(\sigma) \setminus D_{k-1}(\sigma).$$

If $(q_{Z\setminus\operatorname{Sing}Z})^{-1}$ does not have dense image then $D_k(\sigma)$ is empty for general σ. Otherwise, $D_k(\sigma)$ is smooth by generic smoothness theorem and thus

$$\operatorname{Sing}(D_k(\sigma)) \subseteq D_{k-1}(\sigma).$$

By using the fact that the determinantal variety Y_k is Cohen-Macaulay [13, Theorem 14.4 (c)], we can show that the singular locus of $D_k(\sigma)$ is exactly $D_{k-1}(\sigma)$ (and omitting the argument because we do not use it in the sequel). $\square$

Remark 2.37. Let D be a Cartier divisor on M. Assume that the linear system $\Lambda := |\mathscr{O}(D)|$ is base point free. Since the (-1)-th degeneracy locus is empty, the classical Bertini's second theorem follows from Theorem 2.36 by taking $k = 0$, $\mathscr{E} = \mathscr{O}$ and $\mathscr{F} = \mathscr{O}(D)$. Namely, a general member of Λ is smooth. We also know, by the Bertini's first theorem, that if Λ is not composed of a pencil then its general member is irreducible. However, the general degeneracy locus $D_k(\sigma)$ may not be connected.

Definition 2.38. A closed subscheme Z of a smooth variety M is called *superficial* if $\operatorname{codim}(Z, M) \geqslant 2$ and the closed subset $F \subseteq Z$ of points at which the embedding dimension of Z is equal to $\dim M$ satisfies $\operatorname{codim}(F, M) \geqslant 3$.

Proposition 2.39. *[2, (2.1)] Let $\mathscr{L}$ be a line bundle on a smooth variety M. Let Z be the scheme-theoretic base locus of a linear system $V \subseteq H^0(M, \mathscr{L})$. If Z is superficial then a general member of $|V|$ is normal.*

Sketch of Proof. If the codimension of the superficial base scheme Z is greater than two, then a general member of $|V|$ is regular in codimension one. The proposition follows from the fact that divisors on a smooth variety satisfy Serre's S_2 condition.

Let $f : Bl_Z M \to M$ be the blowing up of M along Z and assume that Z has codimension two. Let $F \subseteq Z$ be the closed subset of points at which the embedding dimension of Z equals $\dim M$. Then Z is a codimension two local complete intersection away from F. (Indeed, for $z \in Z \backslash F$, let (A, m) be the regular local ring $\mathscr{O}_{M,z}$ and $\mathscr{O}_{Z,z} = A/I$. From the short exact sequence of A/m-vector spaces

$$0 \to (I + m^2)/m^2 \to m/m^2 \to m/(I + m^2) \to 0,$$

$\operatorname{codim}(Z, M) = 2$ and $z \notin F$, we get $0 < \dim(I + m^2/m^2) \leqslant 2$. By Nakayama's Lemma, I is generated by two elements.)

Thus $Bl_Z M \backslash f^{-1}(F) \to M \backslash F$ is a $\mathbb{P}^1$-bundle over $Z \backslash F$ and an isomorphism elsewhere. We can prove that a general member of $|V|$ is regular in codimension one away from F. Combining $\operatorname{codim}(F, M) > 2$ with Serre's S_2 condition, Proposition 2.39 follows. For more details, see [2, (2.1), p.1226]. $\square$

2.6 Mixed Hodge Structures on Varieties with Normal Crossings

In this section we follow the point of view of the article of Griffiths and Schmid [14, Section 4], which stays as close as possible to classical Hodge theory, and proves a Lefschetz theorem for reducible ample hyperplane sections.

A compact complex analytic variety V is called a variety with normal crossings of dimension n if for each point $x \in V$ there exists a neighborhood U, which can be realized as the union of coordinate hyperplanes:

$$U \cong \{(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1 \cdot z_2 \cdots z_k = 0, |z_i| < \varepsilon\}. \tag{2.6.1}$$

We assume moreover that globally $V = D_1 \cup \cdots \cup D_N$, where the D_i are compact Kähler manifolds meeting transversely, as in (2.6.1).

A mixed Hodge structure on $H^k(V, \mathbb{C})$ consists of an increasing *weight filtration* $W_\bullet$ which shall be defined over $\mathbb{Q}$, and a decreasing *Hodge filtration* $F^\bullet$ such that it induces a pure Hodge structure of weight m on $\mathrm{Gr}_m^W = W_m \otimes \mathbb{C}/W_{m-1} \otimes \mathbb{C}$. More precisely, the induced filtration is given by

$$F^p \mathrm{Gr}_m^W H^k(V, \mathbb{C}) = (F^p \cap W_m \otimes \mathbb{C} + W_{m-1} \otimes \mathbb{C})/W_{m-1} \otimes \mathbb{C}.$$

There exists a functorial mixed Hodge structure on each variety with normal crossings.

Theorem 2.40. *Let V be a variety with normal crossings, $V = \bigcup_{i=1}^N D_i$. Then the cohomology group $H^k(V, \mathbb{C})$ carries a functorial mixed Hodge structure, with weights varying between* 0 *to k.*

We say that mixed Hodge structures on the cohomology of varieties with normal crossings are functorial if, for a morphism $f : X \to Y$ of varieties with normal crossings, $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ is a morphism of mixed Hodge structures of weight 0, i.e. $f^*(W_p H^k(Y)) \subseteq W_p H^k(X)$ and $f^*(F^p H^k(Y)) \subseteq F^p H^k(X)$.

We remark that the Theorem 2.40 is actually true even for rational coefficients. For computing Betti numbers we will only deal with the theorem special to complex coefficients.

To construct a mixed Hodge structure, we need to find a weight filtration $W_\bullet$ on $H^k(V, \mathbb{C})$ verifying

$$(0) \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{k-1} \subseteq W_k = H^k(V, \mathbb{C}).$$

For each multi-index $I = (i_1, \cdots, i_q)$ with $1 \leqslant i_1 < \cdots < i_q \leqslant N$, we let $|I| = q$, the length of I, and $D_I = D_{i_1} \bigcap \cdots \bigcap D_{i_q}$. For fixed q, we define the disjoint union

$$D^{[q]} = \coprod_{|I|=q} D_I.$$

Each $D^{[q+1]}$ is compact Kähler, and we define $A^{p,q} = A^p(D^{[q+1]})$, where $A^*(D^{[q+1]})$ is the usual de Rham complex.

Let $d : A^{p,q} \to A^{p+1,q}$ be the usual exterior derivative and let $\delta : A^{p,q} \to A^{p,q+1}$ be given by the formula, $\delta\varphi = \sum_{|J|=q+2}(\delta\varphi)_J$ and $J = (j_1, \cdots, j_{q+2})$,

$$(\delta\varphi)_J = \sum_{l=1}^{q+2}(-1)^{p+l}\varphi_{j_1,\cdots,\widehat{j_l},\cdots,j_{q+2}}|_{D_J}.$$

Then $\{A^{\bullet,\bullet}, d, \delta\}$ is a double complex, i.e. $d^2 = 0$, $\delta^2 = 0$ and $d\delta + \delta d = 0$. We associate to this double complex a simple complex $(A^\bullet, D)$, where $A^s = \oplus_{p+q=s}A^{p,q}$, $D = d + \delta$.

Theorem 2.41 (De Rham theorem for varieties with normal crossings)**.**

$$H^k(A^\bullet, D) \cong H^k(V, \mathbb{C}).$$

To define a mixed Hodge structure on $H^k(V, \mathbb{C})$ by defining two filtrations W and F, it turns out that these filtrations can be introduced already at the level of the double complex $\{A^{\bullet,\bullet}, d, \delta\}$.

The complex $A^\bullet(D^{[s+1]})$ has a Hodge filtration, so we can take $F^P A^{r,s}$ to be the usual Hodge filtration $F^p A^r(D^{[s+1]})$ on differential r-forms, and define

$$F^p A^\bullet = \bigoplus_{r,s} F^P A^{r,s}.$$

Let us consider the filtered complex $\{A^\bullet, \widetilde{W}\}$, where

$$\widetilde{W}_m := \bigoplus_{r\in\mathbb{Z}} \bigoplus_{s\geqslant -m} A^{r,s},$$

with associated spectral sequence $\{{}_{\widetilde{W}}E_r^{p,q}\}$. By Theorem 2.41, we get

$${}_{\widetilde{W}}E_1^{p,q} = H^q(D^{[p+1]}) \Rightarrow H^{p+q}(V, \mathbb{C}).$$

Note that $H^q(D^{[p+1]})$ has a Hodge structure of weight q, induced by the filtration F.

Now consider the first differential of the spectral sequence:

$$\begin{array}{ccc} {}_{\widetilde{W}}E_1^{p,q} & \xrightarrow{d_1} & {}_{\widetilde{W}}E_1^{p+1,q} \\ \| & & \| \\ H^q(D^{[p+1]}) & \xrightarrow{\delta} & H^q(D^{[p+2]}) \end{array},$$

and

$$ {}_{\widetilde{W}}E_2^{p,q} = H({}_{\widetilde{W}}E_1^{p-1,q} \xrightarrow{d_1} {}_{\widetilde{W}}E_1^{p,q} \xrightarrow{d_1} {}_{\widetilde{W}}E_1^{p+1,q}). \tag{2.6.2} $$

Since d_1 is a morphism of Hodge structures, ${}_{\widetilde{W}}E_2^{p,q}$ has a Hodge structure of weight q. We do not need to consider any further space, because we have the following result.

Lemma 2.42. *The spectral sequence $\{{}_{\widetilde{W}}E_r^{p,q}\}$ degenerates at ${}_{\widetilde{W}}E_2$, that is,*

$$ {}_{\widetilde{W}}E_2^{p,q} \cong {}_{\widetilde{W}}E_\infty^{p,q} \cong \mathrm{Gr}_{-p}^{\widetilde{W}} H^{p+q}(V,\mathbb{C}). $$

As a consequence, the graded piece carries a Hodge structure of weight q induced by the filtration F.

Let $k = p+q$ be fixed. We define the shifted filtration $W_p = \widetilde{W}_{p-k}$ and thus

$$ \mathrm{Gr}_q^W H^k(V,\mathbb{C}) = \mathrm{Gr}_{-p}^{\widetilde{W}} H^k(V,\mathbb{C}). $$

Then $\{H^k(V,\mathbb{C}), W, F\}$ is a mixed Hodge structure with weights varying from 0 to k, which completes the proof of Theorem 2.40.

Recall that, Lefschetz hyperplane theorem, if Y is a smooth projective variety and V is an ample hypersurface then $H^k(Y,\mathbb{C}) \to H^k(V,\mathbb{C})$ is an isomorphism for $k < \dim Y - 1$ and injective for $k = \dim Y - 1$. From Lefschetz's theorem, we have the following result:

Corollary 2.43. *Let Y be a smooth projective variety, and let $V = \bigcup_{i=1}^N D_i$ be an ample divisor on Y with normal crossings. Then, for $k < \dim Y - 1$, the mixed Hodge structure on $H^k(V,\mathbb{C})$ is pure of weight k, i.e. $W_j H^k(V,\mathbb{C}) = 0$ for all $j < k$.*

Proof. By Lefschetz hyperplane theorem, the maps $H^k(Y,\mathbb{C}) \to H^k(V,\mathbb{C})$ induced by the inclusion are isomorphisms for $k < \dim Y - 1$. According to Theorem 2.40, it follows that these isomorphisms are morphisms of the mixed Hodge structures of weight 0. Since $H^k(Y,\mathbb{C})$ is pure of weight k, we get $W_{k-1}H^k(V,\mathbb{C}) = 0$ for $k < \dim Y - 1$. □

Theorem 2.44. *[4, (2.1)] Let Y, V and D_i ($1 \leqslant i \leqslant N$) be as in Corollary 2.43. Then for $i + k - 1 < \dim Y$*

$$ 0 \to H^k(Y,\mathbb{C}) \to \bigoplus_{|I|=1} H^k(D_I,\mathbb{C}) \to \cdots \to \bigoplus_{|I|=i} H^k(D_I,\mathbb{C}) $$

is exact with the convention that $H^k(D_I,\mathbb{C}) = 0$ if $|I| > N$.

Proof. From (2.6.2), we know that the sequence

$$\bigoplus_{|I|=r} H^k(D_I,\mathbb{C}) \to \bigoplus_{|I|=r+1} H^k(D_I,\mathbb{C}) \to \bigoplus_{|I|=r+2} H^k(D_I,\mathbb{C})$$

is exact if and only if $\mathrm{Gr}_k^W H^{r+k}(V,\mathbb{C}) = 0$. According to Corollary 2.43 and our assumption, it follows that $\mathrm{Gr}_k^W H^{r+k}$ vanishes if $1 \leqslant r \leqslant i-2$. Note that

$$0 \to H^k(V,\mathbb{C}) \to \bigoplus_{|I|=1} H^k(D_I,\mathbb{C}) \to \bigoplus_{|I|=2} H^k(D_I,\mathbb{C})$$

is exact because $\mathrm{Gr}_k^W H^k = H^k(V,\mathbb{C})$. Thus the theorem follows from Lefschetz hyperplane theorem. □

Chapter 3

Deformation Theory of Calabi–Yau threefolds

3.1 Unobstructedness Theorem

First we recall the definition of Calabi–Yau 3-fold.

Definition 3.1. A variety X is called a Calabi–Yau 3-fold if it is a projective Gorenstein 3-fold with at worst terminal singularities, such that the canonical Cartier divisor is trivial $K_X \sim 0$ and $H^1(\mathscr{O}_X) = 0$. If the singularities of X are ordinary double points, we will say that X is a Calabi–Yau conifold.

Remark 3.2. The cohomology $H^1(\mathscr{O})$ is well preserved under resolutions and deformations of Calabi–Yau 3-folds. Indeed, let $\pi : \widehat{X} \to X$ be a rational resolution, i.e. $\pi_* \mathscr{O}_{\widehat{X}} \simeq \mathscr{O}_X$ and $R^i \pi_* \mathscr{O}_{\widehat{X}}$ vanishes for all $i > 0$. Using Leray spectral sequence, we get an isomorphism

$$0 \to H^1(\mathscr{O}_X) \to H^1(\mathscr{O}_{\widehat{X}}) \to H^0(R^1 \pi_* \mathscr{O}_{\widehat{X}}) = 0$$

In particular, $H^1(\mathscr{O}_{\widehat{X}}) = 0$ for any resolution $\widehat{X}$ of a Calabi–Yau 3-fold X. On the other hand, $H^1(\mathscr{O}_{X_t}) = 0$ for a general deformation X_t of X by the semicontinuity theorem [19, III Thm.12.8].

Let X be a complex manifold. We can vary the complex structure on X. In general, the miniversal deformation (or Kuranishi) space $\mathrm{Def}(X)$ might be singular if X is arbitrary. But in the case of Calabi–Yau manifolds, we have the following well-known result:

Theorem 3.3 (Bogomolov-Tian-Todorov Unobstructedness Theorem)**.** *If X is a Calabi–Yau manifold, then* $\mathrm{Def}(X)$ *is smooth.*

For singular Calabi–Yau, we have:

Theorem 3.4. *[37, Theorem 1]. Let X be a Calabi-Yau 3-fold with isolated rational complete intersection singularities. Then* $\mathrm{Def}(X)$ *is smooth.*

Given a proper birational morphism $\pi : \widehat{X} \to X$ with $\pi_* \mathscr{O}_{\widehat{X}} \simeq \mathscr{O}_X$ and $R^1\pi_*\mathscr{O}_{\widehat{X}} = 0$, there is a natural map of germs $\mathrm{Def}(\widehat{X}) \to \mathrm{Def}(X)$ (c.f. [25, (11.4)] or [48, (1.4)] on the level of deformation functors). We have the following result.

Proposition 3.5. *[48, (1.8)], [38, (2.3)] Let $\widehat{X}$ be a small projective partial resolution of a Calabi–Yau 3-fold X and* $\mathrm{Def}(\widehat{X})$ *the Kuranishi space of $\widehat{X}$. Then there is a natural closed immersion of* $\mathrm{Def}(\widehat{X})$ *into* $\mathrm{Def}(X)$.

3.2 Smoothings

Theorem 3.6. *[11, (8.7)] Let X be a Calabi–Yau 3-fold with only ordinary double points $p_1, \cdots, p_k$, and $\pi : \widehat{X} \to X$ be a small (not necessarily projective) resolution of X such that $C_i := \pi^{-1}(p_i) \simeq \mathbb{P}^1$. Then X is smoothable if and only if there is a relation $\sum_{i=1}^k \lambda_i [C]_i = 0$ in $H_2(\widehat{X}, \mathbb{Z})$ with $\lambda_i \neq 0$ for all i.*

Definition 3.7. A Calabi–Yau 3-fold X is called maximal if, for any small projective partial resolution $\widehat{X}(\neq X)$ of X, the natural inclusion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X)$ is not surjective.

Theorem 3.8. *[38, (2.5)] Let* $\{p_1, \cdots, p_k\} \subseteq \mathrm{Sing}(X)$ *be the ordinary double points on X and $f : Z \to X$ be a small (not necessarily projective) partial resolution of X such that $C_i := f^{-1}(p_i) \simeq \mathbb{P}^1$ and that f is an isomorphism over $X \backslash \{p_1, \cdots, p_k\}$. Then the following three conditions are equivalent:*

(1) X is maximal;
(2) X is smoothable by a flat deformation;
(3) there is a relation in $H_2(Z, \mathbb{C})$: $\sum \lambda_i [C_i] = 0$ with $\lambda_i \neq 0$ for all i.

The following corollary was proved in [15, (5.1)]. We will generalize it to the case of the relative Picard number greater than one (cf. Theorem 4.10).

Corollary 3.9. *Suppose that $\pi : \widehat{X} \to X$ is a small projective resolution of X with the relative Picard number one. Then X is smoothable unless π is the contraction of a single $\mathbb{P}^1$ to an ordinary double point.*

Proof. If X has only ordinary double points, and $C_1, \cdots, C_k$ are the exceptional curves of π, then the homology classes $[C_i]$'s in $H_2(\widehat{X}, \mathbb{Z})$ coincide, by hypothesis. Hence there is a non-trivial linear dependence relation on $[C_1], \cdots, [C_k]$ unless $n = 1$. Thus by Theorem 3.6, X is smoothable.

Suppose that X does not have only ordinary double points. Let $f : Z \to X$ be a small resolution of the ordinary double points of X (cf. Example 2.15). Let $C_1', \cdots, C_k' \subseteq Z$ be the exceptional curves. Since X has singularities other than ordinary double points and π has the relative Picard number one, the small partial resolution f is non-projective and thus $H_2(Z, \mathbb{C}) = H_2(X, \mathbb{C})$. Hence $[C_1'], \cdots, [C_k'] = 0$ in $H_2(Z, \mathbb{Z})$, and so by Theorem 3.8, X is smoothable. □

There is a simple relation between topological Euler numbers and Milnor numbers of singularities, which will be used in Proposition 4.8.

Proposition 3.10. *Let $\widehat{X} \to X$ be a small resolution of a Gorenstein terminal 3-fold X and $\widetilde{X}$ a smoothing of X. Let $\operatorname{Sing}(X) = \{p_1, \cdots, p_m\}$ and C_i the exceptional curve over p_i. Then*

$$e(\widehat{X}) - e(\widetilde{X}) = \sum_i m(p_i) + \sum_i (e(C_i) - 1),$$

where $e(-)$ denotes the topological Euler number and $m(p_i)$ is the Milnor number of p_i.

Sketch of Proof. By Mayer–Vietoris sequence argument, we can show that $e(\widehat{X}) = e(X) + \sum_i (e(C_i) - 1)$. Using similar methods in [7, (5.4.4)], we can prove that $e(X) = e(\widetilde{X}) + \sum_i m(p_i)$. □

Chapter 4

Decompositions of Small Transitions

4.1 Criteria for small reduced fibers

The aim of this section is to prove our main result which gives a description of irreducible fibers of 3-dimensional flopping contractions.

Theorem 4.1. *Let $\pi : \widehat{X} \to X$ be a small projective resolution of a normal variety X of dimension 3. Suppose that $K_{\widehat{X}}$ is π-trivial and that there is an irreducible normal surface D in $\widehat{X}$ such that $-D$ is π-ample. If the fiber over a singular point $P \in X$ is irreducible, then the analytic type of the singular point P is*

$$x^2 + y^2 + z^2 + w^{2m} = 0 \textit{ for some } m \in \mathbb{N}.$$

Furthermore, the singular point P is an ODP if and only if the surface D is smooth in a neighborhood of the fiber.

Proof. If we can prove that the normal bundle of such irreducible fiber in $\widehat{X}$ is of type $(-1, -1)$ or $(0, -2)$ then the theorem follows immediately (cf. Example 2.21).

Set E be the irreducible surface $\pi(D)$ (since π is small). Since $-D$ is π-ample, E is non-$\mathbb{Q}$-Cartier and D contains exceptional curves $\mathrm{Exc}(\pi)$. Then $\mathrm{Sing}(X) \subseteq E$.

Now we consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r] \arrow[d, "\cong"] & \mathscr{O}_E \arrow[r] \arrow[d] & 0 \\
\pi_*\mathscr{O}_{\widehat{X}} \arrow[r] & \pi_*\mathscr{O}_D \arrow[r] & R^1\pi_*\mathscr{O}_{\widehat{X}}(-D).
\end{tikzcd}$$

If we can prove $\pi_*\mathscr{O}_D \cong \mathscr{O}_E$, then E is normal. According to that D is irreducible and π is surjective, it follows that $\mathscr{O}_E \to \pi_*\mathscr{O}_D$ is injective. To show that it is indeed an isomorphism, we shall prove $R^1\pi_*\mathscr{O}_{\widehat{X}}(-D) = 0$.

Since $\widehat{X}$ is smooth, the pair $(\widehat{X}, \varepsilon D)$ is klt for $0 < \varepsilon \ll 1$. By the relative version of Kodaira vanishing theorem and $-(K_{\widehat{X}} + \varepsilon D)$ and $-D - (K_{\widehat{X}} + \varepsilon D)$ are π-ample, we get $R^i\pi_*\mathscr{O}_{\widehat{X}} = 0$ and $R^i\pi_*\mathscr{O}_{\widehat{X}}(-D) = 0$ for $i > 0$. Hence E is normal and D is rational Gorenstein.

Suppose that the fiber $C = \pi^{-1}(P)$ over a singular point $P \in X$ is irreducible. We claim that $P \in E$ is a smooth surface point. Indeed, since the exceptional curve C can be contracted to the normal surface point $P \in E$, we get $C^2 < 0$ [45, (1.2)]. On the other hand, by adjunction, we have $K_D = (K_{\widehat{X}} + D)|_D$. According to $(K_{\widehat{X}} + D).C < 0$ and $\deg_C(K_D|_C) = \deg_C((K_{\widehat{X}} + D)|_C)$, it follows that $K_D.C < 0$. Thus C is an exceptional curve of the first kind on D. Then, by Proposition 2.27, $p \in E$ is smooth.

If C does not meet any surface singularities in D, then $N_{C/\widehat{X}}$ is $(-1,-1)$ by Lemma 2.23. Otherwise, by Corollary 2.32, the normal bundle $N_{C/\widehat{X}}$ is $(0,-2)$. This completes the proof. □

Corollary 4.2. *With notation as in Theorem 4.1, the scheme theoretical fiber structure on $\pi^{-1}(P)$ is reduced, that is, $\pi^{-1}m_P \cdot \mathscr{O}_{\widehat{X}} = \mathscr{I}$, where $\mathscr{I}$ is the ideal sheaf of the fiber with the reduced structure.*

Proof. By Theorem 4.1, the conormal bundle of the fiber is generated by global sections (because it is either $(1,1)$ or $(0,2)$). The corollary follows from Proposition 2.14. □

Corollary 4.3. *Let $\pi : Y \to X$ be a pl flipping contraction for a 3-dimensional plt pair $(Y, S+B)$. If the fiber C over a singular point $P \in X$ is irreducible, Y is smooth in a neighborhood of C and $K_Y.C = 0$, then*

(1) The scheme theoretical fiber structure on $\pi^{-1}(P)$ is reduced,
(2) The analytic type of the singular point P is

$$x^2 + y^2 + z^2 + w^{2m} = 0 \text{ for some } m \in \mathbb{N}.$$

Furthermore, the singular point P is an ODP if and only if the surface S is smooth in a neighborhood of the fiber C.

Proof. This is a local statement, so we may assume Y is smooth and K_Y is π-trivial. By [26, (5.51)] and $(X, S+B)$ is a plt pair, $\lfloor S+B \rfloor = S$ is a normal surface. Then the theorem follows from the definition of pl flipping contractions and Theorem 4.1. $\square$

Theorem 4.4. *Let $\pi : \widehat{X} \to X$ be a small projective resolution of a Calabi–Yau 3-fold X. Then*

(1) Given a singular point $P \in X$, then the following are equivalent:
 - *(a) The fiber over the singular point P is irreducible;*
 - *(b) The scheme theoretical fiber $\pi^{-1}(P)$ is integral;*
 - *(c) The analytic type of the singular point P is*
$$x^2 + y^2 + z^2 + w^{2m} = 0 \text{ for some } m \in \mathbb{N};$$

 More generally, the same conclusion holds if X is a projective Gorenstein terminal 3-fold.

(2) The singularities of X are of type cA_1 if and only if there is a smooth Weil divisor S containing the singular locus such that $\mathrm{Bl}_S X$ is $\mathbb{Q}$-factorial. In this case, π is isomorphic to the blowing up of X along a smooth Weil divisor.

Furthermore, if S is as in part (2), the singularities of X are ODPs if and only if the normal surface $\pi^{-1}(S)$ is smooth.

Proof. To apply Theorem 4.1, we have to find an irreducible normal surface E in the smooth Calabi–Yau 3-fold $\widehat{X}$ with $\mathscr{O}_{\widehat{X}}(-E)$ being π-ample.

Set $D = \pi^* A_0$ where A_0 ie an ample divisor on X. Obviously, it is big and nef. By Kodaira's Lemma, the linear system $|mD - A|$ is nonempty for any ample divisor A on $\widehat{X}$ and $m \gg 0$. Using the relative Kleiman's criterion for ampleness, we reduce the proof to showing that the linear system $|mD - A|$ contains a normal surface (for a suitable m and A).

Take an ample divisor B on $\widehat{X}$. By definition,

$$\mathbf{B}_+(D) = \mathrm{Bs}(|m_1 m_2 D - m_2 B|)$$

for sufficiently large and divisible m_1 and m_2. On the other hand, by Proposition 2.35, the augmented base locus $\mathbf{B}_+(D)$ is $\mathrm{Null}(D)$. From the definition of D, a curve that has zero intersection number with D must be contained in $\mathrm{Exc}(\pi)$, and thus $\mathbf{B}_+(D) = \mathrm{Exc}(\pi)$.

Set $D_1 = m_1 m_2 D - m_2 B$ for divisible $m_1, m_2 \gg 0$. Since the birational morphism π is induced by $|mD|$ for $m \gg 0$, the linear system $|D_1|$ is not composed of a pencil. By Bertini's first Theorem, a general member of $|D_1|$ is irreducible.

Since X is a projective Gorenstein terminal 3-fold and π is small, there is a very ample divisor $H \in |lA_0|$ such that $H \supseteq \mathrm{Sing}(X)$ and $\widehat{H} = \pi^* H$ is a normal surface (with at worst Du Val singularities) by [42, (1.14)]. We may take $l > m_1 m_2$ and thus $D_1 - \widehat{H}$ is an *antiample* divisor on $\widehat{X}$.

Let F be the closed subset of the base scheme $\mathrm{Bs}(|D_1|)$ consisting of points at which the embedding dimension of $\mathrm{Bs}(|D_1|)$ equals 3. If we can prove $\mathrm{codim}(F, \widehat{X}) = 3$ then the theorem follows form Proposition 2.39.

By Kodaira's vanishing theorem, we have $H^i(\mathscr{O}_{\widehat{X}}(D_1 - H)) = 0$ for $i < 3$. Then we obtain $\mathrm{Bs}(|D_1|) = \mathrm{Bs}(|D_1|_{\widehat{H}}|)$ from the exact sequence

$$0 \to \mathscr{O}_{\widehat{X}}(D_1 - \widehat{H}) \to \mathscr{O}_{\widehat{X}}(D_1) \to \mathscr{O}_{\widehat{H}}(D_1|_{\widehat{H}}) \to 0.$$

Since the base scheme $\mathrm{Bs}(|D_1|_{\widehat{H}}|)$ is contained in the normal surface $\widehat{H}$, we get $F \subseteq \mathrm{Sing}(\widehat{H})$, which are finite sets. Thus the base scheme $\mathrm{Bs}(|D_1|)$ is superficial (cf. Definition 2.38), and we infer that a general member E in $|D_1|$ is normal. Then first case follows immediately from Theorem 4.1.

Suppose that the singularities of X are cA_1 and thus each fiber of π is irreducible. Set $S = \pi(E)$. As in the proof of Theorem 4.1, the surface $S \supseteq \mathrm{Sing}(X)$ is smooth. From the universal property of blowing up, we have a unique morphism from $\widehat{X}$ onto $\mathrm{Bl}_S X$ factoring π, say $\pi = g \circ f$. Since fibers of f have dimension zero, f is an isomorphism, by Zariski's main theorem; in particular $\mathrm{Bl}_S X$ is $\mathbb{Q}$-factorial since it is smooth.

Conversely, assume that there is such a smooth divisor S. Since $\widehat{X}$ and $\mathrm{Bl}_S X$ are $\mathbb{Q}$-factorial Calabi–Yau 3-folds, they are connected by flops [23, (4.9)]. Notice that all fibers of $\mathrm{Bl}_S X \to X$ are irreducible (because S is a smooth surface). By [24, (2.1.12)], all fibers of π are also irreducible. This complete the remainder of the proof by Theorem 4.1. $\square$

Remark 4.5. Since $\widehat{X}$ and $\mathrm{Bl}_S X$ are connected by flops, they have the same analytic singularities [23, (4.11)]. Thus $\mathrm{Bl}_S X$ is also smooth.

The following example is Calabi–Yau 3-folds with cA_1-singularities different from ODPs which admits a small projective resolution.

Example 4.6. Let $\eta_1 : S_1 \to \mathbb{P}^1$ and $\eta_2 : S_2 \to \mathbb{P}^1$ be relatively minimal, rational elliptic surfaces with sections. Fix a point t_0 in $\mathbb{P}^1$ and assume that, in the notation of Kodaira,

(1) η_1 has a singular fiber of type I_n (for $n > 1$) over t_0;
(2) η_2 has a singular fiber of type III over t_0;
(3) either $\eta_1^{-1}(t)$ or $\eta_2^{-1}(t)$ is smooth for every point $t \neq t_0$.

Let $X = S_1 \times_{\mathbb{P}^1} S_2$. Observe that the singularities of X occur precisely at the points (p_1, p_2) in $\eta_1^{-1}(t_0) \times \eta_2^{-1}(t_0)$ where both p_1 and p_2 are singular points in the fibers of the S_1 and S_2 respectively. By construction, X has n singular points and each singularity is analytically isomorphic to a singularity defined by $xy = u(v^2 - u)$, an $A_1(3)$-singularity (cf. Example 2.6). Every irreducible component (considered as a reduced scheme) of $\eta_1^{-1}(t_0) \times \eta_2^{-1}(t_0)$ is a smooth surface, which is in fact isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Every $A_1(3)$-singularity of X lies on such a Weil divisor. A small projective resolution of X is obtained by successively blowing up any sequence of irreducible components of $\eta_1^{-1}(t_0) \times \eta_2^{-1}(t_0)$ which contains all $A_1(3)$-singularities. Note that the local description of such blowing up has computed in Example 2.21.

4.2 Decomposition Process of Small Transition

We recall the definition of (projective) small transitions.

Definition 4.7. Let $\widehat{X} \to X$ be a projective small resolution of a Calabi–Yau 3-fold X, which has terminal singularities. If X can be smoothed to a Calabi–Yau manifold $\widetilde{X}$, then the process of going from $\widehat{X}$ to $\widetilde{X}$ is called a small transition and denoted by a diagram $\widehat{X} \to X \rightsquigarrow \widetilde{X}$. It is called a conifold transition if all singularities of X are ODPs.

There is a simple criterion for ODPs for small transitions.

Proposition 4.8. *Let $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ be a small transition. Then the difference of the topological Euler numbers $e(\widehat{X}) - e(\widetilde{X})$ equals the number $2\,|\mathrm{Sing}(X)|$ if and only if the singularities of X are ODPs.*

Proof. Let $\mathrm{Sing}(X) = \{p_1, \cdots, p_k\}$ and C_i the exceptional curve over an isolated hypersurface singularity p_i. We have the identity of the topological Euler numbers

$$e(\widehat{X}) - e(\widetilde{X}) = \sum m(p_i) + \sum (e(C_i) - 1)\,,$$

where $m(p_i)$ is the Milnor number of p_i. According to Proposition 2.20, the exception curve C_i is a union of smooth rational curves which meet transversally and thus the number $e(C_i) - 1$ is equal to n_i the number of irreducible components of C_i. Observe that $m(p_i)$ and n_i are greater than or equal to one. Then

$$e(\widehat{X}) - e(\widetilde{X}) = \sum m(p_i) + \sum n_i \geqslant 2k,$$

and the equality holds if and only if $n_i = m(p_i) = 1$ for all i. $\square$

Conifold transitions play a fundamental role in Reid's fantasy [43, Section 8] (cf. §1), which conjectures that all the moduli spaces of smooth Calabi–Yau 3-folds are connected through conifold transitions.

For a Calabi–Yau 3-fold X, Namikawa and Steenbrink proved that X can be deformed to a Calabi–Yau 3-fold with at worst ODPs [36]. In view of this result, it seems that one may possibly answer Question 1 affirmatively by finding a deformation direction of $\widehat{X}$ which deforms $\widehat{X} \to X$ into $\widehat{X}_1 \to X_1$ with X_1 being a Calabi–Yau conifold. Unfortunately, Namikawa produced a counterexample to this in [38, Remark 2.8]. We recall it briefly as follows:

Choose a suitable rational elliptic surface S with six singular fibers of type II (i.e. cuspidal rational curves). Let $X = S \times_{\mathbb{P}^1} S$. Then X is a Calabi–Yau 3-fold with six singular points of cA_2 type:

$$x^2 - y^3 = u^2 - v^3,$$

which admits smoothings to $\widetilde{X} = S_1 \times_{\mathbb{P}^1} S_2$ with $S_i \to \mathbb{P}^1$ having disjoint discriminant loci. A small resolution $\pi : \widehat{X} \to X$ can also be constructed (see below). Namikawa observed that the exceptional loci should not be deformed to a disjoint union of $(-1,-1)$-curves. The reason is that a singular fiber of type II splits up into at most two singular fibers of type I, and a general fiber of small deformation of a singularity of X which preserves small resolutions has three ODPs.

To search for a modification of Question 1, we need to study Namikawa's construction of the small resolution π carefully. Notice that the diagonal

$D \cong S$ in X is a smooth Weil divisor which contains the six singular points and is thus not $\mathbb{Q}$-Cartier. On the other hand, there is a $\tau \in \mathrm{Aut}(X)$ such that $D_\tau := \tau(D)$ has the same properties as D. Then $X' := \mathrm{Bl}_D X$ has six ODPs and the exceptional locus of $X' \to X$ consists of six mutually disjoint $\mathbb{P}^1$s, with each of them passing through one of the six ODPs. Now the small resolution can be constructed as the blowing up of X' along the proper transform $\widetilde{D}_\tau$ of D_τ, with π being composed of morphisms $\widehat{X} \to X' \to X$. It admits exceptional trees, composed of couples of rational curves intersecting at one point.

Now comes the key point. Using Friedman's criterion, $X' \to X$ can be deformed to a small resolution $Y' \to Y$ where Y' is smooth and Y has only ODPs. Thus we have decomposed the small transition $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ into two conifold transitions $\widehat{X} \to X' \rightsquigarrow Y'$ and $Y' \to Y \rightsquigarrow \widetilde{X}$:

$$\begin{tikzcd}
\widehat{X} = \mathrm{Bl}_{\widetilde{D}_\tau} X' \arrow[d] & Y' \arrow[d] & \\
X' = \mathrm{Bl}_D X \arrow[d] \arrow[ur, rightsquigarrow] & Y \arrow[dr, leftrightsquigarrow] & \\
X \arrow[ur, rightsquigarrow] \arrow[rr, rightsquigarrow] & & \widetilde{X}.
\end{tikzcd}$$

Combining the above discussions, we modify Question 1 as follows:

Question 2. Let $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ be a small transition. Up to deformations of contractions, is that true $\widehat{X}$ can be connected to $\widetilde{X}$ through *a sequence of conifold transitions*?

To attack Question 2, we introduce primitive small transitions:

Definition 4.9. A small transition $\widehat{X} \xrightarrow{\pi} X \rightsquigarrow \widetilde{X}$ is said to be primitive if it satisfies the following two conditions (up to flops of π):

(1) If $\widehat{X} \to X$ can be deformed to another small resolution $\widehat{Y} \to Y$, then the analytic type of singularities of X and Y are the same.
(2) For any factorization $\widehat{X} \to X' \to X$ of the resolution $\widehat{X} \to X$ with $X' \neq X$ and $X' \neq \widehat{X}$, the closed immersion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X')$ of Kuranishi spaces is an isomorphism.

Evidently, every small transition can be decompose into primitive small transitions up to deformations. If we want to approach Question 2, understanding primitive small transitions becomes essential. The following theorem provides the first step towards this problem:

Theorem 4.10. *Let $\pi : \widehat{X} \to X$ be a small resolution of a Calabi–Yau 3-fold X. If the natural closed immersion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X)$ of Kuranishi spaces is an isomorphism then the singularities of X are ODPs. Moreover, the number of ODPs is equal to the relative Picard number $\rho(\widehat{X}/X)$.*

We note that Theorem 4.10 is a generalization of [15, (5.1)].

Proof. The proof is by induction on the relative Picard number $\rho = \rho(\widehat{X}/X)$. Suppose that $\rho = 1$. The result follows from [15, (5.1)].

To prove the case $\rho \geqslant 2$, we recall some facts about extremal rays. Let D be the pullback of an ample divisor under the morphism π. By Kodaira's Lemma, a linear system $|mD - A|$ is nonempty for any ample divisor A on $\widehat{X}$ and $m \gg 0$. Pick a divisor $E \in |mD - A|$, which is relatively antiample by the relative Kleiman's criterion for ampleness. Let $\overline{NE}(\widehat{X}/X)$ be the relative Mori cone. It is a convex (polyhedral) cone generated by (finitely many) exceptional curves of π. Using the Cone Theorem [26, (3.25)], we have a klt pair $(\widehat{X}, \varepsilon E)$ for $0 < \varepsilon \ll 1$ with $\mathscr{O}_{\widehat{X}}(-E)$ being π-ample such that

$$\overline{NE}(\widehat{X}/X) = \textstyle\sum_{i=1}^{k} \mathbb{R}_{\geqslant 0}[C_i],$$

where $\mathbb{R}_{\geqslant 0}[C_i]$ are different extremal rays and $k \geqslant \rho$. Notice that the Calabi-Yau condition implies that every face of $\overline{NE}(\widehat{X}/X)$ is a $(K_{\widehat{X}} + \varepsilon E)$-negative extremal face. It is also evident that the number of irreducible components of $\mathrm{Exc}(\pi)$ is at least ρ.

Suppose that our assertion is valid for small resolutions with the relative Picard number less than ρ, and let $\pi : \widehat{X} \to X$ be a small resolution with $\rho(\widehat{X}/X) = \rho$. We first claim that the number of irreducible components of $\mathrm{Exc}(\pi)$ is ρ.

Let $U = \widehat{X} \backslash \pi^{-1}(\mathrm{Sing}(X))$. Consider the following long exact sequence

$$0 \to H^1(\Omega^2_{\widehat{X}}) \to H^1(U, \Omega^2_U) \to \bigoplus_{p \in \mathrm{Sing}(X)} H^2_{\pi^{-1}(p)}(\Omega^2_{\widehat{X}}) \xrightarrow{\alpha} H^2(\Omega^2_{\widehat{X}}) \quad (4.2.1)$$

where $H^1_{\pi^{-1}(p)}(\Omega^2_{\widehat{X}})$ is vanishing for all $p \in \mathrm{Sing}(X)$ by the depth argument (cf. Lemma 2.24). Note that $\mathrm{Def}(X)$ is smooth [37] and the tangent space of $\mathrm{Def}(X)$ is isomorphic to $H^1(U, \Omega^2_U)$, by Schlessinger's result

[10, 47]. According to the assumption of the theorem, the dimension of $\mathrm{Def}(\widehat{X})$ and $\mathrm{Def}(X)$ are the same. Then we get $h^1(\Omega^2_{\widehat{X}}) = h^1(U, \Omega^2_U)$ and thus α is injective. Since the image of α is just the vector space generated by the fundamental classes of irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$, we get $\mathrm{rank}(\alpha) = \rho$. According to Lemma 2.24, it follows that the dimension of $\bigoplus_p H^2_{\pi^{-1}(p)}(\Omega^2_{\widehat{X}})$ is greater than or equal to the number of irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$ which is at least ρ. Hence we conclude that the number of irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$ is exactly ρ.

Notice that now we have

$$\overline{NE}(\widehat{X}/X) = \bigoplus_{i=1}^{\rho} \mathbb{R}_{\geqslant 0}[C_i].$$

If any two curves have non-empty intersection, say C_1 and C_2, we let F be the cone generated by $[C_1]$ and $[C_2]$. It is indeed a face since there are precisely ρ generators of the ρ-dimensional cone $\overline{NE}(\widehat{X}/X)$. Let $\pi' : \widehat{X} \to X'$ be the contraction of the $(K_{\widehat{X}} + \varepsilon E)$-negative extremal face F. By the induction hypothesis, the singularities of X' consist of exactly two ODPs and $\mathrm{Exc}(\pi') = C_1 \coprod C_2$. This contradicts to that $C_1 \cap C_2 \neq \varnothing$, and thus $\mathrm{Exc}(\pi)$ is a disjoint union of irreducible rational curves.

By the induction hypothesis and the Cone theorem, we infer that the normal bundle of an irreducible exception curve in $\widehat{X}$ is $(-1, -1)$. Hence the singularities of X are ODPs. $\square$

As an immediate consequence of Theorem 4.10, we have:

Corollary 4.11. *For a primitive small transition $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ and any nontrivial factorization $\widehat{X} \to Y \to X$, the singularities of Y are ODPs.*

Corollary 4.12. *Let $\pi : \widehat{X} \to X$ be a small resolution of a Calabi–Yau 3-fold X. Suppose that, for any Calabi–Yau 3-fold $\widehat{X}'$ which is birationally equivalent to $\widehat{X}$ and any factorization $\widehat{X}' \to X' \to X$ with $X' \neq \widehat{X}'$, the Calabi–Yau 3-fold X' is not smoothable. Then the singularities of X are ODPs. Moreover, the number of ODPs is equal to the relative Picard number $\rho(\widehat{X}/X)$.*

Proof. According to that $\widehat{X}$ and $\widehat{X}'$ are connected by a sequence of flops [21, 23] and the Kuranishi spaces are unchanged under flops [25, (12.6)], Corollary 4.12 follows from Theorem 3.8 and Theorem 4.10. $\square$

Question 3. Can one classifies primitive transitions? Or more ambitiously, is that true a primitive transition is necessarily a conifold transition?

Chapter 5

A Connectedness Theorem of Moduli spaces

5.1 Configurations and Parameter Spaces

We start by introducing the configuration of complete intersections and constructing their parameter spaces.

A configuration is a pair $[V\|\mathfrak{L}]$ of a smooth projective variety V and a sequence of line bundles $\mathfrak{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_m)$, where $\mathscr{L}_j$ is generated by global sections. In this thesis, we always assume that $\dim V = m+3$. A variety X is called a member of the configuration and write $X \in [V\|\mathfrak{L}]$ if it is defined by global sections σ_j of $\mathscr{L}_j$ for $1 \leqslant j \leqslant m$ and of $\dim X = 3$.

If

$$V = \textstyle\prod_{i=1}^k \mathbb{P}^{n_i} \text{ and } \mathscr{L}_j = \textstyle\bigotimes_{i=1}^k pr_i^* \mathscr{O}_{\mathbb{P}^{n_i}}(q_j^i),$$

where $pr_i : V \to \mathbb{P}^{n_i}$ is the natural projection and $q_j^i \geqslant 0$ for all i, j, then we define a configuration matrix

$$[\mathbf{n}\|\mathbf{q}] = \left[\begin{array}{c\|ccc} n_1 & q_1^1 & \cdots & q_m^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_k & q_1^k & \cdots & q_m^k \end{array}\right].$$

Here the $(q_j^1, \cdots, q_j^k)$ and are called the multidegree of $\mathscr{L}_j$ and $X \in [\mathbf{n}\|\mathbf{q}]$ respectively. We may assume that

$$\textstyle\sum_{i=1}^k q_j^i \geqslant 2$$

for all $1 \leqslant j \leqslant m$ (otherwise a hyperplane section of only one factor $\mathbb{P}^n$ reduces the factor to $\mathbb{P}^{n-1}$). Note that the global sections of $\mathscr{L}_j$ are multi-homogeneous polynomials of multidegree $(q_j^1, \cdots, q_j^k)$.

Definition 5.1. Two configuration matrices are said to represent the same configuration if one can go from one to the other by a permutation of the rows or of the columns other than first. We say that $[\mathbf{n}_1 \| \mathbf{q}_1]$ is a sub-configuration matrix of $[\mathbf{n} \| \mathbf{q}]$ if

$$\left[\begin{array}{c\|cc} \mathbf{n}_1 & \mathbf{q}_1 & \mathbf{a} \\ \mathbf{m} & \mathbf{0} & \mathbf{b} \end{array} \right]$$

and $[\mathbf{n} \| \mathbf{q}]$ represent the same configuration.

We can explain the meaning of a complete intersection $X \in [\mathbf{n} \| \mathbf{q}]$ precisely by defining a projective family for the configuration $[\mathbf{n} \| \mathbf{q}]$ whose fibers are complete intersections of multidegree $\mathbf{q}$.

In the following we will write $\underline{X}_i$ and $\underline{u}$ as a short form for indeterminates $X_{i0}, \cdots, X_{in_i}$ and $u_1, \cdots, u_d$ respectively.

Let X be defined by a sequence of multi-homogeneous polynomials $\sigma = (\sigma_j)$ of multidegree $\mathbf{q}$ and of dimension three. Let

$$\Phi^{(1)}, \cdots, \Phi^{(d)}$$

be a basis of $\bigoplus_{j=1}^m H^0(V, \mathscr{L}_j)$ and write $\Phi^{(h)} = (\phi_j^{(h)})$ where $\phi_j^{(h)}$'s belong $\mathbb{C}[\underline{X}_1; \cdots; \underline{X}_k]$ with multidegree $(q_j^1, \cdots, q_j^k)$.

Let $K_\bullet := K_\bullet(\sigma + \sum_{h=1}^d u_h \Phi^{(h)})$ be the Koszul complex and

$$D := \mathrm{Supp}(H_1(K_\bullet)) \subseteq \mathbb{A}^{d+N} = \mathrm{Spec}\left(\mathbb{C}[\underline{u}; \underline{X}_1; \cdots; \underline{X}_k]\right)$$

where $N = \sum_i n_i + k$ is the dimension of V.

Let q be the projection from $\mathbb{A}^{d+N}$ onto $\mathbb{A}^d$, and let $U := \mathbb{A}^d \backslash q(D)$ be the set of points such that the $H_1(K_\bullet)$ vanishes, i.e. the Koszul complex $K_\bullet$ is exact on such points. Obviously, U contains the origin. Let

$$\mathscr{X} = \mathrm{Proj}\left(\mathbb{C}[\underline{u}; \underline{X}_1; \cdots; \underline{X}_k] / (\sigma_1 + \sum_h u_h \phi_1^{(h)}, \cdots, \sigma_m + \sum_h u_h \phi_m^{(h)}) \right).$$

Consider the projection $P : \mathscr{X} \subseteq V \times \mathbb{A}^d \to \mathbb{A}^d$ and its restriction $P_U : \mathscr{X}_U \to U$. Since the the Koszul complex $K_\bullet$ is exact on U, all fibers of

P_U are complete intersections of multidegree $\mathbf{q}$ and have the same Hilbert polynomial $P(t)$ which is computed by the Koszul resolution and depends on its multidegree. Hence P_U is a flat family with the fiber $\mathscr{X}_0 = X$ [19, III Thm.9.9]. From this we know that such complete intersections are parameterized by an open subset of $\mathrm{Hilb}^N_{P(t)}$.

To summarize what we have proved, we get the following proposition:

Proposition 5.2. *Given $X \in [\mathbf{n}\|\mathbf{q}]$. Then there is a Zariski open set $0 \in U$ in $\bigoplus_{j=1}^m H^0(V, \mathscr{L}_j)$ and a flat projective morphism $P_U : \mathscr{X}_U \to U$ with the fiber $\mathscr{X}_0 = X$ such that all complete intersections in V of multidegree $\mathbf{q}$ are parameterized by the pair (U, P_U).*

Hence we may use the configuration $[\mathbf{n}\|\mathbf{q}]$ to denote the parameter space of complete intersections in V of multidegree $\mathbf{q}$ and dimensional three.

The following result is a well known result in [16]. In order to point out what the fundamental cycle of a smooth member is, we are going to use Theorem 2.36 to give a proof.

Proposition 5.3. *A general member X of a configuration $[V\|\mathfrak{L}]$ is smooth and of dimension three, where $\mathfrak{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_m)$ is a sequence of globally generated line bundles over V. Moreover, the normal bundle of X in V is $\bigoplus_{j=1}^m \mathscr{L}_j|_X$ and the fundamental class $[X]$ in $A_3(V)$ is the top Chern class of $\bigoplus_{j=1}^m \mathscr{L}_j$.*

Proof. Apply Theorem 2.36 to the case $k = 0$, $\mathscr{E} = \mathscr{O}_V$ and $\mathscr{F} = \bigoplus_{j=1}^m \mathscr{L}_j$, the zero locus $Z(\sigma)$ is smooth and has the expected codimension m for a general $\sigma : \mathscr{E} \to \mathscr{F}$. Namely, a general $(s_j) \in \bigoplus_{j=1}^m H^0(V, \mathscr{L}_j)$ defines a smooth member $X \in [V\|\mathfrak{L}]$ of dimension $\dim V - m = 3$. By [13, Example 14.4.1], the fundamental class of a general member in $A_3(V)$ is the Chern class $c_m(\mathscr{F}) \cap [V]$. $\square$

In order to connect two configurations, we shall define a formal correspondence on matrices which is introduced in [3].

Given a configuration

$$\widehat{\mathscr{C}} = \left[\begin{array}{c\|cccccc} n & 1 & \cdots & 1 & 0 & \cdots & 0 \\ P & \mathscr{L}_1 & \cdots & \mathscr{L}_{n+1} & \mathscr{L}_{n+2} & \cdots & \mathscr{L}_m \end{array}\right]$$

where P is a smooth projective variety and $\mathscr{L}_j$'s are line bundles on P, we introduce a new configuration

$$\mathscr{C} = \left[\begin{array}{c\|cccc} P & \bigotimes_{i=1}^{n+1} \mathscr{L}_i & \mathscr{L}_{n+2} & \cdots & \mathscr{L}_m \end{array}\right]$$

so as to remove the $\mathbb{P}^n$ factor and denote the correspondence by

$$\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}. \tag{5.1.1}$$

We refer to the correspondence of passing from the right hand side to the left as *formal splitting* and the reverse process as *formal contraction*. A configuration connects to other formally if, after a finite formal splitting and contractions, one represents the same configuration as other one.

We are going to compute the difference of topological Euler numbers of smooth members between the formal correspondence.

Proposition 5.4. *Let $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}$ be as in (5.1.1). Given smooth members $\widehat{X} \in \widehat{\mathscr{C}}$ and $\widetilde{X} \in \mathscr{C}$, we have*

$$e(\widehat{X}) - e(\widetilde{X}) = 2\int_P \Big(c_2(\mathscr{E})^2 - c_1(\mathscr{E})c_3(\mathscr{E})\Big)\, c_{m-n-1}(\mathscr{F})$$

where $\mathscr{E} = \bigoplus_{i=1}^{n+1} \mathscr{L}_i$ and $\mathscr{F} = \bigoplus_{i=n+2}^{m} \mathscr{L}_i$, which are vector bundles of rank $n+1$ and $m-n-1$ respectively, and $e(-)$ denotes the topological Euler number.

Proof. From Proposition 5.3 and the normal sequence

$$0 \to T_{\widetilde{X}} \to T_P|_{\widetilde{X}} \to N_{\widetilde{X}} \to 0,$$

we calculate

$$p(t) := \iota_* c_t(T_{\widetilde{X}}) = c_t(T_P)s_t(\otimes_{i=1}^{n+1}\mathscr{L}_i)s_t(\mathscr{F}),$$

where $\iota : \widetilde{X} \hookrightarrow P$ is the inclusion, $c_t(\mathscr{V})$ denotes the Chern polynomial of a vector bundle $\mathscr{V}$ and $c_t(\mathscr{V})s_t(\mathscr{V}) = 1$. Observe that $\otimes_{j=1}^{n+1}\mathscr{L}_j$ and $\mathscr{E}$ have same first Chern class $\sum_{j=1}^{n+1} c_1(\mathscr{L}_j)$. Then $s_t(\otimes_{j=1}^{n+1}\mathscr{L}_j) = \sum_{i=0}^{\infty} s_i(\mathscr{E})^i t^i$ and fundamental class of $\widetilde{X}$ in $A_3(P)$ is

$$(c_1(\mathscr{E})c_{m-n-1}(\mathscr{F})) \cap [P]. \tag{5.1.2}$$

We are going to calculate $e(\widetilde{X})$. Set $c_t(P) = c_t(T_P)$. By a direct computation,

$$p'''(0) = s_1(\mathscr{E})^3 + C_1 s_1(\mathscr{E})^2 + C_s,$$

where

$$\begin{aligned} C_1 :=& c_1(P) + s_1(\mathscr{F}), \\ C_s :=& s_3(\mathscr{F}) + [c_1(P) + s_1(\mathscr{E})]\, s_2(\mathscr{F}) \\ &+ [c_2(P) + c_1(P)s_1(\mathscr{E})]\, s_1(\mathscr{F}) + c_2(P)s_1(\mathscr{E}) + c_3(P). \end{aligned} \tag{5.1.3}$$

Using the above identity, (5.1.2) and the Gauss-Bonnet theorem, we get

$$e(\widetilde{X}) = \int_{\widetilde{X}} c_3(\widetilde{X}) = \int_P p'''(0)c_1(\mathscr{E})c_{m-n-1}(\mathscr{F}).$$

To compute $e(\widehat{X})$, we identify the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ with $pr^*\mathscr{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n \times P$ (similarly for vector bundles on P), where pr is the projection from $\mathbb{P}^n \times P$ onto $\mathbb{P}^n$.

According to

$$\iota'_* N_{\widehat{X}} = (\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^n}(1)) \bigoplus \mathscr{F},$$

where $\iota' : \widehat{X} \hookrightarrow \mathbb{P}^n \times P$ is the inclusion, it follows that

$$q(t) := \iota'_* c_t(T_{\widehat{X}}) = c_t(T_{\mathbb{P}^n} \oplus T_P)s_t(\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^n}(1))s_t(\mathscr{F})$$

and the fundamental class of $\widehat{X}$ in $A_3(\mathbb{P}^n \times P)$ is

$$(c_{n+1}(\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^n}(1))c_{m-n-1}(\mathscr{F})) \cap [\mathbb{P}^n \times P].$$

Set $H = c_1(\mathscr{O}_{\mathbb{P}^n}(1))$ in $A_1(\mathbb{P}^n \times P)$. An explicit computation show that

$$q'''(0) = s_1(\mathscr{E})H^2 - [2s_2(\mathscr{E}) + C_1 s_1(\mathscr{E})]\, H + [s_3(\mathscr{E}) + C_1 s_2(\mathscr{E}) + C_s]\,,$$

where C_1 and C_s are classes as defined in (5.1.3).

We regard the class $q'''(0)c_{n+1}(\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^n}(1))$ as a polynomial in H, denoted it by $Q(H)$. Then $Q^{(n)}(H)$ is equal to the H^n-term in the class

$$q'''(0)\left(c_1(\mathscr{E})H^n + c_2(\mathscr{E})H^{n-1} + c_3(\mathscr{E})H^{n-2}\right).$$

Let $P(H)$ be the class $p'''(0)c_1(\mathscr{E})H^n$ in $A_*(\mathbb{P}^n \times P)$. Using the recurrence relation $c_l(\mathscr{E}) = -\sum_{i=1}^{l} s_i(\mathscr{E})c_{n-i}(\mathscr{E})$ between Chern classes and Segre classes, we get

$$\begin{aligned} e(\widehat{X}) - e(\widetilde{X}) &= \int_{\mathbb{P}^n \times P} \left(Q^{(n)}(H) - P(H)\right) c_{m-n-1}(\mathscr{F}) \\ &= \int_P 2\left(s_2(\mathscr{E})^2 - s_1(\mathscr{E})s_3(\mathscr{E})\right) c_{m-n-1}(\mathscr{F}) \\ &= \int_P 2\left(c_2(\mathscr{E})^2 - c_1(\mathscr{E})c_3(\mathscr{E})\right) c_{m-n-1}(\mathscr{F}). \end{aligned}$$

Example 5.5. We consider

$$\widehat{\mathscr{C}} := \left[\begin{array}{c\|ccc} 3 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 1 \end{array}\right] \longleftrightarrow \mathscr{C} := \left[\begin{array}{c\|c} 3 & 4 \\ 1 & 2 \end{array}\right].$$

For smooth member $\widehat{X} \in \widehat{\mathscr{C}}$ and $\widetilde{X} \in \mathscr{C}$, the topological Euler numbers $e(\widehat{X})$ and $e(\widetilde{X})$ are -112 and -168 respectively. Let s (resp. t) be the class of a hyperplane on $\mathbb{P}^3$ (resp. $\mathbb{P}^1$), and let $\mathscr{E}$ be the vector bundle $\mathscr{O}(1,0) \bigoplus \mathscr{O}(1,0) \bigoplus \mathscr{O}(2,2)$ of rank three on $\mathbb{P}^3 \times \mathbb{P}^1$. Then the Chern classes of $\mathscr{E}$ are

$$\begin{cases} c_1(\mathscr{E}) = 4s + 2t, \\ c_2(\mathscr{E}) = 5s^2 + 4st, \\ c_3(\mathscr{E}) = 2s^3 + 2s^2t. \end{cases}$$

and the coefficient of s^3t in $c_2(\mathscr{E})^2 - c_1(\mathscr{E}).c_3(\mathscr{E})$ is 28.

Definition 5.6. A configuration matrix $[\mathbf{n}\|\mathbf{q}]$ is called a complete intersection Calabi–Yau (CICY) configuration if it satisfy the Calabi–Yau condtion

$$\textstyle\sum_{j=1}^{m} q_j^i = n_i + 1$$

for all $1 \leqslant i \leqslant k$.

It is easy to see that CICY configuration matrices are preserved under formal splitting and contractions. Note that the topological Euler number of a smooth member which belongs to a CICY configuration matrix is non-positive [3, (2.28)].

Remark 5.7. We do not allow that a Calaba–Yau 3-fold X is a product of three elliptic curves or of an elliptic curve and $K3$ surface since $H^1(\mathscr{O}_X) = 0$. Further we are not interested in a configuration matrix which contains the sub-configuration $[1\|2]$ because the sub-configuration describes two points (counted with multiplicity) in $\mathbb{P}^1$. To exclude such cases, we only treat *non block-diagonal* CICY configuration matrices.

Let us consider the simple case for all $n_i = 1$ and $q_j^i = 0$ or 2.

Example 5.8. Given a CICY configuration $k \times (m+1)$-matrix $[\mathbf{n}\|\mathbf{q}]$ with $n_i = 1$ and $q^i_j = 0$ or 2 for all i, j. Then $k = m + 3$. By Remark 5.7, we know that $[\mathbf{n}\|\mathbf{q}]$ is non block-diagonal and thus

$$\textstyle\sum_{i=1}^{k} q^i_j \geqslant 4$$

for each column of $\mathbf{q}$. According to the Calabi–Yau condition it follows that

$$4(k-3) \leqslant \textstyle\sum_{i,j} q^i_j = 2k$$

and therefore $4 \leqslant k \leqslant 6$. When k equals 5 or 6, we get a product of an elliptic curve and $K3$ surface or of three elliptic curves respectively. By Remark 5.7, the CICY configuration matrix must be

$$\left[\begin{array}{c||c} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{array}\right]$$

in this simple case. We denote this configuration matrix by $\mathscr{C}_{1111}$.

The following proposition were proved in [17, Lemma 2], for the convenience of the readers we give a proof here.

Proposition 5.9. *Every CICY configuration matrices can be connected formally.*

Proof. Given a (non block-diagonal) CICY configuration matrix

$$[\mathbf{n}\|\mathbf{q}] = \left[\begin{array}{c||ccc} n_1 & q^1_1 & \cdots & q^1_m \\ \vdots & \vdots & \ddots & \vdots \\ n_k & q^k_1 & \cdots & q^k_m \end{array}\right].$$

We perform formal splitting iteratively until we arrive at a configuration matrix for which each row entries q^i_j with $n_i > 1$ are 0 or 1 (for example, introducing a sub-configuration matrix $[1\|11]$ to split it). Perform next formal contractions in a way that finally leaves each $n_i = 1$ and $q^i_j = 0$ or 2. Notice that non block-diagonal CICY configuration matrices are preserved under formal splitting and contractions. According to Remark 5.8, it follows that the configuration matrix is the simple configuration $\mathscr{C}_{1111}$. Hence every CICY configuration matrices connect as claimed. $\square$

We conclude this section by the existence of a smooth Calabi–Yau in every CICY configuration matrices. The remaining task is to prove that a general member is irreducible and $H^1(\mathscr{O}) = 0$ by using a suitable Lefschetz-type theorem for an ample reducible divisor, cf. Theorem 2.44.

Proposition 5.10. *A general member of a CICY configuration matrix is a smooth Calabi–Yau 3-fold.*

Proof. Given a CICY configuration matrix $[\mathbf{n}\|\mathbf{q}]$, and let $V = \prod_{i=1}^m \mathbb{P}^{n_i}$. Let $\mathscr{L}_j$ be the line bundle with the multidegree $(q_j^1, \cdots, q_j^k)$. By Proposition 5.3, it suffices to prove that a general smooth member $X \in [\mathbf{n}\|\mathbf{q}]$ is connected and $H^1(\mathscr{O}_X) = 0$. Note that the canonical bundle of X is trivial by the adjunction formula. Therefore we only need to prove that $H^0(X, \mathbb{C})$ and $H^1(X, \mathbb{C})$ have dimension one and zero respectively.

Pick a general section $(s_j) \in \bigoplus_{j=1}^m H^0(V, \mathscr{L}_j)$ for which the divisor $D_j := Z(s_j)$ is a smooth and connected with all subsets of the D_j's meeting transversely. By Theorem 2.44 and $\sum_{j=1}^m D_j$ is ample divisor, we get exact sequences, for $i = 0, 1$,

$$0 \to H^i(V, \mathbb{C}) \cdots \to \bigoplus\nolimits_{|J|=r} H^i(D_J, \mathbb{C}) \to \cdots H^i(X, \mathbb{C}) \to 0 \qquad (5.1.4)$$

where $D_J := D_{j_1} \cap \cdots \cap D_{j_r}$ for a multi-index $J = (j_1, \cdots, j_r)$ of length $|J| = r$ with $1 \leqslant j_1 < \cdots < j_r \leqslant m$ and $X = \bigcap_{|J|=m} D_J$. Note that $i + m < \dim V$ for $i = 0, 1$.

We notice that if all $q_j^{i_s} = 0$ for some i_s then D_J is of the form $D_J' \times \mathbb{P}^{n_{i_s}}$ where D_J' is a complete intersection in $\prod_{i \neq i_s} \mathbb{P}^{n_i}$. In particular, $H^1(D_j, \mathbb{C})$ is zero by Lefschetz hyperplane theorem for all $1 \leqslant j \leqslant m$.

By induction, it follows that the dimension of $\bigoplus_{|J|=r} H^0(D_J, \mathbb{C})$ is $\binom{m}{r}$ and of $\bigoplus_{|J|=r} H^1(D_J, \mathbb{C})$ is zero for the length $r < m$. We remark that the induction process works because every D_J has the form $D_J' \times \prod \mathbb{P}^{n_l}$ with $D' = \sum D_j'$ is ample. Hence the connectedness and simple connectedness of D_J can be proved in the similar way as shown before. Using the sequence (5.1.4) and dimension counting, we get the dimension of $H^0(X, \mathbb{C})$ and $H^1(X, \mathbb{C})$ are one and zero respectively. $\square$

As a byproduct of Theorem 2.44, we obtain the following second Betti number formula:

Proposition 5.11. *With the notation as in the proof of Proposition 5.10,*

$$b_2(X, \mathbb{C}) = (-1)^m \left(m + \sum_{r=1}^{m-1} (-1)^r \sum_{|J|=r} b_2(D_J, \mathbb{C}) \right).$$

Moreover, the second Betti number of X equals the second Betti number of the ambient space V if $b_2(D_J, \mathbb{C}) = b_2(V, \mathbb{C})$ for each $1 \leqslant |J| < m$.

Proof. By $V = \prod_{i=1}^m \mathbb{P}^{n_i}$ and Künneth formula, the second Betti number of V equals m. Since $\dim V > m + 2$, the exact sequence (5.1.4) holds for $i = 2$ and the proposition follows. □

To know the full topological data of X, including the Hodge number $h^{1,1}(X)$, $h^{2,1}(X)$ and the Euler number $e(X) = 2(h^{1,1}(X) - h^{2,1}(X))$, in a CICY configuration matrix $[\mathbf{n} \| \mathbf{q}]$, it suffices to compute either one of these two Hodge numbers. These calculated in [18] for the 7868 CICY matrices constructed in [3]. Finding those Hodge numbers corresponding to a given matrix is, in principle, just a matter of looking up the *relevant* matrix in the list.

Proposition 5.11 gives a direct calculation of $h^{1,1}(X) = b_2(X, \mathbb{C})$ for X in *any* given CICY configuration matrix.

Example 5.12. Consider

$$X \in \left[\begin{array}{c\|ccccc} 4 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 \end{array} \right],$$

which was given in the appendix of [18]. Applying Lefschetz hyperplane theorem, Künneth formula and Proposition 5.11, we get

$$b_2(X, \mathbb{C}) = b_2(\mathbb{P}^4, \mathbb{C}) + b_2(D, \mathbb{C})$$

where $D \in \left[\begin{array}{c\|cc} 2 & 1 & 1 \\ 2 & 1 & 1 \end{array} \right]$ is a smooth surface with Euler number 6. Therefore $b_2(D) = e(D) - 2 = 4$ and the second Betti number of X is 5.

5.2 Determinantal Contractions

In this section, we will recall the definition of determinantal contractions between CICY 3-folds in products of projective spaces which is introduced in [3]. The main tool used in the section is the Bertini-type theorem introduced in the preliminary section.

Let $\widehat{\mathscr{C}}$ be a CICY configuration $k \times (m+1)$-matrix of the type

$$\left[\begin{array}{c\|cccc} n & 1 & \cdots & 1 & O \\ \mathbf{N} & c_1 & \cdots & c_{n+1} & \mathbf{M} \end{array}\right]$$

where $\mathbf{M}$ is a matrix and $\mathbf{N}$, c_j's are column vectors. We have the formal contraction

$$\widehat{\mathscr{C}} \twoheadleftarrow\!\!\!\longrightarrow \mathscr{C} = \left[\begin{array}{c\|cc} \mathbf{N} & \sum_{j=1}^{n+1} c_j & \mathbf{M} \end{array}\right].$$

We are going to define a determinantal contraction for the formal contraction $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\!\longrightarrow \mathscr{C}$ and find a morphism $\pi : \widehat{X} \to X$ with each fiber is a point or a projective line in $\mathbb{P}^n$ where $\pi : \mathbb{P}^n \times Y \to Y$ is the projection and $\widehat{X}$, $X := \pi(\widehat{X})$ is a member of the configuration $\widehat{\mathscr{C}}, \mathscr{C}$ respectively.

Let $\mathbf{N} = [n_l]$ and $P = \prod \mathbb{P}^{n_l}$. We rewrite the configuration

$$\widehat{\mathscr{C}} = \left[\begin{array}{c\|cccccc} n & 1 & \cdots & 1 & 0 & \cdots & 0 \\ P & \mathscr{L}_1 & \cdots & \mathscr{L}_{n+1} & \mathscr{L}_{n+2} & \cdots & \mathscr{L}_m \end{array}\right]$$

where $\mathscr{L}_j$'s are line bundles on P corresponding to c_l's and $\mathbf{M}$. Writing $[z_0; \cdots : z_n] \in \mathbb{P}^n$ and let $\widehat{X} \in \widehat{\mathscr{C}}$ be defined by global sections

$$\sum_{i=0}^{n} s_j^i(p) z_i = 0$$

and $t_l(p) = 0$ where $s_j^i \in H^0(P, \mathscr{L}_j)$ and $t_l \in H^0(P, \mathscr{L}_l)$ for $1 \leqslant j \leqslant n+1$ and $n+2 \leqslant l \leqslant m$. Set

$$\Delta(p) = \det(s_j^i(p))$$

which is a global section of the line bundle $\bigotimes_{j=1}^{n+1} \mathscr{L}_j$ on P. Since z_i cannot all vanish simultaneously, we have $\Delta(p) = 0$ for $(z, p) \in \mathbb{P}^n \times P$.

Obviously, the $X = \pi(\widehat{X})$ is defined by global sections

$$\Delta(p) = 0 \text{ and } t_l(p) = 0$$

for $n+2 \leqslant l \leqslant m$ and thus X belong to the configuration $\mathscr{C}$.

Definition 5.13. We say that a formal contraction $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}$ gives a determinantal contraction if there is a smooth irreducible member $\widehat{X}$ in $\widehat{\mathscr{C}}$ such that the morphism $\pi : \widehat{X} \to X$ given in the above process is an isomorphism or a small resolution of a normal variety $X \in \mathscr{C}$ with only isolated singularities.

It is easy to show that the determinantal contraction $\pi : \widehat{X} \to X$ is an isomorphism if an only if X is smooth (since X, $\widehat{X}$ are smooth minimal models).

The following theorem is the main result in this section. The proof will follow the idea outlined in [3] and apply Theorem 2.36.

Theorem 5.14. *Given a CICY configuration matrices $\widehat{\mathscr{C}}$ and $\mathscr{C}$ as above, the formal contraction $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}$ gives a determinantal contraction.*

Proof. Let

$$\mathscr{L}'_j = \begin{cases} p^*\mathscr{O}_{\mathbb{P}^n}(1) \otimes \pi^*\mathscr{L}_j & \text{if } 1 \leqslant j \leqslant n+1, \\ \pi^*\mathscr{L}_j & \text{if } n+2 \leqslant j \leqslant m, \end{cases}$$

where p and π are the projections from $\mathbb{P}^n \times P$ onto $\mathbb{P}^n$ and P respectively.

By Proposition 5.10, there are Zariski open sets U_j in $H^0(\mathbb{P}^n \times P, \mathscr{L}'_j)$ for each $1 \leqslant j \leqslant m$ such that $\widehat{X} = Z(\sigma)$ is smooth and irreducible for $\sigma \in \prod_{j=1}^m U_j$. Since $\mathscr{L}'_j$ is a line bundle on the product of projective spaces, we have, for $1 \leqslant j \leqslant n+1$,

$$\begin{aligned} H^0(\mathbb{P}^n \times P, \mathscr{L}'_j) &\simeq H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \otimes H^0(P, \mathscr{L}_j) \\ &\simeq \bigoplus_{i=1}^{n+1} \left(H^0(P, \mathscr{L}_j) \cdot z_{i-1} \right) \end{aligned}$$

and, for $n+2 \leqslant j \leqslant m$,

$$H^0(\mathbb{P}^n \times P, \mathscr{L}'_j) \simeq H^0(P, \mathscr{L}_j)$$

where $\{z_0, \cdots, z_n\}$ is a basis of $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$. Therefore we can identify Zariski open sets

$$U_j \simeq \prod_{i=1}^{n+1} U_{ij} \cdot z_{i-1}$$

where U_{ij}'s are Zariski open sets of $H^0(P, \mathscr{L}_j)$ for $1 \leqslant j \leqslant n+1$.

On the other hand, we consider $\tau : \bigoplus_1^{n+1} \mathscr{O}_P \to \bigoplus_{j=1}^{n+1} \mathscr{L}_j$ a morphism of vector bundles. By Theorem 2.36, there are Zariski open sets V_{ij} of $H^0(P, \mathscr{L}_j)$ for $1 \leqslant i, j \leqslant n+1$ such that the expected codimension of the degeneracy loci $D_{n-1}(\tau)$ and $D_{n-2}(\tau)$ in P are four and nine for sections $\tau = [s_{ij}] \in \prod_{i,j=1}^{n+1} V_{ij}$.

Now we define a Zariski open subset

$$U = \prod_{j=1}^{n+1} \left(\prod_{i=1}^{n+1} \left((U_{ij} \cap V_{ij}) \cdot z_{i-1} \right) \right) \times \prod_{j=n+2}^{m} U_j$$

of the space $\bigoplus_{j=1}^m H^0(\mathbb{P}^n \times P, \mathscr{L}_j')$. It remains to show that there is a determinantal contraction π of $\widehat{X} = Z(\sigma)$ for some section $\sigma \in U$.

Pick a section $\sigma = (\sum_i s_j^i z_i, t_j) \in U$, we notice that, for $p \in P$, the dimension of $\pi^{-1}(p)$ is less than two if and only if the corank of the matrix $[s_j^i(p)]$ is less than or equal to two, i.e. $\operatorname{rank}[s_j^i(p)] \geqslant n-1$. Since the number of sections t_j's is equal to $\dim P - 4$ and the expected codimension $D_{n-2}([s_j^i])$ and $D_{n-1}([s_j^i])$ are nine and four, we may assume that $Y := Z(t_{n+2}, \cdots, t_m)$ is smooth and the intersection of Y with $D_{n-2}([s_j^i])$ and $D_{n-1}([s_j^i])$ are empty and isolated points respectively (by taking a suitable $(t_l) \in \prod U_l$).

According to that $X = \pi(\widehat{X})$ is defined by $\Delta|_Y = \det(s_j^i)|_Y$ on the smooth variety Y and is irreducible, it follows that X is integral. Since $\widehat{X} = Z(\sigma)$ is a smooth variey, we have now derived that, for such section $\sigma = (\sum_i s_j^i z_i, t_j)$, the morphism $\pi : \widehat{X} \to X$ is a small resolution of the normal variety X with only isolated singularities (which equals $Y \cap D_{n-1}([s_j^i])$). Hence the formal contraction $\widehat{\mathscr{C}} \longleftrightarrow \mathscr{C}$ gives a determinantal contraction. $\square$

Remark 5.15. If corank of $[s_j^i(p)]$ is 1 or 2 then the solution space of the matrix defines a point or a projective line in $\mathbb{P}^n$ respectively. Namely, each fiber of π is a point or a projective line in $\mathbb{P}^n$.

Corollary 5.16. *With notation as in the proof of Theorem 5.14. For the determinantal contraction $\pi : \widehat{X} \to X$, the number of singularities of X is equal to*

$$\int_P \left(c_2(\mathscr{E})^2 - c_1(\mathscr{E}) c_3(\mathscr{E}) \right) c_{m-n-1}(\mathscr{F})$$

where $\mathscr{E} = \bigoplus_{i=1}^{n+1} \mathscr{L}_i$ and $\mathscr{F} = \bigoplus_{i=n+2}^{m} \mathscr{L}_i$ on P, which are vector bundles of rank $n+1$ and $m-n-1$ respectively.

Proof. As in the proof of Theorem 5.14, the number of singularities of X equals the intersection number $[D_{n-1}([s_j^i])] \cap [Z(t_{n+2}, \cdots, t_m)] \cap [P]$. By [13, Theorem 14.4, Example 14.4.1], for the smooth general member $\widehat{X}$ which is defined by a general section $\sigma = (\sum_i s_j^i z_i, t_j)$, the fundamental classes $[D_{n-1}([s_j^i])]$ and $[Z(t_{n+2}, \cdots, t_m)]$ are $\big(c_2(\mathscr{E})^2 - c_1(\mathscr{E})c_3(\mathscr{E})\big) \cap [P]$ and $c_{m-n-1}(\mathscr{F}) \cap [P]$ respectively. This completes the proof. □

Remark 5.17. If $\pi : \widehat{X} \to X$ is an isomorphism, that is $\mathrm{Sing}(X) = \varnothing$, the $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\!\rightarrow \mathscr{C}$ is referred to as an ineffective splitting in [3, p.512]. By Proposition 3.10, it is ineffective iff X and $\widehat{X}$ have the same Euler characteristic iff the intersection

$$D_{n-1}([s_j^i]) \cap Z(t_{n+2}, \cdots, t_m)$$

is empty. In the case $n = 1$, the intersection is defined by $\dim P$ sections s_j^i and t_l. Therefore the (formal) splitting is ineffective iff the intersection number

$$c_2(\mathscr{E})^2 \cap [P] = D_1. \cdots .D_{\dim P} = 0$$

where $D_1, \cdots, D_{\dim P}$ is Cartier divisors defined by $s_1^0, s_1^1, s_2^0, s_2^1, t_l$'s respectively.

5.3 Connecting the CICY Web

We are now ready to prove the connectedness of parameter spaces of CICY configuration matrices.

Theorem 5.18. *Any two (parameter spaces of) complete intersection Calabi–Yau 3-folds in products of projective spaces are connected by a finite sequence of conifold transitions.*

Proof. By Proposition 5.9 and Theorem 5.14, every CICY configuration matrices connect formally and each formal contraction gives a determinantal contraction $\widehat{X} \to X$, which is an isomorphism or a small projective resolution, say $X \in \mathscr{C}$. According to Proposition 5.4 and Corollary 5.16, it follows that $e(\widehat{X}) - e(\widetilde{X}) = 2\,|\mathrm{Sing}(X)|$, where $\widetilde{X} \in \mathscr{C}$ is a general smooth member. By Proposition 4.8, the singularities of X are ODPs. Hence each parameter space $[\mathbf{n}\|\mathbf{q}]$ connects to the simple one $\mathscr{C}_{1111}$ by conifold transitions. $\square$

Example 5.19. Consider the smooth CICY 3-fold $\widehat{X}$ in $\mathbb{P}^1 \times \mathbb{P}^4$ defined by

$$p_j^0(z)t_0 + p_j^1(z)t_1 = 0 \text{ for } j = 1, 2,$$

where t_0, t_1 are homogeneous coordinates on $\mathbb{P}^1$, $p_1^0(z), p_1^1(z)$ are two general quartic polynomials and $p_2^0(z), p_2^1(z)$ are two linear polynomials on $\mathbb{P}^4$. Since t_i's can not both vanish, it must be the case that the determinant $\Delta(z) := \det(p_j^i(z))$ resulting from the projection along $\mathbb{P}^1$ vanishes. If we take $p_2^i(z) = z_i$ for $i = 0, 1$ and suitable quartic polynomials $p_1^0(z), p_1^1(z)$, then the quintic X defined by $\Delta(z)$ has 16 ODPs, where $p_j^i(z)$'s vanish simultaneously, along a projective plane in $\mathbb{P}^4$. Let $\widetilde{X}$ be a smooth quntic in $\mathbb{P}^4$. Note that all quntics in $\mathbb{P}^4$ are deformation equivalent inside a flat family (c.f. Proposition 5.2). Hence we get a conifold transition $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ which connects parameter spaces of $\widehat{X}$ and X.

Example 5.20 (Fiber products of elliptic surfaces). We consider

$$\widehat{\mathscr{C}} := \left[\begin{array}{c\|cc} 2 & 3 & 0 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{array}\right] \longleftrightarrow \mathscr{C} := \left[\begin{array}{c\|c} 2 & 3 \\ 2 & 3 \end{array}\right].$$

It shall be related to the fiber products of rational elliptic surfaces which was investigated in [47].

Let $f_i : S_i \to \mathbb{P}^1$ be a relatively minimal, rational, elliptic surface with section for $i = 1, 2$. Then S_i is the 9-fold blowing up of $\mathbb{P}^2$ at the base points of a cubic pencil which induces the fibration f_i [32, IV.1.2], that is, there are generic homogeneous cubic polynomials a_i and b_i such that $S_i \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ is a resolution of indeterminacy of the rational map $C_i : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by $C_i(x) = [a_i(x) : b_i(x)]$. Obviously, S_i is defined by

$$P_i(z, x) = z_1 a_i(x) - z_0 b_i(x) = 0$$

where $[z_0 : z_1] \in \mathbb{P}^1$ and $x \in \mathbb{P}^2$.

Let $W = S_1 \times_{\mathbb{P}^1} S_2$. It is well known that W is a Calabi–Yau 3-fold [47]. It is easy to see that W can be obtained as a CICY in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ defined by P_1 and P_2. Therefore $W \in \widehat{\mathscr{C}}$ and is birational to a member in $\mathscr{C}$ which is defined by the bicubic polynomial $a_0(x)b_1(x) - a_1(x)b_0(x) = 0$.

The final remark is that we prefer to identify Calabi–Yau 3-folds which can be connected by a sequence of flops. The reason is that many invariants are preserved under flops, e.g. quantum invariance [30, 29] (see also [49] for a survey on recent development), the miniversal deformation spaces [25, (12.6)], analytic type of singularities [23, (4.11)], integral cohomology groups, etc. (see [24, (3.2.2)]). Here is an example to illustrate the principle.

Example 5.21 (Double solid). Consider the CICY configuration matrix

$$\mathscr{C} := \left[\begin{array}{c\|c} 3 & 4 \\ 1 & 2 \end{array}\right].$$

Let x, y be a basis of $H^0(\mathscr{O}_{\mathbb{P}^1}(1))$. Let $\mathscr{L}$ be the line bundle of multidegree $(3,2)$ on $P := \mathbb{P}^3 \times \mathbb{P}^1$ and $\Gamma = H^0(\mathscr{O}_{\mathbb{P}^3}(3))$. By Proposition 5.10, there is a Zariski open subset of

$$H^0(P, \mathscr{L}) \simeq (\Gamma \cdot x^2) \oplus (\Gamma \cdot xy) \oplus (\Gamma \cdot y^2)$$

such that each section in the open set defines a smooth Calabi–Yau 3-fold.

Choose general cubic polynomials A, B and C on $\mathbb{P}^3$ so that the Calabi–Yau $X \in \mathscr{C}$ defined by

$$Ax^2 + Bxy + Cy^2 = 0$$

is smooth and the octic hypersurface S in $\mathbb{P}^3$ defined by $\Delta := B^2 - 4AC \in H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(8))$ is smooth. Let $\widehat{X}$ be the double cover of $\mathbb{P}^3$ branched over S. Since S is smooth, we can show that $\widehat{X}$ is a smooth Calabi–Yau 3-fold. The Calabi–Yau $\widehat{X}$, called double solids, was firstly studied by Clemens.

We are going to prove that the Calabi–Yau X and the double solid $\widehat{X}$ are birational smooth Calabi–Yau 3-folds. Indeed, we choose a open set U in $\mathbb{P}^3$ such that $\mathscr{O}_{\mathbb{P}^3}(4)|_U \simeq \mathscr{O}_U$ and $A|_U$, $\Delta|_U$ are nowhere zero. Let $V = \{[x : y] \in \mathbb{P}^1 \mid y \neq 1\}$. On $W := U \times V$, we rewrite the equation

$$Ax^2 + Bxy + Cy^2 = \frac{A}{4}\left[(2x + \frac{B}{A}y)^2 - \Delta y^2\right].$$

Then we get a commutative diagram

$$\begin{tikzcd}
X & W \cap X \arrow[l, hook'] \arrow[rr, equal] \arrow[dr] & & \operatorname{Spec}_U \frac{\mathscr{O}_{\mathbb{P}^3}(U)[T]}{(T^2-\Delta)} \arrow[r, hook] \arrow[dl] & \widehat{X} \\
 & & U & &
\end{tikzcd}$$

and thus X and $\widehat{X}$ are birational, smooth minimal models. We know that birational, $\mathbb{Q}$-factorial 3- dimensional minimal models can be connected by a sequence of flops [21, 23]. Hence we do not distinguish between the smooth Calabi-Yau hypersurface X of multidegree $(4,2)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ and the double solid $\widehat{X}$.

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