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從早期宇宙到晚期宇宙與粒子物理學之關聯

Connecting Particle Physics with the Universe from Early Time to Late Time

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摘要

本論文主要將討論粒子物理與宇宙中兩個加速膨脹時期的關聯:第一為早期宇宙中的暴漲模型，第二是晚期宇宙的暗能量模型。關於暴漲模型，我們知道暴漲理論為現今宇宙物理學一個重要的假說，儘管尚未被觀測所證實，但因暴漲理論可以良好地解釋一些大爆炸理論未能解釋的宇宙學現象，因此普遍被宇宙學家所接受。然而暴漲理論本身的來源至今仍未有定論。偉恩伯格博士於 1979 年提出之漸進安全重力論，該理論認為重力常數跟宇宙常數可能並非常數，而會隨能量尺度改變而改變的參數，藉由量子場論中重整群的技巧，重力常數等參數將在高能量時流動到一個固定點，如此一來可以避免量子重力理論所遭遇到在高能量時發散產生奇異點的問題。等效地來說，漸進重力論是一個高能量完備且可重整的重力理論。該理論將誘導出一個新的純量場，我們考慮將此純量場作為引發暴漲的暴漲子，並同時考慮有希格子存在效應下所引發之雙場暴漲模型的物理，包含宇宙學背景演化、膨脹規模、以及暴漲退場進入輻射支配時期的機制。我們認為此暴漲子在暴漲末期將衰變成其他粒子，其衰變率由希格子的場值所控制，衰變結束後，牛頓重力常數及宇宙常數將回歸到符合觀測的觀測值。此外我們也研究此雙場模型的非線性量子起伏所產生的太初曲率微擾之頻譜與非高斯項，並且與近期普朗克衛星所發表的資料做比較。雖然觀測資料限制了本模型的參數空間，但本模型仍然提供了一個可能的關於粒子物理與宇宙學暴漲理論的有趣連結。

關於暗能量部份,暗能量主要是解釋我們所觀測到晚期宇宙的加速膨脹現象。與暴漲理論類似，暗能量的來源至今未有定論。本論文將討論一個可能的解釋暗能量來源的模型。我們考慮一個非準模型的旋量場，稱作 ELKO 旋量場或暗旋量場，該旋量場為 2005 年 Dr. Ahluwalia-Khalilova 和 Dr. Grumiller 所提出。不同於狄拉克旋量場，暗旋量場可以與撓場有更多交互作用，該交互作用即可能是暗能量的來源。我們考慮暗旋量場在愛因斯坦-卡當重力底下與撓場的交互作用，並研究其宇宙學演化。儘管假設該場的動能項具有幻能量的形式，我們發現該模型並不像其他幻能量模型一樣遭遇到各種暗能量奇異點的問題。此模型並且滿足能量條件定理，因此在量子層面也是穩定的。而我們的研究顯示最終擾場將消失，宇宙將進入德希特宇宙時期。

為求完整性,本論文將儘可能介紹所用到的宇宙學知識,從廣義相對論開始，接著暴漲理論的基本知識、宇宙學微擾、非高斯性、漸進安全重力論、漸進安全重力所引發的暴漲模型與普朗克衛星觀測資料的比較、愛因斯坦卡當重力論、暗能量、暗旋量場及其暗能量模型與觀測之比較等等。最後我們將總結本論文並且討論其未來相關的發展。

關鍵字：暴漲、希格子、漸進安全重力、暗旋量場、暗能量

Abstract

In this dissertation, we will study mainly two models, the first one is on inflation and second is on the dark energy. For the inflationary model, we consider a model inspired on asymptotic safe gravity which can induce a scalar field and we identify it as the inflaton. We also study the presence of another scalar field which can be interpreted as the Higgs field. We assume the reheating of the inflaton is controlled by the Higgs field. Firstly, we study the background trajectories of this model and it shows that our model may provide sufficient inflationary e-folds and a graceful exit to a radiation dominated phase. Then we study the possibility of generating primordial curvature perturbations through the Standard Model Higgs boson. This can be achieved under the choice of finely tuned parameters by making use of the modulated reheating mechanism. The primordial non-Gaussianity is expected to be sizable in this model. Though tightly constrained by the newly released Planck cosmic microwave background data, this model provides a potentially interesting connection between collider and early Universe physics.

As for the dark energy, we consider a class of dynamical dark energy models which are constructed through an extended version of fermion fields called the Elko spinors, which are spin one half with mass dimension one. We find that if the Elko spinor interacts with torsion fields in a homogeneous and isotropic universe, then we do not expect quantum instability in this kind of dark energy model even though the fermion possesses a negative kinetic energy. In other words, this dark energy model will asymptotically approach the equation of state $w = -1$ from above without crossing the phantom divide. Therefore, the stability is preserved, i.e. no phantom field will be created. Furthermore, we analyze as well the presence of some pressureless cold dark matter, and the result is unchanged, in this two components system. At late time, the torsion fields will vanish as the Elko spinors dilute, the equation of state will still converge to $w = -1$ and the Hubble parameter will approach a constant, the universe will eventually enter a de Sitter phase with or without the presence of this dark matter.

To make it as self-contained as possible, this dissertation will contain the essential knowledge and relative important issues about these two models, including the general relativity, Einstein-Cartan theory, the cosmological inflation, the cosmological perturbations, the asymptotic safe gravity, the Higgs-modulated inflation model, the dark energy in cosmology, the Elko spinors, the dark energy of phantom dark spinor with torsion. Finally, we

will briefly conclude this dissertation and discuss their future perspectives.

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Keywords— Inflation, Higgs, Asymptotic safe gravity, Dark spinor, Dark energy

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Chapter 1

Introduction

1.1 A Brief History of the Universe

First of all, let's start to briefly review the essential knowledge of the history of our Universe based on the Big Bang Cosmology, including the speculative era before nucleosynthesis. The central premise of modern cosmology is that, at least on large scales, our Universe is homogeneous and isotropic. This is supported by a variety of observations, most spectacularly the nearly identical temperature of cosmic microwave background (CMB) radiation coming from different parts of the sky. Despite the belief in homogeneity on large scales, it is all too apparent that in nearby regions our Universe is highly inhomogeneous. The temperature variations of CMB radiation bear testimony of minute fluctuations in the density of the primordial universe. These fluctuations grew via gravitational instability into the large-scale structures (LSS) that we observe in the universe today. It is believed that these irregularities have grown over time from a distribution that was more homogeneous in the past. Besides, there is undeniable evidence of the expansion of our Universe: the light from distant galaxies is systematically red-shifted, the observed abundance of the light elements (H, He, and Li) matches the predictions of Big Bang Nucleosynthesis (BBN), and the only convincing explanation for the CMB is a relic radiation from a hot early universe.

From 10^{-10} seconds to today the history is based on well understood and experimentally tested theories of particle physics, nuclear and atomic physics and gravity. We are therefore justified to have some confidence about the events shaping our Universe during that time. Let us enter the Universe at 100 GeV, the time of electroweak phase transition (10^{-10}). Above 100 GeV the electroweak symmetry is restored and the Z and $W^{\pm}$ bosons are massless. Interactions are strong enough to keep quarks and leptons in thermal equilibrium. Below 100 GeV the symmetry between electromagnetic and weak is broken, Z and $W^{\pm}$ bosons acquire mass and the cross-section of weak interaction decreases as the temperature of the Universe drops. As a result, at 1 MeV, neutrinos decouple from the rest of the matter. Shortly after, at 1 second, the temperature drops below the electron rest mass and electrons and positrons annihilate efficiently. Only an initial matter-antimatter asymmetry of one part in as billion survives. The resulting photon-baryon fluid is in equilibrium. Around 0.1 MeV the strong interaction becomes important and protons and neutrinos combine into the light elements (H, He, and Li) during BBN ($\sim 200s$). The successful prediction of the H, He, and Li abundance is one of the most striking consequence of the Big Bang theory. The matter and radiation densities are equal around 1 eV ($10^{11}s$). Charged matter particles and photons are strongly coupled in the plasma and fluctuations in the density propagate as cosmic "sound waves". Around 0.1 eV (380,000 yrs) protons and electrons combine into neutral hydrogen atoms. Photons decouple and form the free-streaming cosmic microwave background. 13.7 billion years later these photons gives us the earliest snapshot of our Universe. Anisotropies in the CMB temperature provide evidence for fluctuations in the primordial matter density.

These small density perturbations, $\rho(\vec{x},t) = \bar{\rho}(t)[1+\delta(\vec{x},t)]$, grow via gravitational instability to form large-scale structures observed in the late universe. A competition between the background pressure and the universal attraction of gravities determines the details of the growth of the structure. During radiation domination the growth is slow, $\delta \sim \ln a$ (where $a(t)$ is the scale factor describing the expansion of space). Clustering becomes more efficient after matter dominates the background density (and the pressure drops to zero), $\delta \sim a$. Small scales become non-linear first, $\delta \gtrsim 1$, and form gravitationally bound objects that decouple from the overall expansion. This leads to a picture of hierarchical structure formation with small-scale structures (like stars and galaxies) forming first and then merging into larger structures (clusters and superclusters of galaxies). Around redshift $z \sim 25$ ($1+z = a^{-1}$), high energy photons from the first stars begin to ionize the hydrogen in the inter-galactic medium. This process of "reionization" is completed at $z \approx 6$. Meanwhile, the most massive stars run out of nuclear fuel and explode as "supernovae". In these explosions the heavy elements (C, O, . . .) necessary for the formation of life are created, leading to the slogan "we are all stardust". At $z \approx 1$, a negative pressure "dark energy" comes to dominate the universe. The background spacetime is accelerating and the growth of structure ceases, $\delta \sim const.$

1.2 The First 10^{-10} Seconds

The history of the universe from 10^{10} seconds (1 TeV) to today is based on observational facts and tested physical theories like the Standard Model of particle physics, general relativity and fluid dynamics, e.g. the fundamental laws of high energy physics are well-established up to the energies reached by current particle accelerators ($\sim$ 1 TeV). Before 10^{10} seconds, the energy of the universe exceeds 1 TeV and we lose the comfort of direct experimental guidance. The physics of that era is therefore as speculative as it is fascinating.

To explain the fluctuations seen in the CMB temperature requires an input of primordial seed fluctuations. In these lectures we will explain the conjecture that these primordial fluctuations were generated in the very early universe ($\sim 10^{34}$ seconds) during a period of inflation. We will explain how microscopic quantum fluctuations in the energy density get stretched by the inflationary expansion to macroscopic scales, larger than the physical horizon at that time. After a perturbation exits the horizon no causal physics can affect it and it remains frozen with constant amplitude until it re-enters the horizon at a later time during the conventional (non-accelerating) Big Bang expansion. The fluctuations associated with cosmological structures re-enter the horizon when the universe is about 100,000 years olds, a short time before the decoupling of the CMB photons. Inside the horizon causal physics can affect the perturbation amplitudes and in fact leads to the acoustic peak structure of the CMB and the collapse of high-density fluctuations into galaxies and clusters of galaxies. Since we understand (and can calculate) the evolution of perturbations after they re-enter the horizon we can use the late time observations of the CMB and the LSS to infer the primordial input spectrum. Assuming this spectrum was produced by inflation, this gives us an observational probe of the physical conditions when the universe was 10^{34} seconds old. This fascinating opportunity to use cosmology to probe physics at the highest energies will be part of the subject of this thesis.

Chapter 2

General Relativity

General Relativity (GR) is Einstein's theory of space, time and gravitation. At heart, it is a very simple subject (compared, for example, to anything involving quantum mechanics); the essential idea is straightforward: while most forces of nature are represented by fields defined on spacetime (such as electromegnatic field, or the short-range fields characteristic of subnuclear forces), gravity is inherent in spacetime itself. In particular, what we experience as gravity is a manifestation of the curvature of spacetime. This leads to a slogan "Gravity is Geometry." In the context of GR, the dynamical field giving rise to gravitation is the metric tensor describing the curvature of spacetime itself, rather than some additional field propagating through spacetime; this was Einstein's insight. Follow this insight, we will introduce the field equation of the metric, which is the Einstein equation. Einstein's GR opens a door to the study of gravitation, and cosmology. Therefore, we will briefly review some basic knowledge of GR.

2.1 The Metric

We will assume our spacetime is a 4-dimensional Riemannian differentiable manifold, each point of spacetime can be labelled by a coordinate x^k with $k = 0, 1, 2, 3$. Every Riemannian manifold is equipped with a metric tensor $g_{\mu\nu}$, which defines the length of line elements:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.1}$$

For example, in the Euclidean 3-dimensional space, the line element is $ds^2 = dx^2 + dy^2 + dz^2$, the metric tensor is thus $g_{ij} = \mathrm{diag}(1, 1, 1)$; similarly, in the theory of special relativity, Minkowski spacetime is assumed, the line element is $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, the metric tensor is $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

In GR, arbitrary metric is allowed, except some few conditions on the metric tensor is symmetric and (usually, but not always) nondegenerate, i.e. its determinant $g = \det(g_{\mu\nu}) \neq 0$. This allows us to define the inverse metric $g^{\mu\nu}$ via

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma}, \tag{2.2}$$

where δ^{μ}_{σ} is the Kronecker delta, $\delta^{\mu}_{\sigma} \equiv \mathrm{diag}(1, 1, 1, 1)$, standing for the identity. The symmetry of $g_{\mu\nu}$ implies that $g^{\mu\nu}$ is also symmetric. Just as in special relativity, the metric and its inverse can be used to raise or lower indices on tensors. Given two vectors V^{μ} and W^{ν}, we can define the inner product of them by

$$g(V, W) = g_{\mu\nu} V^{\mu} W^{\nu}. \tag{2.3}$$

A simple example of a nontrivial metric is provided by a 4-dimensional expanding spacetime,

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{2.4}$$

This describes a universe for which “space at a fixed moment of time” is a flat three dimensional Euclidean space, which is expanding as a function of time. This is a special case of a Robertson-Walker metric, one in which special slices are geometrically flat.

2.2 Geodesics

Given a generic metric $g_{\mu\nu}$ for a manifold, one can define the proper time for a test particle in a curve parameterized by $x^{\mu}(\lambda)$. The proper time (for a time-like path) is defined by the functional:

$$\tau = \int \left(-g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right)^{1/2} d\lambda, \tag{2.5}$$

where the integral is over the path. Take variation of the functional, one obtains

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^{\rho}}{d\lambda} \frac{dx^{\sigma}}{d\lambda} = 0. \tag{2.6}$$

This is the geodesic equation. In other words, the geodesic equation is the extremum of the proper time. The quantity, $\Gamma^{\mu}_{\rho\sigma}$, is called the Christoffel symbols, which is important in defining the connection of a metric. It is straightforward to solve the Christoffel symbols for the metric, the result is

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\alpha} (\partial_{\rho} g_{\sigma\alpha} + \partial_{\sigma} g_{\rho\alpha} - \partial_{\alpha} g_{\rho\sigma}). \tag{2.7}$$

Next, we will introduce the idea of covariant derivatives which is generalization of partial derivatives in the flat space. An covariant derivative is an operator that reduces to the partial derivative in flat space with inertial coordinates, but transforms as a tensor on an arbitrary manifold. In fact, the need of covariant derivative is obvious; equations such as $\partial_\mu T^{\mu\nu} = 0$ must be generalized to curved space somehow. We begin by requiring that a covariant derivative ∇ be a map from (k, l) tensor to $(k, l+1)$ tensor which has the following tow properties: (1) Linearity: $\nabla(T + S) = \nabla T + \nabla S$; (2) Leibnitz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$. If ∇ is going to obey the Leibnitz rule, it can always be written as the partial derivative plus some linear transformation. That is, to take the covariant derivative we first take the partial derivative, and then apply a correction to make the result covariant. It means that, for each direction μ, the covariant derivative ∇_μ will be given by the partial derivative ∂_μ plus a correction specified by a set of $n \times n$ matrices $(\Gamma_\mu)^\rho{}_\sigma$. For a vector V^ν, we therefore have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \tag{2.8}$$

Notice that in the second term the index originally on V has moved to Γ, and a new index is summed over. If this is the expression for the covariant derivative of a vector in terms of the partial derivative, we should be able to determine the transformation property of $\Gamma^\nu_{\mu\rho}$ by demanding that the left-hand side be a (1,1) tensor. That is, we want the transformation law to be

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu. \tag{2.9}$$

Combine Eqs. (3.1) and (3.2), we can obtain the transformation rule for the connection coefficients:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{2.10}$$

This is of course not a tensor transformation law; the second term on the rhs spoils it. This is because the connection coefficients are not the components of a tensor. They are constructed in such a way that the combination of Eq. (3.1) transforms like a tensor, therefore the extra terms in the transformation law of the partial derivative cancels exactly with the Γ's. If we further demand the covariant derivative to have additional two properties, such that: (3) it commutes with contractions: $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$, and (4) it reduces to the partial derivative on scalars: $\nabla_\mu \phi = \partial_\mu \phi$. Then, one can deduce the covariant derivative of a one-form ω_ν by using the fact that $\omega_\lambda V^\lambda$ is a scalar and $\nabla_\mu(\omega_\lambda V^\lambda) = \partial_\mu(\omega_\lambda V^\lambda)$, thus one has

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda. \tag{2.11}$$

Notice that covariant derivative is not unique in a manifold, that is to say, given a Riemannian manifold with a metric $g_{\mu\nu}$, there are still many choices of connection coefficients implying distinct notion of covariant differentiation. However, if we require that (5) the covariant derivative to be torsion-free: $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$, and (6) metric compatible: $\nabla_\rho g_{\mu\nu} = 0$, then the covariant derivative is unique, i.e. only one set of connection coefficients satisfies conditions (1) (6), such a set of connection coefficients is called the "Levi-Civita" connection. It is straightforward to solve the Levi-Civita connection coefficients with the metric tensor components:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{2.12}$$

We see that the Levi-Civita connection coefficients are exactly the same as the Christoffel symbol (Γ's) in the geodesic equation, Eq. (), that is why we use the same symbol for these two coefficients.

Now we can define the directional covariant derivative of a given curve $x^\mu(\lambda)$ to be

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu. \tag{2.13}$$

This is a map, defined only along the path, from a (k, l) tensor to a (k, l) tensor. One can define parallel transport of the tensor T along the path $x^\mu(\lambda)$ to be the requirement that the covariant derivative of T along the path vanishes:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \equiv \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = 0. \tag{2.14}$$

This equation is well-defined and known as the equation of parallel transport. For a vector it takes the form

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0. \tag{2.15}$$

If we take V^μ to be the tangent vector of the path $x^\mu(\lambda)$, which is $dx^\mu/d\lambda$, then a curve along which the tangent vector is parallel transported will satisfy the condition:

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{2.16}$$

Then, we see that it is exactly the geodesic equation, Eq. (2.6). Hence, a curve is geodesic if it parallel-transports its own tangent vector, in fact, this property is usually taken as the alternative definition of a geodesics.

2.3 Curvature

Roughly speaking, the concept of curvature is to measure how the "non-flatness" of a manifold is. In fact, parallel transport around a closed loop leaves a vector unchanged in a "flat" manifold, however, parallel transport of a vector around a closed loop in a curved space will lead to a transformation of the vector; the resulting transformation depends on the total curvature enclosed by the loop. It would be more useful to have a local description of the curvature at each point, which is what the Riemann curvature tensor is supposed to provide. Given two vector fields A^μ and B^ν, we imagine taking parallel transport of a vector V^μ by first moving it in the direction of A^μ, then along B^ν, then backward along A^μ, and then B^ν, to return to the starting point. We know the action is coordinate independent, so there should be a tensor tells us how the vector changes when it comes back to its starting point; it will be a linear transformation on a vector. Thus, we expect that this linear map, the change of this vector, δV^ρ, will depend on A, B, and V, we can write

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu, \tag{2.17}$$

where $R^\rho{}_{\sigma\mu\nu}$ is a $(1,3)$ tensor known as the Riemann tensor. Recall that the covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it had been parallel transported, since the covariant derivative of a tensor in a direction along which it is parallel transported is zero. The commutator of two covariant derivatives, then, measures the difference between parallel transporting the tensor first one way and then the other, versus the opposite ordering. Therefore, one obtains that

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_\mu \nu \nabla_\lambda V^\rho, \tag{2.18}$$

$$= (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho, \tag{2.19}$$

where we identify the first term as the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{2.20}$$

and the second term as the torsion tensor

$$T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda_{[\mu\nu]}. \tag{2.21}$$

For the torsion-free Levi-Civita connection, the torsion tensor simply vanishes. We can see that Riemann tensor measures the part of the commutator of covariant derivatives that is proportional to the vector field, while the torsion tensor measures the part that is proportional to the covariant derivative of the vector field; the second derivative doesn't enter at all. Thinking of the Riemann tensor as a map from three vector fields to a forth one, we have

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.22}$$

where $\nabla_X = X^\mu \nabla_\mu$. Similarly, thinking of the torsion tensor as a map from two vector fields to a third one, we have

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.23}$$

We summarize some properties of the Riemann tensor here (without proof), firstly, we lower the index, $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu}$, then Riemann tensor is invariant under interchange of the first pair of indices with the second:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \tag{2.24}$$

it is antisymmetric in its first and last two indices:

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} = -R_{\sigma\rho\mu\nu}. \tag{2.25}$$

The sum of cyclic permutations of the last three indices vanishes:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0, \tag{2.26}$$

which is equivalent to

$$R_{\rho[\sigma\mu\nu]} = 0. \tag{2.27}$$

With some work, we can prove further

$$R_{[\rho\sigma\mu\nu]} = 0. \tag{2.28}$$

With these symmetric properties, the number of independent components of Riemann tensor is $\frac{1}{12}n^2(n^2-1)$. In four dimensions, therefore the Riemann tensor has 20 independent components.

In addition to the algebraic symmetries, the Riemann tensor also obeys a differential identity, which constrains its relative value at different points:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0. \tag{2.29}$$

This is known as the Bianchi identity. Take trace of the first and third indices of the Riemann tensor, we can define the Ricci tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}. \tag{2.30}$$

The Ricci tensor associated with the Levi-Civita connection is automatically symmetric: $R_{\mu\nu} = R_{\nu\mu}$, as a consequence of the Riemann tensor. The trace of the Ricci tensor is callled the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}. \tag{2.31}$$

The Ricci tensor and Ricci scalar contain all information of the trace of the Riemann tensor, leaving us the trace-free parts. The trace free part of the Riemann tensor is called the Weyl tensor, which is defined by

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma}R. \tag{2.32}$$

This messy formula is designed so that all possible contractions of $C_{\rho\sigma\mu\nu}$ vanish, while it retains the symmetry of the Riemann tensor:

$$C_{\rho\sigma\mu\nu} = C_{\nu\rho\sigma}, \tag{2.33}$$
$$C_{\rho\sigma\mu\nu} = C_{[\rho\sigma][\mu\nu]}, \tag{2.34}$$
$$C_{\rho[\sigma\mu\nu]} = 0. \tag{2.35}$$

The Weyl tensor is only defined in three or more dimensions, and in three dimensions it vanishes identically. One of the most important property of the Weyl tensor is that it is invariant inder conformal transformations. For this reason, it is often known as the conformal tensor.

An especially important form of the Bianchi identity comes from contracting twice on Eq. (2.29):

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R. \tag{2.36}$$

We define the Einstein tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{2.37}$$

Then, the Bianchi identity, Eq. (2.36) gives

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{2.38}$$

The Einstein tensor will play the key role in GR and cosmology.

2.4 Einstein's Equation

Consider a (classical) field theory in which the dynamical variables are a set of fields ϕ_i, the action S generally expressed as in integral of a lagrangian $\mathscr{L}$,

$$S = \int \mathscr{L}(\phi_i, \nabla_{\mu}\phi_i)\sqrt{-g}\,d^n x. \tag{2.39}$$

For example, a scalar field theory ϕ in the curved spacetime can be written as

$$S_\phi = \int \left[-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right] \sqrt{-g} d^n x, \tag{2.40}$$

which would lead to an equation of motion

$$\Box \phi - \frac{dV}{d\phi} = 0, \tag{2.41}$$

where the covariant d'Alembertian is $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = \nabla^\mu\nabla_\mu$.

To construct the action for general relativity, note that the dynamical variable is now the metric $g_{\mu\nu}$. Since we know one can choose a coordinate such that the metric is in its canonical form and its first derivatives vanish at each point, the lagrangian scalar should contain at least second order derivatives of the metric for a non-trivial field theory. The Riemann tensor itself is second order derivative in the metric and we know that the Ricci scalar is the only independent scalar we can construct from the Riemann tensor. Therefore, the simplest independent scalar constructed from the metric, which is no higher than second in its derivatives, is the Ricci scalar. Hilbert proposed this simplest possible choice foe a lagrangian for GR,

$$S_H = \int \sqrt{-g} R \, d^n x, \tag{2.42}$$

which is known as the Hilbert action (or Einstein-Hilbert action). The equation of motion for the Hilbert action come from variation the action with the metric. By using the facts, $g^{\mu\nu}\delta g_{\nu\rho} = -g_{\mu\nu}\delta g^{\nu\rho}$, and the trace formula,

$$\det(M) = \exp \mathrm{Tr}(\ln(M)), \tag{2.43}$$

where M is arbitrary matrix, and the variation of the Christoffel symbol:

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}[2g_{\lambda(\mu}\nabla_{\nu)}(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})], \tag{2.44}$$

we obtain the variation of the Hilbert action with respect to the metric:

$$\delta S_H = \int d^n x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \delta g^{\mu\nu}. \tag{2.45}$$

Therefore, we arrive the equation of motion of the Hilbert action, the Einstein equation in vacuum, is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{2.46}$$

We derived the Einstein equation in "vacuum" because we only included the gravitational part of the action, no additional term for matter part. To get full Einstein equation, we consider

$$S = \frac{1}{16\pi G} S_H + S_M, \tag{2.47}$$

where S_M is the action for matter. Take similar procedure, which leads to

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}},$$

then, one obtains the complete Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = G_{\mu\nu} = 8\pi G \, T_{\mu\nu}, \tag{2.48}$$

where the energy-momentum tensor for matter is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.49}$$

For example, for the action of the single scalar field S_ϕ, Eq. (2.40), the energy-momentum tensor is

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi). \tag{2.50}$$

Note that the conservation law $\nabla^\mu T_{\mu\nu} = 0$ now is consistent with the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$.

2.5 Einstein-Cartan Gravity

In this section, we will introduce a simple generalization of GR, the Einstein-Cartan(-Sciama-Kibble) (EC) theory of gravity. EC theory is one, and maybe the simplest one of the modified gravitational theories, which is also based on the Einstein-Hilbert action [184], just like GR. It relaxes, however, the GR constraint on the affine connection, $\tilde{\Gamma}^k_{ij}$, to be symmetric in its lower two indices. Hence the anti-symmetric part of the affine connection, i.e. the Cartan torsion tensor $S_{ij}{}^k = \tilde{\Gamma}^k_{[ij]} = \frac{1}{2}(\tilde{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ji})$, which is a dynamical variable, independent of the Riemannian metric g_{ij} is also allowed [184]. The notation $[ij]$ stands for the anti-symmetrization of the tensor indices, defined by $T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji})$ for any tensor T_{ij}; similarly, the notation (ij) means symmetrization of the tensor indices, $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$. Quantities denoted with a tilde always take torsion into account. The torsion tensor has 24 independent components in general. Note that we still require the metric compactibility condition $\tilde{\nabla}^\rho g_{\mu\nu} = 0$, and the metric compactible affine connection with torsion can be written as [184]

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} - K_{ij}{}^k, \tag{2.51}$$

where Γ^k_{ij} is the usual Christoffel symbol, defined by $\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij})$, and $K_{ij}{}^k$ is called the contortion tensor, defined by [184]

$$K_{ij}{}^k = -S_{ij}{}^k - 2S^k{}_{(ij)} = -S_{ij}{}^k - S^k{}_{ij} - S^k{}_{ji}. \tag{2.52}$$

Note that the Cartan torsion tensor is anti-symmetric in its first two indices, $S_{ij}{}^k = -S_{ji}{}^k$, by definition; however, the contortion tensor is anti-symmetric in its last two indices, $K_{ij}{}^k = -K_i{}^k{}_j$. By virtue of the last two equations, the inverse relation between the torsion and the contortion tensor reads $S_{ij}{}^k = -K_{[ij]}{}^k$.

After introducing the Cartan torsion and the contortion tensor, we can now define the action of the Einstein-Cartan theory of gravity which is simply the Einstein-Hilbert action with torsion and metric which are regarded as independent variables:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}\tilde{R} + \tilde{\mathscr{L}}_m\right), \tag{2.53}$$

where we set the speed of light to be unity, $c = 1$, the gravitational coupling constant $\kappa = 8\pi G$, and $\tilde{\mathscr{L}}_m$ is the lagrangian density of matter minimally coupled to gravity. Before taking the variation of the action, it should be noted that the independent variables are the metric tensor g_{ij} and the torsion tensor $S_{ij}{}^k$, the contortion tensor $K_{ij}{}^k$ actually depends on the metric since we lower and rise some indices via g_{ij} [184]. Even though, in principle we should do the variation with respect to the metric and the torsion tensors, it is more convenient to vary with respect to the contortion tensors instead, since the affine connection can be separated into the torsion-free Christoffel symbol and the contortion tensor, and the relation between torsion and contortion is only algebraic. Thus, we will vary the total action with respect to the metric and the contortion tensors, and we obtain two field equations:

$$\tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij} = \kappa\tilde{\Sigma}_{ij}, \tag{2.54}$$

$$S^{ij}{}_k + \delta^i_k S^j{}_l{}^l - \delta^j_k S^i{}_l{}^l = \kappa\tau^{ij}{}_k, \tag{2.55}$$

where the first field equation is similar to the original Einstein equation, we define $\tilde{G}_{ij} \equiv \tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij}$, which is the Einstein tensor with torsion, $\tilde{\Sigma}_{ij}$ is the canonical energy-momentum tensor, and the second one is called the Cartan equation. Note that in general, $\tilde{R}_{ij}$ is no longer symmetric, so as the $\tilde{G}_{ij}$ due to the fact that affine connection is asymmetric $\tilde{\Gamma}^k_{ij} \neq \tilde{\Gamma}^k_{ji}$. We define the modified torsion tensor to be $T^{ij}{}_k \equiv S^{ij}{}_k + \delta^i_k S^j{}_l{}^l - \delta^j_k S^i{}_l{}^l$. The right hand side (rhs) of Eq.(2.5) is the spin tensor $\tau^{ij}{}_k$, which is defined by

$$\tau_k{}^{ji} = \frac{\delta \tilde{\mathscr{L}}_m}{\delta K_{ij}{}^k}. \tag{2.56}$$

The canonical energy-momentum tensor is given by

$$\tilde{\Sigma}_{ij} = \tilde{\sigma}_{ij} + \left(\widetilde{\nabla} + K_{lk}{}^l\right)\left(\tau_{ij}{}^k - \tau_j{}^k{}_i + \tau^k{}_{ij}\right), \tag{2.57}$$

where $\tilde{\sigma}_{ij}$ is the metric energy-momentum tensor, defined by

$$\tilde{\sigma}_{ij} = \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\,\tilde{\mathscr{L}}_m\right)}{\delta g^{ij}}, \tag{2.58}$$

and the second term in Eq. (2.7) is the correction to the energy-momentum tensor generated by spin-torsion interaction. Since the Cartan equation is, in general, a set of 24 algebraic equations rather than differential relations between torsion and spin fields, it means that there would be no torsion outside matter distribution with spin source. In other words, torsion cannot propagate through the spacetime outside the matter distribution with spin source [184]. Furthermore, we are able to substitute the torsion everywhere by the spin and eliminate the torsion from the formalism. It then leads to the so-called Einstein-Cartan equation,

$$G_{ij} = \kappa \widehat{\sigma}_{ij}, \tag{2.59}$$

where the effective energy-momentum tensor on rhs is given by [184, 226]

$$\begin{aligned} \widehat{\sigma}_{ij} &\equiv \tilde{\sigma}_{ij} + \kappa\left(-4\tau_i{}^k{}_{[l}\tau_{|j|}{}^l{}_{k]} - 2\tau_i{}^{kl}\tau_{jkl} + \tau^{kl}{}_i\tau_{klj}\right) \\ &\quad + \frac{1}{2} g_{ij}\left(4\tau_m{}^k{}_{[l}\tau^{ml}{}_{k]} + \tau^{klm}\tau_{klm}\right) \\ &\equiv \tilde{\sigma}_{ij} + \kappa u_{ij}, \end{aligned} \tag{2.10}$$

which is symmetric and obeys the usual conservation law $\nabla^j \hat{\sigma}_{ij} = 0$. In fact, note that the Einstein-Cartan equation can be rewritten without including any torsion term by simply substituting all the torsion terms with the spin tensor terms. For example, Eq. (2.10) can be rewritten without any torsion term as the metric energy-momentum tensor can be split as a pure metric term plus a spin tensor term. One can interpret Eqs. (2.9) and (2.10) as that the geometry is a result from the contribution of the matter field plus some spin-spin interaction. In summary, all the torsion terms disappear in both side of Eq. (2.9), however, torsion exists on both sides of Eq. (2.4).

Chapter 3

A Review of Inflation in Standard Cosmology

3.1 Big Bang Puzzles

In the conventional Big-Bang model, the universe is taken to be radiation-dominated at early times and matter-dominated at latter times, with a very late transition to vacuum-domination, as we have mentioned in the introduction. Although the Big-Bang model is successful in interpretation of observational data, such as BBN and CMB, one may still ask a philosophical question whether initial conditions giving rise such a universe we see now. The conventional Big-Bang model requires precisely such a fine-tuned set of initial condition to allow the universe to evolve to its current state. Within the conventional picture, the early universe need finely tuned to incredible precision to arrive our current status. In particular, two features of our universe seem highly nongeneric: its spatial flatness, and its high degree of isotropy and homogeneity. One of the major achievements of inflationary scenario provides such a mechanism that it explains the initial conditions of the universe. Via inflation, the universe could grow out of generic initial conditions. Before discussing inflation, we first describe three puzzles of the Big-Bang model which the inflation claims to solve.

3.1.1 The homogeneity problem

A first question is why the approximation of homogeneity and isotropy turns out to be so good. Indeed, inhomogeneities are unstable, because of gravitation, and they tend to grow with time. It can be verified for instance with the CMB that inhomogeneities were much smaller at the last scattering epoch than today. One thus expects that these homogeneities were still smaller further back in time. How to explain a universe so smooth in its past ?

3.1.2 The flatness problem

The flatness problem comes from considering the Fredmann equation (which is the Einstein equation in the FRW metric, we will introduce latter) in a universe with matter and radiation but no vacuum energy, which can be written as

$$H^2 = \frac{1}{3m_p}(\rho_R + \rho_M) - \frac{\kappa}{a^2}, \tag{3.1}$$

where ρ_R and ρ_M are the energy densities for the radiation and the matter, respectively, and κ/a^2 is called the curvature term and κ is a constant taking the values $1, 0$, or -1. The curvature term is proportional to a^{-2}, while the energy density is proportional to the scale factor $a(t)$, $\rho_M \propto a^{-3}$ and $\rho_R \propto a^{-4}$. This raises the question of why the ratio $(\kappa a^{-2})/(\rho/3m_p)$ is not much larger than unity, given that a has increased by a factor of 10^{30} since the Planck epoch. In other words, the density parameter $\Omega = 1$ is a repulsive fixed point in a matter/radition dominated universe, so why do we observe $\Omega \sim 1$ today?

3.1.3 The horizon problem

In FRW cosmology, the particle horizon is defined as the maximum distance that light can propagate between an initial time t_1 to some later time t:

$$\chi_p(t) = \int_{t_i}^{t} \frac{dt}{a(t)}. \tag{3.2}$$

The physical size of the particle horizon is $d_p(t) = a(t)\chi_p$. The particle horizon exists because there is finite amount of time since the Big-Bang singularity, and thus only a finite distance that photons can travel within the age of the universe. Assume, for simplicity, we are in a matter-dominated universe, for which $a \propto t^{2/3}$, assume $a_0 = 1$. The Hubble parameter is therefore given by $H = \frac{2}{3}t^{-1} = a^{-3/2}H_0$. Then the photon travels a comoving distance

$$\triangle r = 2H_0^{-1}(\sqrt{a_2} - \sqrt{a_1}). \tag{3.3}$$

The comoving horizon size at any fixed value of the scale factor $a = a_*$ is the distance a photon travels since the Big-Bang,

$$r_{hor}(a_*) = 2H_0^{-1}\sqrt{a_*}. \tag{3.4}$$

The physical horizon size at some a_* is simply $d_{hor}(a_*) = a_* r_{hor}(a_*) = 2H_*^{-1}$. The horizon problem is simply the fact that CMB is isotropic to a high degree of precision, even though widely separated points on the last scattering surface are completely outside each other's horizons. When we look at the CMB we are observing the universe at the scale factor $a_{CMB} \sim 1/1200$, the comoving distance between a point on the CMB and an observer on Earth is

$$\triangle r = 2H_0^{-1}(1 - \sqrt{a_{CMB}}) \approx 2H_0^{-1}. \tag{3.5}$$

However, the comoving horizon distance for such a point is

$$r_{hor}(a_{CMB}) = 2H_0^{-1}\sqrt{a_{CMB}} \approx 6 \times 10^{-2} H_0^{-1}. \tag{3.6}$$

Hence, if we observe two widely separated parts of the CMB, they will have non-overlapping horizons; distinct patches of the CMB sky were causally disconnected at recombination. Nevertheless, they are observed to be at the same temperature at high precision. To question then is, how did they know ahead of time to coordinate their evolution in the right way, even though they were never in causal contact?

3.2 The Physics of Inflation

A solution to the horizon problem and to the other puzzles is provided by the inflationary scenario, which we will examine in the this section. The basic idea is to "decouple" the causal size from the Hubble radius, so that the real size of the horizon region in the standard radiation dominated era is much larger than the Hubble radius. Such a situation occurs if the comoving Hubble radius decreases sufficiently in the very early universe. The corresponding condition is

$$\ddot{a} > 0, \tag{3.7}$$

i.e. the Universe undergoes a phase of acceleration.

3.3 The FRW Universe

Recall that modern cosmology is based on the theory of general relativity, according to which our Universe is described by a four-dimensional geometry g that satisfies Einsteins equations, Eq. (2.48),

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.48}$$

One of the main assumptions of cosmology, which has been confirmed by observations so far, is to consider, as a first approximation, the universe as being homogeneous and isotropic. Note that these symmetries define implicitly a particular "slicing" of spacetime, in which the space-like hypersurfaces are homogeneous and isotropic. A different slicing of the same spacetime would give space-like hypersurfaces that are not homogeneous and isotropic.

Homogeneity and isotropy turn out to be very restrictive and the only geometries compatible with these requirements are the FRW (Friedmann-Robertson-Walker) spacetimes, with metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{3.8}$$

where $\kappa = 0, 1, 1$ determines the curvature of spatial hypersurfaces: respectively flat, elliptic or hyperbolic. Moreover, the matter content compatible with homogeneity and isotropy is necessarily characterized by an energy-momentum tensor of the form

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho(t), p(t), p(t), p(t)), \tag{3.9}$$

where ρ stands for the energy density and p for the pressure.

Substituting the metric, Eq. (3.8) and the energy-momentum tensor, Eq. (3.9) into Einsteins equations gives the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right) = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}, \tag{3.10}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{3.11}$$

which govern the time evolution of the scale factor $a(t)$.

An immediate consequence of the two above equations is the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{3.12}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. The continuity equation can also be obtained directly from the energy-momentum conservation $\nabla_\mu T^{\mu}_{\nu} = 0$. The cosmological evolution can be described by the equation of state for the matter once it is specified. Let us define

$$p = w\rho, \tag{3.13}$$

with w constant, which includes the two main types of matter that play an important role in cosmology, namely non relativistic matter ($w \simeq 0$) and a gas of relativistic particles ($w = 1/3$). The conservation equation can be integrated to give

$$\rho \propto a^{-3(1+w)}. \tag{3.14}$$

For $\kappa = 0$, one finds

$$a \propto t^{\frac{2}{3(1+w)}}, \tag{3.15}$$

which implies $a(t) \propto t^{1/2}$ for relativistic matter and $a(t) \propto t^{2/3}$ for non-relativistic matter. The present cosmological observations seem to indicate that our Universe is currently accelerating. The simplest way to account for this acceleration is to assume the presence of a cosmological constant Λ in Einsteins equations, i.e. an additional term $\lambda g_{\mu\nu}$ on the left-hand side of Eq. (2.48). By moving this term on the right hand side of Einsteins equations it can also be interpreted as an energy-momentum tensor with equation of state $P = \rho$, where ρ is time-independent. This leads, for $\kappa = 0$ and without any other matter, to an exponential evolution of the scale factor

$$a(t) \propto \exp(Ht). \tag{3.16}$$

In our universe, several species with different equations of state coexist, and it has become customary to characterize their relative contributions by the dimensionless parameters

$$\Omega_{(i)} \equiv \frac{8\pi G\rho_0^{(i)}}{3H_0^2}, \tag{3.17}$$

where the $\rho_0^{(i)}$ denote the present energy densities of the various species, and H_0 is the present Hubble parameter. The first Friedmann equation, Eq. (3.10), evaluated at the present time, implies

$$\Omega_0 = \Sigma_{(i)}\Omega_{(i)} = 1 + \frac{\kappa}{a_0^2 H_0^2} \tag{3.18}$$

One can infer from present observations the following parameters: $\Omega_m \simeq 0.3$ for non-relativistic matter (which includes a small baryonic component $\Omega_b \simeq 0.05$), $\Omega_\Lambda \simeq 0.7$ for a "dark energy" component (compatible with a cosmological constant), $\Omega_\gamma \simeq 5 \times 10^5$ for the photons, and a total Ω_0 close to 1, i.e. no detectable deviation from flatness.

3.4 Inflation

The broadest definition of inflation is that it corresponds to a phase of acceleration of the universe,

$$\ddot{a} > 0. \tag{3.19}$$

In this sense, the current cosmological observations, if correctly interpreted, mean that our present universe is undergoing an inflationary phase. It is worth noting that many of the models suggested for inflation have been adapted to account for the present acceleration. We are however interested here in an inflationary phase taking place in the early universe, thus characterized by very different energy scales. Another difference is that inflation in the early universe must end to leave room to the standard radiation dominated cosmological phase. Cosmological acceleration requires, according to the second Friedmann equation, Eq. (3.10), an equation of state satisfying

$$p < -\frac{1}{3}\rho, \tag{3.20}$$

condition which looks at first view rather exotic. A very simple example giving such an equation of state is a cosmological constant, corresponding to a cosmological fluid with the equation of state

$$p = -\rho. \tag{3.21}$$

However, a strict cosmological constant leads to exponential inflation forever which cannot be followed by a radiation era. Another possibility is a scalar field, which we now discuss in some details.

3.4.1 Single scalar field inflation

Recall that the dynamics of a scalar field minimally coupled to gravity is governed by the action

$$S_\phi = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right), \tag{3.22}$$

where $g\,\det(g_{\mu\nu})$ and $V(\phi)$ is the potential of the scalar field. The corresponding energy-momentum tensor, obtained by varying the action Eq. (3.22) with respect to the metric, is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right). \tag{3.23}$$

In the homogeneous and isotropic geometry Eq. (3.8), the energy-momentum tensor is of the perfect fluid form, with the energy density $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, where one recognizes the sum of a kinetic energy and of a potential energy, and the pressure $p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. The equation of motion for the scalar field is the Klein-Gordon equation, obtained by taking the variation of the above action Eq. (3.22) with respect to the scalar field, $\nabla^\mu \nabla_\mu \phi = \frac{dV}{d\phi}$, which reduces to

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{3.24}$$

in a homogeneous and isotropic universe. The system of equations governing the dynamics of the scalar field and of the cosmological geometry is thus given by

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{3.25}$$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{3.26}$$

$$\dot{H} = -4\pi G\dot{\phi}^2. \tag{3.27}$$

The last equation can be derived from the first two and is therefore redundant.

3.4.2 The slow-roll conditions

The dynamics of Eqs. (3.25) (3.27) does not always give an accelerated expansion but it does so in the so-called slow-roll regime when the potential energy of the scalar field dominates over its kinetic energy. More specifically, the slow-roll approximation consists in neglecting the kinetic energy of the scalar field, $\dot{\phi}^2$, in Eq. (3.25) and its acceleration, $\ddot{\phi}$, in the Klein-Gordon equation, Eq. (3.25). One then gets the simplified system

$$H^2 \simeq \frac{8\pi G}{3}V, \tag{3.28}$$

$$3H\dot{\phi} + V' \simeq 0. \tag{3.29}$$

Let us now examine in which regime this approximation is valid. From Eq. (3.29), the velocity of the scalar field is given by

$$\dot{\phi} \simeq -\frac{V'}{3H}. \tag{3.30}$$

Substituting this relation into the condition $(\dot{\phi}^2/2) \ll V$ yields the requirement

$$\epsilon_V \equiv \frac{m_p^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1, \tag{3.31}$$

where we have introduced the reduced Planck mass $m_p \equiv 1/\sqrt{8\pi G}$ Alternatively, one can use the parameter

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}, \tag{3.32}$$

which coincides with ϵ_V at leading order in slow-roll, since $\varepsilon = \dot{\phi}^2/(2m_p^2 H^2)$. Similarly, $\ddot{\phi} \ll V$ implies, after using the time derivative of Eq. (3.30) and (3.28), the condition

$$\eta_V \equiv m_p^2 \frac{V''}{V} \ll 1. \tag{3.33}$$

In summary, the slow-roll approximation is valid when the conditions $\varepsilon_V, \eta_V \ll 1$ are satisfied by the potential, which means that the slope and the curvature of the potential, in Planck units, must be sufficiently small.

3.4.3 Number of e-folds

Inflation must last long enough, in order to solve the problems of the Hot Big Bang model. To investigate this question, one usually introduces the number of e-folds before the end of inflation, denoted N, and simply defined by

$$N = \ln \frac{a_{end}}{a}, \tag{3.34}$$

where a_{end} is the value of the scale factor at the end of inflation and a is a fiducial value for the scale factor during inflation. By definition, N decreases during the inflationary phase and reaches zero at its end.

In the slow-roll approximation, it is possible to express N as a function of the scalar field. Since $dN = d\ln a = Hdt = (H/\dot{\phi})d\phi$, one easily finds, using Eqs. (3.30) and (3.28), that

$$N(\phi) \simeq \int_{\phi}^{\phi_{end}} \frac{V}{m_p^2 V'} d\phi. \tag{3.35}$$

Given an explicit potential $V(\phi)$, one can in principle integrate the above expression to obtain N in terms of ϕ. This will be illustrated by our model for inflation. Let us now discuss the link between N and the present cosmological scales. If one considers a given scale characterized by its comoving wavenumber $k = 2\pi/\lambda$, this scale crossed out the Hubble radius, during inflation, at an instant $t_*(k)$ defined by

$$k = a(t_*)H(t_*). \tag{3.36}$$

To get a rough estimate of the number of e-foldings of inflation that are needed to solve the horizon problem, let us first ignore the transition from a radiation era to a matter era and assume for simplicity that the inflationary phase was followed instantaneously by a radiation phase that has lasted until now. During the radiation phase, the comoving Hubble radius $(aH)^{-1}$

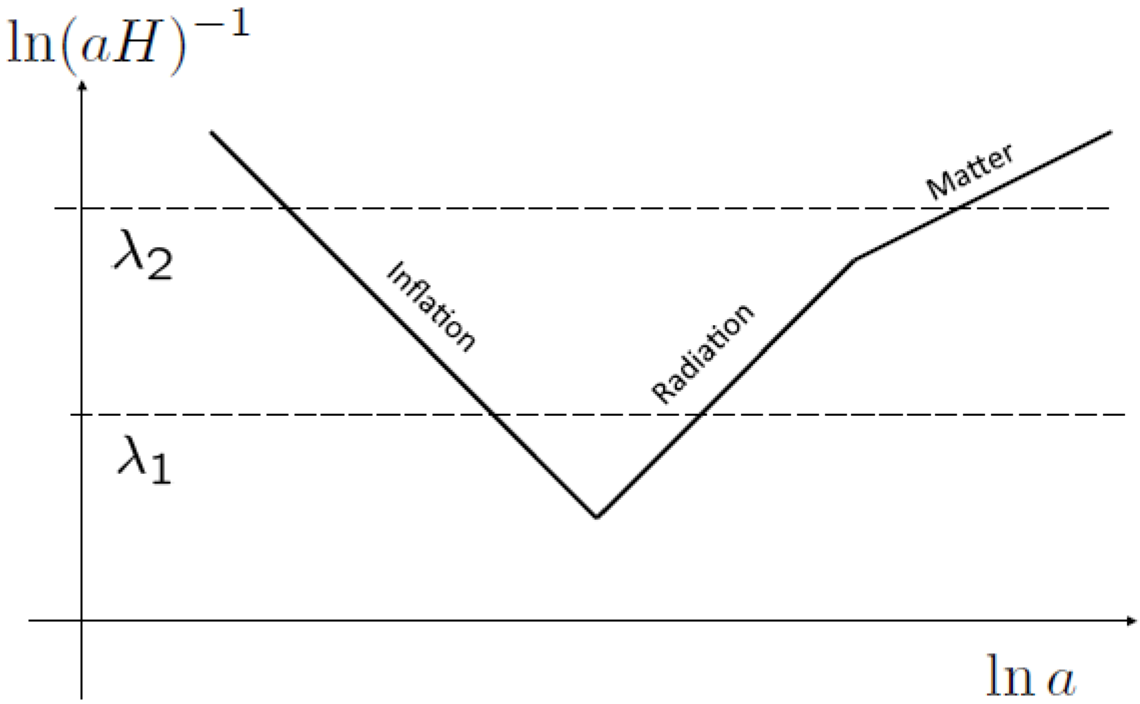

Figure 3.1: Evolution of the comoving Hubble radius $\lambda H = (aH)^{-1}$, during inflation, radiation dominated era and matter dominated era. The horizontal dashed lines correspond to two different comoving lengthscales: the larger scales cross out the Hubble radius earlier during inflation and re-enter the Hubble radius later in the standard cosmological era.

increases like a. In order to solve the horizon problem, the increase of the comoving Hubble radius during the standard evolution must be compensated by at least a decrease of the same amount during inflation. Since the comoving Hubble radius roughly scales like a^1 during inflation, the minimum amount of inflation is simply given by the number of e-folds between the end of inflation and today

$$\ln(a_0/a_{end}) = \ln(T_{end}/T_0) \sim \ln(10^{29}(T_{end}/10^{16}GeV)), \tag{3.37}$$

i.e. around 60 e-folds for a temperature $T \sim 10^{16}$GeV at the beginning of the radiation era. As we will see later, this energy scale is typical of inflation in the simplest models.

This determines roughly the number of e-folds $N(k_0)$ between the moment when the scale corresponding to our present Hubble radius $k_0 = a_0 H_0$

exited the Hubble radius during inflation and the end of inflation. The other lengthscales of cosmological interest are smaller than k_0^{-1} and therefore exited the Hubble radius during inflation after the scale k_0, whereas they entered the Hubble radius during the standard cosmological phase (either in the radiation era for the smaller scales or in the matter era for the larger scales) before the scale k_0 (see Fig. (3.1)).

A more detailed calculation, which distinguishes between the energy scales at the end of inflation and after the reheating, gives for the number of e-folds between the exit of the mode k and the end of inflation

$$N(k) \simeq 62 - \ln \frac{k}{a_0 H_0} + \ln \frac{V_k^{1/4}}{10^{16} GeV} + \ln \frac{V_k^{1/4}}{V_{end^{1/4}}} + \frac{1}{3} \ln \frac{\rho_{reh}^{1/4}}{V_{end}^{1/4}}. \tag{3.38}$$

Since the smallest scale of cosmological relevance is of the order of 1 Mpc, the range of cosmological scales covers about 9 e-folds. The above number of e-folds is altered if one changes the thermal history of the universe between inflation and the present time by including for instance a period of so-called thermal inflation.

3.5 Reheating

After inflation ends the scalar field begins to oscillate around the minimum of the potential. During this phase of coherent oscillations the scalar field acts like pressureless matter

$$\frac{d\bar{\rho}_\phi}{dt} + 3H\bar{\rho}_\phi = 0. \tag{3.39}$$

The coupling of the inflaton field to other particles leads to a decay of the inflaton energy

$$\frac{d\bar{\rho}_\phi}{dt} + (3H + \Gamma_\phi)\bar{\rho}_\phi = 0. \tag{3.40}$$

The coupling parameter Γ_ϕ depends on the model of physical processes. In our thesis, we will consider a reheating mechanism called Higgs modulated reheating, which the decay rate of the inflaton is controlled by the Higgs boson. Eventually, the inflationary energy density is converted into standard model degrees of freedom and the hot Big Bang commences.

3.6 Quantum Fluctuations and Cosmological Perturbations

In cosmology, inhomogeneities grow because of the attractive nature of gravity, which implies that inhomogeneities were much smaller in the past. As a consequence, for most of their evolution, inhomogeneities can be treated as linear perturbations. The linear treatment ceases to be valid on small scales in our recent past, hence the difficulty to reconstruct the primordial inhomogeneities from large-scale structure, but it is quite adequate to describe the fluctuations of the CMB at the time of last scattering. This is the reason why the CMB is currently the best observational probe of primordial inhomogeneities. In this section, we concentrate on the perturbations of the inflaton and show how the accelerated expansion during inflation converts its initial vacuum quantum fluctuations into“macroscopic” cosmological perturbations. In this sense, inflation provides us with “natural” initial conditions. We will also see how the perturbations of the inflaton can be translated into perturbations of the geometry.

Let us now move to the perturbed inflaton field living in a perturbed cosmological geometry. In fact, Einsteins equations imply that scalar field fluctuations must necessarily coexist with metric fluctuations. A correct treatment, either classical or quantum, must therefore involve both the scalar field perturbations and metric perturbations. We thus need to resort to the theory of relativistic cosmological perturbations

3.6.1 Metric perturbation

The most general linear perturbation about the homogenous metric can be expressed as

$$ds^2 = a^2\{-(1+2A)d\tau^2 + 2B_i dx^i d\tau + (\delta_{ij} + h_{ij})dx^i dx^j\}, \tag{3.41}$$

where we have assumed, for simplicity, a spatially flat background metric[1]. We have introduced a time plus space decomposition of the perturbations. The indices i, j stand for spatial indices and the perturbed quantities defined in Eq. (3.41) can be seen as three-dimensional tensors, for which the indices can be lowered (or raised) by the spatial metric δ_{ij} (or its inverse). It is very convenient to separate the perturbations into three categories, the so called

[1] This is all the more justified given that the metric in the early Universe was closer to a spatially flat metric than our present metric, which is itself indistiguishible from a flat geometry, according to observations.

"scalar", "vector" and "tensor" modes. For example, a spatial vector field B^i can be decomposed uniquely into a longitudinal part and a transverse part,

$$B^i = \partial_i B + \bar{B}_i, \quad \partial_i \bar{B}^i = 0, \tag{3.42}$$

where the longitudinal part is curl-free and can thus be expressed as a gradient, and the transverse part is divergenceless. This yields one "scalar" mode, B , and two "vector" modes $\bar{B}^i$ (the index i takes three values but the divergenceless condition implies that only two components are independent). A similar procedure applies to the symmetric tensor h_{ij}, which can be decomposed as

$$h_{ij} = 2C\delta_{ij} + 2\partial_i \partial_j E + 2\partial_{(i} E_{j)} + \bar{E}_{ij}, \tag{3.43}$$

with $\bar{E}^{ij}$ transverse and traceless (TT), i.e. $\partial_i \bar{E}^{ij} = 0$ (transverse) and $\delta_{ij}\bar{E}^{ij} = 0$ (traceless), and E_i transverse. The parentheses around the indices denote symmetrization, namely $2\partial_{(i}E_{j)} \equiv \partial_i E_j + \partial_j E_i$. We have thus defined two scalar modes, C and E, two vector modes, E_i, and two tensor modes, $\bar{E}_{ij}$.

Coordinate Transformations

The metric perturbations, introduced in Eq. (3.41), are modified in a coordinate transformation of the form

$$x^\alpha \to x^\alpha + \xi^\alpha, \quad \xi^\alpha = (\xi^0, \xi^i). \tag{3.44}$$

It can be shown that the change of the metric components can be expressed as

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)}, \tag{3.45}$$

using the symbol ∇ for the four-dimensional covariant derivative, where the variation due the coordinate transformation is defined for the same old and new coordinates (and thus at different physical points). The above variation can be decomposed into individual variations for the various components of the metric defined earlier. One finds

$$A \to A - \xi^{0\prime} - \mathscr{H}\xi^0, \tag{3.46}$$

$$B \to B_i + \partial_i \xi^0 - \xi_i', \tag{3.47}$$

$$h_{ij} \to h_{ij} - 2\left(\partial_{(i}\xi_{j)} - \mathscr{H}\xi^0\delta_{ij}\right), \tag{3.48}$$

where $\mathscr{H} \equiv a'/a$. The effect of a coordinate transformation can also be decomposed along the scalar, vector and tensor sectors introduced earlier. The generator xi^α of the coordinate transformation can indeed be written as

$$\xi^\alpha = (\xi^\alpha, \partial^i\xi + \bar{\xi}^i), \tag{3.49}$$

with $\bar{\xi}^i$ transverse, which shows explicitly that ξ^α contains two scalar components, ξ^0 and ξ, and two vector components, $\bar{\xi}^i$. The transformations Eqs. (3.47) and (3.48) are then decomposed into:

$$B \to B + \xi^0 - \xi', \tag{3.50}$$

$$C \to C - \mathscr{H}\xi^0, \tag{3.51}$$

$$E \to E - \xi, \tag{3.52}$$

$$\bar{B}^i \to \bar{B}^i - \bar{\xi}^{i\prime}, \tag{3.53}$$

$$E^i \to E^i - \bar{\xi}^i. \tag{3.54}$$

The tensor perturbations remain unchanged since ξ^α does not contain any tensor component. To summarize, the whole system scalar field plus gravitation is described by eleven perturbations. They can be decomposed into five scalar quantities: A, B, C and E from the metric and $\delta\phi$; four vector quantities $\bar{B}^i$ and $\bar{E}^i$; two tensor quantities: the two polarizations of E_{ij}^{TT}. However, these quantities are physically redundant since the same physical situation can be described by different sets of values of these perturbations, provided they are related by the coordinate transformations described above. One would thus like to identify the true degrees of freedom, i.e. the physically independent quantities characterizing the system. One can reduce the effective number of degrees of freedom by using the four coordinate transformations, which consist of two scalar transformations and two vector transformations as we saw earlier. Moreover, Einsteins equations contain nondynamical equations, i.e. constraints, which are also the consequence of the invariance by coordinate transformations. They can be decomposed into two scalar constraints and two vector constraints. By taking into account the coordinate changes and the constraints, one finds three true degrees of freedom: two polarizations of the gravitational waves and one scalar degree of freedom. If matter was composed of N scalar fields, one would get N scalar degrees of freedom in addition to the two tensor modes.

In a coordinate transformation, the scalar field perturbation is also modified, according to

$$\delta\phi \to \delta\phi - \phi'\xi^0. \tag{3.55}$$

In single-field inflation, there are thus two natural choices of gauge to describe the scalar perturbation. The first is to work with hypersurfaces that are flat, i.e. $C = 0$, in which case we will denote the scalar field perturbation by Q, i.e.

$$Q = \delta\phi_{C=0}. \tag{3.56}$$

The other choice is to work with hypersurfaces where the scalar field is uniform, i.e. $\delta\phi = 0$, in which case the scalar degree of freedom is embodied by the metric perturbation $C_{\delta\phi} = 0$. In other words, the true scalar degree of freedom can be represented either as a pure matter perturbation or a pure metric perturbation. In the general case, we have

$$Q = \delta\phi - \frac{\phi'}{\mathscr{H}}C, \tag{3.57}$$

which is a gauge-invariant combination (often called the Mukhanov-Sasaki variable.

3.6.2 Quantizing the scalar degree of freedom

In order to quantize the true scalar degree of freedom, one needs the action that governs its dynamics. Let us first note that the linearized equations of motion for the coupled system (gravity + scalar field) are obtained from the expansion of the full action at second-order in the perturbations. Indeed the equations for the linear perturbations correspond to the Euler-Lagrange equations derived from a quadratic Lagrangian. In our case, the difficulty is that there are several scalar perturbations that are not independent. In order to quantize this coupled system, one can work directly with the second-order Lagrangian, or resort to a Hamiltonian approach.

The modern approach, introduced by Maldacena to study perturbations beyond linear order, is based on the Arnowitt-Deser-Misner (ADM) formalism. In the ADM approach, the metric is written in the form

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.58}$$

where N is called the lapse function and N^i the shift function. The full action for the scalar field and gravity

$$S = \int d^4x\sqrt{-g}\left[\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) + \frac{m_p^2}{2}R\right] \tag{3.59}$$

becomes, after substitution of Eq. (3.58),

$$S = \int d^3x dt\sqrt{h}N\left[\frac{\mathscr{V}^2}{2N^2} - \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi - V(\phi)\right] + \frac{m_p^2}{2}\int dt d^3x\frac{\sqrt{h}}{N}\left(E_{ij}E^{ij} - E^2\right), \tag{3.60}$$

where $h = \det h_{ij}$,

$$\mathscr{V} \equiv \dot{\phi} - N^j\partial_j\phi. \tag{3.61}$$

and the symmetric tensor E_{ij}, defined by

$$E_{ij} \equiv \frac{1}{2}\dot{h}_{ij} - N_{(i|j)}, \tag{3.62}$$

(the symbol $|$ denotes the spatial covariant derivative associated with the spatial metric h_{ij}) is proportional to the extrinsic curvature of the spatial slices.

The variation of the action with respect to N yields the energy constraint,

$$\frac{\mathscr{V}^2}{2N^2} + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + V(\phi) + \frac{m_p^2}{2N^2}(E_{ij}E^{ij} - E^2) = 0, \tag{3.63}$$

while e the variation of the action with respect to the shift N^i gives the momentum constraint,

$$m_p^2 \left(\frac{1}{N}(E_i^j - E\delta_i^j)\right)_{|j} = \frac{\mathscr{V}}{N}\partial_i\phi. \tag{3.64}$$

In order to study the linear perturbations about the FRW background, we now restrict ourselves to the flat gauge, which corresponds to the choice

$$h_{ij} = a^2(t)\delta_{ij}. \tag{3.65}$$

The scalar fields on the corresponding flat hypersurfaces can be decomposed as

$$\phi = \bar{\phi} + Q, \tag{3.66}$$

where $\bar{\phi}$ is the spatially homogeneous background value of the scalar field and Q represents its perturbation (on flat hypersurfaces). In the following, we will often omit the bar and simply write the homogeneous value as ϕ, unless this generates ambiguities.

We can also write the (scalarly) perturbed lapse and shift as

$$N = 1 + \alpha, \quad N_i = \beta_{,i}, \tag{3.67}$$

where the linear perturbations α and β are determined in terms of the scalar field perturbation Q by solving the linearized constraints. At first-order, the momentum constraint implies

$$\alpha = \frac{\dot{\phi}}{2m_p^2 H}Q, \tag{3.68}$$

while the energy constraint gives $\partial^2\beta$ in terms of Q and $\dot{Q}$.

3.6.3 Second order action

We now expand the action, up to quadratic order, in terms of the linear perturbations. This action can be written solely in terms of the physical degree of freedom Q by substituting the expression Eq. (3.68) for α (it turns out that β disappears of the second order action, after an integration by parts). The second order action can be written in the rather simple form

$$S_{(2)} = \frac{1}{2}\int dt d^3x \left[\dot{Q}^2 - \frac{1}{a^2}\partial_i Q \partial^i Q - \mathscr{M}^2 Q^2\right], \tag{3.69}$$

with the effective (squared) mass

$$\mathscr{M}^2 = V'' - \frac{1}{a^3}\frac{d}{dt}\left(\frac{a^3}{H}\dot{\phi}^2\right). \tag{3.70}$$

As we did earlier, it is convenient to use the conformal time τ and to introduce the canonical degree of freedom $v = aQ$ which leads to the action

$$S_v = \frac{1}{2}\int d\tau d^3x \left[v'^2 + \partial_i v \partial^i v + \frac{z''}{z}v^2\right], \tag{3.71}$$

with $z = a\frac{\phi'}{\mathscr{H}}$. This action is analogous to that of a scalar field in Minkowski spacetime with a timedependent mass.

The quantity we will be eventually interested in is the comoving curvature perturbation $\mathscr{R}$, which is related to the canonical variable v by the relation $v = z\mathscr{R}$. The power spectrum for v is given by

$$2\pi^2 k^{-3}\mathscr{P}_v(k) = |v_k|^2, \tag{3.72}$$

the corresponding power spectrum for $\mathscr{R}$ is found to be

$$2\pi^2 k^{-3}\mathscr{P}_{\mathscr{R}(k)} = \frac{|v_k|^2}{z^2}. \tag{3.73}$$

In the case of an inflationary phase in the slow-roll approximation, the evolution of ϕ and of H is much slower than that of the scale factor a. Consequently, one gets approximately

$$\frac{z''}{z} \simeq \frac{a''}{a}, \quad (slow-roll) \tag{3.74}$$

and all results obtained previously for u apply directly to our variable v in the slow-roll approximation. This implies that the properly normalized function corresponding to the Bunch-Davies vacuum is approximately given by

$$v_k \simeq \sqrt{\frac{\hbar}{2k}}e^{-ik\tau}(1 - \frac{i}{k\tau}). \tag{3.75}$$

In the super-Hubble limit $k|\tau| \ll 1$ the function v_k behaves like

$$v_k \simeq -\sqrt{\frac{\hbar}{2k}}\frac{i}{k\tau} \simeq i\sqrt{\frac{\hbar}{2k}}\frac{aH}{k}, \tag{3.76}$$

where we have used $a \simeq 1/(H\tau)$. Consequently, combining (113), (110) and (115) and reintroducing the cosmic time gives the power spectrum for R, on scales larger than the Hubble radius,

$$\mathscr{P}_{\mathscr{R}} \simeq \frac{\hbar}{4\pi^2}\left(\frac{H^4}{\dot{\phi}^2}\right)_{k=aH} = \frac{\hbar}{2m_p^2\varepsilon_*}\left(\frac{H_*}{2\pi}\right)^2, \tag{3.77}$$

where we have used $\varepsilon \equiv \dot{H}/H^2$ in the second equality, and the subscript $*$ means that the quantity is evaluated at Hubble crossing ($k = aH$). This is the main result for the spectrum of scalar cosmological perturbations generated from vacuum fluctuations during a slow-roll inflation phase.

3.6.4 Covariant approach

Instead of the traditional metric-based approach, we use here a more geometrical approach to cosmological perturbations, which will enable us to recover easily and intuitively the main useful results, not only for linear perturbations but also for non-linear perturbations. Let us consider a spacetime with metric gab and some perfect fluid characterized by its energy density ρ, its pressure P and its four-velocity u^a. The corresponding energy momentum-tensor is given by

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b).(123) \tag{3.78}$$

Let us also introduce the expansion along the fluid worldlines,

$$\Theta = \nabla_a u^a, \tag{3.79}$$

and the integrated expansion

$$\alpha = \frac{1}{3}\int d\tau_p \Theta, \tag{3.80}$$

where τ_p is the proper time defined along the fluid worldlines. In a FRW spacetime, one would find $\Theta = 3H$. Therefore, in the general case, one can interpret $\Theta/3$ as a local Hubble parameter and $S = \exp(\alpha)$ as a local scale factor, while α represents the local number of e-folds. Then, the conservation law for the energy-momentum tensor, $\nabla_a T^a_b = 0$, implies that the covector

$$\zeta_a \equiv \nabla_a \alpha - \frac{\dot{\alpha}}{\dot{\rho}}\nabla_a \rho \tag{3.81}$$

satisfies the relation

$$\dot{\zeta}_a \equiv \mathscr{L}_u \zeta_a = -\frac{\Theta}{3(\rho+p)} \left(\nabla_a p - \frac{\dot{p}}{\dot{\rho} \nabla_a \rho} \right), \tag{3.82}$$

where a dot denotes the time derivative defined as the Lie derivative along u^a[2]. This result is valid for any spacetime geometry and does not depend on Einsteins equations.

The covector ζ_a can be defined for the global cosmological fluid or for any of the individual cosmological fluids. Using the non-linear conservation equation

$$\dot{\rho} = -3\dot{\alpha}(\rho+p), \tag{3.83}$$

which follows from $u^b \nabla_a T^a_b = 0$, one can re-express ζ_a in the form

$$\zeta_a = \nabla_a \alpha + \frac{\nabla_a \rho}{3(\rho+p)}. \tag{3.84}$$

If $w \equiv P/\rho$ is constant, the above covector is a total gradient and can be written as

$$\zeta_a = \nabla_a \left[\alpha + \frac{1}{3(1+w)} \ln \rho \right]. \tag{3.85}$$

On scales larger than the Hubble radius, our definition agrees with the non-linear curvature perturbation on uniform density hypersurfaces which is defined as

$$\zeta = \delta N - \int_{\bar{\rho}}^{\rho} H \frac{d\tilde{\rho}}{\dot{\tilde{\rho}}} = \delta N + \frac{1}{3} \int_{\bar{\rho}}^{\rho} \frac{d\tilde{\rho}}{(1+w)\tilde{\rho}}, \tag{3.86}$$

where $N = \alpha$. The above equation is simply the integrated version of Eq. (3.81), or of Eq. (3.84).

3.6.5 Linear conserved quantities

Let us now introduce a coordinate system, in which the metric (with only scalar perturbations) reads

$$ds^2 = a^2\{-(1+2A)d\tau^2 + 2\partial_i B dx^i d\tau + [(1+2C)\,\delta_{ij} + 2\partial_i\partial_j E]\, dx^i dx^j\}. \tag{3.87}$$

We decompose the fluid four-velocty as

$$u^\mu = \bar{u}^\mu + \delta u^\mu, \quad \delta u^\mu = \{-A/a, v^i/a\}, \quad v_i = \partial_i v + \bar{v}_i, \tag{3.88}$$

[2]For scalar quantities, this is equivalent to an ordinary derivative along u^a (e.g. $\dot{\rho} \equiv u^a \nabla_a \rho$), but for ζ_a, one has $\dot{\zeta}_a \equiv u^b \nabla_b \zeta_a + \zeta_b \nabla_a u^b$

where $\bar{v}_i$ is transverse. At linear order, the spatial components of ζ_a are simply

$$\zeta_i^{(1)} = \partial_i \zeta^{(1)}, \quad \zeta^{(1)} \equiv \delta\alpha - \frac{\bar{\alpha}'}{\bar{\rho}'}\delta\rho, \tag{3.89}$$

where a prime denotes a derivative with respect to . Linearizing Eq. (3.82) implies that the curvature perturbation on uniform-energy-density hypersurfaces, defined by

$$\zeta = C - \mathscr{H}\frac{\delta\rho}{\rho'} = C + \frac{\delta\rho}{3(\rho + p)}, \tag{3.90}$$

obeys the evolution equation

$$\zeta' = -\frac{\mathscr{H}}{\rho + p}\delta P_{nad} - \frac{1}{3}\nabla^2 \left(E' + v\right), \tag{3.91}$$

where δP_{nad} d is the non-adiabatic part of the pressure perturbation, defined by

$$\delta P_{nad} = \delta P - c_s^2 \delta\rho. \tag{3.92}$$

Note that $\zeta^{(1)}$ differs from ζ but they coincide when the spatial gradients can be neglected, for instance on large scales. The expression Eq. (3.91) shows that ζ is conserved on super-Hubble scales in the case of adiabatic perturbations. Another convenient quantity, which is sometimes used in the literature instead of ζ , is the curvature perturbation on comoving hypersurfaces, which can be written in any gauge as

$$\mathscr{R} = -C - \frac{\mathscr{H}}{\rho + p}\delta q, \quad \partial_i \delta q \equiv \delta_{(s)} T_i^0, \tag{3.93}$$

where the subscript (S) denotes the perturbations of scalar type. For a perfect fluid, $\delta q = (\rho + P)v$, where v has been defined in Eq. (3.88). One can relate the two quantities ζ and R by using the energy and momentum constraints, which were derived earlier in the ADM formalism. Linearizing Eq. (3.63) and Eq. (3.64) yields, respectively,

$$3\mathscr{H}^2 \delta N + a\mathscr{H}\partial^2\beta = -\frac{a^3}{2m_p^2}\delta\rho, \tag{3.94}$$

$$\mathscr{H}\delta N = -\frac{a^3}{2m_p^2}\delta q. \tag{3.95}$$

Combining these two equations yields the relativistic analog of the Poisson equation, namely

$$\partial^2 \Psi = \frac{a^2}{2m_p^2}(\delta\rho - 3\mathscr{H}\delta q) \equiv \frac{a^2}{2m_p^2}\delta\rho_c, \tag{3.96}$$

where we have replaced β by the Bardeen potential $\Psi \equiv -C - \mathscr{H}(B - E') = -\mathscr{H}\beta$ and introduced the comoving energy density $\delta\rho_c \equiv \delta\rho - 3\mathscr{H}\delta q$. Since

$$\zeta = -\mathscr{R} + \frac{\delta\rho_c}{\rho + p} = -\mathscr{R} - \frac{2\rho}{3(\rho + p)}\left(\frac{k}{aH}\right)^2 \Psi, \tag{3.97}$$

one finds that ζ and $\mathscr{R}$ coincide in the super-Hubble limit $k \ll aH$.

3.7 Initial Conditions for Standard Cosmology

In standard cosmology, the initial conditions for the perturbations are usually defined in the radiation dominated era around the time of nucleosynthesis, when the main cosmological components are restricted to the usual photons, baryons, neutrinos and cold dark matter (CDM) particles. The scales that are cosmologically relevant today correspond to length-scales much larger than the Hubble radius at that time. Before the invention of inflation, “initial” conditions were put “by hand”, with the restriction that their late time consequences should be compatible with observations. Inflation now provides a precise prescription to determine these “initial” conditions[3] . Since several species are present, one needs to specify the density perturbation of each species. A simplification arises in the case of single field inflation, since exactly the same cosmological history, i.e. inflation followed by the decay of the inflaton into the usual species, occurs in all parts of our Universe, up to a small time shift depending on the perturbation of the inflaton in each region. As a consequence, even if the number densities of the various species vary from point to point, their ratio should be fixed, i.e.

$$\delta\left(n_A/n_B\right) = 0, \tag{3.98}$$

for any pair of species denoted A and B. This is not necessarily true in multi-field inflation, as the perturbations in the radiation era may depend on different combinations of the scalar field perturbations. The variation in the relative particle number densities between two species can be quantified by the quantity

$$S_{A,B} \equiv \frac{\delta n_A}{n_A} - \frac{\delta n_B}{n_B}, \tag{3.99}$$

[3] although one must be aware that present cosmological scales can correspond to scales smaller than the Planck scale during inflation, suggesting the possibility of trans-Planckian effects

which is usually called the entropy perturbation between A and B. When the equation of state for a given species is such that $w \equiv P/\rho = const$, one can reexpress the entropy perturbation in terms of the density contrast, in the form

$$S_{A,B} = \frac{\delta_A}{1+w} - \frac{\delta_B}{1+w_B}. \tag{3.100}$$

It is convenient to choose a species of reference, for instance the photons, and to define the entropy perturbations of the other species relative to it. The quantities

$$S_b \equiv \delta_b - \frac{3}{4}\delta_\gamma, \tag{3.101}$$

$$S_c \equiv \delta_c - \frac{3}{4}\delta_\gamma, \tag{3.102}$$

$$S_\nu \equiv \delta_\nu - \frac{3}{4}\delta_\gamma, \tag{3.103}$$

thus define respectively the baryon, CDM and neutrino entropy perturbations.

For single field inflation, all these entropy perturbations vanish, $S_b = S_c = S_\nu = 0$, and the primordial perturbations are said to be adiabatic. An adiabatic primordial perturbation is thus characterized by

$$\frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu = \frac{1}{3}\delta_b = \frac{1}{3}\delta_c. \tag{3.104}$$

Only one density constrast needs to be specified. However, since it is a gauge-dependent quantity, it is better to use the gauge-invariant quantity ζ , i.e. the uniform density curvature perturbation, which is also equivalent to $\mathscr{R}$, since we are on super-Hubble scales here. Note that the adiabatic mode, which is directly related to the curvature perturbation, is also called curvature mode. By contrast, the entropy perturbations can be non-zero even if the curvature is zero, and the corresponding modes are called isocurvature modes.

3.8 Modulation

In the multiple fields inflation, a scenario is called modulation when the primordial perturbations are due to the perturbations of a scalar field, which has never dominated the matter content of the universe but has played a crucial role during some cosmological transition, while in the curvaton scenario the curvaton dominates the energy density of the Universe at some epoch in order to give the main contribution to the primordial perturbations.

The best example is the modulated reheating scenario where the decay rate of the inflaton, Γ , depends on a modulaton σ, which has acquired classical fluctuations during inflation. The decay rate is thus slightly different from one super-Hubble patch to another, which generates a curvature perturbation. A simple way to quantify this effect is to compute the number of e-folds between some initial time t_i during inflation, when the scale of interest crossed out the Hubble radius, and some final time t_f. The curvature perturbation is then directly related to the fluctuations of the number of e-folds.

For simplicity, we will assume that, just after the end of inflation at time t_e, the inflaton behaves like pressureless matter (as is the case for a quadratic potential) until it decays instantaneously at the time t_d characterized by $H_d = \Gamma$. At the decay, the energy density is thus $\rho_d = \rho_e \exp\left[3\left(N_d - N_e\right)\right]$ and is transferred into radiation, so that, at time t_f , one gets

$$\rho_f = \rho_d e^{-4(N_f - N_d)} = \rho_e e^{-3(N_f - N_e)-(N_f - N_d)}. \tag{3.105}$$

Using the relation $\Gamma = H_d = H_f \exp[2(N_f N_d)]$ to eliminate $(N_f N_d)$ in Eq. (3.105), we finally obtain

$$N_f = N_e - \frac{1}{3}\ln\frac{\rho_f}{\rho_e} - \frac{1}{6}\ln\frac{\Gamma}{H_f}. \tag{3.106}$$

If one ignores the inflaton fluctuations, the final curvature perturbation is therefore

$$\zeta = N_{,\sigma}\delta\sigma_* = -\frac{1}{6}\frac{\Gamma_{,\sigma}}{\Gamma}\delta\sigma_*, \tag{3.107}$$

which yields the curvature power spectrum

$$\mathscr{P}_\zeta = \frac{1}{36}\left(\frac{\Gamma_{,\sigma}}{\Gamma}\right)^2\left(\frac{H_*}{2\pi}\right)^2. \tag{3.108}$$

The dependence on the modulaton can alternatively show up in the mass of the particles created by the decay of the inflaton. The modulaton can also affect the cosmological evolution during inflation, as in the modulated trapping scenario, which is based on the resonant production of particles during inflation (see also for other recent scenarios based on particle production). If the inflaton is coupled to some particles, whose effective mass becomes zero for a critical value of the inflaton, then there will be a burst of production of these particles when the inflaton crosses the critical value. These particles will be quickly diluted but they will slow down the inflaton. This effect, which increases the number of e-folds until the end of inflation, can depend on a modulaton, for example via the coupling between the inflaton and the particles, and a significant curvature perturbation might be generated.

3.9 Non-Gaussianities

One of the most promising probes of the early Universe, which has been investigately very actively in the last few years, is the non-Gaussianity of the primordial perturbations. Whereas the simplest models of inflation, based on a slow-rolling single field with standard kinetic term, generate undetectable levels of non-Gaussianity, a significant amount of non-Gaussianity can be produced in scenarios with i) non-standard kinetic terms; ii) multiple fields; iii) a non standard vacuum; iv) a non slow-roll evolution.

3.9.1 Higher order correlation functions

The most used estimate of non-Gaussianity is the bispectrum defined, in Fourier space, by

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \equiv (2\pi)^3 \delta^{(3)}(\Sigma_i \mathbf{k}_i) B_\zeta(k_1, k_2, k_3), \tag{3.109}$$

where the Fourier modes are defined by

$$\zeta_{\mathbf{k}} = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \zeta(\mathbf{x}). \tag{3.110}$$

Equivalently, one often uses the so-called f_{NL} parameter, which can be defined in general by

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \frac{6}{5} f_{NL} [P_\zeta(k_1) P_\zeta(k_2) + 2 \text{ perm}], \tag{3.111}$$

where P_ζ is the power spectrum5 defined by

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) P(k_1). \tag{3.112}$$

The f_{NL} parameter was initially introduced for a very specific type of non-Gaussianity characterized by

$$\zeta(\mathbf{x}) = \zeta_G(\mathbf{x}) + \frac{3}{5} f_{NL} \zeta_G^2(\mathbf{x}), \tag{3.113}$$

in the physical space, where ζ_G is Gaussian and the factor $3/5$ appears because f_{NL} was originally defined with respect to the gravitational potential $\Phi = (3/5)\zeta$, instead of ζ. In this particular case, f_{NL}, as defined in Eq. (3.111), is independent of the vectors $\mathbf{k}_i$. In general, f_{NL} is a function of the norm of the three vectors $\mathbf{k}_i$ (which define a triangle in Fourier space since they are constrained by $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ as a consequence of homogeneity), and the shape of the three-point function is an important characterization of how non-Gaussianity was generated. In the context of multi-field inflation, the so-called δN-formalism is particularly useful to evaluate the primordial non-Gaussianity generated on large scales. The idea is to describe, on scales larger than the Hubble radius, the nonlinear evolution of perturbations generated during inflation in terms of the perturbed expansion from an initial flat hypersurface (usually taken at Hubble crossing during inflation) up to a final uniform-density hypersurface (usually during the radiation-dominated era). Using the Taylor expansion of the number of e-folds given as a function of the initial values of the scalar fields,

$$\zeta \simeq \sum_I N_{,I}\delta\phi_*^I + \frac{1}{2}\sum_{IJ} N_{,IJ}\delta\phi_*^I\delta_*^J, \tag{3.114}$$

in Fourier space,

$$\begin{aligned}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle &= \Sigma_{IJK}N_{,I}N_{,J}N_{,K}\langle\delta\phi^I_{\mathbf{k}_1}\delta\phi^J_{\mathbf{k}_2}\delta\phi^K_{\mathbf{k}_3}\rangle \\ &+\frac{1}{2}\Sigma_{IJKL}N_{,I}N_{,J}N_{,K}N_{,L}\langle\delta\phi^I_{\mathbf{k}_1}\delta\phi^J_{\mathbf{k}_2}(\delta\phi^K\star\delta\phi^L)_{\mathbf{k}_3}\rangle + \text{perms.}\end{aligned} \tag{3.115}$$

The first term on the right hand side corresponds to non-Gaussianities arising from nonvanishing three-point function(s) of the scalar field(s). This is the case for models with non-standard kinetic terms, leading to a specific shape of non-Gaussianities, usually called equilateral, where the dominant contribution comes from configurations with three wave vectors of similar length $k_1 \simeq k_2 \simeq k_3$.

The terms appearing in second line of Eq. (3.115) can also lead to sizable non-Gaussianities. Indeed, substituting

$$\langle\delta\phi^I_{\mathbf{k}_1}\delta\phi^I_{\mathbf{k}_2}\rangle = (2\pi)^3\delta_{IJ}\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\frac{2\pi^2}{k_1^3}\mathscr{P}_*(k_1), \quad \mathscr{P}_* \equiv \frac{H_*^2}{4\pi^2}, \tag{3.116}$$

in Eq. (3.115), one finds

$$\frac{6}{5}f_{NL} = \frac{N_{,I}N_{,J}N^{,IJ}}{(N_{,K}N^{,K})^2}, \tag{3.117}$$

where we use Einsteins summation convention for the field indices, which are raised with δ^{IJ}. This corresponds to another type of non-Gaussianity, usually called local or squeezed, for which the dominant contribution comes from configurations where the three wavevectors form a squeezed triangle.

Extending the Taylor expansion Eq. (3.114) up to third order, one can compute in a similar way the tri-spectrum, i.e. the Fourier transform of the connected four-point function defined by

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\zeta_{\mathbf{k}_4}\rangle \equiv (2\pi)^3\delta^{(3)}(\Sigma_i\mathbf{k}_i)T_\zeta(k_1,k_2,k_3,k_4). \tag{3.118}$$

Assuming the scalar field perturbations to be quasi-Gaussian, the trispectrum can be written in the form

$$
\begin{aligned}
T_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = & \tau_{NL}[P(k_{13})P(k_3)P(k_4) + 11 \text{ perms}] \\
& + \frac{54}{25} g_{NL}[P(k_2)P(k_3)P(k_4) + 3 \text{ perms}],
\end{aligned}
\tag{3.119}
$$

with

$$
\tau_{NL} = \frac{N_{IJ}N^{IK}N^{J}N_{K}}{(N_{,L}N^{,L})^3}, \quad g_{NL} = \frac{25}{54}\frac{N_{IJK}N^{I}N^{J}N^{K}}{(N_{L}N^{L})^3}, \tag{3.120}
$$

and where $k_{13} = |\mathbf{k}_1 + \mathbf{k}_3|$.

Chapter 4

Asymptotic Safe Gravity

4.1 Asymptotic Safety

A useful and efficient way of analyzing quantum effects on the low energy scale physics is the renormalization group (RG). An effective theory is obtained by integrating out the quantum fluctuations with higher energy scales than a certain cutoff scale. It contains a number of parameters that run along with the cutoff scale, called the RG flows. One can then incorporate the quantum effects using classical equations of motion from the effective action. The main problem in applying the RG approach to cosmology is that we do not know the complete quantum gravity theory that governs the UV (Planck) scale physics. The notion of asymptotic safety was introduced by Weinberg in 1979 and is connected with the fact that the coupling parameters quantum field theory are energy-dependent due to renormalization. In this scenario the renormalization group (RG) flows approach a fixed point in the UV limit, and a finite dimensional critical surface of trajectories evolves to this point at short distance scales. This picture suggests a nonperturbative UV completion for gravity, where the metric fields remain the fundamental degrees of freedom. Moreover, the low energy regime of classical general relativity is linked with the high energy regime by a well-defined, finite, RG trajectory. A theory is called asymptotically safe if all essential coupling parameters g_i approach a non-trivia fixed point for energies $k \to \infty$. The scenario of AS gravity has been studied extensively in the literature [5, 139, 140, 142, 9, 143, 146, 55, 13, 14]. There is evidence that black hole solutions in an AS gravity with Einstein-Hilbert truncation may be nonsingular [15, 16, 17]; however, the study of black hole physics in an AS gravity theory including higher derivative terms [145] showed that, while the metric factor may be everywhere finite, curvature invariants may still diverge at the origin. The implications of AS gravity with Einstein truncation and Friedmann-Robertson-Walker (FRW) cosmologies were analyzed with respect to late time cosmological acceleration in Refs. [19, 141, 21, 106, 24, 22]. The relation between Brans-Dicke theory and AS gravity was discussed in [25]. It is also possible that AS gravity might drive inflation at early times [138, 26, 27, 28].

A covariant gravitational effective theory (for simplicity matter will be ignored here) involving running coupling with a cutoff p can be expressed as

$$S_p[g_{\mu\nu}] = \int d^4x\sqrt{-g}[p^4 g_0(p) + p^2 g_1(p) R + \mathscr{O}(R^2) + \cdots], \tag{4.1}$$

where g is the determinant of the metric $g_{\mu\nu}$ and R is the Ricci sclalr. The coefficients $g_i(p)$, $(i = 0, 1, \ldots)$ are dimensionless coupling parameters and are functions of the dimension-full UV cutoff. In particular, we have

$$g_0(p) = -\frac{\Lambda(p)}{8\pi G(p)}\, p^{-4}\,, \qquad g_1(p) = \frac{1}{8\pi G(p)}\, p^{-2}\,, \tag{4.2}$$

where $G(p)$ and $\Lambda(p)$ are the quantum corrected gravitational and cosmological constants. The couplings satisfy the following RG equations,

$$\frac{d}{d\ln p}\, g_i(p) = \beta_i[g(p)]\ . \tag{4.3}$$

According to Ref. [2], all beta functions vanish when the coupling parameters g_i approach a fixed point g_i^* in the scenario of asymptotical safety. If $g_i^* = 0$, the fixed point is Gaussian; if $g_i^* \neq 0$, the fixed point is Non-Gaussian (NG). For the NG fixed point, all the coupling parameters are fixed, the cutoff p becomes irrelevant as $p \to \infty$, and the theory is adequately described by a finite number of higher order counter-terms included in the effective action. Near the fixed point we may Taylor expand the beta functions in a matrix form

$$\beta_i[g] = \sum_j \mathscr{B}_{ij}(g_j - g_j^*)\ \ , \tag{4.4}$$

where the elements of the matrix are defined by $\mathscr{B}_{ij} \equiv \frac{\partial\beta_i[g]}{\partial g_j}^*$ at the fixed point. Solving the RG equations (4.3) in the neighborhood of the fixed point we find

$$g_i(p) = g_i^* + \sum_m e_i^n \left(\frac{p}{M_*}\right)^{v_n}\ \ , \tag{4.5}$$

where e^n and v_n are the suitably normalized eigenvectors and corresponding eigenvalues of the matrix $\mathscr{B}_{ij}$. Since $\mathscr{B}$ is a general real matrix with symmetry determined by a particular gravity model, its eigenvalues can be either real or in pairs of complex conjugates. As a consequence, the dimensionality of the UV critical surface is equal to the number of eigenvalues of the matrix $\mathscr{B}$, of which the real parts take negative values. The above solution involves an arbitrary mass scale M_*. By requiring the largest eigenvector of order unity, M_* is typically identified with the energy scale at which the coupling parameters are just beginning to approach the fixed point.

The low energy effective action can, in principle, be obtained from the RG equation. This is however a highly nontrivial functional differential equation with respect to the RG scale k [29] that is virtually impossible to solve exactly. As a simple approximation, we (in agreement with much of the literature) shall adopt the Einstein-Hilbert truncation [30] in the gravity part by neglecting higher derivative terms. In the matter part, the kinetic term of the scalar field is taken to be canonical (i.e. no running since there is no coupling parameter) while the potential is allowed to vary as the RG scale k changes. In this approximation, the practical effect of RG flow is then an evolution of the gravitational coupling G_k (generalizing Newton's constant) and the scalar field potential $V_k(\phi)$. In particular the equations of motion will take the same form as the classical ones.

4.2 Functional Renormalization Group

Whether or not a non-trivial fixed point is realised in quantum gravity can be assessed once explicit renormalisation group equations for the scale-dependent gravitational couplings are available. To that end, we recall the set-up of Wilsons (functional) renormalization group, which is used below for the case of quantum gravity. Wilsonian flows are based on the notion of a cutoff effective action k, where the propagation of fields with momenta smaller than k is suppressed. A Wilsonian cutoff is realised by adding $\Delta S_k = \frac{1}{2}\int \varphi(-q)\, R_k(q)\, \varphi(q)$ within the Schwinger functional

$$\ln\, Z_k[J] = \ln \int [D\varphi]_{\text{ren.}} \exp\left(-S[\varphi] - \Delta S_k[\varphi] + \int J \cdot \varphi\right) \tag{4.6}$$

and the requirement that R_k obeys (i) $R_k(q) \to 0$ for $k^2/q^2 \to 0$, (ii) $R_k(q) > 0$ for $q^2/k^2 \to 0$, and (iii) $R_k(q) \to \infty$ for $k \to \Lambda$ (for examples and plots of R_k, see [82]) Note that the Wilsonian momentum scale k takes the role of the renormalisation group scale μ introduced in the previous section. Under infinitesimal changes $k \to k - \Delta k$, the Schwinger functional obeys $\partial_t \ln Z_k = -\langle \partial_t \Delta S_k \rangle_J$; $t = \ln k$. We also introduce its Legendre transform, the scale-dependent effective action $\Gamma_k[\phi] = \sup_J \left(\int J \cdot \phi - \ln Z_k[J] \right) - \frac{1}{2} \int \phi R_k \phi$, $\phi = \langle \varphi \rangle_J$. It obeys an exact functional differential equation introduced by Wetterich [83]

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k \,, \tag{4.7}$$

which relates the change in Γ_k with a one-loop type integral over the full field-dependent cutoff propagator. Here, the trace Tr denotes an integration over all momenta and summation over all fields, and $\Gamma_k^{(2)}[\phi](p,q) \equiv \delta^2 \Gamma_k / \delta\phi(p) \delta\phi(q)$. A number of comments are in order:

- **Finiteness and interpolation property.** By construction, the flow equation 4.7 is well-defined and finite, and interpolates between an initial condition Γ_Λ for $k \to \Lambda$ and the full effective action $\Gamma \equiv \Gamma_{k=0}$. The endpoint is independent of the regularisation, whereas the trajectories $k \to \Gamma_k$ depend on it.

- **Locality.** The integrand of 4.7 is peaked for field configurations with momentum squared $q^2 \approx k^2$, and suppressed for large momenta [due to condition (i) on R_k] and for small momenta [due to condition (ii)]. Therefore, the flow equation is essentially local in momentum and field space [82, 85].

- **Approximations.** Systematic approximations for Γ_k and $\partial_t \Gamma_k$ are required to integrate 4.7. These include (a) perturbation theory, (b) expansions in powers of the fields (vertex functions), (c) expansion in powers of derivative operators (derivative expansion), and (d) combinations thereof. The iterative structure of perturbation theory is fully reproduced to all orders, independently of R_k [86, 87]. The expansions (b) - (d) are genuinely non-perturbative and lead, via 4.7, to coupled flow equations for the coefficient functions. Convergence is then checked by extending the approximation to higher order.

- **Stability.** The stability and convergence of approximations is, additionally, controlled by R_k [82, 84]. Here, powerful optimisation techniques are available to maximise the physics content and the reliability through well-adapted choices of R_k [82, 84, 85, 89, 80]. These ideas have been explicitly tested in *e.g.* scalar [88] and gauge theories [89].

- **Symmetries.** Global or local (gauge/diffeomorphism) symmetries of the underlying theory can be expressed as Ward-Takahashi identities for n-point functions of Γ. Ward-Takahashi identities are maintained for all k if the insertion ΔS_k is compatible with the symmetry. In general, this is not the case for non-linear symmetries such as in non-Abelian gauge theories or gravity. Then the requirements of gauge symmtry for Γ are preserved by either (a) imposing modified Ward identities which ensure that standard Ward identities are obeyed in the the physical limit when $k \to 0$, or by (b) introducing background fields into the regulator R_k and taking advantage of the background field method, or by (c) using gauge-covariant variables rather than the gauge fields or the metric field [90]. For a discussion of benefits and shortcomings of these options see [76, 80]. For gravity, most implementations presently employ option (b) together with optimisation techniques to control the symmetry [53, 54].

- **Integral representation.** The physical theory described by Γ can be defined without explicit reference to an underlying path integral representation, using only the (finite) initial condition Γ_Λ, and the (finite) flow equation 4.7

$$\Gamma = \Gamma_\Lambda + \int_\Lambda^0 \frac{dk}{k} frac12 \, \mathrm{Tr} \left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k \,. \tag{4.8}$$

This provides an implicit regularisation of the path integral underlying 4.6. It should be compared with the standard representation for Γ via a functional integro-differential equation

$$e^{-\Gamma} = \int [D\varphi]_{\text{ren.}} \exp\left(-S[\phi+\varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi} \cdot \varphi\right) \tag{4.9}$$

which is at the basis of *e.g.* the hierarchy of Dyson-Schwinger equations.

- **Renormalisability.** In renormalisable theories, the cutoff Λ in 4.8 can be removed, $\Lambda \to \infty$, and $\Gamma_\Lambda \to \Gamma_*$ remains well-defined for arbitrarily short distances. In perturbatively renormalisable theories, Γ_* is given by the classical action S, such as in QCD. In this case, illustrated in a), the high energy behaviour of the theory is simple, given mainly by the classical action, and the challenge consists in deriving the physics of the strongly coupled low energy limit. In perturbatively non-renormalisable theories such as quantum gravity, proving the existence (or non-existence) of a short distance limit Γ_* is more difficult. b), experiments indicate that the low energy theory is simple, mainly given by the Einstein Hilbert theory. The challenge consists in identifying a possible high energy fixed point action Γ_*, which upon integration matches with the known physics at low energies. In principle, any Γ_* with the above properties qualifies as fundamental action for quantum gravity. In non-renormalisable theories the cutoff Λ cannot be removed. Still, the flow equation allows to access the physics at all scales $k < \Lambda$ analogous to standard reasoning within effective field theory [69].

- **Link with Callan-Symanzik equation.** The well-known Callan-Symanzik equation describes a flow $k\frac{d}{dk}$ driven by a mass insertion $\sim k^2\phi^2$. In 4.7, this corresponds to the choice $R_k(q^2) = k^2$, which does not fulfill condition (i). Consequently, the corresponding flow is no longer local in momentum space, and requires an additional UV regularisation. This highlights a crucial difference between the Callan-Symanzik equation and functional flows 4.7. In this light, the flow equation 4.7 could be interpreted as a functional Callan-Symanzik equation with momentum-dependent mass term insertion [91].

Now we are in a position to implement these ideas for quantum gravity [139]. A Wilsonian effective action for gravity Γ_k should contain the Ricci scalar $R(g_{\mu\nu})$ with a running gravitational coupling G_k, a running cosmological constant Λ_k (with canonical mass dimension $[\Lambda_k] = 2$), possibly higher order interactions in the metric field such as powers, derivatives, or functions of *e.g.* the Ricci scalar, the Ricci tensor, the Riemann tensor, and, possibly, non-local operators in the metric field. The effective action should also contain a standard gauge-fixing term S_{gf}, a ghost term S_{gh} and matter interactions S_{matter}. Altogether,

$$\Gamma_k = \int d^dx \sqrt{\det g_{\mu\nu}} \left[\frac{1}{16\pi G_k} (-R + 2\Lambda_k) + \cdots + S_{\text{gf}} + S_{\text{gh}} + S_{\text{matter}} \right] , \tag{4.10}$$

and explicit flow equations for the coefficient functions such as G_k, Λ_k or vertex functions, are obtained by appropriate projections after inserting 4.10 into 4.7. All couplings in 4.10 become running couplings as functions of the momentum scale k. For k much smaller than the d-dimensional Planck scale M_*, the gravitational sector is well approximated by the Einstein-Hilbert action with $G_k \approx G_{k=0}$, and similarly for the gravity-matter couplings. At $k \approx M_*$ and above, the RG running of gravitational couplings becomes important. This is the topic of the following sections.

A few technical comments are in order: To ensure gauge symmetry within this set-up, we take advantage of the background field formalism and add a non-propagating background field $\bar{g_{\mu\nu}}$ [139, 76, 92, 93, 94, 95, 96]. This way, the extended effective action $\Gamma_k[g_{\mu\nu}, \bar{g_{\mu\nu}}]$ becomes gauge-invariant under the combined symmetry transformations of the physical and the background field. A second benefit of this is that the background field can be used to construct a covariant Laplacean $-\bar{D}^2$, or similar, to define a mode cutoff at momentum scale $k^2 = -\bar{D}^2$. This implies that the mode cutoff R_k will depend on the background fields. The background field is then eliminated from the final equations by identifying it with the physical mean field. This procedure, which dynamically readjusts the background field, implements the requirements of "background independence" for quantum gravity. For a detailed evaluation of Wilsonian background field flows, see [95]. Finally, we note that the operator traces Tr in Eq. (4.7) are evaluated using heat kernel techniques.

Chapter 5

Higgs Modulated Reheating of RG improved Inflation

In this chapter, we will study the inflationary cosmology of asymptotically safe gravity with Einstein-Hilbert truncation taking into account the renormalization group (RG) running of both gravitational and cosmological constants. We also consider the dynamics of a scalar field which can be interpreted as the Higgs field. The Higgs plays a role of modulating the decay rate of reheating. The background trajectories of this model can provide sufficient inflationary e-folds and a graceful exit to a radiation dominated phase. Due to the compatibility with general relativity (GR) requires some constraints on the running constants and their contributions to the stress energy tensor can be taken into account in the perturbation analysis.

5.1 A Model of Asymptotic Safe Gravity

We start from a RG inspired effective gravitational Lagrangian with Einstein-Hilbert truncation,

$$\mathscr{L}_{AS} = \frac{R - 2\Lambda(p)}{16\pi G(p)}, \tag{5.1}$$

where p is the RG cutoff scale. Just like the ordinary effective field theory, quantum fluctuations beyond the cutoff scale are integrated out. By asymptotic safe, we assume the effective lagrangian automatically connects with ordinary Einstein gravity in the IR regime where the gravitational and cosmological constants flow to some present values that can be constrained by observations. In the UV limit, these "constants" flow to a UV fixed point according to their beta functions. Quantum corrections are therefore described by the evolution of the coupling constants as functions of the cutoff scale p, whose beta functions can be extracted from the RG equations.

We first define the dimensionless gravitational and cosmological constants as follows:

$$g(p) \equiv \frac{p^2}{24\pi} G(p), \quad \lambda(p) \equiv \frac{\Lambda(p)}{p^2}. \tag{5.2}$$

If we had known the exact forms of RG equations, we can follow the flows of g and λ along with the cutoff scale and obtain a fixed point in the UV limit. The AS scenario suggests that this UV fixed point is attractive. Note that the explicit forms of beta functions depend on the choice of the cutoff function and the relevant gauges. In Ref. [138], it was observed that the UV fixed point often corresponds to a de Sitter solution; however neither the energy scale of the background nor the amplitude of quantum fluctuations provide a successful application to early Universe inflationary cosmology [150].

If the RG improved gravity theory is viable, its RG trajectory should connect smoothly with standard Einstein gravity in the IR limit so as to be consistent with astronomical and cosmological observations. We would therefore like to study the RG improved gravity theory in the regime that is sufficiently close to GR while still retaining some quantum corrections by the beta functions at linear order. To begin with, we assume the linearized beta functions for dimensionless coupling constants as follows:

$$\beta_\lambda \equiv p\partial_p \lambda = -2\lambda + 2\alpha g, \tag{5.3}$$

$$\beta_g \equiv p\partial_p g = 2g - 2\beta^2 g^2/3, \tag{5.4}$$

which include next-to-leading order corrections to g. The coefficients α and β are cutoff function dependent. Under this parametrization, one can obtain approximate forms of the dimensionless couplings, which are given by

$$g(p) \simeq \frac{3G_N p^2}{72\pi + \beta^2 G_N p^2}, \tag{5.5}$$

$$\lambda(p) \simeq \frac{\Lambda_{IR}}{p^2} + \frac{3\alpha}{\beta^2} - \frac{216\pi\alpha}{\beta^4 G_N p^2} \ln\left(\frac{72\pi}{G_N} + \beta^2 p^2\right), \tag{5.6}$$

where G_N and Λ_{IR} are Newtons constant and the cosmological constant in the infrared limit and thus correspond to those in GR. The above two couplings approach nonvanishing constant values in the $p \ll M_p$ limit and therefore have the expected AS behavior. Note also that the parameters α and β should in principle be calculated from the theory of quantum theory of gravity rather than free model parameters. However, because it is not known how to do this calculation, there can be many different possibilities and thus we can treat them as free parameters effectively. The corresponding parameter choice can determine the way g and λ approach their fixed point, and thus could impose a possible constraint on the theory. Therefore, we can choose the values of α and β such that our model can fit the observational data.

The corresponding RG improved gravitational and cosmological constants obey the following relations:

$$G(p) \simeq \frac{G_N}{1+\xi_G p^2}, \tag{5.7}$$

$$\Lambda(p) \simeq \Lambda_{IR} + \xi_\Lambda p^2 - \frac{\xi_\Lambda}{\xi_G} G_N^{-1} \ln\left(1+\xi_G G_N p^2\right), \tag{5.8}$$

where ξ_G and ξ_Λ are the model parameters determined by RG flow coefficients through

$$\xi_G = \frac{\beta^2}{72\pi} \quad \text{and} \quad \xi_\Lambda = \frac{3\alpha}{\beta^2}. \tag{5.9}$$

When $p \to 0$, approach the classical values determined by observations and thus GR is recovered in the IR limit. Conversely, in the extreme UV regime the value of G approaches zero, which implies a weakly coupled gravitational system at extremely high energy scale. In between, we expect a period of sufficiently slow variation of Λ and thus the occurrence of an inflationary phase at early times of cosmological evolution. We note that if ξ_G is chosen to be much smaller than unity, one can Taylor expand the last term of Eq. (6.8) and the simplified expression

$$\Lambda \simeq \Lambda_{IR} + \frac{1}{2}\xi_\Lambda \xi_G G_N p^4, \tag{5.10}$$

even if p is of order M_p. We will find that this condition is necessary in order to achieve a viable inflationary phase.

5.2 The $f(R)$ Correspondence

We consider minimal coupling between the AS gravity and the matter field,

$$S_{AS} = \int d^4x \sqrt{-g}[\mathscr{L}_{AS} + \mathscr{L}_m], \tag{5.11}$$

where $\mathscr{L}_m$ is the Lagrangian of the matter field. As the gravitational constant varies along the cutoff scale p which can be a function of spacetime, taking variation of the Lagrangian with respect to the metric $g_{\mu\nu}$ yields the generalized Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^{(m)}_{\mu\nu} + G\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right)G^{-1}, \tag{5.12}$$

where we have introduced the covariant derivative ∇_μ with respect to $g_{\mu\nu}$ by $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda$, and the affine connection is

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right).$$

We also define the operator $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. We see that, Eq. (6.12) is the same as the Einstein equation in classical general relativity, plus an extra contribution due to the RG running of the gravitational constant. Since both G and Λ are no longer constants but functions of the cutoff scale p, it is important to specify the distribution of p in AS gravity. One way to achieve this is by writing down the generalized Bianchi identity of AS gravity. Assuming the conservation of the stress energy tensor of matter fields $\nabla^\mu T^{(m)}_{\mu\nu} = 0$, we can derive the following useful equation requiring the running of the cutoff scale to obey the constraint

$$\frac{R - 2\Lambda(p)}{G(p)}\nabla_\mu G(p) + \nabla_\mu \Lambda(p) = 0. \tag{5.13}$$

Since $\nabla_\mu G(p) = G_{,p}\cdot(\nabla_\mu p)$, and $\nabla_\mu\Lambda(p) = \Lambda_{,p}\cdot(\nabla_\mu p)$, we can simplify the above equation to

$$\frac{R - 2\Lambda(p)}{G(p)}G_{,p} + \Lambda_{,p} = 0, \tag{5.14}$$

if $\nabla_\mu p \neq 0$.

The continuity equation of energy density determines the dynamics of matter components and allows derivation of the equations of motion by varying the Lagrangian with respect to matter fields. Therefore, the dynamics of this cosmological system are completely determined. Inserting the forms of RG modified gravitational constant G ,Eq. (6.7) and cosmological constant Λ ,Eq. (6.10) into Eq. (6.14), one can identify the relation between the Ricci scalar and the cutoff scale,

$$p^2 \simeq \frac{R - 2\Lambda_{IR}}{2\xi_\Lambda} - \frac{3\xi_G G_N R^2}{8\xi_\Lambda^2}. \tag{5.15}$$

We see that the original theory may be reformulated as an effective $f(R)$ model, where $\mathscr{L}_{AS}$ is replaced by

$$f(R) = -\frac{\Lambda_{,p}}{8\pi G_{,p}}R \simeq \frac{R - 2\Lambda_{IR}}{16\pi G_N} + \frac{\xi_G}{32\pi\xi_\Lambda}\left(R - 2\Lambda_{IR}\right)^2. \tag{5.16}$$

The subscript $,p$ denotes the derivative with respect to p. In general, the correspondence between the AS gravity and $f(R)$ theory holds if the Einstein-Hilbert truncation is applied. However, the detailed expression of $f(R)$ depends strongly on the specific forms of RG functions as well as the identification between the cutoff scale and Ricci scalar. We refer to [152] for a general discussion on this issue.

5.3 Classical Equivalence to the JBD Theory

The Jordan-Brans-Dicke (JBD) theory is one of a scalar-tensor theory in which the gravitational interaction is modified by introducing a scalar field coupled to the tensor field of general relativity. In the JBD theory, the gravitational constant G is allowed not to be a constant but instead is replaced by a scalar field which can vary from place to place and with time. Compared to classical GR, this theory contains an extra dimensionless constant, ω, which is the so-called Brans-Dicke parameter. The JBD theory with a scalar degree of freedom φ and potential $U(\phi)$ can be described by the action

$$S_{JBD} = \int d^x \sqrt{-g} \left[\frac{\varphi^2 R - \omega \partial_\mu \varphi \partial^\mu \varphi}{16\pi\varphi} - U(\varphi) + \mathscr{L}_m \right]. \tag{5.17}$$

If φ varies slightly from point to point in spacetime, it would be interpreted as a spacetime-dependent Newton's constant. This theory could include a variety of gravitational theory by different choices of ω and $U(\varphi)$. In our AS gravity, since the gravitational constant also varies with spacetime points, we expect that it would correspond to some kind of the JBD theory. Varying with the metric, we obtain the generalized Einstein equation

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} =& \frac{8\pi}{\varphi} \left(T^{(m)}_{\mu\nu} - U(\varphi) g_{\mu\nu} \right) + \varphi^{-1} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) \varphi \\ &+ \frac{\omega}{\varphi^2} \left(\nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} \nabla_\rho \varphi \nabla^\rho \varphi \right). \end{aligned} \tag{6.18}$$

Varying with φ, we obtain the equation of motion of φ

$$R - 16 U_{,\varphi} + \frac{2\omega}{\varphi} \Box \varphi - \frac{\omega}{\varphi^2} \nabla_\rho \varphi \nabla^\rho \varphi = 0, \tag{5.19}$$

where the subscript $,\varphi$ represents differentiation with respect to φ. Combine Eqs. (6.18) and (6.19), and take the trace, one gets

$$(2\omega + 3) \Box \varphi - 8\pi T^{(m)} + 32\pi U - 16\pi \varphi U_{,\varphi} = 0. \tag{5.20}$$

where $T^{(m)}$ is the trace of the stress energy tensor of matter components. Now, we can see our AS theory is equivalent to one class of JBD theory if we set

$$\varphi = G^{-1}, \quad \omega = 0, \quad U(\varphi) = \frac{\Lambda(p)}{8\pi G(p)}. \tag{5.21}$$

Making use of Eq. (6.14) and the trace of Eq. (6.12), we find

$$\Box G^{-1} = \frac{8\pi}{3} T^{(m)} - \frac{2\Lambda}{3G} - \frac{2\Lambda_{,p}}{3G_{,p}}. \tag{5.22}$$

We see that Eq. (6.22) is consistent with Eq. (6.20) if we substitute Eq. (6.21) into Eq. (6.20). Therefore, we have shown that the AS gravity is equivalent to one class of the JBD theory. This equivalence provides a convenient way of studying the cosmology of the AS gravity and telling us whether us or not the AS gravity is a viable theory of gravity.

5.4 R^2 Inflationary Cosmology

In a standard cosmological model, the value of Λ_{IR} is determined by observations pertaining to late-time acceleration which are typically of the order $O(10^{-121} M_p)$. Since we are considering the inflation occurring in early universe when the typical energy scale is $O(10^{-3} M_p)$, the contribution of Λ_{IR} to the early Universe background dynamics is totally negligible. For the time being, we neglect it and our model reduces to an R^2 inflationary cosmology [116], which is

$$f(R) = \frac{R}{16\pi G_N} + \frac{\xi_G}{32\pi\xi_\Lambda} R^2. \tag{5.23}$$

In addition, for the matter field Lagrangian we focus on the SM Higgs scalar. We use the unitary gauge for the Higgs boson and temporarily neglect all gauge interactions. As a consequence, the Lagrangian of the matter field is given by

$$\mathscr{L}_m \supset -\frac{1}{2}\partial_\mu h \partial^\mu h - V(h) - V_{int}, \tag{5.24}$$

where $V(h)$ is the potential of the Higgs boson and V_{int} represents the interactions between the Higgs and other particles in the Standard Model of particle physics. Without considering interactions with other particles, the form of the potential is approximately,

$$V(h) \simeq \frac{\lambda}{4}(h^2 - v^2)^2, \tag{5.25}$$

in which v is the vacuum expectation value (VEV) of the Higgs boson, with associated Higgs mass, $m_H = \sqrt{2\lambda} v$.

5.5 Background Evolution

Now we turn our attention to inflationary solutions. Recall the action of our theory is now Eq. (6.11) with matter Lagrangian given by in Eq. (6.24):

$$S_{AS} = \int d^4x\sqrt{-g}\left(\frac{R-2\Lambda(p)}{16\pi G(p)} - \frac{1}{2}\partial_\mu h\partial^\mu h - V(h)\right).$$

It is convenient to perform a conformal transformation into the Einstein frame:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{5.26}$$

where Ω is the conformal factor and we can define a new introduce scalar field ϕ as follows:

$$\Omega^2(\phi) \equiv \frac{G_N}{G} \equiv e^{\frac{2\phi}{\sqrt{6}M_p}}. \tag{5.27}$$

As a result, the original AS system is equivalent to a two-scalar-field system minimally coupled to Einstein gravity without RG running, for which the effective Lagrangian in the Einstein frame is given by

$$\mathscr{L} \supset \frac{\tilde{R}}{16\pi G_N} - \frac{(\tilde{\nabla}\phi)^2}{2} - \frac{(\tilde{\nabla}h)^2}{2\Omega^2(\phi)} - \tilde{V}(\phi, h), \tag{5.28}$$

with

$$\tilde{V}(\phi, h) = U(\phi) + \frac{V(h)}{\Omega^4(\phi)}, \tag{5.29}$$

where the potential of the new scalar field takes the form

$$U(\phi) = 2\pi M_p^4 \frac{\xi_\Lambda}{\xi_G}\left(1 - e^{-\frac{2\phi}{\sqrt{6}M_p}}\right). \tag{5.30}$$

This potential is sufficiently flat in the regime where $\phi \ll M_p$ and has the quadratic form around $\phi = 0$. This scalar field ϕ can thus play the role of the inflaton under a careful selection of RG running parameters.

We consider an isotropic and homogeneous universe, described by the flat Friedmann-Robertson-Walker (FRW) metric,

$$ds^2 = -dt^2 + a(t)^2 d^2\mathbf{x}, \tag{5.31}$$

then the Fredmann and Raychaudhuri equations are

$$H^2 = \frac{1}{3M_P^2}\tilde{\rho}, \tag{5.32}$$

$$\dot{H} = -\frac{1}{2M_P^2}(\tilde{\rho} + \tilde{P}), \tag{5.33}$$

where $H \equiv \frac{\dot{a}}{a}$ and the dot denotes the time derivative in the Einstein frame. The energy density and the pressure in the Einstein frame (we dropout the tilde) are

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2\Omega^2(\phi)}\dot{h}^2 + U(\phi) + \frac{V(h)}{\Omega^4(\phi)}, \tag{5.34}$$

$$P = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2\Omega^2(\phi)}\dot{h}^2 - U(\phi) - \frac{V(h)}{\Omega^4(\phi)}. \tag{5.35}$$

By varying the Lagrangian with respect to ϕ and h, we obtain the equations of motion for the scalar fields:

$$\ddot{\phi} + 3H\dot{\phi} + U_{,\phi} - 2\frac{\Omega_{,\phi}}{\Omega^5}V + \frac{\Omega_{,\phi}}{\Omega^3}\dot{h}^3 = 0, \tag{5.36}$$

$$\ddot{h} + 3H\dot{h} - 2\frac{\Omega_{,\phi}}{\Omega}\dot{\phi}\dot{h} + \frac{V_{,h}}{\Omega^2} = 0. \tag{5.37}$$

5.6 Slow-roll Inflation

Equations (6.36) and (6.37) show that the inflaton and the Higgs fields are coupled with each other and the system is rather complicated. Fortunately, the coupling terms can be greatly suppressed during inflation by using the slow-roll condition. We now introduce the series of slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \epsilon_\phi \equiv \frac{\dot{\phi}^2}{2M_pH^2}, \quad \epsilon \equiv \frac{\dot{h}^2}{2\Omega^2 M_p^2 H^2}, \tag{5.38}$$

$$\eta_\phi \equiv \frac{\tilde{V}_{,\phi\phi}}{3H^2}, \quad \eta_h \equiv \frac{\tilde{V}_{,hh}}{3H^2}, \quad \eta_{\phi h} \equiv \frac{\tilde{V}_{,\phi h}}{3H^2}, \tag{5.39}$$

for a cosmological system of coupled double fields. We note that the potential of ϕ becomes very flat when ϕ is larger than M_p and in comparison the parameters ϵ_ϕ and η_ϕ are relatively small. Simultaneously, other parameters are also very small due to the suppression by the large value of the conformal factor Ω.

As a consequence, under the slow-roll conditions, the background dynamics are determined by the following solutions:

$$\dot{\phi} \simeq -\frac{U_{,\phi}}{3H}, \quad \dot{h} \simeq -\frac{V_{,h}}{3\Omega^2 H}, \quad H^2 \simeq \frac{U}{3M_p^2}, \tag{5.40}$$

which implies a quasi-exponential expansion at early times. Since inflation ends when the slow-roll condition breaks, i.e., $\epsilon_\phi = 1$, the substitution of the background solution for $\dot{\phi}$ in Eq. (6.39) into the slow-roll parameter ϵ_ϕ in Eq. (6.38) yields the value of ϕ at the end of inflation:

$$\phi_f = \frac{\sqrt{6}}{2} M_p \ln(1 + \frac{2}{\sqrt{3}}). \tag{5.41}$$

As the contribution of the Higgs field during inflation is negligible, the number of e-folding of inflation mainly depends on ϕ through the relation

$$N = -\int_i^f \frac{U}{M_p^2 U_{,\phi}} d\phi,$$

which is given by

$$N(\phi) \simeq \frac{3}{4} e^{\frac{2\phi}{\sqrt{6}M_p}} - \frac{3}{2}\frac{\phi}{\sqrt{6}M_p} - 1.04 \tag{5.42}$$

It is easy to achieve $N = 60$ if initially the inflaton is placed at $\phi_i \approx 5.46 M_p$. Applying the slow-roll condition, one obtains the Hubble rate

$$H_I \simeq \sqrt{\frac{2\pi\xi_\Lambda}{3\xi_G}} M_p, \tag{5.43}$$

during inflation.

Moreover, the slow-roll parameters for the Higgs field h are automatically small due to the suppression of the large value of the conformal factor $\Omega \sim O(10)$. Thus during inflation h also varies slowly. As is well known, the inflationary phase is an attractor solution in an expanding Universe, and thus it is expected that other matter fields would be dominant in the pre-inflationary phase. Thus, in our model, we assume the Universe was dominated by the Higgs field in the pre-inflationary era and at that moment the slow-roll condition was not satisfied. As a result, the parameter η_h can be larger or of order of unity. Then, we can make use of the relation $\eta_h \simeq 1$ to estimate the amplitude of the Higgs field at the initial moment of inflation, which requires

$$\eta_h = \frac{V_{,hh}}{3H^2\Omega^4} \lesssim 1.$$

Therefore, one can estimate the initial amplitude of the Higgs at the beginning of inflation as

$$h_i \simeq \frac{\Omega_i^2}{\sqrt{\lambda}} H_I. \tag{5.44}$$

At the end of inflation there is no more suppression on slow-roll parameters of h and the values of η_h becomes of the order of unity simultaneously. As a consequence, by using Eq. (6.41), the conformal factor at the end of inflation is $\Omega_f^2 \simeq 2$, and the amplitude of the Higgs field at the end of inflation is estimated as

$$h_f \simeq \frac{2}{\sqrt{\lambda}} H_I. \tag{5.45}$$

Combining Eqs. (6.44) and (6.45), we easily verify that during inflation the Higgs boson h_I is required to satisfy the inequality

$$h_f < h_I < h_i. \tag{5.46}$$

Note that the Higgs field during inflation is not required to be less than the Hubble rate since its background energy density contributed is suppressed by the large value of the conformal factor. Our model therefore evades the theoretical constraint suggested in [159], where they study the inflation which model Higgs field plays as a source for the primordial curvature perturbation, in the curvaton and modulated reheating scenario and conclude that the contribution of the Higgs field to the primordial curvature perturbation must be less than 8%. We will see that this is the key to realizing Higgs modulated reheating.

In the above, we presented analytic solutions to inflationary dynamics. We now verify the results with numerical computations. The results are shown in Figs. 6.1, 6.2 and 6.3. Figure 6.1 shows that the Hubble parameter varies very slowly in the middle region, which corresponds to the inflationary period. On the other hand, the Higgs boson oscillates dramatically at the beginning of the evolution, which implies that the Universe is dominated by the Higgs field in the pre-inflation phase. The transition from the Higgs dominated pre-inflation phase to the inflation phase follows an attractor solution that does not strongly depend on the choice of initial conditions. However, this result also implies that such a scenario has to meet the big bang singularity if one traces backwards in the cosmic evolution. From Fig. 6.1, one can see that inflation ends when the value of ϕ decreases below M_p. The corresponding e-folding number is roughly 65. Subsequently, the inflaton field oscillates around $\phi = 0$ which corresponds to an IR fixed point of AS gravity. Therefore, GR is recovered at the end of inflation.

Figures 6.2 and 6.3 show the evolutions of slow-roll parameters defined in Eqs. (6.38) and (6.39) along with the cosmic expansion. In Fig. 2, the background slow-roll parameter ϵ almost coincides with that for inflaton, ϵ_ϕ, it is not surprising since inflation is driven by the inflaton ϕ which areinduced effects of RG-modified gravitational and cosmological constants. Among those parameters associated with the Higgs boson, the value of η_h is the first to break the slow-roll condition after inflation. Consequently, the method of determining the value of h during inflation by requiring $\epsilon_\phi \simeq 1$ is reliable. By

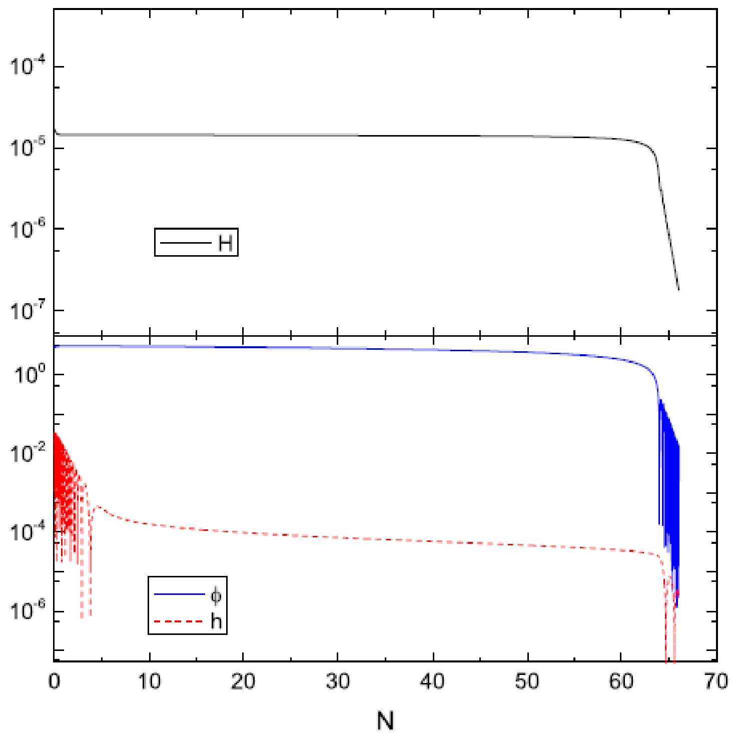

Figure 5.1: Evolution of the Hubble parameter H and two scalar fields ϕ and h as functions of the e-folding number N. In the solutions, the model parameters are $\xi_G = 0.72$ and $\xi_\Lambda = 10^{-10}\xi_G$. The parameters of the potential for the Higgs are taken as $\lambda = 0.13$ and $v = 246$ GeV according to particle physics observations. Initial field values are taken as $\phi_i = 5.46M_p$ and $h_i = 10^{-2}M_p$. Planck units are adopted in the figure.

substituting the parameter choices provided in Fig. 6.1 into the expression Eq. (6.46), we find it consistent with the numerical result shown in the lower panel of Fig. 6.1.

5.7 Higgs Dependent Decay after Inflation

After the inflaton field rolls below the critical value ϕ_f, it starts to oscillate around its fixed IR point which corresponds to the GR limit. One can perform a Taylor expansion of the potential in Eq. (6.29) around $\phi = 0$ up to the ϕ^2 order:

$$\tilde{V}(\phi, h) \simeq \frac{1}{2}M_\phi^2\phi^2 + V(h) + V_{int}\left(1 - \frac{4\phi}{\sqrt{6}M_p} + \frac{4\phi^2}{3M_p^2}\right) + \mathscr{O}(\phi^3) \tag{5.47}$$

In the above expression, we have introduced an effective mass for the inflaton defined by

$$M_\phi^2 \equiv \frac{2\pi}{3}M_p^2\frac{\xi_\Lambda}{\xi_G}. \tag{5.48}$$

The last term of Eq. (6.48) shows that ϕ interacts with other particles through the expansion of the conformal factor. Thus, if V_{int} contains interactions of the Higgs boson with other particles, the same interactions provide channels for the inflaton to decay into them and the corresponding decay rate is expected to be a function of the Higgs field value. A reheating mechanism of such scenario is called Higgs modulated reheating.

In our case, the last term of Eq. (6.47) is responsible for the Higgs dependent inflaton decay. Following Ref. [165], one can generally take the following Higgs dependent interactions:

$$V_{int}^\phi \supset y_a(h)\phi\bar{\psi}_a\psi_a + M_a(h)\phi\chi_a^2 + g_a(h)\phi^2\chi_a^2, \tag{5.49}$$

where χ_a and ψ_a are the scalar and spinor fields which constitute radiation in the early Universe; the subscript $_a$ represents the species of particles. The oscillations of the inflaton can be regarded as periodically oscillating external field on the χ_a and ψ_a fields. The energy is stored in the oscillation of the inflaton. This oscillating inflaton can produce χ_a or ψ_a particles out of the vacuum, i.e. it decays into χ_a or ψ_a. To achieve the modulated reheating scenario in our case, the coupling constants y_a, M_a, and g_a must be functions of the Higgs field. Under this assumption the decay rate of the inflaton to the lowest order in coupling constants is given by

$$\Gamma(h) = \frac{y_a^2(h)}{8\pi}M_\phi + \frac{M_a^2(h)}{8\pi M_\phi} + \frac{g_a^2(h)}{16\pi^3 M_\phi^3}\rho_\phi, \tag{5.50}$$

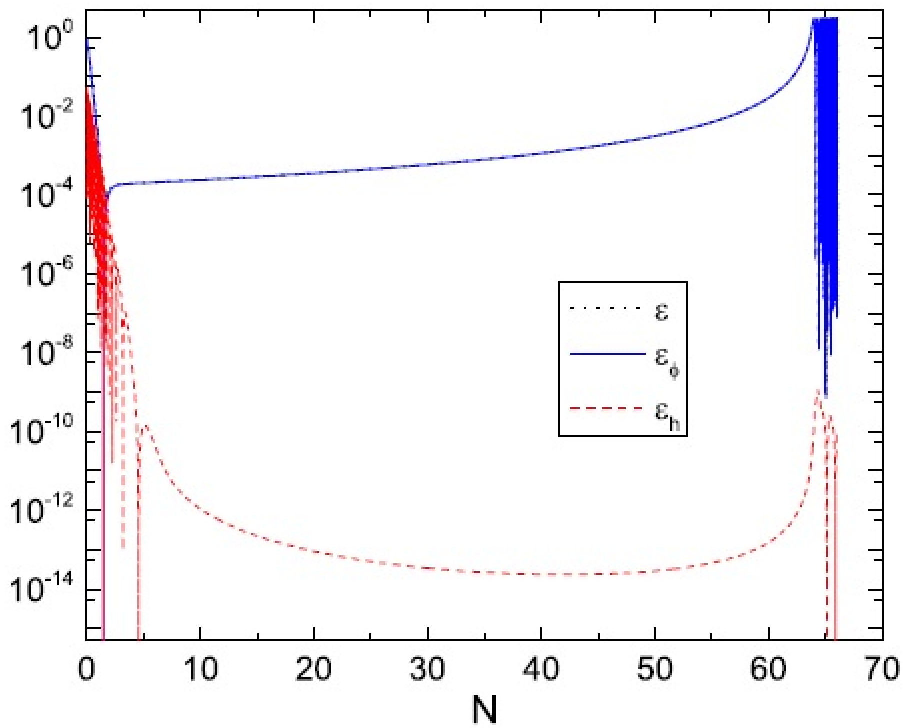

Figure 5.2: Evolution of the slow-roll parameters ϵ, ϵ_ϕ and ϵ_h as functions of the e-folding number N. The model parameters and initial conditions for this plot are the same as those for Fig. 5.1.

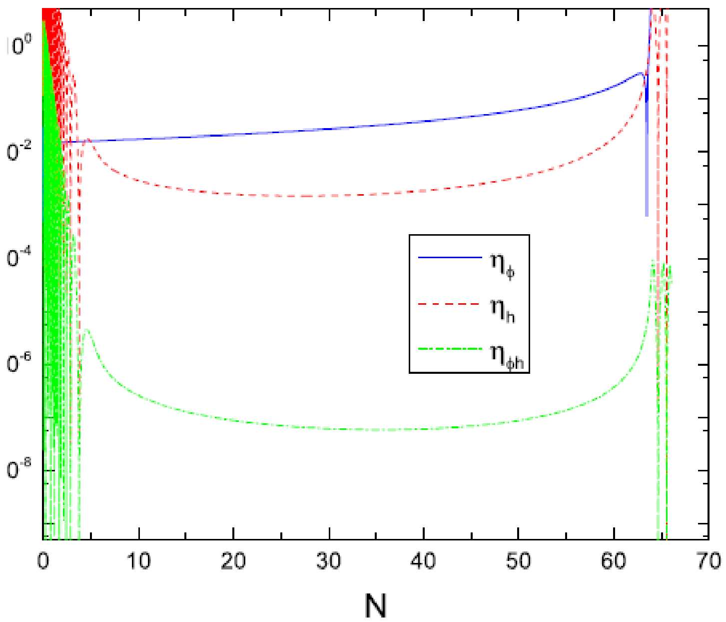

Figure 5.3: Evolution of the slow-roll parameters η_ϕ η_h, and $\eta_{\phi h}$ as functions of the e-folding number N. The model parameters and initial conditions for this plot are the same as those for Fig. 5.1.

where the quadratic potential for ϕ has been applied. Once we obtained the decay rate of the inflaton, we can calculate the time when the inflaton field decays completely into other particles, then the Universe enters the radiation dominant phase. In the following, we consider a specific example to illustrate one possible way of these decay processes. We consider

$$V_{int}^{\phi} \supset \frac{\kappa}{M_p^2} h^2 \chi^2 \phi^2, \tag{5.51}$$

which could arise from the term $V_{int}\phi^2/M_p^2$ appearing in the last term of Eq. (6.47). The corresponding decay rate is given by [**?**]

$$\Gamma(h) \simeq \frac{\kappa^2 h^4}{16\pi M_\phi^3 M_p^4} \rho_\phi. \tag{5.52}$$

When the inflaton domination phase turns into the radiation domination phase, the expansion rate is at the same order of the decay rate of inflaton, thus we have the condition $H \simeq \Gamma$. At this reheating surface, $\rho_\phi \simeq 3M_p^2\Gamma^2$. By making use of Eq. (6.45), we obtain the value of inflaton decay,

$$\phi_D \simeq \frac{\sqrt{3}\lambda^2 \xi_G}{2\kappa\xi_\Lambda} M_p^2, \tag{5.53}$$

at the reheating surface. In order to connect the reheating phase with the inflationary phase smoothly, we expect $\phi_D \leq \phi_f$. As a consequence, it imposes an additional severe constraint, which requires the coefficient κ needs to be finely tuned to satisfy $\kappa \geq \lambda\sqrt{\frac{\xi_G}{\xi_\Lambda}}$. One may take into account the first and second terms in the interaction (39) as well. The corresponding values of inflaton decay are estimated as

$$\frac{c_1^2\xi_\Lambda^2}{\lambda^2\xi_G^2} M_p^2, \quad \text{and} \quad \frac{c_2^2\xi_\Lambda}{\lambda^2\xi_G} M_p^2,$$

respectively, with c_1 and c_2 being the coefficients in front of these interaction terms. We can easily find that both values are much smaller than the result obtained in (6.53) due to the fact that $\frac{\xi_\Lambda}{\xi_G} \ll 1$. Therefore, one can conclude that the decay channel through the term (6.51) is generally dominant.

5.8 Adiabatic and Entropy Perturbations During Inflation

In this section, we briefly review the standard calculations of the primordial power spectrum, the bispectrum and the trispectrum for the mechanism of modulated reheating, with the assumption that the inflaton ϕ decays with a variable decay rate. Then, we will apply this mechanism to our model in which the decay rate is a function of the Higgs field and study its cosmological implications.

We analyze primordial perturbations in a double-field inflation model involving kinetic couplings. We refer to [166, 167, 168] for earlier studies of inflation models in terms of kinetically mixed double fields and [169, 170] for the paradigm of double-field inflation. Also, the topic of primordial perturbations in multiple-field inflation models was recently reviewed in [171, 172].

During inflation, it is convenient to decompose the field space of our model to directions parallel and orthogonal to the trajectory of background evolution. Along these two directions, one can define the adiabatic field, σ, and the entropy field, s, as follows:

$$\dot{\sigma} = \cos\theta\dot{\phi} + \Omega^{-1}\sin\theta\dot{h}, \tag{5.54}$$

$$\dot{s} = -\sin\theta\dot{\phi} + \Omega^{-1}\cos\theta\dot{h}, \tag{5.55}$$

where the rotation angle is given by

$$\cos\theta = \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + \Omega^{-2}\dot{h}^2}}, \quad \sin\theta = \frac{\Omega^{-1}\dot{h}}{\sqrt{\dot{\phi}^2 + \Omega^{-2}\dot{h}^2}}. \tag{5.56}$$

After that, we can perturb the metric and fields up to linear order. One can introduce the field fluctuations along the adiabatic and entropy directions as follows:

$$\delta\sigma = \cos\theta\delta\phi + \Omega^{-1}\sin\theta\delta h, \tag{5.57}$$

$$\delta s = -\sin\theta\delta\phi + \Omega^{-1}\cos\theta\delta h. \tag{5.58}$$

The potential can also be decomposed as

$$V_\sigma = V_\phi\cos\theta + \Omega V_h\sin\theta, \tag{5.59}$$

$$V_s = -V_\phi\sin\theta + \Omega V_h\cos\theta. \tag{5.60}$$

When $\dot{\theta} = 0$, the adiabatic and entropy perturbations decouple. During inflation, by using the slow-roll conditions, we have $\dot{\theta} \simeq 0$. Therefore, we can neglect interactions between these two modes, then we solve for the amplitude of entropy perturbation, $\delta s_* = H_*/2\pi$, during inflation and therefore the field fluctuation for the Higgs boson, which is given by

$$\delta h_* = \Omega_*\delta s_* = \Omega_*\frac{H_*}{2\pi}. \tag{5.61}$$

where the subscript $_*$ denotes the moment of Hubble crossing.

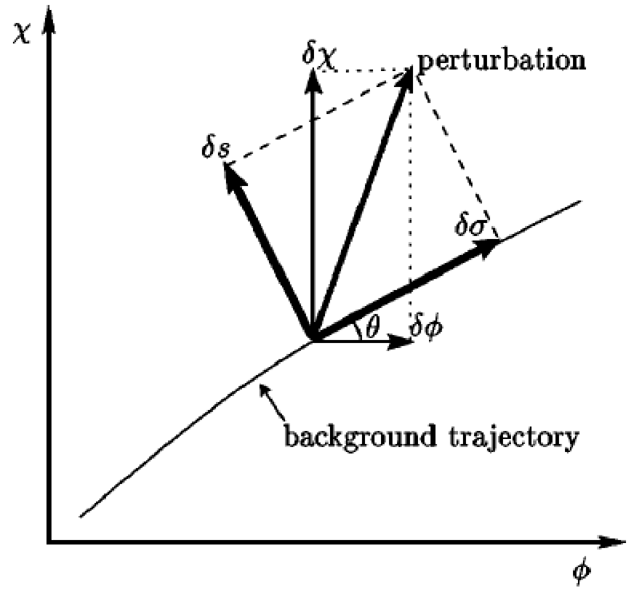

Figure 5.4: An illustration of the decomposition of an arbitrary perturbation into an adiabatic $\delta\sigma$ and entropy δs component. The angle of the tangent to the background trajectory is denoted by θ. The usual perturbation decomposition, along the σ and χ axes, is also shown.

5.9 Higgs Modulated Reheating

The generation of curvature perturbation via Higgs modulated reheating in a canonical model was recently suggested in [160, 161], and soon it was pointed out in [159] that the Higgs dependent interaction potential of the inflaton would be severely constrained by an upper bound on the value of the Higgs during inflation. In the present paper, we extend the paradigm into the noncanonical model under consideration. We show that this upper bound can be greatly relaxed by the relatively large value of the conformal factor and thus the corresponding parameter space is dramatically enlarged. Our scenario is easily extended to nonminimal inflation models.

In the treatment of local non-Gaussianity, the curvature perturbation can be expanded order by order as follows:

$$\begin{aligned}\zeta(x) &= \zeta_1(x) + \frac{3}{5} f_{NL}\zeta_1^2(x) + \frac{9}{25} g_{NL}\zeta_1^3(x) + \mathscr{O}(\zeta_1^4(x)),\\ &= \Sigma_{n=1}^{\infty}\frac{\zeta_n(x)}{n!},\end{aligned} \tag{5.62}$$

where $\zeta_1(x)$ is the Gaussian fluctuation and $\zeta_n(x)$ are the non-Gaussian components of order ζ_1^n. The relation between ζ_n and the non-Gaussian parameters yields the following non-Gaussian estimators:

$$f_{NL} = \frac{5}{6}\frac{\zeta_2}{\zeta_1^2}, \qquad g_{NL} = \frac{25}{54}\frac{\zeta_3}{\zeta_1^3}. \tag{5.63}$$

The correlation functions are defined as

$$\begin{aligned}\left\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\right\rangle &= (2\pi)^3 P(k_1)\delta^3\left(\sum_{n=1}^{2}\vec{k}_n\right),\\ \left\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\right\rangle &= (2\pi)^3 B(\vec{k}_1,\vec{k}_2,\vec{k}_3)\delta^3\left(\sum_{n=1}^{3}\vec{k}_n\right),\\ \left\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\zeta(\vec{k}_4)\right\rangle &= (2\pi)^3 T(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\delta^3\left(\sum_{n=1}^{4}\vec{k}_n\right),\end{aligned} \tag{5.64}$$

where $P(k_1)$ is related to the dimensionless power spectrum as

$$\mathscr{P}_\zeta(k_1) \equiv \frac{k^3}{2\pi^2}P(k_1). \tag{5.65}$$

Inserting the ansatz Eq. (6.60) into Eq. (6.62), one can relate the bispectrum B and the trispectrum T with P as follows:

$$B(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \frac{6}{5} f_{NL} [P(k_1)P(k_2) + 2 \text{ perm}], \tag{5.66}$$

$$\begin{aligned} T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = & \frac{54}{25} g_{NL} [P(k_1)P(k_2)P(k_3) + 3 \text{ perm}] \\ & + \tau_{NL} [P(k_1)P(k_2)P(|\vec{k}_1 + \vec{k}_2|) + 11 \text{ perm}]. \end{aligned} \tag{5.67}$$

Note that if we neglect the nonlinear perturbations induced by self-interactions during inflation, which will be treated, there exists, in this case, a simple relation $\tau_{NL} = \frac{36}{25} f_{NL}^2$.

In the modulated reheating scenario, the decay of the inflaton occurs on a spatial hypersurface with a varying local decay rate Γ, which is assumed to be a function of the Higgs boson in our model. Thus, the local Hubble parameter on the slice of modulated decay satisfies the condition $H = \Gamma(h)$[1]. On super-Hubble scales, the curvature perturbation arising from modulated decay can be written as

$$\zeta_h(x) \simeq -\Theta_1 \frac{\delta h}{h} - \frac{1}{2}\Theta_2 \left(\frac{\delta h}{h}\right)^2 - \frac{1}{6}\Theta_3 \left(\frac{\delta h}{h}\right)^3 |_D, \tag{5.68}$$

where the subscript $_D$ denotes the moment of modulated decay. In our model the potential is dominated by its mass term after inflation, as shown in Eq. (6.47). We therefore obtain the coefficients as follows:

$$\Theta_1 = \frac{h}{6} \frac{\Gamma_{,h}}{\Gamma}, \tag{5.69}$$

$$\Theta_2 = \frac{h^2}{6} \left(\frac{\Gamma_{,hh}}{\Gamma} - \frac{\Gamma_{,h}^2}{\Gamma^2} \right), \tag{5.70}$$

$$\Theta_3 = \frac{h^3}{6} \left(\frac{\Gamma_{,hhh}}{\Gamma} - 3\frac{\Gamma_{,h}\Gamma_{,hh}}{\Gamma^2} + 3\frac{\Gamma_{,h}^3}{\Gamma^3} \right). \tag{5.71}$$

At linear order in the curvature perturbation, the coefficient Θ_1 is typically of the order $O(1)$ and thus ζ is mainly determined by $\delta h/h$ at the moment of modulated decay. In the conventional scenario of modulated reheating, one can approximately take δh_D to be the amplitude of entropy field at the

[1]Within the framework of the multi-field inflationary cosmology, there exist many interesting scenarios for generating primordial curvature perturbation based on different choices of decay slices, such as the modulated curvaton decay mechanism [173, 174, 175] and the uniform curvaton decay mechanism [176]. All these scenarios are well established based on the validity of the δN formalism.

moment of Hubble-crossing during inflation. However for the model of AS inflation, the Higgs boson and the inflaton are coupled through a conformal factor in front of its kinetic term. Making use of Eq. (6.59), one finds

$$\delta h_D = \frac{\Omega_D}{\Omega_*}\delta h_* = \frac{\Omega_D H_*}{2\pi}. \tag{5.72}$$

Moreover, the value of the Higgs at the slice of modulated decay can be related to the Hubble-crossing value by introducing a general function

$$h_D = g(h_*), \tag{5.73}$$

where its detailed form is determined by the explicit potential of the entropy field. For example, in the model under consideration, $g(h_*) \propto h_* \sim h_I$. For simplicity, we assume that $g(h_*)$ is linear.

As a consequence, the power spectrum of curvature perturbation due to modulated reheating is given by

$$P_{\zeta_h} = \Theta_1^2 \frac{\delta h_D^2}{h_D^2} \simeq \Theta_1^2 \Omega_D^2 \frac{H_*^2}{4\pi^2 h_*^2}, \tag{5.74}$$

where we have applied the field fluctuation Eq. (6.70) to obtain the second, approximate equality, expression.

5.10 Observables at Linear Order

If we further neglect the variation of the Hubble parameter during inflation, then we can obtain an approximate power spectrum from Higgs modulated reheating,

$$P_{\zeta_h} \simeq \frac{\lambda}{8\pi^2}\Theta_1^2 \frac{\Omega_D^2}{\Omega_I^4}, \tag{5.75}$$

by inserting the approximate relation Eq. (6.44). Note that the usual decay rate is a power-law function of the Higgs boson such as that in Eq. (6.52), considered in the previous section. Thus Θ_1^2 is typically of the order $O(0.01 \sim 1)$. The coefficient Ω_D is totally determined by ϕ_D as provided by Eq. (6.53), and it therefore depends only on λ and κ; numerically Ω_D is of the order $O(1)$. Finally, we note that the power spectrum generated from the Higgs modulated reheating is determined by the Higgs coupling λ, the interaction coupling κ, and the conformal factor during inflation Ω_I (or, equivalently, the value of inflaton ϕ_I).

In addition to the curvature perturbation generated by modulated reheating, there exists the intrinsic curvature perturbation due to the inflaton fluctuation, which takes the form

$$P_{\zeta_\phi} = \frac{H_I^2}{8\pi^2 \epsilon M_P^2}. \tag{5.76}$$

It is convenient to define a Higgs-to-curvature ratio

$$q_h = \frac{P_{\zeta_h}}{P_{\zeta_h} + P_{\zeta_\phi}} = \frac{\epsilon\lambda\Theta_1^2\Omega_D^2 M_p^2}{\epsilon\lambda\Theta_1^2\Omega_D^2 M_p^2 + \Omega_I^4 H_I^2}, \tag{5.77}$$

to characterize the relative contribution of Higgs fluctuations. If the main contribution to generating primordial curvature perturbation is due to the modulated reheating, then we expect $g_h \simeq 1$. By choosing a group of values for the model parameters such as that provided in Fig. 1 and the decay rate given by Eq. (6.52), one finds $\epsilon \sim 10^{-4}$, $\Omega_I \sim 10$, $\Omega_D \sim 1$, $\Theta_1^2 \sim 0.10$, and $\lambda \sim 0.13$. Under this particular parameter choice, we find that the mechanism of Higgs modulated reheating dominates as long as $H_I^2 < 10^{-9} M_p^2$ without any fine-tuning.

These two power spectra actually show different signatures on their spectral indices. Specifically, their spectral indices are given by

$$n_{\zeta_\phi} - 1 = -6\epsilon + \frac{2U_{,\phi\phi}}{3H^2}, \tag{5.78}$$

$$n_{\zeta_h} - 1 = -2\epsilon + \frac{2V_{,hh}}{3\Omega_I^4 H^2}, \tag{5.79}$$

which are calculated at the moment of Hubble-crossing. In addition, the primordial tensor perturbations are only dependent on the inflationary Hubble parameter, whose spectrum is given by

$$P_T = \frac{2H_I^2}{\pi^2 M_p^2}. \tag{5.80}$$

As usual, the spectral tilt is given by

$$n_T = -2\epsilon. \tag{5.81}$$

In the modulated reheating scenario the conventional tensor-to-scalar ratio r_T is now defined as

$$r_T \equiv \frac{P_T}{P_{\zeta_h} + P_{\zeta_\phi}} = 16\epsilon(1 - q_h), \tag{5.82}$$

which indicates that the amplitude of a primordial gravitational wave is doubly suppressed in the Higgs modulated reheating mechanism since both ϵ and $1 - q_h$ are small quantities.

5.11 Non-Gaussianities

In contrast to the prediction of a canonical single-field inflation model [177], a salient feature of the modulated reheating mechanism is that sizable amplitudes of primordial non-Gaussianities can be obtained under suitable parameter choices. In this section we study the curvature perturbation beyond linear level. As we expected, the curvature perturbation is mainly sourced by the Higgs fluctuations. For the time being, we ignore the nonlinear effects of inflaton, which are generally suppressed by slow-roll parameters. For the nonlinear fluctuations seeded by the Higgs fluctuations, there exist two categories of seeds, with one being proportional to the connected correlators of the Higgs and the other being an intrinsically non-Gaussian distribution [161].

5.11.1 Non-Gaussianities from modulated reheating

The first type of non-Gaussianity originates from the field fluctuations at super-Hubble scales during the process of postinflation modulated reheating. In this era the Higgs field is considered Gaussian while the non-Gaussianity is induced by the nonlinear conversion from δh to ζ. One can insert the second and third order curvature perturbations in Eq. (6.66) into the non-Gaussian estimator Eq. (6.61) and obtain this part of the "universal" nonlinearity parameters:

$$f_{NL,un}^{local} = 5q_h^2\left(1-\frac{\Gamma\Gamma_{,hh}}{\Gamma_{,h}^2}\right), \tag{5.83}$$

$$g_{NL,un}^{local} = \frac{50}{3}q_h^3\left(2-3\frac{\Gamma\Gamma_{,hh}}{\Gamma_{,h}^2}+\frac{\Gamma^2\Gamma_{,hhh}}{\Gamma_{,h}^2}\right), \tag{5.84}$$

which are of local type.

In particular, for the interaction term considered in Eq. (6.51), the decay rate is proportional to h^4 and $q_h \simeq 1$ can be obtained under a reasonable set of values of model parameters. As a consequence, one obtains $f_{NL,un}^{local} \simeq \frac{5}{4}$ and $g_{NL,un}^{local} \simeq \frac{25}{12}$. These nonlinear parameters are sizable when compared with those in slow-roll inflation models, but the corresponding non-Gaussianities are still difficult to test observationally.

5.11.2 Non-Gaussianities from Higgs self-interaction during inflation

The second type of non-Gaussianity originates from the nonquadratic potential of the lighter field, which in our model corresponds to the Higgs potential: $V \simeq \lambda h^4/4$. In fact, this self-interaction of the scalar field can also generate primordial non-Gaussian fluctuations during inflation.

Following from [161] (see also [178]), the n-point correlation function of δh is evaluated by

$$\begin{aligned} &\left\langle \delta h_{\vec{k}_1}(\tau)\delta h_{\vec{k}_2}(\tau)\cdots\delta h_{\vec{k}_n}(\tau), \right\rangle \\ &\quad = -i\left\langle \left| \int_{-\infty}^{\tau} a d\tau' \left[\delta h_{\vec{k}_1}(\tau)\delta h_{\vec{k}_2}(\tau)\cdots\delta h_{\vec{k}_n}(\tau), H_{int}^{(n)}(h(\tau'))\right]\right| \right\rangle, \end{aligned} \quad (5.85)$$

where H^n_{int} int is the nth order interaction Hamiltonian and $\langle\cdots\rangle$ is the ensemble average. Here by nth order we mean the part of H_{int} that is of the order $\mathscr{O}(\delta h^n)$. In our model the Higgs field is conformally coupled to the inflaton due to the RG running gravitational constant. The corresponding field fluctuation is expressed as

$$\delta h_{\vec{k}} = \frac{i\Omega H}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}, \quad (5.86)$$

during inflation. In addition, the interaction Hamiltonian of the Higgs field takes the form

$$\begin{aligned} H_{int}^{(n)} &= \int d^3x a^3 \mathscr{H}_{int}^{(n)}, \\ &= \int d^3x a^3\, \Omega^{-4}(\phi)\frac{1}{n!}V^{(n)}(h)\delta h^n, \end{aligned} \quad (5.87)$$

where $V^{(n)}(h) \equiv \frac{\partial^n V}{\partial h^n}$is the nth derivative of the potential $V(h)$ with respect to the field h, and $V(h)$ as well as $\Omega(\phi)$ are given by Eqs. (6.25) and (6.27), respectively.

We perform the integrals appearing in the correlation functions and find

$$\begin{aligned} \left\langle \delta h_{\vec{k}_1}(\tau)\delta h_{\vec{k}_2}(\tau)\cdots\delta h_{\vec{k}_n}(\tau)\right\rangle &= \frac{(\Omega_* H_*)^{2n-4} V_*^{(n)} K^3}{\prod_{i=1}^n (2k_1^3)}\delta^3\left(\sum_{i=1}^n \vec{k}_i\right) \\ &\quad \times (2\pi)^3 I_n(\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n) \end{aligned} \quad (5.88)$$

where a kernel integral function has been introduced as follows:

$$I_n \equiv 2Re\left[-i\int_{-\infty}^{\tau_{end}} d^3x \frac{d\tau'}{K^3\tau'^4}\prod_{i=1}^n (1-ik_i\tau)e^{iK\tau}\right], \quad (5.89)$$

with K defined as $K = \sum_{i=1}^{n} \vec{k}_i$. For 3− and 4−point correlation functions which are of observable interest, we identify, according to the definitions in Eq. (6.62), the following expressions:

$$
\begin{aligned}
B_{\delta h}^{n-un}(\vec{k}_1, \vec{k}_2, \vec{k}_3) &= \frac{\Omega_*^2 H_*^2 V_*^{(3)} K^3}{\prod_{i=1}^{3}(2k_i^3)} I_3, \\
&= \frac{3\lambda h_* \Omega_*^2 H_*^2 K^3}{4\prod_{i=1}^{n} k_i^3}\left[\frac{8}{9} - \frac{2\sum_{i<j} k_i k_j}{K^2} - \frac{2}{3}(\gamma_E + N_k)\frac{\sum_i k_i^3}{K^3}\right],
\end{aligned}
\tag{5.90}
$$

and

$$
\begin{aligned}
T_{\delta h}^{n-un}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) &= \frac{\Omega_*^4 H_*^4 V_*^{(4)} K^3}{\prod_{i=1}^{4}(2k_i^3)} I_4, \\
&= \frac{3\lambda \Omega_*^2 H_*^2 K^3}{8\prod_{i=1}^{n} k_i^3}\left[\frac{8}{9} - \frac{2\sum_{i<j} k_i k_j}{K^2} + 2\frac{\prod_i k_i}{K^4} - \frac{2}{3}(\gamma_E + N_k)\frac{\sum_i k_i^3}{K^3}\right],
\end{aligned}
\tag{5.91}
$$

where $\gamma_E \simeq 0.58$ is the Euler-Masheroni constant and N_K is the e-folding number for the perturbation mode with a fixed K crossing the Hubble radius until the end of inflation τ_{end}. As introduced in the previous section, the subscript $_*$ indicates the values at Hubble-crossing.

We first calculate the non-Gaussianities of *equilateral* type. One can estimate the correlation functions $B_{\delta h}^{n-un}$ and $T_{\delta h}^{n-un}$ under the particular limit that all the k_i's are of the same value. As a result, substituting Eqs. (6.88) and (6.89) into the expressions Eq. (6.62) yields the nonlinearity parameters of equilateral type as follows:

$$
f_{\mathrm{NL,int}}^{\mathrm{equil}} \simeq -\frac{5\lambda h_*^2}{3\Theta_1 \Omega_*^2 H_*^2} q_h^2 (N_k + \Gamma_E - 3), \tag{5.92}
$$

$$
g_{\mathrm{NL,int}}^{\mathrm{equil}} \simeq -\frac{25\lambda h_*^2}{27\Theta_1^2 \Omega_*^2 H_*^2} q_h^3 (N_k + \Gamma_E - \frac{169}{48}). \tag{5.93}
$$

Next we study the non-Gaussianities originating from the self-interaction of the Higgs field during inflation in the squeezed limit where we assume $k_1 \ll k_2, k_3$ (for bispectrum) ,and $k_1 \ll k_2, k_3, k_4$ (for trispectrum). The same scenario in the framework of GR was discussed in [161]. Here we directly calculate the correlation function of the curvature perturbation and then derive the nonlinearity parameters:

$$f_{\rm NL,int}^{\rm equil} \simeq -\frac{5\lambda h_*^2}{3\Theta_1\Omega_*^2 H_*^2} q_h^2 (N_k + \Gamma_E - \frac{7}{3}), \tag{5.94}$$

$$g_{\rm NL,int}^{\rm equil} \simeq -\frac{25\lambda h_*^2}{27\Theta_1^2\Omega_*^2 H_*^2} q_h^3 (N_k + \Gamma_E - 3). \tag{5.95}$$

From the above results, we can immediately see that the primordial non-Gaussianities due to the self-interaction of the Higgs field during inflation are negative. This is a novel feature in the Higgs modulated reheating scenario. A similar feature was observed in [161] in the framework of standard GR, but in our model the amplitude of nonlinearity parameters involves a new parameter which is the conformal factor Ω.

Consider for the moment the primordial curvature perturbation due solely to the modulated reheating, $q_h \simeq 1$. By choosing a group of reasonable values for the model parameters such as that provided in the previous section, we find $\epsilon \sim 10^{-4}$, $\Omega_I \sim 10$, $\Omega_D \sim 1$, $\Theta_1^2 \sim 0.10$, and $\lambda \sim 0.13$. In addition, there is a theoretical lower bound: $h_I > \frac{2H_I}{\sqrt{\lambda}}$. By assuming $N_K \sim 50$, one obtains $f_{\rm NL,int}^{\rm local} \lesssim -10$ We see that this particular parameter choice appears to be incompatible with the newly released Planck data. This points to the necessity of performing an analysis of the observational constraints on our model. This is the main content of the next section.

5.12 Constraints on Model Parameters by Planck

We compare our results with Planck mission has released data on CMB anisotropy. The results highly constrain cosmological parameters with unprecedented accuracy. Specifically, the amplitude and spectral index of primordial curvature perturbation are determined to be $10^9 P_\zeta = 2.23 \pm 0.16$, and $n_s = 0.9603 \pm 0.0073$ (68% C.L.) at the pivot scale $k = 0.002 \rm Mpc^{-1}$ [120]. Moreover, there is no significant evidence for primordial curvature perturbation deviating from Gaussian distribution. In particular, the bounds on nonlinearity parameters are quoted as $f_{\rm NL}^{\rm local} = 2.7 \pm 5.8$, $f_{\rm NL}^{\rm equal} = -42 \pm 75$ (68% C.L.) [121]. In addition, the upper bound on the tensor-to-scalar ratio is given by $r_T < 0.11$ at 2σ level. In our model of RG improved Higgs modulated reheating, there are eight model parameters. Among these parameters, H_* Ω_* ϵ are associated with the background model; h_* and λ are are related to the details of the Higgs model; and Θ_1 Θ_2 and Θ_3 are determined by specific forms of the decay process, respectively. To be more explicit, λ is basically constrained by particle physics experiments such as those at LHC,

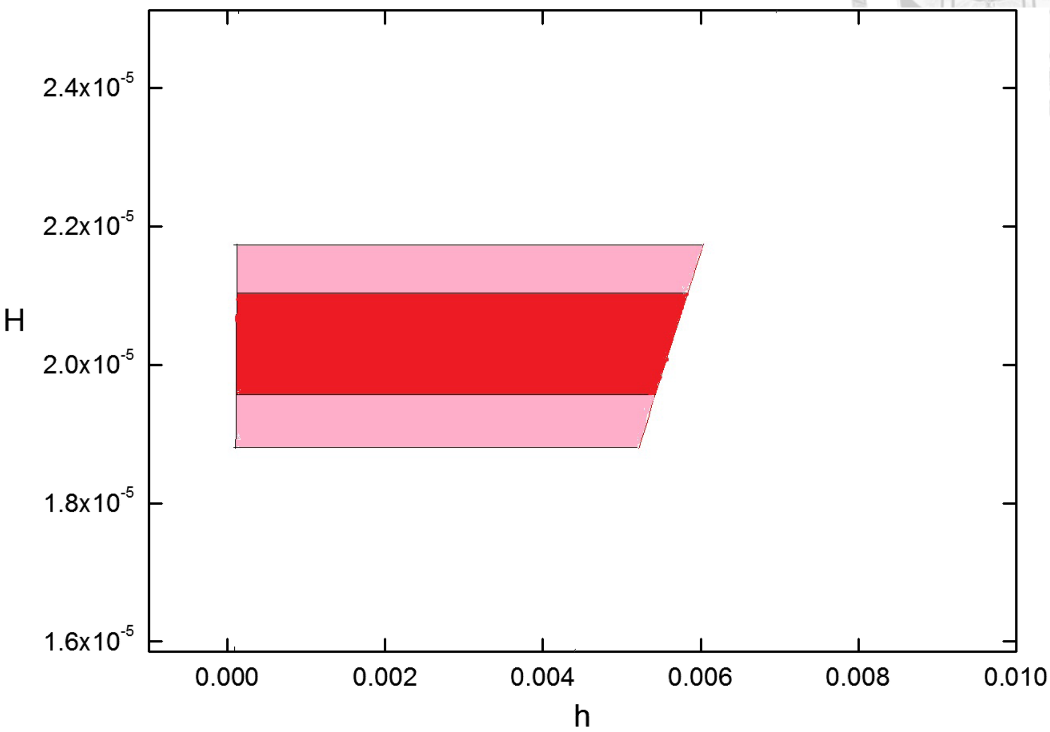

Figure 5.5: Observational and theoretical constraints on h_* and H_* for $q_h = 0$. The viable parameter space is within the red region (C.L. 68%) and the light red region (C.L. 95%).

which have determined that $\lambda \simeq 0.13$. At present the Planck data has not yet imposed strong constraints on the trispectrum and thus Θ_3 is free free. We therefore only need to analyze the combined constraints on the remaining parameters.

Allowing all plausible values for ϵ and Θ_1, we obtain a combined constraint on the inflationary Hubble rate H_* and the amplitude of the Higgs field h_* for different choices of q_h as depicted in Figs. 6.5, 6.6, and 6.7. One can read from the figures that H_* is constrained to be of the order $O(10^{-5} \sim 10^{-6})M_p$ and h_* is allowed to vary between $10^{-4}M_p$ and $10^{-2}M_p$ during inflation. One can also see from the figures that the larger q_h is, the smaller the central value of H will be, consistent with the definition of q_h, namely Eq. (6.75). This indicates that the more Higgs contributes to the final power spectrum, the lower scale inflation we can get.

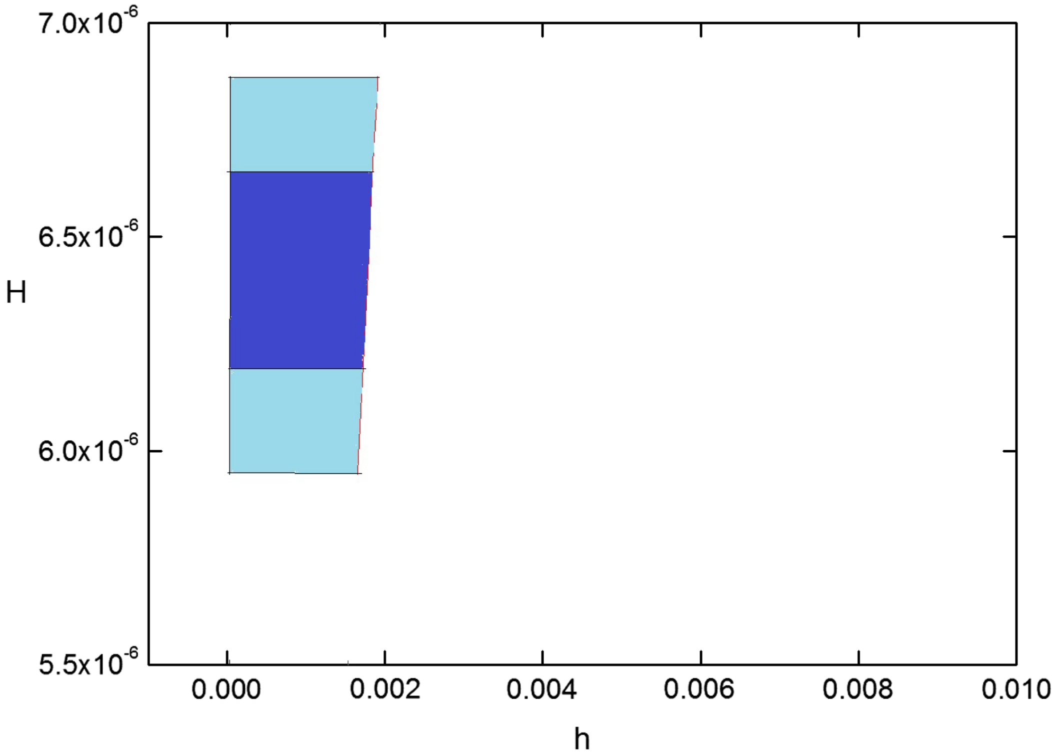

Figure 5.6: Observational and theoretical constraints on h_* and H_* for $q_h = 0.9$. The viable parameter space is within the blue region (C.L. 68%) and the light blue region (C.L. 95%).

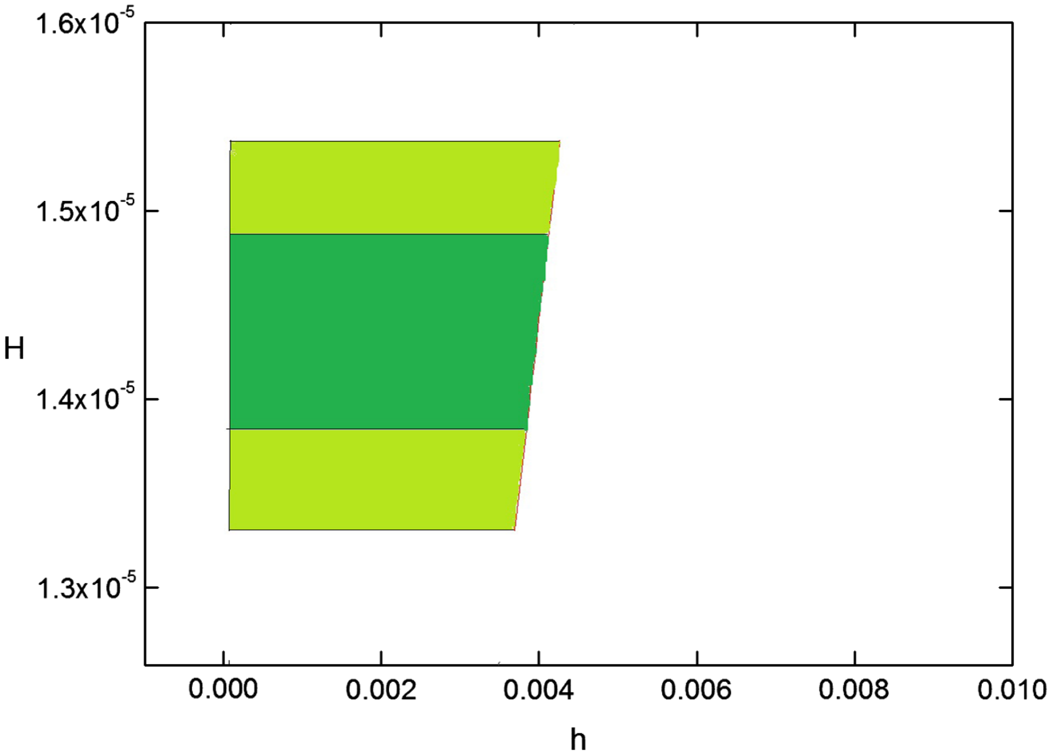

Figure 5.7: Observational and theoretical constraints on h_* and H_* for $q_h = 0.5$. The viable parameter space is within the green region (C.L. 68%) and the light green region (C.L. 95%).

Chapter 6

A Review of Dark Energy in Cosmology

It has been shown that our universe is spatially flat and started accelerating in the recent past. This conclusion has been backed up by many observational data such as type Ia supernovae [192], baryon acoustic oscillation [193], cosmic microwave background radiation [194]. To explain the recent accelerated expansion, an unknown component with negative pressure is usually assumed, and called dark energy. In the last years, answering what dark energy really is has become one of the most challenging questions in cosmology. The simplest candidate for dark energy is a small positive cosmological constant Λ which gives the equation of state $w \equiv p/\rho = -1$ where p stands for pressure and ρ for the dark energy density. Although the cosmological constant with cold dark matter; i.e. ΛCDM model, can explain pretty well the observational data, it suffers, however, from fine-tuning and coincidence problems, in other words, why the cosmological constant is so small and only became dominating almost at present? To address these issues, cosmologists have considered dynamical dark energy models, such as quintessence [195], phantom [196], quintom [197]. In these models, the equation of state w is not necessarily a constant and may evolve with time. Most dark energy models are constructed by scalar fields, having $w \geq -1$, converging to $w = -1$, and the quantum stability of such theories is guaranteed by the energy conditions [198]. However, recent models with equation of state $w < -1$ and converging to $w = -1$ from below, generally referred to as phantom, have drawn lots of attention. The equation of state $w < -1$ is usually realized by a negative kinetic energy, and this counter intuitive assumption violates all the energy conditions, resulting usually in singularities, such as the big rip [199, 200, 196, 201, 202], the sudden [203, 204, 205, 206] the big freeze [205, 206, 208, 209], the type-IV singularity [205, 207, 208, 209, 210, 211], the little rip [212, 213, 214, 215].

In this chapter we briefly review a number of approaches that have been adopted to try and explain the remarkable observation of our late time accelerating Universe. In particular we discuss the arguments for and recent progress made towards understanding the nature of dark energy. We review the observational evidence for the current accelerated expansion of the universe and present a number of dark energy models in addition to the conventional cosmological constant, paying particular attention to scalar field models such as quintessence, phantom, and Quintom.

6.1 The Cosmological Constant

The cosmological constant Λ, was originally introduced by Einstein in 1917 to achieve a static universe. After Hubble's discovery of the expansion of the universe in 1929, it was dropped by Einstein as it was no longer required. From the point of view of particle physics, however, the cosmological constant naturally arises as an energy density of the vacuum. Moreover, the energy scale of Λ should be much larger than that of the present Hubble constant H_0, if it originates from the vacuum energy density. This is the "cosmological constant problem" and was well known to exist long before the discovery of the accelerated expansion of the universe in 1998.

The Einstein tensor $G^{\mu\nu}$ and the energy momentum tensor $T^{\mu\nu}$ satisfy the Bianchi identities $\nabla_\nu G^{\mu\nu} = 0$ and energy conservation $\nabla_\nu T^{\mu\nu} = 0$. Since the metric $g^{\mu\nu}$ is constant with respect to covariant derivatives ($\nabla_\alpha g^{\mu\nu} = 0$), there is a freedom to add a term $\Lambda g_{\mu\nu}$ in the Einstein equations Then the modified Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{6.1}$$

By taking a trace of this equation, we find that $-R+4\Lambda = 8\pi G T$. Combining this relation with Eq. (6.1), we obtain

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) \,. \tag{6.2}$$

Let us consider Newtonian gravity with metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the perturbation around the Minkowski metric $\eta_{\mu\nu}$. If we neglect the time-variation and rotational effect of the metric, R_{00} can be written by a gravitational potential Φ, as $R_{00} \simeq -(1/2)\Delta h_{00} = \Delta\Phi$. Note that g_{00} is given by $g_{00} = -1 - 2\Phi$. In the relativistic limit with $|p| \ll \rho$, we have

$T_{00} \simeq -T \simeq \rho$. Then the 00 component of Eq. (6.2) gives

$$\Delta\Phi = 4\pi G\rho - \Lambda\,. \tag{6.3}$$

In order to reproduce the Poisson equation in Newtonian gravity, we require that $\Lambda = 0$ or Λ is sufficiently small relative to the $4\pi G\rho$ term in Eq. (6.3). Since Λ has dimensions of $[\text{Length}]^{-2}$, the scale corresponding to the cosmological constant needs to be much larger than the scale of stellar objects on which Newtonian gravity works well. In other words the cosmological constant becomes important on very large scales.

In the FRW background, the modified Einstein equations give

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}\,, \tag{6.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}\,. \tag{6.5}$$

This clearly demonstrates that the cosmological constant contributes negatively to the pressure term and hence exhibits a repulsive effect.

Let us consider a static universe ($a = \text{const}$) in the absence of Λ. Setting $H = 0$ and $\ddot{a}/a = 0$ in Eqs. (6.4) and (6.5), we find

$$\rho = -3p = \frac{3K}{8\pi G a^2}\,. \tag{6.6}$$

Equation (6.6) shows that either ρ or p needs to be negative. When Einstein first tried to construct a static universe, he considered that the above solution is not physical[1] and so added the cosmological constant to the original field equations, Eq. (2.48).

Using the modified field equations (6.4) and (6.5) in a dust-dominated universe ($p = 0$), we find that the static universe obtained by Einstein corresponds to

$$\rho = \frac{\Lambda}{4\pi G}\,, \quad \frac{K}{a^2} = \Lambda\,. \tag{6.7}$$

Since $\rho > 0$ we require that Λ is positive. This means that the static universe is a closed one ($K = +1$) with a radius $a = 1/\sqrt{\Lambda}$. Equation (6.7) shows that the energy density ρ is determined by Λ.

The requirement of a cosmological constant to achieve a static universe can be understood by having a look at the Newton's equation of motion (**??**). Since gravity pulls the point particle toward the center of the sphere, we need a repulsive force to realize a situation in which a is constant. This

[1]We note however that the negative pressure can be realized by scalar fields.

corresponds to adding a cosmological constant term $\Lambda/3$ on the right hand side of Eq. (**??**).

The above description of the static universe was abandoned with the discovery of the redshift of distant stars, but it is intriguing that such a cosmological constant should return in the 1990's to explain the observed acceleration of the universe.

Introducing the modified energy density and pressure

$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi G}\,, \quad \tilde{p} = p - \frac{\Lambda}{8\pi G}\,, \tag{6.8}$$

we find that Eqs. (6.4) and (6.5) reduce to the original Fredmann Eqs. In the subsequent sections we shall use the field equations (3.10) and (3.11) when we study the dynamics of dark energy.

6.1.1 fine-tuning problem

If the cosmological constant originates from a vacuum energy density, then this suffers from a severe fine-tuning problem. Observationally we know that Λ is of order the present value of the Hubble parameter H_0, that is

$$\Lambda \approx H_0^2 = (2.13h \times 10^{-42}\,\mathrm{GeV})^2\,. \tag{6.9}$$

This corresponds to a critical density ρ_Λ,

$$\rho_\Lambda = \frac{\Lambda m_{\mathrm{pl}}^2}{8\pi} \approx 10^{-47}\,\mathrm{GeV}^4\,. \tag{6.10}$$

Meanwhile the vacuum energy density evaluated by the sum of zero-point energies of quantum fields with mass m is given by

$$\begin{aligned}\rho_{vac} &= \frac{1}{2}\int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{k^2+m^2} \\ &= \frac{1}{4\pi^2}\int_0^\infty dk\, k^2\sqrt{k^2+m^2}\,. \end{aligned}\tag{6.11}$$

This exhibits an ultraviolet divergence: $\rho_{vac} \propto k^4$. However we expect that quantum field theory is valid up to some cut-off scale $k_{\max}$ in which case the integral (6.11) is finite:

$$\rho_{\mathrm{vac}} \approx \frac{k_{\max}^4}{16\pi^2}\,. \tag{6.12}$$

For the extreme case of General Relativity we expect it to be valid to just below the Planck scale: $m_{\mathrm{pl}} = 1.22 \times 10^{19}\,\mathrm{GeV}$. Hence if we pick up $k_{\max} = m_{\mathrm{pl}}$, we find that the vacuum energy density in this case is estimated as

$$\rho_{\mathrm{vac}} \approx 10^{74}\,\mathrm{GeV}^4\,, \tag{6.13}$$

which is about 10^{121} orders of magnitude larger than the observed value given by Eq. (6.10). Even if we take an energy scale of QCD for $k_{\max}$, we obtain $\rho_{\rm vac} \approx 10^{-3}\,{\rm GeV}^4$ which is still much larger than ρ_Λ.

We note that this contribution is related to the ordering ambiguity of fields and disappears when normal ordering is adopted. Since this procedure of throwing away the vacuum energy is ad hoc, one may try to cancel it by introducing counter terms. However this requires a fine-tuning to adjust ρ_Λ to the present energy density of the universe. Whether or not the zero point energy in field theory is realistic is still a debatable question.

6.2 Quintessence

Quintessence is described by an ordinary scalar field ϕ minimally coupled to gravity, but as we will see with particular potentials that lead to late time inflation. The action for Quintessence is given by

$$S = \int {\rm d}^4x\sqrt{-g}\left[-\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] , \tag{6.14}$$

where $(\nabla\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and $V(\phi)$ is the potential of the field. In a flat FRW spacetime the variation of the action (6.14) with respect to ϕ gives

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{\phi} = 0 . \tag{6.15}$$

The energy momentum tensor of the field is derived by varying the action (6.14) in terms of $g^{\mu\nu}$:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} . \tag{6.16}$$

Taking note that $\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, we find

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] . \tag{6.17}$$

In the flat Friedmann background we obtain the energy density and pressure density of the scalar field:

$$\rho = -T_0^0 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \;\; p = T_i^i = \frac{1}{2}\dot{\phi}^2 - V(\phi) . \tag{6.18}$$

Then the Fredmann eqs yield

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] , \tag{6.19}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\dot{\phi}^2 - V(\phi)\right] . \tag{6.20}$$

We recall that the continuity equation is derived by combining these equations.

From Eq. (6.20) we find that the universe accelerates for $\dot{\phi}^2 < V(\phi)$. This means that one requires a flat potential to give rise to an accelerated expansion. In the context of inflation the slow-roll parameters

$$\epsilon = \frac{m_{\rm pl}^2}{16\pi}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \quad \eta = \frac{m_{\rm pl}^2}{8\pi}\frac{1}{V}\frac{d^2V}{d\phi^2}\,, \tag{6.21}$$

are often used to check the existence of an inflationary solution for the model (6.14). Inflation occurs if the slow-roll conditions, $\epsilon \ll 1$ and $|\eta| \ll 1$, are satisfied. In the context of dark energy these slow-roll conditions are not completely trustworthy, since there exists dark matter as well as dark energy. However they still provide a good measure to check the existence of a solution with an accelerated expansion. If we define slow-roll parameters in terms of the time-derivatives of H such as $\epsilon = -\dot{H}/H^2$, this is a good measure to check the existence of an accelerated expansion since they implement the contributions of both dark energy and dark matter.

The equation of state for the field ϕ is given by

$$w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}\,. \tag{6.22}$$

In this case the continuity equation can be written in an integrated form:

$$\rho = \rho_0 \exp\left[-\int 3(1+w_\phi)\frac{\mathrm{d}a}{a}\right]\,, \tag{6.23}$$

where ρ_0 is an integration constant. We note that the equation of state for the field ϕ ranges in the region $-1 \le w_\phi \le 1$. The slow-roll limit, $\dot{\phi}^2 \ll V(\phi)$, corresponds to $w_\phi = -1$, thus giving $\rho = \text{const}$ from Eq. (6.23). In the case of a stiff matter characterized by $\dot{\phi}^2 \gg V(\phi)$ we have $w_\phi = 1$, in which case the energy density evolves as $\rho \propto a^{-6}$ from Eq. (6.23). In other cases the energy density behaves as

$$\rho \propto a^{-m} \qquad 0 < m < 6\,. \tag{6.24}$$

Since $w_\phi = -1/3$ is the border of acceleration and deceleration, the universe exhibits an accelerated expansion for $0 \le m < 2$.

It is of interest to derive a scalar-field potential that gives rise to a power-law expansion:

$$a(t) \propto t^p\,. \tag{6.25}$$

The accelerated expansion occurs for $p > 1$, we obtain the relation $\dot{H} = -4\pi G\dot{\phi}^2$. Then we find that $V(\phi)$ and $\dot{\phi}$ can be expressed in terms of H and $\dot{H}$:

$$V = \frac{3H^2}{8\pi G}\left(1 + \frac{\dot{H}}{3H^2}\right), \tag{6.26}$$

$$\phi = \int \mathrm{d}t \left[-\frac{\dot{H}}{4\pi G}\right]^{1/2}. \tag{6.27}$$

Here we chose the positive sign of $\dot{\phi}$. Hence the potential giving the power-law expansion (6.25) corresponds to

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi}{m_{\mathrm{pl}}}\right), \tag{6.28}$$

where V_0 is a constant. The field evolves as $\phi \propto \ln t$. The above result shows that the exponential potential may be used for dark energy provided that $p > 1$.

In addition to the fact that exponential potentials can give rise to an accelerated expansion, they possess cosmological*scaling solutions* in which the field energy density (ρ_ϕ) is proportional to the fluid energy density (ρ_m). Exponential potentials were used in one of the earliest models which could accommodate a period of acceleration today within it, the loitering universe.

The above discussion shows that scalar-field potentials which are not steep compared to exponential potentials can lead to an accelerated expansion. In fact the original quintessence models are described by the power-law type potential

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha}, \tag{6.29}$$

where α is a positive number (it could actually also be negative) and M is constant. Where does the fine tuning arise in these models? Recall that we need to match the energy density in the quintessence field to the current critical energy density, that is

$$\rho_\phi^{(0)} \approx m_{\mathrm{pl}}^2 H_0^2 \approx 10^{-47}\ \mathrm{GeV}^4. \tag{6.30}$$

The mass squared of the field ϕ is given by $m_\phi^2 = \frac{d^2V}{d\phi^2} \approx \rho_\phi/\phi^2$, whereas the Hubble expansion rate is given by $H^2 \approx \rho_\phi/m_{\mathrm{pl}}^2$. The universe enters a tracking regime in which the energy density of the field ϕ catches up that of the background fluid when m_ϕ^2 decreases to of order H^2. This shows that the field value at present is of order the Planck mass ($\phi_0 \sim m_{\rm pl}$), which is typical of most of the quintessence models. Since $\rho_\phi^{(0)} \approx V(\phi_0)$, we obtain the mass scale

$$M = \left(\rho_\phi^{(0)} m_{\rm pl}^\alpha\right)^{\frac{1}{4+\alpha}} . \tag{6.31}$$

This then constrains the allowed combination of α and M. For example the constraint implies $M = 1\,\mathrm{GeV}$ for $\alpha = 2$. This energy scale can be compatible with the one in particle physics, which means that the severe fine-tuning problem of the cosmological constant is alleviated. Nevertheless a general problem we always have to tackle is finding such quintessence potentials in particle physics.

The Quintessence field must couple to ordinary matter, which even if suppressed by the Planck scale, will lead to long range forces and time dependence of the constants of nature. There are tight constraints on such forces and variations and any successful model must satisfy them.

6.3 Phantom

Recent observational data indicates that the equation of state parameter w lies in a narrow strip around $w = -1$ and is quite consistent with being below this value. The scalar field models discussed in the previous subsections correspond to an equation of state $w \geq -1$. The region where the equation of state is less than -1 is typically referred to as a being due to some form of phantom (ghost) dark energy. Specific models in braneworlds or Brans-Dicke scalar-tensor gravity can lead to phantom energy. Meanwhile the simplest explanation for the phantom dark energy is provided by a scalar field with a negative kinetic energy. Such a field may be motivated from S-brane constructions in string theory.

Historically, phantom fields were first introduced in Hoyle's version of the steady state theory. In adherence to the perfect cosmological principle, a creation field (C-field) was introduced by Hoyle to reconcile the model with the homogeneous density of the universe by the creation of new matter in the voids caused by the expansion of the universe. It was further refined and reformulated in the Hoyle and Narlikar theory of gravitation. The action of the phantom field minimally coupled to gravity is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] , \tag{6.32}$$

where the sign of the kinetic term is opposite compared to the action (6.14) for an ordinary scalar field. Since the energy density and pressure density are given by $\rho = -\dot{\phi}^2/2 + V(\phi)$ and $p = -\dot{\phi}^2/2 - V(\phi)$ respectively, the equation of state of the field is

$$w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)}\,. \tag{6.33}$$

Then we obtain $w_\phi < -1$ for $\dot{\phi}^2/2 < V(\phi)$.

As discussed in previous sections, the curvature of the universe grows toward infinity within a finite time in the universe dominated by a phantom fluid. In the case of a phantom scalar field this Big Rip singularity may be avoided if the potential has a maximum, e.g.,

$$V(\phi) = V_0 \left[\cosh\left(\frac{\alpha\phi}{m_{\rm pl}}\right)\right]^{-1}, \tag{6.34}$$

where α is constant. Due to its peculiar properties, the phantom field evolves towards the top of the potential and crosses over to the other side. It turns back to execute a period of damped oscillations about the maximum of the potential at $\phi = 0$. After a certain period of time the motion ceases and the field settles at the top of the potential to mimic the de-Sitter like behavior ($w_\phi = -1$). This behavior is generic if the potential has a maximum. In the case of exponential potentials the system approaches a constant equation of state with $w_\phi < -1$.

Although the above behavior of the phantom field is intriguing as a "classical cosmological" field, unfortunately phantom fields are generally plagued by severe Ultra-Violet (UV) quantum instabilities. Since the energy density of a phantom field is unbounded from below, the vacuum becomes unstable against the production of ghosts and normal (positive energy) fields. Even when ghosts are decoupled from matter fields, they couple to gravitons which mediate vacuum decay processes of the type: vacuum $\rightarrow$ 2 ghosts + 2γ. It was shown by Cline *et al.* that we require an unnatural Lorenz invariance breaking term with cut off of order $\sim$ MeV to prevent an overproduction of cosmic gamma rays. Hence the fundamental origin of the phantom field still poses an interesting challenge for theoreticians. covering various cosmological aspects of phantom fields.

6.4 Chaplygin Gas

So far we have discussed a number of scalar-field models of dark energy. There exist another interesting class of dark energy models involving a fluid known as a Chaplygin gas [180]. This fluid also leads to the acceleration of the universe at late times, and in its simplest form has the following specific equation of state:

$$p = -\frac{A}{\rho}\,, \tag{6.35}$$

where A is a positive constant. The equation of state for the Chaplygin gas can be derived from the Nambu-Goto action for a D-brane moving in the $D+1$ dimensional bulk. For the case of the moving brane (via the Born-Infeld Lagrangian), the derivation of the Chaplygin gas equation of state was first discussed in the context of braneworld cosmologies in [181].

With the equation of state (6.35) the continuity equation can be integrated to give

$$\rho = \sqrt{A + \frac{B}{a^6}}\,, \tag{6.36}$$

where B is a constant. Then we find the following asymptotic behavior:

$$\rho \sim \frac{\sqrt{B}}{a^3}\,, \qquad a \ll (B/A)^{1/6}\,, \tag{6.37}$$

$$\rho \sim -p \sim \sqrt{A} \qquad a \gg (B/A)^{1/6}\,. \tag{6.38}$$

This is the intriguing result for the Chaplygin gas. At early times when a is small, the gas behaves as a pressureless dust. Meanwhile it behaves as a cosmological constant at late times, thus leading to an accelerated expansion.

One can obtain a corresponding potential for the Chaplygin gas by treating it as an ordinary scalar field ϕ. Using Eqs. (6.35) and (6.36) together with $\rho = \dot{\phi}^2/2 + V(\phi)$ and $p = \dot{\phi}^2/2 - V(\phi)$, we find

$$\dot{\phi}^2 = \frac{B}{a^6\sqrt{A + B/a^6}}\,, \tag{6.39}$$

$$V = \frac{1}{2}\left[\sqrt{A + B/a^6} + \frac{A}{\sqrt{A + B/a^6}}\right]\,. \tag{6.40}$$

We note that this procedure is analogous to the reconstruction methods we adopted for the quintessence and tachyon potentials. Since the Hubble expansion rate is given by $H = (8\pi\rho/3m_{\rm pl}^2)^{1/2}$, we can rewrite Eq. (6.39) in terms of the derivative of a:

$$\frac{\kappa}{\sqrt{3}}\frac{\mathrm{d}\phi}{\mathrm{d}a} = \frac{\sqrt{B}}{a\sqrt{Aa^6 + B}}\,. \tag{6.41}$$

This is easily integrated to give

$$a^6 = \frac{4Be^{2\sqrt{3}\kappa\phi}}{A(1 - e^{2\sqrt{3}\kappa\phi})^2}\,. \tag{6.42}$$

Substituting this for Eq. (6.40) we obtain the following potential:

$$V(\phi) = \frac{\sqrt{A}}{2}\left(\cosh\sqrt{3}\kappa\phi + \frac{1}{\cosh\sqrt{3}\kappa\phi}\right) . \tag{6.43}$$

Hence, a minimally coupled field with this potential is equivalent to the Chaplygin gas model.

Chaplygin gas provides an interesting possibility for the unification of dark energy and dark matter. However it was shown that the Chaplygin gas models are under strong observational pressure from CMB anisotropies (see Ref. [182]). This comes from the fact that the Jeans instability of perturbations in Chaplygin gas models behaves similarly to cold dark matter fluctuations in the dust-dominant stage given by (6.37) but disappears in the acceleration stage given by (6.38). The combined effect of the suppression of perturbations and the presence of a non-zero Jeans length gives rise to a strong integrated Sachs-Wolfe (ISW) effect, thereby leading to the loss of power in CMB anisotropies. This situation can be alleviated in the generalized Chaplygin gas model with $p = -A/\rho^\alpha$, $0 < \alpha < 1$. However, even in this case the parameter α is rather severely constrained, i.e., $0 \leq \alpha < 0.2$ at the 95% confidence level.

Chapter 7

Dark Energy Model by Dark Spinor with Torsion

In this chapter, we propose a dark energy model with the phantom dark spinor in Einstein-Cartan theory. We will first introduce the definition of the dark spinor and regard it as the origin of dark energy, then we analyze the cosmological evolution of this kind of dark energy model in the existence of torsion.

7.1 The ELKO Spinors

In this section, we will briefly summarize the essential definitions and properties of the ELKO spinors [1]. As mentioned in the introduction, ELKO spinors are the eigenspinors of the charge congugation operator in momentum space with spin one half. To be explicit, we define the ELKO spinor as [185, 186]

$$\lambda(\mathbf{p}) = \begin{pmatrix} \pm\sigma_2\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}, \tag{7.1}$$

where σ_2 is the second Pauli matrix, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, ϕ_L is the left-handed Weyl spinor, and ϕ_L^* is its complex conjugate. Recall that the left-handed and right-handed Weyl spinors are 2-dimensional objects in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation of Lorentz group, which transform under the infinitesi-

[1] Readers who are interested in more details, please consult Refs [185, 186]

mal rotations $\boldsymbol{\theta}$ and boosts $\boldsymbol{\beta}$ as [227]

$$\phi_L \to (1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2})\phi_L,$$
$$\phi_R \to (1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2})\phi_R.$$

The charge conjugation operator is given by

$$\mathscr{C} = \begin{pmatrix} \mathbb{0} & i\Theta \\ -i\Theta & \mathbb{0} \end{pmatrix} K, \tag{7.2}$$

where $\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is called the Wigner time reversal operator for spin one-half fields and K is an operator that complex conjugates any Weyl spinor that appears on its right. Then the ELKO spinor satisfies the eigenstate equation:

$$\mathscr{C}\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p}). \tag{7.3}$$

The plus sign is called the *self-conjugate spinor*, denoted by $\lambda^S(\mathbf{p})$ and the minus sign is called the *anti-self-conjugate spinor*, denoted by $\lambda^A(\mathbf{p})$ [228]. The *helicity* is defined as the component of the spin angular momentum in the direction of the three momentum. Thus, the helicity operator can be written as $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$, where $\boldsymbol{\sigma}$ are the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\hat{\boldsymbol{p}}$ is the unit vector of the three momentum. If we choose a Weyl spinor $\phi_L(k^\mu)$ in its rest frame to be the eigenspinor of the helicity operator, that is

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\phi_L^{\pm}(k^\mu) = \pm\phi_L^{\pm}(k^\mu), \tag{7.4}$$

where $k^\mu = \left(m, \lim_{p\to 0}\frac{\boldsymbol{p}}{p}\right)$ and $p = |\boldsymbol{p}|$, then from the property [185]

$$\Theta\boldsymbol{\sigma}\Theta^{-1} = -\boldsymbol{\sigma}^*, \tag{7.5}$$

we have

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,(\Theta[\phi_L^{\pm}(k^\mu)]^*) = \mp(\Theta[\phi_L^{\pm}(k^\mu)]^*). \tag{7.6}$$

This shows that $\Theta[\phi_L^{\pm}(k^\mu)]^*$ has an opposite helicity to $\phi_L^{\pm}(k^\mu)$, or equivalently, due to $\sigma_2 = i\Theta$, $\sigma_2[\phi_L^{\pm}(k^\mu)]^*$ also has opposite helicity to $\phi_L^{\pm}(k^\mu)$. Therefore, we have two helicity states of the Weyl left-handed spinors, one has positive helicity ϕ_L^+, the other has negative helicity ϕ_L^-, both of them are eigenspinors of the helicity operator. Hence, for each self-conjugate and anti-self-conjugate ELKO spinor, this would give us two classes of spinors which can be explicitly defined as

$$\lambda^S_{\{-,+\}}(k^\mu) = +\begin{pmatrix} \sigma_2[\phi^+_L(k^\mu)]^* \\ \phi^+_L(k^\mu) \end{pmatrix}, \tag{7.7a}$$

$$\lambda^S_{\{+,-\}}(k^\mu) = +\begin{pmatrix} \sigma_2[\phi^-_L(k^\mu)]^* \\ \phi^-_L(k^\mu) \end{pmatrix}, \tag{7.7b}$$

and

$$\lambda^A_{\{+,-\}}(k^\mu) = +\begin{pmatrix} -\sigma_2[\phi^-_L(k^\mu)]^* \\ \phi^-_L(k^\mu) \end{pmatrix}, \tag{7.7c}$$

$$\lambda^A_{\{-,+\}}(k^\mu) = -\begin{pmatrix} -\sigma_2[\phi^+_L(k^\mu)]^* \\ \phi^+_L(k^\mu) \end{pmatrix}. \tag{7.7d}$$

By using the boost operator in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$-representation for any given four-momentum p^μ,

$$e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\varphi}} = \begin{pmatrix} e^{\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}} & \mathbf{0} \\ \mathbf{0} & e^{-\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}} \end{pmatrix}, \tag{7.8}$$

where $\boldsymbol{\varphi}$ is the boost parameter, defined by $\cosh\varphi = \frac{E}{m}$, $\sinh\varphi = \frac{p}{m}$, and $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{|p|}$, then we are able to boost the rest spinor $\lambda(k^\mu)$ to its moving frame,

$$\lambda(p^\mu) = e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\varphi}}\lambda(k^\mu). \tag{7.9}$$

To be explicit, we define the self-conjugate and anti-self-conjugate ELKO spinors as follows:

$$\lambda^S_{\{-,+\}}(p^\mu) = \sqrt{\frac{E+m}{2m}}\left(1-\frac{p}{E+m}\right)\lambda^S_{\{-,+\}}(k^\mu), \tag{7.10a}$$

$$\lambda^S_{\{+,-\}}(p^\mu) = \sqrt{\frac{E+m}{2m}}\left(1+\frac{p}{E+m}\right)\lambda^S_{\{+,-\}}(k^\mu), \tag{7.10b}$$

and

$$\lambda^A_{\{+,-\}}(p^\mu) = \sqrt{\frac{E+m}{2m}}\left(1+\frac{p}{E+m}\right)\lambda^A_{\{+,-\}}(k^\mu), \tag{7.10c}$$

$$\lambda^A_{\{-,+\}}(p^\mu) = \sqrt{\frac{E+m}{2m}}\left(1-\frac{p}{E+m}\right)\lambda^A_{\{-,+\}}(k^\mu). \tag{7.10d}$$

Note that the double-helicity structure of the ELKO spinor is the crucial character different from the standard Majorana spinor, which is a set of two Weyl spinors, and both of them are the eigenspinors of the charge conjugation operator with eigenvalue being unity. It means Majorana spinor is only self-conjugate under the charge conjugation operator, where as the ELKO spinors contain both the self-conjugate and the anti-self-conjugate eigenspinors of charge conjugation operator. Therefore the ELKO spinors could form a basis for the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$-representation of spin one-half particles.

Next, note that if one uses the usual Dirac dual, $\bar{\psi}=\psi^\dagger\gamma^0$, for the ELKO spinors, one will obtain imaginary norms in some combinations of the four ELKO spinors, for instance, $\bar{\lambda}^S_{\{+,-\}}(\mathbf{p})\lambda^S_{\{-,+\}}(\mathbf{p})=2im$. In order to construct the lagrangian, one needs a dual for the ELKO spinors to yield real norms. From this point of view, one defines a new dual for the ELKO spinors, which is similar to the Dirac dual [185]:

$$\bar{\lambda}_\alpha(\mathbf{p})=i\varepsilon^\beta_\alpha\lambda^\dagger_\beta(\mathbf{p})\gamma^0, \tag{7.11}$$

with the antisymmetric symbol $\varepsilon^{\{-,+\}}_{\{+,-\}}=-1=\varepsilon^{\{+,-\}}_{\{-,+\}}$. Equation (3.11) holds for self-conjugate as well as anti-self-conjugate $\lambda(\mathbf{p})$. The new dual is unique and has the following properties: (i) it yields an invariant real definite norm, and, (ii) it yields a positive-definite norm for two of the four ELKO spinors, and negative-definite norm for the remaining two. With the definition of the new dual for the ELKO spinors, one has [185]:

$$\bar{\lambda}^S_\alpha(\mathbf{p})\lambda^S_{\alpha'}(\mathbf{p})=2m\delta_{\alpha\alpha'}, \tag{7.12a}$$
$$\bar{\lambda}^A_\alpha(\mathbf{p})\lambda^A_{\alpha'}(\mathbf{p})=-2m\delta_{\alpha\alpha'}, \tag{7.12b}$$
$$\bar{\lambda}^S_\alpha(\mathbf{p})\lambda^A_{\alpha'}(\mathbf{p})=0, \tag{7.12c}$$
$$\bar{\lambda}^A_\alpha(\mathbf{p})\lambda^S_{\alpha'}(\mathbf{p})=0, \tag{7.12d}$$

where α takes two possibilites, $\{-,+\}$ and $\{+,-\}$. Both self-conjugate and anti-self-conjugate ELKO spinors constitute a complete basis for the four-component spinors in $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$-representation. The completeness relation is given by [185]

$$\frac{1}{2m}\sum_\alpha\Big(\lambda^S_\alpha(\mathbf{p})\bar{\lambda}^S_\alpha(\mathbf{p})-\lambda^A_\alpha(\mathbf{p})\bar{\lambda}^A_\alpha(\mathbf{p})\Big)=\mathbb{1}. \tag{7.13}$$

Since the Dirac spinors also form a complete basis of the four-component spinors, the ELKO spinors can be expressed as a linear combination of the Dirac spinors. Projecting the ELKO spinors onto the Dirac spinors and by means of the field equation of the Dirac spinors, i.e. the Dirac equation, one can deduce the equations of motion for the ELKO spinors (in momentum space) [185]:

$$\left(\gamma_\mu p^\mu \delta_\alpha^\beta + im\mathbb{1}\varepsilon_\alpha^\beta\right)\lambda_\beta^S(\mathbf{p}) = 0, \tag{7.14a}$$

$$\left(\gamma_\mu p^\mu \delta_\alpha^\beta - im\mathbb{1}\varepsilon_\alpha^\beta\right)\lambda_\beta^A(\mathbf{p}) = 0. \tag{7.14b}$$

To obtain the field equation in the configuration space, we make the substitution $p^\mu \to i\partial^\mu$, and define

$$\lambda^{S/A}(x) = \lambda^{S/A}(\mathbf{p})\exp\left[\epsilon^{S/A}\cdot(ix_\mu p^\mu)\right], \tag{7.15}$$

where, by Eqs. (7.14a) and (7.14b), $\epsilon^S = -1$ and $\epsilon^A = +1$, we have

$$\left(i\gamma_\mu \partial^\mu \delta_\alpha^\beta + im\mathbb{1}\varepsilon_\alpha^\beta\right)\lambda_\beta^{S/A}(x) = 0. \tag{7.16}$$

From Eq. (7.14) and (7.16), it is clear that the presence of the anti-symmetric symbol ε_α^β in the mass term makes the ELKO spinors not satisfying the Dirac equation as opposed to that for the Dirac and the Majorana spinors. Nevertheless, since one can "square" the equation of motion, Eq. (7.14), $(\gamma_\mu p^\mu \delta_\alpha^\beta + im\mathbb{1}\varepsilon_\alpha^\beta)(\gamma_\mu p^\mu \delta_\alpha^\beta - im\mathbb{1}\varepsilon_\alpha^\beta) = (p_\mu p^\mu - m^2)\mathbb{1}\delta_\alpha^\beta$, the ELKO spinors do satisfy the Klein-Gordon equation (in momentum space):

$$\left(\eta_{\mu\nu}p^\mu p^\nu - m^2\mathbb{1}\right)\lambda^{S/A}(\mathbf{p}) = 0. \tag{7.17}$$

Or equivalently, in field configuration space, we have:

$$\left(\eta_{\mu\nu}\partial^\mu \partial^\nu + m^2\mathbb{1}\right)\lambda^{S/A}(x) = 0. \tag{7.18}$$

Based on this field equation, one may construct the action of the free ELKO spinors in flat spacetime as:

$$\begin{aligned} S[\stackrel{\neg}{\lambda}(x), \lambda(x)] &= \int d^4x \mathscr{L}\left(\stackrel{\neg}{\lambda}(x), \lambda(x)\right), \\ &= \int d^4x \left(\frac{1}{2}\partial^\mu \stackrel{\neg}{\lambda}(x)\partial_\mu \lambda(x) - \frac{1}{2}m^2 \stackrel{\neg}{\lambda}(x)\lambda(x)\right). \end{aligned} \tag{7.19}$$

Although one may read off directly from the above action that the ELKO spinor has the dimension of mass, it should be noted that it is not a rigorous proof about the dimensionality. For a careful discussion see Refs. [185, 228].

7.2 A Dark Energy Model of Phantom ELKO Spinors with Torsion

In this section, we consider a dynamical dark energy model constructed from the ELKO spinors in Einstein-Cartan theory. To begin with, since it is sometimes more convenient to work in an orthonormal basis, let us introduce the vielbein e^μ_a, defined by

$$g_{\mu\nu}e^\mu_a e^\nu_b = \eta_{ab}, \tag{7.20}$$

where $g_{\mu\nu}$ is the spacetime metric and η_{ab} is the metric of the local inertial frame given by $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$. The Greek letters $(\mu,\nu,\ldots)$ take values $(t,x,\ldots)$ and are called the *holonomic* indices representing the spacetime frame, the Latin letters $(a,b,\ldots)$ taking values $(0,1,\ldots)$ are called the *anholonomic* indices representing the local inertial (orthonormal) frame. We choose the anholonomic γ-matrices, γ^a, in the Weyl representation [188]

$$\gamma^0 = \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbb{0} & -\sigma^i \\ \sigma^i & \mathbb{0} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}, \tag{7.21}$$

where $i = 1,2,3$, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The γ-matrices satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{7.22}$$

We define $\gamma^\mu = e^\mu_a\gamma^a$, then $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$. The anti-commutator of two matrices is defined as: $\{A,B\} = AB+BA$ while the commutator as $[A,B] = AB-BA$.

The covariant derivatives of the ELKO spinor λ and its (ELKO) dual $\bar{\lambda}$ in the local inertial frame are defined in the same way as for the ordinary spinors, i.e.

$$\nabla_\mu\lambda = \partial_\mu\lambda - \Gamma_\mu\lambda, \tag{7.23a}$$

$$\nabla_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} + \bar{\lambda}\Gamma_\mu, \tag{7.23b}$$

where Γ_μ is called the spin connection which is used to make the covariant derivative of a spinor transform correctly under both local Lorentz transformation and general coordinate transformation. By further requiring that $\nabla_\mu e^a_\nu = 0$, the relation between the spin connection and the affine connection can be obtained in the following form [188, 224]

$$\Gamma_\mu = \frac{i}{2}\omega^{ab}_\mu f_{ab}, \tag{7.24a}$$

$$\omega^{ab}_\mu = e^a_\nu\partial_\mu e^{\nu b} + e^a_\nu e^{\sigma b}\Gamma^\nu_{\mu\sigma}, \tag{7.24b}$$

where $f^{ab} = \frac{i}{4}[\gamma^a,\gamma^b]$ is the generator of the local Lorentz group. Within the presence of torsion fields, we now need to extend the definition of the covariant derivatives on spinors to include torsions. According to Eq. (2.51), we may separate the non-torsion free affine connection into a torsion-free

Christoffel symbol plus a contortion tensor. Applying this relation into the spin connection Eq. (7.24) and after some algebra, we obtain:

$$\tilde{\nabla}_a \lambda = \nabla_a \lambda + \frac{1}{4} K_{abc} \gamma^b \gamma^c \lambda. \tag{7.25}$$

Since $\stackrel{\neg}{\lambda}\lambda$ is a real scalar, the covariant derivative on the dual spinor $\stackrel{\neg}{\lambda}$ can be obtained from the Leibnitz rule. We obtain then

$$\tilde{\nabla}_a \stackrel{\neg}{\lambda} = \nabla_a \stackrel{\neg}{\lambda} - \frac{1}{4} K_{abc} \stackrel{\neg}{\lambda} \gamma^b \gamma^c. \tag{7.26}$$

After defining the covariant derivatives of the ELKO spinors with torsion, we can construct our dark energy model by considering the ELKO spinors with a negative kinetic energy in an Einstein-Cartan theory, here we define our lagrangian density to be

$$\tilde{\mathscr{L}}_{ELKO} = -\frac{1}{2} g^{ab} \tilde{\nabla}_{(a} \stackrel{\neg}{\lambda} \tilde{\nabla}_{b)} \lambda - V(\stackrel{\neg}{\lambda}\lambda), \tag{7.27}$$

where $V(\stackrel{\neg}{\lambda}\lambda)$ is an arbitrary potential. Notice the different definitions of the kinetic term in Eqs. (7.19) and (7.27). Besides, we should mention that the main difference between Ref. [189] and our work is that here we consider a negative kinetic term regarding it as a dynamical dark energy model and we analyze if the model would lead to instabilities or not. Note that if we only use $g^{ab} \tilde{\nabla}_a \stackrel{\neg}{\lambda} \tilde{\nabla}_b \lambda$ in our lagrangian, after taking the variation with respect to the metric, we have the term $\tilde{\nabla}_a \stackrel{\neg}{\lambda} \tilde{\nabla}_b \lambda$, which is not necessarily symmetric since the spin connection does not commute with each other in general, i.e. $\Gamma_a \Gamma_b \neq \Gamma_b \Gamma_a$, even if there is no torsion. Therefore, we have to symmetrize the kinetic term to ensure the symmetric property of the field equation. Although the lagrangian density is somewhat similar to the one of a complex scalar field, we emphasize that a complex scalar field is a spin-0 field, and hence cannot interact with torsion as a spinor field does. Taking the variation with respect to the metric, we obtain the metric energy-momentum tensor

$$\tilde{\sigma}_{ij} = -2 \tilde{\nabla}_{(i} \stackrel{\neg}{\lambda} \tilde{\nabla}_{j)} \lambda - g_{ij} \tilde{\mathscr{L}}_{ELKO}. \tag{7.28}$$

The spin tensor can be obtained by taking the variation of the action with respect to the contortion tensor:

$$\tau^{kj}{}_i = \frac{\delta \tilde{\mathscr{L}}_{ELKO}}{\delta K^i{}_{jk}} = -\frac{1}{4} \tilde{\nabla}_i \stackrel{\neg}{\lambda} \gamma^j \gamma^k \lambda + \frac{1}{4} \stackrel{\neg}{\lambda} \gamma^j \gamma^k \tilde{\nabla}_i \lambda, \tag{7.29}$$

which can be separated into torsion-free and non-torsion free parts,

$$\tau^{ij}{}_k = -\frac{1}{4}\nabla_k \overset{\neg}{\lambda}\gamma^j\gamma^i\lambda + \frac{1}{4}\overset{\neg}{\lambda}\gamma^j\gamma^i\nabla_k\lambda + \frac{1}{16}K_{kab}\overset{\neg}{\lambda}\gamma^a\gamma^b\gamma^j\gamma^i\lambda + \frac{1}{16}K_{kab}\overset{\neg}{\lambda}\gamma^j\gamma^i\gamma^a\gamma^b\lambda, \tag{7.30}$$

where the first two terms on rhs in Eq. (7.30) are torsion free while the last two terms are non-torsion free. From this, we can see that the spin angular momentum tensor indeed depends on the contortion tensor and cannot be expressed as an axial vector of the torsion tensor as the Dirac spinor does [184, 190]. Therefore, the ELKO spinor can possibly couple to all the irreducible parts of the torsion tensor, and give richer implications in Einstein-Cartan cosmology than the ordinary Dirac spinors [188].

From Sec. 2.5, we know that the gravitational action in Einstein-Cartan theory is similar to GR, the difference lies in the Ricci scalar $\tilde{R}$, where we treat the metric and the non-torsion free affine connection to be independent variables. It follows that the full action of our model reads

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}\tilde{R} + \tilde{\mathscr{L}}_{ELKO}\right). \tag{7.31}$$

In a spatially homogeneous and isotropic universe, we use the flat Friedman-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{7.32}$$

where $a(t)$ is the scale factor. Accordingly, the vielbein e^μ_a are easy to obtain

$$e^\mu_0 = \delta^\mu_0, \;\; e^\mu_i = \frac{1}{a}\delta^\mu_i, \tag{7.33a}$$

and the inverse vielbein e^a_μ reads

$$e^a_0 = \delta^a_0, \;\; e^a_i = a\delta^a_0. \tag{7.33b}$$

In this background, the non-vanishing torsion free Christoffel symbols are [189]

$$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \frac{\dot{a}}{a}, \tag{7.34a}$$

$$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\dot{a}, \tag{7.34b}$$

where the dot denotes differentiation with respect to the cosmic time t. The corresponding spin connection coefficients in the holonomic frame Γ_μ can be obtained by Eq. (7.24), which are [189, 224]

$$\Gamma_t = 0, \;\; \Gamma_{x^i} = -\frac{1}{2}(a\dot{a})\gamma^t\gamma^{x^i}, \; x^i = x, y, z. \tag{7.35}$$

It follows that we can compute the spin connection in the anholonomic frame, Γ_a, the non-vanishing terms are

$$\Gamma_0 = 0, \;\; \Gamma_i = -\frac{1}{2}\left(\frac{\dot{a}}{a}\right)\gamma^0\gamma^i, \; i = 1, 2, 3. \tag{7.36}$$

If the cosmological principle is assumed, it can greatly reduce the degrees of freedom of the torsion, in other words, the only not necessarily vanishing components torsion tensor in the anholonomic frame are [189]

$$S_{ijk} = f(t)\epsilon_{ijk}, \tag{7.37a}$$
$$S_{i0i} = -h(t), \;\; i = 1, 2, 3, \tag{7.37b}$$

where $f(t)$ and $h(t)$ are called the torsion functions, which depend only on t due to the homogeneous and isotropic assumption, and ϵ_{ijk} is the anti-symmetric Levi-Civita symbol with $\epsilon_{123} = 1$.

With the above expression, once we know the non-vanishing torsion terms, we can obtain the non-vanishing contortion terms by means of Eq. (2.52). Then, by using Eq. (2.1) we can determine the connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ and finally compute the Einstein tensor with torsion $\tilde{G}_{ij}$ directly using the definition of the Ricci tensor, $\tilde{R}_{\sigma\mu} = \partial_\mu\tilde{\Gamma}^{\mu}_{\nu\sigma} - \partial_\nu\tilde{\Gamma}^{\nu}_{\mu\sigma} + \tilde{\Gamma}^{\mu}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \tilde{\Gamma}^{\mu}_{\nu\lambda}\tilde{\Gamma}^{\lambda}_{\mu\sigma}$, and the Ricci scalar, $\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}$. Then, we obtain [189]

$$\tilde{G}_{tt} = 3\left(\frac{\dot{a}}{a}\right)^2 + 12\left(\frac{\dot{a}}{a}\right)h + 12h^2 - 3f^2, \tag{7.38}$$

$$\tilde{G}_{xx} = a^2\left[-2\left(\frac{\ddot{a}}{a}\right) - \frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + 8h\right) - 4\dot{h} - 4h^2 + f^2\right], \tag{7.39}$$

$$\tilde{G}_{xx} = \tilde{G}_{yy} = \tilde{G}_{zz}. \tag{7.40}$$

On the other hand, to obtain the complete field equation, one also has to know the energy-momentum tensor, the rhs of Eq. (2.54), $\tilde{\Sigma}_{ij}$. Since the cosmological principle has to be applied not only to the geometrical side but also to the matter side, the matter distribution should be also homogeneous and isotropic. Therefore, we can assume that the ELKO spinor fields in our model depend only on time, t, writing $\lambda(t) = \varphi(t)\xi$ and $\stackrel{\neg}{\lambda} = \varphi(t)\stackrel{\neg}{\xi}$, where $\varphi(t)$ is a real function and ξ is a constant ELKO spinor and its ELKO dual $\stackrel{\neg}{\xi}$ is defined by Eq. (3.11). Since the cosmological principle implies the off-diagonal components of the Einstein tensor to vanish, for example $\tilde{G}_{tx} = \tilde{G}_{xy} = 0$, it naturally constrains the energy-momentum tensor on the rhs of the field equation, Eq. (2.54). That is to say, the off-diagonal components of the energy-momentum tensor should also vanish even in absence of torsion.

To be precise, this means that the ELKO spinor is required to satisfy the condition that the off-diagonal components of the metric energy-momentum tensor should also vanish, i.e. $\tilde{\sigma}_{tx} = \tilde{\sigma}_{xy} = 0$. The simplest way to satisfy this condition is to assume the so-called dark ghost spinor which is the ELKO spinor with zero norm, $\stackrel{\neg}{\lambda}\lambda = 0$ [229]. The word "ghost" refers to the fact that it has no contribution to the metric energy-momentum tensor and thus has no effect on the curvature of spacetime in the absence of torsion [230, 231, 232]. A cosmological dark ghost spinor can be given by [189, 229]

$$\lambda = \varphi(t)\xi, \quad \xi = \begin{pmatrix} 0 \\ \pm i \\ 1 \\ 0 \end{pmatrix} \tag{7.41}$$

with its corresponding dual spinor

$$\stackrel{\neg}{\lambda} = \varphi(t)\stackrel{\neg}{\xi}, \quad \stackrel{\neg}{\xi} = i\,(0, i, \pm 1, 0)\,. \tag{7.42}$$

Since the norm of the dark ghost spinor vanishes, the potential V which is a function of $\stackrel{\neg}{\lambda}\lambda$ plays a role similar to that of the cosmological constant. However, a standard ELKO spinor with non-vanishing norm may also satisfy the condition that the off-diagonal components of the metric energy-momentum tensor vanish, in that case, higher order self-interactions are allowed.

The Cartan equation (2.55) is in general a set of 24 algebraic equations, however, using Eq. (7.18), it reduces to two independent equations relating torsion and spin tensors. Using Eq. (7.22), we can solve the torsion functions in terms of the matter field $\varphi(t)$. Indeed, after some lengthy calculation we get

$$h(t) = -\frac{1}{4}\kappa\varphi^2 f = -\frac{\frac{1}{2}\kappa^2\varphi^4}{4+\kappa^2\varphi^4}\left(\frac{\dot{a}}{a}\right), \tag{7.43}$$

$$f(t) = \frac{2\kappa\varphi^2}{4+\kappa^2\varphi^4}\left(\frac{\dot{a}}{a}\right). \tag{7.44}$$

Here, we can see that the dark ghost spinor indeed has non-trivial contributions to both the spatial axial components and to the temporal components of the torsion tensor as compared with the Dirac spinor which has only a contribution to the spatial axial vector components of the torsion tensor [190, 188]. Moreover, the non-trivial components of the spin angular momentum tensor in our model are $\tau_{123} = \frac{1}{2}\frac{\dot{a}}{a}\varphi^2 + h\varphi^2$ and $\tau_{101} = -\frac{1}{2}f\varphi^2 = \tau_{202} = \tau_{303}$, which are of ocurse homogeneous and isotropic in agreement with the cosmological principle. To obtain the canonical energy-momentum tensor, we need to compute the contributions of the spin angular momentum taking into account the torsion interactions, $(\tilde{\nabla}_k + 2S_{kl}{}^l)(\tau_{ij}{}^k - \tau_j{}^k{}_i + \tau^k{}_{ij})$. Finally, we obtain that the non-vanishing components reads

$$\Sigma_{tt} = V_0 + 3\left(\frac{\dot{a}}{a}\right) f\varphi^2 + 6fh\varphi^2, \tag{7.45}$$

$$\Sigma_{xx} = -a^2 V_0 - a^2\varphi^2 f\left(6h - 2\frac{\dot{\varphi}}{\varphi} - \frac{\dot{f}}{f}\right), \tag{7.46}$$

$$\Sigma_{xx} = \Sigma_{yy} = \Sigma_{zz}, \tag{7.47}$$

where $V_0 = V(0)$. We will analyze the dynamics of our dark energy model in the next section.

7.3 Cosmological Evolution of the Phantom ELKO Spinor

The evolution of the Hubble parameter, $H = \dot{a}/a$, can be determined from Einstein equation (2.54). The corresponding Friedmann and Raychaudhuri equations read

$$H = \frac{\sqrt{\kappa V_0}}{2\sqrt{3}}\frac{4+\kappa^2\varphi^4}{\sqrt{4-3\kappa^2\varphi^4}}, \tag{7.48}$$

$$\dot{H} = -\frac{\kappa V_0}{12}\frac{20\kappa^2\varphi^4 + 3\kappa^4\varphi^8}{4-3\kappa^2\varphi^4}. \tag{7.49}$$

The evolution of the matter field $\varphi(t)$ can be obtained by taking the time derivative of Eq. (7.48) and equating it to Eq. (7.49), then

$$\frac{\dot{\varphi}}{\varphi} = -\frac{\sqrt{\kappa V_0}}{4\sqrt{3}}\frac{20+3\kappa^2\varphi^4}{20-3\kappa^2\varphi^4}\sqrt{4-3\kappa^2\varphi^4}. \tag{7.50}$$

Eq. (7.48) gives the evolution of the matter field, combining it with Eq. (7.49), one obtains a differential equation for the scale factor in terms of the matter field $\varphi(t)$

$$\frac{d\ln a}{d\ln\varphi} = -2\frac{4+\kappa^2\varphi^4}{4-3\kappa^2\varphi^4}\frac{20-3\kappa^2\varphi^4}{20+3\kappa^2\varphi^4}. \tag{7.51}$$

Solving the above differential equation we obtain

$$a(\varphi) = \frac{a_0}{\varphi^2}\left[\frac{(4-3\kappa^2\varphi^4)^4}{20+3\kappa^2\varphi^4}\right]^{\frac{1}{9}}, \tag{7.52}$$

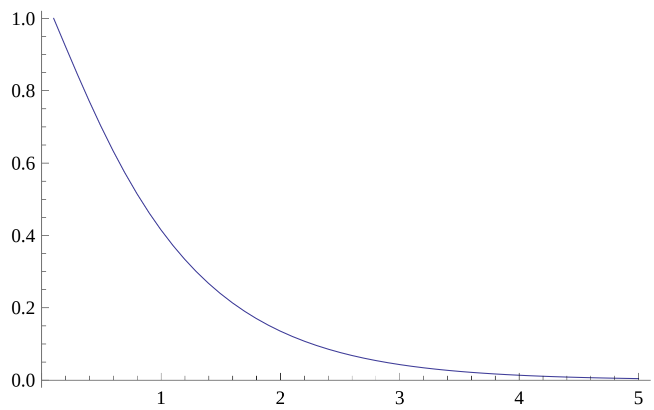

Figure 7.1: Numerical plot of $\varphi(t)$ in Eq. (7.3) from $t = 0$ to $t = 5$ with $\varphi(0) = 1$, and $\kappa = V_0 = 1$.

where a_0 is an integration constant.

As we previously mentioned, the Einstein-Cartan equation (2.59) can be interpreted as that the geometry is the result from the contribution of the matter fields plus some spin-spin interaction. Therefore for a homogeneous and isotropic Universe, we can define for example an equation of state for dark energy, w_d, related to the ELKO spinor from the metric energy-momentum tensor given in Eq. (2.58) where $\tilde{\sigma}^i{}_j = \text{diag}(\rho_d, -p_d, -p_d, -p_d)$, and $w_d \equiv p_d/\rho_d$, then we have

$$w_d = -1 + \frac{2\kappa^2\varphi^4}{12 - 3\kappa^2\varphi^4}. \tag{7.53}$$

This equation of state does not take into account the spin-spin interaction; i.e. the energy momentum tensor u_{ij} defined in Eq. (2.60). We could equally define a spin-spin effective equation of state related to u_{ij} which we omit here for simplicity.

Since φ is constrained by Eq.(7.53) to satisfy the condition $0 \leq \varphi^2 < \sqrt{\frac{4}{3\kappa^2}}$, the time derivative of φ is always negative (please c.f. again Eq. (7.53)), then φ will decrease. In fact, φ will monotonically decrease to its lower bound, $\varphi = 0$, as time goes to infinity, as we next show in Eq. (7.58). Then, w_d goes to -1 asymptotically, the Hubble parameter is almost a constant, and the scale factor expands as $a(t) \propto \exp(Ht)$, therefore our universe enters a de Sitter phase at late time. From the positivity of the second term on rhs in Eq. (7.53), we see that the equation of state will be always larger than -1. And if we consider the contribution of spin-spin interaction, we can define the total equation of state $w_{tot} \equiv p_{tot}/\rho_{tot}$ from

$\hat{\sigma}^i{}_j = \mathrm{diag}(\rho_{tot}, -p_{tot}, -p_{tot}, -p_{tot})$, then

$$w_{tot} = -1 + \frac{2}{3}\frac{20\kappa^2\varphi^4 + 3\kappa^4\varphi^8}{(4+\kappa^2\varphi^4)^2}. \tag{7.54}$$

Since both definitions of equation of state show that our dynamical dark energy model does not cross the phantom divide, we do not expect quantum instabilities even though the kinetic energy is negative, cf. Eq. (7.27). Note that in a finite cosmic time, φ will never become zero, therefore, neither the Hubble parameter nor its time derivative diverge at a finite cosmic time, hence this model is free from the big rip singularity [199, 200, 196, 201, 202]. Indeed, the universe would be asymptotically de Sitter in this model.

We can expand Eqs. (7.48) and (7.50) around $\varphi = 0$ to the first few orders to see its qualitative behavior,

$$H = \frac{\sqrt{\kappa V_0}}{2\sqrt{3}}\left(1 + \frac{5}{4}\kappa^2\varphi^4 + \mathscr{O}(\varphi^8)\right), \tag{7.55}$$

$$\frac{\dot{\varphi}}{\varphi} = -\frac{\sqrt{\kappa V_0}}{2\sqrt{3}}\left(1 - \frac{3}{40}\kappa^2\varphi^4 + \mathscr{O}(\varphi^8)\right). \tag{7.56}$$

Solving these differential equations to the first order, we obtain

$$a(t) = a_0 \exp\left(\frac{\sqrt{\kappa V_0}}{2\sqrt{3}}t\right), \tag{7.57}$$

$$\varphi(t) = \varphi_0 \exp\left(-\frac{\sqrt{\kappa V_0}}{2\sqrt{3}}t\right). \tag{7.58}$$

We see that as time goes to infinity, $\varphi(t)$ exponentially decays, so do the torsion functions h and f; cf. Eqs. (7.43) and (7.44). It is not surprising because as the spin sources dilutes the torsion will vanish accordingly [189].

We next consider the existence of some kind of cold dark matter given by a perfect fluid of a spin-0 particle with the energy-momentum tensor given by $\sigma_{(m)}{}^{\mu}{}_{\nu} = \mathrm{diag}(\rho_m, 0, 0, 0)$, where ρ_m is its energy density. Since it has spin zero, it has no extra contribution to the torsion by the Cartan equation, Eq.(2.55), it only has an additive contribution to the total energy-momentum tensor, $\hat{\sigma}_{ij}$ in Eq. (2.59), that is

$$\hat{\sigma}_{ij} = \sigma^{(m)}_{ij} + \tilde{\sigma}^{(de)}_{ij} + \kappa u_{ij}, \tag{7.59}$$

where $\tilde{\sigma}^{(de)}$ is the metric energy-momentum tensor of the ELKO spinor defined in Eq. (2.58). Then Eqs. (7.48) and (7.50), are modified as

$$H^2 = \frac{\kappa V_0}{12}(1+\beta)\frac{(4+\kappa^2\varphi^4)^2}{4-3\kappa^2\varphi^4}, \tag{7.60}$$

$$\begin{aligned}\frac{\dot{\varphi}}{\varphi} &= -\frac{\sqrt{\kappa V_0}}{4\sqrt{3}}\frac{1}{\sqrt{1+\beta}}\frac{20+3\kappa^2\varphi^4}{20-3\kappa^2\varphi^4}\sqrt{4-3\kappa^2\varphi^4} \\ &\quad -\frac{1}{4}\frac{(4+\kappa^2\varphi^4)(4-3\kappa^2\varphi^4)}{20\kappa^2\varphi^4-3\kappa^4\varphi^8}\frac{\dot{\beta}}{1+\beta}, \end{aligned}\tag{7.61}$$

where $\beta \equiv \frac{\rho_m}{V_0}$ while Eq. (7.49) remain unchanged. We define the total equation of state of the universe by using again $\hat{\sigma}^i{}_j = \mathrm{diag}(\rho_{tot}, -p_{tot}, -p_{tot}, -p_{tot})$, which gives

$$w_{tot} \equiv \frac{p_{tot}}{\rho_{tot}} = -1 + \frac{2}{3}\frac{20\kappa^2\varphi^4 + 3\kappa^4\varphi^8}{(4+\kappa^2\varphi^4)^2}(1+\beta)^{-1}. \tag{7.62}$$

The conservation of the energy-momentum tensor $\nabla_i \sigma^{(m)i}{}_j = 0$ reads

$$\frac{\dot{\beta}}{\beta} = -3\left(\frac{\dot{a}}{a}\right). \tag{7.63}$$

Substituting $\frac{\dot{a}}{a}$ using Eq. (7.60) into Eq. (7.63), we get

$$\dot{\beta} = -\frac{\sqrt{3\kappa V_0}}{2}\frac{4+\kappa^2\varphi^4}{\sqrt{4-3\kappa^2\varphi^4}}\beta\sqrt{\beta+1}. \tag{7.64}$$

To see the stability of the late time behavior, we analyze the autonomous (φ, β) system , which is

$$\begin{aligned}\dot{\varphi} &= -\frac{\sqrt{\kappa V_0}}{4\sqrt{3}}\frac{\varphi\sqrt{4-3\kappa^2\varphi^4}}{\sqrt{1+\beta}}\frac{20+3\kappa^2\varphi^4}{20-3\kappa^2\varphi^4} \\ &\quad + \frac{\sqrt{3\kappa V_0}}{8}\frac{\varphi(4+\kappa^2\varphi^4)^2\sqrt{4-3\kappa^2\varphi^4}}{20\kappa^2\varphi^4-3\kappa^4\varphi^8}\frac{\beta}{\sqrt{1+\beta}}, \end{aligned}\tag{7.65}$$

$$\dot{\beta} = -\frac{\sqrt{3\kappa V_0}}{2}\frac{4+\kappa^2\varphi^4}{\sqrt{4-3\kappa^2\varphi^4}}\beta\sqrt{\beta+1}. \tag{7.66}$$

The only fixed point is $(\varphi_0, \beta_0) = (0, 0)$. We linearize the system around the fixed point, by expanding $(\varphi, \beta) = (\varphi_0 + \delta\varphi, \beta_0 + \delta\beta)$, and we obtain that

$$\begin{pmatrix}\delta\dot{\varphi} \\ \delta\dot{\beta}\end{pmatrix} = \frac{\sqrt{\kappa V_0}}{2\sqrt{3}}\begin{pmatrix}-1 & 0 \\ 0 & -6\end{pmatrix}\begin{pmatrix}\delta\varphi \\ \delta\beta\end{pmatrix}. \tag{7.67}$$

The linearized system is automatically diagonal, one can easily read off its eigenvalues, both are real and negative. Therefore, $(\varphi, \beta) = (0, 0)$ is an attractive fixed point, and this would give us $w_{tot} \to -1$ in the future. As the universe expands, the torsion will vanish. When both φ and β are small, the Hubble parameter will be nearly constant, the scale factor $a(t)$ grows exponentially which means the universe will again enter a de Sitter phase.

The numerical evolution of the equation of state, $w_{tot}(z)$, and the Hubble parameter $H(z)$ of the universe with redshift $z \equiv -1 + \frac{a_0}{a}$ shown in Fig. 2 and 3 respectively where a_0 stands for the present value of the scale factor.

Note that for $\tilde{\sigma}^{(de)i}{}_j = \mathrm{diag}(\rho_{de}, -p_{de}, -p_{de}, -p_{de})$, the conservation equation, $\nabla_i(\tilde{\sigma}^{(de)i}{}_j + \kappa u^i{}_j) = 0$, can be interpreted as the continuity equation of the energy density of the ELKO spinor with a source term, $\dot{\rho}_{de} + 3H(\rho_{de} + p_{de}) = Q$ where $Q > 0$ means energy is transferred from torsion to ELKO fields, and $Q = 0$ means no interaction between torsion and ELKO fields.

We can as well define an equation of state for dark energy again as $w_{de} \equiv \frac{p_{de}}{\rho_{de}}$, then[2]

$$w_{de} = -1 + \frac{2\kappa^2\varphi^4(1+\beta)}{3(4 - 3\kappa^2\varphi^4) + 6\kappa^2\varphi^4(1+\beta)}. \tag{7.68}$$

We can as well define an effective equation of state for dark energy $w_{de}^{eff} \equiv \frac{p_{de}}{\rho_{de}} - \frac{Q}{3H\rho_{de}}$, then

$$\begin{aligned} w_{de}^{eff} &= -1 + \frac{2\beta\kappa^2\varphi^4}{4 - (1-2\beta)\kappa^2\varphi^4} \\ &\quad + \frac{8}{3}\frac{(6-3\beta)\kappa^4\varphi^8 + (40-24\beta)\kappa^2\varphi^4 + 48}{(4+\kappa^2\varphi^4)(4-(1-2\beta)\kappa^2\varphi^4)(20-3\kappa^2\varphi^4)}. \end{aligned} \tag{7.69}$$

The term, Q, is equally present when $\beta = 0$; i.e. in the absence of dark matter. The numerical evolution of w_{de} and w_{de}^{eff} with redshift z are given in Figs. 4, and 5. Note that at early time, $w_{de}^{eff} < w_{de}$ which means $Q > 0$, thus energy is transferred from torsion to the ELKO fields, and at late time, $w_{de}^{eff} \approx w_{de}$ which means $Q \approx 0$ as is expected since torsion will eventually vanish.

[2]Please note that Eq. (7.68) is different from Eq. (7.53). The reason is that by adding cold dark matter into the model, we modify the spin connection Eq. (7.35) and (7.36), therefore we modify the ELKO enengy momentum tensor in Eq. (7.28).

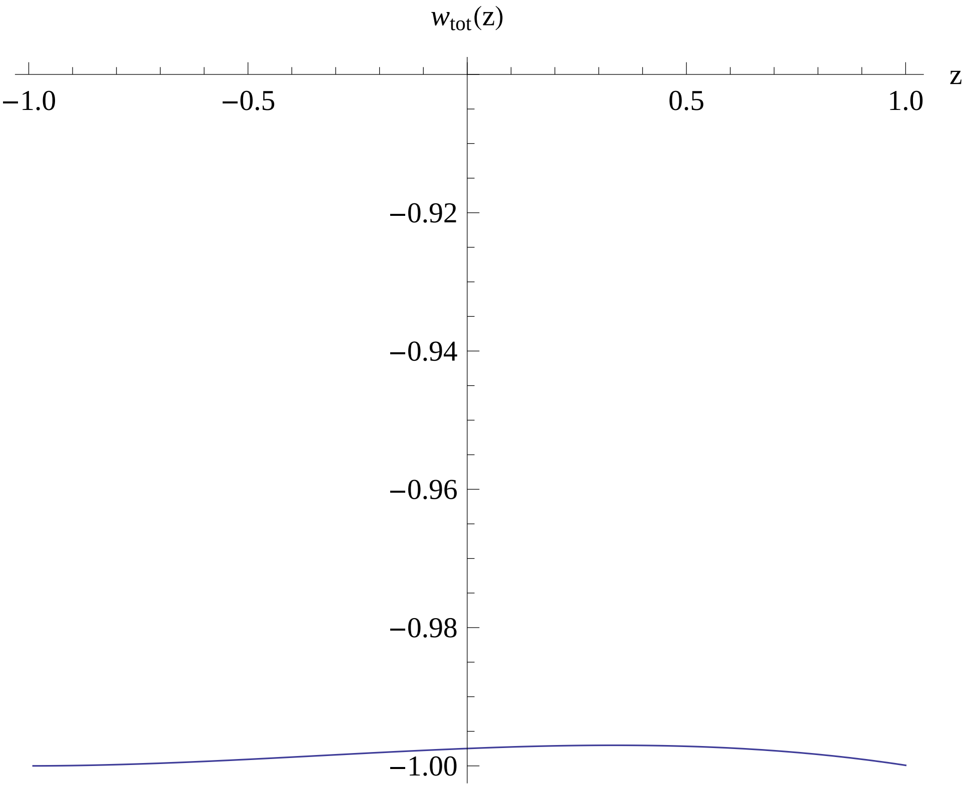

Figure 7.2: $w_{\text{tot}}(z)$ defined in Eq. (7.62) from $z = 1$ to $z = -1$ with $\varphi(1) = 0.1$, $\beta(1) = 0.01$, $\kappa = 1$, and $\Omega_{m_0} \approx 0.3$.

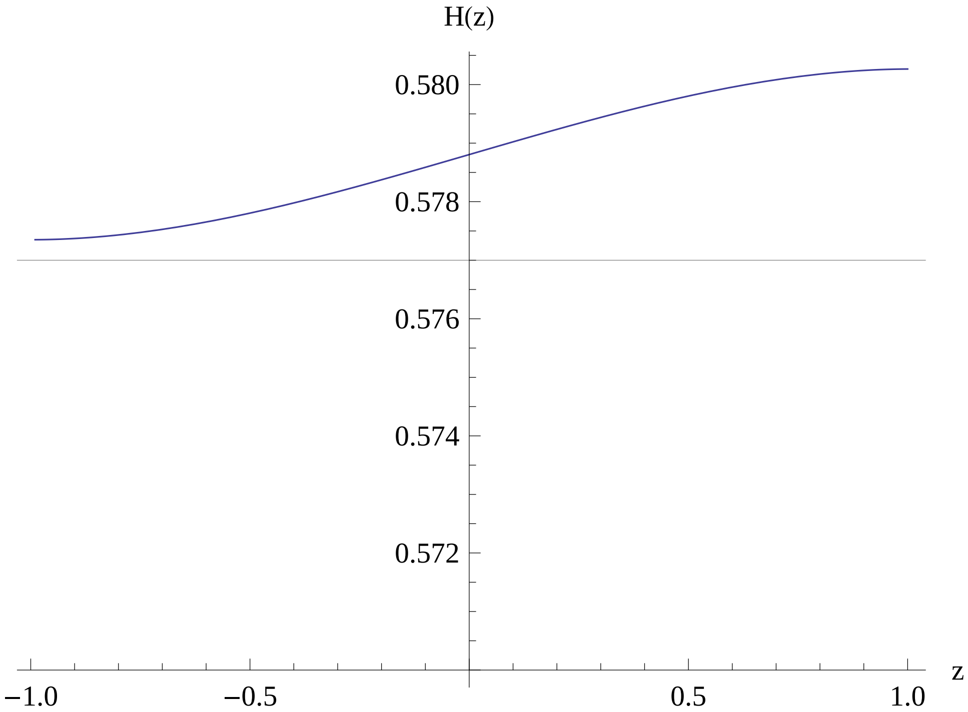

Figure 7.3: $H(z)$ gievn in Eq. (7.60) from $z = 1$ to $z = -1$ with $\varphi(1) = 0.1$, $\beta(1) = 0.01$, and $\kappa = 1$. The asymptotic line is $H(z) = \frac{\sqrt{3}}{3} \approx 0.577$, and $\Omega_{m_0} \approx 0.3$.

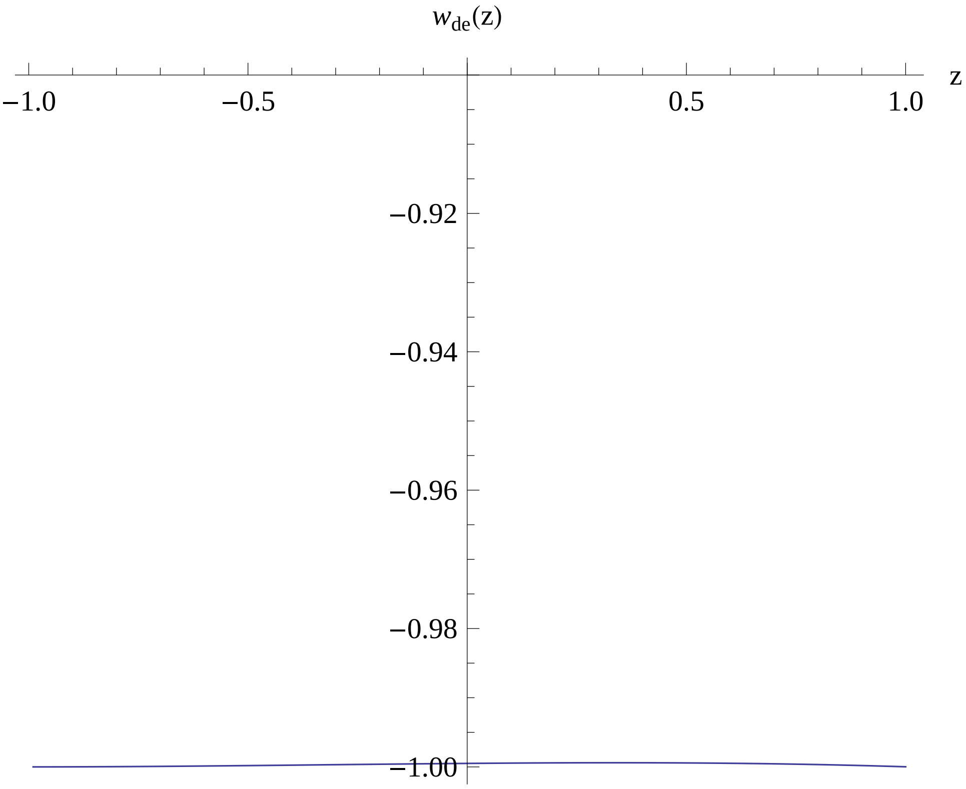

Figure 7.4: $w_{de}(z)$ given in Eq. (7.68) from $z = 1$ to $z = -1$ with $\varphi(1) = 0.1$, $\beta(1) = 0.01$, $\kappa = 1$, and $\Omega_{m_0} \approx 0.3$.

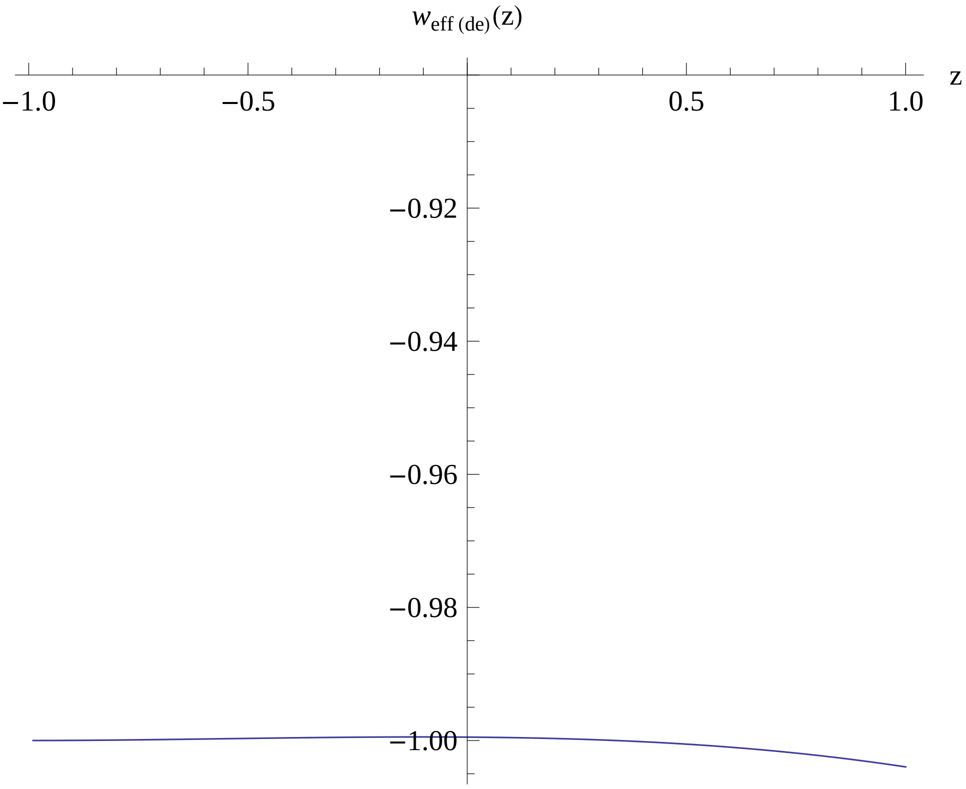

Figure 7.5: $w_{de}^{eff}(z)$ given in Eq. (7.69) from $z = 1$ to $z = -1$ with $\varphi(1) = 0.1$, $\beta(1) = 0.01$, $\kappa = 1$, and $\Omega_{m_0} \approx 0.3$.

Chapter 8

Conclusions and Future Perspectives

In this chapter, we summarize the main conclusion of this thesis. We studied the RG improved inflationary cosmology where the decay rate of inflaton is modulated by a second scalar field, which we identified as a Higgs. In this model the background evolution is driven by a RG running cosmological constant and gravitational constant with their RG flows satisfying the AS behavior. By choosing Einstein-Hilbert truncation, we find this model is classically equivalent to a model of R^2 gravity. The elegant property of this model is that it can give rise to a sufficiently long inflationary phase at high energy scale and smoothly exit to standard GR after inflation. Moreover, a RG running gravitational constant can provide a second scalar field to vary slowly without an extremely flat potential since the slow-roll parameters associated with this field are greatly suppressed by a large value of the conformal factor. As a consequence, this scalar field seeds isocurvature perturbations during inflation which can be converted into primordial curvature perturbation under a suitable mechanism. We consider this mechanism as the process of modulated reheating. Based on this mechanism, we performed a detailed analysis on the power spectrum and non-Gaussianities of primordial cosmological perturbations. We then confronted our model with the Planck data and concluded that a viable parameter space exists, although it is highly constrained. Although this model suffers from the fine-tuning problem, the scenario under present study points to a new possible connection between particle physics and early Universe cosmology. We conclude by mentioning that the mechanism of Higgs modulated reheating can be generalized to an arbitrary nonminimal inflationary model or a model of $f(R)$ inflation. By relaxing theoretical requirements of AS gravity, the parameter space available to such a mechanism is increased.

As for the dark energy model, we consider Einstein-Cartan theory which is a simple generalization of ordinary general relativity, incorporating the torsion fields as the anti-symmetric part of the affine connection. In this theory, there are two field equations, one is like the traditional Einstein equation and the other is an algebraic relation between the torsion fields and the spin fields of the matter sources. We introduce a new kind of spin one-half particle called the Elko spinor or dark spinor which is the eigenspinor of the charge conjugation operator, and different from the Majorana spinor due to the double-helicity structure [186]. The equation of motion of the Elko spinor is the Klein-Gordon equation rather than the Dirac equation. Then, we propose a dark energy model with a negative kinetic energy constructed from the Elko spinor which is interacting with the torsion fields in the FLRW universe. Although the kinetic energy is negative, the equation of states w_{de} and w_{tot} do not cross the phantom divide and approaches to -1 asymptotically, satisfying the weak energy condition, hence we expect the model to be stable at the quantum level. No big rip singularity will occur at a finite cosmic time in this setup. And torsion will vanish at late time, the Hubble parameter will become nearly a constant. Furthermore, we consider the existence of some cold dark matter which is assumed to be a pressureless scalar particle without contribution to the torsion fields. In this two components system, we find that there is a unique attractive fixed point, which is simply $(\varphi, \beta) = (0, 0)$, and all of the equations of state w_{tot}, $w_{(de)}$, and $w_{eff(de)}$ will converge to -1 from above no matter what the initial condition is. Therefore, the universe will eventually enter the de Sitter phase at late time with or without dark matter.

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