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Master Thesis

超幾何算子之 ZETA 行列式

ZETA Determinant of Hypergeomertic Equation via Ramanujan's Identity

吳宗堂

Zhong Tang Wu

指導教授：謝春忠 博士, 陳其誠 博士

Advisor: Chun-Chung Hsieh, Ph.D and Ki-Seng Tan, Ph.D

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中文摘要

本文研究高斯超幾何方程的施圖姆-劉維爾形式及其 Zeta 行列式.基於 Lesch 給出以 wronskian 表示正則奇異施圖姆-劉維爾算子之 Zeta 行列式的公式,以及拉馬努金的某項公式,便得出本文的主要結果: 超幾何方程之施圖姆-劉維爾形式的 Zeta 行列式之 closed form.

關鍵詞: 高斯超幾何方程, Zeta 行列式, 正則奇異施圖姆-劉維爾算子, 拉馬努金公式, closed form.

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Abstract

In the thesis, the eignevalue probelm of the Hypergeometric equation(for short: HGE or E(a,b,c)) is discussed. There are three parts in this thesis. First of all, I introduce some heuristic backgrounds and motivations of the eigenvalue problem of HGE. The second part is a survey about the theory of the HGE. Finally, based on Lesch's formula of zeta determinant of Regular Singular Sturm-Liouville Operators, I calculate the zeta determinant with repect to the Sturm-Liouville form of HGE operator on a closed interval, by using Ramanujan's identities.

Key words: Hypergeometric equation, Sturm-Liouville form, zeta determinant, Ramanujan's identities, Regular Singular Sturm-Liouville Operators,

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Chapter 1

Introduction

1.1 Motivation and main results.

Guass Hypergeometric equation (in this thesis, we call it the Hypergeometric equation)

$$x(1-x)\frac{d^2 f(x)}{dx^2} + (c - (a+b+1)x)\frac{df(x)}{dx} - abf(x) = 0,$$

is a kind of Euler's Hypergeometric differential equations, and it is a second-order ordinary differential equation. Here we denote it as $E(a,b,c)$ for both the operator and the equation. The Hypergeometric series

$${}_2F_1(a,b;c;x) := \sum_{n=1}^{\infty} \frac{(a;n)(b;n)}{(c;n)(1;n)} x^n$$

is one of the solutions of this equation. They appear in many fields of "Exact Science" from the era of Euler to now.

In this thesis, we ask a question: "How does the operator structure of the Hypergeometric equation look like? Can we do its spectral resolutions?" We give a partially positive answer:"We can do that on the closed interval $[0,1]$." (For other domain, see the problem in section 1.1.1 .) In particular, the question and the answer can be described as in the following dialogues:

"Does any potential application for this spectral resolution exist?" My short answer to this is : "Why, sir, there is every probability that you will soon be able to tax it." (by Michael Faraday,).

The dream hidden in this thesis is an analogue of "from hamonic function to eigenfunction" in the context of Riemannian geometry and Hodge theory. Especially, the dream can also be considered as the classical philosophy in functional analysis "ker and eigen" of nice operators deeply involved with geometry and arithmetic, such as Laplacian operator and, in our case, Hypergeometric equation.

We object is the zeta determinant of a one-parameter family of a regular singular Sturm-Liouville operator. The closed form of it can be calculated by using the formulaes

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given by Matthias Lesch and a Ramanujan's formula on the Hypergeometric series that related to the derivative of the Schwarz map. In case $a = b = \frac{1}{2}, c = 1$, the Schwarz map is the inverse of Modular lambda function:

$$\tau = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} = \frac{\int_1^\infty \frac{dt}{\sqrt{t(t-1)(t-(1-x))}}}{\int_1^\infty \frac{dt}{\sqrt{t(t-1)(t-x)}}}.$$

where our regular singular Sturm-Liouville operator:

$$\mathrm{L} = \frac{d^2}{dx^2} + \frac{a(x)}{x^2(1-x)^2}$$

is the Sturm-Liouville form of the Hypergeometric operator (and the associated equation is called the Sturm-Liouville form of the Hypergeometric equation):

$$\mathrm{H}_G^{(\mu_0,\mu_1,\mu_\infty)} = \frac{d^2}{dx^2} + \frac{1}{4}(\frac{(1-\mu_0^2) + (-1+\mu_0^2-\mu_1^2+\mu_\infty^2)x + (1-\mu_\infty^2)x^2}{x^2(1-x)^2}).$$

For classical Laplacian operator $\triangle$ of a compact Reimannian manifold M, we defines the zeta determinant by

$$\det \triangle := \exp(-\zeta_\triangle'(0)),$$

where the Laplacian $\triangle$ has the discrete spectrum $0 < \lambda_1 < \lambda_2 \cdots$, and we define the spectral zeta function of $\triangle$ by

$$\zeta_\triangle(s) := \sum_{n=1}^\infty \frac{1}{\lambda_n^s}.$$

The absolute convergence of $\zeta_\triangle(s)$ on $\{\mathrm{Re}(s) > \sigma\}$ (for some σ) depends on the geometry of M(c.f.Weyl asympotic formula). Moreover, We can use Mellin transform by the apriori estimate of the trace of heat kernel of $\triangle$: $\sum_{n=1}^\infty \exp(-\lambda t)$ and the identity

$$\sum_{n=1}^\infty \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=1}^\infty \exp(-\lambda_n t) dt,$$

to obtain the analytical continuation of $\zeta_\triangle$ at $s = 0$, so the $\zeta_\triangle'(0)$ and $\det \triangle$ are well-defined.

When the coefficient of the operator is smooth on whole domain, the estimate of it heat kernel is relatively easy. However, when the coefficients of operator have strong singularities in the domain, such as $-\frac{d^2}{dx^2} - \frac{1}{4x^2}$ on $[0, \alpha)$ for any $\alpha > 0$, the foundation of the study of the heat kernel need to be reconstructed. One of the fundamental work on this is the "singular asymptotics lemma" of Bruning and Seeley, see [7](see also [1]).

On the other-hand, the difficulties of evaluation of the zeta determinant of an operator may be regarded as the ignorance of the exact form of eigenvalues and eigenfunctions. However, in the case of regular singular Sturm-Liouville operators

$$\mathrm{L} = \frac{d^2}{dx^2} + \frac{a(x)}{x^2(1-x)^2}$$

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on $[0,1]$(where $a(x)$ is a smooth function defined on the cloesd interval $[0,1]$), Lesch gives a clear expression of the zeta determinant via only the $a(0), a(1)$ and the pair of normalized solutions of L. For the whole details, we refer the readers to the papers of Bruning-Seeley's[7] and Lesch's [2].

The Lesch's formula of the zeta determinant is

Theorem 1.1.1. $\det_\zeta(\mathrm{L}) = \exp(-\zeta'_{\mathrm{L}}(0))$,

$$\det_\zeta(\mathrm{L}) = \frac{\pi \mathrm{W}(\psi, \varphi)}{2^{\nu_0+\nu_1}\Gamma(\nu_0+1)\Gamma(\nu_1+1)} \tag{1.1}$$

where ψ *(φ,resp.) is the normalized solution of* L *at 0(at 1,resp.),* $\nu_0 = \sqrt{\frac{1}{4} + -\frac{1}{4}(1-\mu_0^2)}$ *(ν_1, μ_1, resp.),* $\mathrm{W}(\psi,\varphi) = \psi\varphi_x - \varphi\psi_x$*=the Wronskian of* ψ, φ.

Based on this, a particular case of the main result of this thesis is

$$\det_\zeta \mathrm{H}_{\mathrm{G}}^{(0,0,0)} = 1.$$

1.1.1 Motivation

I start this study since my advisor Chun-Chung Hsieh asked Masaaki Yoshida: "Why you study Schrodinger equation?" in the lectures of Yoshida on his lovely book <Hypergeometric Functions, My Love> (see [3]) invited by Chang-Shou Lin in the autumn of 2014. If my understaning is correct, Yoshida has not considered the term "Schrodinger equation" in his lecture ever. In fact, Yoshida is considering the normal form(it this this we call it Sturm-Liouville from) of Hypergeometric equation at that time. For me, however, the question of Hsieh is not totally a nonsense. I dream on this diection, if the Hypergeometric equation have discrete spectrum

$$\{0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots\}$$

and assign the form the spectrum, we can see the eigenspaces

$$E_k = \{g \in F(\mathbb{CP}^1 \setminus \{0,1,\infty\}) : \mathrm{H}_G g = \lambda_k g\}.$$

of $\mathrm{H}_G = \mathrm{H}_G^{(\mu_0,\mu_1,\mu_\infty)}$ are dimension two. (Here $F(\mathbb{CP}^1 \setminus \{0,1,\infty\})$ is a unknown function space that we want to find it but not success yet.) Therefore, we may get a series of linear monodromy representations if we can do analytic continuation for the eigenfuctions,

$$\rho_k : \pi_1(\mathbb{CP}^1 \setminus \{0,1,\infty\}) \to \mathrm{GL}(E_k) \cong \mathrm{GL}_2(\mathbb{C}).$$

Formally, we have

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{0,1,\infty\}) \to \mathrm{GL}(\bigoplus_k E_k) \cong \mathrm{GL}_\infty(\mathbb{C}),$$

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by using the naive direct sum of the monodromy matrices. One of the intention of the author is to try to use this hypothetical construction to study the Belyi embedding

$$\mathrm{Gal}(\frac{\overline{\mathbb{Q}}}{\mathbb{Q}}) \hookrightarrow \mathrm{Out}(\pi_1(\mathbb{CP}^1 \setminus \{0, 1, \infty\})),$$

and the graded Grothendieck-Teichmuller Lie algebra. The beliefs of the author are based on the deep-interconnections between the Hypergeometric function and the conformal field theory [4].And the recent work of Takashi Ichikawa: [6],[5], shows that there is a rigorous relation between Conformal theory and Grothendieck-Teichmuller lego game, this work is on the same line of the beliefs of the author.

However, this dream is still in the air. It has no any concrete result yet. As an opportunity and challenge, we use the spectral resolution of regular singular Sturm-Liouville operators in this thesis. For example, the spectral resolution on the domain of H_G is still not clear. Hence, we use a framework over the closed interval $[0, 1]$. There is a question: how would the eigenvalue or eigenfunction change when the domain are replaced as $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

So here we have a challenge:

Remark 1.1.2. Can we find a functional space that allow us to do a spectral resolution of H_G over $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$? Which one is a "suitable" functional space?

1.2 Heat Kernel and Mellin Transform in the Case of Riemann Zeta.

The eigenvalues of $-\frac{d^2}{dx^2}$ over $\frac{\mathbb{R}}{2\pi\mathbb{Z}}$ are $\{n^2\}_{n\in\mathbb{Z}}$. That is,

$$-\frac{d^2}{dx^2}e^{inx} = n^2 e^{inx}.$$

We see $\{e^{inx}\}_{n\in\mathbb{Z}}$ is the set of eigenfunctions.In fact, it forms an orthonormal basis of $L^2(\frac{\mathbb{R}}{2\pi\mathbb{Z}}, \mathbb{C})$.

The layout below is taken from <A geometric glance at zeta functions, L-functions, and automorphic forms >, by Ken Richardson.

1.2.1 Mellin Tranform

First of all, as Riemann did, we turn our subject into an integral:

Lemma.

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \exp(-\lambda t)dt,$$

for $\lambda > 0$.

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Proof. $\int_0^\infty t^{s-1}\exp(-\lambda t)dt \underset{\lambda t=u}{=} \frac{\lambda^{-1}}{\lambda^{s-1}}\int_0^\infty u^{s-1}\exp(-u)du = \lambda^{-s}\Gamma(s)$ □

Corollary.

$$\sum_n \lambda_n^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_n \exp(-\lambda_n t)dt$$

and

$$2\zeta_{\mathrm{RIE}}(2s) = \sum_{n\in\mathbb{Z}\backslash 0} n^{-2s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{n\in\mathbb{Z}\backslash 0}\exp(-n^2 t)dt$$

1.2.2 Heat Kernel.

We observe

$$1 + \sum_{n\in\mathbb{Z}\backslash 0}\exp(-n^2 t) = \mathrm{K}(x,x,t).$$

where $\mathrm{K}(x,y,t)$ is the heat kernel of S^1:

$$\sum_{n\in\mathbb{Z}}\exp(-n^2 t)e^{inx}e^{-iny} = 1 + 2\sum_{n=1}^{\infty}\exp(-n^2 t)\cos(n(x-y)),$$

$\mathrm{K}(x,x,t)$ is the trace of heat operator. That is, $\mathrm{K}(x,x,t) = \sum_\rho \rho$, where ρ is the eigenvalue of heat operator K_t.

We recall some facts about the heat equation(operator). Our heat kernel is the fundamental solution for the following problem onS^1:

$$(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})\mathrm{K}(x,y,t) = 0,\ t > 0$$

$$\lim_{t\to 0}\mathrm{K}(x,y,t) = \delta(x-y).$$

Apply the defintion of $\delta(x-y)$ (the Dirca function), we can solve the following initial condition problem:

$$(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})u_f(x,t) = 0,\ \text{if } t > 0$$

$$\lim_{t\to 0}u_f(x,t) = f(x).$$

That means (and also justify the "kernel" in the name):

$$\int_{S^1}\mathrm{K}(x,y,t)f(y)dy$$

is the unique solution for the problem. That is,

$$u_f(x,t) = \int_{S^1}\mathrm{K}(x,y,t)f(y)dy.$$

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We can check this fact by applying $\lim_{t\to 0}\mathrm{K}(x,y,t) = \delta(x-y)$. uniqueness can be derived with the fact: heat operator K_t is well defined:

$$K_t : L^2(S^1) \longrightarrow C^\infty(S^1)$$

$$f \longmapsto u_f.$$

In the case of S^1, we can obtain the closed form of the eigenvalues of K_t since we knew an explicit expression of an orthonormal basis in $L^2(S^1)$ and how the heat operator K_t acts on it:

$$K_t(e^{inx}) = \exp(-n^2t)e^{inx}.$$

Therefore, we get the trace formula which we claim in the beginning:

$$\sum_{n\in\mathbb{Z}} \exp(-n^2t) = \mathrm{Tr}(K_t)$$

Remark. In some papers or books, K_t may be witten as $e^{-\triangle t}$, which the $\triangle$ is the Laplacian operator on a Riemannian manifold (M,g).

1.3 Some Backgrounds

The link (up to day is still unfinished) between the regular singular Sturm-Liouville operators and the Hypergeometric equation can be summerized by a word: conical singularity. A basic example is $-\frac{d^2}{dx^2} - \frac{1}{4x^2}$. It is the radial part in the polar coordinate of the Laplacian of $\mathbb{R}^2$. In [1],[7],[8], the operator

$$-\frac{d^2}{dx^2} + \frac{a(x)}{x^2},$$

where $a(x)$ is smooth on $[0,\infty)$ with $a(0) \geq \frac{-1}{4}$ which has been studied and motivated by considering the Laplacian on “manifold with an asymptotically conical singularity as the model. For the detail we refer to their papers.

In this thesis, we provide a new example of this direction:

$$\mathrm{H_G}(0,0,0) = -\frac{d^2}{dx^2} - \frac{1}{4}\left(\frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}\right),$$

It is a Sturm-Liouville form of the Hypergeometric equation

$$E(\frac{1}{2},\frac{1}{2};1) : x(1-x)\frac{d^2z}{dx^2} + (1-2x)\frac{dz}{dx} - \frac{1}{4}z = 0), \qquad (1.2.1)$$

The necessary explanations and backgrounds of this operator are the main ingredient of this thesis. $\mathrm{H_G}(0,0,0)$ has strong parallelism with $-\frac{d^2}{dx^2} - \frac{1}{4x^2}$. Moreover, we will

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see $\mathrm{H_G}(0,0,0)$ may be called the hyperbolic version of $-\frac{d^2}{dx^2} - \frac{1}{4x^2}$, since the part $x(1-x)\frac{d^2 f(x)}{dx^2} + (c-(a+b+1)x)\frac{df(x)}{dx}$ of hypergeometric equation is the radial part in the polar coordinate of the Laplacian of hyperbolic plane. (See Appendix)

And, not only the special case $\mathrm{E}(\frac{1}{2}, \frac{1}{2}; 1)$, but also this Ramanujan identity are used in the studies of the metrics of with three conical singularities on the 2-sphere of constant curvature. For example,[9],[10]. Moreover, the relation between hypergeometric function/equation and the Schwarzian derivative of Schwarz map plays the dominant role of their works. However, the rigorous formulation of this link between the analysis of heat trace of hypergeometric equation and the geometry of the metrics with three conical singularities on the 2-sphere of constant curvature is still totally open. Finding a good functional space over 2-sphere minus three point may be regarded as the ground of the formulation of this propose.

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Chapter 2

A Survey of the Hypergeometric Equation

We will give equivalence among hypergeomteric series, differential equation, and integral representation, under certain conditions of parameters.

2.1 Hypergeomteric Series.

Here we define Hypergeomteric series by

$$ {}_2F_1(a,b;c;x) := \sum_{n=1}^{\infty} \frac{(a;n)(b;n)}{(c;n)(1;n)} x^n, $$

where $c \neq 0, -1, -2, -3, ...$, and $(a;n) := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$. Sometime we denote $(a;n) \equiv (a)_n$. The radius of convergence is one (unless a or b is a nonpositive integer, in which case the series is a polynomial):

$$ \frac{A(n+1)}{A(n)} = \frac{(a+n)(b+n)}{(c+n)(1+n)} \underset{n\to\infty}{\longrightarrow} 1. $$

Now we have a holomorphic function ${}_2F_1(a,b;c;x)$, and we may study it with the differential equation which it satisfies.

For this, recall the fact described at the first chapter for Euler operator:$\theta := x\frac{d}{dx}$, we have

$$ \theta \cdot x^\lambda = \lambda x^\lambda, $$

Fomally speaking, $x^\lambda$and λ are the eigenfunction and the eigenvalue of the operator θ.

2.2 Hypergeomteric Equation.

We can check ${}_2F_1(a,b;c;x)$ is a solution of $(a+\theta)(b+\theta)f - (c+\theta)(1+\theta)\frac{1}{x}f = 0$.

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Proposition. $\mathrm{E}(a, b; c)$ *is equivalent to*

$$(a+\theta)(b+\theta)-(c+\theta)(1+\theta)\frac{1}{x}$$

Proof. Let f be a test function, then we check the claim in two part:

$$(a+\theta)(b+\theta)\cdot f = abf + x(a+b+1)f' + x^2 f'';$$

$$[(c+\theta)(1+\theta)\frac{1}{x}]\cdot f = [(c+\theta)(1+\theta)]\cdot(\frac{1}{x}f)$$

$$= [c+(1+c)\theta+\theta^2]\cdot(\frac{1}{x}f) = cf' + xf''$$

□

2.3 Integral representation of Hypergeomteric series.

Proposition 2.3.1.

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt,$$

for $\Re(c) > \Re(a) > 0$.

The proof is a computaion which uses the following identities.

Lemma. $(1-tx)^{-b} = \sum_{n=0}^{\infty}\frac{(b;n)}{(1;n)}(tx)^n$ *and* $\int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

Proof. To prove our propostion, we just need to observe that $\frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1) = (a; n)$ and the summation in the lemma can interchage with the integral. □

2.4 Projective Equivalence implies Sturm-Liouville form of Hypergeomteric equation.

Here, we show how to change a second order ordinary differential equation into it's Sturm-Liouville form, then apply it to the case of hypergeometric equation. Let u be a solution of

$$u'' + pu' + qu = 0,$$

Consider $u = fv$,, then

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$$v'' + (p + 2\frac{f'}{f})v' + (q + p\frac{f'}{f} + \frac{f''}{f})v = 0.$$

Now let f be a solution of $p + 2\frac{f'}{f} = 0$, then

$$q + p\frac{f'}{f} + \frac{f''}{f} = q - \frac{1}{2}p' - \frac{1}{4}p^2.$$

In the case of the hypergeometric equation

$$[x(1-x)\frac{d^2}{dx^2} + (c - (a+b+1)x)\frac{d}{dx} - ab]u = 0,$$

we have

$$f(x) = x^{-\frac{c}{2}}(1-x)^{-\frac{(a+b-c+1)}{2}},$$

and

$$v'' + \frac{1}{4}(\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1-\mu_0^2-\mu_1^2+\mu_\infty^2}{x(1-x)})v = 0,$$

where

$$\mu_0 = 1-c, \mu_1 = c-a-b, \mu_\infty = b-a.$$

With the notion of Schwarz derivative, we can give a geometric interpretation of

$$q - \frac{1}{2}p' - \frac{1}{4}p^2 = \frac{1}{4}(\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1-\mu_0^2-\mu_1^2+\mu_\infty^2}{x(1-x)}). \qquad (4.2.1)$$

Remark. We regard

$$f(x) = x^{-\frac{c}{2}}(1-x)^{-\frac{(a+b-c+1)}{2}}$$

as the integration factor of the original ODE $u'' + pu' + qu = 0$.

2.5 Schwarz derivative of Schwarz map.

Let $u(x)$ be an analytic function in x. The Schwarz derivative of u is denoted by$\{u;x\}$, and it is defined by

$$(\frac{u''}{u'})' - \frac{1}{2}(\frac{u''}{u'})^2. \qquad (4.3.1)$$

Schwarz discovered the following result:be fundemental solutions of $u'' + pu' + qu = 0$, and

$$s = \frac{u_2}{u_1}, \qquad (4.3.2)$$

Then we have

$$\{s;x\} = 2q - \frac{1}{2}p^2 - \frac{dp}{dx}. \qquad (4.3.3)$$

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Proof. If we put $u_2' = s'u_1 + su_1'$ and $u_2'' = s''u_1 + 2s'u_1' + su_1''$ into $u_2'' + pu_2' + qu_2 = 0.$, we get

$$2s'u_1' + (s'' + ps')u_1 = 0,$$

and

$$p + \frac{s''}{s'} = -2\frac{u_1'}{u_1}. \qquad (4.3.4)$$

Differentiate the equation of above, and we can get

$$p' + (\frac{s''}{s'})' = -2\frac{u_1u_1'' - (u_1')^2}{u_1^2} = -2\frac{u_1(-pu_1' - qu_1) - (u_1')^2}{u_1^2}$$

$$= 2q + 2\left(p\frac{u_1'}{u_1} + (\frac{u_1'}{u_1})^2\right).$$

Use $p + \frac{s''}{s'} = -2\frac{u_1'}{u_1}$ again and we can obtain

$$p' + (\frac{s''}{s'})' = 2q - \frac{1}{2}p^2 + \frac{1}{2}(\frac{s''}{s'})^2.$$

□

Since we have $q - \frac{1}{2}p' - \frac{1}{4}p^2 = \frac{1}{4}(\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1-\mu_0^2-\mu_1^2+\mu_\infty^2}{x(1-x)})$, we can rewrite

$$v'' + \frac{1}{4}(\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1-\mu_0^2-\mu_1^2+\mu_\infty^2}{x(1-x)})v = 0$$

into

$$v'' + \frac{1}{2}\{s;x\}v = 0, \qquad (4.3.5)$$

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Chapter 3

Calculation of the Zeta Determinant.

The calculation of the zeta determinant is based on the following identity of Ramanujan. (In [11] P.87 Entry 30)

Theorem 3.0.1. *Let* $a+b+1=c+d$, $\Gamma=\Gamma(a,b;c,d)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}$,*and*

$$y=\Gamma\frac{{}_2F_1(a,b;d;1-x)}{{}_2F_1(a,b;c;x)}.$$

then

$$y'=-\frac{x^{-c}(1-x)^{-d}}{{}_2F_1^2(a,b;c;x)}. \tag{3.1}$$

Directly compute the derivative of y, we have

$$y'=\Gamma\frac{\begin{vmatrix} {}_2F_1(a,b;c;x) & \frac{d}{dx}\,{}_2F_1(a,b;c;x) \\ {}_2F_1(a,b;d;1-x) & \frac{d}{dx}\,{}_2F_1(a,b;d;1-x) \end{vmatrix}}{{}_2F_1^2(a,b;c;x)}. \tag{3.2}$$

Combine (3.1) and (3.2), and get the following

Corollary.

$$\begin{vmatrix} {}_2F_1(a,b;c;x) & \frac{d}{dx}\,{}_2F_1(a,b;c;x) \\ {}_2F_1(a,b;d;1-x) & \frac{d}{dx}\,{}_2F_1(a,b;d;1-x) \end{vmatrix}=-\frac{x^{-c}(1-x)^{-d}}{\Gamma} \tag{3.3}$$

Consider the case

$$a=s,\ b=1-s,\ c=1,$$

that is, $\mu_0=0,\ \mu_1=0,\ \mu_\infty=1-2s$. Then ψ (φ,resp.), the normalized solution of $\mathrm{H}_{\mathrm{G}}^{(0,0,1-2s)}$ at 0(at 1,resp.), is

$$\varphi(x)=\sqrt{x(1-x)}\,{}_2F_1(s,1-s;1;x),$$

$$\psi(x)=\sqrt{x(1-x)}\,{}_2F_1(s,1-s;1;1-x).$$

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Lemma.
$$W(\psi,\varphi)=\begin{vmatrix}\psi(x) & \frac{d}{dx}\psi(x)\\ \varphi(x) & \frac{d}{dx}\varphi(x)\end{vmatrix}=\frac{1}{\Gamma(s)\Gamma(1-s)}=\frac{\sin(\pi s)}{\pi}.$$

Proof. Denote ${}_2F_1(s,1-s;1;x)$ by $F_0(x)$ and ${}_2F_1(s,1-s;1;1-x)$ by $F_1(x)$. Since $\frac{d}{dx}\varphi(x)=\sqrt{x(1-x)}\frac{d}{dx}F_0(x)+F_0(x)\frac{d}{dx}\sqrt{x(1-x)}$, we have similar form for $\frac{d}{dx}\psi(x)$. Then

$$\begin{vmatrix}\psi(x) & \frac{d}{dx}\psi(x)\\ \varphi(x) & \frac{d}{dx}\varphi(x)\end{vmatrix}=\begin{vmatrix}\psi(x) & F_1(x)\frac{d}{dx}\sqrt{x(1-x)}\\ \varphi(x) & F_0(x)\frac{d}{dx}\sqrt{x(1-x)}\end{vmatrix}+\begin{vmatrix}\psi(x) & \sqrt{x(1-x)}\frac{d}{dx}F_1(x)\\ \varphi(x) & \sqrt{x(1-x)}\frac{d}{dx}F_0(x)\end{vmatrix}$$

$$=F_0(x)F_1(x)\begin{vmatrix}\sqrt{x(1-x)} & \frac{d}{dx}\sqrt{x(1-x)}\\ \sqrt{x(1-x)} & \frac{d}{dx}\sqrt{x(1-x)}\end{vmatrix}+x(1-x)\begin{vmatrix}F_1(x) & \frac{d}{dx}F_1(x)\\ F_0(x) & \frac{d}{dx}F_0(x)\end{vmatrix}$$

$$=0+x(1-x)\begin{vmatrix}F_1(x) & \frac{d}{dx}F_1(x)\\ F_0(x) & \frac{d}{dx}F_0(x)\end{vmatrix}.$$

By (3.3) and the definition of Γ, we get

$$x(1-x)\begin{vmatrix}F_1(x) & \frac{d}{dx}F_1(x)\\ F_0(x) & \frac{d}{dx}F_0(x)\end{vmatrix}=\frac{1}{\Gamma(s)\Gamma(1-s)}.$$

Fianlly, apply the reflection formula of gamma function:$\frac{1}{\Gamma(s)\Gamma(1-s)}=\frac{\sin(\pi\alpha)}{\pi}$, and we are done. ☐

Now, we apply this lemma into (1.1) to finish the calculation of

Theorem 3.0.2. *For $a=s$, $b=1-s$, $c=1$, that is $\mu_0=0$, $\mu_1=0$, $\mu_\infty=1-2s$, we have $\nu_i=\sqrt{\frac{1}{4}+-\frac{1}{4}(1-\mu_i^2)}=0$ for $i=0,1$ and*

$$\mathrm{det}_\zeta \mathrm{H}_\mathrm{G}^{(0,0,1-2s)}=\sin(\pi \mathrm{s}).$$

In particular,

$$\mathrm{det}_\zeta \mathrm{H}_\mathrm{G}^{(0,0,0)}=1.$$

Also, for those $\mathrm{H}_G^{(\mu_0,\mu_1,\mu_\infty)}$ correspond to $\mathrm{E}(\frac{1}{r},\frac{r-1}{r};1)$, we see

$$a+b=\frac{1}{r}+\frac{r-1}{r}=1=c,$$

that is

$$\mu_0=0,\mu_1=0,\mu_\infty=\frac{r-2}{r}.$$

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Therefore, for $r = 3, 4, 6$, the zeta determinants of $\mathrm{H}_G^{(0,0,\frac{r-2}{r})}$ are

$$\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2},$$

that are the inverse of the constants $\csc(\frac{\pi}{r})$ in Ramanujan's alternative bases [12]

$$\exp(-\pi \csc(\frac{\pi}{r}) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)})$$

respectively. The author doesn't know any theoretical explanation of this coincidence, it will be welcome if some can tell.

Remark. In a fancy way, we can rewrite $\det_\zeta \mathrm{H}_{\mathrm{G}}^{(0,0,1-2\mathrm{s})} = \sin(\pi \mathrm{s})$ as

$$\det{}_\zeta \mathrm{H}_{\mathrm{G}}^{(0,0,1-2\mathrm{s})} = \mathrm{s} \prod_{\mathrm{k}=1}^{\infty} (1 - \frac{\mathrm{s}^2}{\mathrm{k}^2 \pi^2}),$$

by the Euler's product formula of sine function.

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Chapter 4

Appendix

4.1 Laplacian in polar coordinates of half plane and Disk.

$$ds^2_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}$$

$$\triangle_{\mathbb{H}} = y^2(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2})$$

For simplicity, let us skip the isometry from $\mathbb{H}$ to $\mathbb{D}$. And wirte down the metric and its Laplacian as

$$ds^2_{\mathbb{D}} = dr^2 + \sinh(r)d\theta^2,$$

$$\triangle_{\mathbb{D}} = \frac{\partial^2}{\partial r^2} + \frac{1}{\tanh(r)}\frac{\partial}{\partial r} + \frac{1}{\sinh^2(r)}\frac{\partial^2}{\partial \theta^2}.$$

Consider

$$\cosh(r) = 2u + 1,$$

so $u = \frac{\cosh(r)-1}{2}$ and $u(u+1) = \frac{\sinh^2(r)}{4}$. We have :

$$\frac{\partial^2}{\partial r^2} + \frac{1}{\tanh(r)}\frac{\partial}{\partial r} = u(u+1)\frac{\partial^2}{\partial u^2} + (2u+1)\frac{\partial}{\partial u}$$

4.2 Eigenproblem of Laplacian implies hypergeometric.

For $f(u,\theta) = f(u)$ such that

$$\triangle_{\mathbb{D}} f(u) = s(1-s)f(u),$$

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we have

$$[u(u+1)\frac{\partial^2}{\partial u^2}+(2u+1)\frac{\partial}{\partial u}+s(1-s)]f(u)=0$$

which means

$$f(u)={}_2F_1(s,1-s;1;-u),$$

where $f(u)$ is also known as the zero order Legendre functions with $-s$ degree.

On the other hand, if we consider

$$\triangle_{\mathbb{D}}g(u)=(\frac{1}{4}+s^2)g(u),$$

i.e.

$$[u(u+1)\frac{\partial^2}{\partial u^2}+(2u+1)\frac{\partial}{\partial u}+(\frac{1}{4}+s^2)]g(u)=0.$$

Hence we have

$$g(u)={}_2F_1(\frac{1}{2}+is,\frac{1}{2}-is;1;-u),$$

because $(\frac{1}{2}+is)(\frac{1}{2}-is)=\frac{1}{4}+s^2$and $(\frac{1}{2}+is)+(\frac{1}{2}-is)=1$.

Following from our interest on

$$\triangle_{\mathbb{D}}F(x,y)=(\frac{1}{4}+s^2)F(x,y),$$

we give a berifly study on

$${}_2F_1(\frac{1}{2}+is,\frac{1}{2}-is;1;x)=\sum_{n=1}^{\infty}\frac{(\frac{1}{2}+is,n)(\frac{1}{2}-is,n)}{(n!)^2}x^n$$

where $(a,n):=a(a+1)\cdots(a+(n-1))$. By definition we have

$$(\frac{1}{2}+is,n)=(\frac{1}{2}+is)(\frac{1}{2}+is+1)\cdots(\frac{1}{2}+is+(n-1)),$$

and

$$(\frac{1}{2}-is,n)=(\frac{1}{2}-is)(\frac{1}{2}-is+1)\cdots(\frac{1}{2}-is+(n-1)).$$

Thus

$$(\frac{1}{2}+is,n)(\frac{1}{2}-is,n)=\prod_{k=1}^{n}[(\frac{2k-1}{2})^2+s^2],$$

and then

$${}_2F_1(\frac{1}{2}+is,\frac{1}{2}-is;1;x)=\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}[(\frac{2k-1}{2})^2+s^2]}{(n!)^2}x^n.$$

There is a beautiful estimate on the coefficients of ${}_2F_1(\frac{1}{2}+is,\frac{1}{2}-is;1;x)$, it is given by my freind Da-wei Yang

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Proposition.

$$\frac{\prod_{k=1}^{n}[(\frac{2k-1}{2})^2+s^2]}{(n!)^2} \leq \frac{e^{-(\sum_{k=1}^{n}\frac{1}{k})+\ln n}}{n} e^{(\frac{1}{4}+s^2)\frac{\pi^2}{6}}$$

Proof. First we observe

$$\frac{\prod_{k=1}^{n}[(\frac{2k-1}{2})^2+s^2]}{(n!)^2} = \frac{\prod_{k=1}^{n}[k^2-k+\frac{1}{4}+s^2]}{(\prod_{k=1}^{n}k)^2} = \prod_{k=1}^{n}[1-\frac{1}{k}+(\frac{1}{4}+s^2)\frac{1}{k^2}].$$

Since$(1+x)\leq e^x$, we have

$$\prod_{k=1}^{n}[1-\frac{1}{k}+(\frac{1}{4}+s^2)\frac{1}{k^2}] \leq \prod_{k=1}^{n}\exp(-\frac{1}{k}+(\frac{1}{4}+s^2)\frac{1}{k^2})$$

$$= \exp\sum_{k=1}^{n}(-\frac{1}{k}+(\frac{1}{4}+s^2)\frac{1}{k^2}) = \left(\exp(\sum_{k=1}^{n}-\frac{1}{k})\right)\exp(\sum_{k=1}^{n}(\frac{1}{4}+s^2)\frac{1}{k^2}).$$

Then by observe $\sum_{k=1}^{n}(\frac{1}{4}+s^2)\frac{1}{k^2}$ is monotone increase to $(\frac{1}{4}+s^2)\frac{\pi^2}{6}$ with the factexp$(-\ln n)=\frac{1}{n}$, we get our result.

□

Remark. Recall $\sum_{k=1}^{n}\frac{1}{k}-\ln n \underset{n\to\infty}{\longrightarrow} \gamma$ =the Euler constant. Hence for large n, we have

$$\frac{e^{-(\sum_{k=1}^{n}\frac{1}{k})+\ln n}}{n} e^{(\frac{1}{4}+s^2)\frac{\pi^2}{6}} \approx \frac{\gamma}{n} e^{(\frac{1}{4}+s^2)\frac{\pi^2}{6}}.$$

Problem. Can we compute the monodromy group of ${}_2F_1(\frac{1}{2}+is,\frac{1}{2}-is;1;-u)$?

Since compare with ${}_2F_1(s,1-s;1;-u)$, which we known it well in sense of Ramanujan's alternative bases[12], the family ${}_2F_1(\frac{1}{2}+is,\frac{1}{2}-is;1;-u)$ is unexplored up to the knowledge of the author. And from the point of view of eigenproblem of Laplacian of upper half plane, they may have close or complementary relationship.

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