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保角緊緻流形之相關探討
A survey on conformally compact manifolds

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完成一篇論文是漫長的工作。首先，我想感謝我的指導教授。謝謝李瑩英老師給予我研究方向，並每週陪我不斷地討論，給予我許多珍貴的建議。在與教授討論的過程中，我能以不同的觀點來看待我正在做的課題並得以檢視自己的疏漏。也很感謝蔡忠潤老師，鄭日新老師，蕭欽玉老師和王振男老師在我研究有困惑時給予我不少建議，真的很有幫助。此外，感謝蔡忠潤老師、崔茂培老師和鄭日新老師願意當我的口試委員，並協助我修正論文內容。最後，謝謝我的家人給予我經濟支持，讓我能在這裡安靜地就讀，並完成這篇論文。

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中文摘要

在這篇文章裡，我探討了保角緊緻流形的重要結果及其關聯性。這些主題包含了重整化體積，GJMS 算子，Q 曲率和基礎的散射理論。這篇文章的主旨是從不同的歷史發展出發來看保角緊緻流形的研究，並探討這些不同的歷史發展交匯時的結果。

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Abstract

In this paper, I survey several important results for conformally compact manifolds and relate these different objects together. These topics includes renormalized volume, GJMS operators, Q-curvature, and basic scattering theory. The main goal of this paper is to survey conformally compact manifolds from different historical developments and discuss how these developments are related.

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Chapter 1

Introduction

A conformal manifold M is a manifold equipped with an equivalence class of Riemannian metrics, where two metrics g, h are (conformally) equivalent if $g = uh$ for some smooth positive function u on M. A diffeomorphism between two Riemannian manifolds is conformal if the pull back metric is conformally equivalent to the original metric. Conformal diffeomorphisms of the Euclidean sphere is one of the most basic starting points for studying conformal manifolds. One can view S^n as a line of light cone in $(n+2)$ dimensional Minkowski space, and the orthogonal transformation preserving the light cone gives rise to a conformal diffeomorphism of S^n. In other words, assume the coordinates in $\mathbb{R}^{n+2}$ are $(y_0, y_1, ..., y_{n+1})$, and the light cone is $G = \{\Sigma_{1\leq k\leq n+1} y_k^2 - y_0^2 = 0\}$. Then G projectives into the sphere $\Sigma_{1\leq k\leq n+1} x_k^2 - 1 = 0$, where $x_k = y_k/y_0$. The Lorentz metric restricts on G is $\Sigma_{1\leq k\leq n+1} dy_k^2 - dy_0^2 = y_0^2\, \Sigma_{1\leq k\leq n+1} dx_k^2$. So the orthogonal transformation preserving light cones induces a conformal map of the sphere. From this observation, Fefferman and Graham proposed the ambient metric (see chapter 4 for more details) in order to study conformal invariants [1]. In the same paper, they also proposed a different kind of metric called Poincaré metric (see chapter 4) based on another viewpoint for conformal diffeomorphism of S^n. It considers S^n as the sphere at infinity of H^{n+1}. Using the Poincaré disk model, it is known that the conformal diffeomorphism of S^n can be uniquely extended to H^{n+1} as a hyperbolic isometry, and also, by restricting hyperbolic isometry to the boundary we get the conformal diffeomorphism of S^n (see [2] for the details). This relation between S^n and H^{n+1} in some sense connects the conformal geometry of the

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boundary to the Riemannian geometry of the interior. This viewpoint between boundary and interior has many important generalizations in physics and mathematics. For instance, Penrose proposes the idea of conformal infinity in his Penrose diagram, and it leads us to consider conformally compact manifolds.

Let X be a compact smooth $(n+1)$ dimensional manifold with smooth boundary $\partial X = M$. Let r be a smooth nonnegative function on X. We say that r *is a defining function for* M if $M \equiv \{p \in X | r(p) = 0\}$ and $dr \neq 0$ on M. Note that if r is a *defining function,* then for any positive smooth function f, fr is also a defining funtion for M.

Let $[h]$ be a conformal class of metrics on M. We say that a Riemannian metric g on interior of X is *conformally compact* with conformal infinity $[h]$ if $\overline{g} = r^2 g$ extends as a continuous (or some smoothness conditions) Riemannian metric on X and $\overline{g}|_{TM} \in [h]$.

The basic example is the hyperbolic space H^{n+1}. Consider the Poincaré disk model with the hyperbolic metric $g = \frac{4}{\left(1-|x|^2\right)^2} \sum_{1 \leq i \leq n+1} \left(dx^i\right)^2$ on the unit ball. If take $r = \frac{\left(1-|x|^2\right)}{2}$, then $\overline{g}$ is the standard Euclidean metric. In this case, we see that hyperbolic metric is conformally compact on $X = B^{n+1}$ with $\partial X = M = S^n$. Note that by rescaling r, $\overline{g}|_{TS^n}$ changes, and we get a conformal class of metric on S^n.

The developments of conformally compact manifolds have many different aspects. In this survey, I would like to introduce some basic theory for conformally compact manifolds. These developments from different starting points turns out to have some interesting connections.

In chapter 2, we start from the curvature tensor, and deduce some general properties for conformally compact manifolds. In chapter 3, we introduce the idea of renormalized volume. The concept of renormalized volume is proposed by physicists based on Fefferman and Graham's result [1], and it is a conformal invariant in odd dimensions and have been studied by many mathematicians. In this part, we follow Graham's work [3], impose some regularity condition, and define the renormalized volume on conformally compact Einstein manifolds.

While renormalized volume is more physics-oriented, in chapter 4, we consider two related subjects: GJMS operators and scattering theory. To get the whole picture, we'll

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briefly introduce the ambient metric and Poincaré metric proposed by Fefferman and Graham. Originally, they propose these two metrics to study conformal invariants, and in paper [4], they continue the study from [1], introducing a family of conformally invariant operators with leading terms Δ^k by using ambient metric. These operators generalize conformal Laplacian, and give definition for Q-curvatures. Since [1] has shown that there is one to one correspondence between ambient metric and Poincaré metric, instead of deriving GJMS operators from ambient metric, it's natural to ask whether we can derive GJMS operators from Poincaré metric. What bridges this gap is scattering theory. This work is done by Graham and Zworski [5]. The development of scattering theory is due to mathematicians studying eigenvalue spectrums for Laplacian operators on asymptotic hyperbolic manifolds. In this chapter we'll state the result relating the scattering matrix for a Poincaré metric and conformally invariant operators with leading terms in Δ^k.

The last chapter is about application. We state some results in [6], and show that the huge machinery from different branches can be related by equations. Especially, we will introduce an equality relates Q-curvature and renormalized volume via theorem proved by scattering theory [5]. The main work in [6] is that the authors prove that Q-curvature can be regarded as a coefficient in the solution to a boundary problem for Laplace operator. This gives an alternative definition for Q-curvature.

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Chapter 2

Basic properties

Since conformally compact manifolds are generalization from hyperboic model, they also behave similary near the boundary. Note that conformally compact manifolds is automatically complete since $g = r^{-2}\overline{g}$ and the boundary is pushed to infinity. In fact, we have following characterizations for conformally compact manifolds.

Lemma 1. *[7] [8] Let $(X^{n+1},\ g)$ be a conformally compact manifold, then we have the following,*

(i) Suppose $\overline{g}$ is a C^2 metric, then $Ric\,(g) + ng = O\,(r^{-1})$ near M iff $|dr|^2_{\overline{g}} = 1$ on M.

(ii) If $|dr|^2_{\overline{g}} = 1$ on M, then g has asymptotic sectional curvature -1 near M.

Proof. To prove the first property, we have to consider the conformal transformation of curvature tensor. Let R_{jk} and $\overline{R}_{jk}$ be Ricci tensors for g and $\overline{g}$ respectively. Denote $\overline{r}_{ij} = \partial_j\partial_i r - \overline{\Gamma}^k_{ij}\partial_k r$. Then via formulas for conformal transfomation (see, for instance [9]), we have

$$\begin{aligned} R_{jk} &= -r^{-2}(n\overline{g^{il}}r_i r_l)\overline{g_{jk}} + r^{-1}\left(\overline{g^{il}}\,\overline{r_{il}}\,\overline{g_{jk}} + (n-1)\,\overline{r_{jk}}\right) + \overline{R_{jk}} \\ &= -r^{-2}(n\,|dr|^2_{\overline{g}})\overline{g_{jk}} + O\left(r^{-1}\right) \text{ near } M. \end{aligned}$$

Therefore, $Ric\,(g) + ng = O\,(r^{-1})$ near M iff $|dr|^2_{\overline{g}} = 1$ on M.

As for the second property, by formula for conformal transformation, we have

$$R_{ijkl} = -\left(|dr|^2_{\overline{g}}\right)(g_{ik}g_{jl} - g_{il}g_{jk}) + O\left(r^{-3}\right).$$

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We know that a manifold has constant sectional curvature κ iff $R_{ijkl} = -\kappa\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$. Therefore, $-\left|dr\right|_{\overline{g}}^{2} = -1$ on M is the asymptotic sectional curvature of g. □

From lemma 1, we see that conformally compact manifolds satisfying $\left|dr\right|_{\overline{g}}^{2} = 1$ on M behave like the hyperbolic metric. Therefore, we call a conformally compact metric g *asymptotically hyperbolic if* $\left|dr\right|_{\overline{g}}^{2} = 1$ on M. Espeicially, lemma 1 tells us that a conformally compact asymptotically Einstein metric is asymptotically hyperbolic.

When we fix a defining function r, it determines the conformal representative $r^2 g|_{TM}$. But given a conformal representative doesn't fix a defining function as it just specifies values on M. Next, we introduce a special defining function for a given conformal representative, which will be useful for later calculations.

Lemma 2. *[8] Let g be an asymptotically hyperbolic metric on X with $\partial X = M$. Then given a boundary metric $h \in [h]$, there exists a unique defining function r such that $\overline{g}|_{TM} = r^2 g|_{TM} = h$, and $\left|dr\right|_{\overline{g}}^{2} = 1$ in a neighborhood of M. Moreover, the metric $\overline{g}$ takes the form $\overline{g} = g_r + dr^2$ near M; g_r is a 1-parameter family of metrics on M.*

Proof. Fix an defining function s with $\overline{g^s} = s^2 g$ and $\left|ds\right|_{\overline{g^s}}^{2} = 1$ on M Our goal is to find some ω, such that for the defining function $t = se^{\omega}$ and $\overline{g^t} = t^2 g$. We have $\left|dt\right|_{\overline{g^t}}^{2} = 1$ in a neighborhood of M.

As $\overline{g^t} = e^{2\omega}\overline{g^s}$, and $dt = e^{\omega}\left(ds + sd\omega\right)$, it gives

$$\begin{aligned} \left|dt\right|_{\overline{g^t}}^{2} &= \left|e^{\omega}\left(ds + sd\omega\right)\right|_{e^{2\omega}\overline{g^s}}^{2} = \left|ds + sd\omega\right|_{\overline{g^s}}^{2} \\ &= \left|ds\right|_{\overline{g^s}}^{2} + 2s(\nabla^{\overline{g^s}} s)(\omega) + s^2\left|d\omega\right|_{\overline{g^s}}^{2}, \end{aligned}$$

where the gradient $\left(\nabla^{\overline{g^s}} s\right)^i$ is given by $\left(\overline{g^s}\right)^{ij}\partial_j s$.

Plugging into the condtion $\left|dt\right|_{\overline{g^t}}^{2} = 1$, we have

$$2\nabla^{\overline{g^s}} s(\omega) + s^2\left|d\omega\right|_{\overline{g^s}}^{2} = \frac{1 - \left|ds\right|_{\overline{g^s}}^{2}}{s}.$$

Since $\nabla^{\overline{g^s}} s$ is transverse to M. Above equation is a non-charcteristic first order PDE for ω, so by theory from PDE (see, for instance [10]), there exists solution in a neighborhood

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of M with $\omega|_M$ prescribed. The first part of lemma is thus proved.

Next, we want to identify $M \times [0, \varepsilon)$ with a neighborhood of M in X by a defining function r. Fix the special defining function r in the lemma.We now denote $\nabla^{\overline{g}}$ as ∇. Let $(p, \lambda) \in M \times [0, \varepsilon)$ be the integral curve $s(t)$ of ∇r, starting from p, and go λ unit time. Say, $s(\lambda) = (x_1, ..., x_n, \lambda)$, and $\nabla r (s(t)) = s'(t) = (0, ..., 0, 1)$.

$$\begin{aligned} r(s(\lambda)) - r(s(0)) &= \int_0^\lambda \frac{d}{dt} r(s(t)) = \int_0^\lambda \langle \nabla r, s' \rangle \, dt = \\ \int_0^\lambda \langle \nabla r, \nabla r \rangle \, dt &= \int_0^\lambda \overline{g}_{ij} \overline{g}^{ik} \partial_k r \overline{g}^{jl} \partial_l r = \int_0^\lambda |dr|^2_{\overline{g}} = \lambda. \end{aligned}$$

So the λ coordinates is just r. Hence $\partial_j r = 0$ for $1 \leq j \leq n$.

$$(\nabla r)^i = (\overline{g})^{ij} \partial_j r = (\overline{g})^{i,n+1} .$$

Since $(\nabla r)^i = 0$ for $1 \leq i \leq n$, it follows that $(\overline{g})^{i,n+1} = 0$ for $1 \leq i \leq n$. Therefore, $\overline{g} = g_r + dr^2$ for some tensors g_r on M. $\square$

The paper [3] provides a concrete example for lemma 2. The example is again the hyperbolic metric $g = \frac{4}{\left(1-|x|^2\right)^2} \sum_{1 \leq i \leq n+1} \left(dx^i\right)^2$. To find the special defining function, we have to solve $|dr|^2_{\overline{g}} = \left| d\left(\log \frac{1}{r}\right) \right|^2_g = 1$. For equation looks like $|dF|^2_g = 1$, it is known that the distance function is one of its solution. Therefore, one of the solution is $\log \frac{1}{r} = d(x) = \log \frac{1+|x|}{1-|x|}$. So $r = \frac{1-|x|}{1+|x|}$. Therefore, $\overline{g} = r^2 g = \frac{4}{(1+|x|)^4} \sum_{1 \leq i \leq n+1} \left(dx^i\right)^2$, and the conformal representative g_0 is $1/4 \sum_{1 \leq i \leq n+1} \left(dx^i\right)^2$. So we can express $\overline{g}$ as $\left(1 - r^2\right)^2 g_0 + dr^2 = g_r + dr^2$. The metric expansion will play a key role in the following chapters.

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Chapter 3

Renormalized volume

In this chapter, we introduce a volume-related quantity for conformally compact manifolds. A conformally compact manifold $(X^{n+1},\ g)$ has infinite volume. A compact submanifold Y in the interior of X has finite volume, but as ∂Y approaches ∂X, the volume of Y tends to infinity. Physicists suffer from a similar scenario as above [11]. They consider some stress tensor T on given spacetime region K, and the tensor diverges as ∂K tends to infinity. For some specific tensor, they observe that the divergence part of T depends only on the intrinsic geometry of the boundary. By using tools in quantum field theory, they are able to add some counter terms to T and substract the divergence. After substraction, they can take ∂K tends to infinity. The above technique is a kind of *renormalization* in quantum field theory.

Motivated by above, we may imagine T as $Vol\,(Y)$, ∂K as ∂Y, infinity as ∂X, and try to renormalize the volume. First of all, we define the regulated volume of X [12].

The regulated volume $Vol_\varepsilon\,(X)$ *is defined as* $\int_{r>\varepsilon} dvol_g$.

As ε tends to 0, $Vol_\varepsilon\,(X)$ tends to $Vol\,(X)$. To analyze $Vol_\varepsilon\,(X)$, we would like to do asymptotic expansion of regulated volume in terms of ε. To do this, we have to consider the asymptotic expansion of the metric. Below, we always use the special defining function stated in lemma 2.

Lemma 3. *[3] Let* $(X^{n+1},\ g)$ *be a conformally compact Einstein manifold. Suppose* g *has asymptotic expansions in* r *to high enough order, then*

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1. *For n odd,* $g_r = g^{(0)} + g^{(2)}r^2 + (\textit{even powers}) + g^{(n-1)}r^{n-1} + g^{(n)}r^n + \ldots$

2. *For n even,* $g_r = g^{(0)} + g^{(2)}r^2 + (\textit{even powers}) + kr^n \log r + g^{(n)}r^n + \ldots$

Note that in the odd case, if we fix a conformal representative h, the terms $g^{(j)}$ are locally determined tensors on M for $0 \leq j \leq n-1$, $g^{(n)}$ is undetermined but $trace_h\left(g^{(n)}\right) = 0$.

As for the even case, $g^{(j)}$ are locally determined for $0 \leq j \leq n-2$, k is locally determined and trace-free, and only trace of $g^{(n)}$ is locally determined.

Proof. **(Sketch)** Let $T = Ric(g) + ng$. Consider the tensor component of T. Using $M \times (0, \varepsilon)$ as local coordinates, express g as $r^{-2}\left(g_r + dr^2\right)$ by lemma 2. For $1 \leq i, j \leq n$, we have

$$-2rT_{ij} = rg_{ij}^{''} + (1-n)\, g_{ij}^{'} - g^{kl}g_{kl}^{'}g_{ij} - rg^{kl}g_{ik}^{'}g_{jl}^{'} + \frac{r}{2}g^{kl}g_{kl}^{'}g_{ij}^{'} - 2rRic_{ij}\left(g_r\right) = 0. \quad (3.1)$$

The differentiation is with respect to r, and $Ric\left(g_r\right)$ means the Ricci curvature of g_r with fixed r. Setting $r = 0$, we have

$$(1-n)\, g_{ij}^{'} - g^{kl}g_{kl}^{'}g_{ij} = 0.$$

Observe that above equation can be regarded as an invertible operator acting on the column vector $g_{\mu v}^{'}$. Therefore, $g_{\mu v}^{'}|_{r=0} = 0$. Similarly, by differentiating $(t-1)$ times on $-2rT_{ij}$, we have

$$(t-n)\, \partial_r^t g_{ij} - g^{kl}\partial_r^t g_{kl}g_{ij} = \left(terms\ involving\ \partial_r^u g_{ij},\ u < t\right).$$

As long as $t < n$, the LHS is an invertible operator acting on $\partial_r^t g_{\mu v}$. Therefore, $\partial_r^t g_{\mu v}$ is solvable. By induction, we can show that all odd derivatives are zero.
For $t = n$, if n is odd, one can conclude that the RHS vanishes at $r = 0$ by counting the order of derivatives (odd or even). Therefore, we only know that $g^{kl}\partial_r^n g_{kl}g_{ij} = 0$. If n is even, the RHS may not be zero. We may assume $g_r = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) +$

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$pr^n \log r + qr^n$, where p and q are smooth functions. Then plug it into equation (3.1) to observe the properties of p and q. $\square$

With above metric expansion, one can calculate $Vol_\varepsilon(X)$ and obtain

Lemma 4. *[3] The regulated volume $Vol_\varepsilon(X)$ has the following expansions:*

1. *For n odd, $Vol_\varepsilon(X) = c_0\varepsilon^{-n} + c_2\varepsilon^{-n+2} + (\text{odd powers}) + c_{n-1}\varepsilon^{-1} + V + o(1)$.*

2. *For n even,*

$$Vol_\varepsilon(X) = c_0\varepsilon^{-n} + c_2\varepsilon^{-n+2} + (\text{even powers}) + c_{n-2}\varepsilon^{-2} + L\log\frac{1}{\varepsilon} + V + o(1),$$

where c_i and L are integrals over M for a conformal representative h, and V is the constant term.

Proof. Let $h = g_0$ be the chosen representative metric on M, and $(x_1, x_2, \ldots x_n)$ be coordinates on M. Note that $g = r^{-2}(g_r + dr^2)$, and $h = g_0$. We have

$$dvol_g = (\det g)^{1/2} dx_1 dx_2 \ldots dx_n dr = r^{-n-1}\left(\frac{\det g_r}{\det h}\right)^{\frac{1}{2}} dvol_h dr. \tag{3.2}$$

By metric expansions for g_r in lemma 3, we have

$$\left(\frac{\det g_r}{\det g_0}\right)^{\frac{1}{2}} = 1 + v^{(2)}r^2 + (even\ powers) + v^{(n)}r^n + \ldots, \tag{3.3}$$

where $v^{(j)}$ are locally determined functions on M.

Fix a small number r_0, then

$$Vol_\varepsilon(X) = \int_{r>\varepsilon} dvol_g = c + \int_{r_0>r>\varepsilon} dvol_g.$$

Now, apply equation (3.2) and (3.3) for $dvol_g$, we get the desired results. In particular, $c_0 = \frac{1}{n} Vol_h(M)$ and $L = \int_M v^{(n)} dvol_h$. $\square$

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The renormalized volume is defined as the constant term V in the expansion for $Vol_\varepsilon(X)$

For the renormalized volume, Graham has the following theorem.

Theorem 1. *[3] If n is odd, then V is independent of the choice of conformal representatives on M. If n is even, then L is independent of the choice of conformal representatives on M.*

Here we just give the ideas of the proof. For the details, please consult theorem 3.1 in [3]. First of all, we state a useful lemma.

Lemma 5. *[3] Let h and $\widehat{h}$ be two different representative metrics on M associated with two special defining function r and $\widehat{r}$ satisfying the condition in Lemma 2. Then $r = \widehat{r}e^w$, where w is a function on $M \times [0, \varepsilon)$, and the Taylor expansion of w at 0 up to $\widehat{r}^{n+1}$ terms consists only of even powers of $\widehat{r}$.*

Proof. **(Sketch)** Recall that $\overline{g^r} = r^2 g$. Since $|d\widehat{r}|^2_{\overline{g^{\widehat{r}}}} = |dr|^2_{\overline{g^r}} = 1$, similar to the proof of lemma 2, we have

$$2w_{\widehat{r}} + \widehat{r}\left(w^2_{\widehat{r}} + |d_M w|^2_{g_{\widehat{r}}}\right) = 0.$$

When $\widehat{r} = 0$, $w_{\widehat{r}} = 0$. Note that $|d_M w|^2_{g_{\widehat{r}}} = g^{ij}_{\widehat{r}} w_i w_j$. Now, consider $\partial^{k+1}_{\widehat{r}} w$ for k even by differentiating above equation. We have

$$2\partial^{k+1}_{\widehat{r}} w = 2\left(\partial^p_{\widehat{r}} w_{\widehat{r}}\right)\left(\partial^q_{\widehat{r}} w_{\widehat{r}\widehat{r}}\right) + \left(\partial^a_{\widehat{r}} g^{ij}_{\widehat{r}}\right)\left(\partial^b_{\widehat{r}} w_i\right)\left(\partial^c_{\widehat{r}} w_j\right) \text{ at } r = 0,$$

where $p + q = k - 2$, and $a + b + c = k - 1$. By induction, if w is differentiated odd times, then it's zero. And $g^{ij}_{\widehat{r}}$ is zero when differentiated odd times due to lemma 3. Therefore, we can prove the results by considering the parity. □

Note that via lemma 5, we know that if $\widehat{h} = e^{2u}h$, then $\widehat{r} = e^u r + O(r^2)$. Next, we sketch the proof for the theorem. The main idea is to calculate $Vol(\{r > \varepsilon\}) - Vol(\{\widehat{r} > \varepsilon\})$, where r and $\widehat{r}$ are the special function associated with different conformal representatives. If there is no constant term in above expression, it means that the renormalized volume V for different conformal representatives is cancelled. Therefore, V is a conformal invariant.

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Write e^w as $b(\hat{r}, x_1, x_{2,}...x_n)$. Note that when $\hat{r} = \varepsilon$, $r = \hat{r} b(\hat{r}, x) = \varepsilon b(\varepsilon, x) \triangleq \hat{\varepsilon}$. So, $Vol(\{r > \varepsilon\}) - Vol(\{\hat{r} > \varepsilon\}) = \int_M \int_{\varepsilon}^{\hat{\varepsilon}} dvol_g$. Using equation (3.2), the integral becomes

$$\Sigma_{0 \leq j \leq n-1, j\ even} \varepsilon^{-n+j} \int_M \frac{v^{(j)}}{-n+j} \left(b(x, \varepsilon)^{-n+j} - 1 \right) dvol_h + o(1) \text{ for n odd.}$$

The remaining thing is to argue that this integral does not have constant term when n is odd (or no $\log \frac{1}{\varepsilon}$ term when n is even). From lemma 5, we see that Taylor expansion of $b(\varepsilon, x)$ only consists of even powers of ε up to ε^{n+1} terms. Write down the expression and take $\varepsilon \to 0$, then we are done.

In [3], Graham explicitly calculated the renormalized volume for hyperbolic spaces. He proved that $V = (-1)^{\frac{n+1}{2}} \frac{\pi^{\frac{n+2}{2}}}{\Gamma\left(\frac{n+2}{2}\right)}$ for n odd and $L = (-1)^{\frac{n}{2}} \frac{(2\pi)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}$ for n even. We remark that the renormalized volume is in strong connection to Euler characteristics. For example, when $n = 2$, $L = -\pi\chi(M)$. Furthermore, in [13], it is showed that for conformally compact hyperbolic manifolds, L and V are just multiples of Euler characteristics in even and odd dimensions respectively.

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Chapter 4

GJMS operators and scattering theory

This chapter will divide into two sections, and serve as a brief review for some important facts in [4] and [5].

4.1 GJMS operators

GJMS operator is a kind of conformally invariant operators, and GJMS stands for Graham, Jenne, Mason and Sparling. To begin with, we give the definition of conformally invariant operators.

Let M be a compact n-dimensional Riemannian manifold with metric h. Let $\widehat{h} = e^{2u}h$. A metric-dependent operator P is called conformally invariant if there exist constants a and b, such that $P\left(\widehat{h}\right)(\phi) = e^{bu}P(h)\left(e^{-au}\phi\right)$ for all smooth functions ϕ on M.

The conformal Laplacian $L = -\Delta + \frac{n-2}{4(n-1)}R$ is the most well-known conformally invariant operator. In these case, a and b are $\frac{2-n}{2}$ and $\frac{-2-n}{2}$ respectively. There is a formal point of view by considering that conformal Laplacian acts on spaces with different *weighted functions*. It is known as *conformal densities*, and we say that the conformal Laplacians are a map from $\varepsilon\left[1-\frac{n}{2}\right]$ to $\varepsilon\left[-1-\frac{n}{2}\right]$, where $\varepsilon[w]$ denotes conformal densities of weight w. We will introduce it briefly after we define ambient metric. For more details, we refer to [14] for more details.

Besides conformal Laplacian, Paneitz [15] found a fourth order conformally invari-

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ant operator on a Riemannian manifold with leading term Δ^2. This operator maps from $\varepsilon\left[2-\frac{n}{2}\right]$ to $\varepsilon\left[-2-\frac{n}{2}\right]$, and is now known as Paneitz operator. Branson and Østed [16] use the constant term of Paneitz operator to define Q curvature in 4 dimensions. Note that the constant term of conformal Laplacian is basically the scalar curvature.

GJMS operators are generalizations of conformal Laplacian and Paneitz operator. In [4], the authors prove the following theorem regarding these kind of operators.

Theorem 2. *Suppose M is a conformal manifold of dimension $n > 2$. If n is odd, for each positive integer k, there exists a conformal invariant operator of order $2k$, mapping from $\varepsilon\left[k-\frac{n}{2}\right]$ to $\varepsilon\left[-k-\frac{n}{2}\right]$ with leading term Δ^k. If n is even, the same result is true with the restriction $1 \leq k \leq \frac{n}{2}$.*

These conformally invariant operators with leading terms Δ^k form a family P_k, *the GJMS operators*. Given two conformal representatives h and $\widehat{h}$, if $\widehat{h} = e^{2u}h$, then $P_k(h)$ and $\widehat{P_k} \triangleq P_k\left(\widehat{h}\right)$is related as

$$\widehat{P_k} = e^{(-n/2-k)u} P_k e^{(n/2-k)u}.$$

These operators are constructed through usages of ambient metric [1], and they can be expressed via curvature and covariant derivatives for a given representative metric. Instead of giving an outline of this proof, I would rather give a concrete example from [4] to demonstrate the construction. To begin with, it's necessary to introduce the *ambient metric*.

In [17] it was shown that a conformally flat (n) space of signature (p, q) can be viewed as the quadric in P^{n+1}. The quadric is the projectivization of the light cone in flat $(n+2)$ space of signature $(p+1, q+1)$. The most standard example is S^n. Every point in S^n can be viewed as a line of light cone in n+2 dimensional Minkowski space. The ambient metric is a generalization of this to a curved version. It associated a $conformal$some partial differential equations with inital data from conformal structure. To show an example for GJMS operator, let's define $\widetilde{G}$ precisely.

Suppose M is a conformal manifold of signature (p, q) . $p+q = n > 2$. Let g be an rep-

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resentative metric in the conformal class. Define the metric bundle $G = \{(t,x)\,|x \in M, t \in \mathbb{R}^+\}$. For each t, we can associate a metric $t^2 g(x)$, and therefore, sections of G are representatives for conformal class. There is a nature symmetric 2-tensor g_0 on G by natural projection π from G to M. Namely, for $(t,x) \in G$ and $A, B \in T_{(t,x)}G$, we define

$$g_0(A,B) = g(\pi_* A, \pi_* B).$$

Locally, if $g = g_{ij}(x)\,dx^i dx^j$, then $g_0 = t^2 g_{ij}(x)\,dx^i dx^j$.

Besides, for $s > 0$ we can define dilation δ_s on G as $\delta_s(t,x) = (st,x)$ and have $\delta_s^* g_0 = s^2 g_0$. Now, consider the product manifold $\widetilde{G} = G \times (-1,1)$ as our ambient space. Define the inclusion map $\iota : G \hookrightarrow \widetilde{G}$ as $\iota(z) = (z,0)$ for $z \in G$. We identify G with $\iota(G)$. Denote points in $\widetilde{G}$ as (z,ρ), where $z = (t,x) \in G$. We can extend δ_s on $\widetilde{G}$ by acting on the G component only. Now, we would like to extend the tensor g_0, and find an ambient metric $\widetilde{g}$ of signature $(p+1,q+1)$ in $\widetilde{G}$. We require that $\delta_s^* \widetilde{g} = s^2 \widetilde{g}$, and since it's an extension, $\iota^* \widetilde{g} = g_0$. Then [1] shows that locally

$$\widetilde{g} = t^2 g_{ij}(x,\rho)\,dx^i dx^j + 2\rho dt^2 + 2t dt d\rho, \tag{4.1}$$

where $g_{ij}(x,0) = g_{ij}(x)$ is the represented metric on M. Here, we give the partial statements of [1]'s theorem.

Theorem 3. *Assume dimension of M is n, and G and $\widetilde{G}$ are defined as above. Then we have the following.*

1. *Suppose n is odd, then there is a formal power series solution $\widetilde{g}$ to the equation $Ric(\widetilde{g}) = 0$. Furthermore, the solution is unique up to a R_+-equivariant diffeomorphism of $\widetilde{G}$ fixing G.*

2. *Suppose n is even, then in general, there doesn't exist a formal power series solution $\widetilde{g}$ to the equation $Ric(\widetilde{g}) = 0$. We can only find $\widetilde{g}$ such that $Ric(\widetilde{g})$ vanishes to order $\frac{n}{2} - 2$ on G and the components tangent to G vanish to order $\frac{n}{2} - 1$. The solution is unique up to addition of terms vanishing to order $\frac{n}{2}$ and a R_+-equivariant*

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diffeomorphism of $\widetilde{G}$ fixing G.

Note that in above two statements, the order refers to the power of ρ in Taylor series expansion.

Same as equation (3.1), we can determine the power series expansion for $g_{ij}(x,\rho)$ in ρ by induction. First of all, write down each component of $Ric(\tilde{g})$. We have

$$\rho g_{ij}'' + \left(1-\frac{n}{2}\right)g_{ij}' - \frac{1}{2}g^{kl}g_{kl}'g_{ij} - \rho g^{kl}g_{ik}'g_{jl}' + \frac{\rho}{2}g^{kl}g_{kl}'g_{ij}' + Ric_{ij}(g_\rho) = 0.$$

The differentiation is with respect to ρ, and $Ric(g_\rho)$ means the Ricci curvature with fixed ρ.

Set $\rho = 0$, we can solve $g_{ij}^{(r)}$ inductively. For example, we have

$$g_{ij}'(x,0) = 2P_{ij}(x) \text{ and } g^{ij}g_{ij}''(x,0) = 2P_{ij}P^{ij}, \tag{4.2}$$

where

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)}g_{ij}\right).$$

Definition 1. *The metric $g_{ij}(x,\rho)$ is the ambient metric associated to the ambient manifold $\widetilde{G}$.*

With ambient metric, we can calculate the GJMS operators via propositions in [4]. First, let's define some notation. In above, we have introduced $\varepsilon[w]$, the idea of weighted function. Basically, it's a function depends on conformal representatives. After fixing a conformal representative, it becomes a function on M. To be more precisely, we may denote $f \in \varepsilon[w]$ as $f = f(g,x)$, where $x \in M$, and g is a conformal representative. If $g_u = e^{2u}g$, then $f(g_u,x) = (e^u)^w f(g,x)$. Naturely, it can be view as a function on the metric bundle G. Abusing the notation, we may denote $f = f(t,x)$, where t corresponds to t^2g. For the dilation δ_s on G, $\delta_s^* f = f(st,x) = s^w f$. Define X as $\frac{d}{ds}\delta_s|_{s=1}$, and locally, $X = t\partial_t$. We say a function u on G is homogeneous of degree w if $X(u) = wu$. Observe that $f \in \varepsilon[w]$ is homogeneous of degree w on G. Similarly, we can define a function on $\widetilde{G}$ to be homogeneous of degree w by same criterion.

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From above viewpoint, we know that the conformal Laplacian is in fact an operator acting on functions which are homogeneous of degree w on G. Given $f \in \varepsilon[w]$, the main idea is to find an extension of f on $\widetilde{G}$ such that $\widetilde{\Delta}\widetilde{f} = 0$, where $\widetilde{\Delta}$ is the Laplace operator for ambient metric, and $\widetilde{f}$ denotes the extension of f. One try to solve it by power series expansion; however, it turns out that there is an obstruction at certain order, and the obstruction is a conformally invariant operator.

The procedure is somewhat mysterious, but it's inspired by [1]. When [1] do the formal power series expansion of $Ric(\widetilde{g}) = 0$ in even dimensions, there is an obstruction in certain orders. This obstruction is conformally invariant, and is known as the ambient obstruction. So it is sometimes fruitful when we have obstruction in power series expansion. Here's the lemma.

Lemma 6. *Let* $w = -\frac{1}{2}n + k$, *where* $k \in \mathbb{N}$. *Suppose* $f \in \varepsilon[w]$. *Let* $\widetilde{f}$ *be an extenision of* f *to* $\widetilde{G}$, *and* $\widetilde{f}$ *is homogeneous of degree* w. *Denote the Laplace operator on* $\widetilde{G}$ *as* $\widetilde{\Delta}$, *and* $Q = \widetilde{g}(X, X)$. *The extension* $\widetilde{f}$ *modulo* Q^k *is uniquely determined by* $\widetilde{\Delta}\widetilde{f} = 0$ *modulo* Q^{k-1}. *It's impossible to solve for* $\widetilde{\Delta}\widetilde{f} = 0$ *since there is an obstruction at order* Q^k. *The obstruction is* $Q^{1-k}\widetilde{\Delta}\widetilde{f}|_G$, *which is a conformal invariant operator from* $\varepsilon[w]$ *to* $\varepsilon[w - 2k]$.

We'll calculate the most simple example for $k = 1$.

The lemma seems to be weird, but note that $Q = 0$ on G, and that Q is homogeneous of degree 2 with respect to δ_s. So, Q is a defining function for G. In fact, $\widetilde{g}(X, X) = \widetilde{g}(t\partial t, t\partial t) = 2\rho t^2$ in local coordinates. Note that since t is postive, $Q = 0$ if and only if $\rho = 0$. The Taylor expansion around $Q = 0$ is basically the Taylor expansion around $\rho = 0$. Then what the lemma states can be viewed as concerning the Taylor expansion of $\widetilde{\Delta}\widetilde{f}$ with respect to ρ.

Let $f \in \varepsilon[w]$ and fix a conformal representative g. Denote $\phi(x) = f(1, x) \in C^\infty(M)$, then $f(t, x) = t^w\phi(x)$. An extension $\widetilde{f}$ homogeneous of degree w can be written as $\widetilde{f}(t, x, \rho) = t^w\phi(x, \rho)$, where $\phi(x, 0) = \phi(x)$. In terms of metric $\widetilde{g}$ in equation

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(4.1), we have

$$\widetilde{\Delta}\widetilde{f} = \widetilde{\Delta}\left(t^{w}\phi\right) = t^{w-2}\left[-2\rho\phi'' + \left(2w+n-2-\rho g^{ij}g'_{ij}\right)\phi' + \Delta\phi + \frac{1}{2}wg^{ij}g'_{ij}\phi\right]. \tag{4.3}$$

The differentiation is with respect to ρ, and $\phi = \phi(x,\rho)$. The metric defining Δ is $g_{ij}(x,\rho)\,dx^{i}dx^{j}$ with ρ fixed, so Δ is an operator on x alone.

For $k=1$, $Q^{1-k}\widetilde{\Delta}\widetilde{f}|_{G} = \widetilde{\Delta}\left(t^{w}\phi\right)|_{\rho=0}$. Apply equation (4.2), we have

$$\widetilde{\Delta}\left(t^{w}\phi\right)|_{\rho=0} = -t^{w-2}\left[-\Delta\phi + \frac{n-2}{4(n-1)}R\phi\right].$$

So we get the conformal Laplacian, which is a a conformal invariant operator from $\varepsilon[w]$ to $\varepsilon[w-2]$. In general, it's quite difficult to compute higher order GJMS operators, but for the flat case, it's simple. If g_{ij} are constant, then for $w = -\frac{1}{2}n + k$, we have

$$\widetilde{\Delta}\left(t^{w}\phi\right) = t^{w-2}\left[-2\rho\phi'' + 2(k-1)\phi' + \Delta\phi\right].$$

For $k=2$, $\widetilde{\Delta}\left(t^{w}\phi\right) = 0$ modulo Q^{1}. Therefore, $\widetilde{\Delta}\left(t^{w}\phi\right)|_{\rho=0} = 0$, we get $\Delta\phi = 2\phi'|_{\rho=0}$. The conformally invariant operator is an obstruction at order Q^{2}. In other words, $\widetilde{\Delta}\left(t^{w}\phi\right)$ modulo Q^{2} is nonzero. The obstruction is the first order Taylor coefficient. $\partial_{\rho}\left(\widetilde{\Delta}\left(t^{w}\phi\right)\right)|_{\rho=0} = t^{w-2}\left[\partial_{\rho}(\Delta\phi)\right]|_{\rho=0} = t^{w-2}\left[\frac{1}{2}\left(\Delta^{2}\phi\right)\right]$. So the conformally invariant operator at $k=2$ for flat metric is $\Delta^{2}\phi$. Similarly, by induction, one can show that $\Delta^{k}\phi$ is a GJMS operator for every $k \in \mathbb{N}$.

Finally, we'd like to verify that Paneitz operator is the GJMS operator for $k=2$. Paneitz operator [15] has the following form:

$$\Delta^{2}\phi + \nabla_{j}\left[\left(\frac{-4}{n-2}R^{ij} + \frac{n^{2}-4n+8}{2(n-1)(n-2)}g^{ij}R\right)\nabla_{i}\phi\right] +$$
$$\left[\frac{n-4}{4(n-1)}\Delta R - \frac{n-4}{(n-2)^{2}}R^{ij}R_{ij} + \frac{(n-4)\left(n^{3}-4n^{2}+16n-16\right)}{16(n-1)^{2}(n-2)^{2}}R^{2}\right]\phi.$$

From the lemma, for $k=2$, $\widetilde{\Delta}\widetilde{f} = 0$ modulo Q^{1}. Therefore, $\widetilde{\Delta}\widetilde{f}|_{\rho=0} = 0$. By equation

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(4.3), we have

$$\phi'(x,0) = -\frac{1}{2}\left[\Delta\phi - \frac{n-4}{4(n-1)}R\phi\right]. \tag{4.4}$$

We know it's impossible to solve for $\widetilde{\Delta}\widetilde{f} = 0$ modulo Q^2 by lemma, and the obstruction is the conformally invariant operator. The obstruction is

$$\partial_\rho\left(\widetilde{\Delta}\left(t^w\phi\right)\right)|_{\rho=0} = t^{w-2}\left[-g^{ij}g_{ij}^{'}\phi' + \left[\partial_\rho\left(\Delta\phi\right)\right] + \frac{1}{2}w\partial_\rho\left(g^{ij}g_{ij}^{'}\phi\right)\right]|_{\rho=0}.$$

By equation (4.4), $\partial_\rho\left(\Delta\phi\right)|_{\rho=0} = -\frac{1}{2}\left[\Delta^2\phi - \frac{n-4}{4(n-1)}\Delta\left(R\phi\right)\right]$. Apply equation (4.4) and (4.2), and via Bianchi identity, we have $\nabla_j R_k^j = \frac{1}{2}\nabla_k R$. From these equations, we can arrange the conformally invariant operator into the form of Paneitz operator.

At last, we remark that Branson first defined Q-curvature by GJMS operators. In n dimensions, the constant term of P_k can be denoted as $(n/2-k)\,Q_k$, and the Q-curvature in even dimension n is then defined as $Q_{n/2}$. Moreover, from the transformation law for GJMS operators, it can be deduced that if $\widehat{h} = e^{2u}h$, and the corresponding Q curvatures are $\widehat{Q_1}$ and Q_2. Then

$$P_{n/2}u + Q_2 = e^{nu}\widehat{Q_1}.$$

4.2 Scattering theory

First of all, we would like to introduce the idea of Poincaré metric. Poincaré metric is introduced with ambient metric in [1]. It is a higher dimension metric which is constructed from a conformal manifold $(M,[h])$. Basically,in [1], it is showed that the Poincaré metric associated to a conformal manifold can be constructed from the regarding ambient metic, and vice versa. Since we can construct GJMS operators from the ambient metric, it is natural to ask whether we can construct it directly from the Poincaré metrc. To do it, the key role involves some scattering theory.

Let $(M,[h])$ be a conformal manifold. We can construct a $(n+1)$ dimensional manifold X with $\partial X = M$, and r is a definig function for M. For any conformal representative $h \in [h]$, there exists a metric g on X, such that g is an asymptotic solution to the Einstein

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equation $Ric\,(g) = -ng$. To be more specific, one can find a conformally compact metric g on X with conformal infinity $[h]$ satisfying

$$Ric\,(g) + ng = \left\{ \begin{array}{ll} O\,(r^{\infty}) & n \text{ odd} \\ O\,(r^{n-2}) & n \text{ even} \end{array} \right. .$$

By fixing the conformal representative, the solution g mod $O\,(r^{\infty})$ for n odd (or g mod $O\,(r^{n-2})$ for n even) is unique up to diffeomorphism. These metrics are called as Poincaré metrics associated to $[h]$.

The construction of Poincaré metric is basically the same as the metric expansion we have introduced in previous section. Since [1] has proved the existence, we can simply assume g as $r^{-2}\,(g_r + dr^2)$, and solve g_r in terms of a given boundary metric just as before.

Next, we introduce the idea of scattering matrix.

A scattering matrix is an operator originated from physics. The idea is that every wave can be decomposed into incoming waves and outgoing waves. When a beam of waves E collides into a energy barrier, E will scatter into incoming waves F and outgoing waves G. It is known that a solution under a Hamiltonian operator can be decomposed into linear combinations of eigenstates. Therefore, in above picture, we define the scattering matrix $S\,(s)$ as the operator maps the eigenstate at energy s^2 in F to the eigenstate at energy s^2 in G. For example, consider the Laplace operator in R^n. For $(\Delta - s^2)\,u\,(x) = 0$, there is a unique solution to this equation such that as $|x| \to \infty$, $u\,(x) = Fe^{is|x|} + Ge^{-is|x|} + O\left(|x|^{-\frac{n+1}{2}}\right)$ [18]. Then the scattering matrix is defined by $S\,(s)\,F = G$.

With above preliminaries, we now state some important results done by [5]. The main focus of their works is studying the asymptotic solutions of

$$(\Delta_g - s\,(n-s))\,u = O\,(r^{\infty})\,, \tag{4.5}$$

where g is a given Poincaré metric. To solve above equations, they found a special family of operators $\wp\,(s)$. Take $f \in \varepsilon\,[s-n]$. They construct a meromorphic family of operators

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$\wp(s)$ from $\varepsilon[s-n]$ to $C^{\infty}(X^{o})$ for Re s> $n/2$ with following properties.

$$(\Delta_g - s(n-s))\wp(s) f = 0$$

$$\begin{aligned} \wp(s) f &= r^{n-s}F + r^{s}G && \text{if } s \notin n/2+\mathbb{N} \\ \wp(s) f &= r^{n/2-k}F + Hr^{n/2+k}\log r && \text{if } s = n/2+k,\, k\in\mathbb{N}, \end{aligned}$$

where $F, G, H \in C^{\infty}(X)$ and $F|_M = f$. Besides, $\wp(n)1 = 1$.

Motivated by the solution for $(\Delta - s^2)u$, one may regard F as the incoming data, and G as the outgoing data. Then define the scattering matrix by $S(s) f = G|_M$. From above setup, they have following results:

1. Derive GJMS operators from a given Poincaré metric as a coefficient in the solution to equation (4.5).
2. Find the relationship between scattering matrix and GJMS operators.
3. Define Q-curvature via scattering matrix.

The corresponding theorems for above three results are stated below.

Lemma 7. *Let (X, g) be a Poincare matric associated to $(M, [h])$ and let $f \in C^{\infty}(M)$. k is a positive integer, and $k \leq \frac{n}{2}$ if n is even. There exists solution to equation (4.5) for $s = n/2+k$ of the following form*

$$u = r^{n/2-k}\left(F + Hr^{2k}\log r\right),$$

where F, $H \in C^{\infty}(X)$, and $F|_M = f$. F mod $O\left(r^{2k}\right)$ and $H|_M$ are determined by a conformal representative formally.

$$H|_M = -2c_k P_k f, \quad c_{k=}(-1)^k\left[2^{2k}k!(k-1)!\right]^{-1},$$

where P_k are GJMS operators.

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Theorem 4. *Let $S(s)$ be the scattering matrix of (X, g). With same assumptions as above, and $(n/2)^2 - k^2$ is not an L^2 eigenvalue of Δ_g. Then $S(s)$ has simple pole at $s = n/2 + k$, and*

$$c_k P_k = -\textit{Residue}_{s=n/2+k} S(s).$$

Theorem 5. *With same notations as above, the scattering matrix has following properties:*

$$S(n)\,1 = 0 \qquad \textit{if } n \textit{ odd.}$$
$$S(n)\,1 = c_{n/2} Q \qquad \textit{if } n \textit{ even.}$$

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Chapter 5

Application

This chapter serves as a brief application for previous chapters. Recall that in first chapter, we introduce the special defining function to construct a suitable coordinate, and then, we use this coordinate to construct renormalized volume. Later, we state some some basic facts for GJMS operators and scattering theory. Now based on what we have, we show an application from [6], in which they give a new definition for Branson's Q-curvature in even dimensional conformal geometry and prove a theorem concerning renormalized volume and Q-curvature from [5]. They have the following theorems.

Theorem 6. *[6] Let (X, g) be be a compact $(n+1)$-manifold with $\partial X = M$, where g is the Poincaré metric on X with conformal infinity $(M, [h])$. Fix a conformal representative h with defining function r. Then there is a unique function $U \in C^\infty(X^o)$ solving*

$$\Delta_g U = n$$

with the following asymptotics

$$U = \begin{cases} \log r + A + Br^n & n \text{ odd} \\ \log r + A + Br^n \log r & n \text{ even,} \end{cases}$$

where A and B are smooth functions on X, and even functions in r after mod $O(r^\infty)$.

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Besides,

$$A|_M = 0$$
$$B|_M = -2c_{n/2}Q \quad if\ n \text{ is even,}$$

where $c_{k=}(-1)^k \left[2^{2k}k!\,(k-1)!\right]^{-1}$.

The idea of proving this theorem can be divided into two steps. First of all, one considers the solution of $\Delta_g U = n$ near the boundary M mod some finite order terms. In first step, they show that the asymptotic expansions of U can be determined by fixing a conformal representative. In second step, they use solutions of $(\Delta_g - s(n-s))\,u$ from [5] to deduce theorem 6. We now briefly sketch the proof in the following.

Step 1 In this step, we only discuss near the boundary, so we take $X = M \times [0,1)$. First, one proves the following lemma.

Lemma 8. *[6] Let g be the Poincaré metric on X with conformal infinity $(M,[h])$. Fix a conformal representative h with defining function r.*

1. *Suppose n is even, then there is a unique function U mod $O(r^n)$ solving*

$$\Delta_g U = n + O\left(r^{n+1}\log r\right)$$

with the following form

$$U = \log r + A + Br^n \log r + O(r^n),$$

where $A, B \in C^\infty(X)$, $A|_M = 0$. The function A mod $O(r^n)$ and $B|_M$ are formally determined by h.

2. *Suppose n is odd, then there is a unique function U mod $O(r^\infty)$ solving*

$$\Delta_g U = n + O(r^\infty)$$

with the following form

$$U = \log r + A,$$

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where $A \in C^{\infty}(X), A|_M = 0$. The function A mod $O(r^{\infty})$ is formally determined by h.

Proof. (sketch) Fix a conformal representative h, and recall that $g = r^{-2}(h_r + dr^2)$ with $h_0 = h$. Then

$$\Delta_g = -(r\partial_r)^2 + \left(n - \frac{r}{2}tr_{h_r}(\partial_r h_r)\right) r\partial r + r^2\Delta_{h_r}.$$

Note that $\Delta_g U = n$ is equivalent to $\Delta_g(U - \log r) = \frac{r}{2}tr_{h_r}(\partial_r h_r)$. Observe that for $a_j \in C^{\infty}(M)$, we have $\Delta_g(a_j r^j) = j(n-j)a_j r^j + O(r^{j+1})$. Take a_0 and $a_1 = 0$, by induction, we can find a_j such that

$$\Delta_g\left(\Sigma_{1<j<n}a_j r^j\right) = \frac{r}{2}tr_{h_r}(\partial_r h_r) + r^n E,$$

where $E \in C^{\infty}(X)$. Finally, we may eliminate $r^n E$ by introducing the term $Br^n \log r$. □

Step 2 We now use some of the facts in chapter 3 from [5]. For convenience, we summarize it below. We have a family of operators $\wp(s)$ such that

$$(\Delta_g - s(n-s))\wp(s)f = 0.$$

If $s \notin n/2 + \mathbb{N}$, $\wp(s)f = r^{n-s}F + r^s G$, where $F|_M = f$ and $\wp(n)1 = 1$. $S(s)f = G|_M$. Furthurmore, $S(n)1 = 0$ if n is odd, and $S(n)1 = c_{n/2}Q$ if n is even.

Now, take $f = 1$, consider $(\Delta_g - s(n-s))\wp(s)1 = 0$. Differentiation with respect to s at $s = n$. Get

$$\Delta_g\left[-\frac{d}{ds}\wp(s)1|_{s=n}\right] = n.$$

Therefore, $-\frac{d}{ds}\wp(s)1|_{s=n}$ is a solution to $\Delta_g U = n$.

Since $\wp(s)f = r^{n-s}F + r^s G$,

$$U = (F\log r - F' - Gr^n \log r - r^n G')|_{s=n}.$$

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Via above properties, the form of U follows. Due to step 1, up to r^n terms, U is formally determined by h. So the uniqueness follows from that the fact that there is no L^2 harmonic functions.

With theorem 6, we have following applications to the coefficient L in the volume expansion.

Theorem 7. *[6] If n is even, then*

$$L = 2c_{n/2} \int_M Q dvol_h.$$

Proof. Take a small number $r_0 > 0$. Use Green's identity, we have

$$\int_{\varepsilon<r<r_0} \Delta_g U dvol_g = \varepsilon^{1-n} \int_{r=\varepsilon} \partial_r U dvol_{h_\varepsilon} - r_0^{1-n} \int_{r=r_0} \partial_r U dvol_{h_{r_0}}.$$

Recall that

$$dvol_{h_r} = \left(1 + v^{(2)} r^2 + (even\ powers) + v^{(n)} r^n + ...\right) dvol_h.$$

Substitute $U = \log r + A + B r^n \log r + O(r^n)$ and compare the coefficient of $\log \epsilon$ on both sides of equation. We have

$$-nL = n \int_M B dvol_h.$$

The rest follows from theorem 6. □

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