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漢米爾頓瑞曲流與可微分球定理
A Survey on Hamilton's Ricci Flow and Differentiable Sphere Theorem

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國立臺灣大學碩士學位論文
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A survey on Hamilton's Ricci Flow and Differentiable Sphere Theorem

本論文係 陳柏傑 君（學號 R02221004）在國立臺灣大學數學系完成之碩士學位論文，於民國 104年 06月24日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

在黎曼幾何中，一個令人關切的問題是如何分類有正截曲率的流形。在 1951 年，H.E. Rauch 介紹了 curvature pinch 這個想法，並提出如果一個簡單連通的流形他的截曲率都界在 (1, 4] 之間，那是否同胚於一個 n 維球。這問題在 1960 年，被 M.Berger 和 W. Klingenberg 利用比較技巧解決了。而之後留下了另一個問題是這種流形是否會微分同胚於一個 n 維球，這猜想又稱為可微分球定理。本文主要是在整理 S. Brendle 和 R. Schoen 在 2009 年利用 Hailton's Ricci flow 解決可微分球定理的工作。

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英文摘要

A central problem in Riemannian geometry concerns the classification of manifolds of positive sectional curvature. In 1951, H.E. Rauch introduced the notion of curvature pinching for Riemannian manifolds and posed the question of whether a simply connected manifold M^n whose sectional curvatures all lie in the interval $(1, 4]$ is necessarily homeomorphic to the sphere $\mathbf{S}^n$. This was proven by using comparison techniques due to M. Berger and W. Klingenberg around 1960. However, this theorem leaves open the question of whether M is diffeomorphic to $\mathbf{S}^n$. This conjecture is known as the Differentiable Sphere Theorem. The goal of this survey is to present this work via Hamilton's Ricci flow due to S. Brendle and R. Schoen around 2009.

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1 Introduction

In this survey, we will follow the work of S. Berndle and R. Schoen to prove the Differentiable Sphere Theorem [1]. First, we will prove the Hamilton's convergence result for the Ricci flow. Second, we will use the Hamilton's convergence result to prove an example in dimension three. In the end, we will consider some curvature tensor which has nonnegative isotropic curvature to construct the pinching set, and then we can prove the Differentiable Sphere Theorem by appling the Hamilton's convergence result.

2 Preliminary and Background Knowledge

The main idea to prove the Differentiable Sphere Theorem is to construct a pinching set, and so we can apply the Hamilton's convergence criterion. Here, we give the definition of Piniching set first.

Definition 2.1. *Let R be an algebraic curvature tensor on $\mathbb{R}^n$, and let $\delta \in (0,1)$. We say that R is strictly δ-pinched if $0 < \delta K(\pi_1) < K(\pi_2)$ for all two-dimensional planes $\pi_1, \pi_2 \subset \mathbb{R}^n$. Moreover, we say that R is weakly δ-pinched if $0 \leq \delta K(\pi_1) \leq K(\pi_2)$ for all two-planes $\pi_1, \pi_2 \subset \mathbb{R}^n$*

Definition 2.2. *A set $F \subset \mathscr{C}_B(\mathbb{R}^n)$ is called a pinching set if the following conditions are met:*

(1) *F is closed, convex, and $O(n)$-invariant.*

(2) *F is invariant under the Hamilton ODE $\frac{d}{dt}R = R^2 + R^\#$. Where*

$$\begin{aligned} R^2(X,Y,Z,W) &= \sum_{p,q=1}^{n} R(X,Y,e_p,e_q)R(Z,W,e_p,e_q) \\ R^\#(X,Y,Z,W) &= 2\sum_{p,q=1}^{n} R(X,e_p,Z,e_q)R(Y,e_p,W,e_q) \\ &- 2\sum_{p,q=1}^{n} R(X,e_p,W,e_q)R(Y,e_p,Z,e_q) \end{aligned}$$

for all vectors $X,Y,Z,W \in V$. For simplify, we denote $Q(R) = R^2 + R^\#$

(3) *For each $\delta \in (0,1)$, the set $\{R \in F : R$ is not weakly δ-piniched $\}$ is bounded.*

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We now describe a theorem about the curvature condition which is preserved by Ricci flow whenever the corresponding set $F \subset \mathscr{C}_B(\mathbb{R}^n)$ is convex and invariant under the Hamilton's ODE $\frac{d}{dt}R = Q(R)$.

Theorem 2.1 (R. Hamilton [2])**.** *Assume that $F \subset \mathscr{C}_B(\mathbb{R}^n)$ is closed, convex, $O(n)$-invariant, and invariant under the Hamilton ODE. Moreover, suppose that M is a compact manifold of dimension n, and $g(t), t \in [0,T)$, is a solution to the Ricci flow on M with the property that $R_{(p,0)} \in F_{(p,0)}$ for all points $p \in M$. Then $R_{(p,t)} \in F_{(p,t)}$ for all points $p \in M$ and all $t \in [0,T)$.*

By Theorem 2.1, we have the following two lemmas.

Lemma 2.1. *Given any real number $\delta \in (0,1)$, we can find a positive constant C such that*

$$K_{\min}(p,t) \geq \delta K_{\max}(p,t) - C$$

for all points $p \in M$ and all $t \in [0,T)$.

Lemma 2.2. $T < \infty$ *and* $\limsup_{t\to T} K_{\max}(t) = \infty$.

In section 3, we need some derivative estimates so we introduce the following corollary.

Corollary 2.1. *Let M be a compact manifold of dimension n. Let $g(t)$, $t \in [0,\tau]$, be a solution to the Ricci flow on M satisfying*

$$\sup_M |R_{g(t)}| \leq \tau^{-1}$$

for all $t \in [0,\tau]$. Moreover, let H be a smooth tensor field satisfying

$$\frac{\partial}{\partial t} H = \Delta H + R * H$$

and

$$\sup_M |H| \leq \Lambda$$

for all $t \in [0,\tau]$. Given any integral $m \geq 1$, we can find a positive constant C such that

$$\sup_M |D^m H|^2 \leq C\Lambda^2 \tau^{-m}$$

for all $t \in [\frac{\tau}{2},\tau]$.

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In section 5, to constructing pinching set, we will consider some nonnegative isotropic curvatures.

Definition 2.3. *Assume that V is a vector space of dimension $n \geq 4$ equipped with an inner product. Then, an algebraic curvature tensor $R \in \mathscr{C}_B(V)$ is said to have nonnegative isotropic curvature if*

$$R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4) +$$
$$R(e_2, e_3, e_2, e_3) + R(e_2, e_4, e_2, e_4) - 2R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset V$.

Proposition 2.1 (M. Micallef, J.D. Moore [3])**.** *Let $R \in \mathscr{C}_B(V)$ be a n algebraic curvature tensor on V. Then the following statements are equivalent:*

(i) R has nonnegative isotorpic curvature.

(ii) We have $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$ for all vectors $\zeta, \eta \in V^{\mathbb{C}}$ satisfying $g(\zeta, \zeta) = g(\zeta, \eta) = g(\eta, \eta) = 0$.

By the definition of nonnegative isotropoic curvature, we can easily show that this curvature condition implies nonnegative scalar curvature.

Proposition 2.2 (M. Micallef, M, Wang [4])**.** *Let V be a vector space of dimension $n \geq 4$ equipped with an inner product. Moreover, let R be a algebraic curvature tensor on V with nonnegarive isotropic curvatue. Then the scalar curvature of R is nonnegative. Moreover, if the Ricci tensor of R is equal to zero, then $R = 0$.*

Proof. Let $\{e_1, \ldots, e_n\}$ be an arbitrary orthonormal basis of V. Since R has nonnegative isotropic curvature, we have

$$R(e_i, e_k, e_i, e_k) + R(e_i, e_l, e_i, e_l) + R(e_j, e_k, e_j, e_k) + R(e_j, e_l, e_j, e_l) \geq 0$$

whenever $i, j, k, l \in \{1, \ldots, n\}$ are mutually distinct. Summation over $l \in \{1, \ldots, n\} \backslash \{i, j, k\}$ yields

$$\mathrm{Ric}(e_i, e_i) + \mathrm{Ric}(e_j, e_j) - 2R(e_i, e_j, e_i, e_j)$$
$$+(n-4)(R(e_i, e_k, e_i, e_k) + R(e_j, e_k, e_j, e_k)) \geq 0$$

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if $i, j, k \in \{1, \ldots, n\}$ are mutually distinct. In the next step, we take the sum over $k \in \{1, \ldots, n\} \backslash \{i, j\}$. This implies

$$\mathrm{Ric}(e_i, e_i) + \mathrm{Ric}(e_j, e_j) - 2R(e_i, e_j, e_i, e_j) \geq 0$$

if $i, j \in \{1, \ldots, n\}$ are distinct. Summation over $j \in \{i, \ldots, n\} \backslash \{i\}$ yields

$$(n-4)\,\mathrm{Ric}(e_i, e_i) + \mathrm{scal} \geq 0$$

for all $i \in \{1, \ldots, n\}$. This implies scal ≥ 0. From this, the first statement follows.

In order to prove the second statement, we assume that Ric $= 0$, then we conclude that R has nonpositive sectional curvature. From this, we deduce that $R = 0$. □

3 Hamilton's convergence criterion

In this section, we consider M be a compact manifold of dimension $n \geq 3$, and let g_0 be a metric on M with positive scalar curvature. Let $g(t)$, $t \in [0, T)$, be the unique maximal solution to the Ricci flow with initial metric g_0. We also assume there exists a pinching set $F \subset \mathscr{C}_B(\mathbb{R}^n)$ such that $R_{(p,0)} \in F_{(p,0)}$ for all $p \in M$. Then by Theorem 2.1 we can get the following result.

Lemma 3.1. *Let t_k be a sequence of times such that $\lim_{k\to\infty} t_k = T$ and $K_{\max}(t_k) \geq \frac{1}{2}\sup_{t\in[0,t_k]} K_{\max}(t)$ for all k. Then*

$$\liminf_{k\to\infty} \frac{K_{\min}(t_k)}{K_{\max}(t_k)} \geq 1,$$

where K_{min}, $K_{\max}$ is defined by

$$K_{\max}(t) = \sup_{p\in M} K_{\max}(p, t)$$
$$K_{\min}(t) = \inf_{p\in M} K_{\min}(p, t)$$

Proposition 3.1. *We have*

$$\frac{K_{\min}(t)}{K_{\max}(t)} \to 1$$

as $t \to T$.

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Proof. Suppose the assertion is false. Then there exists a sequence of time τ_k such that $\lim_{k\to\infty}\tau_k = T$ and

$$\liminf_{k\to\infty}\frac{K_{\text{min}}(t)}{K_{\text{max}}(t)} < 1. \tag{1}$$

For each k, there exist a real number $t_k \in [0,\tau_k]$ such that

$$K_{\text{max}}(t_k) = \sup_{t\in[0,\tau_k]} K_{\text{max}}(t).$$

It follows from Lemma 2.2 that $\lim_{k\to\infty} K_{\text{max}}(t_k) = \infty$. From this, we deduce that $lim_{k\to\infty}t_k = T$. Using Lemma 3.1, we obtain

$$\liminf_{k\to\infty}\frac{K_{\text{min}}(t_k)}{K_{\text{max}}(t_k)} \geq 1.$$

In particular, we have $K_{\text{min}}(t_k) \geq \frac{1}{2}K_{\text{max}}(t_k)$ if k is sufficiently large. Since the minimum of the curvature is monotone increasing in time, we have

$$\inf_{x\in M}\text{scal}_{g(\tau_k)}(x) \geq \inf_{x\in M}\text{scal}_{g(t_k)}(x),$$

hence,

$$\inf_{x\in M} K_{\text{max}}(x,\tau_k) \geq \inf_{x\in M} K_{\text{min}}(x,t_k).$$

Putting these facts together, we conclude that

$$K_{\text{max}}(\tau_k) \geq K_{\text{min}}(t_k) \geq \frac{1}{2}K_{\text{max}}(t_k) = \frac{1}{2}\sup_{t\in[0,\tau_k]} K_{\text{max}}(t)$$

if k is sufficently large. Consequently, we have

$$\liminf_{k\to\infty}\frac{K_{\text{min}}(\tau_k)}{K_{\text{max}}(\tau_k)} \geq 1$$

by Lemma 3.1. This contradicts to the equation (1). □

We now estimate the scalar curvature of $g(t)$ in terms of time to blow-up.

Lemma 3.2. *We have*

$$(T-t)\sup_M \text{scal}_{g(t)} \to \frac{n}{2}$$

and

$$(T-t)\inf_M \text{scal}_{g(t)} \to \frac{n}{2}$$

as $t \to T$

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Proof. Fix a real number $\varepsilon > 0$. By Proposition 3.1, there exist a number η such that $|\mathring{\mathrm{Ric}}|^2 \leq \frac{\varepsilon}{n}\,\mathrm{scal}^2$ on $M \times [T-\eta, T)$. Using the evolution equation of scalar curvature, we obtain

$$\frac{\partial}{\partial t}\,\mathrm{scal} = \Delta\,\mathrm{scal} + |\,\mathrm{Ric}\,|^2 \leq \Delta\,\mathrm{scal} + \frac{2(1+\varepsilon)}{n}\,\mathrm{scal}^2$$

on $M \times [T-\eta, T)$. Hence, it follows from the maximum principle that

$$\frac{n}{2\sup_M \mathrm{scal}_{g(\tau)}} + (1+\varepsilon)(\tau - t) \geq \frac{n}{2\sup_M \mathrm{scal}_{g(t)}}$$

for all $t \in [T-\eta, T)$ and all $\tau \in [t, T)$. We now pass to the limit as $\tau \to T$.

By Lemma 2.2, we have $\limsup_{\tau\to T} \sup_M \mathrm{scal}_{g(\tau)} = \infty$. This implies

$$(1+\varepsilon)(T-t) \geq \frac{n}{2\sup_M \mathrm{scal}_{g(t)}}$$

for all $t \in [T-\eta, T)$. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\liminf_{t\to T}\left[(T-t)\sup_M \mathrm{scal}_{g(t)}\right] \geq \frac{n}{2}.$$

Using Proposition 3.1, we obtain

$$\liminf_{t\to T}\left[(T-t)\inf_M \mathrm{scal}_{g(t)}\right] \geq \frac{n}{2}. \tag{2}$$

On the other hand, if we have the following inequality

$$T - t \leq \frac{n}{2\inf_M \mathrm{scal}_{g(t)}} \tag{3}$$

for all $t \in [0, T)$. This implies

$$\limsup_{t\to T}\left[(T-t)\inf_M \mathrm{scal}_{g(t)}\right] \leq \frac{n}{2}.$$

Using Proposition 3.1, we conclude that

$$\limsup_{t\to T}\left[(T-t)\sup_M \mathrm{scal}_{g(t)}\right] \leq \frac{n}{2}. \tag{4}$$

Combining (2) and(4) together, the assertion follows.

Now we need to prove the inequality (3).

For any fix $t_0 \in [0, T)$, let $\inf_M \mathrm{scal}_{g(t_0)} = \alpha > 0$ and $\tau = \min\{T, \frac{n}{2\alpha} + t_0\}$. We define a function $h : M \times [t_0, \tau) \to \mathbb{R}$ by

$$h = \mathrm{scal} - \frac{n\alpha}{n - 2\alpha(t - t_0)}.$$

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Using the evolution equation of scalar curvature, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}h &= \Delta h + 2|\operatorname{Ric}|^2 - \frac{2}{n}\left(\frac{n\alpha}{n-2\alpha(t-t_0)}\right)^2 \\ &\geq \Delta h + \frac{2}{n}\operatorname{scal}_{g(t)}^2 - \frac{2}{n}\left(\frac{n\alpha}{n-2\alpha(t-t_0)}\right)^2 \\ &= \Delta h + \frac{2}{n}\left(\operatorname{scal}_{g(t)} + \frac{n\alpha}{n-2\alpha(t-t_0)}\right)h\end{aligned}$$

on $M \times [t_0, \tau)$. By the definition of α, we have $h(p, t_0) \geq 0$ for all $p \in M$. Hence, the maximum principle implies that $h(p, t) \geq 0$ for all $p \in [0, \tau)$. From this we deduce that $T - t_0 \leq \frac{n}{2\alpha}$ and we complete the proof. $\square$

Lemma 3.3. *Fix a real number $\alpha \in (0, \frac{1}{n-1})$. Given any integer $m \geq 1$, we can find a positive constant C such that*

$$\sup_M |D^m \overset{\circ}{\operatorname{Ric}}_{g(t)}|^2 \leq C(T-t)^{2\alpha-m-2}$$

for all $t \in [0, T)$. Moreover, we have

$$(D_X \operatorname{Ric})(Y, Z) = (D_X \overset{\circ}{\operatorname{Ric}})(Y, Z) + \frac{2}{n-2}\sum_{k=1}^{n}(D_{e_k}\overset{\circ}{\operatorname{Ric}})(X, e_x)g(Y, Z)$$

for all vector fields X, Y, Z. We replace the trace Ricci tensor to the Ricci tensor, that is

$$\sup_M |D^m \operatorname{Ric}_{g(t)}|^2 \leq C(T-t)^{2\alpha-m-2}$$

for all $t \in [0, T)$.

Proof. we first claim the following inequality

$$\sup_M |\overset{\circ}{\operatorname{Ric}}_{g(t)}| \leq C_1(T-t)^{\alpha-1}$$

for some positive constant C_1.

proof of the claim. Fix a positive real number ε such that $(1 - \frac{1}{n-1} + n\varepsilon) \leq 1 - \alpha$. By Lemma 3.2, we can find a real number $\eta \in (0, T)$ such that

$$(T-t)\operatorname{scal} \leq \frac{n(1+\varepsilon)}{2}$$

then by Proposition 3.1, we have

$$\left|R_{ijkl} - \frac{1}{n(n-1)}\operatorname{scal}(g_{ik}g_{jl} - g_{il}g_{jk})\right|^2 \leq \varepsilon^2 \operatorname{scal}^2$$

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on $M \times [T-\eta, T)$. This implies

$$\sum_{i,j,k,l=1}^{n} \left(R_{ijkl} - \frac{1}{n(n-1)}\,\mathrm{scal}(g_{ik}g_{jl} - g_{il}g_{jk}\right) \mathring{\mathrm{Ric}}^{ik}\mathring{\mathrm{Ric}}^{jl} \leq \varepsilon\,\mathrm{scal}\,|\mathring{\mathrm{Ric}}|^2$$

hence

$$\sum_{i,j,k,l=1}^{n} R_{ijkl}\mathring{\mathrm{Ric}}^{ik}\mathring{\mathrm{Ric}}^{jl} \leq \left(-\frac{1}{n(n-1)} + \varepsilon\right) \mathrm{scal}\,|\mathring{\mathrm{Ric}}|^2$$

on $M \times [T-\eta, T)$. Consequently, we have

$$\begin{aligned}
\sum_{i,j,k,l=1}^{n} R_{ijkl}\mathring{\mathrm{Ric}}^{ik}\,\mathrm{Ric}^{jl} &\leq \left(\frac{1}{n} - \frac{1}{n(n-1)} + \varepsilon\right) \mathrm{scal}\,|\mathring{\mathrm{Ric}}|^2 \\
&\leq \left(1 - \frac{1}{1-n} + n\varepsilon\right) \frac{1+\varepsilon}{2(T-t)}|\mathring{\mathrm{Ric}}|^2 \\
&\leq \frac{1-\alpha}{2(T-t)}|\mathring{\mathrm{Ric}}|^2
\end{aligned}$$

on $M \times [T-\eta, T)$. Using the evolution equation of Ricci tensor, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathring{\mathrm{Ric}}|^2) &= \Delta(|\mathring{\mathrm{Ric}}|^2) - 2|D\mathring{\mathrm{Ric}}|^2 + 4\sum_{i,j,k,l=1}^{n} R_{ijkl}\mathring{\mathrm{Ric}}^{ik}\,\mathrm{Ric}^{jl} \\
&\leq \Delta(|\mathring{\mathrm{Ric}}|^2) - 2|D\mathring{\mathrm{Ric}}|^2 + \frac{1-\alpha}{2(T-t)}|\mathring{\mathrm{Ric}}|^2
\end{aligned}$$

on $M \times [T-\eta, T)$. By the maximum principle, the function $(T-t)^{2-2\alpha}|\mathring{\mathrm{Ric}}|^2$ is uniformly bounded from above. □

We have the trace-free Ricci tensor satisfies the evolution equation of the form

$$\frac{\partial}{\partial t}\mathring{\mathrm{Ric}}_{g(t)} = \Delta\mathring{\mathrm{Ric}}_{g(t)} + R * \mathring{\mathrm{Ric}}_{g(t)}.$$

Hence, Corollary 2.1 implies that

$$\sup_M |D^m\mathring{\mathrm{Ric}}_{g(t)}|^2 \leq C_2(T-t)^{2\alpha-m-2}$$

for all $t \in [0, T)$. □

Lemma 3.4. *Fix a real number $\alpha \in (0, \frac{1}{n-1})$. There exists a positive constants C such that*

$$\sup_M \left| \mathrm{Ric}_{g(t)} - \frac{1}{2(T-t)}g(t) \right|^2 \leq C(T-t)^{2\alpha-2}$$

for all $t \in [0, T)$.

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Proof. By Lemma 3.3, we can find a positive constant C_1 such that $|\mathring{\mathrm{Ric}}| \leq C_1(T-t)^{2\alpha-2}$. Moreover, it follows from Lemma 3.3 that

$$|\Delta\,\mathrm{scal}\,| \leq C_2(T-t)^{\alpha-2}$$

for some positive constant C_2. And we have

$$\begin{aligned} &\frac{\partial}{\partial t}\,\mathrm{scal} = \Delta\,\mathrm{scal} + 2|\,\mathrm{Ric}\,|^2 \\ \Rightarrow\quad &\left|\frac{\partial}{\partial t}\,\mathrm{scal} - \frac{2}{n}\right| \leq |\Delta\,\mathrm{scal}\,| + |\mathring{\mathrm{Ric}}|^2 \leq C_3(T-t)^{\alpha-2}. \end{aligned}$$

Using Lemma 3.2, we obtain

$$\left|\frac{\partial}{\partial t}\left(\frac{1}{\mathrm{scal}}\right) + \frac{2}{n}\right| \leq C_4(T-t)^{\alpha}$$

for some positive constant C_4. This implies

$$\left|\frac{1}{\mathrm{scal}} + \frac{2}{n}(T-t)\right| \leq \frac{C_4}{\alpha+1}(T-t)^{\alpha+1}$$

Applying Lemma 3.2 again, we conclude that

$$\left|\,\mathrm{scal} + \frac{n}{2(T-t)}\right| \leq C_5(T-t)^{\alpha-1}$$

for some positive constant C_5. Since $|\mathring{\mathrm{Ric}}|^2 \leq C_1(T-t)^{2\alpha-2}$, the assertion follows. □

Proposition 3.2. *Assume $\frac{\partial}{\partial t}\tilde{g}(t) = \omega(t)$ and satisfy $\int_0^T \sup_M |D^m\omega(t)|_{\tilde{g}(t)}dt < \infty$ for $m = 0, 1, 2, \ldots$. Then, as $t \to T$, the metrics $\tilde{g}(t)$ converge in C^∞ to a smooth limit metirc $\bar{g}$.*

Now we can prove the Hamilton's convergence criterion.

Theorem 3.1 (R.Hamilton [2])**.** *Let M be a compact manifold of dimension $n \geq 3$, and let g_0 be a Riemannian metric on M with positive scalar curvature. Suppose that there exists a pinching set $F \subset \mathscr{C}_B(\mathbb{R}^n)$ such that the curvature tensor of g_0 lies in F for all points $p \in M$. Finally, let $g(t)$, $t \in [0, T)$, be the unique maximal solution to the Ricci flow with initial metric g_0. Then, as $t \to T$, the metrics $\frac{1}{2(n-1)(T-t)}g(t)$ converge in C^∞ to a metric of constant sectional curvature* 1.

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Proof. Consider the rescaled metrics $\tilde{g} = \frac{1}{2(n-1)(T-t)} g(t)$.

Then $\frac{\partial}{\partial t}\tilde{g}(t) = \omega(t)$,where

$$\omega(t) = -\frac{1}{(n-1)(T-t)}\left(\mathrm{Ric}_{g(t)} - \frac{1}{2(T-t)}g(t)\right),$$

and

$$\Big|\mathrm{Ric}_{g(t)} - \frac{1}{2(T-t)}g(t)\Big|^2 = \Big|-(n-1)(T-t)\omega(t)\Big|^2 = (n-1)^2(T-t)^2\Big|\omega(t)\Big|^2 = \frac{1}{4}\Big|\omega(t)\Big|^2_{\tilde{g}(t)}.$$

We now fix a real number $\alpha \in (0, \frac{1}{n-1})$. By Lemma 3.4, we have

$$\sup_M \Big|\omega(t)\Big|^2_{\tilde{g}(t)} \leq C(T-t)^{2\alpha-2}$$

so

$$\sup_{t\in[0,T)}\left[(T-t)^{1-\alpha}\sup_M |\omega(t)|_{\tilde{g}(t)}\right] < \infty$$

Moreover, it follows from Lemma 3.3 that

$$\sup_{t\in[0,T)}\left[(T-t)^{1-\alpha}\sup_M |D^m\omega(t)|_{\tilde{g}(t)}\right] < \infty$$

for $m = 1, 2, \ldots$. By Proposition 3.2, the metrics $\tilde{g}(t)$ converge in C^∞ to a limit metirc $\bar{g}$ on M. By Lemma 3.2, the scalar curvature of $\bar{g}$ is equal to $n(n-1)$. Moreover, it follows from Proposition 3.1 that $\bar{g}$ has constant sectional curvature 1. □

4 Curvature Pinching in Dimension Three

In this section, we give an example in three-manifold with positive Ricci curvature. Let V be a three-dimensional vector space equipped with an inner product, and let R be an algebraic curvature tensor on V. Then

$$R_{ijkl} = (\mathrm{Ric}_{ik}\, g_{jl} - \mathrm{Ric}_{il}\, g_{jk} - \mathrm{Ric}_{jk}\, g_{il} + \mathrm{Ric}_{jl}\, g_{ik}) - \frac{1}{2}\,\mathrm{scal}(g_{ik}g_{jl} - g_{il}g_{jk}),$$

Define

$$A_{ij} = \mathrm{scal}\,\delta_{ij} - 2\,\mathrm{Ric}_{ij}\,.$$

Let $R(t)$, $t \in [0, T)$, is a solution of the ODE $\frac{d}{dt}R(t) = Q(R)$ in $\mathscr{C}_B(\mathbb{R}^3)$.

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Proposition 4.1.

$$\frac{d}{dt}\operatorname{Ric}_{ij} = -4\operatorname{Ric}_{ij}^2 + 3\operatorname{scal}\ \operatorname{Ric}_{ij} + 2|\operatorname{Ric}|^2\delta_{ij} - \operatorname{scal}^2\delta_{ij} \tag{5}$$

$$\frac{d}{dt}A_{ij} = 2A_{ij}^2 - \operatorname{tr}(A)A_{ij} - \frac{1}{2}|A|^2\delta_{ij} + \frac{1}{2}\operatorname{tr}(A)^2\delta_{ij} \tag{6}$$

Proof. Since $R(t)$ is the solution of Hamilton's ODE, we have

$$\frac{d}{dt}\operatorname{Ric}_{ij} = 2\sum_{p,q=1}^{3} R_{ipjq}\operatorname{Ric}_{pq},$$

and putting the fact that

$$R_{ipjq} = (\operatorname{Ric}_{ij}\delta_{pq} - \operatorname{Ric}_{iq}\delta_{pj} - \operatorname{Ric}_{pj}\delta_{iq} + \operatorname{Ric}_{pq}\delta_{ij}) - \frac{1}{2}\operatorname{scal}(\delta_{ij}\delta_{pq} - \delta_{iq}\delta_{pj}).$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\operatorname{Ric}_{ij} &= 2\sum_{p,q=1}^{3}\left[(\operatorname{Ric}_{ij}\delta_{pq} - \operatorname{Ric}_{iq}\delta_{pj} - \operatorname{Ric}_{pj}\delta_{iq} + \operatorname{Ric}_{pq}\delta_{ij}) - \frac{1}{2}\operatorname{scal}(\delta_{ij}\delta_{pq} - \delta_{iq}\delta_{pj})\right]\operatorname{Ric}_{pq} \\
&= 2\operatorname{Ric}_{ij}\operatorname{scal} - 4\sum_{p,q=1}^{3}\operatorname{Ric}_{iq}\delta_{jp} + 2|\operatorname{Ric}|^2\delta_{ij} - \operatorname{scal}^2\delta_{ij} + \operatorname{Ric}_{ij}\operatorname{scal} \\
&= -4\operatorname{Ric}_{ij}^2 + 3\operatorname{scal}\ \operatorname{Ric}_{ij} + 2|\operatorname{Ric}|^2\delta_{ij} - \operatorname{scal}^2\delta_{ij}
\end{aligned}$$

Combining equation (5) and

$$\frac{d}{dt}\operatorname{scal} = 2|\operatorname{Ric}|^2,$$

we have

$$\begin{aligned}
\frac{d}{dt}A_{ij} &= \frac{d}{dt}(\operatorname{scal}\delta_{ij} - 2\operatorname{Ric}_{ij}) \\
&= 2|\operatorname{Ric}|^2\delta_{ij} - 2\left(-4\operatorname{Ric}_{ij}^2 + 3\operatorname{scal}\ \operatorname{Ric}_{ij} + 2|\operatorname{Ric}|^2\delta_{ij} - \operatorname{scal}^2\delta_{ij}\right) \\
&= 8\operatorname{Ric}_{ij}^2 - 6\operatorname{scal}\operatorname{Ric}_{ij} - 2|\operatorname{Ric}|^2\delta_{ij} + 2\operatorname{scal}^2\delta_{ij} \\
&= 2A_{ij}^2 - \operatorname{scal}A_{ij} - \frac{1}{2}|A|^2\delta_{ij} + \frac{1}{2}\operatorname{scal}^2\delta_{ij}.
\end{aligned}$$

Since $\operatorname{tr}(A) = \operatorname{scal}$,we prove the equation (6). □

Corollary 4.1. *The eigenvalues of $A(t)$ satisfy*

$$\begin{aligned}
\frac{d}{dt}\lambda_1 &= \lambda_1(t)^2 + \lambda_2(t)\lambda_3(t), \\
\frac{d}{dt}\lambda_2 &= \lambda_2(t)^2 + \lambda_3(t)\lambda_1(t), \\
\frac{d}{dt}\lambda_3 &= \lambda_3(t)^2 + \lambda_1(t)\lambda_2(t)
\end{aligned}$$

for all $t \in [0, T)$.

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Now we construct a set which is invariant under Hamilton's ODE.

Proposition 4.2. *Fix a real number* $\delta \in [0,1]$. *Then the set*

$$\{R \in \mathscr{C}_B(\mathbb{R}^3) : \lambda_1 + \lambda_2 \geq 2\delta\lambda_3 \text{ and } (\lambda_3 - \lambda_1)^{1+\delta} \leq N(\lambda_1 + \lambda_2)\}$$

is invariant under the ODE $\frac{d}{dt}R = Q(R)$.

Proof. Suppose that $\lambda_1 + \lambda_2 = 2\delta\lambda_3$. Using Corollary 4.1, we obtain

$$\begin{aligned} \frac{d}{dt}(\lambda_1 + \lambda_2 - 2\delta\lambda_3) &= \lambda_1^2 + \lambda_2^2 - 2\delta\lambda_1\lambda_2 + (\lambda_1 + \lambda_2 - 2\delta\lambda)\lambda_3 \\ &= (1-\delta)(\lambda_1^2 + \lambda_2^2) + \delta(\lambda_1 - \lambda_2)^2 \geq 0, \end{aligned}$$

so $\lambda_1 + \lambda_2 \geq 2\delta\lambda_3$ is preserved by the Hamilton ODE. Using Corollary 4.1 again, we obtain

$$\frac{d}{dt}\log(\lambda_3 - \lambda_1) = \lambda_1 - \lambda_2 + \lambda_3 \leq \lambda_3$$

and

$$\frac{d}{dt}\log(\lambda_1 + \lambda_2) = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 + \lambda_2} + \lambda_3 \geq \frac{1}{2}(\lambda_1 + \lambda_2) + \lambda_3 \geq (1+\delta)\lambda_3$$

Putting these facts together, we conclude that

$$\frac{d}{dt}[(1+\delta)\log(\lambda_3 - \lambda_1) - \log(\lambda_1 + \lambda_2)] \leq 0$$

Therefore, the ratio $\dfrac{(\lambda_3 - \lambda_1)^{1+\delta}}{\lambda_1 + \lambda_2}$ is monotone decreasing. From this, the assertion follows. □

Proposition 4.3. *Let* K *be a compact subset of* $\mathscr{C}_B(\mathbb{R}^3)$. *We assume every algebraic curvature tensor* $R \in K$ *has positive Ricci curvature. Then there exist a pinching set* $F \subset \mathscr{C}_B(\mathbb{R}^3)$ *such that* $K \subset F$.

Proof. By assumption, we have

$$K \subset \{R \in \mathscr{C}_B(\mathbb{R}^3) : \lambda_1 + \lambda_2 > 0\}.$$

Since K is compact, we can find a real number $\delta \in (0,1)$ and $N > 0$ such that

$$K \subset \{R \in \mathscr{C}_B(\mathbb{R}^3) : \lambda_1 + \lambda_2 \geq 2\delta\lambda_3 \text{ and } (\lambda_3 - \lambda_1)^{1+\delta} \leq N(\lambda_1 + \lambda_2)\}$$

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We now define

$$F = \{R \in \mathscr{C}_B(\mathbb{R}^3) : \lambda_1 + \lambda_2 \geq 2\delta\lambda_3 \text{ and } (\lambda_3 - \lambda_1)^{1+\delta} \leq N(\lambda_1 + \lambda_2)\}$$

The functions $R \mapsto \lambda_1 + \lambda_2 - 2\delta\lambda_3$ and $R \mapsto N(\lambda_1 + \lambda_2) - (\lambda_3 - \lambda_1)^{1+\delta}$ are concave. Consequently, the set F is convex. Moreover, the set F is invariant under the ODE $\frac{d}{dt}R = Q(R)$ by Proposition 4.2. Therefore, F is a pinching set. □

By Proposition 4.3, we can find a pinching set that contains every algebraic curvature tensor with positive Ricci curvature. By Theorem 3.1, we can prove the convergence result in dimension three.

Theorem 4.1 (R.Hamilton [5]). *Let M be a compact three-manifold and let g_0 be a Riemannian metric on M with positive Ricci curvature. Let $g(t)$, $t \in [0, T)$, be the unique maximal solution to the Ricci flow with initial metric g_0. Then, as $t \to T$, the metrics $\frac{1}{4(T-t)}g(t)$ converge in C^∞ to a metirc of constant sectional curvature 1.*

5 1/4-pinched Differentiable Sphere Theorem

In this section, we consider some curvature condition which can be used to prove the Differentiable Sphere Theorem.

Theorem 5.1 (S.Brendle,R. Schoen [6]). *Let M be a compact manifold of dimension $n \geq 4$, and let g_0 be a Riemannian metric on M. Assume that (M, g_0) is strictly $1/4$-pinched in the pointwise sense. Let $g(t)$, $t \in [0, T)$, be the unique maximal solution to the Ricci flow with initial metrics g_0. Then, as $t \to T$, the metric $\frac{1}{2(n-1)(T-t)}g(t)$ converge in C^∞ to a metric of constant sectional curvature 1.*

Definition 5.1. *Let $R \in \mathscr{C}_B(V)$ be an algebraic curvature tensor on V, then define algebraic tensors $\tilde{R} \in \mathscr{C}_B(V \times \mathbb{R})$ and $\hat{R}, S \in \mathscr{C}_B(V \times \mathbb{R}^2)$ by*

$$\begin{aligned} \tilde{R}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) &= R(v_1, v_2, v_3, v_4) \\ \hat{R}(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) &= R(u_1, u_2, u_3, u_4) \\ S(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) &= R(u_1, u_2, u_3, u_4) + \langle z_1, z_3\rangle\langle z_2, z_4\rangle - \langle z_1, z_4\rangle\langle z_2, z_3\rangle \end{aligned}$$

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for all vectors $\tilde{v}_j = (v_j, y_j) \in V \times \mathbb{R}$ *and* $\hat{u}_j = (u_j, z_j) \in V \times \mathbb{R}^2$.

According to Definition 2.3 and Proposition 2.1, we can have the following propositions.

Proposition 5.1. *The following statements are equivalent:*

(i) $\tilde{R}$ *has nonnegative isotropic curvature.*

(ii) We have

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4)$$
$$+R(e_2, e_3, e_2, e_3) + \lambda^2 R(e_2, e_4, e_2, e_4) - 2\lambda R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthnormal four-frames $\{e_1, e_2, e_3, e_4\} \subset V$ *and all* $\lambda \in [0, 1]$.

(iii) We have $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$ *for all vectors* $\zeta, \eta \in V^{\mathbb{C}}$ *satisfying* $g(\zeta, \zeta) g(\eta, \eta) - g(\zeta, \eta)^2 = 0$.

Proposition 5.2. *The following statements are equivalent:*

(i) $\hat{R}$ *has nonnegative isotropic curvature.*

(ii) We have

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4)$$
$$+\mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) - 2\mu\lambda R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthnormal four-frames $\{e_1, e_2, e_3, e_4\} \subset V$ *and all* $\lambda, \mu \in [0, 1]$.

(iii) We have $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$ *for all vectors* $\zeta, \eta \in V^{\mathbb{C}}$.

Proposition 5.3. *The following statements are equivalent:*

(i) S *has nonnegative isotropic curvature.*

(ii) We have

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4)$$
$$+\mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4)$$
$$-2\lambda\mu R(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2) \geq 0$$

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for all orthnormal four-frames $\{e_1, e_2, e_3, e_4\} \subset V$ *and all* $\lambda \in [0, 1]$.

(iii) We have $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) + |g(\zeta, \zeta)g(\eta, \eta) - g(\zeta, \eta)^2| \geq 0$ *for all vectors* $\zeta, \eta \in V^{\mathbb{C}}$.

Proposition 5.4. *The following cones*

$$\tilde{C} = \{R \in \mathscr{C}_B(\mathbb{R}^n) : \tilde{R} \text{ has nonegative isotropic curvature}\}$$

$$\hat{C} = \{R \in \mathscr{C}_B(\mathbb{R}^n) : \hat{R} \text{ has nonegative isotropic curvature}\}$$

$$G = \{R \in \mathscr{C}_B(\mathbb{R}^n) : S \text{ has nonegative isotropic curvature}\}$$

are invariant under the Hamilton ODE $\frac{d}{dt}R = Q(R)$.

Proposition 5.5. *We have* $\hat{C} \subset G \subset \tilde{C}$ *and* $G \subset \{R \in \mathscr{C}_B(\mathbb{R}^2) : R + I \in \hat{C}\}$

Before proving the Differentiable Sphere Theorem, we recall a fact that can help us to prove the relation between 1/4-pinched and the cone of $\tilde{C}$ first.

Property 5.1 (M. Berger [7]). *Let* (M, g) *be a Riemannian manifold, and let* p *be an arbitrary point in* M. *Moreover, suppose that* $\underline{\kappa} \leq K(\pi) \leq \overline{\kappa}$ *for all two-dimensional planes* $\pi \subset T_pM$. *Then*

$$R(e_1, e_2, e_3, e_4) \leq \frac{2}{3}(\overline{\kappa} - \underline{\kappa})$$

for all orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_pM$.

Proposition 5.6. *Let* (M, g) *be a Riemannian manifold of dimension* $n \geq 4$. *Then:*

(i) If (M, g) *is weakly* $1/4$ *- pinched in the pointwise sense, then the curvature tensor of* (M, g) *lies in the cone* $\hat{C}$ *for all points* $p \in M$.

(ii) If (M, g) *is strictly* $1/4$ *- pinched in the pointwise sense, then the curvature tensor of* (M, g) *lies in the interior of cone* $\hat{C}$ *for all points* $p \in M$.

Proof. For each point $p \in M$, we deonte $K_{\max}(p)$ as the maximum sectional curvature at the point p. Similary, we denote $K_{min}(p)$ as the minimum sectional curvature at point p. If (M, g) is weakly $1/4$ - pinched in the pointwise sense, we have $0 \leq K_{\max}(p) \leq K_{min}(p)$ for all points $p \in M$. Then using the Property 5.1, we obtain

$$R(e_1, e_2, e_3, e_4) \leq \frac{2}{3}(K_{\max}(p) - K_{min}(p)) \leq 2K_{min}(p)$$

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for all points $p \in M$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$.

Thus, we conclude that

$$\begin{aligned} & R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ & + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2\mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda\mu R(e_1, e_2, e_3, e_4) \\ & \geq (1 + \lambda^2 + \mu^2 + \lambda^2\mu^2 - 4\lambda\mu) K_{min}(p) \\ & = ((1-\lambda\mu)^2 + (\lambda-\mu)^2) K_{min}(p) \geq 0 \end{aligned}$$

for all points $p \in M$, all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$, and all $\lambda, \mu \in [0, 1]$. By Proposition 5.2, the curvature tensor of (M, g) lies in the cone $\hat{C}$ for all points $p \in M$. This proves the first statement. The second statement can be proved by using the similarly argument. □

By Proposition 5.6, we only need to construct a pinching set which lies in the interior of the cone $\hat{C}$.

Definition 5.2. *For any fix integer $n \geq 4$ and each $s > 0$, we define a cone $\hat{C}(s) \subset \mathscr{C}_B(\mathbb{R}^n)$ by*

$$\hat{C}(s) = \left\{ \ell_{a(s),b(s)}(R) : R \in \hat{C} \text{ and } \mathrm{Ric} \geq \frac{\delta(s)}{n} \mathrm{scal}\,\mathrm{id} \right\},$$

where $\ell_{a,b} : \mathscr{C}_B(\mathbb{R}^n) \to \mathscr{C}_B(\mathbb{R}^n)$ is a linear transformation defined by

$$\ell_{a,b}(R) = R + b\,\mathrm{Ric} \owedge \mathrm{id} + \frac{1}{n}(a-b)\,\mathrm{scal}\,\mathrm{id} \owedge \mathrm{id}$$

and

$$2a(s) = \begin{cases} \frac{2s+(n-2)s^2}{1+(n-2)s^2} & \text{for } 0 < s \leq \frac{1}{2}, \\ 2s & \text{for } s > \frac{1}{2}, \end{cases}$$

$$2b(s) = \begin{cases} 2s & \text{for } 0 < s \leq \frac{1}{2}, \\ 1 & \text{for } s > \frac{1}{2}, \end{cases}$$

$$\delta(s) = \begin{cases} 1 - \frac{1}{1+(n-2)s^2} & \text{for } 0 < s \leq \frac{1}{2}, \\ 1 - \frac{4}{n-2+8s} & \text{for } s > \frac{1}{2}. \end{cases}$$

Note that $a(s)$, $b(s)$, and $\delta(s)$ are continuous functions of s.

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Proposition 5.7. *Let K be a compact subset of $\mathscr{C}_B(\mathbb{R}^n)$. Moreover, let $F \subset \mathscr{C}_B(\mathbb{R}^n)$ be the smallest set containing K which is closed, convex, $O(n)$-invariant, and invariant under the Hamilton ODE. If*

$$F \subset \{R \in \mathscr{C}_B(R^n) : R + h_0 I \in \hat{C}(s_0)\}$$

for suitable real numbers $s_0 > 0$ and $h_0 > 0$, then F is a pinching set.

Corollary 5.1. *Let K be a compact subset of $\mathscr{C}_B(\mathbb{R}^n)$ which is contained in the interior of the cone $\hat{C}$. Then there exists a pinching set $F \subset \mathscr{C}_B(\mathbb{R}^n)$ such that $K \subset F$.*

***Proof*.** Let $F \subset \mathscr{C}_B(\mathbb{R}^n)$ be the smallest set containing K which is closed, convex, $O(n)$-invariant, and invariant under the Hamilton ODE. Since K is contained in the interior of the cone $\hat{C}$, we can find a real number $s_0 > 0$ such that $K \subset \hat{C}(s_0)$. The cone $\hat{C}(s_0)$ is closed convex, $O(n)$-invariant, and invariant under the Hamilton ODE. Consequently, we have $F \subset \hat{C}(s_0)$. Hence, Proposition 5.7 implies that F is a pinching set. □

***Proof of Theorem 5.1*.** By Proposition 5.6, so we know the curvature tensor of (M, g) lies in the interior of cone $\hat{C}$ for all point $p \in M$. Then by Proposition 6. there exists a pinching set $F \subset \mathscr{C}_B(\mathbb{R}^n)$ such that $K \subset F$. Final, by Theorem 3.1, we can finish the proof. □

6 An Improved Convergence Theroem

From Proposition 5.6, we can improve the Differentiable Sphere Theroem if there exists a pinching set contained in the cone $\tilde{C}$. In this section, our work is to construct the pinching set.

Proposition 6.1. *Consider a pair of real numbers a, b such that $2a = 2b + (n-2)b^2$ and $b \in (0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}]$. Then the set $\ell_{a,b}(G)$ is invariant under the ODE $\frac{d}{dt}R = Q(R)$.*

Proposition 6.2. *Let K be a compact subset of $\mathscr{C}_B(\mathbb{R}^n)$ which is contained in the interior of the cone $\tilde{C}$. Then there exists a pinching set $F \subset \mathscr{C}_B(\mathbb{R}^n)$ such that $K \subset F$.*

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Proof. Let $F \subset \mathscr{C}_B(\mathbb{R}^n)$ be the smallest set containing K which is closed, convex, $O(n)$-invariant, and invariant under the Hamilton ODE. By assumption, the set K is contained in the interior of the cone $\tilde{C}$. Using Proposition 5.1, we obtain

$$R(e_1,e_3,e_1,e_3)+\lambda^2 R(e_1,e_4,e_1,e_4)$$
$$+\mu^2 R(e_2,e_3,e_2,e_3)+\lambda^2\mu^2 R(e_2,e_4,e_2,e_4)-2\lambda\mu R(e_1,e_2,e_3,e_4)>0$$

for all $R \in K$, all orthonormal four-frames $\{e_1,e_2,e_3,e_4\} \subset \mathbb{R}^n$, and all pairs $\lambda,\mu \in [0,1]$ satisfying $(1-\lambda^2)(1-\mu^2)=0$. Consequently, we can find a positive real number N such that

$$R(e_1,e_3,e_1,e_3)+\lambda^2 R(e_1,e_4,e_1,e_4)+\mu^2 R(e_2,e_3,e_2,e_3)+\lambda^2\mu^2 R(e_2,e_4,e_2,e_4)$$
$$-2\lambda\mu R(e_1,e_2,e_3,e_4)+N(1-\lambda^2)(1-\mu^2)>0$$

for all $R \in K$, all orthonormal four-frames $\{e_1,e_2,e_3,e_4\} \subset \mathbb{R}^n$, and all $\lambda,\mu \in [0,1]$.

Without loss of generality, we may assume that $N=1$. By Proposition 5.3, K is contained in the interior of the set G. We next consider a pair of real numbers a,b such that $2a=2b+(n-2)b^2$ and $b \in (0,\frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}]$. By continuity, we can choose b sufficiently small so that $K \subset \ell_{a,b}(G)$. The set $\ell_{a,b}(G)$ is closed, convex, and $O(n)$-invariant. Moreover, the set $\ell_{a,b}(G)$ is the invariant set under the Hamilton ODE by Proposition 6.1. Thus, we conclude that $F \subset \ell_{a,b}(G)$.

We now consider the cones $\hat{C}(s)$, $s>0$. We can find a real number $s_0>0$ such that $\ell_{a,b}(\hat{C}) \subset \hat{C}(s_0)$. Using Proposition 5.5, we obatin

$$\ell_{a,b}(R)+(1+2(n-1)b)I=\ell_{a,b}(R+I)\in \ell_{a,b}(\hat{C})\subset \hat{C}(s_0)$$

for all $R \in G$. Thus, we conclude that

$$F \subset \ell_{a,b}(G) \subset \{R \in \mathscr{C}_B(\mathbb{R}^n) : R+(1+2(n-1)a)I \in \hat{C}(s_0)\}.$$

Hence, it follows from Proposition 5.7 that F is a pinching set. □

Moreover, with the Proposition 6.2, then we can have the following Theorem.

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Theorem 6.1 (S.Brendle [8]). *Let M be a compact manifold of dimension $n \geq 4$, and let g_0 be a Riemannian metric on M. We assume that the curvature tensor of g_0 lies in the interior of the cone $\tilde{C}$ for all points $p \in M$. Let $g(t)$, $t \in [0, T)$, be the unique maximal solution to the Ricci flow with initial metric g_0. Then, as $t \to T$, the metrics $\frac{1}{2(n-1)(T-t)} g(t)$ converge in C^∞ to a metric of constant sectional curvature* 1.

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