國立臺灣大學理學院物理學研究所

碩士論文

Department of Physics
College of Science
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Master Thesis

藉由宇宙大尺度結構形成對 f(R)重力理論之制約

Constraining f(R) Gravity via Structure Formation of the Universe

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中華民國 100 年 6 月
June, 2011

誌謝

能完成這篇論文，我要特別感謝我的指導教授陳丕燊老師，在台大物理所碩士班這三年來，我除了跟著他接觸了宇宙學的多種領域外，還在他身上見識到一位專家該有的學養，老師身兼數職，在那麼忙的情況下，依然堅持盡量空出時間與我們那麼多學生討論，此外往往都會工作到相當晚，我在耳濡目染之下，多少有學到老師的研究態度，老師謝謝您！再來是要感謝顧哲安老師，他邏輯性的思考對我影響甚深，一切都從「原理原則」出發。最重要的是，他願意每周都花數小時的時間與我們學生討論，不管是作物理、導算式，甚至是寫程式，這份論文內的工作有一半以上都是出自他的點子，真是非常感謝！另外要感謝杜蕙慈老師在程式以及重力透鏡知識方面的協助，當初是她建議我去修改 CMBFAST，而程式本身也多得她協助，真是相當感激！另外感謝師大的李沃龍教授，在數值與宇宙微擾方面的各種建議，大大地幫助了我的研究進展！另外感謝淡江劉國欽教授，雖然之前沒機會讓他直接指導，但曾聽韋廷同學轉述了不少劉老師的意見，對我都相當實用！接下來是交大齊祖康教授，齊老師在我之前上台報告時常給一些建議，這些都是我今天能完成論文的助力之一。謝謝各位老師！

另外在同學方面，首先要感謝的當然是韋廷，沒有他之前做 f(R)打下的基礎，那麼我的研究過程必然寸步難行。再來是學弟則鈞，許多文獻發掘、程式技術的改進，都是他的功勞，且有他的幫忙，我才敢肯定我的筆算是可信的。再來是念恩學長，在重力組時多得他指點，在電腦程式系統方面也是有他才能順利進行。再來謝謝建廷、寶瑜在暗能組對我的協助。再來是彥緯學長，感謝你不嫌棄地對我的物理知識多方指導！感謝哲民，從大學認識你到現在，受了你不少幫助，謝謝你拉我進這領域！感謝玉樵學長與政道，教了我許多物理與雜學知識。感謝偉駿，協助我處理很多雜事。感謝承德學長，我的數值觀念和程式能力有一大半是靠他建立起來的，更不用說聽他講物理觀念，那更是受益匪淺。謝謝各位同學！

最後感謝我的父母、外婆與妹妹，有你們一路上在背後支持，才有今天的我。

中文摘要

我們研究 f(R)重力理論，一個替代暗能量去解釋晚期宇宙加速膨脹的理論，這之下的宇宙微擾演化。使用 GR 當作計算 f(R)重力下的宇宙微擾的早期近似是一個常規的方法，對於晚期宇宙，則使用辻川(稱作 Tsu)提出的近似方程去計算物質密度微擾。用 GR 加上 Tsu 去計算物質密度微擾及物質功率譜是常規的方法。在這篇論文中我們提出一個新的近似，「雙重微擾」(稱作 DP)，去計算早期宇宙微擾，而在晚期，我們使用 Tsu。對不同的 f(R)設計者模型和不同傅立葉模式下，我們研究其在方法 I(GR 加上 Tsu)和方法 II(DP 加上 Tsu)之間物質密度微擾與物質功率譜之差異。我們發現早期 f(R)重力的重力修正效應或許不可忽略。因此，我們的近似可以改善常規的方法。

關鍵字：重力修正理論、f(R)重力理論、f(R)設計者模型、大尺度結構、宇宙微擾

ABSTRACT

We investigate the evolution of the cosmological perturbations in $f(R)$ gravity, an alternative to dark energy for explaining the late-time cosmic acceleration. It is conventional to use GR as the approximation to calculate cosmological perturbations in $f(R)$ gravity at early times. For the late-time universe, it is to use the approximate formula proposed by Tsujikawa (termed *Tsu*) to calculate the matter density perturbation. The method with GR and *Tsu* is conventional to calculate the matter density perturbation and the matter power spectrum. In this thesis we propose a new approach, "double perturbation" (*DP*), to calculate cosmological perturbations at early times. For the late times, we use *Tsu*. For different designer $f(R)$ models, we study the difference between Method I (GR+*Tsu*) and Method II (*DP*+*Tsu*) in matter density perturbations and matter power spectra for different Fourier modes. For the shorter-wavelength Fourier modes we find that the effect of the gravity modification at early times in $f(R)$ gravity may not be negligible. We conclude that to be self-consistent, in the $f(R)$ theory one should employ the approximation presented in this thesis instead of that of GR in the treatment of the early-time evolution.

Keywords: modified gravity, $f(R)$ theory, designer $f(R)$, large scale structure, cosmological perturbations

Contents

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List of Figures

Chapter 1

Introduction

1.1 The Accelerating Universe

In 1998 the observations of type Ia supernovae (SNIa) suggested that the expansion of the Universe was decelerating at early times but accelerating since around the redshift z=0.5 [1–3]. Subsequent observations, including more detailed studies of supernovae and independent evidence from the cosmic microwave background (CMB) radiation, large-scale structure, and clusters of galaxies, have confirmed and firmly established this remarkable finding. Models attempting to explain the accelerating expansion include some form of dark energy, modified gravity [5] and the large-scale inhomogeneities [6]. In this chapter we will discuss dark energy [4] and $f(R)$ modified gravity [7], but not inhomogeneous cosmology.

The observations [8–12] suggest that our universe should be nearly flat and consist of 73% dark energy (DE) [13], 23% dark matter, 4% ordinary matter, and approximated 0.008% radiation. To explain the late-time cosmic acceleration, we consider that the dark energy has a negative pressure. At large scales the dark energy is "gravitationally repulsive" because of the negative pressure. So far, the ΛCDM model is the best-fit dark energy model for the observations of the cosmological evolution at the background level. However, the ΛCDM model still the coincidence and fine-tuning problems [14, 15]. To avoid these problem, people develop dynamical dark energy models, such as quintessence [16] and phantom energy [18]. These models involve the dynamical and potential energy of a scalar field. However, there is no dark energy model which can avoid all the problems [19]. See [20, 21] for a review.

1.2 $f(R)$ Modified Gravity

Another possibility to explain the late-time cosmic acceleration is the modified gravity. There are several models of modification to GR [22–25] (also see [5] for a review). In this thesis, we focus on $f(R)$ modified gravity (see [7] for a review). We modify the Einstein-Hilbert action of GR, a modification primarily manifesting only on cosmologically large scales.

For the viability of the $f(R)$ models the cosmological solutions therein should have a late-time de Sitter attractor at $R = R_1(> 0)$. The following conditions need to be satisfied [26]

- (i) $f_R > -1$ for $R \geq R_1$ (> 0), where $f_R \equiv df/dR$. This is required to avoid anti-gravity from ordinary matter.

- (ii) $f_{RR} > 0$ for $R \geq R_1$, $f_R \equiv df^2/dR^2$. This is required for the stability of cosmological perturbations, for the presence of a matter era, and for consistency with local gravity tests.

- (ii) $f(R) \to -2\Lambda$ for $R \gg R_0$, where R_0 is the Ricci scalar today. This is required for the consistency with local gravity tests and for the presence of the radiation and matter eras.

- (iv) $0 < Rf_{RR}/(1 + f_R)(r = -2) < 1$, where $r \equiv R(1 + f_R)/(f - R)$. This is required for the stability of the late-time de Sitter point.

In [26], the authors provide a general form of $f(R)$ models which can be consistent with both cosmological and local gravity constraints.

$$f(R) = R - \lambda R_c f_1(x)\,, \qquad x \equiv R/R_c\,, \tag{1.1}$$

where R_c (> 0) defines a characteristic value of the Ricci scalar R and λ is some positive free parameter.

The following models have been well studied:

- (A) $f_1(x) = x^p \quad (0 < p < 1)$ [28],
- (B) $f_1(x) = x^{2n}/(x^{2n}+1) \quad (n > 0)$ [29],
- (C) $f_1(x) = 1 - (1 + x^2)^{-n} \quad (n > 0)$ [30],
- (D) $f_1(x) = 1 - e^{-x}$ [31–33],
- (E) $f_1(x) = \tanh(x)$ [34].

The models in [29, 30, 34, 35] can satisfy both the cosmological and the local gravity constraints by using so-called chameleon mechanism [36]. The models in [29, 30] are cosmologically and locally viable and are more distinguishable from ΛCDM. The model in [29] is popular and has been discussed and constrained by many works. For the models in [31, 37, 38], both the inflation in the early universe and the onset of the recent accelerated expansion arise in these models in a natural, unified way. These three models [31, 37, 38] easily can pass local tests, such as the Newton law, the stability of the Earth-like gravitational solution and the very heavy mass for an additional scalar degree of freedom. By the cosmological and solar-system tests, [39] gives new constraints of $f(R)$ gravity.

1.3 Our work on $f(R)$ Theories

In our work we focus on the cosmological perturbations in the $f(R)$ theory. The background information from the observations, such as type Ia supernova, large-scale structure and CMB, is treated as the inputs, but not the output in the $f(R)$ theory test.

In practice, each of the $f(R)$ models should be close to ΛCDM at the background level. Consequently, measurements of cosmic expansion alone cannot distinguish f(R) gravity from dark energy, and additional independent measurements such as the cosmic structures are indispensable.

In this thesis we focus on the designer $f(R)$ model [27, 40]. Once we know the expansion history of the universe, we can design a $f(R)$ model which can mimic the required cosmic expansion at the background level.

To calculate the cosmological perturbations in $f(R)$ gravity, it is conventional to take GR as the approximation at early times. At late times, [41, 42] use matter-dominated and sub-horizon approximations to calculate the matter density perturbation. In our works, we calculate the cosmological perturbations in $f(R)$ gravity from the early universe to now. For the early times , we calculate the cosmological perturbations by two methods: 1. We use GR as an approximation to calculate the cosmological perturbations via CMBFAST, a code, written by U. Seljak and M. Zaldarriaga [43], for calculating the linear CMB anisotropy spectra based on integration over the sources along the photon past light cone. 2. We develop our numerical tool based on CMBFAST, which can solve the $f(R)$ field equations and the Boltzmann equations with our early-time approximation. For the late times, we use the matter-dominated and subhorizon approximations [41] to calculate the matter density perturbation. We also obtain the prediction of the matter power spectrum at z=0. Then we compare both the matter density perturbations and the matter power spectra from different approximations. And we discuss that if the conventional method is a good approximation or not. A part of our work is based on the master thesis [40] by Wei-Ting Lin.

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Chapter 2

Background Expansion

2.1 Dark Energy

In general relativity (GR), the Einstein-Hilbert action is[1]

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\, R + \sum_{fluid} S_{fluid}\,, \tag{2.1}$$

and taking the variation respect to $g^{\mu\nu}$,

$$\delta S = \frac{1}{16\pi G_N}\int d^4x\left\{\delta\left(\sqrt{-g}\,\right)R + \sqrt{-g}\;\delta R\right\} + \sum_{fluid}\delta S_{fluid}\,, \tag{2.2}$$

we use the least action principle to derive the field equation for GR, i.e.,

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0\,. \tag{2.3}$$

Then we define the energy-momentum tensor for cosmic fluids as

$$T_{\mu\nu(fluid)} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{(fluid)}}{\delta g^{\mu\nu}}\,, \tag{2.4}$$

[1]The speed of light c is set to unity.

and the Einstein field equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \sum_a T_{\mu\nu(a)} \ , \tag{2.5}$$

where the index a runs over the particle species. According to Eq. (2.5), the expansion should be decelerating if the subscript a only include the baryons, cold dark matter (CDM) and radiation. Since we find the early universe can be described well by GR. It implies us that there might be some modifications to GR (ie. modified gravity) or a new substance (ie. dark energy) in the subscript a.

For the background expansion of the universe we consider a homogeneous and isotropic space-time described by the flat Robertson-Walker metric

$$ds^2 = a^2(\tau)\{-d\tau^2 + d\vec{x}^2\}, \tag{2.6}$$

where the conformal time $\tau = t/a(\tau)$.

Dark energy posit a form of "gravitationally repulsive" stress-energy $T_{\mu\nu(DE)}$ in the universe. Eq. (2.5) becomes

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu(fluid)} + T_{\mu\nu(DE)}\right) \ . \tag{2.7}$$

For perfect fluid,

$$T^{\mu}{}_{\nu} = \mathrm{diag}\left(-\rho, P, P, P\right) \ . \tag{2.8}$$

Taking the trace part of Eq. (2.7), we obtain

$$\frac{\frac{d^2a}{dt^2}}{a} = \frac{-4\pi G_N}{3}\left(\rho_m + 3P_m + \rho_r + 3P_r + \rho_{DE} + 3P_{DE}\right) \ , \tag{2.9}$$

where the subscript m means the matter part, and r means the radiation part. The L.H.S. of Eq. (2.9) describes the acceleration of the universe. In the R.H.S., if the the DE term $(\rho_{DE} + 3P_{DE})$ negative enough at late time, this kind of dark energy model could explain the late-time acceleration.

2.2 $f(R)$ Theory

Starting from the modified Einstein-Hilbert action,

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\ \left[R + f(R)\right] + \sum_{fluid} S_{fluid}\ . \tag{2.10}$$

Following the steps as what we had done in Eq. (2.2-2.4), we obtain the modified Einstein field equations in the $f(R)$ theory,

$$\left(1+f_R\right)R_{\mu\nu} - \frac{1}{2}\left(R+f\right)g_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f_R = 8\pi G_N\sum_a T_{\mu\nu(a)}\ , \tag{2.11}$$

where $f_R \equiv df/dR$, and $f_{RR} \equiv df_R/dR$. To simplify the field equation, we define an effective energy-momentum tensor contributed from the deviation of GR. The effective energy-momentum relates to the effective dark energy. It originates from the modification to the geometry,

$$T_{\mu\nu(eff)} \equiv \frac{1}{8\pi G_N}\left[\frac{1}{2}fg_{\mu\nu} - \left(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f_R\right]\ . \tag{2.12}$$

Then the Einstein's field equations in the $f(R)$ theory is similar to that in GR,

$$G_{\mu\nu} = 8\pi G_N\left(T_{\mu\nu} + T_{\mu\nu(eff)}\right)\ . \tag{2.13}$$

The effective energy density is

$$\rho_{eff} \equiv \frac{1}{8\pi G_N}\left(\frac{1}{2}Rf_R - 3H^2f_R - \frac{f}{2} - \frac{3H}{a}\dot{f}_R\right)\ . \tag{2.14}$$

Dots denote derivatives with respect to τ, here $\dot{f}_R \equiv \partial f_R/\partial \tau$. The effective pressure is

$$P_{eff} \equiv \frac{1}{8\pi G_N}\left(\frac{1}{a^2}\ddot{f}_R + \frac{H}{a}\dot{f}_R - \frac{1}{6}Rf_R - H^2 f_R + \frac{f}{2}\right) . \tag{2.15}$$

We define the effective equation of state,

$$w_{eff} \equiv \frac{P_{eff}}{\rho_{eff}} = -\frac{1}{3} - \frac{2}{3}\,\frac{-\frac{1}{2a^2}\ddot{f}_R - \frac{1}{6}f + H^2 f_R}{-\frac{H}{a}\dot{f}_R - H^2 f_R + \frac{1}{6}Rf_R - \frac{1}{6}f} . \tag{2.16}$$

Taking the 00 part of Eq. (2.13), then we obtain the first effective Friedmann equation,

$$H^2 + H^2 f_R + \frac{f}{6} + \frac{H}{a}\dot{f}_R - \frac{1}{6}Rf_R = \frac{8\pi G_N}{3}\left(\rho_m + \rho_r\right) . \tag{2.17}$$

Taking the ii part of (2.13), then we obtain

$$\left(H^2 - \frac{R}{3}\right) + f_R\left(\frac{R}{6} + H^2\right) - \frac{f}{2} - \frac{1}{a^2}\ddot{f}_R - \frac{H}{a}\dot{f}_R = 8\pi G_N\left(P_m + P_r\right) . \tag{2.18}$$

After the linear combination of the above two equations, we obtain the second effective Friedmann equation,

$$\frac{\frac{d^2a}{dt^2}}{a} + \frac{f}{6} + \frac{1}{2a^2}\ddot{f}_R - H^2 f_R = \frac{-4\pi G_N}{3}\left(\rho_m + 3P_m + \rho_r + 3P_r\right) . \tag{2.19}$$

which describes the acceleration in the FLRW cosmology. Thus, if the part

$$\frac{f}{6} + \frac{1}{2a^2}\ddot{f}_R - H^2 f_R + \frac{4\pi G_N}{3}\left(\rho_m + 3P_m + \rho_r + 3P_r\right)$$

is negative at the present epoch, the $f(R)$ theory may explain the late-time acceleration.

2.3 Designer $f(R)$

In this thesis, we only test the designer $f(R)$ models. Here we give the following informations [27,40]:

- differential equation:

$$\frac{f''}{H_0^2} - \left(1 + \frac{E'}{2E} + \frac{R''}{R'}\right)\frac{f'}{H_0^2} + \left(\frac{12E'}{E} + \frac{3E''}{E}\right)\left(\frac{f}{6H_0^2} + E_{eff}\right) = 0 \ . \quad (2.20)$$

$$E \equiv \frac{H^2}{H_0^2} = \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + E_{eff}, \ \Omega_{m0} \equiv \frac{\rho_m}{\rho_{cr0}}, \ \Omega_{r0} \equiv \frac{\rho_r}{\rho_{cr0}}, \ E_{eff} \equiv \frac{\rho_{eff}}{\rho_{cr0}} \ .$$

The subscript zero means today's quantities, and the subscript i means the initial ones. The prime means d/d lna.

- initial conditions:

$$\frac{f(a_i)}{H_0^2} \simeq \frac{f_h(a_i)}{H_0^2} + \frac{f_p(a_i)}{H_0^2} = A_+ a_i^{p_+} + A_p E_{eff}(a_i)$$

$$\frac{f'(a_i)}{H_0^2} \simeq p_+ A_+ a_i^{p_+} - 3\left[1 + w_{eff}(a_i)\right] A_p E_{eff}(a_i)$$

$$p_+ = \frac{-b + \sqrt{b^2 - 4c}}{2} \ , E_{eff}(a_i) \equiv \frac{\rho_{eff}(a_i)}{\rho_{cr0}}$$

$$A_p = \frac{-6c}{-3w'_{eff}(a_i) + 9w_{eff}^2(a_i) + (18 - 3b)\, w_{eff}(a_i) + 9 - 3b + c}$$

$$b \equiv \frac{7 + 8r}{2\,(1 + r)} \ , \ c \equiv \frac{-3}{2\,(1 + r)} \ , \ r \equiv \frac{1}{a_i} \frac{\Omega_{r0}}{\Omega_{m0}} \ .$$

With given Ω_{m0}, Ω_{r0}, $w_{eff}(a)$, and f_{Ri}, A_+ is the only degree of freedom that should be fixed to obtain f. Here we do not consider the decaying mode solution of the differential equation, because that mode decays with time which means it grows backward in time. We should set the decaying mode to be close to zero in order to avoid large deviation from GR at early times. However, to consider full $f(R)$ solutions, we need to consider the growing mode. From the numerical solution, we find that the background evolution grows backward in time, which is not consistent with the background evolution that grows forward in time.

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Chapter 3

Perturbed $f(R)$ Evolution Equations

Recall that we have chosen FLRW metric and flat spacetime as our background metric. We consider the scalar metric perturbations in the synchronous gauge. Because we develop the numerical tool base on CMBFAST. In CMBFAST, all quantities are calculated in the synchronous gauge. The line element is given by

$$ds^2 = a^2(\tau)\{-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\}\,. \tag{3.1}$$

Notice that, in every gauge the scalar modes only have two degrees of freedom. Here we introduce two fields $h(\vec{k},\tau)$ and $\eta(\vec{k},\tau)$ in k-space and write the scalar modes of h_{ij} as a Fourier integral

$$h_{ij}(\vec{x},\tau) = \int d^3k e^{i\vec{k}\cdot\vec{x}} \left\{ \hat{k}_i\hat{k}_j h(\vec{k},\tau) + (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\, 6\eta(\vec{k},\tau) \right\}\,, \quad \vec{k} = k\hat{k}\,. \tag{3.2}$$

Let $\sum_a \rho_a\delta_a \equiv \sum_a \delta\rho_a = -\delta T^0{}_0$. The variables θ_a and σ_a are defined as

$$\sum_a (\rho_a + P_a)\theta_a \equiv ik^j\delta T^0{}_j\,, \qquad \sum_a (\rho_a + P_a)\sigma_a \equiv -(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\Sigma^i{}_j\,, \tag{3.3}$$

and $\Sigma^i_j \equiv T^i_j - \delta^i_j T^k_k/3$ denotes the traceless component of T^i_j. δ_a are the energy density perturbations. θ_a are the peculiar velocity divergences. σ_a are the shear perturbations. Each δ_a, θ_a and σ_a obeys the Boltzmann equations in the synchronous gauge [44].

3.1 Boltzmann Equations

This section shows the Boltzmann equations in the synchronous gauge [44], here we do not consider massive neutrino. The calculation of massive neutrino is so complicated that it is viewed as our future work. In numerical calculation, we only consider cold dark matter, baryon, photon and massless neutrino. Each equation is wring in Fourier space. These part is the same with the Boltzmann equations in *GR*.

3.1.1 Cold Dark Matter

$$\dot{\delta}_c = -\frac{1}{2}\dot{h} \,. \tag{3.4}$$

$$\theta_c = 0,\ \sigma_c = 0 \,.$$

3.1.2 Baryon

$$\begin{aligned} \dot{\delta}_b &= -\theta_b - \frac{1}{2}\dot{h} \,, \\ \dot{\theta}_b &= -\frac{\dot{a}}{a}\theta_b + c_s^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b) \,, \\ \sigma_b &= 0 \,. \end{aligned} \tag{3.5}$$

3.1.3 Photon

$$
\begin{aligned}
\dot{\delta}_\gamma &= -\frac{4}{3}\theta_\gamma - \frac{2}{3}\dot{h}\,, \\
\dot{\theta}_\gamma &= k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + an_e\sigma_T(\theta_b - \theta_\gamma)\,, \\
\dot{F}_{\gamma\,2} &= 2\dot{\sigma}_\gamma = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma\,3} + \frac{4}{15}\dot{h} + \frac{8}{5}\dot{\eta} - \frac{9}{5}an_e\sigma_T\sigma_\gamma + \frac{1}{10}an_e\sigma_T\left(G_{\gamma\,0} + G_{\gamma\,2}\right)\,, \\
\dot{F}_{\gamma\,l} &= \frac{k}{2l+1}\left[lF_{\gamma\,(l-1)} - (l+1)F_{\gamma\,(l+1)}\right] - an_e\sigma_T F_{\gamma\,l}\,, \quad l \geq 3\,, \\
\dot{G}_{\gamma\,l} &= \frac{k}{2l+1}\left[lG_{\gamma\,(l-1)} - (l+1)G_{\gamma\,(l+1)}\right] + an_e\sigma_T\left[-G_{\gamma\,l} + \frac{1}{2}\left(F_{\gamma\,2} + G_{\gamma\,0} + G_{\gamma\,2}\right)\left(\delta_{l0} + \frac{\delta_{l2}}{5}\right)\right]\,,
\end{aligned}
\tag{3.6}
$$

3.1.4 Massless Neutrino

$$
\begin{aligned}
\dot{\delta}_\nu &= -\frac{4}{3}\theta_\nu - \frac{2}{3}\dot{h}\,, \\
\dot{\theta}_\nu &= k^2\left(\frac{1}{4}\delta_\nu - \sigma_\nu\right)\,, \\
\dot{F}_{\nu\,2} &= 2\dot{\sigma}_\nu = \frac{8}{15}\theta_\nu - \frac{3}{5}kF_{\nu\,3} + \frac{4}{15}\dot{h} + \frac{8}{5}\dot{\eta}\,, \\
\dot{F}_{\nu\,l} &= \frac{k}{2l+1}\left[lF_{\nu\,(l-1)} - (l+1)F_{\nu\,(l+1)}\right]\,, \quad l \geq 3\,.
\end{aligned}
\tag{3.7}
$$

3.2 Metric Perturbations

We write down the field equations Eq. (2.11) in the synchronous gauge, where the perturbation quantities $\delta Q = \delta Q(\vec{k}, \tau)$.

For the zero-zero component, we let μ and ν equal to zero in Eq. (2.11). After a Fourier transformation,

$$
-\frac{24\pi G_N}{H^2}\sum_a \rho_a\delta_a = \left[3\left(1 + \frac{\dot{H}}{aH^2}\right) - \frac{k^2}{a^2H^2}\right]\chi - \frac{3}{aH}\dot{\chi} + (1 + f_R)\,\frac{2k^2}{a^2H^2}\eta
$$

$$+\left[\frac{3\dot{f}_R}{2k^2}\left(1-\frac{\dot{H}}{aH^2}\right)-3\frac{f_R\dot{H}}{k^2H}-\frac{12\pi G_N a}{k^2H}\rho_{eff}\left(1+w_{eff}\right)-\frac{1+f_R}{aH}\right]q$$

$$+\left(6\frac{1+f_R}{aH}+\frac{3\dot{f}_R}{a^2H^2}\right)\dot{\eta}+\frac{3\dot{f}_R}{2k^2aH}\dot{q}\ . \tag{3.8}$$

For the zero-j component, we let μ equal to zero and ν equal to j respectively in Eq. (2.11). After a Fourier transformation,

$$\frac{8\pi G_N a}{H}\sum_a \rho_a\theta_a\left(1+w_a\right)=k^2\chi-\frac{k^2\dot{\chi}}{aH}+\left[\frac{\dot{f}_R}{2}-f_R\frac{\dot{H}}{H}+\frac{4\pi G_N a}{H}\rho_{eff}\left(1+w_{eff}\right)\right]q$$

$$+\left(1+f_R\right)\frac{2k^2}{aH}\dot{\eta}+\frac{\dot{f}_R}{2aH}\dot{q}\ . \tag{3.9}$$

For the $i\neq j$ component, we let μ equal to i and ν equal to j respectively in Eq. (2.11). After a Fourier transformation,

$$-\frac{12\pi G_N a^2}{2k^2\left(1+f_R\right)}\sum_a \rho_a\sigma_a\left(1+w_a\right)=\frac{1}{2k^2}\dot{q}+\frac{aH}{k^2}q-\eta+\frac{\chi}{1+f_R}\ . \tag{3.10}$$

Where $q\equiv\dot{h}+6\dot{\eta}$, and

$$\chi\equiv f_{RR}\delta R_N\ , \tag{3.11}$$

where δR_N is defined as the perturbation of the Ricci scalar in the conformal Newtonian gauge,

$$\delta R_N\equiv\left(g^{\mu\nu}\delta R_{\mu\nu}\right)_N\ . \tag{3.12}$$

In the synchronous gauge,

$$\begin{aligned}\delta R_N=&-\frac{6}{a^2}\left[\ddot{\eta}-\frac{1}{2k^2}\left(3a^2H\dot{H}+a\ddot{H}\right)q\right]\\&-\frac{18H}{a}\dot{\eta}+\frac{2k^2}{a^2}\left[\frac{1}{2k^2}\left(\dot{q}+3aHq\right)-2\eta\right].\end{aligned} \tag{3.13}$$

In principle to solve the two scalar modes q and η, we only need two field equations. Here we introduce a new dynamical variable χ, it is just for convenient, or we need to solve the higher order derivative equations. In fact, there are some intractable terms such as $\ddot{\eta}$ and $\ddot{q}$ in χ. For solving the three scalar modes, q, η and χ, we need three field equations, Eq. (3.8), Eq. (3.9) and Eq. (3.10).

3.3 Rearranged Formulae

To solve the scalar modes by numerical, we need to rearrange the field equations. For solving the evolutions of η ,q and χ, we need three equations only has $\dot{\eta}$ or $\dot{q}$ or $\dot{\chi}$ in the L.H.S.. After some linear combinations of the equations in the previous section, we derive the rearranged field equations

$$\dot{q} = -2aHq + 2k^2\eta - \frac{2k^2\chi}{1+f_R} - \frac{12\pi G_N a^2}{1+f_R}\sum_a \rho_a\sigma_a(1+w_a) \ , \tag{3.14}$$

$$\dot{\eta} = \left(\frac{a^2H^2}{\dot{f}_R k^2}\right)\left\{\left[\frac{k^2}{3aH}(1+f_R) + \frac{\dot{f}_R\dot{H}}{2aH^2}\right]q - (1+f_R)\frac{2k^4}{3a^2H^2}\eta + \left(\frac{k^4}{3a^2H^2} - \frac{k^2\dot{H}}{aH^2}\right)\chi \right.$$
$$\left. - \frac{8\pi G_N a}{H}\sum_a \rho_a\theta_a(1+w_a) - \frac{8\pi G_N k^2}{3H^2}\sum_a \rho_a\delta_a\right\} \ , \tag{3.15}$$

$$\dot{\chi} = -\frac{aH}{k^2}\left[\frac{\dot{f}_R}{2} + f_R\frac{\dot{H}}{H} - \frac{4\pi G_N a}{H}\rho_{eff}(1+w_{eff})\right]q + 2(1+f_R)\dot{\eta} + \dot{f}_R\eta$$
$$+\left(aH - \frac{\dot{f}_R}{1+f_R}\right)\chi - \frac{8\pi G_N a^2}{k^2}\sum_a \rho_a\theta_a(1+w_a) - \frac{12\pi G_N a^2\dot{f}_R}{k^2(1+f_R)}\sum_a \rho_a\sigma_a(1+w_a) \ . \tag{3.16}$$

These are the full $f(R)$ field equations in the synchronous gauge.

Chapter 4

Approximations

4.1 Two-Scale Problem

The "two-scale problem" might happens when we numerically solve the differential equation (3.15).

Here we define "f-terms" which proportional to the derivatives of f. And we also call the other "GR-terms" which do not proportional to the derivatives of f. Each of the scale of f-terms should be much smaller then the scale of GR-terms when the $f(R)$ theory very similar to GR.

Let us observe the structure of Eq. (3.15),

$$\dot{\eta} = (f\text{-term})^{-1}\left[\text{GR-terms} + f\text{-terms}\right] . \tag{4.1}$$

In the case of GR, both the summations of the GR-terms and the f-terms in the square bracket of Eq. (3.15) should be zero. When the $f(R)$ theory is very close to GR, the summations should be tiny and non-zero values. However, in the square bracket the order of the numerical error of the summation of the GR-terms may be close to or greater than the order of the f-terms . Furthermore, the f-term in the parentheses would enlarge this numerical error very much. This enlarged numerical error would make the calculation wrong. Here we call this situation "two-scale problem".

GR can describe the cosmic microwave background (CMB) anisotropies very well before the CMB decoupling. Thus, the $f(R)$ effect was very small, and the two scales of f-terms and GR-terms are very different at early times. In other words, the two-scale problem would happen at early times.

4.2 Our Approximation (a solution to the two-scale problem)

For solving the two-scale problem at early times, we provide our approximation. We decompose η into two parts, $\eta^{(0)}$ and $\eta^{(1)}$ by two different orders,

$$\eta = \eta^{(0)} + \eta^{(1)} \tag{4.2}$$

Where $\eta^{(0)}$ corresponds to the order of GR-terms, and $\eta^{(1)}$ corresponds to the order of f-terms. We obtain the evolution equations respectively of $\eta^{(0)}$ and $\eta^{(1)}$ via dividing Eq. (3.16) by the two orders with Eq. (4.2),

$$\dot{\eta}^{(0)} \equiv \frac{4\pi G_N a^2}{k^2} \sum_a \rho_a \theta_a (1 + w_a) - \frac{2\pi G_N a^2}{(1+f_R)\, k^2} \rho_{eff} (1 + w_{eff})\, q\,, \tag{4.3}$$

$$\begin{aligned}\dot{\eta}^{(1)} =& \frac{1}{1+f_R}\left[\frac{1}{2}\dot{\chi} + \left(\frac{aH}{4k^2}\dot{f}_R + \frac{a\dot{H}}{2k^2} f_R\right) q - f_R \dot{\eta}^{(0)} - \frac{1}{2}\dot{f}_R \left(\eta^{(0)} + \eta^{(1)}\right)\right. \\ & \left. -\frac{1}{2}\left(aH - \frac{\dot{f}_R}{1+f_R}\right)\chi + \frac{6\pi G_N a^2 \dot{f}_R}{k^2(1+f_R)} \sum_a \rho_a \sigma_a (1 + w_a)\right].\end{aligned} \tag{4.4}$$

At early times, we neglect the f-terms in δR_N. Therefore, we have an approximation form for δR_N from Eqs. (2.11),(3.12),

$$\delta R_N = 3\Box\chi + R\chi - f_R \delta R_N - 8\pi G_N \delta T_N \approx -8\pi G_N \delta T_N\,, \tag{4.5}$$

where δT_N is defined as the perturbation of the trace of the stress-energy tensor in the conformal Newtonian gauge,

$$\delta T_N \equiv (g^{\mu\nu}\delta T_{\mu\nu})_N \ . \tag{4.6}$$

Thus, in the early universe, we have

$$\chi \approx \chi_{(\text{approx})} \equiv -8\pi G_N f_{RR}\delta T_N = 8\pi G_N f_{RR}\sum_a \rho_a(1-3w_a)\delta_{N,a} \,, \tag{4.7}$$

where $\delta_{N,a}$ is the density perturbation of fluid a in the conformal Newtonian gauge.

The second approximation is to neglect the terms related to $\dot{\eta}^{(1)}$ in the relation of $\dot{\chi}$ which is obtained from the derivative of Eq. (4.7),

$$\begin{aligned}\dot{\chi} \approx \frac{d}{d\tau}\chi_{(\text{approx})} \approx \dot{\chi}_{(\text{approx})} \equiv 8\pi G_N \sum_a \rho_a \left[\dot{f}_{RR} - 3aH(1+w_a)f_{RR}\right](1-3w_a)\delta_{N,a} \\ + 8\pi G_N f_{RR}\sum_a \rho_a(1-3w_a)\dot{\delta}^{(0)}_{N,a} \,,\end{aligned} \tag{4.8}$$

where

$$\dot{\delta}^{(0)}_{N,a} \equiv (1+w_a)\left(-\theta_a - \frac{q}{2} + 3\dot{\eta}^{(0)} - \frac{3aH}{2k^2}\dot{q} - \frac{3a^2H^2 + 3a\dot{H}}{2k^2}q\right) . \tag{4.9}$$

To calculate the evolution of the perturbations in the $f(R)$ theory, we need Eqs. (3.14),(4.2),(4.3),(4.4),(4.7),(4.8) and the Boltzmann equations. We provide our approximation to obtain Eqs. (4.2),(4.3),(4.4),(4.7),(4.8) which have no two-scale problem.

However, we expect that the approximation could not be satisfied at late times. In principle, we need to solve the perturbations by the original $f(R)$ equations at late time. Unfortunately, even when the f-term in the parentheses in Eq. 4.1 at late times is much bigger than it at the early times, the two-scale problem still exist. Maybe the designer $f(R)$ model is so close to ΛCDM that the the f-term in the parentheses in Eq. 4.1 is still not big enough at late times. One possible solution is choose other conventional $f(R)$ models. Another safe is using other approximations to calculate the late-time perturbations. Because we take the order of f-terms as new perturbations quantities to divided Eq. (3.16) by two orders. For convenient, we call our early-time approximations "double perturbation" (DP).

4.3 GR Approximation

In numerical one takes the evolution of the perturbations in the ΛCDM as an conventional approximation of solving the perturbations of the $f(R)$ theory at early times. We call this approximation "GR approximation". Because in GR approximation one neglects the effect of the modification of the $f(R)$ theory to GR, the validity of this approximation might need to be examined.

Our approximate equations in Sec.4.2 are the same as the evolution equations of the perturbations in GR, when we neglect the f-terms. Therefore, in GR approximation one does not consider the effect of the f-terms which are very small at early times. Therefore, we claim that our approximation contains more $f(R)$ effect than GR approximation at early time.

4.4 Late-Time and Subhorizon Approximations

In this section, we use the approximate formula derived by Tsujikawa [41] . At late times, our universe is matter dominated, so we can approximately neglect the contribution from radiation. For the subhorizon approximation, we only consider the modes which are much smaller than the comoving hubble radius, $k \gg aH$. Consider the conformal Newtonian gauge,

$$ds^2 = a^2(\tau) \left\{ -\left[1 + 2\Psi(\vec{x}, \tau)\right] d\tau^2 + \left[1 - 2\Phi(\vec{x}, \tau)\right] d\vec{x}^2 \right\} . \qquad (4.10)$$

The following perturbed quantities are defined in conformal Newtonian gauge,

$$\delta''_{m(New)} \simeq -\left(2 + \frac{H'}{H}\right)\delta'_{m(New)} + \frac{4\pi G_{eff}\rho_m}{H^2}\delta_{m(New)} \ , \tag{4.11}$$

where

$$G_{eff} \equiv \frac{G_N}{1+f_R}\frac{1+\frac{4k^2}{a^2}\frac{f_{RR}}{1+f_R}}{1+\frac{3k^2}{a^2}\frac{f_{RR}}{1+f_R}} \ . \tag{4.12}$$

For convenient, we call the approximations Tsu.

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Chapter 5

Comparison of the Approximations

5.1 Early-time Approximations GR and Ours(DP)

In order to compare the results obtained from the different early-time approximations, we utilize the designer $f(R)$ model [27]. The designer $f(R)$ model is one of the $f(R)$ models. It can generate any required expansion history of the universe.

Our numerical tool to solve the cosmological perturbations is based on CMBFAST public code. We choose the cosmological parameters from Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) observations [12], namely, the effective number of neutrino species, $N_{eff} = 4.34$; the mass fraction of helium, $Y_{He} = 3.26$; the hubble constant, $H_0 = 73.8(\text{Mpc}^{-1}\cdot\text{km/s})$; the abundance of baryon $\Omega_{b0} = 0.0455$; the abundance of cold dark matter $\Omega_{c0} = 0.226$; the abundance of effective dark energy $\Omega_{eff0} = 0.728$; the matter-radiation equality time, $z_{eq} = 4828$.

On the other hand, our numerical tool to solve the background evolution is the designer $f(R)$ code developed by Wei-Ting Lin. We choose the initial conditions to design a certain $f(R)$ model. The initial time, $a_i = 10^{-8}$; the initial value of f_R, $f_{Ri} \equiv f_R(a_i) = -1.3923016 \text{ x } 10^{-39}$; the constant equation of state, $w_{eff} = -1$.

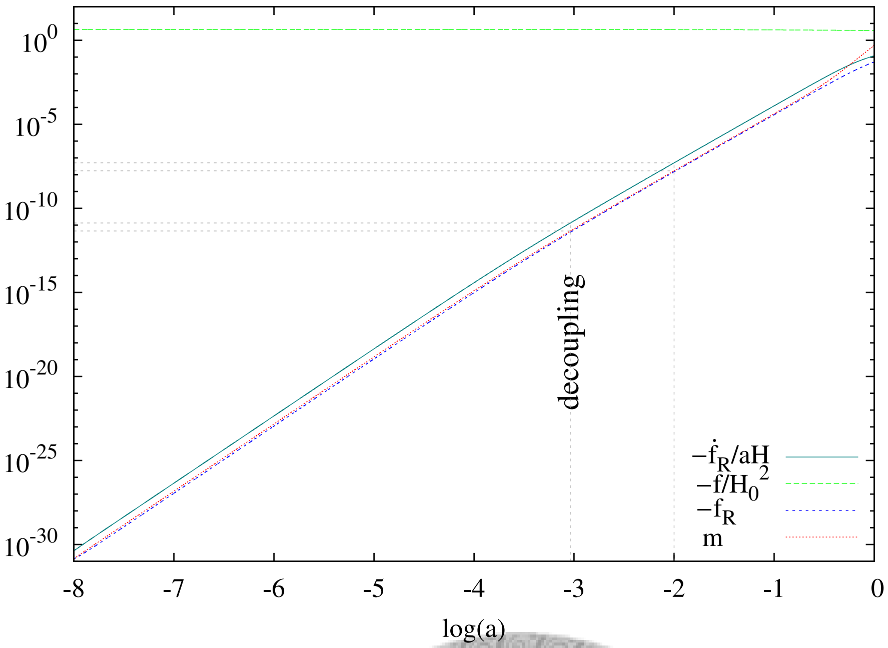

Figure 5.1: The background evolutions of the designer $f(R)$ model. The initial conditions for designer $f(R)$ model $a_i = 10^{-8}$, $f_{Ri} = -1.3923016 \text{ x } 10^{-39}$, $w_{eff} = -1$. m is defined by $Rf_{RR}/(1 + f_R)$, some workers may utilize it to analyze the deviation between a $f(R)$ model and ΛCDM. The CMB decoupling time is z=1090.

This $f(R)$ model has passed the large scale structure tests [39].

Fig. 5.1 shows the background evolutions of the designer $f(R)$ model. Because the derivatives of f are small, f/H_0^2 is almost a constant, where $f \approx -2\Lambda$. f_{Ri} should be a very small value, such that f_R is small enough at early times. The parameter m is defined by $Rf_{RR}/(1 + f_R)$ [28]. Some workers may utilize it to analyze the deviation between a $f(R)$ model and ΛCDM [26, 41, 42]. We can see that m is close to order one at late times, and f_R may not be viewed as a small quantity.

Most of the late-time approximations of the evolution equations of the perturbations in $f(R)$ are built in the conformal Newtonian gauge [26, 41, 42]. Therefore, for the future works we show the following results in the conformal Newtonian gauge.

5.1.1 Results

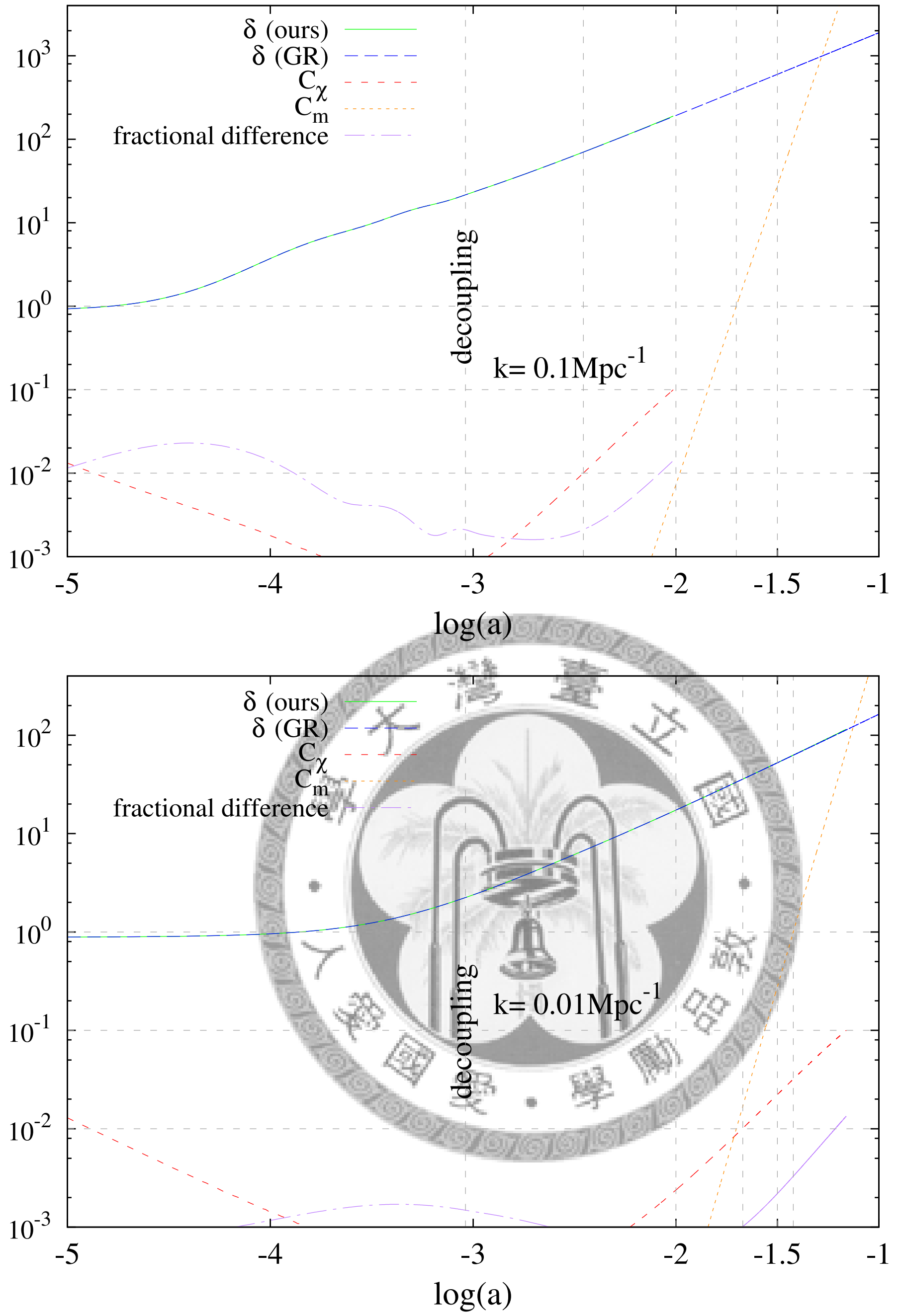

Figure 5.2: The comparison of the matter density perturbations calculated by our approximation and GR approximation. The upper figure is for the case k=0.1Mpc^{-1}, and the lower figure is for the case k=0.01Mpc^{-1}. δ(ours) is the matter perturbation in the conformal Newtonian gauge calculated by our approximation. δ(GR) is the matter perturbation in the conformal Newtonian gauge calculated by GR approximation. The fractional difference is defined by $|\delta(\text{ours}) - \delta(\text{GR})|/(|\delta(\text{ours})| + |\delta(\text{GR})|)$. c_χ is defined by $|\chi_{(\text{approx})} - \chi|/(|\chi_{(\text{approx})}| + |\chi|)$, it is the criterion of the validity of our approximation. c_m is defined by $(aH/k)^2 R f_{RR}/(1 + f_R)$, some workers may utilize it as a criterion of the validity of the GR approximation. The CMB decoupling time is z=1090.

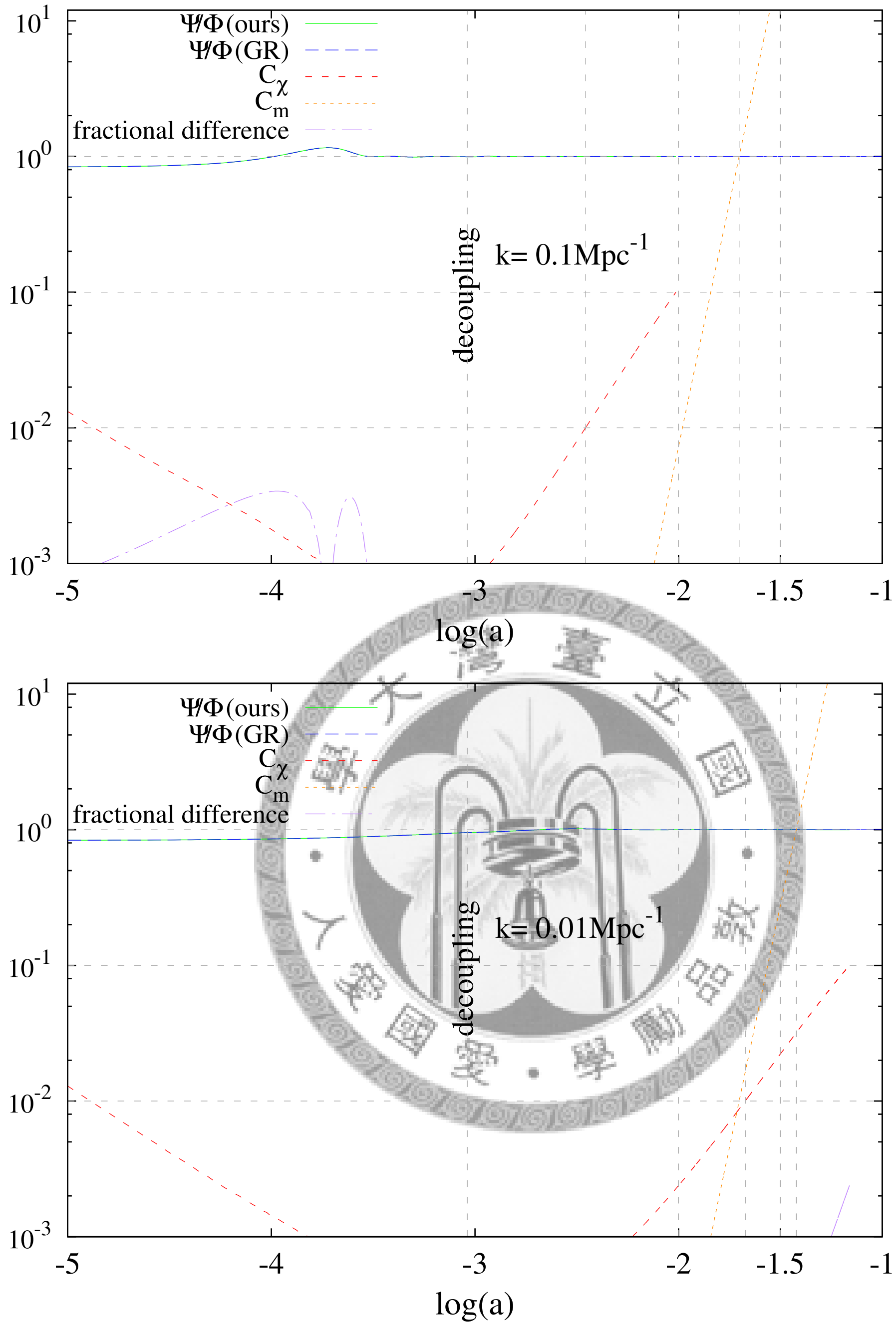

Figure 5.3: The comparison of the Ψ/Φ calculated by our approximation and GR approximation. The upper figure is for the case k=0.1Mpc^{-1}, and the lower figure is for the case k=0.01Mpc^{-1}. Ψ/Φ(ours) is Ψ/Φ calculated by our approximation. Ψ/Φ(GR) is Ψ/Φ calculated by GR approximation. The fractional difference is defined by $|\Psi/\Phi(\text{ours}) - \Psi/\Phi(\text{GR})|/(|\Psi/\Phi(\text{ours})| + |\Psi/\Phi(\text{GR})|)$. c_χ is defined by $|\chi_{(\text{approx})} - \chi|/(|\chi_{(\text{approx})}| + |\chi|)$, it is the criterion of the validity of our approximation. c_m is defined by $(aH/k)^2 R f_{RR}/(1 + f_R)$, some workers may utilize it as a criterion of the validity of the GR approximation. The CMB decoupling time is z=1090.

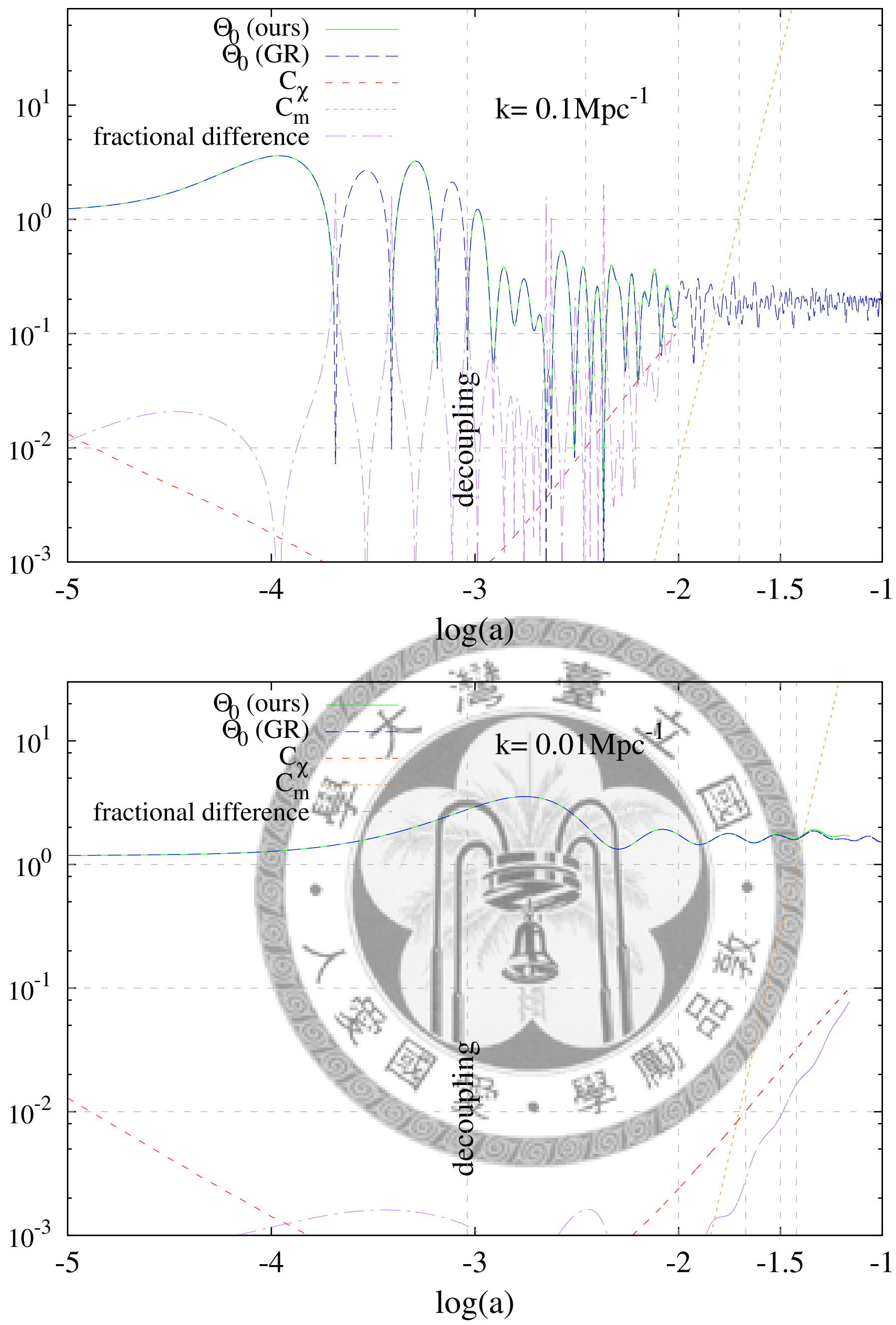

Figure 5.4: The comparison of the CMB photons density perturbations calculated by our approximation and GR approximation. The upper figure is for the case k=0.1Mpc^{-1}, and the lower figure is for the case k=0.01Mpc^{-1}. Θ_0(ours) is the CMB photons density perturbations calculated by our approximation. Θ_0(GR) is the CMB photons density perturbations calculated by GR approximation. The fractional difference is defined by $|\Theta_0(\text{ours}) - \Theta_0(\text{GR})|/(|\Theta_0(\text{ours})| + |\Theta_0(\text{GR})|)$. c_χ is defined by $|\chi_{(\text{approx})} - \chi|/(|\chi_{(\text{approx})}| + |\chi|)$, it is the criterion of the validity of our approximation. c_m is defined by $(aH/k)^2 R f_{RR}/(1 + f_R)$, some workers may utilize it as a criterion of the validity of the GR approximation. The CMB decoupling time is z=1090.

In Fig. 5.2-5.4, we calculate the cosmological perturbations for comoving wave number k=0.1Mpc^{-1} and k=0.01Mpc^{-1}. The fractional difference c_χ is defined by $|\chi_{(\text{approx})}-\chi|/(|\chi_{(\text{approx})}|+|\chi|)$, it is the criterion of the validity of our approximation. When c_χ is close to order one, Eqs. (4.7),(4.8) may not be good approximations. Thus, we show the calculations from our approximation until $c_\chi = 0.1$. c_m is defined by $(aH/k)^2 m$, some workers may utilize it as a criterion of the validity of the GR approximation when they also consider the subhorizon approximation at late times. The CMB decoupling time is z=1090.

In Fig. 5.2, we present the matter perturbation in the conformal Newtonian gauge calculated by our approximation, δ(ours),and that calculated by GR approximation, δ(GR). The fractional difference between δ(ours) and δ(GR) is defined by $|\delta(\text{ours}) - \delta(\text{GR})|/(|\delta(\text{ours})| + |\delta(\text{GR})|)$.

In Fig. 5.3, we present Ψ/Φ calculated by our approximation, Ψ/Φ(ours), and that calculated by GR approximation, Ψ/Φ(GR). The fractional difference between Ψ/Φ(ours) and Ψ/Φ(GR) is defined by $|\Psi/\Phi(\text{ours}) - \Psi/\Phi(\text{GR})|/(|\Psi/\Phi(\text{ours})| + |\Psi/\Phi(\text{GR})|)$.

In Fig. 5.4, we present the CMB photons density perturbation calculated by our approximation, Θ_0(ours), and that calculated by GR approximation, Θ_0(GR). The fractional difference between Θ_0(ours) and Θ_0(GR) is defined by $|\Theta_0(\text{ours}) - \Theta_0(\text{GR})|/(|\Theta_0(\text{ours})| + |\Theta_0(\text{GR})|)$.

Figure 5.4 shows that for the Fourier mode with $k = 0.1\,\text{Mpc}^{-1}$ the fractional difference in the CMB photon density perturbation is about 1% around the photon-baryon decoupling time, $z_{\text{dec}} = 1090$ ($a \sim 10^{-3}$), and reaches as large as 10% around $a = 10^{-2}$. For $k = 0.1\,\text{Mpc}^{-1}$ the fractional difference is about one order of magnitude smaller: $\lesssim 0.1\%$ around the decoupling time; $\sim 1\%$ around $a = 10^{-1.5} \simeq 0.03$. This result indicates that the effect of the gravity modification at early times in the $f(R)$ theory may not be negligible compared to the accuracy of the CMB observations. With regard to the matter density perturbation in Fig. 5.2, for $k = 0.1\,\mathrm{Mpc}^{-1}$ the fractional difference is about 1% around $a = 10^{-2}$, which is marginally negligible when compared to the current observational accuracy, while for $k = 0.01\,\mathrm{Mpc}^{-1}$ it is smaller: $\lesssim 10^{-3}$ before $a = 10^{-1.5} \simeq 0.03$.

5.2 Two Methods of Solving Cosmological Perturbations

There are two methods to calculate the cosmological perturbations from early times to now. Method I is the conventional method, and Method II is our new method in this thesis.

- Method I : GR→ Tsu:
  We use the field equations in GR to solve the perturbations until some time for some k. On this timing we still think DP are good approximations. However, the late-time and subhorizon approximations can be applied. Then we calculate matter density perturbation by Tsu.

- Method II : DP→ Tsu:
  We use DP to solve the perturbations until some time for some k. Similarly, we calculate the matter density perturbation by Tsu.

We will compare the matter density perturbations δ_m and the matter power spectra P_m predicted by these two methods.

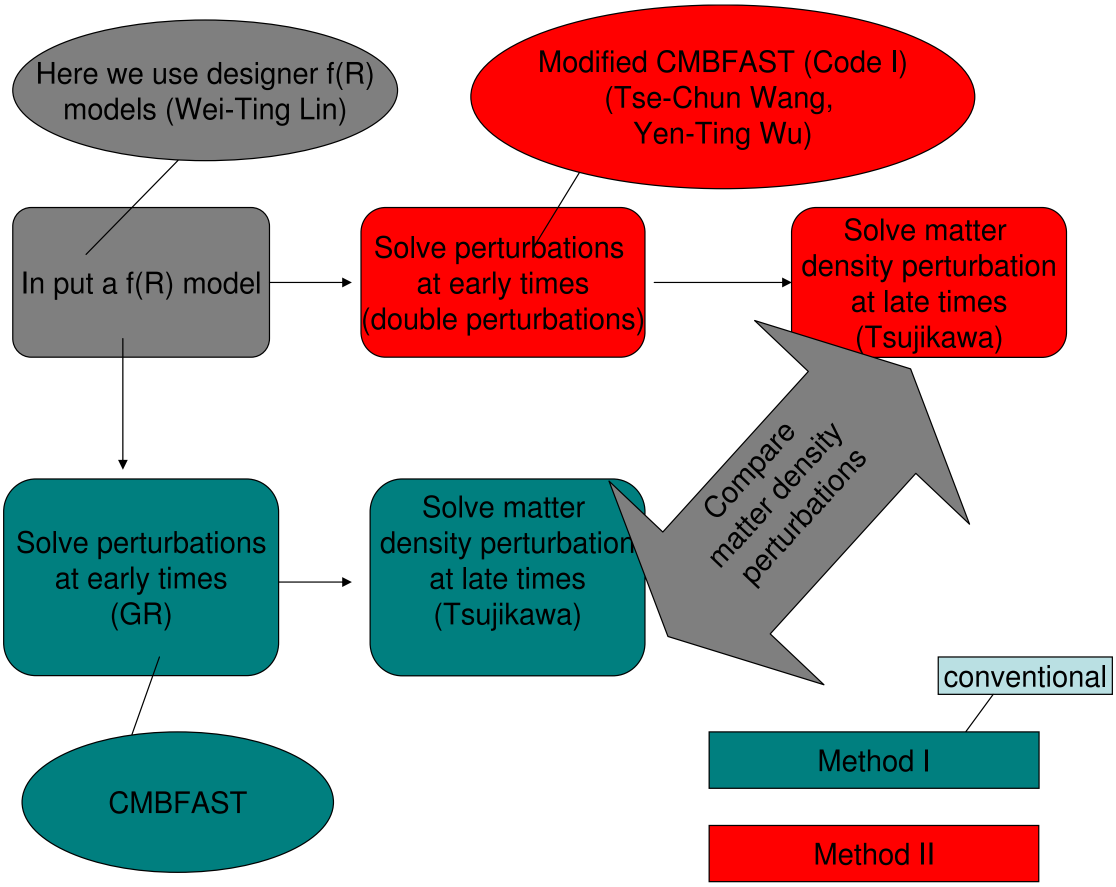

Figure 5.5: The flowchart of our strategy.

5.2.1 Criteria

Most of the other people use GR to solve the early-time perturbations and solve the late-time and subhorizon matter density perturbation by Tsu. To reproduce their work, we need to make sure that in the traslation point, both GR and Tsu are viable. Then we choose a criterion to decide the timing a_c to switch from GR to Tsu. The parameter $m \equiv Rf_{RR}/(1+f_R)$ was first introduced in [28]. It characterizes the deviation from the ΛCDM model ($f(R) = R/8kG - \Lambda$). If $m < (aH/k)^2$, the deviation is small. This regime is called "GR regime". Some other workers solve perturbations by GR in the GR regime. If $m > (aH/k)^2$, the deviation is big. This regime is called "scalar-tensor regime". The other workers solve the matter density perturbation by Tsu in the scalar-tensor regime. Of course, they need to check if both subhorizon and matter-dominated are satisfied. We take the subhorizon criterion as $k/aH = c_k$. For subhorizon case, we need to care about that if the model is in the GR regime ($m < (aH/k)^2$) and matter-dominated. We think it is much better to switch from GR to Tsu in the GR regime when subhorizon and matter-dominated are satisfied. So we take a_c to judge when we switch from GR to Tsu.

In our work, we also use DP to solve the perturbations in the GR regime. Then we use a_c to judge when we switch from DP to Tsu. However, we know DP is an early-time approximation and it is invalid at late times. At a_c, both DP and Tsu are viable. We use $\eta^{(1)}/\eta^{(0)} = c_\eta$ to judge if DP is viable. Unfortunately, these two periods can not overlap when we use the criterion $c_\eta < 0.01$. We choose another criterion c_η which is not too rigorous to find the overlap. Even though the calculation by DP is not so accurate near some a_η, where $\eta^{(1)}(a_\eta)/\eta^{(0)}(a_\eta) \approx 1$, we still believe Method II is better Method I. Because we have considered the $f(R)$ corrections to GR, even though the f-terms cannot be viewed as new perturbation quantities, the calculation by DP is still closer to the exact solution than the calculation by GR at late times. It is to say, if Method II is not good, then Method I is worse. If the result from Method II is very different to the result from Method I, Method I may not be a good approximation. We use $m/(aH/k)^2 = c_m$ as our criterion. However, we give the priority to c_m. So $a_c = (k/H)(m/c_m)^{1/2}$

5.2.2 Results

In our work, we choose the designer $f(R)$ model in Sec.5.1 : $f_{Ri} = -1.3923016$ x 10^{-30}, $w = -1.0$. This model is cosmological viable [39, 40].

In the following works, we take the criteria, $c_k < 10, c_\eta < 3.0$. The k we choose here satisfy $a_\eta > a_c$. Recall that $c_m = 0.01$ determine if the calculations translate from GR (or DP) to Tsu.

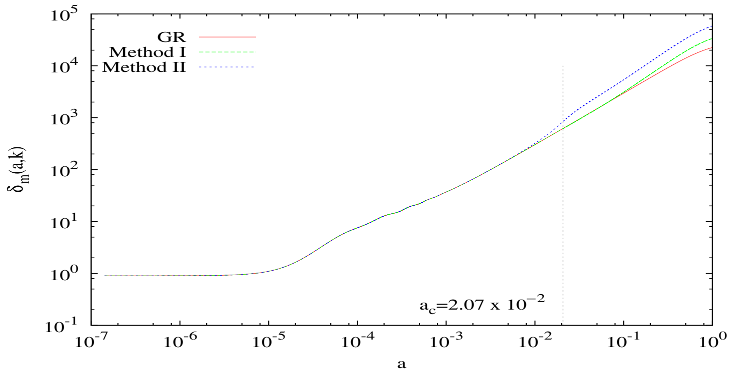

Figure 5.6: The matter density perturbation $\delta_m(a)$ in the conformal Newtonian gauge for $k = 0.20\text{Mpc}^{-1}$. The designer $f(R)$ model $f_{Ri} = -1.3923016 \text{ x } 10^{-30}$, $w_{eff} = -1.0$. The criterions $a_c = 0.0207016$, $c_k > 10$, $c_\eta < 3.0$, $c_m = 0.01$.

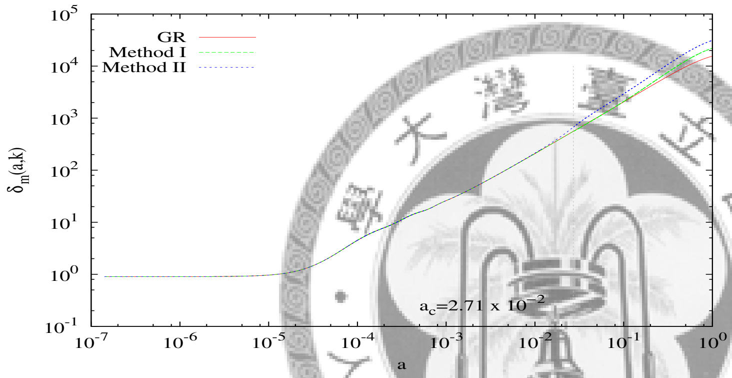

Figure 5.7: The matter density perturbation $\delta_m(a)$ in the conformal Newtonian gauge for $k = 0.11\text{Mpc}^{-1}$. The designer $f(R)$ model $f_{Ri} = -1.3923016 \text{ x } 10^{-30}$, $w_{eff} = -1.0$. The criterions $a_c = 0.0270897$, $c_k > 10$, $c_\eta < 3.0$, $c_m = 0.01$.

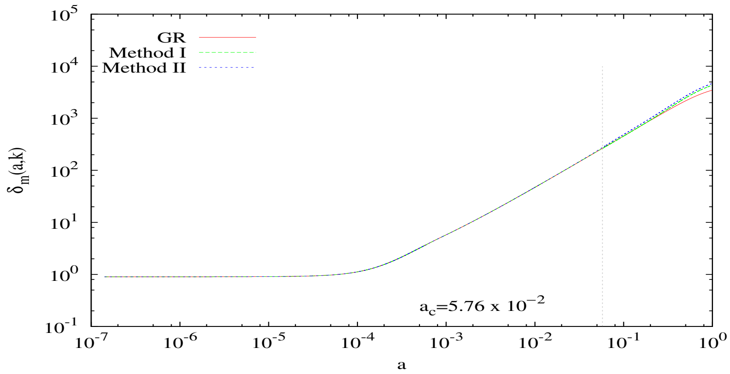

Figure 5.8: The matter density perturbation $\delta_m(a)$ in the conformal Newtonian gauge for $k = 0.02\text{Mpc}^{-1}$. The designer $f(R)$ model $f_{Ri} = -1.3923016 \text{ x } 10^{-30}$, $w_{eff} = -1.0$. The criterions $a_c = 0.0576506$, $c_k > 10$, $c_\eta < 3.0$, $c_m = 0.01$.

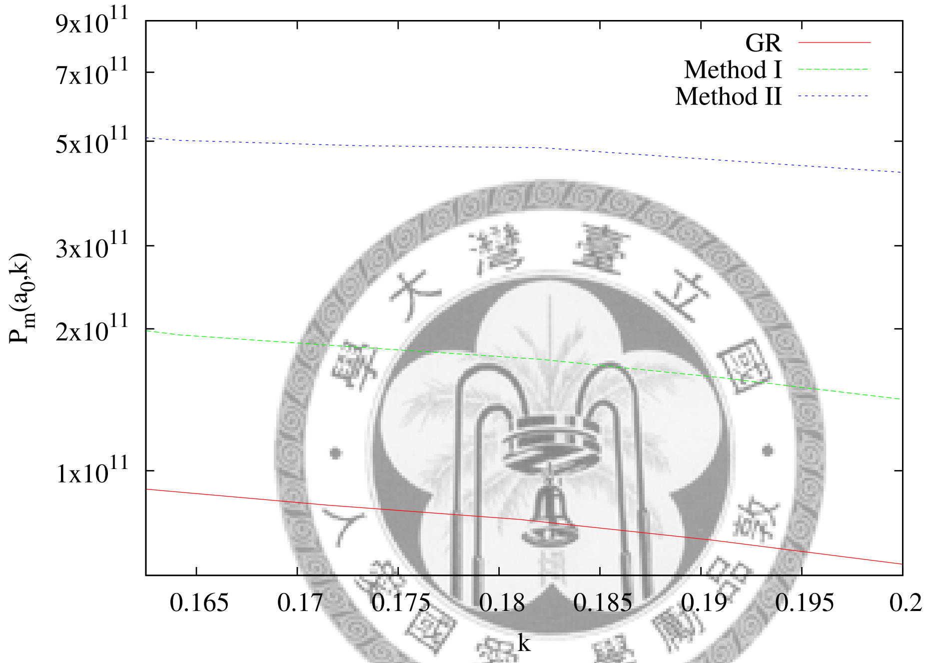

Figure 5.9: The matter power spectrum $P_m(a, k) \equiv \delta_m^2/k^3(\mathrm{Mpc}^{-3})$ in the conformal Newtonian gauge. The dimension of the comoving wave number k is Mpc^{-1}. The designer $f(R)$ model $f_{Ri} = -1.3923016$ x 10^{-30}, $w_{eff} = -1.0$, $c_k > 10$, $c_\eta < 3.0$, $c_m = 0.01$.

In Fig. 5.6-Fig. 5.8 are for different comoving wave number k. Notice that, in this model, the deviation from GR is not small at late times. Therefore, a_η are too small to find small k (when k is smaller, a_c is bigger) which can satisfy $a_\eta > a_c$. We calculate the matter density perturbations $\delta_m(a, k)$ for Method I, Method II and GR.

We can see that,. The fractional difference of matter power spectrum between these two methods is bigger than 10%. In this case, we believe that Method II is better than Method I, because Method II have considered the 1st order f-terms (ie. $f_R, \dot{f}_R, f_{RR}$ and $\dot{f}_R$). The only deficiency of Method II is that it does not consider higher order f-terms (ie. $f_R^2, \dot{f}_R^2, f_{RR}^2, \dot{f}_{RR}^2$ and so on). Even though Method II is not accurate at late times because of the higher order f-terms. We can still say that calculations by Method II is closer to the calculations by the exact $f(R)$ field equations Eq. (3.14), Eq. (3.15) and Eq. (3.16) than Method I. The difference between Method I and Method II is not the only inaccuracy source. However, we can tell the lower bound of the inaccuracy. So under these criteria $c_k > 10$, $c_\eta < 3.0$ and $c_m = 0.01$, we may say that Method I is not a good approximation. To see more results, we try to change the criterion $c_k > 10$ to $c_k > 100$. We choose a big $c_\eta < 3.0$ here. Here c_m is not our prior criterion to determine a_c. Instead, we choose c_k as our criterion to translate to Tsu. So $a_c = (m/c_m)^{1/2}k/H$, and $c_k > 10$ and $c_\eta < 3.0$ also need to be satisfied. We use the same designer $f(R)$ model as before. For the criteria $c_k = 100$ and $c_m < 0.01$, we obtain the corresponding a_c. At a_c, it should be matter-dominated.

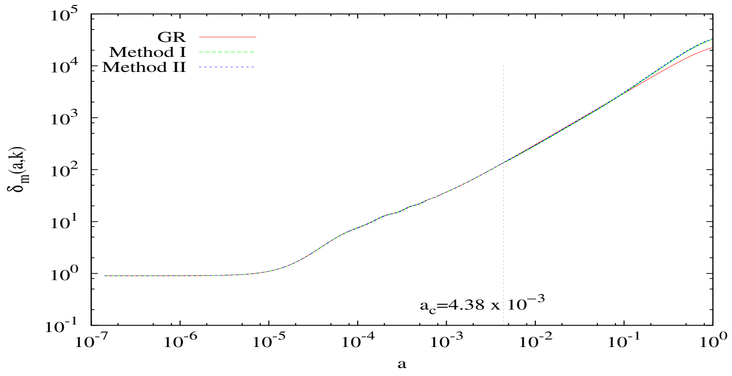

Figure 5.10: The matter density perturbation $\delta_m(a)$ in the conformal Newtonian gauge for $k = 0.20\text{Mpc}^{-1}$. The designer $f(R)$ model $f_{Ri} = -1.3923016$ x 10^{-30}, $w_{eff} = -1.0$. The criterions $a_c = 0.0043854$, $c_k = 100$, $c_\eta < 0.1$, $c_m < 0.01$.

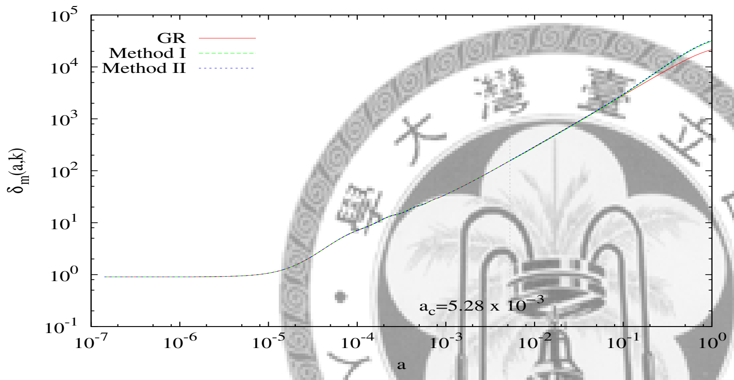

Figure 5.11: The matter density perturbation $\delta_m(a)$ in the conformal Newtonian gauge for $k = 0.18\text{Mpc}^{-1}$. The designer $f(R)$ model $f_{Ri} = -1.3923016$ x 10^{-30}, $w_{eff} = -1.0$. The criterions $a_c = 0.0052803$, $c_k = 100$, $c_\eta < 0.1$, $c_m < 0.01$.

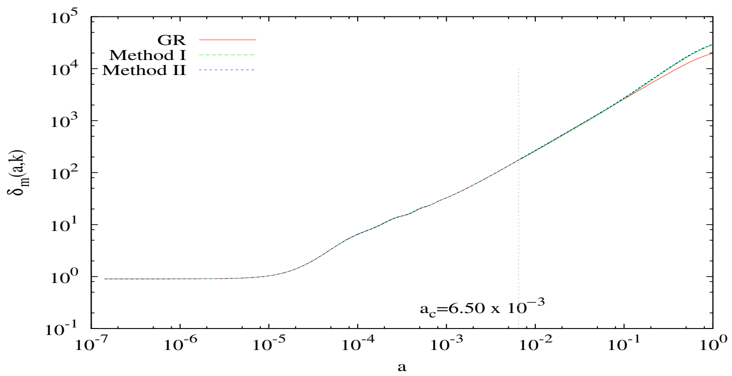

Figure 5.12: The matter density perturbation $\delta_m(a)$ in the conformal Newtonian gauge for $k = 0.16\text{Mpc}^{-1}$. The designer $f(R)$ model $f_{Ri} = -1.3923016$ x 10^{-30}, $w_{eff} = -1.0$. The criterions $a_c = 0.0065091$, $c_k = 100$, $c_\eta < 0.1$, $c_m < 0.01$.

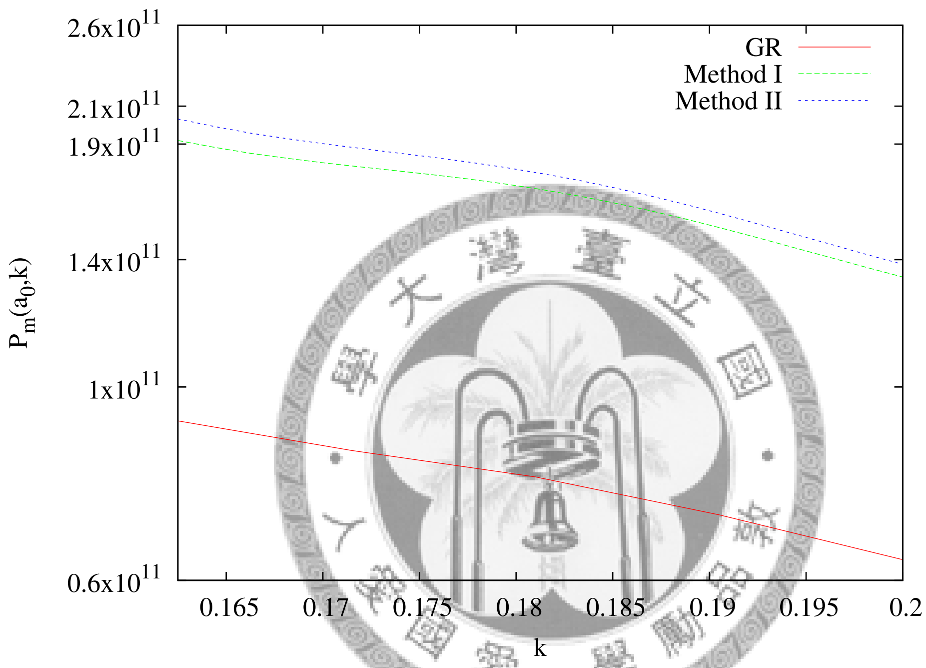

Figure 5.13: The matter power spectrum $P_m(a,k) \equiv \delta_m^2/k^3(\mathrm{Mpc}^{-3})$ in the conformal Newtonian gauge. The dimension of the comoving wave number k is Mpc^{-1}. The designer $f(R)$ model $f_{Ri} = -1.3923016$ x 10^{-30}, $w_{eff} = -1.0$, $c_k = 100$, $c_\eta < 0.1$, $c_m < 0.01$.

In Fig. 5.10-Fig. 5.12 are also for the same designer $f(R)$ model, and we take different $k(\mathrm{Mpc}^{-1})$. Again, in this model the deviation from GR is not small at late times. Therefore, a_η are too small to find small k (when k is smaller, a_c is bigger) which can satisfy $a_\eta > a_c$. We calculate the matter density perturbations $\delta_m(a,k)$ for Method I, Method II and GR. In Fig. 5.10-Fig. 5.12, we calculate the matter density perturbations $\delta_m(a,k)$ for Method I, Method II and GR. Then we can use them to calculate the matter power spectrum $P_m(a_0,k)$ at a_0=1.

We can see that, the differences between Method I, Method II are so small. For this model, the deviation from GR is not big at late times. Because a_η are big enough, both GR and DP are good approximations before a_c. Where the criterion $c_k = 100$, $c_\eta < 0.1$ and $c_m < 0.01$.

Chapter 6

Discussion

6.1 Conclusion

In Chapter 3, we showed the $f(R)$ field equations. In Sec.3.3, we present the equations we need to solve the cosmological perturbations. In Chapter 4, we found the two-scale numerical problem at early times due to the smaller f-terms and bigger GR-terms. Then we propose our approximation "double perturbation"(DP) to deal with the two-scale problem at early times. However, the two-scale problem still exist at late times. We also present the approximations where some other workers may take them to solve the two-scale problem. There are the GR approximation as an early-time approximation, and the late-time and approximation with subhorizon presented by Tsujikawa.

In Sec.5.1, we compared the two early-time approximtaions. We use double perturbation as the early-time approximation to calculate the early-time matter density perturbation. For the conventional method, we used GR early-time approximation with Tsujikawa's approximation to calculate the early-time matter density perturbation and compare it to the result from our method. We found our approximation can improve the conventional methods and provide a new way to constrain the $f(R)$ models.

In Sec.5.2, we proposed two methods to solve the full time cosmological perturbations. We use double perturbation as the early-time approximation and Tsujikawa's late-time approximation. This is our new method to calculate the full-time matter density perturbation and the matter power spectrum without fitting. For the conventional method, we used GR early-time approximation with Tsujikawa's approximation to calculate the full-time matter density perturbation and the matter power spectrum without fitting. Then we compared to the results from these two method. We finally found that the conventional Method (ie. Method I) might not be a good approximation under certain criteria c_k, c_m and c_η. Because of the prediction of the observational matter power spectrum might be accurate at 10% level [45]. The fractional difference of matter power spectrum from these two methods might be smaller than 10%. When $c_k > 10$, $c_\eta = 3.0$ and $c_m = 0.01$. The fractional differences are more than 10%. It seems that calculate early-time perturbations by GR might not be a good choice. We think "double perturbation" is the better choice to calculate early-time perturbations.

Thus for self-consistency's sake, the GR approximation is problematic, and a better treatment for the early-time evolution is necessary, which our approximation may provide.

6.2 The Future Works

The tested model is generated by designer $f(R)$. However, we need to use the other conventional $f(R)$ models which are exact functions of the Ricci scalar. To compare other people's works, we should use their models, not only the designer $f(R)$ models. In [29], there are some useful rearranged equations to solve $f(R)$ evolution in the background level. However, we have found that it is difficult to handle an exact $f(R)$ model by numerical. Even though in background level, the two-scale problem emerges at early times again! Because some higher order derivative terms in $f(R)$

theory do not appear in general relativity. So these terms must be f-terms. At early times, each f-terms are smaller than the inaccuracies of the summation of GR-terms. To solve this two-scale problem, we should use some approximation. In principle, we need to use iteration to solve this problem.

So far, our works can only tell that if other people used good or bad approximations. We have not given the new constraint for $f(R)$ gravity yet. In principle, we need to solve the $f(R)$ field equations at all times. Because the two-scale problem, we use DP to solve perturbations at early times. We have developed a numerical code base on CMBFAST to solve the $f(R)$ perturbations by DP. We call it "CODE I". On the other hand, we also developed a numerical code base on CMBFAST to solve the $f(R)$ perturbations by the exact $f(R)$ field equations. We call it "CODE II"

We have tried to solve the late-time perturbations by the exact $f(R)$ field equations (ie. CODE II) . However, the first derivative of the metric perturbation η (ie. $\dot{\eta}$) is very discontinuous at the translating point a_c. The late-time cosmological perturbations will fiercely oscillate as time goes by. This can not be thought as a correct result. We have tried different designer $f(R)$ models, a_c and k. But the problem is still there. It is because that the two-scale problem still exist at late times. Maybe the other conventional $f(R)$ models could solve this numerical problem.

We have tried to prolong the viable period of CODE I. We use iteration to let the calculation of χ be more close to Eq. 3.11. However, this method does not prolong the viable period of CODE I because of the big c_χ at early times.

Tsu can only calculate subhorizon and matter-dominated cases. We need to choose another method to solve the late-time perturbations. The hopeful candidate is in [42]. Even though in this paper the authors also use matter-dominated and subhorizon approximations, they claim that they have considered some higher order corrections for the subhorizon approximation. It seems their work may give more precisely calculations than Tsu. And we do not need to consider wave length so much smaller than the horizon. We can consider a smaller a_c. At the smaller a_c, DP is more possible a good approximation. Notice that in [26], the authors did not use Eq. (4.11) to calculate matter density perturbations. Instead, they use only matter-dominated approximation but no subhorizon approximation. Their new formula is more general than Eq. (4.11). Because their formula will be reduced to Eq. (4.11) when considering subhorizon approximation.

In [26], the authors also only consider matter-dominated approximation. So far, we believe DP can be adapted to the early matter-dominated period. Thus, it might also be a hopeful candidate to calculate late-time matter density perturbation without numerical problems.

It is another save to consider higher order perturbations in $f(R)$ gravity. We might consider "triple perturbations" or something we haven't thought. If we could the higher order perturbations or [42] to calculate perturbations to develop a new approximation to solve the perturbations at late times (ie. to develop CODE 1.5). We hope when CODE 1.5 is viable, we can find a timing $a_{1.5}$ when CODE 1.5 can connect to CODE II safely. Then we could calculate matter power spectrum by CODE II and obtain our new constraint for $f(R)$ gravity. And we can constrain the $f(R)$ models by the matter density power spectrum.

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