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相依結構對多資產選擇權定價之模擬分析

Bivariate Options Pricing with Copula-GARCH Model - Simulation Analysis

游明翰

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摘要

二元選擇權是由兩個標的資產所衍生出的選擇權，其價格會與兩個資產的變動與相依結構有很大的相關性。但由於其市場透明度不高，平常很難於公開市場觀察二元選擇權的價格。本篇論文將取三種市場上較廣為被交易的二元選擇權來評價，利用 copula-GARCH 模型來檢測在不同的邊際分配參數設定下，二元選擇權價格對 copula 函數選擇的敏感度。

我們的研究結果可整理為三大結論，首先，Frank copula 模型常常會產生較其他 copula 模型差異較大之評價結果。第二點，二元彩虹選擇權的價格，對 copula 模型的選擇最為敏感。最後，copula-GARCH 的二元選擇權評價模型中，對殘插值的分配設定會嚴重影響評價的結果。總結來說，相依結構的設定對二元選擇權的價格會產生顯著的影響，是在評價二元選擇權時不可被忽略的一環。

關鍵字：二元選擇權、多資產選擇權、相依結構。

Abstract

Bivariate option is the contingent claims derives from a pair of underlying assets. The underlying assets can be equity, commodities, foreign exchange rate, interest rate or any index with quotations. In this paper, we present a copula-GARCH model and the Monte Carlo simulation method base on the model. We examine the pricing result of three kinds of bivariate options - digital, rainbow and spread option, in many different cases and find that the choosing of pricing copula may cause a significant difference of the pricing result. Furthermore, the pricing result of rainbow option is most sensitive to the choosing of copulas in the three kinds of bivariate options.

Key Words: Bivariate Option, Copula, Dependent Structure, GARCH, Monte Carlo.

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1 Introduction

In general, bivariate option is the contingent claims derives from a pair of underlying assets. The underlying assets can be equity, commodities, foreign exchange rate, interest rate or any index with quotations. The payoff of the bivariate contingent claim is also various. We can classify them into digital, rainbow and spread options by different payoff functions. These kinds of option are usually traded in over the counter (OTC) market. The transparency makes it difficult to do the empirical comparison of the pricing result. Therefore, we only do the research through discussing the pricing result under different model assumptions.

The process we set for monitor the marginal asset price change is GARCH process. It is one of the famous processes which researchers often set to analyze option value under varied volatility condition. Duan (1995) first developed the GARCH pricing model on stock options. Then the method had been extended to do the pricing of options in many other fields. In this paper, we extend the GARCH option pricing method to bivariate field, and examine the importance of dependent structure in pricing bivariate options under various marginal distribution settings.

The difficulty of extending option valuation model from single underlying asset to multiple underlying assets is that the dependent structure between multi-assets is complicated and hard to describe. There are many models of dependent structures which can describe multivariate process in analytical ways, such as BEKK[1] model, Dynamic Conditional Correlation (DCC) model, or models of copulas functions. Copula function is the most flexible and popular dependent structure model in the present day. We use the copula function as our dependent structure setting to price three kinds of bivariate options and simulate result under many conditions to discuss how the different copula function settings affect the option prices.

In this paper, we present a copula-GARCH model and the Monte Carlo simulation method base on the model. We examine the pricing result of three kinds of bivariate options in many different cases and find that the choosing of pricing copula may cause a significant difference of the pricing result. Furthermore, the pricing result of rainbow option is most sensitive to the choosing of copulas in the three kinds of bivariate options.

The reminder is laid out as follows. Section 2 reviews the important research result done by predecessors. Section 3 introduces the copula-based GARCH bivariate option pricing model. Section 4 is some analysis on the simulation result and the conclusion is showed in section 5.

[1] BEKK model was named by its first developer Yoshi Baba, Robert F. Engle, Dennis Kraft and Ken Kroner. Engle and Kroner coordinated, completed the research and published the model in 1995.

2 Literature Review

There are two mainstream models researchers often use to model the price dynamics with the considering of varied volatility. One follows the continuous time framework which built by Black and Scholes (1973), such as constant elasticity of variance (CEV) model or stochastic volatility model. These models are convenient in analyzing the pattern of price change, simple in calculating option prices, and easy to do application. However, the continuous time framework has to face the difficulty that the variance rate is not observable empirically. Duan (1995) had developed another discrete time option pricing framework follows Bollerslev's (1986) GARCH process. Duan showed that options can be priced by setting the underlying asset follows a GARCH process and the model has some advantages comparing with continuous time framework. First, the GARCH option pricing model includes the price dynamics with considering of risk premium and the risk neutralization by change numeraire. Second, the pricing model is non-Markovian. Last, the model can explain the implied volatility smile bias associated with the B-S model. Duan (1996) had further proved that GARCH option pricing model would converge to stochastic volatility model. Therefore, we can apply the GARCH option pricing model with more complete fundamental theory. Furthermore, Heston and Nandi (2000) followed the same framework to develop a closed-form solution for European option. They proved that the out-of-sample valuation errors from the single lag version of the GARCH model are lower than the Black-Scholes model. Through their contribution, we can see that the ability of discrete time framework on capturing the correlation of volatility with spot returns and the path dependence in volatility are both better than continuous time framework.

The GARCH model was first extended to multivariate setting by Bollerslev, Engle and Wooldridge (1988). They provided a so-called VECH representation which extended GARCH representation in the univariate case to the vectorized conditional variance matrix. VECH model is very general but cannot ensure the conditional variance-covariance matrix to be positive semidefinite. For solving the problem, Engle and Kroner (1995) developed BEKK model. BEKK model is also general and can ensure the conditional variance-covariance matrix to be positive semidefinite. However, BEKK and factor models have some disadvantages such as the parameters cannot be easily interpreted, and the intuitions of the effects of the parameters in a univariate GARCH equation are not readily seen.

In traditional VGARCH model, the parameters have to be re-estimated daily as new observation joint the sample. For computational simplicity, the constant-correlation GARCH model which is relatively easy to ensure the variance-covariance matrix to be positive semidefinite and have no need to re-estimate the matrix as new sample point joints, is popular among empirical researchers. We can see the empirical application researches done by Bollerslev (1990), Kroner and Claessens (1991), Kroner and Sultan (1991, 1993), Park and Switzer (1995) and Lien and Tse (1998). Nevertheless, the constant-correlation model was not good enough. Engle (2002), Engle and Sheppard (2001) proposed a Dynamic Conditional Correlation (DCC) GARCH model. They developed the theoretical and empirical properties of DCC GARCH model capable of estimating large time-varying covariance matrices. Then empirically inferred the model to compare the volatility estimator of S&P 500 Sector indices to the indices volatility and got a great success on multi-asset volatility estimation.

However, correlation coefficient is often insufficient measure for monitoring the dependent structure between different assets. There are many researchers pointing out that correlation model was not good enough to explain some empirical observations such as asset prices have a greater tendency to move together in bad states, see Boyer et al. (1999) and Patton (2003, 2004). Some researchers started to implement copula functions from statistic model into multivariate GARCH model in finance. Copula is a function that joints univariate distribution functions to form multivariate distribution functions. By the work of Sklar (1959) and the introduction to finance field by Nelson (1999) and Joe (1997), copula becomes one of the most important tools on modeling dependent structure between multiple assets especially in the bivariate GARCH model.

In 2000s, there are many researchers start to develop the Copula-GARCH model in many different kind of research fields. Jondeau and Rockinger (2006) applied the Copula-GARCH model on testing the behavior of stock market in different region. They suggested that the Copula-GARCH model can perform well on monitoring stock market returns in the markets which have higher dependency when returns move in the same direction than when they move in the different directions. Hsu, Tseng and Wang (2008) proposed a method estimating the optimal hedging ratio by Copula-based GARCH model and examined the effectiveness of the model by in-sample and out-of-sample empirical analysis. They proved that the Copula-GARCH performs more effectively than CCC-GARCH and DCC-GARCH models in estimating the hedging ratio.

The first application of Copula-GARCH model in bivariate option pricing was done by Goorbergh, Genest and Werker (2005). They developed the pricing model and parameter estimation method of bivariate option and examined the price differences induced by static and dynamic copula parameters. The bivariate option they examined was only 1-month maturity put-on-max option. Results showed that differences between static and dynamic parameter copula-GARCH option price do exist. Moreover, the prices induced by different copula function also have differences in the study.

3 Bivariate Options

As we mentioned before, the bivariate option is the contingent claim which is derived from two underlying assets. They are often traded on the OTC market and had less data for empirical examination. As the reason, most researches focus on the influence of model selecting to the pricing result. Nevertheless, these researches are valuable for providing a guideline on choosing an appropriate pricing model.

In general, the payoff function of a bivariate option is constructed by the pair of prices $(S_{1,t}, S_{2,t})$, such as $\max(S_{1,t} - S_{2,t} - K, 0)$. However, the scale of the initial prices of these two assets may have a big difference. That will cause the pricing result being meaningless. For this reason, we set the payoff function as a function related to $(R_{1,t}, R_{2,t})$, where $R_{i,t} = S_{i,t}/S_{i,0}, i = 1,2$. $R_{i,t}$ is considered as the percentage growth of underlying asset. Our following analyses are all under the price process $(R_{1,t}, R_{2,t})$ of underlying assets instead of $(S_{1,t}, S_{2,t})$. In this way, we can unify the scale and start value of underlying assets. The log return can be also defined as:

$$r_{i,t} \equiv \ln(S_{i,t}/S_{i,t-1}) = \ln(R_{i,t}/R_{i,t-1}).$$

There are various of bivariate options coordinated by their payoff functions in the OTC market. The bivariate options we are going to discuss have the payoff functions which are very popular in the market. They are bivariate-digital, -rainbow and -spread options. The payoff functions of these options at time T are given by:

Digital: $\mathrm{I}\left(R_{1,T} > K, R_{2,T} > K\right)$,

Rainbow: $\max\left[\max\left(R_{1,T}, R_{2,T}\right) - \mathrm{K}, 0\right]$,

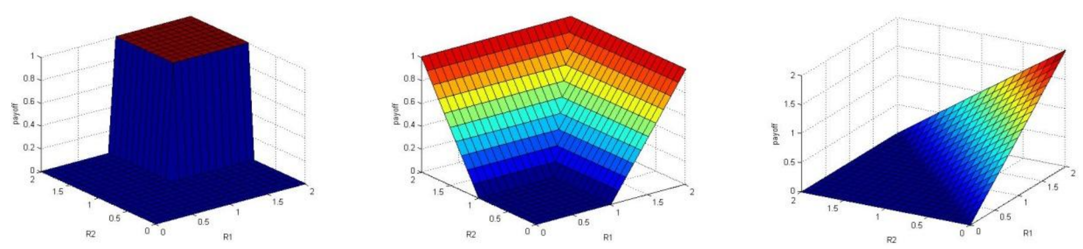
Spread: $\max\left(R_{1,T} - R_{2,T} - K, 0\right)$,

where $\mathrm{I}(\mathrm{A}, \mathrm{B})$ is the indicator which equals to one if A, B condition are both being satisfied and equals to zero otherwise, and the strike price K is set at the same scale with the price process $(R_{1,t}, R_{2,t})$. There are many other similar payoff functions of these three kinds of options. The function form we choose is only one of them.

For example, assuming these three kinds of bivariate options are derived from the same underlying asset A and B. The strike price is set to be 1 for digital and rainbow options, and 0 for spread option. You buy these three options when the prices of A and B are 10 and 5 in the beginning. Finally, (A, B) come to (12, 4) and $(R_{1,T},\ R_{2,T})$ equal to (1.2, 0.8). By the payoff formula, your digital, rainbow and spread option would return 0, 0.2 and 0.4 separately. The payoff graphs of each option are showed in Figure 1.

Figure 1
Payoff Graphs of Bivariate Options

Digital Option | Rainbow Option | Spread Option



4 Methodology

The first step in our valuation model is setting the marginal joint-risk-neutral return process. We set up the marginal return processes follows the GARCH (1, 1) processes with Gaussian innovations, which can be transformed into risk-neutral process. Then use different copula functions to describe the relation between the innovations of each marginal distribution and get the joint-risk-neutral return process. The second step is to simulate as much price paths as possible and calculate the mean of option payoff as the option price by Monte Carlo Simulation method. Details will be shown in followed sections.

4.1 GARCH Model

The specification for marginal distributions is from Bollerslev's (1986) and Duan (1995). With the considering that the options are sensitive to variance of underlying assets returns, we set GARCH process represent the return process since GARCH process was shown to have the ability on capturing the time-varying variances of return process. Duan (1995) provided the LRNVR condition which can stipulate the one-period ahead conditional variance in GARCH process is invariant with respect to the measure transformation to the risk- neutralized pricing measure. Followed the same setting with Goorbergh, Genest and Werker (2005), we set up the marginal distribution follows the GARCH (1, 1) process with Gaussian innovations which is often used by most researchers. For $i \in \{1,2\}$,

$$r_{i,t+1} = \mu_f + \eta_{i,t+1},$$

$$h_{i,t+1} = \omega_i + \beta_i h_{i,t} + \alpha_i \eta_{i,t+1}^2,$$

$$\mathcal{L}_P(\eta_{i,t+1} | \mathcal{F}_t) = N(0, h_{i,t}),$$

where $\omega_i > 0$, $\beta_i > 0, \alpha_i > 0$ and $\mathcal{L}_P(\cdot | \mathcal{F}_t)$ denotes the objective probability law condition on the information set $\mathcal{F}_t$, which includes all realized market information before time t. We can see that the variance of innovation is adapted by the variance of innovation and the residuals of the observations last period. The parameters represent the relations between variance and the asset returns. Under the LRNVR condition, we can achieve the returns process under risk neutral probability measure Q as followed,

$$r_{i,t+1} = r_f - \frac{1}{2} h_{i,t} + \eta_{i,t+1}^*,$$

$$h_{i,t+1} = \omega_i + \beta_i h_{i,t} + \alpha_i (r_{i,t+1} - \mu_i)^2,$$

$$\mathcal{L}_Q(\eta_{i,t+1}^* | \mathcal{F}_t) = N(0, h_{i,t}),$$

where r_f is the risk-free rate, assumed to be a constant in our model. $\eta_{i,t+1}^*$ represents the GARCH innovation under the risk neutral probability measure Q. Then we can simulate the return process under risk neutral assumptions. After we have the marginal distribution of each asset, next step is constructing the dependent structure by Copula functions.

4.2 Copulas Functions

Copula function is usually used to describe the relations between different random variables. Different form DCC or CCC model describing the dependence structure by variance-covariance matrix, copula functions model the dependence structure between multiple random variables by setting the joint density function of them. There are many different kinds of copula functions which exhibit different relations between opposite assets. They can model the assets with many kinds of special interactions, such as the phenomenon that asset returns have higher synchronization when volatility comes large. Therefore, copula structure is more flexible than traditional variance-covariance model. On the other hand, the flexibility of copula functions also made the model selecting becomes more complicated and important. Goorbergh, Genest and Werker (2005) pointed out that wrong copula model setting may induce the wrong pricing result. Our main purpose is to confirm the influence of copula for bivariate option price. Therefore, we only focus on the bivariate copulas. The definition of copula is as followed:

Definition (Copula):

A function C: $[0,1]^2 \rightarrow [0,1]$ *is a copula if it satisfies*

(i) $C(u,v) = 0 \; for \; u = 0 \; or \; v = 0;$

(ii) $\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}C(u_i,v_j) \geq 0, \forall \; (u_i,v_j) \in [0,1]^2 \; with \; u_1 < u_2 \; and \; v_1 < v_2;$

(iii) $C(u,1) = u, C(1,v) = v, \forall \; u,v \in [0,1].$

Follows the definition, we can join the innovations of marginal GARCH process together to generate a multivariate GARCH process. The common seen bivariate copula functions are Gaussian copula, student-t copula and three types of Archimedean copulas – Clayton, Gumbel and Frank. Gaussian and student-t copula model show no tail dependence between assets but student-t copula has more observations in the tails. Clayton, Gumbel and Frank copulas represent lower, upper and two-sided tail dependency respectively. The function forms of these copulas are showed in the Appendix A. We will display the pricing result of option value under these five kinds of copula models and analyze the differences between them.

For each copula functions, there exists a concordance measure Kendal's τ which has a one-to-one relation to defined parameter. We can also calculate the Kendall's τ of each copula functions by following formula:

$$\tau(\theta) = 4EC_{\theta}(U, V) - 1,$$

where C_{θ} joins random vector (U, V) as a joint distribution and expectation is taken with respect to (U, V). Appendix B shows the closed form formula of Kendall's τ for each copula functions. For unifying the pricing condition, we do the valuation under the setting that all the Kendall's tau of different copula functions has the equal value.

4.3 Monte Carlo Simulation

To generate a pair of price process $(R_{1,t}, R_{2,t})$ from $t = 0$ to T, we should first simulate $(r_{1,t}, r_{2,t})$ for every $t = 1 \dots T$. The steps of generating $(r_{1,t}, r_{2,t})$ from time period $t - 1$ to t are listed as follows:

- Generate a pair of observations (u, v) from random vector (U, V) where U, V follow independent uniform distribution.
- Let $u_1 = u$ and calculate $u_2 = c_{u_1}^{-1}(v)$ where $c_u(v)$ is the partial derivative of the copula defined as:

$$c_u(v) = \lim_{\Delta u \to \infty} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = \frac{\partial C}{\partial u} = C_u(v).$$

  The inverse function of $c_u(v)$ of each copula will be shown in Appendix C. After the transformation, the observation (u_1, u_2) will be a pair of observation from (U_1, U_2), which is joint distributed as C.

- $F_i(\cdot), i = 1{,}2$ represents the cumulated density function of the marginal innovations $\eta_{i,t} \sim \text{Normal}(0, h_{i,t})$. Take $\left(N_{1,t}, N_{2,t}\right) = (F^{-1}(u_1), F^{-1}(u_2))$ as the observation of innovations at time t. Calculate $(r_{1,t}, r_{2,t})$ by formula:

$$r_{i,t} = r_f - \frac{1}{2} h_{i,t-1} + N_{i,t}, i \in \{1{,}2\}.$$

After we have all $(r_{1,t}, r_{2,t})$ from $t = 1, \dots, T$, we can get the price process $(R_{1,t}, R_{2,t})$ defined as:

$$R_{\mathrm{i},t} = \exp\left(\sum_{j=1}^{t} r_{i,j}\right), t = 1, \ldots, t, i = 1,2.$$

Finally, we can calculate the payoff of price $(R_{1,T}, R_{2,T})$ and count it in as one of the simulation sample.

5 Result Analysis

We separate the simulation condition into many cases for realizing the price difference between different copula functions. Table 1 shows the conditions we set for each case. We will discuss the parameter settings and results case by case.

Table 1
The pricing conditions of each case

	Distribution of GARCH innovation	GARCH parameter	Initial Volatility (h_0)
Case I	Normal	The same	The same
Case II	Normal	The same	Different
Case III	Normal	Different	The same
Case IV	Student's t	The same	The same

Case I

We set case I in the most basic environment. Assume that parameters of marginal GARCH processes are in the same value. The setting can help us to realize the option price relations between different pricing copula models when underlying assets have the similar pattern. The group of settings is a reasonable GARCH parameter setting related to the empirical estimation results. The Kendal's τ of each copula function is defined to be 0.5 and ν of student's t copula equals to 5.

Table 2

Parameter settings of case I

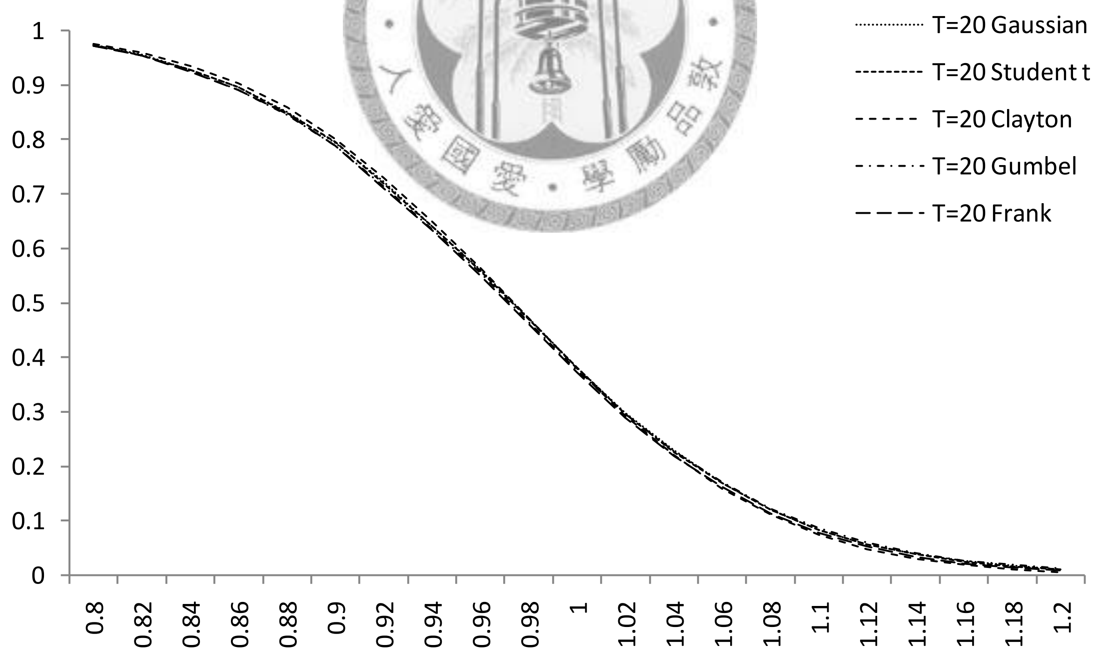
Parameter	R_1	R_2
μ_i	0.0005	0.0005
ω_i	0.00001	0.00001
β_i	0.92	0.92
α_i	0.06	0.06

Each marginal GARCH process starts from $h_{i,0} = \omega_i/(1-\alpha_i-\beta_i)$. The pricing results of 20-day expired digital option relate to different strike prices K after 10000 times simulations are showed in Figure 2. The results for digital options in other maturities are showed in Figure 3. After our coordination, Figure 4 shows the pricing result of digital options with different time to maturities. By calculating the difference between pricing results of different copula models for each simulation path, we can do the T test to examine whether the differences among the results are zeros. Finally, we find that the option price simulated by Frank copula has slightly different to the option prices simulated by other copula functions. Table 3 shows the test results for 1 month[2] and 6 month matured ATM digital options. Comparing the T statistics of 1 month and 6 months digital options, we can observe that T statistics does not increase in all copula pairs. It does not match the observation that the price differences between different pricing copulas are seem to be widen as time to maturity comes longer in Figure 2 and the mean value showed in table 3.

[2] In this paper, every 1 month have 20 days.

In addition, the result can also help us on realizing the option price changing related to different strike prices or time to maturities. We can see that ITM[3] digital option prices decrease as time to maturity comes larger in Figure 3. It is not observed in ATM or OTM digital options and violating the general sense of option. We conjecture that the increasing time to maturity may decrease the probability that R_1, R_2 both stay in the money as time to maturity comes longer. Since the option holder only get paid when R_1, R_2 both stay in the money, the ITM digital option price would decreases as time to maturity comes longer.

Figure 2
Pricing Result vs. Strike Prices of 1 month matured Digital Option.



[3] ITM, ATM, OTM represent in the money, at the money and out of the money options. Since R_1, R_2 are started at the same value 1, ATM indicates that K equals to 1 and so on.

Figure 3

Pricing Result vs. Strike Prices of Digital Option in Different Time to Maturities

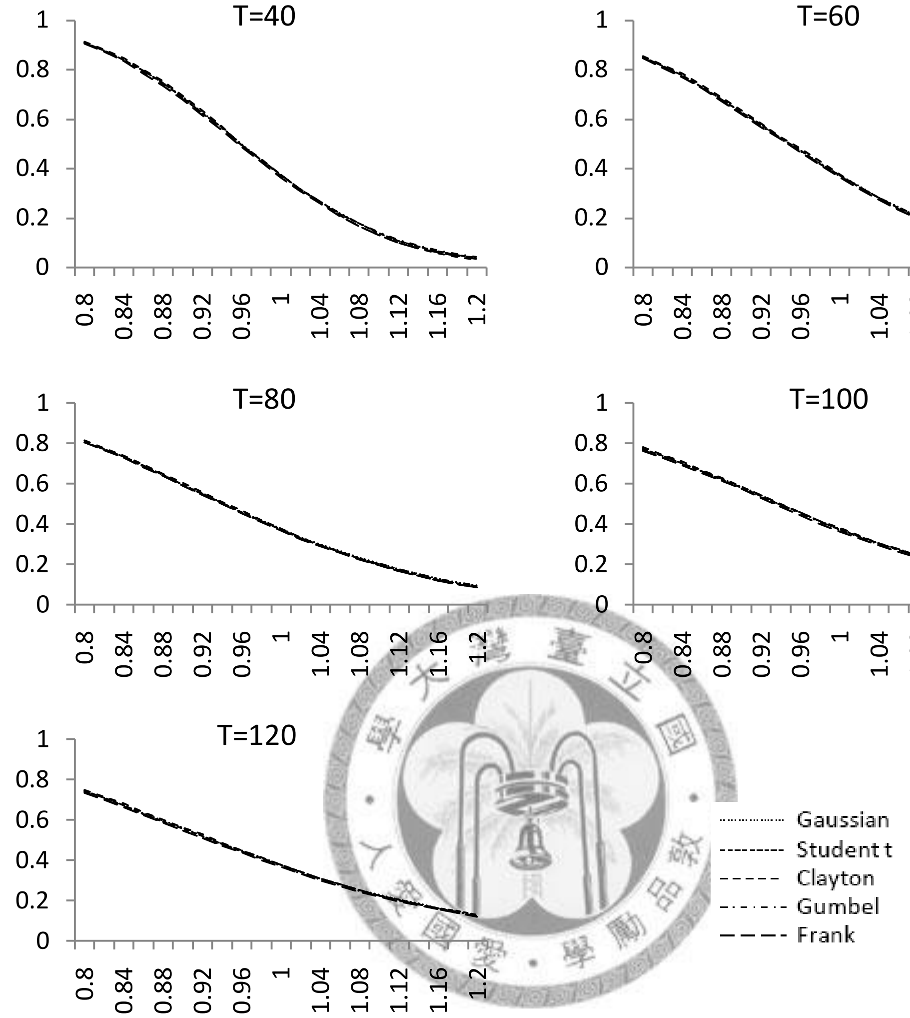


Figure 4

Digital Option Price vs. Time to Maturity

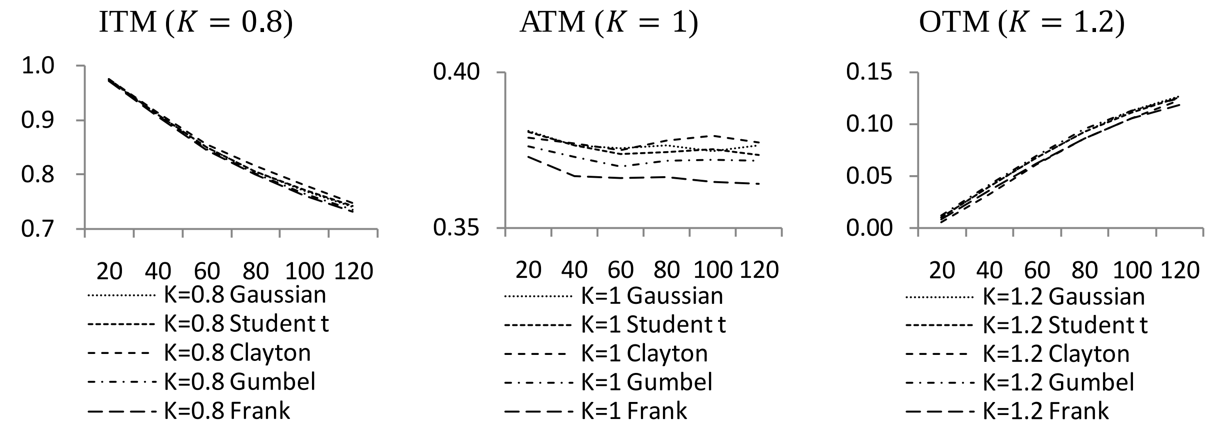


Table 3

Zero test for differences between values of 1 month matured ATM digital option pricing by different copulas

		Gaussian	Student t	Clayton	Gumbel
T=20	Student t	-0.0004 (-0.3266)			
	Clayton	-0.0021 (-0.9889)	-0.0017 (-0.78)		
	Gumbel	-0.005 (-3.1511)	-0.0046 (-2.8429)	-0.0029 (-1.108)	
	Frank	-0.0085 (-5.4833)	-0.0081 (-4.6581)	-0.0064 (-2.9162)	-0.0035 (-1.8124)
		Gaussian	Student t	Clayton	Gumbel
T=120	Student t	-0.0033 (-2.5392)			
	Clayton	0.0009 (0.4365)	0.0042 (2.0305)		
	Gumbel	-0.005 (-3.3276)	-0.0017 (-1.0646)	-0.0059 (-2.3568)	
	Frank	-0.0126 (-7.6619)	-0.0093 (-5.1813)	-0.0135 (-6.0061)	-0.0076 (-4.2258)

Sample points are the differences between payoffs simulated from row copula model and line copula model for every simulation path. Table value represents the mean of the differences and the value in the parentheses is the T statistics. Here only shows the result of 1 and 6 month ATM digital option. The test for other strike prices and maturities shows the similar result.

The similar analysis is done for spread and rainbow options under the same simulation sample. Figure 5 is the pricing result of spread option and Figure 7 is the result of rainbow option under the same random sample as we used during pricing the digital option. We should notice that the strike price setting of spread option is different from the rainbow and digital options. The strike price K of ATM spread option is defined to be zero since the initial spread between underlying assets is zero. The price of spread option and rainbow option vs. time to maturity graph is showed in Figure 6 and Figure 8. We do not observe the pattern that option price decreases as time to maturity increases of digital options. The prices of spread and rainbow option all increase as time to maturity increases.

The zero-test for the differences between pricing results of different copulas is also done in Table 4 and 5. We can see that the pricing result using Frank copula has significantly different to the pricing result using other copula functions in spread and rainbow options although the mean of differences is small. Furthermore, we observe that differences of results from every pair of pricing copulas are almost significantly not to be zero in the zero-test done for rainbow options, except the difference between Gaussian and student’s t copulas. We conjecture that rainbow option maybe more sensitive to the setting of copula models.

Figure 5

Pricing Result vs. Strike Prices of Spread Option in Different Time to Maturities

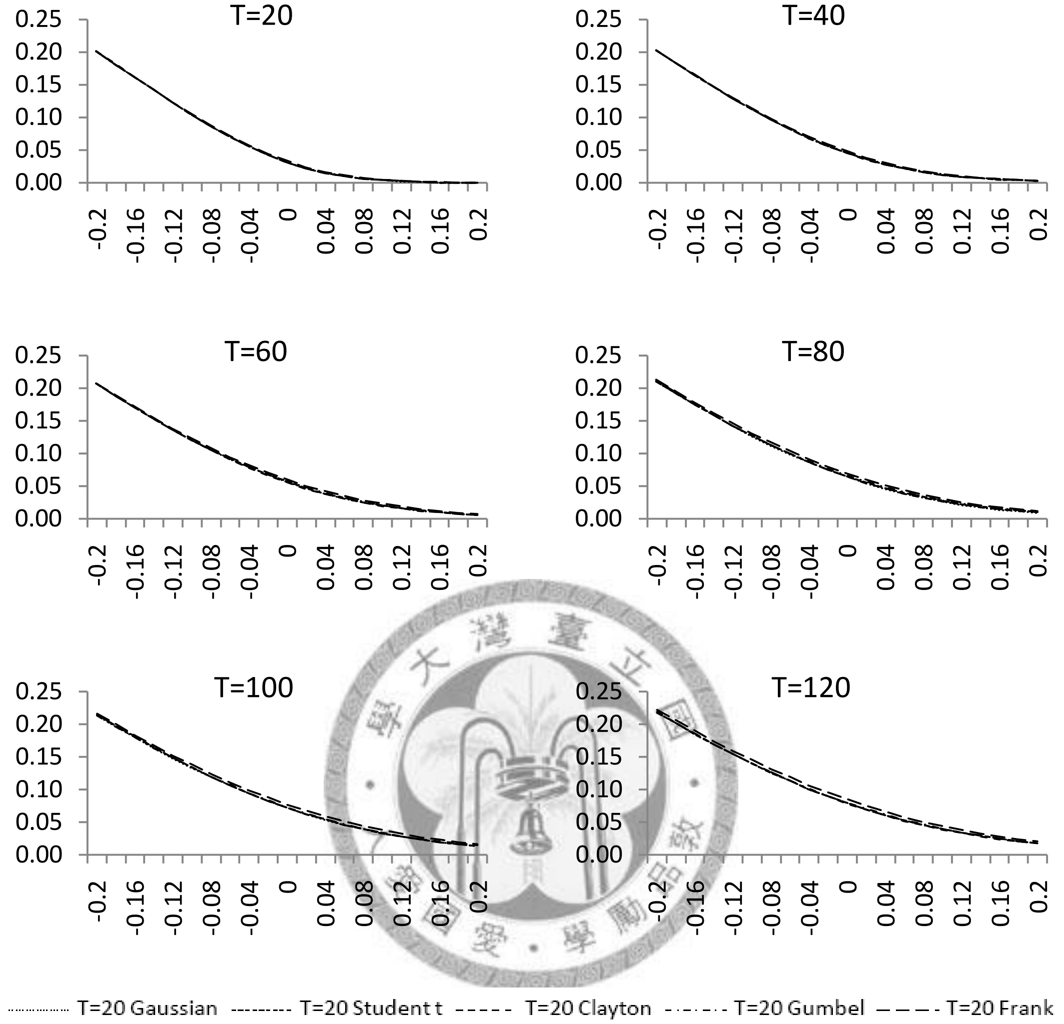


Figure 6

Spread Option Price vs. Time to Maturity

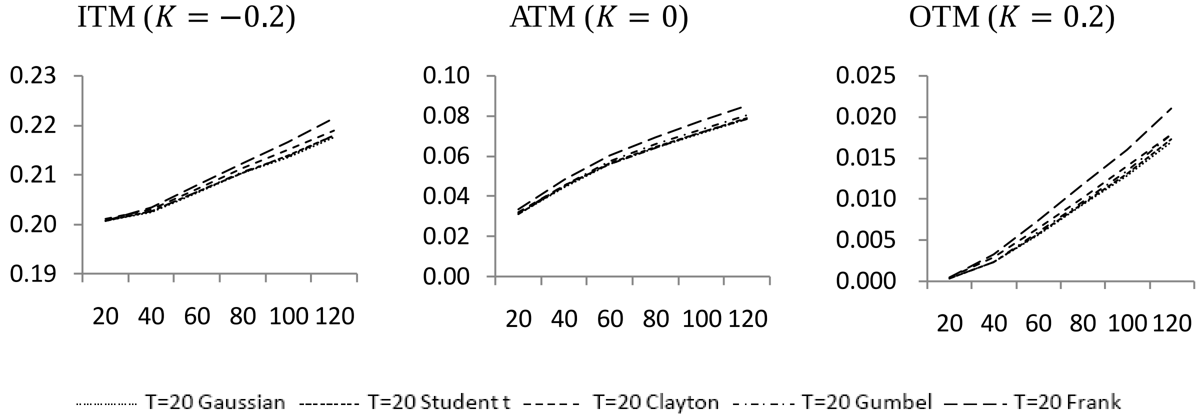


Table 4

Zero test for differences between values of 1 month and 6 month matured Spread Option pricing by different copulas

		Gaussian	Student t	Clayton	Gumbel
T=20	Student t	0.0000 (0.2343)			
	Clayton	0.0005 (2.6417)	0.0005 (2.3716)		
	Gumbel	0.0003 (2.8215)	0.0003 (2.6813)	-0.0002 (-0.7637)	
	Frank	0.0022 (18.3994)	0.0022 (14.8501)	0.0017 (7.5592)	0.0019 (11.9554)
		Gaussian	Student t	Clayton	Gumbel
T=120	Student t	0.0003 (1.5662)			
	Clayton	-0.0001 (-0.1102)	-0.0004 (-0.6494)		
	Gumbel	0.0016 (5.0639)	0.0012 (3.9945)	0.0016 (2.0219)	
	Frank	0.0063 (19.3695)	0.0060 (14.8072)	0.0064 (10.7111)	0.0048 (10.4353)

Sample points are the differences between payoffs simulated from row copula model and line copula model for every simulation path. Table value represents the mean of the differences and the value in the parentheses is the T statistics. Here only shows the result of 1 and 6 month ATM spread option. The test for other strike prices and maturities shows the similar result.

Figure 7

Pricing Result vs. Strike Prices of Rainbow Option in Different Time to Maturities

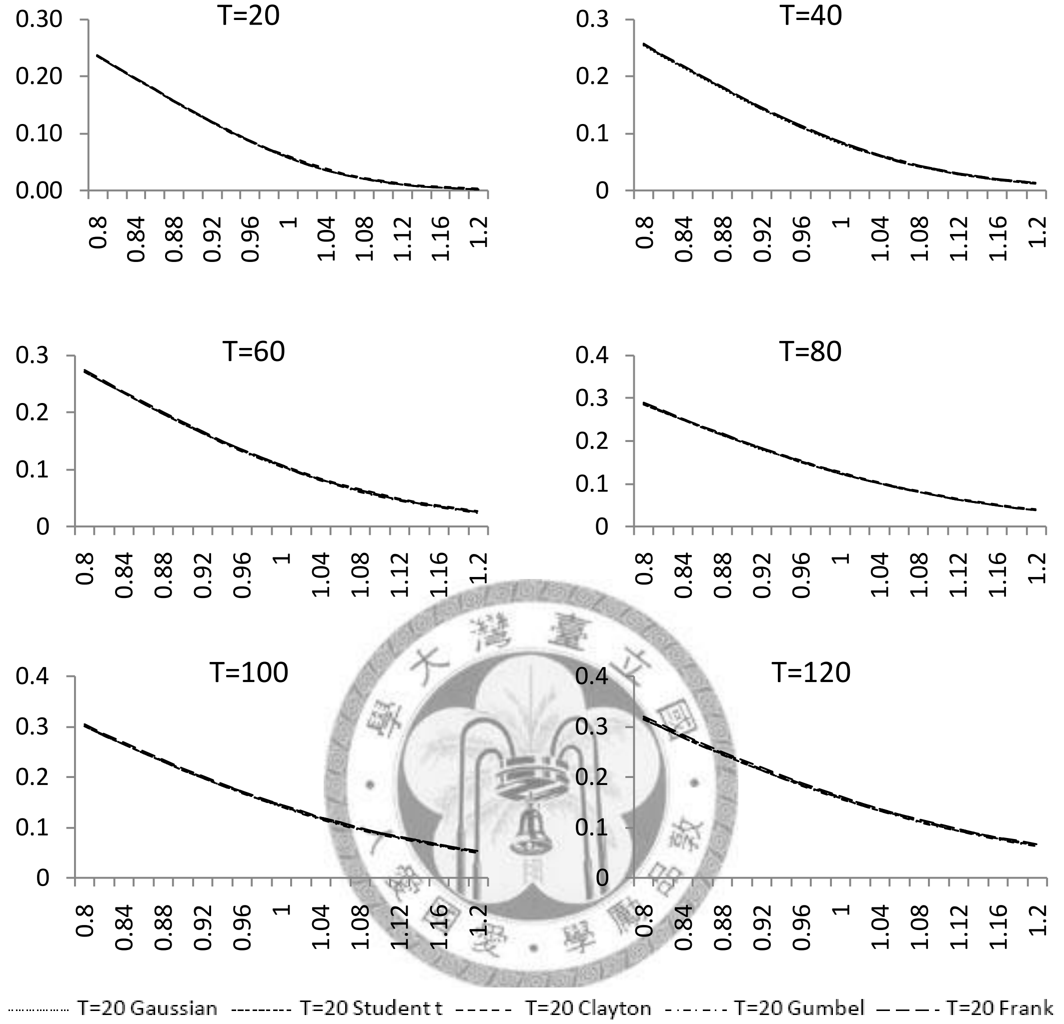


Figure 8

Rainbow Option Price vs. Time to Maturity

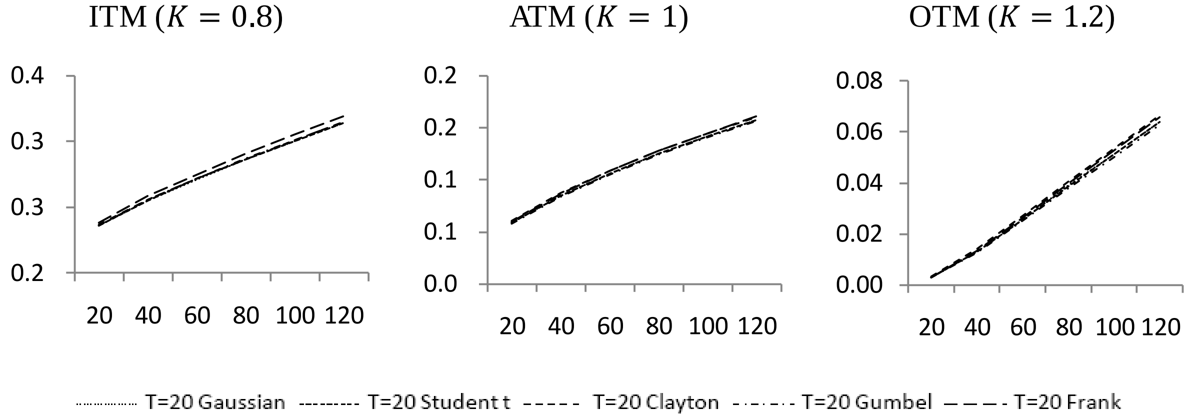


Table 5

Zero test for differences between values of 1 month and 6 month matured Rainbow Option pricing by different copulas

		Gaussian	Student t	Clayton	Gumbel
T=20	Student t	0.0000 (0.0406)			
	Clayton	0.0016 (9.6073)	0.0016 (8.5977)		
	Gumbel	-0.0005 (-5.7921)	-0.0005 (-6.4026)	-0.0021 (-8.8908)	
	Frank	0.0011 (11.3656)	0.0011 (9.2638)	-0.0005 (-2.7898)	0.0016 (11.3927)
		Gaussian	Student t	Clayton	Gumbel
T=120	Student t	0.0004 (2.0045)			
	Clayton	0.0029 (6.1315)	0.0025 (4.8577)		
	Gumbel	-0.0005 (-1.7964)	-0.0009 (-3.2898)	-0.0034 (-4.9202)	
	Frank	0.0036 (12.625)	0.0032 (9.2019)	0.0007 (1.4442)	0.0041 (9.5622)

Sample points are the differences between payoffs simulated from row copula model and line copula model for every simulation path. Table value represents the mean of the differences and the value in the parentheses is the T statistics. Here only shows the result of 1 and 6 month ATM rainbow option. The test for other strike prices and maturities shows the similar result.

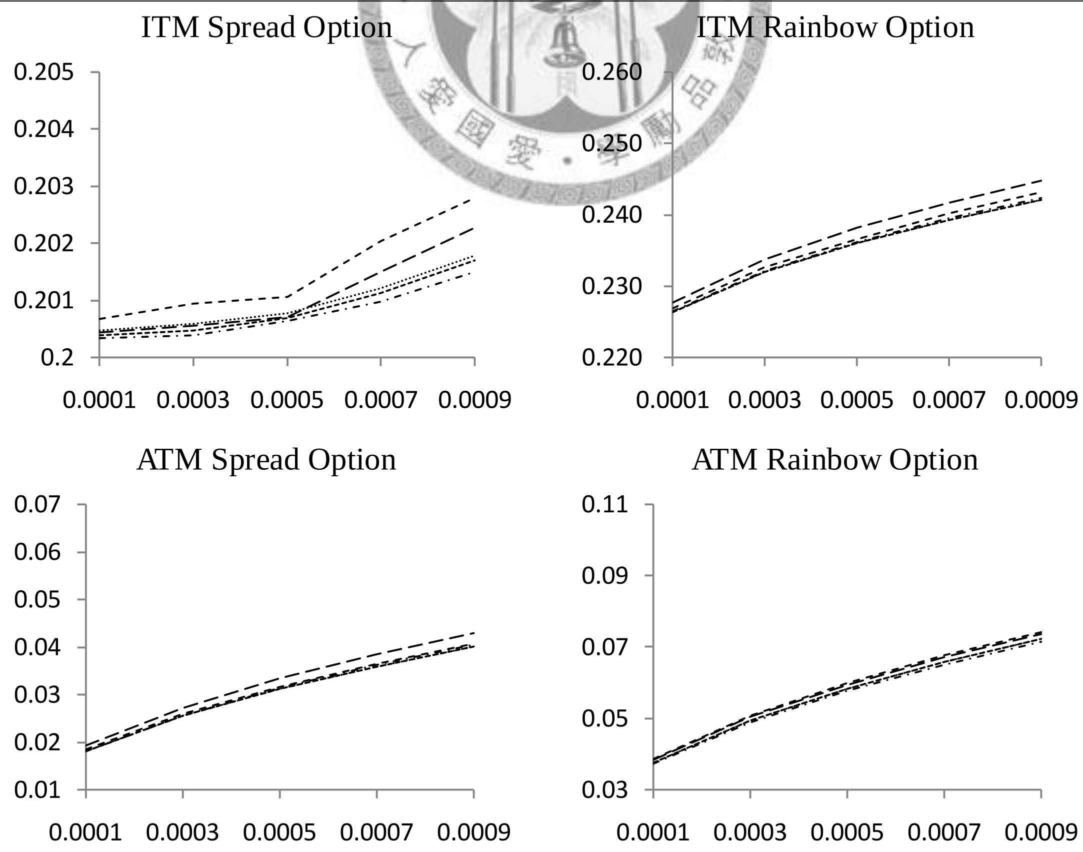
The above results are done under the GARCH parameter settings in table 2 and assumption that the initial volatility h_0 equals to 0.0005, which is the long term mean of volatility. We further simulate the return process by separately setting the initial volatility h_0 equals to 0.0001, 0.0003, 0.0007 and 0.0009, to examine the price changes of different types of bivariate option in different initial volatility. Figure 9 shows the price change with respect to initial volatility for spread and rainbow options. We can see that differences between different pricing copulas are seem to be widen as initial volatility comes larger.

Notwithstanding we observe that the pricing result differences between different copula are all widen as greater initial volatility, the T statistics of zero-test in Table 6 do not show a big improvement by comparing with the results in Table 3, 4 and 5. Therefore, we summarize that the significance of zero-test for the result difference between different pricing copulas would not be affected by the setting of initial volatility.

However, we observe some fundamental characters for spread and rainbow options. These two types of option prices increase when the initial volatility increases. In addition, the influence of initial volatility comes greater if option comes more in the money. On the other hand, we find the pricing result of digital option is different to spread or rainbow option. Figure 10 is the price changes of digital option responds to the initial volatility changes. We observe the pattern which is opposite to the price pattern of spread and rainbow options, the price of ITM and ATM digital option decreases as the initial volatility comes larger. Our conjecture is similar to the idea we give to explain that digital option price decreases as time to maturity comes longer. Higher initial volatility would enhance the uncertainty that ATM or ITM digital option becomes out of money, but would not influences the OTM digital options. The payoff logic of digital option may induce the special pattern.

Figure 9

Price vs. Initial volatility h_0 for Spread and Rainbow Option



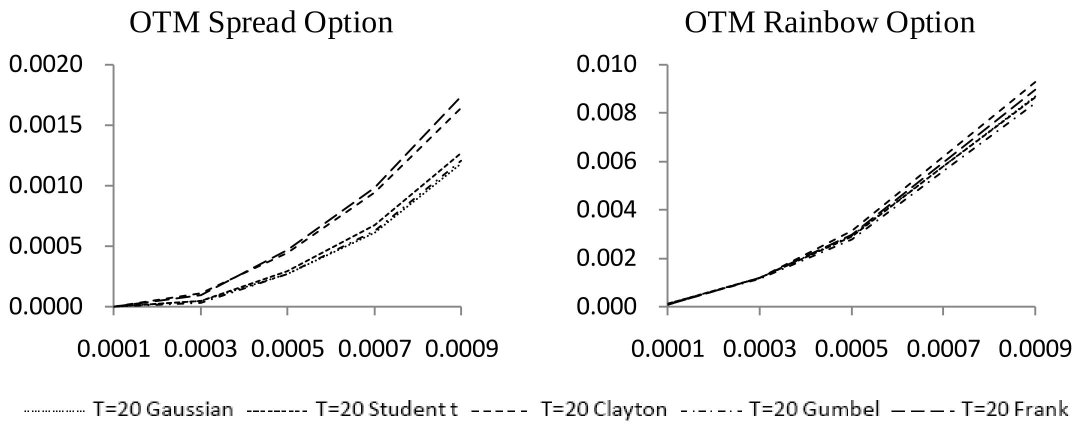
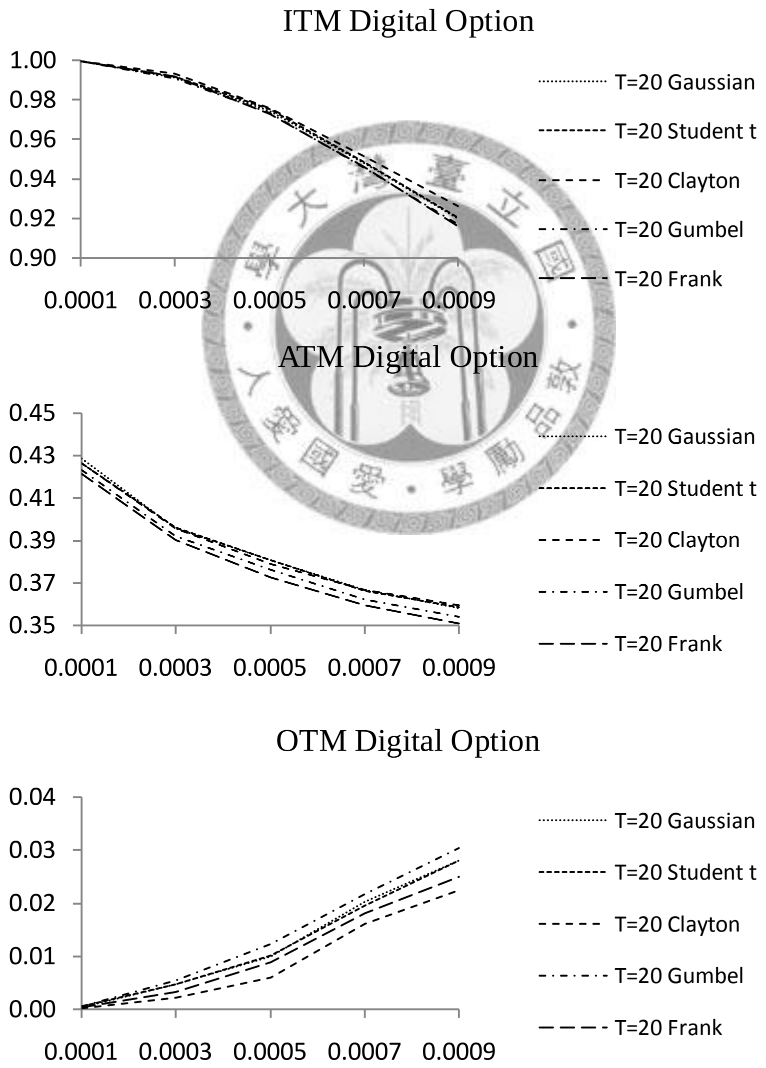


Figure 10

Price vs. Initial volatility h_0 for Digital Option



We only show the result of options matured in 1 month. The behavior of options with longer maturity is very similar.

Table 6

Zero test for differences between values of 1 month matured ATM Option pricing by different copulas

		Gaussian	Student t	Clayton	Gumbel
Digital Option	Student t	0.0001			
		(0.0887)			
	Clayton	0.0013	0.0012		
		(0.6366)	(0.5747)		
	Gumbel	-0.0043	-0.0044	-0.0056	
		(-2.8936)	(-2.995)	(-2.2568)	
	Frank	-0.0072	-0.0073	-0.0085	-0.0029
		(-4.7118)	(-4.3433)	(-4.0144)	(-1.5941)
		Gaussian	Student t	Clayton	Gumbel
Spread Option	Student t	0.0000			
		(0.0332)			
	Clayton	0.0007	0.0007		
		(2.8143)	(2.6024)		
	Gumbel	0.0005	0.0005	-0.0003	
		(3.1388)	(3.1616)	(-0.7578)	
	Frank	0.0029	0.0029	0.0022	0.0024
		(18.6181)	(15.1105)	(7.4643)	(11.5848)
		Gaussian	Student t	Clayton	Gumbel
Rainbow Option	Student t	0.0000			
		(0.2866)			
	Clayton	0.0021	0.0021		
		(9.9282)	(8.8268)		
	Gumbel	-0.0006	-0.0006	-0.0027	
		(-4.9426)	(-5.7222)	(-8.7957)	
	Frank	0.0015	0.0015	-0.0006	0.0021
		(11.7976)	(9.5066)	(-3.0447)	(10.9651)

Sample points are the differences between payoffs simulated from row copula model and line copula model for every simulation path. Table value represents the mean of the differences and the value in the parentheses is the T statistics. Here only shows the result of 1 month ATM option. The test for other strike prices and maturities shows the similar result.

Case II

We have done the multifarious analysis for three types of bivariate options and compare the price difference between pricing copulas in the most simple model settings in Case I. Following the same settings with Case I, we only make the initial volatility h_0 of marginal distributions be different values to examine whether the result of Case I is consistent. We assume h_0 of R_1, R_2 be 0.0001 and 0.0009 for 10,000 times simulation and compare the pricing result with Case I.

The option price vs. strike price graph of Case II is very similar to the graph of Case I. We put the pricing result of 1 month matured options comparing the result of Case I in Figure 11. The price differences between different pricing copulas in Case II are smaller than in the Case I. The zero-test result is put in Table 7. Comparing with Table 4, 5, and 6, we observe that mean value of differences and T statistics are both smaller than the T statistics we get in the previous zero-test done in Case I. The decreasing phenomenon is not evident for digital options, but significantly for spread and rainbow option. Therefore, we presume that the difference between the initial volatility of marginal GARCH process does not have great impact to the pricing result of digital options. However, the difference between the initial volatility of marginal GARCH processes maybe a sensitive factor for spread and rainbow options.

Table 7

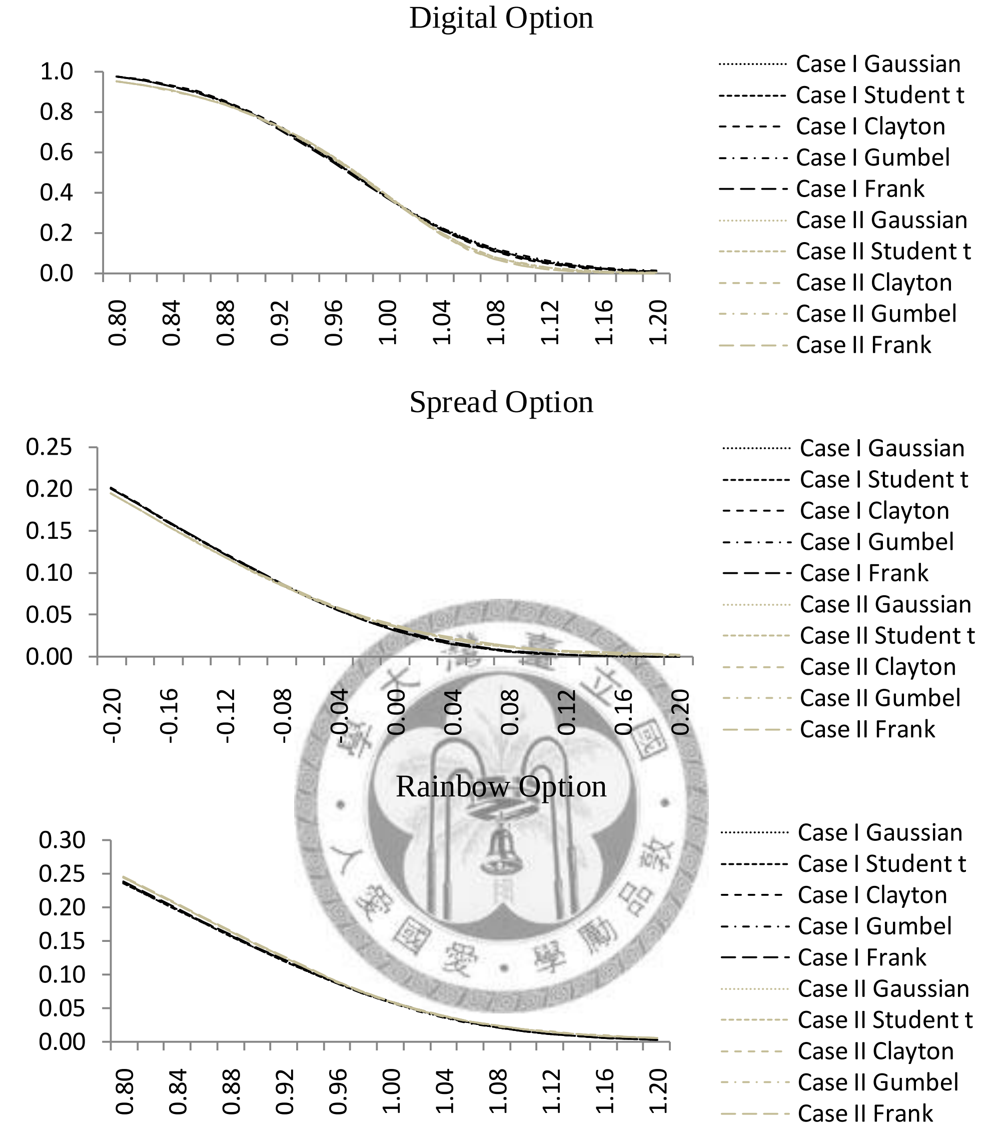
Zero test for differences between values of 1 month matured ATM Option pricing by different copulas

		Gaussian	Student t	Clayton	Gumbel
Digital Option	Student t	0.0009 (0.7423)			
	Clayton	0.0008 (0.3535)	-0.0001 (-0.0434)		
	Gumbel	-0.0044 (-2.6786)	-0.0053 (-3.2697)	-0.0052 (-1.894)	
	Frank	-0.007 (-4.217)	-0.0079 (-4.3727)	-0.0078 (-3.3646)	-0.0026 (-1.3099)
		Gaussian	Student t	Clayton	Gumbel
Spread Option	Student t	0.0000 (0.4278)			
	Clayton	0.0008 (3.2054)	0.0008 (2.757)		
	Gumbel	0.0001 (0.8077)	0.0001 (0.5814)	-0.0007 (-1.8799)	
	Frank	0.0014 (9.0007)	0.0014 (7.1593)	0.0006 (2.3018)	0.0013 (5.5203)
		Gaussian	Student t	Clayton	Gumbel
Rainbow Option	Student t	0.0000 (0.0658)			
	Clayton	0.0012 (4.9325)	0.0012 (4.4132)		
	Gumbel	-0.0002 (-1.6679)	-0.0002 (-1.9012)	-0.0015 (-4.0331)	
	Frank	0.0009 (5.8685)	0.0009 (4.7598)	-0.0003 (-1.5161)	0.0011 (4.8396)

Sample points are the differences between payoffs simulated from row copula model and line copula model for every simulation path. Table value represents the mean of the differences and the value in the parentheses is the T statistics. Here only shows the result of 1 month ATM option. The test for other strike prices and maturities shows the similar result.

Figure 11

1 month matured option prices vs. strike prices conditioned in Case I and II



Here we only show the 1 month matured options. The comparison of pricing result in other maturities shows the similar result.

Case III

In Case I, II, we set the simplest setting of marginal GARCH process and find some significant differences between different pricing copulas. We turn into more realistic setting and let the marginal GARCH process be different. The parameters we set for marginal GARCH process are showed in Table 8. The GARCH parameters of R_1

are unchanged and the parameters of R_2 are another group of reasonable settings. We design the parameter settings of R_1, R_2 to be the different GARCH process but have the same long term mean of volatility. The Kendal's τ of each copula functions is 0.5, which equals to the value we use in Case I and II.

Table 8

Parameter settings of case III

Parameter	R_1	R_2
μ_i	0.0005	0.0005
ω_i	0.00001	0.00005
β_i	0.92	0.90
α_i	0.06	0.09

Since the difference of pricing result using distinct copula functions does not have big differences between different maturities as our observation in Case I, we only put the simplified result which only contains the 1 month matured option price vs. strike prices graph in Figure 12. We can see that the price curves by distinct pricing copulas are almost in the same curve. However, we still can find some significant differences in the results of zero-tests for difference between distinct pricing copulas in Table 9. By comparing with the test done in Case I and II, we observe that T statistics have decreased and shows the results that the difference is not insignificantly to be zero for some price difference between pricing copula pairs.

After the analysis in Case I, II and III, we price the three kinds of bivariate options in three different conditions under the GARCH model with normal innovations and get some significant result supports that the differences between distinct pricing copulas do exist. In Case IV, we will price the bivariate options under the GARCH model with student's t distribution innovations and test that whether the significant result can be sustained.

Figure 12

1 month matured option prices vs. strike prices conditioned in Case III

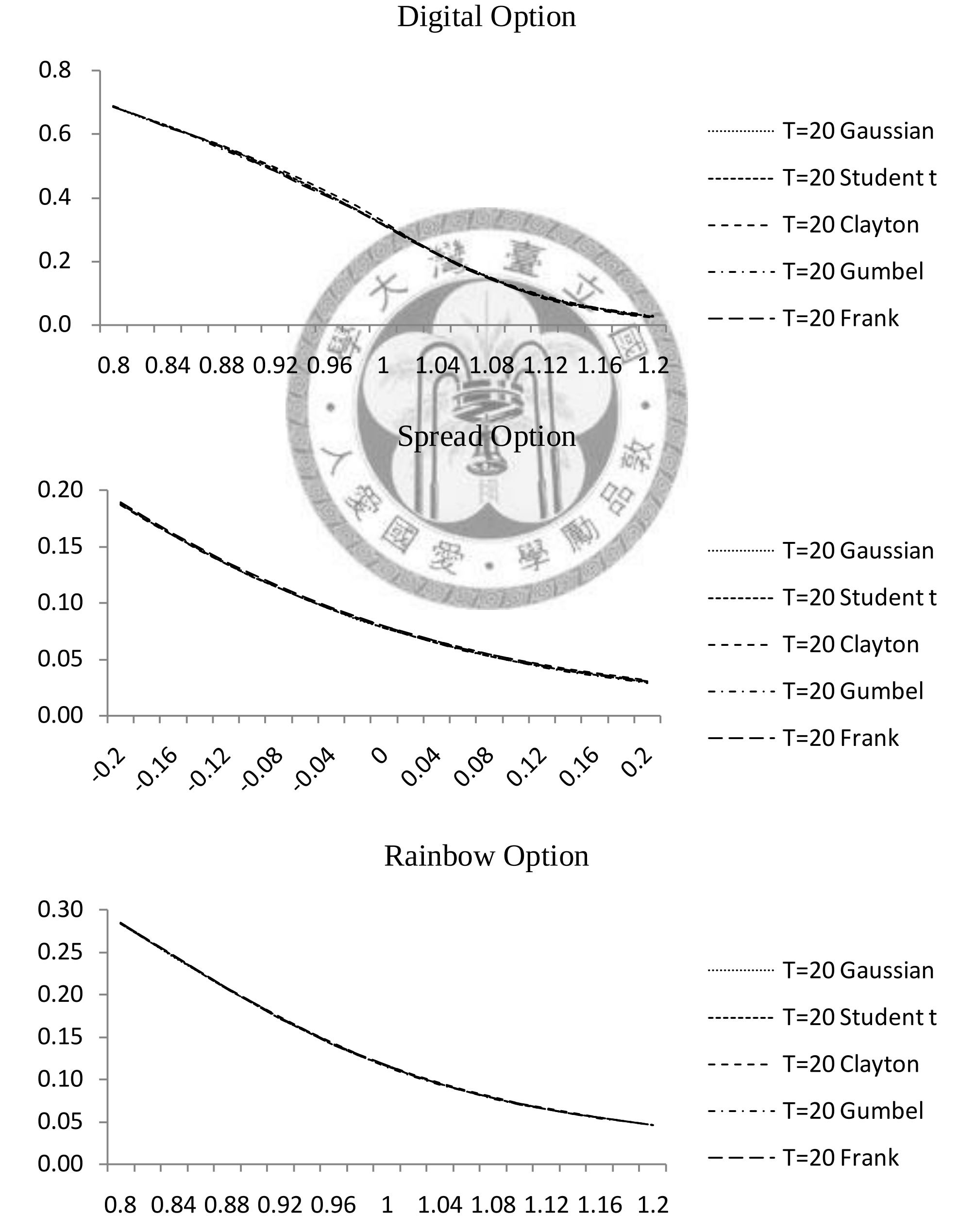


Table 9

Zero test for differences between values of 1 month matured ATM Option pricing by different copulas

		Gaussian	Student t	Clayton	Gumbel
Digital Option	Student t	-0.0024 (-1.863)			
	Clayton	0.0045 (1.9794)	0.0069 (3.0069)		
	Gumbel	-0.0038 (-2.3572)	-0.0014 (-0.8249)	-0.0083 (-3.034)	
	Frank	-0.0056 (-3.4616)	-0.0032 (-1.7835)	-0.0101 (-4.2679)	-0.0018 (-0.9435)
		Gaussian	Student t	Clayton	Gumbel
Spread Option	Student t	0.0001 (0.4325)			
	Clayton	0.0012 (1.6923)	0.0011 (1.3815)		
	Gumbel	0.0003 (0.8696)	0.0002 (0.6613)	-0.0008 (-0.8252)	
	Frank	0.0021 (4.8604)	0.0019 (3.7358)	0.0009 (1.3876)	0.0017 (2.6471)
		Gaussian	Student t	Clayton	Gumbel
Rainbow Option	Student t	0.0000 (0.1796)			
	Clayton	0.0018 (2.6606)	0.0018 (2.3277)		
	Gumbel	-0.0003 (-0.7267)	-0.0003 (-0.9436)	-0.0021 (-2.1152)	
	Frank	0.0013 (3.2653)	0.0013 (2.5533)	-0.0005 (-0.7564)	0.0016 (2.5396)

Sample points are the differences between payoffs simulated from row copula model and line copula model for every simulation path. Table value represents the mean of the differences and the value in the parentheses is the T statistics. Here only shows the result of 1 month ATM option. The test for other strike prices and maturities shows the similar result.

Case IV

The above conditions are all under the assumption that the distribution of GARCH innovation followed normal distribution. In Case IV, we define the distribution of GARCH innovation as student's t distribution to examine the influence of copula model in pricing bivariate options. The marginal GARCH parameter settings follow the same group of settings in table 2. We set the marginal GARCH innovations follow student's t distribution with degrees of freedom $\nu_{m,i}, i = 1,2$. Here we denote the degrees of freedom $\nu_{m,i}$ of marginal student's t distribution with subscript m to separate it from the degrees of freedom ν of student's t copula. In the following simulation, we assume that $\nu_{m,1} = \nu_{m,2} = 10$, and the parameter settings of copula models are the same to the settings in Case I, II and III.

After we simulate the return processes, we calculate the option price by taking the average value of payoffs from return processes simulated by different copula models. We only show the result of 1 month matured option value in Figure 13 since that results of different maturities are similar to 1 month matured result. The results of zero-test for differences between different pricing copulas are in table 10. We can see that the T statistics of many pair of pricing results from different copulas are big and shows the significance of differences. In addition, we observe that not only the T statistics increase but also the means of differences increase when we compare the result with above cases.

If we compare the option value we get in Case I, we can also see that the difference caused by different GARCH marginal distributions is not huge. By the result of Case IV, we induce that the student's t innovation of marginal GARCH process may widen the price differences between different pricing copulas.

Figure 13

1 month matured option prices vs. strike prices conditioned in Case IV

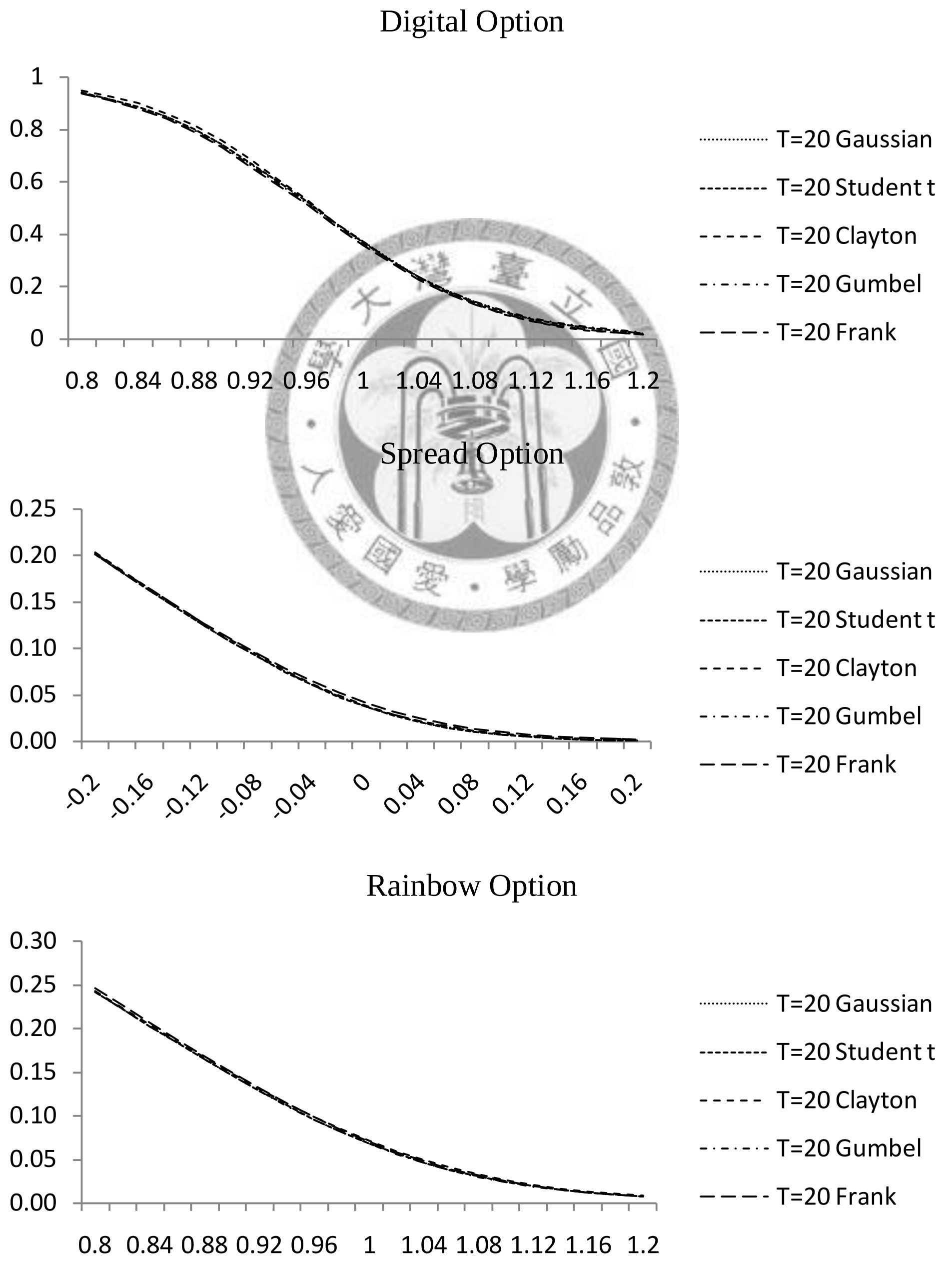


Table 10

Zero test for differences between values of 1 month matured ATM Option pricing by different copulas

		Gaussian	Student t	Clayton	Gumbel
Digital Option	Student t	-0.0008 (-0.6713)			
	Clayton	0.0001 (0.0482)	0.0009 (0.4147)		
	Gumbel	-0.0051 (-3.2466)	-0.0043 (-2.7259)	-0.0052 (-2.0244)	
	Frank	-0.011 (-7.0303)	-0.0102 (-5.784)	-0.0111 (-5.0674)	-0.0059 (-3.1596)
		Gaussian	Student t	Clayton	Gumbel
Spread Option	Student t	-0.0004 (-3.6912)			
	Clayton	0.0008 (2.7943)	0.0011 (3.9174)		
	Gumbel	0.0004 (2.3511)	0.0007 (4.8285)	-0.0004 (-1.0358)	
	Frank	0.0038 (21.7963)	0.0042 (18.9589)	0.003 (10.2034)	0.0034 (14.756)
		Gaussian	Student t	Clayton	Gumbel
Rainbow Option	Student t	-0.0001 (-1.4341)			
	Clayton	0.0025 (11.0135)	0.0026 (10.4745)		
	Gumbel	-0.0009 (-6.4673)	-0.0007 (-6.5128)	-0.0034 (-10.3985)	
	Frank	0.0019 (13.6277)	0.0021 (11.7297)	-0.0006 (-2.7407)	0.0028 (12.912)

Sample points are the differences between payoffs simulated from row copula model and line copula model for every simulation path. Table value represents the mean of the differences and the value in the parentheses is the T statistics. Here only shows the result of 1 month ATM option. The test for other strike prices and maturities shows the similar result.

6 Conclusion

In this paper, we construct the copula-GARCH model by following the methodology of Goorbergh, Genest and Werker (2005), and extend the pricing model to 5 copula functions and 3 different types of bivariate options. We first define the marginal distribution of each asset return follows a GARCH (1, 1) process, and joint the return process by copula functions. Further, we simulate the return process of each asset under the copula-GARCH model and calculate option payoffs for each simulation. Finally, we can take the average value of the payoffs from 10000 times simulations as the option pricing result and analyze the results.

In our simulation, we separate the pricing condition into four cases. Case I represents the bivariate option of assets which have very similar volatility pattern. Case II is the same condition with Case I, except that the initial volatilities of each asset are different. Case III represents the bivariate option of assets with totally different return process. Case IV is similar to Case I, but the marginal GARCH innovations are substituted by student's t distributions.

Under results of all these pricing conditions, we analyze the outcomes and point out some observations of the results. First observation is that option price simulated by Frank copula is always different a lot from the option price simulated by other copulas. Second, the differences between different pricing copulas widen as the maturity of option comes longer or higher initial volatility, but the significance of the test for differences does not increase at the same time (i.e. the T statistics does not increases). Third, the setting of the innovation of marginal GARCH process is a important factor since it do affect the pricing result. The mean and T statistics of price differences between different copula models are wider in the student's t innovation than in the normal innovation.

Summarizing the results in all cases, the zero-test result for the differences suggest that rainbow option may be the type of bivariate option which is most sensitive to the selecting of copula model since there are always greatest amount of pairs of copulas showing that the difference between their pricing result is significantly not zero. We recommend that the copula function selecting on bivariate option pricing under copula-GARCH model is very important. Even though the differences of pricing result between different copula models are small, the differences still significantly exist.

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Appendix A. Common Bivariate Copula Functions

Name	Copula Function
Gaussian	$C_{\rho}^{\mathrm{Ga}}(u,v)=\Phi_{\rho}\left(\Phi^{-1}(u),\Phi^{-1}(v)\right)$
Student's t	$C_{\rho,\nu}^{\mathrm{t}}(u,v)=\mathrm{t}_{\rho,\nu}\left(\mathrm{t}_{\rho}^{-1}(u),\mathrm{t}_{\rho}^{-1}(v)\right)$
Gumbel	$C_{\alpha}^{\mathrm{Gu}}(u,v)=\exp\left\{-[(-\ln(u))^{\alpha}+(-\ln(v))^{\alpha}]^{1/\alpha}\right\}$
Clayton	$C_{\alpha}^{\mathrm{Cl}}(u,v)=\max\left[(u^{-\alpha}+v^{-\alpha}-1)^{-1/\alpha},0\right]$
Frank	$C_{\alpha}^{\mathrm{Fr}}(u,v)=-\frac{1}{\alpha}\ln\left(1+\frac{(\exp(-\alpha u)-1)(\exp(-\alpha v)-1)}{\exp(-\alpha)-1}\right)$

Appendix B. Kendall's τ of each Copulas

Name	Kendall's τ
Gaussian	$\frac{2}{\pi}\arcsin(\rho)$
Student's t	$\frac{2}{\pi}\arcsin(\rho)$
Gumbel	$1-\alpha^{-1}$
Clayton	$\alpha/(\alpha+2)$
Frank	$1+4\,[\mathrm{D}_1(\alpha)-1]/\alpha$

For Frank Copula, $\mathrm{D}_1(\alpha)=\frac{1}{\alpha}\int_0^{\alpha}\frac{t}{\exp(t)-1}\mathrm{d}t$, is called the "Debye" function.

Appendix C. Inverse Function of $c_{u_1}(v)$ of each Copula Models

Name	$c_{u_1}^{-1}(v)$
Gaussian	$\Phi\left(\sqrt{1-\rho^2}\Phi^{-1}(v)+\rho\Phi^{-1}(u_1)\right)$
Student's t	$\mathrm{t}_\nu\left(\rho\,\mathrm{t}_\nu^{-1}(u_1)+\sqrt{\left[\nu+\left(\mathrm{t}_\nu^{-1}(u_1)\right)^2\right]\times\frac{1-\rho^2}{\nu-1}}\,\mathrm{t}_{\nu+1}^{-1}(v)\right)$
Gumbel	No close-form solution
Clayton	$\left[u_1^{-\alpha}\left(v^{-\frac{\alpha}{\alpha+1}}-1\right)+1\right]^{-\frac{1}{\alpha}}$
Frank	$-\frac{1}{\alpha}\ln\left(1+\frac{v(1-e^{-\alpha})}{v(e^{-u_1\alpha}-1)-e^{-u_1\alpha}}\right)$

The inverse function of $c_{u_1}(v)$ of Gumbel copula can be calculated by solving an equation by numerical way. Detail is showed in the book *Copula Methods in Finance*, written by Cherubini, Luciano, and Vecchiato, 2004.