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Master Thesis

以賽局理論解決 IEEE 802.22 網路基地台間的共存問題

A Game Theoretic Resource Allocation for Inter-BS Coexistence in IEEE 802.22

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本論文係柯君翰君（R96942036）在國立臺灣大學電信工程學研究所、所完成之碩士學位論文，於民國 98 年 7 月 9 日承下列考試委員審查通過及口試及格，特此證明

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摘要

以賽局理論解決 IEEE 802.22 網路基地台間的共存問題

柯君翰

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IEEE 802.22是第一個以感知無線電為根基的無線通訊標準。在論文裡，我們將依據IEEE 802.22網路基地台共存機制(inter-BS coexistence mechanism)中的動態資源租賃(dynamic resource renting and offering)與頻道競爭(adaptive on demand channel contention)，提出一個頻譜共享演算法，其目的為達到最有效與最公平的頻譜分配。賽局理論(game theory)將被應用於分析我們的設計。首先，透過畫圖方法，我們對兩個基地台的系統進行分析，並從中得到了一些觀察結果。據此觀察，我們可導證出n個基地台系統將存在奈許均衡(Nash equilibrium)，且在奈許均衡下各基地台的效用(即頻譜分配)將是唯一的。此外我們更證明出，四個與效率、公平相關的衡量指標，包括最有效率的分配、柏拉圖最佳化、加權大-小公平分配、加權比例公平分配等，在奈許均衡下都成立的。最後，以提出的頻譜共享演算法為根基，我們可以設計出另一個中央集權式的頻譜分配機制，使得每個基地台在以增加效用為前提下，都會誠實地說出自己的需求，如此最後將達成最有效、最公平的頻譜分配。

關鍵字: IEEE 802.22; inter-BS coexistence; credit token; game theory; Nash equilibrium; strategy-proofness.

Abstract

A Game Theoretic Resource Allocation for Inter-BS Coexistence in IEEE 802.22

Chun-Han Ko

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IEEE 802.22 is the first cognitive-radio-based wireless communication standard. We propose a spectrum transaction scheme for *dynamic resource renting and offering* (DRRO) and *adaptive on demand channel contention* (AODCC) in IEEE 802.22 inter-BS coexistence mechanism to achieve efficient and fair spectrum sharing. Game theory is applied to formulate and analyze the proposed spectrum sharing algorithm. We first analyze the simplest two-base-station (BS) game through a graphical method to gain insights for the solution. Then, the Nash Equilibrium of the n-BS game is derived and the utility profile at the Nash equilibrium is shown to be unique. We prove several desirable properties, including allocative efficiency, Pareto optimality, weighted max-min fairness, and weighted proportional fairness, are attained at the Nash equilibrium. Lastly, we design a strategy-proof spectrum allocation mechanism based on the proposed spectrum sharing algorithm so that truthful strategies optimize each BS's performance.

Keywords: IEEE 802.22; inter-BS coexistence; credit token; game theory; Nash equilibrium; strategy-proofness.

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Chapter 1

Introduction

IEEE 802.22 is the first cognitive-radio-based standard operating over 54-862 MHz licensed TV bands [1]. IEEE 802.22 systems, as depicted in Figure 1.1, are composed of base stations (BSs) and consumer premise equipments (CPEs). IEEE 802.22 systems are enabled to opportunistically access the licensed spectrum bands on the premise of not interfering with the licensed users, e.g. TV stations and wireless microphones. Cognitive radio technique [2, 3] is applied to perform spectrum sensing [4]. Through spectrum sensing, vacant channels over which IEEE 802.22 systems can operate are discovered. However, an important issue arises under a common scenario that multiple IEEE 802.22 BSs operate in the same vicinity and cause severe interference. This issue is called inter-BS coexistence (or self-coexistence) in IEEE 802.22 standard. To address this issue, an inter-BS coexistence mechanism is defined.

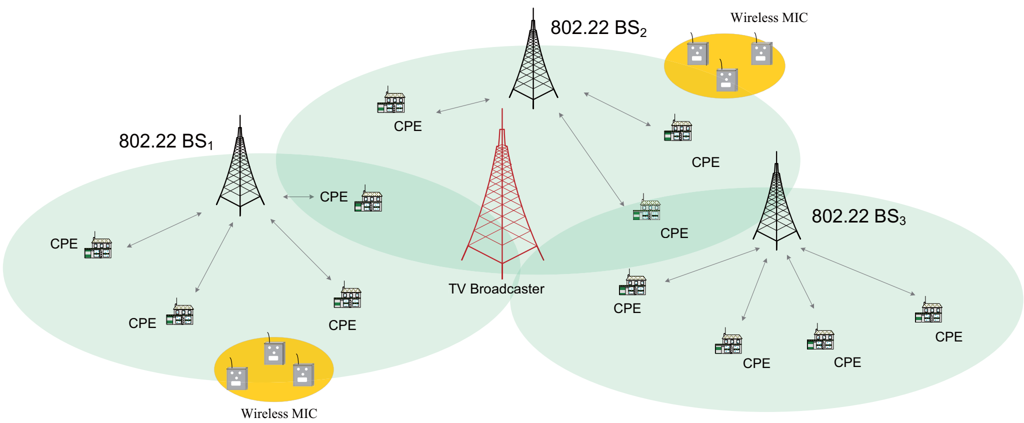


Figure 1.1: IEEE 802.22 Systems

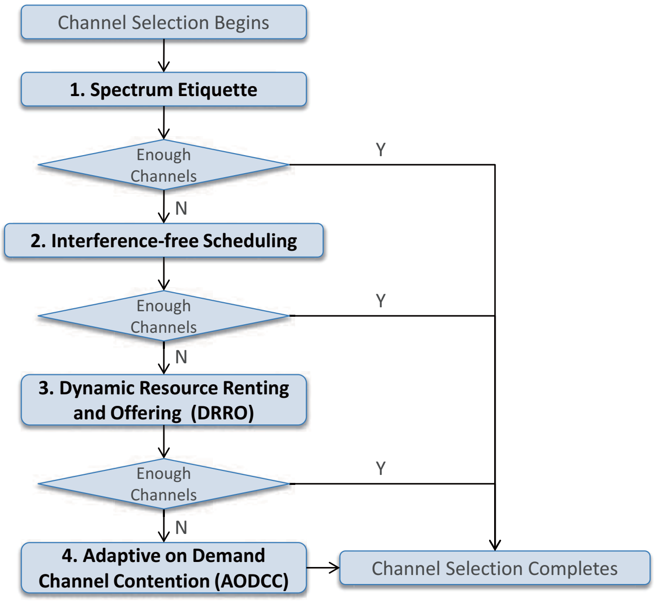


Figure 1.2: IEEE 802.22 Inter-BS Coexistence Mechanism

1.1 IEEE 802.22 Inter-BS Coexistence Mechanism

In IEEE 802.22 standard, the inter-BS coexistence mechanism [1] consists of four stages: spectrum etiquette, interference-free scheduling, *dynamic resource renting and offering* (DRRO), and *adaptive on demand channel contention* (AODCC), as illustrated in Figure 1.2. Spectrum etiquette is the first stage where BSs try to locally find channels that their neighbor BSs cannot or do not use. If no spare channel is available under this rule, BSs will conduct interference-free scheduling.

In interference-free scheduling, BSs share the same channel by scheduling their traffic in a non-interfering manner. It, however, can only occur on the premise that the BS who owns the channel agrees to share it with others. In words, if the owner needs to operate exclusively, interference-free scheduling cannot occur and the inter-BS coexistence mechanism must go to the next stage.

IEEE 802.22 uses credit token for DRRO and AODCC operations. The concept of credit token and its utilization for spectrum sharing are first introduced in [5]. In IEEE 802.22, credit token is similar to money except that credit token can be frozen but cannot be transferred. Each BS is assumed to have a pre-given credit token budget. In DRRO, two entities are defined: offeror is a BS who currently has unused resources; renter is a BS of the counterpart, who currently has an additional resource requirement. An offeror can offer its unused resources by broadcasting the offering information which includes the available resources and the minimum number of credit tokens (MNCT) required. Renters who hear the offering information can send renting requests which include the desired resources and the number of credit tokens (NCT) willing to pay. After receiving and comparing the renting requests, the offeror derives the best (in term of higher credit tokens) renters. These renters are granted to access their requested resources and NCT they are willing to pay is frozen.

AODCC is the final stage of the mechanism. AODCC is triggered when BSs do not get enough resources through the previous three stages. AODCC is very similar to DRRO except that a channel owner, also called contention destination, now passively receives contention requests. When a BS, called contention source, selects a contention destination and makes a contention request, the channel contention procedure occurs at the contention destination. The contention destination compares NCT the contention source is willing to pay with its MNCT required. If the former is larger, the contention destination shall release the requested resources and NCT the contention source is willing to pay is frozen; otherwise, the contention destination replies with rejection.

1.2 Related Work

Recently, game theory has been applied to model IEEE 802.22 operations. S. Sengupta *et al.* applied minority game theory to investigate the problem that whether a BS should stay at the present channel or switch to another channel [6]. They showed a mixed strategy Nash equilibrium existed and the mixed strategy space performed better than the pure strategy space in achieving optimal solution. D. Gao *et al.* modeled the DRRO mechanism as a progressive second price auction [7]. The utilization of this auction mechanism had a major benefit that BSs would make their requests truthfully. D. Niyato *et al.* formulated the transaction of spectrum bands between licensed users and BSs by a sealed-bid double auction [8]. They also introduced a pricing mechanism to model the service between BSs and CPEs.

Nash equilibrium was found through a numerical method. If interested in minority game theory, progressive second price auction, and double auction, readers can refer to [19, 20, 21].

In this thesis, we aim to find a game theoretic solution for IEEE 802.22 inter-BS coexistence. Compared to [6], it concentrated on spectrum etiquette. Compared to [7], it investigated DRRO without taking credit token budgets into consideration. Compared to [8], it focused on service pricing rather than inter-BS coexistence. In contrast, we propose a spectrum sharing algorithm based on the IEEE 802.22 DRRO and ADOCC mechanisms. We formulate the problem with game theory and discover that a Nash equilibrium always exists. The Nash equilibrium has some desirable properties, including allocative efficiency, Pareto optimality, weighted max-min fairness, weighted proportional fairness. Also, by adopting the allocation rule of the spectrum sharing algorithm, we design a strategy-proof mechanism which ensures efficiency and fairness at the truth-revealing dominant-strategy equilibrium.

Chapter 2

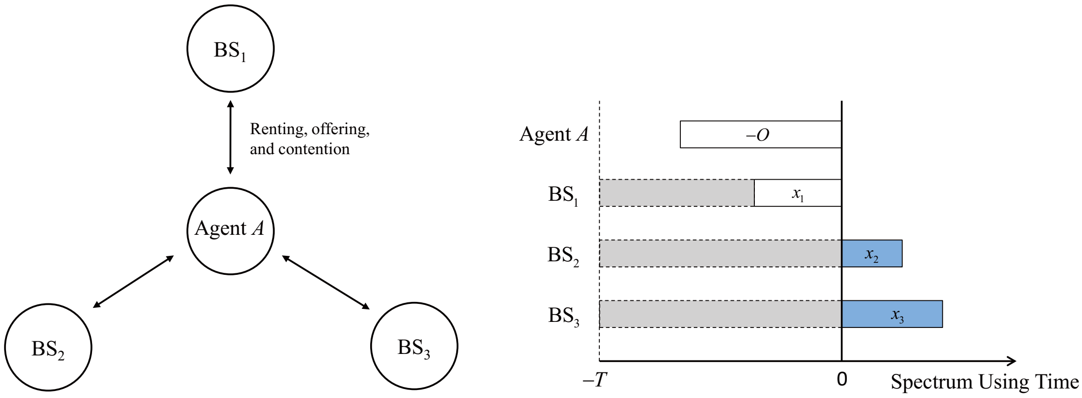
Spectrum Sharing Scheme

2.1 System Model

The system we consider consists of an agent, A, and n BSs, BS_i for $i = 1, 2, ..., n$. Agent A, serving like a marketplace, provides the centralized renting-and-offering and contention procedures for all BSs. Besides, Agent A offers spectrum using time O which is the vacant or to-be-utilized spectrum using time of licensed users. If O is less than zero, "offering O" means "retrieving $-O$." Each BS_i has a single orthogonal spectrum band, spectrum using time T, a credit token budget B_i, and a max traffic requirement x_i (in time) additional to T. All of these are assumed to be public information. In other parts of this thesis, we will use "spectrum" to denote spectrum using time for short. Figure 2.1(a) is an illustration of a system of Agent A and three BSs. Figure 2.1(b) is the corresponding max additional traffic requirements.

2.2 Spectrum Sharing Algorithm

Founded on DRRO and AODCC in the IEEE 802.22 inter-BS coexistence mechanism, we propose a spectrum sharing algorithm. Initially, Agent A broadcasts that the renting-and-offering procedure starts and it wants to provide spectrum O. After broadcasting this information, Agent A receives the acquisition/offering requests from all BSs. According to the type of the request, each BS is called an acquirer or an offeror. Agent A collects the

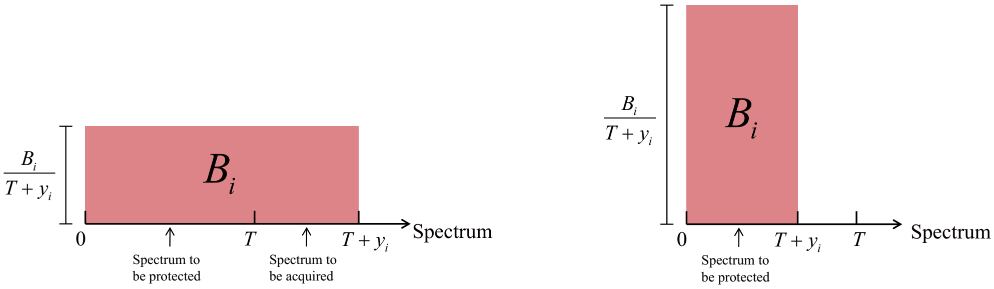


(a) System Diagram (b) Max Additional Traffic Requirements

Figure 2.1: System of Agent A and Three BSs

offered spectrum from the offerors and then assigns O and the collected offered spectrum to the acquirers, in decreasing order of the unit acquisition price, for the requested amount until exhaustion. (More details about the acquisition/offering requests and the unit acquisition price will be explained later.) When multiple acquirers have the same unit acquisition price and there is no enough spectrum for them, the amount assigned to them is assumed equal.

A BS, on the other hand, is assumed to aim to increase their spectrum. To increase the spectrum, each BS must use its credit tokens not only to acquire others' spectrum but also to protect its originally owned spectrum from others' contention. We assume every unit of the spectrum to be acquired and the spectrum to be protected is equally significant for each BS. Hence the credit token budget should be fairly allocated. Specifically, after hearing the renting-and-offering information, each BS$_i$ makes an acquisition/offering request, y_i, which is the spectrum it claims to acquire if $y_i > 0$ or to offer if $y_i < 0$. At the same time, the unit acquisition price for the spectrum to be acquired, $[y_i]^+$, and the unit protection price for the spectrum to be protectd, $T - [-y_i]^+$, are both determined to be equal to $p_i\left(y_i\right) = \dfrac{B_i}{T + y_i}$ as depicted in Figure 2.2. The function $[\cdot]^+$ gives a non-negative value. When Agent A receives the acquisition/offering requests, it assigns O and the offerors' provided spectrum to the acquirers under the previously described renting-and-offering procedure. After the



(a) $y_i > 0$ (b) $y_i \leq 0$

Figure 2.2: Unit Spectrum Acquisition and Protection Price of BS i

renting-and-offering procedure, if the acquirers do not get enough spectrum, they will turn to contention.

While the contention procedure starts, Agent A first collects the spectrum, $T - [-y_i]^+$, each BS$_i$ wants to protect. The collected $\left(T - [-y_i]^+\right)$s are sorted in increasing order of the unit protection price. Afterwards Agent A assigns the sorted $\left(T - [-y_i]^+\right)$s to the acquirers for the inadequate amount in decreasing order of the unit acquisition price. The assignment ends if the unit protection price is greater than or equal to the unit acquisition price. Finally, Agent A returns the unassigned spectrum back to all BSs. When multiple acquirers have the same acquisition price and there is no enough spectrum for them, we assume the amount assigned to them is equal. When multiple BSs have the same protection price and their spectrum is assigned to others, we assume the assigned amount is equally afforded by these BSs. After both renting-and-offering and contention procedures finish, the credit tokens the acquirers spend for spectrum acquisition are frozen and data transmission begins.

Lastly, we show, in Table 2.1, the mathematical expressions of the spectrum BS$_i$ acquires and offers in the renting-and-offering procedure and the spectrum BS$_i$ acquires or loses in the contention procedure. The former is $min\left(y_i, r_i\right)$ where $r_i(\mathbf{y})$ is the amount BS$_i$ can acquire from renting. The latter is $min\left([y_i - r_i]^+, c_i\right)$ where $[y_i - r_i]^+$ represents the inadequate amount after renting and c_i represents the amount BS$_i$ can acquire (the first term) or will lose from (the other two terms) contention. Then the total spectrum BS$_i$ gains or loses in both procedures is $min\left(y_i, r_i\right) + min\left([y_i - r_i]^+, c_i\right)$. For simplicity, we can also use $min(y_i, t_i)$ to represent the total spectrum BS$_i$ acquires or loses in both procedures where t_i is the total spectrum BS$_i$ can acquire (the first term) or will lose (the last two terms.) In Lemma 2.1, we prove $min\,(y_i, r_i) + min\,\left([y_i - r_i]^+, c_i\right) = min(y_i, t_i)$. Besides, the frozen credit tokens of BS$_i$ is $P_i\,(\mathbf{y}) = p_i\,(y_i)\,[min(y_i, t_i)]^+$. All other notations are summarized in Table 2.1 as well.

Lemma 2.1. $min(y_i, r_i) + min([y_i - r_i]^+, c_i) = min(y_i, t_i)\ \forall y_i \geq -T$ *and* $\forall i \in \{1, ..., n\}$.

Proof. See Appendix A. □

2.3 Problem Description

The problem we want to investigate is as below.

Problem: *Given that Agent A provides the spectrum O and that the original spectrum T, the credit token budget B_i, and the max traffic requirement x_i of each BS$_i$ are all public information, if the acquisition/offering request y_i is constrained by $-T$ and x_i, i.e. $-T \leq y_i \leq x_i$, how does each BS$_i$ make the acquisition/offering request in order to increase the spectrum?*

Table 2.1: Notations

Agent A and BS$_i$, $i = 1, 2, ..., n$.	
T	Spectrum owned by each BS.
O	Spectrum offered by Agent A.
B_i	Credit token budget of BS$_i$.
x_i	Max traffic requirement additional to T of BS$_i$.
y_i	Acquisition/offering request of BS$_i$. It is the spectrum BS$_i$ claims to acquire or to offer.
$p_i(y_i)$	Unit acquisition and protection price of BS$_i$. $p_i(y_i) = \frac{B_i}{T+y_i}$
$min\,(y_i, r_i)$	Spectrum BS$_i$ acquires or offers in the renting-and-offering procedure.
$r_i(\mathbf{y})$	Spectrum BS$_i$ can acquire from renting. $r_i(\mathbf{y}) = \frac{[y_i]^+}{\sum\limits_{j;p_j=p_i} [y_j]^+} \left[O + \sum\limits_{j=1}^{n} [-y_j]^+ - \sum\limits_{j;p_j>p_i} [y_j]^+ \right]^+$
$min\left([y_i - r_i]^+, c_i\right)$	Spectrum BS$_i$ acquires or loses in the contention procedure.
$c_i(\mathbf{y})$	Spectrum BS$_i$ can acquire or will lose from contention. $c_i(\mathbf{y}) = \frac{[y_i - r_i]^+}{\sum\limits_{j;p_j=p_i} [y_j - r_j]^+} \left[\sum\limits_{j;p_j<p_i} \left(T - [-y_j]^+\right) - \sum\limits_{j;p_j>p_i} [y_j - r_j]^+ \right]^+ - \left(T - [-y_i]^+\right) + \left[\left(T - [-y_i]^+\right) - \frac{T - [-y_i]^+}{\sum\limits_{j;p_j=p_i} \left(T - [-y_j]^+\right)} \left[- \sum\limits_{j;p_j<p_i} \left(T - [-y_j]^+\right) + \sum\limits_{j;p_j>p_i} [y_j - r_j]^+ \right]^+ \right]^+$
$min\,(y_i, t_i)$	Spectrum BS$_i$ acquires or loses in both procedures. $min\,(y_i, t_i) = min\,(y_i, r_i) + min\left([y_i - r_i]^+, c_i\right)$
$t_i(\mathbf{y})$	Spectrum BS$_i$ can acquire or will lose in both procedures. $t_i(\mathbf{y}) = \frac{[y_i]^+}{\sum\limits_{j;p_j=p_i} [y_j]^+} \left[O + \sum\limits_{j;p_j=p_i} [y_j]^+ - \sum\limits_{j;p_j \geq p_i} y_j + \sum\limits_{j;p_j<p_i} T \right]^+ - T + \left[\left(T - [-y_i]^+\right) - \frac{T - [-y_i]^+}{\sum\limits_{j;p_j=p_i} \left(T - [-y_j]^+\right)} \left[-O - \sum\limits_{j;p_j=p_i} [y_j]^+ + \sum\limits_{j;p_j \geq p_i} y_j - \sum\limits_{j;p_j<p_i} T \right]^+ \right]^+$
$P_i(\mathbf{y})$	Frozen credit tokens of BS$_i$. $P_i(\mathbf{y}) = p_i(y_i)\,[min(y_i, t_i)]^+$

Chapter 3

Game Formulation

Game theory is utilized to deal with the spectrum sharing problem. From the perspective of each BS, spectrum sharing is intrinsically a game that each BS unitarily optimizes its performance by acquiring or offering spectrum according to its credit token budget and max traffic requirement. In the following, we briefly introduce game theory and then construct the spectrum sharing game model.

3.1 Game Theory

Game theory is a set of mathematical tools used to model and analyze interactive decision processes [9, 10]. The core of a game consists of three primary components:

1. A player set N.
2. A strategy space S formed from the Cartesian product of each player's strategy set, $S = \prod_{i \in N} S_i$.
3. A set of utility functions $U = \{u_i(\mathbf{s})\}$ where $\mathbf{s} \in S$ and $u_i(\mathbf{s})$, $i \in N$, represents player i's utility under the strategy profile $\mathbf{s}$.

In a game, each player is assumed to choose the best available strategy. Each player's best available strategy is the one maximizing his utility under the belief that other players do in the same way as well. The collection of all players' best available strategies forms a steady state at which no player has a reason to choose any strategy different from his best available one. Such a steady state is called a Nash equilibrium [11].

Definition 3.1. *A strategy profile* $\mathbf{s}^* = \left(s_i^*, s_{-i}^*\right)$ *is a Nash equilibrium if*

$$u_i\left(\mathbf{s}^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \quad \forall s_i \neq s_i^* \textit{ and } \forall i \in N$$

Though we can find the Nash equilibria of a game where each player has only a few strategies by examining all the possible strategy profiles to see if they satisfy Definition 3.1, it is always full of difficulties in more complicated games. An alternative method is to work with players' best response functions.

Definition 3.2. $BR_i\left(s_{-i}\right)$ *is the best response function of player* i *if*

$$BR_i\left(s_{-i}\right) = \left\{s_i : \ u_i\left(s_i, s_{-i}\right) \geq u_i\left(s_i', s_{-i}\right), \ \forall s_i' \neq s_i\right\}$$

The best response function of any player depicts his best (in term of highest utility) strategy given all possible s_{-i} from other players. A Nash equilibrium can also be defined by best response functions.

Definition 3.3. *A strategy profile* $\mathbf{s}^* = \left(s_i^*, s_{-i}^*\right)$ *is a Nash equilibrium if*

$$s_i^* = BR_i\left(s_{-i}^*\right) \ \forall i \in N$$

3.2 Spectrum Sharing Game

We now apply game theory to construct a model for the spectrum sharing problem. Besides the three main components, the credit token budget and the max traffic requirement of each BS should be taken into account as well:

1. Player set N: Each BS$_i$ is the player of the game. $N = \{1, 2, ..., n\}$.

2. Strategy space $Y = \prod_{i \in N} Y_i$: We treat BS$_i$'s acquisition/offering request y_i as its strategy.

All possible acquisition/offering requests of BS_i compose the strategy set Y_i. Mathematically, $Y_i = \{y_i : -T \leq y_i \leq x_i\}\ \forall i \in N$.

3. Set of utility functions $U = \{u_i(\mathbf{y})\}$: Since the goal of each BS is to increase its spectrum, it is reasonable to set the spectrum as the utility. We ignore the constant term T for the sake of convenience. The utility is therefore the spectrum acquired or lost from renting, offering, and contention. Mathematically, BS_i's utility function is $u_i(\mathbf{y}) = min\,(y_i, t_i)$.

4. Set of credit token budgets (CTB set) $B = \{B_i\}$.

5. Max traffic set $X = \{x_i\}$: Without losing generality, we assume $\{p_i\,(x_i)\}$ is sorted in decreasing order. This assumption will simplify our analysis.

The game model is summarized in Table 3.1.

Table 3.1: Spectrum Sharing Game Model

	$G = (N, Y, U, B, X)$
Player Set	Set of the BSs. $N = \{1, 2, ..., n\}$
Strategy Space	Cartesian product of each player's strategy set which is the set of all possible acquisition/offering requests. $Y = \prod_{i \in N} Y_i$ and $Y_i = \{y_i : -T \leq y_i \leq x_i\}\ \forall i \in N$
Set of Utility Functions	Set of the spectrum each player acquires or loses. $U = \{u_i(\mathbf{y})\}$ and $u_i(\mathbf{y}) = min\,(y_i, t_i)\ \forall i \in N$
CTB Set	$B = \{B_i\}$
Max Traffic Set	$X = \{x_i\}$ with $\{p_i\,(x_i)\}$ arranged in decreasing order.

Chapter 4

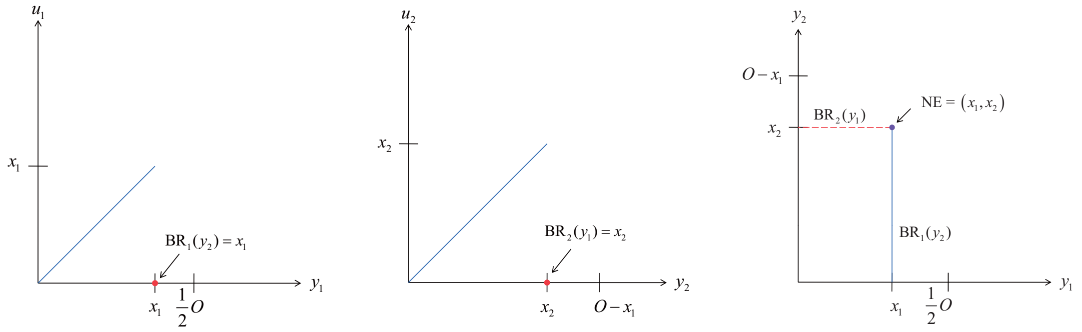
Graphical Analysis - Two Players with Same Budget

A graphical method to derive the Nash equilibrium in the simplest 2-same-budget-player game is presented to gain the insights for the solution of the general n-player game, i.e. Game G. It consists of two main procedures. First the utility functions of both players are drawn to derive their best response functions. Then two best response functions are drawn together. The resulting intersection is Nash equilibrium.

Recall that we have assumed $p_1(x_1) \geq p_2(x_2)$. This assumption reduces to $x_1 \leq x_2$ when both players have the same credit token budget. Accordingly, the traffic requirements can be categorized into three cases. The first case is $x_1 + x_2 \leq O$ which, by applying $x_1 \leq x_2$, can be equivalently expressed as $x_1 \leq \frac{O}{2}$ and $x_2 \leq O - x_1$. The second case is $x_1 \leq \frac{O}{2}$ and $x_1 + x_2 > O$, equivalently $x_1 \leq \frac{O}{2}$ and $x_2 > O - x_1$. The final case is $x_1 > \frac{O}{2}$ and $x_1 + x_2 > O$ which are equivalent to $x_1 > \frac{O}{2}$ and $x_2 > \frac{O}{2}$. In the following, we discuss case by case.

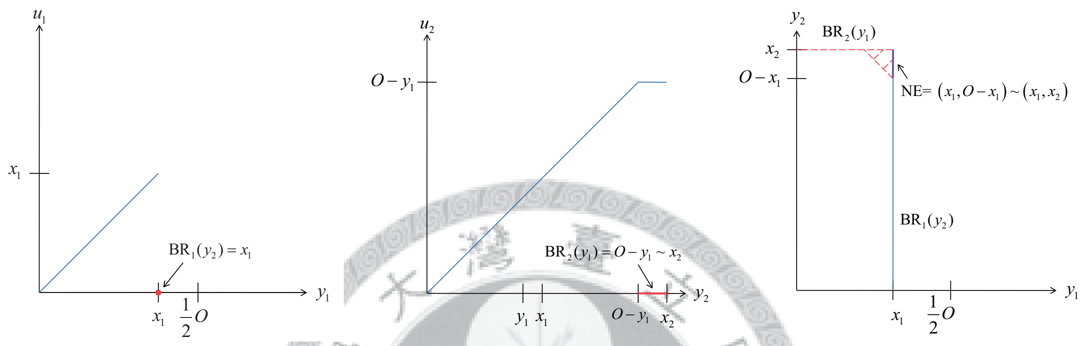
4.1 Traffic Case 1 - $x_1 \leq \frac{O}{2}$ and $x_2 \leq O - x_1$

The best response function of player 1, illustrated in Figure 4.1(a), is uniquely x_1. It means player 1 will always play the unique dominant strategy, $y_1 = x_1$, regardless of player 2's



(a) u_1 and BR_1 (b) u_2 and BR_2 (c) BR_1, BR_2, and NE

Figure 4.1: Best Response Functions and Nash Equilibrium in Traffic Case 1



(a) u_1 and BR_1 (b) u_2 and BR_2 (c) BR_1, BR_2, and NE

Figure 4.2: Best Response Functions and Nash Equilibrium in Traffic Case 2

strategy. We call such a strategy a dominant one since it always dominates (or results in higher utility than) all other strategies. A similar observation is obtained in Figure 4.1(b) that player 2's best response function is x_2 and therefore player 2 plays the unique dominant strategy, $y_2 = x_2$. By drawing two best response functions together in Figure 4.1(c), we find their intersection, (x_1, x_2), a unique Nash equilibrium. The corresponding utility profile is (x_1, x_2) as well.

4.2 Traffic Case 2 - $x_1 \leq \dfrac{O}{2}$ and $x_2 > O - x_1$

We have already derived that player 1 plays the unique dominant strategy, $y_1 = x_1$ when $x_1 \leq \dfrac{O}{2}$. As depicted in Figure 4.2(b), player 2's best response function is $BR_2(y_1) = O - y_1 \backsim x_2$ which implies that the strategy, $y_2 = x_2$, is player 2's unique dominant strategy. However, it is not meaningful to discuss the concept of dominant strategy for player 2 while it is like to play a single-player game. We explain why player 2 is like to play a single-player game as follows: when $x_1 \leq \frac{O}{2}$, player 1 plays the unique dominant strategy, $y_1 = x_1$, and acquires x_1 from O. (When x_1 is less than zero, "acquiring x_1" means "offering $-x_1$.") For player 2, it has $(O - x_1)$ remained to acquire without any other player. Therefore player 2 is like to play a single-player game and it can always acquire $(O - x_1)$ by playing y_2 such that $O - x_1 \leq y_2 \leq x_2$. The single-player effect obviously results in multiple Nash equilibria. This can also be shown by drawing the two best response functions together. The resulting intersection is a line segment between $(x_1, O - x_1)$ and (x_1, x_2) which means multiple Nash equilibria, $(x_1, O - x_1 \backsim x_2)$, exist. Though multiple Nash equilibria exist, the corresponding utility profile is uniquely $(x_1, O - x_1)$.

4.3 Traffic Case 3 - $x_1 > \frac{O}{2}$ and $x_2 > \frac{O}{2}$

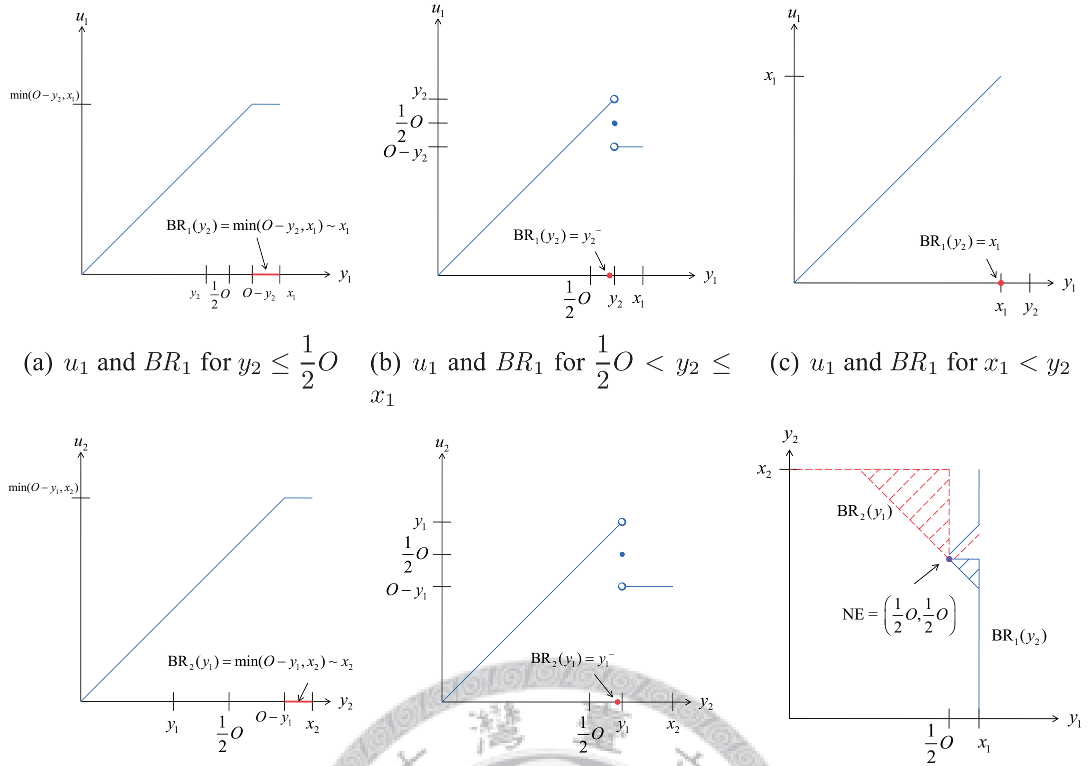
We derive the best response function of player 1 from Figure 4.3(a) to Figure 4.3(c),

$$BR_1(y_2) = \begin{cases} min\,(O - y_2, x_1) \backsim x_1 & if\ y_2 \leq \frac{O}{2} \\ y_2^- & if\ \frac{O}{2} < y_2 \leq x_1 \\ x_1 & if\ x_1 < y_2 \end{cases}$$

and the best response function of player 2 from Figure 4.3(d) and Figure 4.3(e),

$$BR_2(y_1) = \begin{cases} min\,(O - y_1, x_2) \backsim x_2 & if\ y_1 \leq \frac{O}{2} \\ y_1^- & if\ \frac{O}{2} < y_1 \end{cases}$$

We see neither player 1 nor player 2 has dominant strategy. By drawing two best response functions together, we find their intersection, $\left(\frac{O}{2}, \frac{O}{2}\right)$ a unique equal-strategy Nash equilibrium. The corresponding utility profile is $\left(\frac{O}{2}, \frac{O}{2}\right)$.



(a) u_1 and BR_1 for $y_2 \leq \frac{1}{2}O$ (b) u_1 and BR_1 for $\frac{1}{2}O < y_2 \leq x_1$ (c) u_1 and BR_1 for $x_1 < y_2$

(d) u_2 and BR_2 for $y_1 \leq \frac{1}{2}O$ (e) u_2 and BR_2 for $y_1 > \frac{1}{2}O$ (f) BR_1, BR_2, and NE

Figure 4.3: Best Response Functions and Nash Equilibrium in Traffic Case 3

We summarize the observations as follows. As will be shown, the listed items, playing the essential roles in the two-same-budget-player game, can be extended to the general n-different-budget-player game.

1. Condition for unique dominant strategies: When $x_1 \leq \frac{O}{2}$, player 1 plays the unique dominant strategy, $y_1 = x_1$. When $x_2 \leq O - x_1$, player 2 plays the unique dominant strategy, $y_2 = x_2$.

2. Existence of a Nash equilibrium: A Nash equilibrium always exists in all cases.

3. Condition for multiple Nash equilibria: The only case where multiple Nash equilibria exist is $x_1 \leq \frac{O}{2}$ and $x_2 > O - x_1$. We have explained that because player 2 is like to play a single-player game with $(O - x_1)$ offered, it can always acquire $(O - x_1)$ by playing $O - x_1 \leq y_2 \leq x_2$. Multiple Nash equilibria, $(x_1, O - x_1 \backsim x_2)$, hence exist.

4. Unique utility profile at the Nash equilibrium: Even in the multi-Nash-equilibrium case, the corresponding utility profile is unique.

Chapter 5

Mathematical Analysis - n Players

In this chapter, we first extend the two-same-budget-player game to the general n-different-budget-player game, i.e. Game G. The result is summarized in Table 5.1. Afterwards, we do formal derivations for the Nash equilibrium of Game G.

5.1 Extension from Two-Player Game to n-Player Game

Table 5.1: Summary of Extension

	2-Same-Budget-Player Game	n-Same-Budget-Player Game	n-Different-Budget-Player Game
Traffic Threshold	$\left\{\frac{O}{2}, O - x_1\right\}$	$\left\{\frac{-\sum_{l=0}^{j-1} x_l}{n-j+1}\right\}$	$\{e_{j,-(j-1)}\}$
Traffic Case	$x_1 > \frac{O}{2}$ and $x_2 > \frac{O}{2}$; $x_1 \leq \frac{O}{2}$ and $x_2 > O - x_1$; $x_1 \leq \frac{O}{2}$ and $x_2 \leq O - x_1$	$x_j \leq \frac{-\sum_{l=0}^{j-1} x_l}{n-j+1}$ $\forall j \in \{1, ..., k\}$, $x_j > \frac{-\sum_{l=0}^{k} x_l}{n-k}$ $\forall j \in \{k+1, ..., n\}$ where $k \in \{0, N\}$	$x_j \leq e_{j,-(j-1)}$ $\forall j \in \{1, ..., k\}$, $x_j > e_{j,-k}$ $\forall j \in \{k+1, ..., n\}$ where $k \in \{0, N\}$
Nash Equilibrium	$\left(\frac{O}{2}, \frac{O}{2}\right)$; $(x_1, O - x_1 \sim x_2)$; (x_1, x_2)	$\left(x_1, ..., x_k, \frac{-\sum_{l=0}^{k} x_l}{n-k}, ..., \frac{-\sum_{l=0}^{k} x_l}{n-k}\right)$ if $k \neq n-1$; $\left(x_1, ..., x_{n-1}, -\sum_{l=0}^{n-1} x_l \sim x_n\right)$ if $k = n-1$	$(x_1, ..., x_k, e_{k+1}, ..., e_{n,-k})$ if $k \neq n-1$; $(x_1, ..., x_{n-1}, e_{n,-(n-1)} \sim x_n)$ if $k = n-1$

Recall that we have assumed the max traffic requirements are such that $\{p_i(x_i)\}$ is arranged in decreasing order. In the two-same-budget-player game, we see there are two traffic thresholds, $\frac{O}{2}$ and $O = x_1$. Accordingly, the traffic can be categorized into three cases: $x_1 > \frac{O}{2}$ and $x_2 > \frac{O}{2}$; $x_1 \leq \frac{O}{2}$ and $x_2 > O - x_1$; $x_1 \leq \frac{O}{2}$ and $x_2 \leq O - x_1$. The corresponding Nash equilibrium is $\left(\frac{O}{2}, \frac{O}{2}\right)$, $(x_1, O - x_1 \sim x_2)$, and (x_1, x_2).

Extended from the two-same-budget-player game, it is reasonably to guess the n-same-budget-player game has the set of n traffic thresholds, $\left\{ \frac{-\sum_{l=0}^{j-1} x_l}{n-j+1} \right\}$, where $x_0 = -O$. Accordingly, we can categorize the traffic into $(n+1)$ cases. The $(k+1)$-th case, $k \in \{0, N\}$, is $x_j \leq \frac{-\sum_{l=0}^{j-1} x_l}{n-j+1}$ $\forall j \in \{1, ..., k\}$ and $x_j > \frac{-\sum_{l=0}^{k} x_l}{n-k}$ $\forall j \in \{k+1, ..., n\}$. The Nash equilibrium is $\left(x_1, ..., x_k, \frac{-\sum_{l=0}^{k} x_l}{n-k}, ..., \frac{-\sum_{l=0}^{k} x_l}{n-k} \right)$ if $k \neq n-1$ and $\left(x_1, ..., x_{n-1}, -\sum_{l=0}^{n-1} x_l \sim x_n \right)$ if $k = n-1$. By substituting 2 for n, we can check that the n-same-budget-player game really reduces to the two-same-budget-player game.

To further extend to the n-different-budget-player game, we must know what plays the same role as $\frac{-\sum_{l=0}^{k} x_l}{n-k}$ in the n-same-budget-player game.

Definition 5.1. *For Game G, we define, with O denoted as* $-x_0$*,*

$$e_{j,-k} \equiv \frac{B_j}{\frac{1}{n-k} \sum_{l=k+1}^{n} B_l} \left(\frac{-\sum_{l=0}^{k} x_l}{n-k} \right) + \left(\frac{B_j}{\frac{1}{n-k} \sum_{l=k+1}^{n} B_l} - 1 \right) T$$

$$\forall j \in \{k+1, ..., n\} \text{ and } \forall k \in \{0, N\}$$

$e_{j,-k}$ can be interpreted as weighted and translated $\frac{-\sum_{l=0}^{k} x_l}{n-k}$ with the weight $\frac{B_j}{\frac{1}{n-k} \sum_{l=k+1}^{n} B_l}$. The term $-k$ in the subscript indicates that player i, $i \in \{1, ..., k\}$, which has already acquired x_i from O, is excluded. When $k = 0$, $e_{j,-0}$ is denoted as e_j for short. Following the definition, there is a corollary stating some properties of $e_{j,-k}$.

Corollary 5.1. *For Game G, the following statements about* $e_{j,-k}$ *are always true:*

1. $p_j\left(e_{j,-k}\right) = \dfrac{\frac{1}{n-k} \sum_{l=k+1}^{n} B_l}{T + \frac{-\sum_{l=0}^{k} x_l}{n-k}}$ $\forall j \in \{k+1, ..., n\}$.

2. $\sum_{j=1}^{k} x_j + \sum_{j=k+1}^{n} e_{j,-k} = min\left(O, \sum_{j=1}^{n} x_j\right) \forall k \in \{0, N\}$.

3. $x_k \leq e_{k,-(k-1)} \Leftrightarrow x_j \leq e_{j,-(j-1)} \forall j \in \{1, ..., k\}$.

4. $x_{k+1} > e_{k+1,-k} \Leftrightarrow x_j > e_{j,-k} \forall j \in \{k+1, ..., n\}$.

5. $x_k \leq e_{k,-(k-1)} \Rightarrow p_k(e_{k,-(k-1)}) \geq p_j(e_{j,-k}) \forall j \in \{k+1, ..., n\}$.

Proof. See Appendix B. □

Corollary 5.1.1 says that the strategies $e_{i,-k}$ and $e_{j,-k}$ $\forall i, j \in \{k+1, ..., n\}$ are equal-price, i.e. they result in the same price. Besides, player j who plays $y_j = e_{j,-k}$ can be viewed similar to the player having the average credit token budget $\frac{1}{n-k} \sum_{l=k+1}^{n} B_l$ and playing the strategy $\frac{-\sum_{l=0}^{k} x_l}{n-k}$. In Corollary 5.1.2, when $k \neq n$, $\sum_{j=1}^{k} x_j + \sum_{j=k+1}^{n} e_{j,-k} = O$. It can be equivalently represented as $\sum_{j=k+1}^{n} e_{j,-k} = -\sum_{l=1}^{k} x_l$. Therefore the strategy profile, $(e_{k+1,-k}, ..., e_{n,-k})$, is called the the sum-$\left(-\sum_{l=1}^{k} x_l\right)$ equal-price strategy profile. Especially when $k = 0$, the strategy profile, $(e_1, ..., e_n)$, is called the sum-O equal-price strategy profile.

When $B_i = B_j \forall i, j \in N$, the weights for all $e_{j,-k}$ become 1 and $e_{j,-k}$ reduces to $\frac{-\sum_{l=0}^{k} x_l}{n-k}$. It is intuitively to believe that $e_{j,-k}$ play the same roles as $\frac{-\sum_{l=0}^{k} x_l}{n-k}$ in the same-budget case. Hence Game G should have the set of n traffic thresholds, $\{e_{j,-(j-1)}\}$. Besides, we can classify the traffic into $(n+1)$ cases where the $(k+1)$-th case, $k \in \{0, N\}$, is $x_k \leq e_{k,-(k-1)}$ and $x_{k+1} > e_{k+1,-k}$. From Corollary 5.1.3 and 5.1.4, the $(k+1)$-th case can equivalently represented as $x_j \leq e_{j,-(j-1)} \forall j \in \{1, ..., k\}$ and $x_j > e_{j,-k} \forall j \in \{k+1, ..., n\}$.

Definition 5.2. *For Game G and* $\forall k \in \{0, N\}$*, we define*

$$Traffic_k \equiv x_j \leq e_{j,-(j-1)} \quad \forall j \in \{1, ..., k\} \text{ and } x_j > e_{j,-k} \forall j \in \{k+1, ..., n\}$$

The Nash equilibrium under $Traffic_k$ should be $(x_1, ..., x_k, e_{k+1}, ..., e_{n,-k})$ if $k \neq n-1$ and $\left(x_1, ..., x_{n-1}, e_{n,-(n-1)} \sim x_n\right)$ if $k = n-1$. Finally, by letting all credit token budgets be the same, we check the n-different-budget-player game becomes the n-same-budget-player game.

5.2 n-Player Game

The game model for n different-budget players, Game G, is illustrated in Table 3.1. Before starting, we should mention that we will use u_i and t_i to express $u_i(\mathbf{y})$ and $t_i(\mathbf{y})$ at any given strategy profile $\mathbf{y}$ for short. If we need to compare the results between two different strategy profiles, say (y_i, y_{-i}) and (y_i', y_{-i}), we will distinguish by using u_i' and t_i' to express $u_i(y_i', y_{-i})$ and $t_i(y_i', y_{-i})$.

First, let us examine the increasing property of utility functions with respect to strategies.

Lemma 5.1. *For Game G under Traffic$_k$, $k \in \{0, N\}$, the following statements are always true:*

1. $u_i = y_i \ \forall i \in \{1, ..., k\}$.
2. *if* $y_i \leq e_{i,-k}$ *for some* $i \in \{k+1, ..., n\}$, $u_i = y_i$.

Proof. From Corollary 5.1.2, we know $\sum_{j=1}^{k} e_{j,-(j-1)} + \sum_{j=k+1}^{n} e_{j,-k} = min\left(O, -\sum_{j=1}^{n} x_i\right) \leq O$. The derivation below is suitable for both 1) and 2). For player i, we have

$$\begin{aligned}
& O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j\geq p_i} y_j + \sum_{j;p_j<p_i} T \\
\geq & \sum_{j=1}^{k} e_{j,-(j-1)} + \sum_{j=k+1}^{n} e_{j,-k} + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j\geq p_i} y_j + \sum_{j;p_j<p_i} T \\
= & \sum_{j;p_j=p_i} [y_j]^+ + \sum_{j\leq k;p_j\geq p_i} \left(e_{j,-(j-1)} - y_j\right) + \sum_{j>k;p_j\geq p_i} (e_{j,-k} - y_j) \\
+ & \sum_{j\leq k;p_j<p_i} \left(e_{j,-(j-1)} + T\right) + \sum_{j>k;p_j<p_i} (e_{j,-k} + T) \geq \sum_{j;p_j=p_i} [y_j]^+ \geq 0 \qquad (5.1)
\end{aligned}$$

Following Equation (5.1), if $y_i > 0$, we have

$$\begin{aligned}
t_i &= \frac{[y_i]^+}{\sum_{j;p_j=p_i} [y_j]^+} \left[O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j\geq p_i} y_j + \sum_{j;p_j<p_i} T\right]^+ \\
&\geq \frac{y_i}{\sum_{j;p_j=p_i} [y_j]^+} \left(\sum_{j;p_j=p_i} [y_j]^+\right) = y_i \qquad (5.2)
\end{aligned}$$

If $y_i \leq 0$, we have

$$\begin{aligned} t_i &= -T \\ &+ \left[\left(T - [-y_i]^+\right) - \frac{T-[-y_i]^+}{\sum\limits_{j;p_j=p_i}\left(T-[-y_j]^+\right)} \left[-O - \sum_{j;p_j=p_i} [y_j]^+ + \sum_{j;p_j \geq p_i} y_j - \sum_{j;p_j<p_i} T \right]^+ \right]^+ \\ &= -T + (T + y_i) = y_i \end{aligned} \quad (5.3)$$

Equation (5.2) and (5.3) reveals that $t_i \geq y_i$ and consequently $u_i = min(y_i, t_i) = y_i$. □

Lemma 5.1.1 shows that u_i, $i \in \{1, ..., k\}$, is an increasing function of y_i under $Traffic_k$. The condition for unique dominant strategies is then implied.

Theorem 5.1. *For Game G under $Traffic_k$, $k \in \{0, N\}$, player i, $i \in \{1, ..., k\}$, plays the unique dominant strategy, $y_i = x_i$.*

Proof. It is shown in Lemma 5.1.1 that under $Traffic_k$, $u_i = y_i$ $\forall i \in \{1, ..., k\}$. Player i's utility is an increasing function of y_i and uniquely reaches its maximum at $y_i = x_i$. Therefore player i can always play $y_i = x_i$ to get the highest utility. In words, player i, $i \in \{1, ..., k\}$, plays the unique dominant strategy, $y_i = x_i$. □

Recall we have guessed the Nash equilibrium under $Traffic_k$ is $(x_1, ..., x_k, e_{k+1}, ..., e_{n,-k})$ if $k \neq n-1$ and $\left(x_1, ..., x_{n-1}, e_{n,-(n-1)} \sim x_n\right)$ if $k = n-1$. To verify our guess is correct, we prove that all other strategy profiles cannot be a Nash equilibrium. The proof is taken into two parts. The first part is to prove that $y_i < x_i$ for any $i \in \{1, ..., k\}$ or $y_i < e_{i,-k}$ for any $i \in \{k+1, ..., n\}$ is not in any Nash equilibrium. The second part is to prove that $y_i > e_{i,-k}$ for any $i \in \{k+1, ..., n\}$ is not in any Nash equilibrium.

Lemma 5.2. *For Game G under $Traffic_k$, $k \in \{0, N\}$, $y_i < x_i$ for any $i \in \{1, ..., k\}$ or $y_i < e_{i,-k}$ for any $i \in \{k+1, ..., n\}$ is not in any Nash equilibrium.*

Proof. Since Theorem 5.1 reveals that $y_i = x_i$ is the unique dominant strategy for player i $\forall i \in \{1, ..., k\}$, $y_i < x_i$ for any $i \in \{1, ..., k\}$ is not in any Nash equilibrium. Also, from Lemma 5.1.2, we know $u_i = y_i$ if $y_i \leq e_{i,-k}$ for any $i \in \{k+1, ..., n\}$. It means when playing $y_i < e_{i,-k}$, player i can always play $y'_i = e_{i,-k}$ to get higher utility. Therefore $y_i < e_{i,-k}$ for any $i \in \{k+1, ..., n\}$, is not in any Nash equilibrium either. □

Lemma 5.3. *For Game G under* $Traffic_k$*,* $k \in \{0, N\}$ *and* $k \neq n-1$*,* $y_i > e_{i,-k}$ *for any* $i \in \{k+1, ..., n\}$ *is not in any Nash equilibrium.*

Proof. From Theorem 5.1 and Lemma 5.2, we know a Nash equilibrium exists only if $y_i = e_{i,-(i-1)}$ $\forall i \in \{1, ..., k\}$ and $y_i \geq e_{i,-k}$ for $\forall i \in \{k+1, ..., n\}$. Because $\sum_{i=1}^{n} u_i \leq min\left(O, \sum_{i=1}^{n} x_i\right)$, if $u_l > e_{l,-k}$ for some $l \in \{k+1, ..., n\}$, there must exist some other $m \in \{k+1, ..., n\}$ having $u_m < e_{m,-k}$. Then player m can play $y'_m = e_{m,-k}$ to make $u'_m = e_{m,-k} > u_m$ which means a Nash equilibrium does not exist. Therefore a Nash equilibrium exists only if $y_i \geq e_{i,-k}$ and $u_i = e_{i,-k}$ $\forall i \in \{k+1, .., n\}$.

Given this necessary condition for the existence of a Nash equilibrium, we let N_{equal} and $N_{smallest}$ respectively denote the set of player i having $y_i = e_{i,-k}$ and the set of player i having $y_i > e_{i,-k}$ with the smallest price where $i \in \{k+1, ..., n\}$. When $N_{equal} = \emptyset$ or $N_{smallest} \neq \emptyset$, any player m in $N_{smallest}$ or N_{equal} can play $y'_m = e^{+}_{m,-k}$ to get higher utility since

$$\begin{aligned} & O + [y'_m]^+ - \sum_{j;p_j \geq p'_m} y_j + \sum_{j;p_j < p'_m} T \\ \geq \; & \sum_{j=1}^{k} e_{j,-(j-1)} + \sum_{j=k+1}^{n} e_{j,-k} + [y'_m]^+ - \sum_{j;p_j \geq p'_m} y_j + \sum_{j;p_j < p'_m} T \\ = \; & [y'_m]^+ - y'_m + e_{m,-k} + \sum_{j;p_j < p'_m} (e_{j,-k} + T) > [y'_m]^+ - y'_m + e_{m,-k} \end{aligned} \tag{5.4}$$

From Equation (5.4), if $y'_m > 0$, we have

$$t'_m = \left[O + [y'_m]^+ - \sum_{j;p_j \geq p'_m} y_j + \sum_{j;p_j < p'_m} T \right]^+ > [e_{m,-k}]^+ \geq e_{m,-k} \tag{5.5}$$

If $y'_m \leq 0$, we have

$$t'_m > -T + [(T + y'_m) - (y'_m - e_{m,-k})]^+ = -T + (T + e_{m,-k}) = e_{m,-k} \tag{5.6}$$

Therefore $u'_m = min(y'_m, t'_m) > e_{m,-k} = u_m$. Consequently, a Nash equilibrium does not exist when $N_{equal} = \emptyset$ or $N_{smallest} \neq \emptyset$. In words, $y_i > e_{i,-k}$ for any $i \in \{k+1, ..., n\}$ is not in any Nash equilibrium. □

Combing Lemma 5.2 and Lemma 5.3, we have verified that under $Traffic_k$, any strategy profile other than $(x_1, ..., x_k, e_{k+1}, ..., e_{n,-k})$ if $k \neq n-1$ and $\left(x_1, ..., x_{n-1}, e_{n,-(n-1)} \sim x_n\right)$ if $k = n-1$ cannot be a Nash equilibrium. In words, only $(x_1, ..., x_k, e_{k+1}, ..., e_{n,-k})$ if $k \neq n-1$ and $\left(x_1, ..., x_{n-1}, e_{n,-(n-1)} \sim x_n\right)$ if $k = n-1$ can be a Nash equilibrium. We therefore check its property and find it a Nash equilibrium.

Theorem 5.2. *For Game G under $Traffic_k$, $k \in \{0, N\}$ and $k \neq n-1$, it has the unique Nash equilibrium, $NE_k = (x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$.*

Proof. Assume Game G under $Traffic_k$ is at the strategy profile, $(x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$. The corresponding utility profile is also $(x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$. For player i, $i \in \{1, ..., k\}$, if it plays $y'_i < x_i$, then $u'_i = y'_i < u_i$. For player i, $i \in \{k+1, ..., n\}$, if it plays $y'_i < e_{i,-k}$, $u'_i = y'_i < u_i$; if it plays, $y'_i > e_{i,-k}$, $u'_i = e_{i,-k}$. Consequently, $(x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$ meets the definition of Nash equilibrium. Since there is no other possible Nash equilibrium, Game G under $Traffic_k$, $k \in \{0, N\}$ and $k \neq n-1$, has the unique Nash equilibrium $(x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$. □

Theorem 5.3. *For Game G under $Traffic_{n-1}$, it has multiple Nash equilibria, $NE_{n-1} = \left(x_1, ..., x_{n-1}, e_{n,-(n-1)} \backsim x_n\right)$.*

Proof. When $k = n-1$, it is proved in Theorem 5.1 that player i, $i \in \{1, ..., n-1\}$, plays the unique dominant strategy $y_i = x_i$. For player n, it is like to play a single-player game with $-\sum_{j=1}^{n-1} x_j$, equivalently $e_{n,-(n-1)}$, offered. Player n can play $e_{n,-(n-1)} \leq y_n \leq x_n$ such that $u_n = e_{n,-(n-1)}$. Hence G has multiple Nash equilibria, $NE_{n-1} = \left(x_1, ..., x_{n-1}, e_{n,-(n-1)} \backsim x_n\right)$. □

After deriving the Nash equilibrium, we can easily verify that the utility profile at the Nash equilibrium is always unique.

Theorem 5.4. *For Game G under $Traffic_k$, $k \in \{0, N\}$, it has the unique utility profile, $U_k^* = (x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$, at the Nash equilibrium NE_k.*

Proof. It is drawn by substituting all NE_ks into the utility functions. □

Recall we have set spectrum as utilities for all BSs. In system meaning, the utility profile at the Nash equilibrium represents the spectrum allocation at the Nash equilibrium. Theorem 5.4, in words, reveals that our spectrum sharing algorithm always results in the unique traffic-dependent spectrum allocation at the Nash equilibrium, $AR^* = U_k^*$ given $Traffic_k$, $k \in \{0, N\}$.

Chapter 6

Properties Attained at Nash Equilibrium

After deriving the Nash equilibrium, we prove that several desirable properties, including allocative efficiency, Pareto optimality, weighted max-min fairness, and weighted proportional fairness, are attained at the Nash Equilibrium. Correspondingly, the spectrum allocation, AR^*, at the Nash equilibrium possesses these properties as well. In the following, we briefly introduce these properties and give their mathematical definitions in game theory. To conform with the expressions in our game, we use $\mathbf{y}$ and Y instead of $\mathbf{s}$ and S to represent the strategy profile and the strategy space respectively.

Efficiency is one of the key system design issues. Allocative efficiency [12, 13] is an efficient resource allocation in the sense of maximizing total utilities over all players. It is regarded as the most optimality since no other allocations can achieve greater social welfare. Another efficiency named Pareto optimality [12, 14] is defined as an allocation upon which no player can be made happier (in utility) without making at least one other player less happy. It is always true that allocative efficiency implies Perato optimality. The mathematical definitions of allocative efficiency and Pareto optimality are given as below.

Definition 6.1. *A resource allocation game is allocatively efficient if the Nash equilibrium is a solution to the optimization problem*

$$max \sum_{i=1}^{n} u_i(\mathbf{y}) \quad s.t. \; \mathbf{y} \in Y$$

Definition 6.2. *A resource allocation game is Pareto optimal if,* $\mathbf{y}^*$ *is the Nash equilibrium,*

$$\exists \mathbf{y}' \neq \mathbf{y}^*,\ u_i(\mathbf{y}') > u_i(\mathbf{y}^*) \ \Rightarrow\ \exists j \in N,\ u_j(\mathbf{y}') < u_j(\mathbf{y}^*)$$

Fairness is another key system design issue. Two kinds of fairness definitions are considered here. First, we say an allocation satisfies weighted max-min fairness [15, 16] if it is not possible to increase one player's weighted utility without simultaneously decreasing another player's weighted utility which is already smaller. We say an allocation exhibits weighted proportional fairness [15, 17] if it maximizes the weighted sum of logarithmic utilities of all players, or equivalently, it maximizes the product of all players' utilities with weights in exponents.

Definition 6.3. *A resource allocation game is weighted max-min fair with the weights* $\{w_i\}$*, if the Nash equilibrium is a solution to the optimization problem*

$$max\ min\left(\frac{u_1(\mathbf{y})}{w_1}, ..., \frac{u_n(\mathbf{y})}{w_n}\right) \quad s.t.\ \mathbf{y} \in Y$$

Definition 6.4. *A resource allocation game is weighted proportional fair with the weights* $\{w_i\}$*, if the Nash equilibrium is a solution to the optimization problem*

$$max \prod_{i=1}^{n} u_i(\mathbf{y})^{w_i} \quad s.t.\ \mathbf{y} \in Y$$

Remember that we ignore the constant term T for the sake of convenience when setting spectrum as utilities. Upon the later derivations of weighted max-min fairness and weighted proportional fairness, we shall replace u_i with $(T + u_i)\ \forall i \in N$; otherwise, the objective functions will not be correctly characterized.

Deciding the weights is another important issue. It is reasonably to believe that the weight for each player i is positively proportional to B_i. This is because B_i of player i, mainly influencing the unit acquisition and protection price and determining the priority to acquire and to protect the spectrum in system meaning, is the power to increase its utility. We choose

$\hat{B}_i = \frac{B_i}{\frac{1}{n}\sum_{j=1}^{n} B_j}$, the normalized B_i, to be the weight for player i.

Lastly, according to Lemma 6.1 which shows the range of utility functions, we can transform the above property-related optimization problems from strategy domain into utility domain. Consequently, we can prove the properties by verifying that the utility profile at the Nash equilibrium is a solution to the corresponding optimization problems.

Lemma 6.1. *For game G and $\forall \mathbf{y} \in Y$, the following statements about utility functions are always true:*

1. $-T \le u_i(\mathbf{y}) \le x_i \; \forall i \in N$.
2. $-nT \le \sum_{i=1}^{n} u_i(\mathbf{y}) \le min\left(O, \sum_{i=1}^{n} x_i\right)$.

Proof. See Apendix C ☐

6.1 Allocative Efficiency

Theorem 6.1. *Game G is allocatively efficient. Equivalently, the utility profile at the Nash equilibrium is a solution to the optimization problem,*

$$max \sum_{i=1}^{n} u_i \quad s.t. \; -T \le u_i \le x_i \; \forall i \in N \; and \; -nT \le \sum_{i=1}^{n} u_i \le min\left(O, \sum_{i=1}^{n} x_i\right)$$

Proof. Recall in Theorem 5.4 that the utility profile under $Traffic_k$, $k \in \{0, N\}$, is $U_k^* = (x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$. Also, Corollary 5.1.2 shows $\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} e_{i,-k} = min\left(O, \sum_{i=1}^{n} x_i\right) \; \forall k \in \{0, N\}$. Therefore we know $\sum_{i=1}^{n} u_i$ is maximized by $U_k^* \; \forall k \in \{0, N\}$. Game G is allocatively efficient. ☐

6.2 Pareto Optimality

Theorem 6.2. *Game G is Pareto optimal. Equivalently, if, $\mathbf{y}^*$ is the Nash equilibrium,*

$$\exists (u'_1, ..., u'_n) \ne (u_1(\mathbf{y}^*), ..., u_n(\mathbf{y}^*)), \; u'_i > u_i(\mathbf{y}^*) \; \Rightarrow \; \exists j \in N, \; u'_j < u_j(\mathbf{y}^*)$$

Proof. Since allocative efficiency implies Pareto optimality and Game G is allocatively efficient, it is Pareto optimal as well. □

6.3 Weighted Max-Min Fairness

Theorem 6.3. *Game G is weighted max-min fair with the weights $\left\{\hat{B}_i\right\}$. Equivalently, the utility profile at the Nash equilibrium is a solution to the optimization problem,*

$$max\ min\left(\frac{T+u_1}{\hat{B}_1},...\frac{T+u_n}{\hat{B}_n}\right)$$

$$s.t.\ -T \le u_i \le x_i\ \forall i \in N\ and\ -nT \le \sum_{i=1}^{n} u_i \le min\left(O, \sum_{i=1}^{n} x_i\right)$$

Proof. When Game G is under $Traffic_0$, by substituting U_0^* into the objective function and using Corollary 5.1.1, we derive

$$\frac{T+u_i}{\hat{B}_i} = \frac{T+e_i}{\hat{B}_i} = \frac{T+\frac{O}{n}}{\frac{1}{n}\sum\limits_{l=1}^{n} B_l}\left(\frac{1}{n}\sum_{l=1}^{n} B_l\right) = T + \frac{O}{n} \quad \forall i \in N \tag{6.1}$$

$$\begin{aligned} & min\left(\frac{T+u_1}{\hat{B}_1},...,\frac{T+u_n}{\hat{B}_n}\right) = min\left(\frac{T+e_1}{\hat{B}_1},...,\frac{T+e_n}{\hat{B}_n}\right) \\ = \ & min\left(\frac{T+u_1}{\hat{B}_1},...,\frac{T+u_n}{\hat{B}_n}\right) = min\left(T+\frac{O}{n},...,T+\frac{O}{n}\right) = T+\frac{O}{n} \end{aligned} \tag{6.2}$$

Because $\sum\limits_{i=1}^{n} e_i = O$, if $u_j = e_j + \delta_j$ for some player j with $\delta_j > 0$, there must be some player m having $u_m = e_m - \delta_m$ with $\delta_m > 0$. Therefore we have

$$\begin{aligned} & min\left(\frac{T+u_1}{\hat{B}_1},...\frac{T+u_n}{\hat{B}_n}\right) = min\left(...,\frac{T+e_j+\delta_j}{\hat{B}_j},...,\frac{T+e_m-\delta_m}{\hat{B}_m},...\right) \\ = \ & min\left(...,T+\frac{O}{n}+\frac{\delta_j}{\hat{B}_j},...,T+\frac{O}{n}-\frac{\delta_m}{\hat{B}_m},...\right) \le T+\frac{O}{n}-\frac{\delta_m}{\hat{B}_m} < T+\frac{O}{n} \end{aligned} \tag{6.3}$$

Equation (6.3) tells that $min\left(\frac{T+u_1}{\hat{B}_1},...,\frac{T+u_n}{\hat{B}_n}\right)$ is maximized by U_0^*.

When Game G is under $Traffic_k$ where $k \neq 0$, we know, from Corollary 5.1.5, $p_k(e_{k,-(k-1)}) \geq p_j(e_{j,-k})\ \forall j \in \{k+1, ..., n\}$. Then $p_1(x_1) \geq ... \geq p_k(x_k) \geq p_k(e_{k,-(k-1)}) \geq p_j(e_{j,-k})\ \forall j \in \{k+1, ..., n\}$. It is equivalent to

$$\frac{T+x_1}{\hat{B}_1} \leq ... \leq \frac{T+x_k}{\hat{B}_k} \leq \frac{T+e_{k,-(k-1)}}{\hat{B}_k} \leq \frac{T+e_{j,-k}}{\hat{B}_j} \quad \forall j \in \{k+1, ..., n\} \tag{6.4}$$

Equation (6.4) reveals that the max value of $min(\frac{T+u_1}{\hat{B}_1}, ..., \frac{T+u_n}{\hat{B}_n})$ is $\frac{T+x_1}{\hat{B}_1}$ and is reached at $u_1 = x_1$. Since $u_1 = x_1$ is implied by U_k^*, $k \neq 0$, $min(\frac{T+u_1}{\hat{B}_1}, ..., \frac{T+u_n}{\hat{B}_n})$ is maximized by U_k^* where $k \neq 0$.

In summary, $min\left(\frac{T+u_1}{\hat{B}_1}, ...\frac{T+u_n}{\hat{B}_n}\right)$ is maximized by the utility profile at the Nash equilibrium and hence Game G is weighted max-min fair. □

6.4 Weighted Proportional Fairness

Theorem 6.4. *Game G is weighted proportional fair with the weights $\left\{\hat{B}_i\right\}$. Equivalently, the utility profile at the Nash equilibrium is a solution to the optimization problem,*

$$max \prod_{i=1}^{n} (T+u_i)^{\hat{B}_i} \quad s.t. -T \leq u_i \leq x_i\ \forall i \in N\ and\ -nT \leq \sum_{i=1}^{n} u_i \leq min\left(O, \sum_{i=1}^{n} x_i\right)$$

Proof. When Game G is under $Traffic_0$, we have, from *A.M.* $\geq$ *G.M.*,

$$\begin{aligned}
\prod_{i=1}^{n} (T+u_i)^{\hat{B}_i} &= \prod_{i=1}^{n} \left(\frac{T+u_i}{\hat{B}_i}\right)^{\hat{B}_i} \prod_{i=1}^{n} \hat{B}_i^{\hat{B}_i} \leq \left[\frac{\sum_{i=1}^{n} \hat{B}_i \left(\frac{T+u_i}{\hat{B}_i}\right)}{\sum_{i=1}^{n} \hat{B}_i}\right]^{\sum_{i=1}^{n} \hat{B}_i} \prod_{i=1}^{n} \hat{B}_i^{\hat{B}_i} \\
&= \left[\frac{1}{n} \sum_{i=1}^{n} (T+u_i)\right]^n \prod_{i=1}^{n} \hat{B}_i^{\hat{B}_i} = \left(T + \frac{1}{n} \sum_{i=1}^{n} u_i\right)^n \prod_{i=1}^{n} \hat{B}_i^{\hat{B}_i} \\
&\leq \left(T + \frac{O}{n}\right)^n \prod_{i=1}^{n} \hat{B}_i^{\hat{B}_i}
\end{aligned} \tag{6.5}$$

The equality holds iff $\frac{T+u_i}{\hat{B}_i} = \frac{T+u_j}{\hat{B}_j}\ \forall i, j \in N$ and $\sum_{i=1}^{n} u_i = O$ for which U_0^* is the unique solution. Thus $\prod_{i=1}^{n}(T+u_i)^{\hat{B}_i}$ is maximized by U_0^*.

When Game G is under $Traffic_k$, $k \neq 0$, by applying $A.M. \geq G.M.$ to the term $\prod_{i=k+1}^{n}(T+u_i)^{\hat{B}_i}$, we have

$$\prod_{i=1}^{n}(T+u_i)^{\hat{B}_i} \leq \left(\frac{T+\frac{O-\sum_{i=1}^{k}u_i}{n-k}}{\frac{1}{n-k}\sum_{i=k+1}^{n}\hat{B}_i}\right)^{\sum_{i=k+1}^{n}\hat{B}_i} \prod_{i=1}^{k}\left(\frac{T+u_i}{\hat{B}_i}\right)^{\hat{B}_i}\prod_{i=1}^{n}\hat{B}_i^{\hat{B}_i} \tag{6.6}$$

The equality holds iff $\frac{T+u_i}{\hat{B}_i} = \frac{T+\frac{O-\sum_{j=1}^{k}u_j}{n-k}}{\frac{1}{n-k}\sum_{j=k+1}^{n}\hat{B}_j}$ for $i \in \{k+1,...,n\}$.

We denote the right-hand-side term in Equation (6.6) as $H(u_1,...,u_k)$ for short. By differentiating H by u_k and rearranging it, we have

$$\begin{aligned}\frac{\partial H}{\partial u_k} &= \frac{\sum_{i=k}^{n}\hat{B}_i}{\sum_{i=k+1}^{n}\hat{B}_i}\left(\frac{T+\frac{O-\sum_{i=1}^{k-1}u_i}{n-k+1}}{\frac{1}{n-k+1}\sum_{i=k}^{n}\hat{B}_i}-\frac{T+u_k}{\hat{B}_k}\right)\left(\frac{T+\frac{O-\sum_{i=1}^{k}u_i}{n-k}}{\frac{1}{n-k}\sum_{i=k+1}^{n}\hat{B}_i}\right)^{\left(\sum_{i=k+1}^{n}\hat{B}_i-1\right)} \\ &\cdot \left(\frac{T+u_k}{\hat{B}_k}\right)^{\left(\hat{B}_k-1\right)}\prod_{i=1}^{k-1}\left(\frac{T+u_i}{\hat{B}_i}\right)^{\hat{B}_i}\prod_{i=1}^{n}\hat{B}_i^{\hat{B}_i}\end{aligned} \tag{6.7}$$

From Equation (6.7), we know $\frac{\partial H}{\partial u_k}\Big|_{u_k:\frac{T+u_k}{\hat{B}_k}\leq\frac{T+\frac{O-\sum_{j=1}^{k-1}u_j}{n-k+1}}{\frac{1}{n-k+1}\sum_{j=k}^{n}\hat{B}_j}} \geq 0$. Therefore H is a non-decreasing function of u_k iff $\frac{T+u_k}{\hat{B}_k} \leq \frac{T+\frac{O-\sum_{j=1}^{k-1}u_j}{n-k+1}}{\frac{1}{n-k+1}\sum_{j=k}^{n}\hat{B}_j}$. We continue differentiating H by u_{k-1} and apply the condition for that H is a non-decreasing function of u_k. We can derive H is a non-decreasing function of u_{k-1} and u_k iff $\frac{T+u_{k-1}}{\hat{B_{k-1}}} \leq \frac{T+\frac{O-\sum_{j=1}^{k-2}u_j}{n-k+2}}{\frac{1}{n-k+2}\sum_{j=k-1}^{n}\hat{B}_j}$ and $\frac{T+u_k}{\hat{B}_k} \leq \frac{T+\frac{O-\sum_{j=1}^{k-1}u_j}{n-k+1}}{\frac{1}{n-k+1}\sum_{j=k}^{n}\hat{B}_j}$. By applying the same procedure iteratively to u_{k-2}, ..., and u_1, we can derive H

is a non-decreasing function of u_i $\forall i \in \{1, ..., k\}$ iff $\frac{T+u_i}{\hat{B}_i} \leq \frac{T+\frac{O-\sum_{j=1}^{i-1} u_j}{n-i+1}}{\frac{1}{n-i+1}\sum_{j=i}^{n} \hat{B}_j}$ $\forall i \in \{1, ..., k\}$.

Then Equation (6.6) along with the above analysis on H becomes

$$
\begin{aligned}
\prod_{i=1}^{n} (T+u_i)^{\hat{B}_i} &\leq \left(\frac{T+\frac{O-\sum_{i=1}^{k} u_i}{n-k}}{\frac{1}{n-k}\sum_{i=k+1}^{n} \hat{B}_i} \right)^{\sum_{i=k+1}^{n} \hat{B}_i} \prod_{i=1}^{k} \left(\frac{T+u_i}{\hat{B}_i} \right)^{\hat{B}_i} \prod_{i=1}^{n} \hat{B}_i^{\hat{B}_i} \\
&\leq \left(\frac{T+\frac{-\sum_{i=0}^{k} x_i}{n-k}}{\frac{1}{n-k}\sum_{i=k+1}^{n} \hat{B}_i} \right)^{\sum_{i=k+1}^{n} \hat{B}_i} \prod_{i=1}^{k} \left(\frac{T+x_i}{\hat{B}_i} \right)^{\hat{B}_i} \prod_{i=1}^{n} \hat{B}_i^{\hat{B}_i} \qquad (6.8)
\end{aligned}
$$

The equality holds iff $\frac{T+u_i}{\hat{B}_i} = \frac{T+x_i}{\hat{B}_i} \leq \frac{T+\frac{-\sum_{j=0}^{i-1} x_j}{n-i+1}}{\frac{1}{n-i+1}\sum_{j=i}^{n} \hat{B}_j}$ $\forall i \in \{1, ..., k\}$ and $\frac{T+u_i}{\hat{B}_i} = \frac{T+\frac{-\sum_{j=0}^{k} x_j}{n-k}}{\frac{1}{n-k}\sum_{j=k+1}^{n} \hat{B}_j} < \frac{T+x_i}{\hat{B}_i}$ $\forall i \in \{k+1, ..., n\}$. Equivalently, the equality holds iff $u_i = x_i \leq e_{i,-(-1)}$ $\forall i \in \{1, ..., k\}$ and $u_i = e_{i,-k} < x_i$ $\forall i \in \{k+1, ..., n\}$, i.e. U_k^*. Therefore $\prod_{i=1}^{n} (T+u_i)^{\hat{B}_i}$ is maximized by U_k^* where $k \neq n$.

In summary, $\prod_{i=1}^{n} (T+u_i)^{\hat{B}_i}$ is maximized by the utility profile at the Nash equilibrium. Game G is weighted proportional fair. □

Chapter 7

Strategy-Proof Mechanism - Max Traffic Declaration

In the previous content, we have derived the Nash equilibrium, the spectrum allocation result, (i.e. the utility profile at the Nash equilibrium,) and the corresponding properties. If we omit the process of players' making acquisition/offering requests but directly adopt the final spectrum allocation result, the proposed spectrum sharing algorithm can be simplified as the following spectrum allocation rule with the same pricing (credit-token-frozen) rule.

Definition 7.1. *Given the spectrum O of Agent A, the spectrum T, the credit token budget B_i, and the max traffic requirement x_i of each player i as public information, and assuming that $\{p_i(x_i)\}$ is arranged in decreasing order without losing generality, the spectrum allocation is $AR^* = (x_1, ..., x_k, e_{k+1,-k}, ..., e_{n,-k})$ under $Traffic_k$, $k \in \{0, N\}$.*

According to this spectrum allocation rule, we design a mechanism M which can be adopted in the more general case that all players' max traffic requirements are private information. In Mechanism M, each player i declares its max traffic requirement, x_i', which may be different from the true max traffic requirement x_i. Given all players' declarations, Mechanism M applies the spectrum allocation rule to allocate spectrum. Since now each player possibly gains more spectrum than its true max traffic requirement, it is reasonable to add the assumption that when having reached its true max traffic requirement, a player's utility is the true max traffic.

Mechanism M possesses the property of strategy-proofness [12, 18], that is the truth-revelation of the max traffic is a dominant-strategy equilibrium.

Definition 7.2. *A mechanism is strategy-proof if truth-revelation is a dominant-strategy equilibrium.*

Theorem 7.1. *Mechanism M is strategy-proof. Equivalently, the strategy profile, $(x_1, ..., x_n)$, is a dominant-strategy equilibrium.*

Proof. Given any x'_{-i}, we want to prove $x'_i = x_i$ always results in the highest utility for every player i under all traffic cases.

Let $\underline{N} = \{\underline{i}\}$ be the sorted player set N such that $\left\{p_{\underline{i}}\left(x'_{\underline{i}}\right)\right\}$ is in decreasing order. Let $e_{\underline{j,-k}}$, $\forall \underline{k} \in \{0, \underline{N}\}$ and $\forall \underline{j} \in \{\underline{k+1}, ..., \underline{n}\}$, be the same as $e_{j,-k}$ in Definition 5.1 with $\{x_i\}$ replaced by $\left\{x'_{\underline{i}}\right\}$. Also, let *Traffic*$_{\underline{k}}$, $\underline{k} \in \{0, \underline{N}\}$, denote $x'_{\underline{j}} \leq e_{\underline{j,-(j-1)}}$ $\forall \underline{j} \in \{\underline{1}, ..., \underline{k}\}$ and $x'_{\underline{j}} > e_{\underline{j,-k}}$ $\forall \underline{j} \in \{\underline{k+1}, ..., \underline{n}\}$.

Assume that player i now plays $x'_i = x_i$ and has the m-th priority, i.e. $i = \underline{m}$ and $x_i = x'_{\underline{m}}$. When $\underline{m} \leq \underline{k}$, we have $x_i = x'_{\underline{m}} \leq e_{\underline{m,-(m-1)}}$. Correspondingly, $u_i = x_i$ which is the highest utility player i can obtain. When $\underline{m} > \underline{k}$, we have $x_i = x'_{\underline{m}} > e_{\underline{m,-k}}$ and $u_i = e_{\underline{m,-k}}$. If player i plays $x'_i < e_{\underline{m,-k}}$, then $u'_i = x'_i < e_{\underline{m,-k}} = u_i$; if player i plays $x'_i \geq e_{\underline{m,-k}}$, then $u'_i = e_{\underline{m,-k}}$. In words, no other strategy results in higher utility. From the above, $x'_i = x_i$ results in the highest utility under all traffic cases and therefore is a dominant strategy of player i.

Because the derivation above is applicable $\forall i \in N$, $x'_i = x_i$ is a dominant strategy of every player i and the strategy profile, $(x_1, ..., x_n)$, is a dominant-strategy equilibrium. □

Having Mechanism M be at $(x_1, ..., x_n)$, the spectrum allocation result is the same as previous. Thus efficiency and fairness hold. Such a dominant-strategy equilibrium is nevertheless not unique. In fact, any $(x'_1, ..., x'_n)$ where $x'_i \geq x_i$ $\forall i \in N$ is also a dominant-strategy equilibrium. (This is implied in the proof of Theorem 7.1.) Given that Mechanism M is at any dominant-strategy equilibrium other than $(x_1, ..., x_n)$, weighted max-min fairness is the only property attained.

Lastly, the worst performance of Mechanism M is attained when $x'_i = \infty$ $\forall i \in N$, i.e. each player untruthfully declares that its max traffic is infinity. The corresponding utility profile is $(min(x_1, e_1), ..., min(x_n, e_n))$.

Theorem 7.2. *In Mechanism* M*,* $(min(x_1, e_1), ..., min(x_n, e_n))$ *is the worst utility profile for all players.*

Proof. We continue using the notations in Theorem 7.1. Given $\left(x'_{\underline{1}}, ..., x'_{\underline{n}}\right)$ is any dominant-strategy equilibrium, we must have $x'_{\underline{i}} \geq x_{\underline{i}}\ \forall \underline{i} \in \underline{N}$. Assume that $\left(x'_{\underline{1}}, ..., x'_{\underline{n}}\right)$ is such that Mechanism M is under $Traffic_{\underline{k}}$, $\underline{k} \neq 0$. The allocation is $\left(x'_{\underline{1}}, ..., x'_{\underline{k}}, e_{\underline{k+1,-k}}, ..., e_{\underline{n,-k}}\right)$ and the utility profile is $\left(min(x_{\underline{1}}, x'_{\underline{1}}), ..., min(x_{\underline{k}}, x'_{\underline{k}}), min(x_{\underline{k+1}}, e_{\underline{k+1,-k}}), ..., min(x_{\underline{n}}, e_{\underline{n,-k}})\right)$. We notice $min(x_{\underline{i}}, x'_{\underline{i}}) = x_{\underline{i}}\ \forall \underline{i} \in \{\underline{1}, ..., \underline{k}\}$. Besides, by applying Corollary 5.1.5 to $e_{\underline{i,-k}}$ $\forall \underline{i} \in \{\underline{k+1}, ..., \underline{n}\}$, we have $p_{\underline{i}}(e_{\underline{i}}) \geq p_{\underline{i}}(e_{\underline{i,-1}}) \geq ... \geq p_{\underline{i}}(e_{\underline{i,-k}})$, or equivalently $e_{\underline{i}} \leq ... \leq e_{\underline{i,-k}}$. Thus $min(x_{\underline{i}}, e_{\underline{i}}) \leq min(x_{\underline{i}}, e_{\underline{i,-k}})\ \forall \underline{i} \in \{\underline{k+1}, ..., \underline{n}\}$.

Summing up the above, we derive, $\forall \underline{k} \neq 0$, $(min(x_{\underline{1}}, e_{\underline{1}}), ..., min(x_{\underline{n}}, e_{\underline{n}})) \leq \left(min(x_{\underline{1}}, x'_{\underline{1}}), ..., min(x_{\underline{k}}, x'_{\underline{k}}), min(x_{\underline{k+1}}, e_{\underline{k+1,-k}}), ..., min(x_{\underline{n}}, e_{\underline{n,-k}})\right)$. In other words, $(min(x_{\underline{1}}, e_{\underline{1}}), ..., min(x_{\underline{n}}, e_{\underline{n}}))$, or equivalently $(min(x_1, e_1), ..., min(x_n, e_n))$, is the worst utility profile. □

Chapter 8

Conclusions

IEEE 802.22 is the first cognitive-radio-based wireless standard. IEEE 802.22 systems, operating over the licensed TV bands, utilize the spectrum sensing technique and the inter-BS coexistence mechanism to achieve an effective radio resource sharing with licensed users and other coexistent IEEE 802.22 devices as well.

We propose an efficient and fair spectrum sharing scheme for *dynamic resource renting and offering* (DRRO) and *adaptive on demand channel contention* (AODCC) in the IEEE 802.22 inter-BS coexistence mechanism. In our spectrum sharing game, all BSs always reach a Nash equilibrium where the spectrum allocation result is uniquely determined. The spectrum sharing algorithm is desirable because it achieves efficiency and fairness among all BSs. The allocation is efficient as allocative efficiency and Pareto optimality are achieved. It also meets both weighted max-min fair and weighted proportional fair criteria. By adopting this spectrum allocation result, a strategy-proof mechanism, ensuring efficiency and fairness at the truth-revealing dominant-strategy equilibrium, is designed to be applied in the more general case that max traffic requirements are private information.

To further enhance our research, there are still some aspects we need to work on. While setting the system model, we assume all information is available. Although a solution has been given for the extended case that the max traffic demands are private information, the assumption that BSs' credit token budgets are known by each other may seem a little impractical yet. It will be our future work to design incentives for BSs to declare their budgets truthfully.

We also assume every unit of spectrum is equally important for every BS and the credit token budget is fairly allocated. Accordingly, the utility function of every BS is piecewise linear. We will try to generalize this assumption by adopting different forms of utility functions, e.g. an exponential form or a convex form. Furthermore, a common marketplace for all BSs is considered for simplicity in our work. In order to meet the wide-area purpose of IEEE 802.22, we can design a multi-market scenario where each BS can choose to join one or more markets. This will be a very interesting extension. Finally, instead of using credit tokens, we aim to investigate a monetary-based spectrum allocation mechanism, in accordance with the proposed efficient and fair spectrum allocation, to apply in more different resource sharing schemes. With monetary transfer included in utility functions, a unique truth-revealing dominant-strategy equilibrium is possibly drawn.

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Appendix A

Proof of Lemma 2.1

Proof. Assume that player k is the one satisfying Equation (A.1) with smallest price.

$$O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ > 0 \tag{A.1}$$

1) $\forall i$ s.t. $p_i > p_k$, we can derive from Equation (A.1) that

$$\begin{aligned} O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_i} [y_j]^+ &= O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ + \sum_{j;p_i \geq p_j > p_k} [y_j]^+ \\ &> \sum_{j;p_j=p_i} [y_j]^+ \end{aligned} \tag{A.2}$$

Hence $r_i \geq y_i$, $min(y_i, r_i) = y_i$, and $[y_i - r_i]^+ = 0$. Besides, we also have

$$\sum_{j;p_j<p_i} \left(T - [-y_j]^+\right) - \sum_{j;p_j>p_i} [y_j - r_j]^+ = \sum_{j;p_j<p_i} \left(T - [-y_j]^+\right) \geq 0 \tag{A.3}$$

$$\begin{aligned} & O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j \geq p_i} y_j + \sum_{j;p_j<p_i} T \\ = \ & O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_i} [y_j]^+ + \sum_{j;p_j<p_i} \left(T - [-y_j]^+\right) > \sum_{j;p_j=p_i} [y_j]^+ \end{aligned} \tag{A.4}$$

Therefore $c_i \geq 0$, $min([y_i - r_i]^+, c_i) = 0$, $t_i \geq y_i$, and $min(y_i, t_i) = y_i$. Consequently, $min(y_i, r_i) + min([y_i - r_i]^+, c_i) = y_i = min(y_i, t_i)$ $\forall i$ s.t. $p_i > p_k$.

2) For player k, Equation (A.2) implies that Equation (A.1) has the upper constraint $\sum_{j;p_j=p_k} [y_j]^+$, i.e.

$$\sum_{j;p_j=p_k} [y_j]^+ \geq O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ > 0 \tag{A.5}$$

From Equation (A.5),

$$r_k = \frac{[y_k]^+}{\sum_{j;p_j=p_k} [y_j]^+} \left(O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ \right) \tag{A.6}$$

$$[y_k - r_k]^+ = -\frac{[y_k]^+}{\sum_{j;p_j=p_k} [y_j]^+} \left(O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j\geq p_k} [y_j]^+ \right) \tag{A.7}$$

$$\begin{aligned} c_k &= \frac{[y_k - r_k]^+}{\sum_{j;p_j=p_k} [y_j - r_j]^+} \left(\sum_{j;p_j<p_k} \left(T - [-y_j]^+\right) - \sum_{j;p_j>p_k} [y_j - r_j]^+ \right) \\ &= \frac{[y_k - r_k]^+}{\sum_{j;p_j=p_k} [y_j - r_j]^+} \sum_{j;p_j<p_k} \left(T - [-y_j]^+\right) \\ &= \frac{[y_k]^+}{\sum_{j;p_j=p_k} [y_j]^+} \sum_{j;p_j<p_k} \left(T - [-y_j]^+\right) \end{aligned} \tag{A.8}$$

If $\sum_{j;p_j<p_k} \left(T - [-y_j]^+\right) > \sum_{j;p_j=p_k} [y_j - r_j]^+$, then $c_k > [y_k - r_k]^+$. and $min(y_k, r_k) + min([y_k - r_k]^+, c_k) = y_k - [y_k - r_k]^+ + [y_k - r_k]^+ = y_k$. Furthermore, by appplying Equation (A.7), we can derive that

$$
\begin{aligned}
& O + \sum_{j;p_j=p_k} [y_j]^+ - \sum_{j;p_j \geq p_k} y_j + \sum_{j;p_j<p_k} T \\
= \ & O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ + \sum_{j;p_j<p_k} \left(T - [-y_j]^+\right) \\
> \ & O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ + \sum_{j;p_j=p_k} [y_j - r_j]^+ \\
= \ & O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ - O - \sum_{j=1}^{n} [-y_j]^+ + \sum_{j;p_j \geq p_k} [y_j]^+ \\
= \ & \sum_{j;p_j=p_k} [y_j]^+
\end{aligned}
\tag{A.9}
$$

Consequently, $t_k \geq y_k$ and $min(y_k, r_k) + min([y_k - r_k]^+, c_k) = y_k = min(y_k, t_k)$

If $\sum_{j;p_j<p_k} \left(T - [-y_j]^+\right) \leq \sum_{j;p_j=p_k} [y_j - r_j]^+$, we have $c_k \leq [y_k - r_k]^+$ and

$$
\begin{aligned}
0 \ < \ & O + \sum_{j;p_j=p_k} [y_j]^+ - \sum_{j;p_j \geq p_k} y_j + \sum_{j;p_j<p_k} T \\
= \ & O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ + \sum_{j;p_j<p_k} \left(T - [-y_j]^+\right) \\
\leq \ & O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ + \sum_{j;p_j=p_k} [y_j - r_j]^+ \\
= \ & O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j>p_k} [y_j]^+ - O - \sum_{j=1}^{n} [-y_j]^+ + \sum_{j;p_j \geq p_k} [y_j]^+ \\
= \ & \sum_{j;p_j=p_k} [y_j]^+
\end{aligned}
\tag{A.10}
$$

Equation (A.10) tells that $t_k \leq y_k$ and $min(y_k, t_k) = t_k$. Also, from Equation (A.7) and (A.8),

$$
\begin{aligned}
& min(y_k, r_k) + min([y_k - r_k]^+, c_k) = y_k - [y_k - r_k]^+ + c_k \\
= \; & y_k + \frac{[y_k]^+}{\sum\limits_{j;p_j=p_k} [y_j]^+} \left(O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j \geq p_k} [y_j]^+ \right) + \frac{[y_k]^+}{\sum\limits_{j;p_j=p_k} [y_j]^+} \sum_{j;p_j<p_k} (T - [-y_j]^+) \\
= \; & \frac{[y_k]^+}{\sum\limits_{j;p_j=p_k} [y_j]^+} \left(O + \sum_{j=1}^{n} [-y_j]^+ - \sum_{j;p_j > p_k} [y_j]^+ + \sum_{j;p_j<p_k} (T - [-y_j]^+) \right) - [-y_k]^+ \\
= \; & \frac{[y_k]^+}{\sum\limits_{j;p_j=p_k} [y_j]^+} \left(O + \sum_{j;p_j=p_k} [y_j]^+ - \sum_{j;p_j \geq p_k} y_j + \sum_{j;p_j<p_k} T \right) - [-y_k]^+ \\
= \; & t_k = min(y_k, t_k) \qquad \text{(A.11)}
\end{aligned}
$$

From the above, $min(y_k, r_k) + min([y_k - r_k]^+, c_k) = y_k = min(y_k, t_k)$.

3) $\forall i$ s.t. $p_i < p_k$, we have $r_i = 0$ and $[y_i - r_i]^+ = [y_i]^+$. Equation (A.7) along with $[y_i - r_i]^+ = [y_i]^+$ reveals that

$$
\begin{aligned}
& \sum_{j;p_j<p_i} (T - [-y_j]^+) - \sum_{j;p_j>p_i} [y_j - r_j]^+ \\
= \; & \sum_{j;p_j<p_i} (T - [-y_j]^+) - \sum_{j;p_k>p_j>p_i} [y_j]^+ - \sum_{j;p_j=p_k} [y_j - r_j]^+ \\
= \; & O + \sum_{j=1}^{n} [-y_j]^+ + \sum_{j;p_j<p_i} (T - [-y_j]^+) - \sum_{j;p_j>p_i} [y_j]^+ \\
= \; & O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j \geq p_i} y_j + \sum_{j;p_j<p_i} T \qquad \text{(A.12)}
\end{aligned}
$$

Thus $c_i = t_i + [-y_i]^+$ and

$$
\begin{aligned}
& min(y_i, r_i) + min([y_i - r_i]^+, c_i) = min(y_i, 0) + min([y_i]^+, t_i + [-y_i]^+) \\
= \; & -[-y_i]^+ + [y_i]^+ - \left[[y_i]^+ - [-y_i]^+ - t_i \right]^+ = y_i - [y_i - t_i]^+ \\
= \; & min(y_i, t_i) \qquad \text{(A.13)}
\end{aligned}
$$

Summing up 1), 2), and 3), $min(y_i, r_i) + min([y_i - r_i]^+, c_i) = min(y_i, t_i) \; \forall i \in N$. $\square$

Appendix B

Proof of Corollary 5.1

Proof. 1) Starting from Definition 5.1, $\forall j \in \{k+1, ..., n\}$

$$e_{j,-k} = \frac{B_j}{\frac{1}{n-k}\sum\limits_{l=k+1}^{n} B_l}\left(\frac{-\sum\limits_{l=0}^{k} x_l}{n-k}\right) + \left(\frac{B_j}{\frac{1}{n-k}\sum\limits_{l=k+1}^{n} B_l} - 1\right)T$$

$$\Leftrightarrow \quad T + e_{j,-k} = \frac{B_j}{\frac{1}{n-k}\sum\limits_{l=k+1}^{n} B_l}\left(T + \frac{-\sum\limits_{l=0}^{k} x_l}{n-k}\right)$$

$$\Leftrightarrow \quad \frac{T + e_{j,-k}}{B_j} = \frac{T + \frac{-\sum\limits_{l=0}^{k} x_l}{n-k}}{\frac{1}{n-k}\sum\limits_{l=k+1}^{n} B_l}$$

$$\Leftrightarrow \quad p_j\left(e_{j,-k}\right) = \frac{B_j}{T + e_{j,-k}} = \frac{\frac{1}{n-k}\sum\limits_{l=k+1}^{n} B_l}{T + \frac{-\sum\limits_{l=0}^{k} x_l}{n-k}} \tag{B.1}$$

2) $\forall k \in \{0, N\}$,

$$\sum_{j=k+1}^{n} e_{j,-k} = \frac{\sum\limits_{j=k+1}^{n} B_j}{\frac{1}{n-k}\sum\limits_{l=k+1}^{n} B_l}\left(\frac{-\sum\limits_{l=0}^{k} x_l}{n-k}\right) + \left(\frac{\sum\limits_{j=k+1}^{n} B_j}{\frac{1}{n-k}\sum\limits_{l=k+1}^{n} B_l} - (n-k)\right) T$$
$$= (n-k)\left(\frac{-\sum\limits_{l=0}^{k} x_l}{n-k}\right) + [(n-k)-(n-k)]\,T = -\sum_{l=0}^{k} x_l \qquad \text{(B.2)}$$

Hence

$$\sum_{j=1}^{k} x_j + \sum_{j=k+1}^{n} e_{j,-k} = min\left(O, \sum_{j=1}^{n} x_j\right) \qquad \text{(B.3)}$$

3) $x_j \leq e_{j,-(j-1)}$ is equivalent to

$$\frac{B_j}{T + x_j} \geq \frac{B_j}{T + e_{j,-(j-1)}} = \frac{\frac{1}{n-j+1}\sum\limits_{l=j}^{n} B_l}{T + \frac{-\sum\limits_{l=0}^{j-1} x_l}{n-j+1}} \qquad \text{(B.4)}$$

From $\frac{B_{j-1}}{T + x_{j-1}} \geq \frac{B_j}{T + x_j}$ and Equation (B.4),

$$\frac{B_{j-1}}{T + x_{j-1}} \geq \frac{\frac{1}{n-j+1}\sum\limits_{l=j}^{n} B_l}{T + \frac{-\sum\limits_{l=0}^{j-1} x_l}{n-j+1}} \qquad \text{(B.5)}$$

By using the fact that $\frac{E}{F} \geq \frac{G}{H} \Rightarrow \frac{E}{F} \geq \frac{E + mG}{F + mH}$ $\forall F, H$ and $F + mH > 0$, we substitute B_{j-1} for E, $(T + x_{j-1})$ for F, $\frac{1}{n-j+1}\sum\limits_{l=j}^{n} B_l$ for G, $\left(T + \frac{-\sum\limits_{l=0}^{j-1} x_l}{n-j+1}\right)$ for H and $(n-j+1)$ for m and obtain

$$\frac{B_{j-1}}{T+x_{j-1}} \geq \frac{B_{j-1}+\sum_{l=j}^{n} B_l}{T+x_{j-1}+(n-j+1)T-\sum_{l=0}^{j-1} x_l}$$

$$= \frac{\sum_{l=j-1}^{n} B_l}{(n-j+2)T-\sum_{l=0}^{j-2} x_l} = \frac{\frac{1}{n-j+2}\sum_{l=j-1}^{n} B_l}{T-\frac{\sum_{l=0}^{j-2} x_l}{(n-j+2)}} \tag{B.6}$$

It is equivalent to

$$x_{j-1} \leq e_{j-1,-(j-2)} \tag{B.7}$$

Equation (B.7) holds $\forall j \in N$ and therefore implies that $x_k \leq e_{k,-(k-1)} \Leftrightarrow x_j \leq e_{j,-(j-1)}$ $\forall j \in \{1,...,k\}$ and $\forall k \in N$.

4) $x_{k+1} > e_{k+1,-k}$ is equivalent to

$$\frac{B_{k+1}}{T+x_{k+1}} < \frac{B_{k+1}}{T+e_{k+1,-k}} \tag{B.8}$$

By using $\frac{B_n}{T+x_n} \leq ... \leq \frac{B_{k+2}}{T+x_{k+2}} \leq \frac{B_{k+1}}{T+x_{k+1}}$ and $\frac{B_{k+1}}{T+e_{k+1,-k}} = \frac{B_{k+2}}{T+e_{k+2,-k}} = ...\frac{B_n}{T+e_{n,-k}}$, Equation (B.8) can be extened as

$$\frac{B_n}{T+x_n} \leq ... \leq \frac{B_{k+2}}{T+x_{k+2}} \leq \frac{B_{k+1}}{T+x_{k+1}} < \frac{B_{k+1}}{T+e_{k+1,-k}} = \frac{B_{k+2}}{T+e_{k+2,-k}} = ...\frac{B_n}{T+e_{n,-k}} \tag{B.9}$$

Therefore $\frac{B_{k+1}}{T+x_{k+1}} < \frac{B_{k+1}}{T+e_{k+2,-k}}$, ..., and $\frac{B_n}{T+x_n} < \frac{B_n}{T+e_{n,-k}}$. Equivalently, $x_j > e_{j,-k}$ $\forall j \in \{k+1,...,n\}$.

5) $x_k \leq e_{k,-(k-1)}$ is equivalent to

$$\frac{B_k}{T+x_k} \geq \frac{B_k}{T+e_{k,-(k-1)}} = \frac{\frac{1}{n-k+1}\sum_{l=k}^{n} B_l}{T+\frac{-\sum_{l=0}^{k-1} x_l}{n-k+1}} \tag{B.10}$$

By using the fact that $\frac{E}{F} \geq \frac{G}{H} \Rightarrow \frac{G}{H} \geq \frac{mG-E}{mH-F}$ $\forall F, H$ and $mH-F>0$, we subtitute

B_k for E, $(T + x_k)$ for F, $\frac{1}{n-k+1} \sum\limits_{l=k}^{n} B_l$ for G, $\left(T + \frac{- \sum\limits_{l=0}^{k-1} x_l}{n-k+1} \right)$ for H, and $(n - k + 1)$ for m and obtain that

$$\begin{aligned} \frac{\frac{1}{n-k+1} \sum\limits_{l=k}^{n} B_l}{T + \frac{- \sum\limits_{l=0}^{k-1}}{n-k+1}} &\geq \frac{\sum\limits_{l=k}^{n} B_l - B_k}{(n-k+1)T - \sum\limits_{l=0}^{k-1} x_l - T - x_k} \\ &= \frac{\sum\limits_{l=k-1}^{n} B_l}{(n-k)T - \sum\limits_{l=0}^{k} x_l} = \frac{\frac{1}{n-k} \sum\limits_{l=k-1}^{n} B_l}{T - \frac{1}{n-k} \sum\limits_{l=0}^{k} x_l} \end{aligned} \tag{B.11}$$

In words, $p_k(e_{k,-(k-1)}) \geq p_j(e_{j,-k}) \; \forall j \in \{k+1, ..., n\}$. $\square$

Appendix C

Proof of Lemma 6.1

Proof. 1) It is the result directly from the setting of utility functions.

2) Notice that $-nT \leq \sum_{i=1}^{n} y_i \leq \sum_{i=1}^{n} x_i$. If $\sum_{i=1}^{n} y_i \leq min\left(O, \sum_{i=1}^{n} x_i\right)$, then $\forall i \in N$,

$$\begin{aligned} & O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j \geq p_i} y_j + \sum_{j;p_j<p_i} T \\ = \; & O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j=1}^{n} y_j + \sum_{j;p_j<p_i} (T + y_j) \geq \sum_{j;p_j=p_i} [y_j]^+ \end{aligned} \tag{C.1}$$

Equation (C.1) implies that $t_i \geq y_i \; \forall i \in N$. Therefore

$$-nT \leq \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} min(y_i, t_i) = \sum_{i=1}^{n} y_i \leq min\left(O, \sum_{i=1}^{n} x_i\right). \tag{C.2}$$

If $O < \sum_{i=1}^{n} y_i \leq \sum_{i=1}^{n} x_i$, we let player k be the one satisfying Equation (C.3) with smallest price.

$$\begin{aligned} & O + \sum_{j;p_j=p_k} [y_j]^+ - \sum_{j;p_j \geq p_k} y_j + \sum_{j;p_j<p_k} T > - \sum_{j;p_j=p_k} \left(T - [-y_j]^+\right) \\ \Leftrightarrow \; & O + - \sum_{j;p_j=p_k} \left(T - [-y_j]^+\right) - \sum_{j;p_j>p_k} y_j + \sum_{j;p_j \leq p_k} T > - \sum_{j;p_j=p_k} \left(T - [-y_j]^+\right) \\ \Leftrightarrow \; & O + - \sum_{j;p_j>p_k} y_j + \sum_{j;p_j \leq p_k} T > 0 \end{aligned} \tag{C.3}$$

In the following, we want to derive the relationship between t_i and $y_i \; \forall i \in N$. First, we easily know $t_i = -T$ $\forall i$ s.t. $p_i < p_k$. Besides, derived from Equation (C.3), $\forall i$ s.t. $p_i > p_k$,

$$\begin{aligned} & O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j \geq p_i} y_j + \sum_{j;p_j<p_i} T \\ = \; & O + \sum_{j;p_j=p_i} [y_j]^+ - \sum_{j;p_j>p_k} y_j + \sum_{j;p_j \leq p_k} T + \sum_{j;p_i>p_j>p_k} (T + y_j) > \sum_{j;p_j=p_i} [y_j]^+ \end{aligned} \quad \text{(C.4)}$$

Therefore $t_i \geq y_i$ $\forall i$ s.t. $p_i > p_k$. We turn back to derive t_k for player k. If $\sum_{j;p_j=p_k} [y_j]^+ \geq O + \sum_{j;p_j=p_k} [y_j]^+ - \sum_{j;p_j \geq p_k} y_j + \sum_{j;p_j<p_k} T > 0$ (the left sign of the inequality is implied by the lower constraint of Equation (C.4),)

$$t_k = \begin{cases} \dfrac{y_k}{\sum_{j;p_j=p_k} [y_j]^+} \left(O + \sum_{j;p_j=p_k} [y_j]^+ - \sum_{j;p_j \geq p_k} y_j + \sum_{j;p_j<p_k} T \right) \leq y_k & if \ y_k > 0 \\ y_k \quad if \ y_k \leq 0 \end{cases} \quad \text{(C.5)}$$

If $0 \geq O + \sum_{j;p_j=p_k} [y_j]^+ - \sum_{j;p_j \geq p_k} y_j + \sum_{j;p_j<p_k} T > - \sum_{j;p_j=p_k} (T - [-y_j]^+)$,

$$t_k = -[-y_k]^+ - \frac{T-[-y_k]^+}{\sum_{j;p_j=p_k} (T-[-y_j]^+)} \left(-O - \sum_{j;p_j=p_k} [y_j]^+ + \sum_{j;p_j \geq p_k} y_j - \sum_{j;p_j<p_k} T \right) \leq y_k \quad \text{(C.6)}$$

From Equation (C.5) and (C.6), we know $t_k \leq y_k$ and

$$\sum_{i;p_i=p_k} t_i = O - \sum_{i;p_i>p_k} y_i + \sum_{i;p_i<p_k} T \quad \text{(C.7)}$$

Finally, from $t_i = -T$ $\forall i$ s.t. $p_i < p_k$, $t_i \geq y_i$ $\forall i$ s.t. $p_i > p_k$, $t_k \leq y_k$, and Equation (C.7),

$$\begin{aligned} & \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} min(y_i, t_i) = \sum_{i;p_i>p_k} y_i + \sum_{i;p_i=p_k} t_i - \sum_{i;p_i<p_k} T \\ = \; & \sum_{i;p_i>p_k} y_i + O - \sum_{i;p_i>p_k} y_i + \sum_{i;p_i<p_k} T - \sum_{i;p_i<p_k} T = O = min\left(O, \sum_{i=1}^{n} x_i\right) \end{aligned} \quad \text{(C.8)}$$

In summary, $-nT \leq \sum_{i=1}^{n} u_i \leq min\left(O, \sum_{i=1}^{n} x_i\right)$. □