Thomas Precession and the Torque Equation from the Lab Frame Point of View

M.S.Thesis

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Chapter 1

Introduction

1.1 Origin of the Thomas precession

Historically, Thomas precession came about as an attempt to explain why the then-new quantum theory involving electron spins could still correctly explain certain spectra of anomalous Zeeman effect, despite a puzzling missing factor of 1/2 in a standard calculation. To set the stage, we note that, with hindsight, the introduction of electron spin by Uhlenbeck and Goudsmit in 1926[1][2] can correctly explain the results of Stern-Gerlach type experiments and the normal Zeeman effect, provided that one uses a g-factor of 2 for the electron spin. However, similar calculations applied to the anomalous Zeeman effect, which must include the interaction of the magnetic dipole moment of the electron with its orbital motion (essentially the magnetic field the electron experiences while orbiting the positively charged nucleus) yielded a result that was twice the value measured in experiments. The missing factor of 1/2 then became a puzzle.

The difficulty was timely resolved when, in 1926, L.H.Thomas [3][4] showed that people had overlooked a then-little-known relativistic kinematic effect in their calculations. Briefly, an electron moving along a circular orbit actually experiences a precessional motion with respect to the inertial frame in the lab. This "extra" precessional motion happens to partially offset the precessional motion caused by the aforementioned "spin-orbit interaction." (For an electron, this counter-effect turns out to be one half that obtained by a straightforward spin-orbit coupling calculation, thus successfully explains the missing factor of 1/2. But in the more general situation when the orbiting particle does not have a g-factor of exactly 2, the correction is not one half, because the effect is additive, not multiplicative.)

Because of the significance of this work, this extra precessional motion has been termed "Thomas precession," though earlier authors clearly already noticed this phenomenon through the composition of two successive pure boost Lorentz transformations[5].

Though Thomas has settled in a scoop the difficulty facing the original spectral problem, Thomas precession by itself never leaves the spotlight, possibly due to its non-intuitive character. Indeed, several authors have addressed related issues, such as the frame-independent approach, the lab frame viewpoint of the spin-orbit coupling[7], and the geometrical approach[8]. For a standard textbook derivation, see [9]. A critical review of some of the interpretations and derivations of Thomas precession can also be found in[6][10].

1.2 Motivation of the present work

Because Thomas precession can and should be checked against experiments in the lab, it seems reasonable and worthwhile to investigate how a magnetic dipole interacts with a given static electric field directly from the point of view of a lab observer. Indeed, such an approach has been attempted, and it was the main impetus to the work of [7]. In approaches of this type, one's starting point typically involves the realization that a moving magnetic dipole actually is accompanied by an induced electric dipole, which then can interact with the external electric field. To a lab observer, the interaction involves an electric dipole, and one is thus naturally led to setup the equation of motion for the dipole by considering the torque acting on it. All this sounds so straightfoward that one probably will not doubt if any tricky points may be hidden under the idea. However, the author of [7] points out that this may not be the case, because an issue of the so-called "hidden momentum" must be included in one's formulation, not to mention that Thomas precession must still be invoked again to get the final result right.

It was these unexpected and intriguing claims that caused my attention: Why is it that one must be forced to use Thomas precession again when (s)he is already dealing with things entirely from the very point of view of a lab observer right from the beginning? And, as I progressed and began to get a better grasp of the whole problem, I gradually realized that this problem is less trivial than one's intuition might have first suggested. The work presented below summarizes what I have learned from this investigation. Briefly, the following includes three parts: As a prerequisite, Chapter 2 gives a quick review of how the correct energy (and hence the torque equation) for an orbiting electron is derived when Thomas precession is taken into account, Chapter 3 then discusses the problem from the lab frame point of view, with special emphasis on the main argument of Muñoz [7]. Having pointed out what is failing in Reference[7], we stated in Chapter 4 one further weak point of Ref.[7], then discuss how one may remedy it using an explicit example. In Chapter 5 we propose a slightly more rigorous derivation of the torque equation and suggest that our result can still be made consistent with experimental data.

Chapter 2

The spin-orbit energy for an electron: the traditional approach

2.1 Conventions adopted

For a smoother transition to the problem we had in mind, we present in this chapter the elements of Thomas' argument leading to the resolution of the original spectral problem. But before going on, we set straight our convention on the symbols used and the approximation we would like to adhere to. In what follows, both γ and $\boldsymbol{\beta}$ refer to the factors one encounters in the standard Lorentz transformation. That is, $\boldsymbol{\beta}$ is the velocity of an electron with respect to the lab frame, and $\gamma = 1/\sqrt{1-\beta^2}$. We also neglect terms of order higher than c^{-2} (i.e. order higher than β^2) in the final result. This implies that in many of the intermediate steps of our calculations, we will set γ to be unity without explicitly stating this approximation. We also use a g-factor of 2 for the electron. All primed quantities refer to the electron's rest frame, and unprimed ones to the lab frame.

2.2 The original problem Thomas solved

Consider an electron orbiting about an isolated nucleus. Because the electron is in motion, in the rest frame of the electron it experiences a magnetic field $\boldsymbol{B'} = \gamma \boldsymbol{E} \times \boldsymbol{\beta}$

, with $\boldsymbol{v} = c\boldsymbol{\beta}$ being the instantaneous velocity of the electron. Here, $\boldsymbol{E}$ is the static electric field of the positively charged nucleus. Since the electron has a charge and spin, it also has a magnetic moment $\boldsymbol{\mu'} = (e/mc)\boldsymbol{S'}$. The interaction between the magnetic dipole and the magnetic field is via a torque $\boldsymbol{\tau'} = \boldsymbol{\mu'} \times \boldsymbol{B'}$. The spin dynamics is determined by

$$\frac{d\boldsymbol{S'}}{dt'} = \boldsymbol{\mu'} \times \boldsymbol{B'}. \tag{2.1}$$

The above equation tells us that there is an interaction energy given by

$$\begin{aligned} U' &= -\boldsymbol{\mu'} \cdot \boldsymbol{B'} \\ &= -\frac{e\gamma}{mc}\boldsymbol{S'} \cdot (\boldsymbol{E} \times \boldsymbol{\beta}). \end{aligned} \tag{2.2}$$

To convert the above in terms of what the lab frame observes, we first notice that the interaction energy in the lab frame differs from the above just by an additional γ factor. Also, $\boldsymbol{S'} = \boldsymbol{S} + O(\boldsymbol{\beta}^2)$. Hence, if we neglect terms of order higher than $\boldsymbol{\beta}^2$ again ($\boldsymbol{S'} \approx \boldsymbol{S}$ implies that $\boldsymbol{\mu'} \approx \boldsymbol{\mu}$), we obtain

$$U \approx -\frac{e}{mc}\boldsymbol{S} \cdot (\boldsymbol{E} \times \boldsymbol{\beta}). \tag{2.3}$$

But this result turned out to be twice that of the experimentally observed "fine structure" energy. This discrepancy was what motivated Thomas to start his now famous work.

2.3 Summary of the work of Thomas

Thomas was the first one to show that the discrepancy between the above naive theoretical derivation and the experimental observation is originated from a then-little-known relativistic effect. Specifically, Thomas pointed out that all that was required was a correct treatment of the Lorentz transformation connecting the electron rest frame and the lab frame. The most important idea coming out of his study is: When an electron moves around the nucleus while keeping its own coordinate system non-rotating (with respect to itself), its coordinate system still appears to "rotate" with respect to a lab observer.

Assuming the existence of a certain rotation (with respect to an inertial frame) for the electron's rest frame, then, according to classical mechanics, we know that the time rate of change of any vector $\boldsymbol{G}$ appears to differ, depending on whether a reference frame is rotating or not. Indeed, the transformation between the two reference frames is given by the relation[14]

$$\left(\frac{d\boldsymbol{G}}{dt}\right)_{nonrotating\ frame} = \left(\frac{d\boldsymbol{G}}{dt}\right)_{rotating\ frame} + \boldsymbol{\omega} \times \boldsymbol{G}, \tag{2.4}$$

where $\boldsymbol{\omega}$ is the angular velocity of *the rotating frame with respect to the nonrotating frame.*

Next, we introduce a new inertial frame called the boosted lab frame (blf), which is produced by simply boosting the lab frame using the instantaneous velocity of the electron. (Notice that this is an inertial frame instantaneously comoving with the electron only at that particular moment. At the next moment the electron will deviate from it because of the centripetal acceleration it experiences.) And it differs from the electron rest frame by a rotation found by Thomas. Applying Eqn.2.4 to the spin $\boldsymbol{S'}$ of the electron, we get

$$\left(\frac{d\boldsymbol{S'}}{dt'}\right)_{blf} = \frac{d\boldsymbol{S'}}{dt'} + \boldsymbol{\omega} \times \boldsymbol{S'}, \tag{2.5}$$

with $\boldsymbol{\omega}$ being the angular velocity of *the electron rest frame with respect to the blf* . The detailed calculation yields[9]

$$\begin{aligned} \boldsymbol{\omega} &= \frac{\gamma^2}{\gamma+1}\dot{\boldsymbol{\beta}} \times \boldsymbol{\beta} \\ &\approx \frac{1}{2}\dot{\boldsymbol{\beta}} \times \boldsymbol{\beta}, \end{aligned} \tag{2.6}$$

where $\dot{\boldsymbol{\beta}} = c^{-1}d\boldsymbol{v}/dt$. The nonrelativistic approximation, i.e., Newton's equation of motion

$$\dot{\boldsymbol{\beta}} \approx \frac{e}{mc}\boldsymbol{E}$$

then yields

$$\boldsymbol{\omega} \approx \frac{e}{2mc}\boldsymbol{E} \times \boldsymbol{\beta}.$$

Using Eqns.2.1 and 2.6, we see that Eqn.2.5 becomes :

$$\begin{aligned}\left(\frac{d\boldsymbol{S'}}{dt'}\right)_{blf} &\approx \frac{e}{mc}\boldsymbol{S'} \times (\boldsymbol{E} \times \boldsymbol{\beta}) + \frac{e}{2mc}(\boldsymbol{E} \times \boldsymbol{\beta}) \times \boldsymbol{S'} \\ &= \frac{e}{2mc}\boldsymbol{S'} \times (\boldsymbol{E} \times \boldsymbol{\beta}). \end{aligned} \tag{2.7}$$

As a consequence, the correct interaction energy is

$$\begin{aligned} U &\approx U^{blf} \\ &\approx -\frac{e}{2mc}\boldsymbol{S'} \cdot (\boldsymbol{E} \times \boldsymbol{\beta}) \\ &\approx -\frac{e}{2mc}\boldsymbol{S} \cdot (\boldsymbol{E} \times \boldsymbol{\beta}), \end{aligned} \tag{2.8}$$

which is reduced by a factor of 2 from the naive expression of Eqn.2.3.

Before we leave this chapter, it is worthwhile pointing out one important fact concerning the precessional rate of the spin in this interaction. Clearly, Eqn.2.7 implies that the frequency of the precession has a magnitude given by

$$\omega_{precession} \backsim \frac{eE}{m} \cdot \frac{\beta}{c} \backsim v_{orbit}\omega_{orbit} \cdot \frac{\beta}{c} \backsim \beta^2 \omega_{orbit},$$

which says that it is extremely slow compared to the orbital frequency of the electron, because β typically is of the order of 0.01. We will make use of this fact in our later development of the theory.

Chapter 3

Thomas Precession: In the lab frame

Because it is straightforward to derive things first in the electron's rest frame and then transform everything back to the lab frame, less attention was directed to doing things directly from the viewpoint of a lab observer. One exception is the work of Muñoz, who considered the problem from the lab frame and attempted to compare the differences using both approaches[7]. Though Muñoz succeeded in deriving the same results for both approaches, his argument is dubious at best. To make our point, we next give a brief account of how the lab frame point of view was taken to attack the problem.

Thus, starting with the lab frame, we will see a fixed nucleus, which generates a static electric field in the surrounding space. The electron moves around the nucleus, carrying a spin and also a permanent magnetic moment of a fixed value. To a lab observer, a direct interaction between the magnetic dipole with the static electric field is impossible by classical electrodynamics. Thus, something indirect is responsible for the interaction. The next most obvious candidate is via relativistic effect, which predicts that the *moving* magnetic dipole actually carries with it an induced electric dipole. This is briefly reviewed in the following section.

3.1 The electric dipole accompanying a moving magnetic dipole

Several authors have considered how an electric dipole is generated when a magnetic dipole is in motion. In particular, it is known that the induced electric dipole moment $\boldsymbol{p}$ is given by $\boldsymbol{p} = \boldsymbol{\beta} \times \boldsymbol{\mu}'$ [9, 11]. There are several methods to obtain this result. For instance, we can take a vector potential for a magnetic dipole

$$\boldsymbol{A} = \mu_0 \boldsymbol{\mu} \times \boldsymbol{r} / 4\pi r^3$$

and transform the four-vector potential from the moving electron frame to the lab frame. A straightforward calculation gives

$$\boldsymbol{p} = \boldsymbol{\beta} \times \boldsymbol{\mu}.$$

This relationship is important and will be used frequently later.

3.2 The hidden momentum

When a magnetic moment $\boldsymbol{\mu}$ is moving in an electric field $\boldsymbol{E}$, we must introduce a new momentum term called the "hidden momentum,"

$$\boldsymbol{P}_{hidden} \approx \boldsymbol{\mu} \times \boldsymbol{E}/c. \tag{3.1}$$

This extra momentum is not only experimentally measurable but is also required on the theoretical ground if the conservation law of the linear momentum is to hold[12, 13]. Previous authors have demonstrated quite clearly how the hidden momentum may arise for different models of the magnetic moment, but here we will verify this fact using a straightfoward approach: We simply look at the equation of motion from the electron's rest frame and convert everything to the lab frame to check its self-consistency.

Assume a magnetic moment $\boldsymbol{\mu}$ is moving in a static electric field, and there is no free current and charge besides $\boldsymbol{\mu}$ itself. In the lab frame where one only observes a stationary $\boldsymbol{E}$, we have a moving $\boldsymbol{\mu}$ accompanied by an induced electric dipole $\boldsymbol{p} = \boldsymbol{\beta} \times \boldsymbol{\mu}$. The moving $\boldsymbol{\mu}$ has a center-of-mass momentum $\boldsymbol{P}_{cm}$. With hindsight, we blindly introduce the hidden momentum $\boldsymbol{P}_{hidden} = \boldsymbol{\mu} \times \boldsymbol{E}/c$ in our system. Then the total momentum of this system is assumed to be $\boldsymbol{P} = \boldsymbol{P}_{cm} + \boldsymbol{P}_{hidden}$. The dynamical equation of $\boldsymbol{\mu}$ in this frame now reads

$$\begin{aligned}
\frac{d\boldsymbol{P}_{cm}}{dt} &= \frac{d\boldsymbol{P}}{dt} - \frac{d\boldsymbol{P}_{hidden}}{dt} \\
&= (\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{E} - \frac{d}{dt}(\frac{\boldsymbol{\mu} \times \boldsymbol{E}}{c}) \\
&= [(\boldsymbol{\beta} \times \boldsymbol{\mu}) \cdot \boldsymbol{\nabla}]\boldsymbol{E} - \frac{\boldsymbol{\mu}}{c} \times [\frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{E}] \\
&= [(\boldsymbol{\beta} \times \boldsymbol{\mu}) \cdot \boldsymbol{\nabla}]\boldsymbol{E} - \boldsymbol{\mu} \times [(\boldsymbol{\beta} \cdot \boldsymbol{\nabla})\boldsymbol{E}], \qquad (3.2)
\end{aligned}$$

where $\partial \boldsymbol{E}/\partial t = 0$ because electric field is static by assumption.

On the other hand, in the comoving frame with $\boldsymbol{\mu}$, we will see $\boldsymbol{B'} = \boldsymbol{E} \times \boldsymbol{\beta}$. And then the dynamical equation of $\boldsymbol{\mu}$ in this frame is

$$\begin{aligned}
\frac{d\boldsymbol{P}_{cm}}{dt} &= (\boldsymbol{\mu} \cdot \boldsymbol{\nabla})\boldsymbol{B'} \\
&= (\boldsymbol{\mu} \cdot \boldsymbol{\nabla})(\boldsymbol{E} \times \boldsymbol{\beta}) \\
&= [\boldsymbol{\nabla} \times (\boldsymbol{E} \times \boldsymbol{\beta})] \times \boldsymbol{\mu} + \boldsymbol{\nabla}[\boldsymbol{\mu} \cdot (\boldsymbol{E} \times \boldsymbol{\beta})]. \qquad (3.3)
\end{aligned}$$

Now,

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{E} \times \boldsymbol{\beta}) &= (\boldsymbol{\beta} \cdot \boldsymbol{\nabla})\boldsymbol{E} - (\boldsymbol{\nabla} \cdot \boldsymbol{E})\boldsymbol{\beta} + (\boldsymbol{\nabla} \cdot \boldsymbol{\beta})\boldsymbol{E} - (\boldsymbol{E} \cdot \boldsymbol{\nabla})\boldsymbol{\beta} \qquad (3.4) \\
&= (\boldsymbol{\beta} \cdot \boldsymbol{\nabla})\boldsymbol{E}, \qquad (3.5)
\end{aligned}$$

because $\boldsymbol{\beta}$ is space-independent and there is no free charge so that $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$. And the second term of Eqn.3.3 can be rearranged to read

$$\begin{aligned}
\boldsymbol{\nabla}[\boldsymbol{\mu} \cdot (\boldsymbol{E} \times \boldsymbol{\beta})] &= \boldsymbol{\nabla}[\boldsymbol{E} \cdot (\boldsymbol{\beta} \times \boldsymbol{\mu})] \\
&= \boldsymbol{E} \times [\boldsymbol{\nabla} \times (\boldsymbol{\beta} \times \boldsymbol{\mu})] + (\boldsymbol{E} \cdot \boldsymbol{\nabla})(\boldsymbol{\beta} \times \boldsymbol{\mu}) \\
&\quad + (\boldsymbol{\beta} \times \boldsymbol{\mu}) \times (\boldsymbol{\nabla} \times \boldsymbol{E}) + [(\boldsymbol{\beta} \times \boldsymbol{\mu}) \cdot \boldsymbol{\nabla}]\boldsymbol{E} \qquad (3.6) \\
&= [(\boldsymbol{\beta} \times \boldsymbol{\mu}) \cdot \boldsymbol{\nabla}]\boldsymbol{E}, \qquad (3.7)
\end{aligned}$$

where three terms of Eqn.3.6 are equal to zero because $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$ are space-independent, and the field is static so that $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$.

In virtue of Eqns.3.5 and 3.7, Eqn.3.3 becomes

$$\begin{aligned}\frac{d\boldsymbol{P}_{cm}}{dt} &= [(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})\boldsymbol{E}]\times\boldsymbol{\mu} + [(\boldsymbol{\beta}\times\boldsymbol{\mu})\cdot\boldsymbol{\nabla}]\boldsymbol{E} \\ &= [(\boldsymbol{\beta}\times\boldsymbol{\mu})\cdot\boldsymbol{\nabla}]\boldsymbol{E} - \boldsymbol{\mu}\times[(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})\boldsymbol{E}], \qquad (3.8)\end{aligned}$$

which is seen to be identical to the expression of Eqn.3.2. This verifies and justifies the inclusion of the hidden momentum in the formalism. Of course, this also implies that one needs to take extra care in dealing with the momentum, since the hidden momentum is rarely suspected of its existence.

3.3 The inclusion of the hidden momentum by Muñoz

Now let us turn to the work of Muñoz[7]. To begin with, we note that this work has the merit of trying to set things straight entirely in the lab frame. Indeed, this may be desirable in view of the fact that one then does not have to go through the formalism of convoluting Lorentz transformations to first obtain the correction from Thomas precession in order just to get the right answer. Having this said, we would like to point out in advance that, quite unfortunately, something unsatisfactory is present in his argument for the derivation of the torque equation. The line of argument in his work may be summarized as follows.

In the lab frame, the quantity related to the torque $\boldsymbol{\tau}$ is the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$, which includes not just the spin considered by most previous researchers but also the explicit inclusion of the orbital angular momentum as well. The orbital angular momentum is $\boldsymbol{L} = \boldsymbol{r}_{cm} \times \boldsymbol{P}$, with $\boldsymbol{r}_{cm}$ being the position vector of the center of mass and $\boldsymbol{P}$ the *total* momentum of the system. Thus,

$$\begin{aligned}\frac{d\boldsymbol{L}}{dt} &= \frac{d}{dt}[\boldsymbol{r}_{cm}\times(\boldsymbol{P}_{cm}+\boldsymbol{P}_{hidden})] \\ &= \boldsymbol{r}_{cm}\times\frac{d\boldsymbol{P}}{dt} + \boldsymbol{v}_{cm}\times\boldsymbol{P}_{hidden}. \qquad (3.9)\end{aligned}$$

On the other hand, the total torque $\boldsymbol{\tau}$ may be computed as the integral of $\boldsymbol{r}\times\rho\boldsymbol{E}$ over all space. Since ρ is nonvanishing only in a very small region, we can approximate

$\boldsymbol{E}(\boldsymbol{r}) \approx \boldsymbol{E}(\boldsymbol{r}_{cm})$, and, with $\boldsymbol{r} = \boldsymbol{r}_{cm} + \boldsymbol{x}$ we have

$$\begin{aligned}\boldsymbol{\tau} &\approx \int \boldsymbol{r} \times \rho \boldsymbol{E}(\boldsymbol{r}_{cm}) d^3 r \\ &= \boldsymbol{r}_{cm} \times e\boldsymbol{E}(\boldsymbol{r}_{cm}) + \int \boldsymbol{x} \rho d^3 x \times \boldsymbol{E}(\boldsymbol{r}_{cm}). \end{aligned} \tag{3.10}$$

The first term of Eqn.3.10 is equal to the $\boldsymbol{r}_{cm} \times d\boldsymbol{P}/dt$ in Eqn.3.9. The integration of the second term of Eqn.3.10 is just the electric dipole $\boldsymbol{p}$ by definition. Substituting Eqns.3.10 and 3.9 into $d\boldsymbol{J}/dt = \boldsymbol{\tau}$, we obtain

$$\begin{aligned}\frac{d\boldsymbol{S}}{dt} &= \boldsymbol{\tau} - \frac{d\boldsymbol{L}}{dt} \\ &\approx \boldsymbol{p} \times \boldsymbol{E} - \boldsymbol{v}_{cm} \times \boldsymbol{P}_{hidden}. \end{aligned} \tag{3.11}$$

This is the equation central to Muñoz's work. However, the argument following this equation gets a twist and makes his entire approach unsatisfactory. This is discussed in the next section.

3.4 Missing the right turn...

Assuming all that is well up to the point of Eqn.3.11, Muñoz then claimed that "*the Thomas precession is again expected to alter Eqn.3.11 by a factor of one-half* ," which forcifully turns the original equation into

$$\frac{d\boldsymbol{S}}{dt} \approx \frac{1}{2}\boldsymbol{p} \times \boldsymbol{E} - \frac{1}{2}\boldsymbol{v}_{cm} \times \boldsymbol{P}_{hidden}. \tag{3.12}$$

With this assumed form (of Eqn.3.12) and the following

$$\left\{ \begin{array}{ccc} \boldsymbol{p} &=& \boldsymbol{\beta} \times \boldsymbol{\mu} \\ \boldsymbol{P}_{hidden} &=& \frac{\boldsymbol{\mu} \times \boldsymbol{E}}{c} \\ \boldsymbol{\mu} &=& \frac{e}{mc}\boldsymbol{S} \\ \boldsymbol{v}_{cm} &\equiv& c\boldsymbol{\beta} \end{array} \right. ,$$

one easily obtains

$$\begin{aligned}\frac{d\boldsymbol{S}}{dt} &\approx \frac{e}{2mc}[(\boldsymbol{\beta} \times \boldsymbol{S}) \times \boldsymbol{E} - \boldsymbol{\beta} \times (\boldsymbol{S} \times \boldsymbol{E})] \\ &= \frac{e}{2mc}\boldsymbol{S} \times (\boldsymbol{E} \times \boldsymbol{\beta}), \end{aligned} \tag{3.13}$$

which can be viewed as being equivalent to Eqn.2.7, because we only retained terms to first order in β and to this accuracy $dt \approx dt', \boldsymbol{S} \approx \boldsymbol{S'}$.

To summarize, the author got what he wanted, a derivation of the torque equation entirely from the point of view of a lab observer. Or did he?

In our view, his derivation suffers from the fatal error of assuming that a quantity determined purely by a lab observer must still be subjected to the correction of Thomas precession. Though Muñoz spent the latter part of his paper trying to justify his approach, we feel that his efforts are futile because of the wrong turn he has adopted. In the next chapter, we suggest a remedy to his approach and illustrate our point with a concrete example.

Chapter 4

Two wrongs corrected

In this chapter we suggest two important points completely missed out by Muñoz and illustrate with an example to show that actually things can be rectified. In other words, there is nothing wrong with believing that one can strictly adhere to the lab frame point of view. All that is needed is do it the right way.

The first point we would like to bring out is the assumption made in Section 3.3 approximating $\boldsymbol{E}(\boldsymbol{r}) \approx \boldsymbol{E}(\boldsymbol{r}_{cm})$ and expanding only $\boldsymbol{r} = \boldsymbol{r}_{cm} + \boldsymbol{x}$. In fact, to be self-consistent, one should expand not just $\boldsymbol{r}$ to the first order, but must also retain $\boldsymbol{E}$ to the same order. Thus, the correct approximation should be $\boldsymbol{E}(\boldsymbol{r}) \approx \boldsymbol{E}(\boldsymbol{r}_{cm}) + (\boldsymbol{x} \cdot \boldsymbol{\nabla})\boldsymbol{E}$.

Hence, Eqn.3.10 should be approximated as

$$\begin{aligned} \boldsymbol{\tau} &= \int \boldsymbol{r} \times \rho \boldsymbol{E}(\boldsymbol{r}) d^3 r \\ &\approx \boldsymbol{r}_{cm} \times e\boldsymbol{E}(\boldsymbol{r}_{cm}) + \int \boldsymbol{x} \rho d^3 x \times \boldsymbol{E}(\boldsymbol{r}_{cm}) + \boldsymbol{r}_{cm} \times \int d^3 x \rho (\boldsymbol{x} \cdot \boldsymbol{\nabla})\boldsymbol{E}, \end{aligned} \tag{4.1}$$

where we have neglected terms of second order or higher in the small distance $\boldsymbol{x}$. Comparing Eqn.4.1 with Eqn.3.10, we can easily revise Eqn.3.11 to

$$\begin{aligned} \frac{d\boldsymbol{S}}{dt} &= \boldsymbol{\tau} - \frac{d\boldsymbol{L}}{dt} \\ &\approx \boldsymbol{p} \times \boldsymbol{E} - \boldsymbol{v}_{cm} \times \boldsymbol{P}_{hidden} + \boldsymbol{r}_{cm} \times \int d^3 x \rho (\boldsymbol{x} \cdot \boldsymbol{\nabla})\boldsymbol{E}, \end{aligned} \tag{4.2}$$

which now has the extra term $[\boldsymbol{r}_{cm} \times \int \rho (\boldsymbol{x} \cdot \boldsymbol{\nabla})\boldsymbol{E}]$ compared to Eqn.3.10.

This extra term can be identified with

$$\boldsymbol{r}_{cm} \times \int d^3x\rho(\boldsymbol{x} \cdot \boldsymbol{\nabla})\boldsymbol{E} = \boldsymbol{r}_{cm} \times (\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{E},$$

which, however, does not bear any resemblance to the terms preceding it. But if we restrict ourselves to the special case when $\boldsymbol{E}$ is just the ordinary central field

$$\boldsymbol{E} = k\frac{\boldsymbol{r}}{r^3},$$

then

$$\begin{aligned} & \boldsymbol{r} \times (\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{E} \\ = \; & \boldsymbol{r} \times (\boldsymbol{p} \cdot \boldsymbol{\nabla})k\frac{\boldsymbol{r}}{r^3} \\ = \; & k\boldsymbol{r} \times \frac{\boldsymbol{p} - 3\boldsymbol{p} \cdot \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3} \\ = \; & k\frac{\boldsymbol{r}}{r^3} \times \boldsymbol{p} \\ = \; & \boldsymbol{E} \times \boldsymbol{p}, \end{aligned}$$

which cancels the first term of Eqn.4.2 and gives us

$$\frac{d\boldsymbol{S}}{dt} \approx -\boldsymbol{v}_{cm} \times \boldsymbol{P}_{hidden} = \boldsymbol{\beta} \times (\boldsymbol{E} \times \boldsymbol{\mu}), \tag{4.3}$$

where use has been made of Eqn.3.1.

As a concrete example, we may consider an electron on a circular orbit about the nucleus. Then

$$\begin{aligned} \frac{d\boldsymbol{S}}{dt} & \approx \boldsymbol{\beta} \times (\boldsymbol{E} \times \boldsymbol{\mu}) \\ & = (\boldsymbol{\beta} \cdot \boldsymbol{\mu})\,\boldsymbol{E} - (\boldsymbol{\beta} \cdot \boldsymbol{E})\,\boldsymbol{\mu} \\ & = (\boldsymbol{\beta} \cdot \boldsymbol{\mu})\,\boldsymbol{E}, \end{aligned} \tag{4.4}$$

because the velocity vector is orthogonal to the central electric field.

With the previously mentioned fact that the spin only precesses very slowly compared to the orbital motion of the electron, we may treat the $\boldsymbol{\mu}$ in Eqn.4.4 as a fixed vector and perform a time average over one period of the orbital motion of the electron, just to get the averaged precessional rate of the spin. The result is

$$\frac{d\boldsymbol{S}}{dt} \approx \frac{1}{2}\boldsymbol{\mu} \times (\boldsymbol{E} \times \boldsymbol{\beta}), \tag{4.5}$$

which is seen to agree with Eqn.2.7, at least under the approximation we are considering.

Since nowhere did we make use of Thomas precession in our derivation, it is clear that Muñoz has erred when he made the ad hoc assumption that the lab frame observer still had to adopt the dubious Thomas precession in his formulation.

Chapter 5

Deriving it via the Lorentz transformation

We end our investigation in this chapter by considering how to transform physical variables directly from the unambiguious electron's rest frame to the lab frame. In so doing it is hoped that one may gain a better insight into the nature of the physical interpretation of the interaction terms one sees in a lab frame.

5.1 The dynamical equation rederived

We start out with the equation of motion in the electron's rest frame:

$$\frac{d\boldsymbol{S'}}{dt'} = \boldsymbol{\mu'} \times \boldsymbol{B'}, \tag{5.1}$$

$$\left(\frac{d\boldsymbol{S'}}{dt'}\right)_{blf} = \frac{d\boldsymbol{S'}}{dt'} + \boldsymbol{\omega} \times \boldsymbol{S'}. \tag{5.2}$$

Combining them, we obtain

$$\left(\frac{d\boldsymbol{S'}}{dt'}\right)_{blf} = \boldsymbol{\mu'} \times \boldsymbol{B'} + \boldsymbol{\omega} \times \boldsymbol{S'}. \tag{5.3}$$

And then we introduce the Lorentz transformation of any 4-vector (A_0, A_1, A_2, A_3) [9, 14] :

$$\begin{cases} A_0' = \gamma(A_0 - \boldsymbol{\beta}\cdot\boldsymbol{A}), \\ \\ \boldsymbol{A'} = \boldsymbol{A} + \frac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{A})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}A_0. \end{cases}$$

where A_0 is the time-component, and $\boldsymbol{A} \equiv (A_1, A_2, A_3)$ is the spatial-components.

The transformation of $\boldsymbol{S'}$ will be

$$\begin{cases} S_0' = \gamma(S_0 - \boldsymbol{\beta}\cdot\boldsymbol{S}), \\ \\ \boldsymbol{S'} = \boldsymbol{S} + \frac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{S})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}S_0. \end{cases}$$

Because of the covariant constraint $S_0' = 0$, we have $S_0 = \boldsymbol{\beta}\cdot\boldsymbol{S}$. A closer examination of the transformation above, we see that the relation $\boldsymbol{S'} = \boldsymbol{S} + O(\boldsymbol{\beta}^2)$ holds, a fact we have already utilized again and again before.

Before applying the transformation of $\boldsymbol{S}$ on Eqn.5.3, we should explicitly list the assumptions we will make. They are:

1. We still neglect the terms of order higher than β^2 .
2. The term $\boldsymbol{\mu'}\times\boldsymbol{B'}$ is the easiest, because $\boldsymbol{B'}$ just equals $\boldsymbol{E}\times\boldsymbol{\beta}$ and $\boldsymbol{\mu'} = \boldsymbol{\mu} + O(\boldsymbol{\beta}^2)$ (since $\boldsymbol{S'} = \boldsymbol{S} + O(\boldsymbol{\beta}^2)$), so that $\boldsymbol{\mu'} \approx \boldsymbol{\mu}$.
3. Because of $\boldsymbol{\omega} \approx \frac{1}{2}\dot{\boldsymbol{\beta}}\times\boldsymbol{\beta}$, we could obtain $\boldsymbol{\omega}\times\boldsymbol{S'} \approx \boldsymbol{\omega}\times\boldsymbol{S}$, if we neglect terms of order higher than β^2 .
4. Using chain rule, $d\boldsymbol{S'}/dt' = \gamma d\boldsymbol{S'}/dt \approx d\boldsymbol{S'}/dt$, if we neglect the terms order higher than β^2.
5. One very important rule we must keep in mind is that when we evaluate the term $d\boldsymbol{S'}/dt$, we can *not* just blindly set $\boldsymbol{S'} \approx \boldsymbol{S}$! This may appear surprising at first sight if one recalls our earlier remark that $\boldsymbol{S'} \approx \boldsymbol{S}$ is correct to the second order in β^2. However, this is indeed the case, because, after each differentiation we will get a

term proportional to $\dot{\boldsymbol{\beta}}$, which, being proportional to the very strong acceleration the electron experiences, may render the result one order larger!

Having stated the precaution, we now move on to completing our program. Having applied the transformation of $\boldsymbol{S}$ on Eqn.5.3 while keeping the five rules above in check, we obtain

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\mu} \times \boldsymbol{B'} + \boldsymbol{\beta}(\dot{\boldsymbol{\beta}} \cdot \boldsymbol{S}). \tag{5.4}$$

Further substitutions of the variables involved, we can recast the above into

$$\frac{d\boldsymbol{S}}{dt} = \frac{e}{mc}\boldsymbol{E}(\boldsymbol{S} \cdot \boldsymbol{\beta}). \tag{5.5}$$

This dynamical equation in the lab frame can be further massaged to a more manageable form if we apply again the idea that the spin actually can be treated as a fixed vector during one complete circuit of the electron. Then, the above reduces to

$$\frac{d\boldsymbol{S}}{dt} \approx \frac{1}{2}\frac{e}{mc}\boldsymbol{S} \times (\boldsymbol{E} \times \boldsymbol{\beta}) \tag{5.6}$$

after we average it out in one circular orbit. This, of course, assumes the same form of Eqn.2.7, as any sensible theory must reproduce. Needless to say, the energy we calculated using this formalism is the same as the standard spin-orbit interaction value.

5.2 Conclusion

We have pointed out certain shortcomings in the previous work attempting to deal with the spin-orbit interaction completely from the lab point of view. We also proposed certain remedies which are capable of rectifying the weakness we have spotted in the previous work. It seems that generalizations of our approach to cover a broader scope are possible, and we are currently pursuing them.

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