國立臺灣大學社會科學院經濟學系

碩士論文

Department of Economics

College of Social Sciences

National Taiwan University

Master Thesis

可選擇集合的多樣性：以多重偏好所衍伸之附加價值進行評估

Diversity in Opportunity Sets: Assessing by Attached Value Derived from Multiple Preferences

曾玟銨

Wen-An Tseng

指導教授: 林逸軒 博士

Advisor: Yi-Hsuan Lin, Ph.D.

中華民國 112 年 8 月

August, 2023

doi:10.6342/NTU202303269

國立臺灣大學碩士學位論文
口試委員會審定書

可選擇集合的多樣性：以多重偏好所衍伸之附加價值進行評估

Diversity in Opportunity Sets: Assessing by Attached Value Derived from Multiple Preferences

本論文係曾玟銨君（學號 R10323048）在國立臺灣大學經濟學系完成之碩士學位論文，於民國 112 年 7 月 26 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

______________________（指導教授）

______________________　黃貞穎

doi:10.6342/NTU202303269

摘要

這篇論文從多重偏好的角度出發，探討集合中蘊含的多樣性，並提出一種基於滿足這些偏好程度的集合排序。我使用參考點作為具體化「滿足」概念的工具。

首先我將比較我所提出的模型與文獻中其他基於一組給定偏好進行集合排序的模型之間的差異。然後，在潛在偏好不可觀察的情況下，我發現這種集合排序可以用與Kreps (1979) 相同的公理來描述。然而，在我的模型中，我推導出了一種不同的表示方法，其中集合的評估是基於偏好滿足，而非最大化。

關鍵字：集合偏好，集合排序，可選擇集，多樣性，自由，多重偏好

doi:10.6342/NTU202303269

Abstract

This paper explores the diversity present in opportunity sets from the perspective of multiple preferences and proposes a ranking of sets based on the degree to which they satisfy these preferences. I utilize a reference point (or, a default option) as a tool to concretize the concept of satisfaction.

First I compare the differences between my model and other model in the literature that also rank sets based on a given collection of preferences. Then, I show that my model can be characterized by the two axioms when the underlying preferences are unobservable. These axioms are the same as those in Kreps (1979), however, I derived a distinct representation under which a set is evaluated based on preference satisfaction, rather than maximization.

Keywords: Set preference, Set ranking, Preference over sets, Ranking over sets, Opportunity sets, Diversity, Freedom, Multi-preference

doi:10.6342/NTU202303269

Contents

doi:10.6342/NTU202303269

Chapter 1 Introduction

Consider a decision problem involving the selection of a restaurant for dining. Given that members in the group have their own preferences over foods, it is impractical to select an option that maximizes each one' s preference. In such a scenario, it is more reasonable for the members to agree on an option that offers food that satisfies them sufficiently, rather than insisting on finding a restaurant that offers their most favorite food.

Consider another scenario involving a city organizer who is tasked with planning the construction of a sports center. Faced with a set of proposals, each one may contain several facilities such as a basketball court, tennis court, volleyball court, swimming pool, gym, etc., the organizer must take into account the preferences of the city's residents. Because of the limited budget, there does not exist a proposal includes all of the most sumptuous or most luxurious facilities. Additionally, citizens are willing to buy the ticket if the facilities in the sports center is good enough for them.

Therefore, a pragmatic solution to such decision problems is to ranking sets (restaurants, or design proposals) according to the number of preferences (members in the group, or citizens) who are satisfied with their elements (dishes, or facilities). In other words, for a group consists of three spicy-lovers who regard spicy hot pot as their fa-

doi:10.6342/NTU202303269

vorite dish and two people cannot handle any spicy food, although the majority of this group prefer spicy hot pot, an American restaurant which offers everyone an acceptable dish would be a more appropriate restaurant choice, even though no one considers any dish there as their favorite. Similarly, it is a better decision for the organizer to choose the proposal includes a swimming pool and a gym, facilities which a majority of citizens find satisfactory, rather than the one consisting of a basketball court and a volleyball court, which might be strongly preferred only by teenagers.

Generally, I consider the scenario where ranking sets involves several preferences over elements, but not in the sense of maximization, rather in terms of satisfaction. Additionally, there are two restrictions for this scenario. First, whether these preferences are satisfied by a set only depends on the elements within the set, independently of the satisfaction of others. Taking sports center problem as an example, this assumption implies that whether a citizen's willingness to purchase a ticket is not influenced by the choices made by others but is solely determined by the facilities within the sports center. The second assumption posits that the elements will not be ”depleted” or ”exhausted” when satisfying preferences, meaning there is no concept of quantity for element. As still in the sports center example, the facilities is assumed to be non-rivalrous goods.

In my model, I utilize a reference point (or, default option) as a tool to concretize the concept of satisfaction. Specifically, I postulate that there is an alternative, which serves as the reference point, located outside the space of elements. I then extend preferences over elements to preferences over both elements and reference point, and define that a set satisfies a preference if and only if this set contains an element that is ranked above the reference point according to the preference.

doi:10.6342/NTU202303269

I propose to rank sets based on preference satisfaction. Specifically, a set is better if and only if it satisfies more preferences in the collection. The number of preferences satisfied with a set is what I define as the attached value of the set derived from multiple preferences.

When the collection of preferences is observable, I show that my ranking differs from those rankings over opportunity sets in the literature. When the collection of preferences is unobservable, I provide a behavioral characterization for my ranking. Theorem 1 indicates that a complete and transitive ranking over sets satisfies two axioms if and only if there exists a collection of preferences where this ranking prefers the sets which satisfy more preferences in the collection.

These two axioms are the same as those in Kreps (1979). However, his representation is motivated by uncertainty about the future preference of an individual. In particular, his representation postulates a collection of utility functions, only one of which may be realized when this individual must make a choice from a set. No matter which utility function is realized, she always chooses an element to maximize it. Then the ex-ante value of a set is given by the sum of these maximized utilities.

Given a collection of utility functions, Kreps' ranking over sets differs from mine. However, my result shows that if the underlying preferences are unobservable, two models are behaviorally equivalent. A ranking over sets can be generated from two distinct collections of preferences or utility functions, using different mechanisms, respectively.

doi:10.6342/NTU202303269

Chapter 2 Literature Review

This kind of rankings over sets involving a collection of preferences is first proposed to analyze individual welfare in terms of freedom.

Pattanaik and Xu (1990) was the first to argue that the assessment of individual welfare should have considered freedom of choice in addition to agents' utilities. They mentioned two reasons why freedom of choice should have been taken into consideration: the first was that freedom was considered instrumental in achieving a higher level of an individual's utility, and the second was that freedom might have had its intrinsic value in many circumstances.

For the instrumental value of freedom, Kreps (1979) offered a rationale for this concept. He considered a two-stage choice problem, where the agent was unsure about his future preference, therefore the agent had a ”desire for flexibility”. In this context, the freedom of choice allowed the agent to maximize his utility when the uncertain future was realized.

For the intrinsic value of freedom, Pattanaik and Xu (1998) advocated that the consideration of preferences which a reasonable person in the agent's situation could possibly have was important in assessing the ”intrinsic” value of freedom, which was referred to as the multi-preference approach. Moreover, Puppe (2002) further advocated the

doi:10.6342/NTU202303269

multi-preference model of freedom and established its link to traditional social choice theory. He proposed an alternative interpretation of the collection of preferences. By interpreting these preferences as a decision-maker's different multiple selves, the problem of comparing opportunity sets in terms of freedom could be reframed as a social choice problem that involved the preferences of the entire society.

Puppe and Xu (2010) introduced the concept of essential elements in an opportunity set. An element was considered essential if, by deleting it, the reduced opportunity set offered less freedom than the original set. They also showed that the notion of essentiality could be given a natural interpretation in terms of the multi-preference approach.

On the other hand, Pattanaik and Xu (2000) and Bossert, Pattanaik, Xu, et al. (2001) explored the role of diversity in measuring the freedom contained in opportunity sets, using the concept of similarity and the distance between the elements within the opportunity sets, respectively.

Nehring and Puppe (2002) also proposed a multi-attribute approach and showed that this approach could serve as a useful and, in some contexts, perhaps even a canonical conceptual framework for thinking about diversity. The idea of the multi-attribute approach was that, taking biodiversity as an example, the value of diversity consisted in the realization of certain attributes or potentialities of life by some species.

In most of the literature, the focus was on the ranking of "individual" over sets, emphasizing the concept of freedom provided by these sets. It is worth noting that the collection of preferences that a reasonable person in the agent's situation could possibly have and the measurement of diversity are two distinct and independent tools used to quantify the degree of freedom.

doi:10.6342/NTU202303269

When utilizing the former approach to analyze the freedom contained within opportunity sets, it implies that only one preference from the collection will be realized. The individual ultimately needs to choose an element from the opportunity set based on their preferences.

Similarly, in the latter approach, the individual also needs to make a choice from the opportunity set. However, this approach directly examines the property of elements within the opportunity sets, like dissimilarity or kinds of distance between elements. More dissimilar or more ”far” the elements within the opportunity sets are implies that the individual has more choices in the opportunity sets, leading to increased freedom.

On the contrary, when considering the ranking of a ”society” over opportunity sets, a different concept of ”diversity” becomes more relevant in this context. In this scenario, all preferences within the society will realize and should be somehow satisfied. Therefore, diversity becomes an index of the extent to which preferences within the society are being satisfied. Moreover, It is important to measure diversity based on the members, that is, the respective preferences within the society.

doi:10.6342/NTU202303269

Chapter 3 Model

Let X be a finite set of elements and 2^X be the collection of subsets of X including the empty set, often referred to as a collection of menus. 2^X is on which I would like to analyze the ranking in the perspective of satisfaction.

To embody the concept of satisfaction, I induce a reference point or a default option outside X, denoted as r. Let $\mathcal{R}$ denote a finite collection of complete and transitive preferences on $X \cup \{r\}$, which is referred to as the reference set, or can be though as a society of individuals.

In my model, a preference is satisfied by a set if and only if there exists an element in the set that is preferred to the reference point r for it.

Giving the elements space X and the collection of preferences $\mathcal{R}$, I define the function $N_{\mathcal{R}} : 2^X \to 2^{\mathcal{R}}$ as follows: for all $A \in 2^X$,

$$N_{\mathcal{R}}(A) = \left\{R \in \mathcal{R} : \exists x \in A \; s.t. \; x \, R \, r\right\},$$

which is the preferences in the collection that are satisfied by the set A.

Notice that the reference point r is not included in the menus which the society ranks, in order words, the reference point is not an element which can directly satisfy

doi:10.6342/NTU202303269

preferences. Instead, it serves as a benchmark for preferences. In the restaurant example mentioned before, each member in the group can consider ”home dinner” as the reference point. This means that if a restaurant does not offer a dish better than home dinner for some members, they will choose not to attend the dining and instead have dinner at home. The reference point ”home dinner” is an alternative outside any menu of restaurant, yet all individuals utilize it as a benchmark to determine whether a restaurant offers a dish satisfying them.

Furthermore, one can view the reference point solely as a threshold. If the reference point is not adopted, one can still use a more complicated depiction to convey the concept of satisfaction: given a collection of preferences $\mathcal{R}$. In the collection, for each preference R, there exists an a_R in X such that a_R is the worse element for R that can satisfy R. It's worth noting that it might be possible for $a_R = a_{R'}$ for some $R, R' \in \mathcal{R}$, $R \neq R'$. Consequently, a set satisfies a preference R if and only if there exists an element within the set which is preferred to a_R for R. This kind of definition is equivalent to my earlier definition, where for each preference $R \in \mathcal{R}$, a reference point r is embedded between a_R and the subsequent element. My definition can simply the follow-up discussion and the reference point serves as a tool within my definition.

The transitive and complete ranking $\succsim$ on 2^X which this paper aims to analyze then is defined as: for all $A, B \in 2^X$,

$$A \succsim B \iff |N_{\mathcal{R}}(A)| \geq |N_{\mathcal{R}}(B)|. \tag{3.1}$$

doi:10.6342/NTU202303269

Chapter 4 Observable Reference Set

In this chapter, I demonstrate that when the reference set is observable, that is, the collection of preferences in the society is given, this ranking $\succsim$ is distinct from the other rankings studied in the literature.

Consider the collection of preference relations, $\mathcal{R} = \{R_1, R_2, R_3\}$ on $X \cup \{r\}$, where $X = \{a, b, c\}$, and

$$a\, R_1\, r\, R_1\, b\, R_1\, c,$$

$$a\, R_2\, r\, R_2\, b\, R_2\, c,$$

$$b\, R_3\, r\, R_3\, a\, R_3\, c.$$

In this collection of preferences, the elements satisfy preference R_i if and only if it is ranked at the top of whole space X for preference R_i.

One can verify that the ranking $\succsim$ is

$$\{a, b, c\} \sim \{a, b\} \succ \{a, c\} \sim \{a\} \succ \{b, c\} \sim \{b\} \succ \{c\} \sim \emptyset.$$

In the literature, Pattanaik and Xu (1998) first define for $A \in 2^X$,

$$\text{max}(A) = \{x \in A : \exists R \in \mathcal{R}\ s.t.\ x\, R\, y\ \forall y \in A\},$$

doi:10.6342/NTU202303269

which collects the elements in set A for which there exists preference ranking them at the top within set A. Then propose the ranking of opportunity sets $\succsim_M$, which assesses freedom reflected in the opportunity set, is defined as

$$A \succsim_M B \Longleftrightarrow |\max(A)| \geq |\max(B)|.$$

In this case, it is

$$\{a, b, c\} \sim_M \{a, b\} \succ_M \{a, c\} \sim_M \{a\} \sim_M \{b, c\} \sim_M \{b\} \sim_M \{c\} \succ_M \emptyset.$$

On the other hand, Puppe and Xu (2010), analyses freedom based on the notion of essential elements introduced by Puppe (1996), which is defined as, for $A \in 2^X$,

$$E(A) = \big\{x \in A : \; A \succ^* A \setminus \{x\}\big\}.$$

This essential subset has a natural interpretation in terms of multiple preferences: consider $\mathcal{R} = \{R_1, ...R_n\}$ and define

$$E(A) = \bigcup_{i \in \{1,...,n\}} \max_{R_i} A,$$

where $\max_{R_i} A = \{x \in A : \; xR_iy \; \forall y \in A\}$. This definition captures an intuitive and reasonable concept: if an element is the best within the set for some preferences, then removing this element would diminish the attractiveness of the set.

And notice that this definition is equivalent to $\max(A)$. Furthermore, they propose the ranking $\succsim_E$ such that

$$A \succsim_E B \Longleftrightarrow |E(A \cup B) \cap A| \geq |E(A \cup B) \cap B|.$$

doi:10.6342/NTU202303269

Note that $\succsim_E$ is not transitive in general, but coincidentally, it is in this case, as follow:

$$\{a, b, c\} \sim_E \{a, b\} \succ_E \{a, c\} \sim_E \{a\} \sim_E \{b, c\} \sim_E \{b\} \succ_E \{c\} \succ_E \emptyset.$$

One can focus on the comparison between $\{a\}$ and $\{b, c\}$. $\succsim_M$ and $\succsim_E$ focus on the quantity of elements being ranked at top in the set; in contrast, $\succsim$ emphasizes the number of preferences which rank elements in the set at the top of the whole space. That is, both $\succsim_M$ and $\succsim_E$, moreover, the majority of set ranking involving multiple preferences which is characterized in the literature, evaluate sets based on the quantity of elements possessing a particular property (being ranked at top in the set). However, $\succsim$ assesses sets not solely based on the property, but also considers the extent of the property (being ranked at top in the whole space for "how many" preference in the collection). This distinction significantly sets $\succsim$ apart from the rankings in the literature, making it challenging to characterize when the collection of preferences is given.

doi:10.6342/NTU202303269

Chapter 5 Unobservable Reference Set

In this chapter, let us consider the scenario where the reference set is unobservable, given a ranking $\succsim$ on 2^X, is there any condition that ensures the existence of a reference set $\mathcal{R} = \{R_1, R_2, ..., R_n\}$ which satisfies (3.1)?

Before answering this question, there are two important observations about the reference set $\mathcal{R}$.

First, in general, one can write $\mathcal{R} = \{R_1, R_2, ..., R_n\} \cup \{R_1^*, R_2^*, ..., R_k^*\}$, where for all R_i, there exists at least one $x \in X$ such that xR_ir; for all R_j^*, there exists no $x \in X$ such that xR_j^*r. Then $\mathcal{R}$ satisfies (3.1) if and only if $\{R_1, R_2, ..., R_n\}$ satisfies (3.1). Therefore, in the following discussion, I only focus on the case where for all $R \in \mathcal{R}$, there exists $x \in X$ such that $xR\,r$.

Secondly, notice that if there is a reference set $\mathcal{R} = \{R_1, R_2, ..., R_n\}$ satisfies (3.1), then another reference set $\mathcal{R}' = \{R_1', R_2', ..., R_n'\}$ can also satisfy (3.1), where R_i' is generated by rearranging the order of elements above and below the reference point r, respectively, for each $R_i \in \mathcal{R}$. Therefore, the crucial information for the question is not the specific reference set itself, but rather the sets that are ranked above the reference

doi:10.6342/NTU202303269

point r by the preferences in the reference set.

Define $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^k \subseteq 2^X \setminus \{\emptyset\} \times \mathbb{N}$, and function $n_{\mathcal{G}} : 2^X \to \mathbb{N} \cup \{0\}$ such that for all $A \in 2^X$,

$$n_{\mathcal{G}}(A) = \sum_{\{i: A \cap G_i \neq \emptyset\}} g_i.$$

Lemma 1. *Given the ranking $\succsim$ on 2^X. There exists a reference set $\mathcal{R}$ which satisfies (3.1) if and only if there exists $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^k \subseteq 2^X \setminus \{\emptyset\} \times \mathbb{N}$ such that*

$$A \succsim B \Longleftrightarrow n_{\mathcal{G}}(A) \geq n_{\mathcal{G}}(B). \tag{5.1}$$

And denote $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^k$ as the collection of upper contour sets of r (for short, upper contour collection) with respect to $\mathcal{R}$, and $\mathcal{R}$ as the reference set with respect to $\mathcal{G}$.

Proof. For sufficiency, given the reference set $\mathcal{R} = \{R_1, R_2, ..., R_n\}$ such that

$$A \succsim B \Longleftrightarrow |N_{\mathcal{R}}(A)| \geq |N_{\mathcal{R}}(B)|.$$

Without loss of generality, assume $\mathcal{R}$ be arranged in the order

$$\mathcal{R} = \{R_{11}, ..., R_{1l_1}, R_{21}, ...R_{2l_2}, ..., R_{k^*1}, ..., R_{k^*l_{k^*}}\}$$

such that

- $\sum_{j=1}^{k^*} l_j = n$, and
- $\{x \in X : xR_{ij_1} r\} = \{x \in X : xR_{ij_2} r\}$ for all $i = 1, 2, ...k^*$ and $j_1, j_2 = 1, 2, ..., l_i$, denote as $\{x \in X : xR_i^* r\}$, and

doi:10.6342/NTU202303269

- $\{x \in X : xR^*_{i_1} r\} \neq \{x \in X : xR^*_{i_2} r\}$ for all $i_1 \neq i_2$.

Define $\mathcal{G}$ as $\{(G_i, g_i)\}_{i=1}^k$ such that $k = k^*$,and $G_i = \{x \in X : xR^*_i r\}$, $g_i = l_i$ for $i = 1, 2, ..., k^*$. Then for all $A \in 2^X$,

$$|N_{\mathcal{R}}(A)| = |\{R \in \mathcal{R} : \exists x \in A \ s.t. \ xR\, r\}| = \sum_{\{j: A \cap G_j \neq \emptyset\}} g_i = n_{\mathcal{G}}(A).$$

We have

$$A \succsim B \iff |N_{\mathcal{R}}(A)| \geq |N_{\mathcal{R}}(B)| \iff n_{\mathcal{G}}(A) \geq n_{\mathcal{G}}(B).$$

For necessity, give $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^k \subseteq 2^X \setminus \{\emptyset\} \times \mathbb{N}$ such that

$$A \succsim B \iff n_{\mathcal{G}}(A) \geq n_{\mathcal{G}}(B).$$

Define

$$\mathcal{R} = \{R_{11}, R_{12}, ..., R_{1g_1}, ..., R_{k1}, R_{k2}, ..., R_{kg_k}\}$$

such that $\{x \in X : xR_{ij}\, r\} = G_i$ for all $i = 1, 2, ..., k$ and $j = 1, 2, ..., g_i$. Then for all $A \in 2^X$, if $A \cap G_i \neq \emptyset$, then $R_{ij} \in \{R \in \mathcal{R} : \exists x \in A \ s.t. \ xR\, r\} = N(A)$ for all $j = 1, 2, ...g_i$. Therefore,

$$n_{\mathcal{G}}(A) = \sum_{\{j: A \cap G_j \neq \emptyset\}} g_i = |\{R \in \mathcal{R} : \exists x \in A \ s.t. \ xR\, r\}| = |N_{\mathcal{R}}(A)|.$$

We have

$$A \succsim B \iff n_{\mathcal{G}}(A) \geq n_{\mathcal{G}}(B) \iff |N_{\mathcal{R}}(A)| \geq |N_{\mathcal{R}}(B)|.$$

□

Notice that the form of the function $n_{\mathcal{G}}$ is referred as diversity function in Nehring

doi:10.6342/NTU202303269

and Puppe (2002), which is utilized to evaluate the diversity contained by sets. In their model, G_i and g_i are interpreted as representing an ”attitude” and the weight assigned to that attitude, respectively. The diversity function measure the sum of weights of attitudes realizing by sets. It is important to note that, under this interpretation, g_i is not restricted to being a natural number.

By this lemma, one can restate the question as follows: given a ranking $\succsim$ on 2^X, is there any condition that ensures the existence of a a set $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^k \subseteq 2^X \setminus \{\emptyset\} \times \mathbb{N}$ which satisfies (5.1)?

5.1 Characterization Results

The following fact is from Nehring and Puppe (2002).

Fact 1. *If a function $f : 2^X \to \mathbb{R}$ satisfies $f(\emptyset) = 0$, then there exists a unique function $g : 2^X \to \mathbb{R}$, the conjugate Moebius inverse, such that $g(\emptyset) = 0$, and for all $S \in 2^X$,*

$$f(S) = \sum_{A:A\cap S\neq\emptyset} g(A).$$

Moreover, the function g is given by, for all $\emptyset \neq A \in 2^X$,

$$g(A) = \sum_{S\subseteq A} (-1)^{|A\setminus S|+1} f(S^c).$$

The following lemma can be inferred from the lemma 3. in Kreps (1979).

Lemma 2. *Let $F \subseteq 2^X$ with $|X| < \infty$. If a transitive and complete ranking $\succsim$ satisfies that $B \subsetneq A$ implies $A \succ B$. Than there exist non-positive integers $\phi(C)$ for all $C \in F$*

doi:10.6342/NTU202303269

such that for all $A, B \in F$,

$$A \succsim B \iff \sum_{\{C \in F : A \subseteq C\}} \phi(C) \geq \sum_{\{C \in F : B \subseteq C\}} \phi(C).$$

Proof. Following Kreps' notations, let $w(A) = \sum_{\{C \in F : A \subseteq C\}} \phi(C)$, and $w^*(A) = \sum_{\{C \in F : A \subsetneq C\}} \phi(C)$. Then $w(A) = \phi(A) + w^*(A)$. one can progressively define each $\phi(A)$ as going down with the $\sim$-equivalence class. With this process, $w^*(A)$ is given if $\phi(A')$ is defined for all A' with $A' \succ A$ (thus, for all A' with $A \subsetneq A'$). For the $\succsim$-most preferred $\sim$-equivalence class, define the same non-positive integer $\phi(A)$ for all A in the class, then $w(A)$ is the same within the class because for A in this class, there does not exist A' such that $A \subsetneq A'$, thus $w^*(A) = 0$. Then for the $\succsim$-second preferred $\sim$-equivalence class, define non-positive integer $\phi(A)$ for each A in the class, such that $\phi(A) + w^*(A)$ is equal among the class and less than for A' in the first class. Keep doing this for the following class, because F is finite, there are only classes, and the process can give a desired representation.

Notice that specifically, $\phi(A)$ in the second class can only be negative, however, it can be zero in other classes.

For example, let $X = \{x, y, z\}$, $F = \{\{xz\}, \{yz\}, \{x\}, \{y\}, \{z\}\}$, and $\succsim$ such that

$$\{xz\} \succ \{yz\} \succ \{x\} \sim \{y\} \sim \{z\}.$$

$\phi(.)$ such that

$$\phi(\{xz\}) = -1, \quad \phi(\{yz\}) = -2, \quad \phi(\{x\}) = -2, \quad \phi(\{y\}) = -1, \quad \phi(\{z\}) = 0$$

can give the representation. $\square$

doi:10.6342/NTU202303269

Theorem 1. *If X is a finite set. A transitive and complete ranking $\succsim$ on 2^X satisfies (i) $B \subseteq A$ implies that $A \succsim B$, and (ii) $A \cup B \sim A$ implies that $A \cup B \cup C \sim A \cup C$ for all $C \in 2^X$ if and only if there exists an reference set $\mathcal{R}$ such that*

$$A \succsim B \Longleftrightarrow |N_{\mathcal{R}}(A)| \geq |N_{\mathcal{R}}(B)|.$$

Proof. For necessity, observe that if $R \in N_{\mathcal{R}}(A_1) \cup N_{\mathcal{R}}(A_2)$, then it is either $R \in N_{\mathcal{R}}(A_1)$ or $R \in N_{\mathcal{R}}(A_2)$, which implies there exists $x \in A_1$ or $x \in A_2$ such that $xR\,r$. Equivalently, there exists $x \in A_1 \cup A_2$ such that $xR\,r$, by definition, $R \in N_{\mathcal{R}}(A_1 \cup A_2)$, Thus, $N_{\mathcal{R}}(A_1) \cup N_{\mathcal{R}}(A_2) \subseteq N_{\mathcal{R}}(A_1 \cup A_2)$.

If $R \in N_{\mathcal{R}}(A_1 \cup A_2)$, then there exists $x \in A_1 \cup A_2$ such that $xR\,r$, that is, the element x must belong to either A_1 or A_2, therefore, $R \in N_{\mathcal{R}}(A_1)$ or $R \in N_{\mathcal{R}}(A_2)$, which means $R \in N_{\mathcal{R}}(A_1) \cup N_{\mathcal{R}}(A_2)$, thus, $N_{\mathcal{R}}(A_1 \cup A_2) \subseteq N_{\mathcal{R}}(A_1) \cup N_{\mathcal{R}}(A_2)$. Therefore, for all $A_1, A_2 \in 2^X$, $N_{\mathcal{R}}(A_1) \cup N_{\mathcal{R}}(A_2) = N_{\mathcal{R}}(A_1 \cup A_2)$. And this implies that for all $B \subseteq A$, $N_{\mathcal{R}}(B) \subseteq N_{\mathcal{R}}(A)$, then $|N_{\mathcal{R}}(A)| \geq |N_{\mathcal{R}}(B)|$, $A \succsim B$.

If $A \cup B \sim A$, then it must be $N_{\mathcal{R}}(A \cup B) = N_{\mathcal{R}}(A) \cup N_{\mathcal{R}}(B) = N_{\mathcal{R}}(A)$. Thus,

$$N_{\mathcal{R}}(A \cup B \cup C) = N_{\mathcal{R}}(A) \cup N_{\mathcal{R}}(B) \cup N_{\mathcal{R}}(C) = N_{\mathcal{R}}(A) \cup N_{\mathcal{R}}(C) = N_{\mathcal{R}}(A \cup C),$$

then $|N_{\mathcal{R}}(A \cup B \cup C)| = |N_{\mathcal{R}}(A \cup C)|$, $A \cup B \cup C \sim A \cup C$.

For sufficiency, let $\succsim$ be a transitive and complete binary relation satisfying (i) and (ii), for all $A \in 2^X$, define

$$\hat{A} = \bigcup_{\{A' \in 2^X : A \cup A' \sim A\}} A'.$$

And define $F \subseteq 2^X$ as

$$F = \{A \in 2^X : A = \hat{A}\}.$$

doi:10.6342/NTU202303269

Note that if $B \subsetneq A$ and $A, B \in F$, it must be $A \succ B$, otherwise, we have $A \sim B$, thus $A \cup B = A \sim B$, and $B \subsetneq A \subseteq \hat{B}$, that is, $B \notin F$, which is contradictory.

By Lemma 2, there exists non-positive integers $\phi(C)$ for all $C \in F$ such that

$$A \succsim B \iff \sum_{\{C \in F : A \subseteq C\}} \phi(C) \geq \sum_{\{C \in F : B \subseteq C\}} \phi(C).$$

Observe that for all $A \in 2^X$ there exists $A' \in F$ such that $A \sim A'$ because X is finite and then so is $\{B \in 2^X : A \cup B \sim A\}$. And $A \sim \hat{A}$. Therefore, for all $A \in 2^X$, define function $h : 2^X \to \mathbb{N} \cup \{0\}$ such that for all $A \in 2^X$,

$$h(A) = \sum_{\{C \in F : \hat{A} \subseteq C\}} \phi(C).$$

Then the function h gives the representation. And it can be rewritten as

$$h(A) = \sum_{\{C \in 2^X : A \subseteq C\}} \psi(C)$$

where $\psi(C) = \phi(C)$ if $C \in F$, $\psi(C) = 0$ if $C \in 2^X \setminus F$. It is trivial for $A \in F$. For $A \notin F$,

$$\sum_{\{C \in 2^X : A \subseteq C\}} \psi(C) = \sum_{\{C \in 2^X \setminus F : A \subseteq C\}} \psi(C) + \sum_{\{C \in F : A \subseteq C, \hat{A} \subseteq C\}} \psi(C) + \sum_{\{C \in F : A \subseteq C, \hat{A} \not\subseteq C\}} \psi(C)$$

Because the first part equals to 0 by definition of $\psi(C)$, and for the third part, the collection $\{C \in F : A \subseteq C, \hat{A} \not\subseteq C\}$ is empty, to show that, assume $A \subseteq C$ for $C \in F$, note that $A \subseteq \hat{A}$ and $A \sim \hat{A}$, thus

$$A \cup \hat{A} \sim A,$$

doi:10.6342/NTU202303269

by (ii), it must be

$$A \cup \hat{A} \cup C \sim A \cup C.$$

Because $A \subseteq C$, this is equivalent to

$$C \cup \hat{A} \sim C.$$

By definition, $\hat{A} \subseteq \hat{C} = C$. Therefore,

$$\sum_{\{C \in 2^X : A \subseteq C\}} \psi(C) = \sum_{\{C \in F : A \subseteq C, \hat{A} \subseteq C\}} \phi(C) = \sum_{\{C \in F : \hat{A} \subseteq C\}} \phi(C)$$

Moreover, the function $n : 2^X \to \mathbb{N} \cup \{0\}$ such that

$$n(A) = h(A) - h(\emptyset)$$

can also give the representation, and $n(A) \geq 0$ for all $A \in 2^X$, and $n(\emptyset) = 0$.

Observe that for all $A \in 2^X$,

$$\begin{aligned} n(A) &= h(A) - h(\emptyset) \\ &= \sum_{\{C \in 2^X : A \subseteq C\}} \psi(C) - \sum_{\{C \in 2^X : \emptyset \subseteq C\}} \psi(C) \\ &= \sum_{\{C \in 2^X : A \cap C^c \neq \emptyset\}} -\psi(C) \\ &= \sum_{\{C \in 2^X : A \cap C \neq \emptyset\}} g(C), \end{aligned}$$

where $g(C) = -\psi(C^c)$ for all $C \in 2^X \setminus \{\emptyset\}$. Define $g(\emptyset) = 0$, then by Fact 1, the function $g : 2^X \to \mathbb{N} \cup \{0\}$ is the unique conjugate Moebius inverse of n. Then define $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^k \subseteq 2^X \setminus \{\emptyset\} \times \mathbb{N}$, $\{G_i\}_{i=1}^k$ be the collection of all $A \in 2^X$ with

doi:10.6342/NTU202303269

$g(A) > 0$, and $g_i = g(G_i)$ for all $i = 1, 2, ..., k$. The function $n_{\mathcal{G}} : 2^X \to \mathbb{N} \cup \{0\}$ can give the representation of $\succsim$, the proof is complete with lemma 1. □

5.2 Minimum Cardinality of Reference Set

The previous section demonstrates that the ranking $\succsim$ on 2^X satisfying the conditions in Theorem 1 can be represented by a reference set $\mathcal{R}$ in formula (3.1).

Therefore, this section aims to further ask a natural question: Given the ranking $\succsim$ on 2^X satisfying the conditions in Theorem 1, is there a minimum value for the cardinality of reference set $\mathcal{R}$ which represents $\succsim$? The following proposition gives the answer: yes.

Propositin 1. *If a transitive and complete ranking $\succsim$ on 2^X satisfies (i) $B \subseteq A$ implies that $A \succsim B$, and (ii) $A \cup B \sim A$ implies that $A \cup B \cup C \sim A \cup C$ for all $C \in 2^X$, there exists a minimal cardinality for the reference set $\mathcal{R}$ which represents $\succsim$ in formula (3.1).*

Proof. I prove this proposition by directly finding the minimal cardinality for the disrable reference set $\mathcal{R}$.

Revisit the Lemma 1 and investigate the relationship between the upper contour collection $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^k$ and the reference set $\mathcal{R}$ with respect to $\mathcal{G}$. For each $i = 1, 2, ..., k$, one can interpret (G_i, g_i) from the perspective of the reference set $\mathcal{R}$ as follows: there are g_i preferences in $\mathcal{R}$ ranking all elements of G_i above reference point r and all elements of G_i^c (complement of G_i) below reference point r. Therefore, the cardinality of reference set $\mathcal{R}$ is equal to $\sum_{i=1}^k g_i$.

In the proof of Theorem 1, one can observe that the upper contour collection $\mathcal{G} =$

doi:10.6342/NTU202303269

$\{(G_i, g_i)\}_{i=1}^k$ is determined at the step which assigns non-positive integers $\phi(C)$ to all $C \in F$. This is because $\mathcal{G}$ collects the all $A \in 2^X$ with $g(A) > 0$ and $0 < g(A) = -\phi(A^c)$ only if $A^c \in F$. Thus $\mathcal{R}$ and its cardinality are simultaneously determined .

To obtain the minimum cardinality of reference set $\mathcal{R}$, we look back at the first step of sufficiency proof (or equivalently, Lemma 2) and take $F = \{A \in 2^X : A = \hat{A}\}$, and choose non-negative integers $\phi(C)$ as large as possible for all $C \in F$ (because $0 < g(A) = -\phi(A^c)$) to represent the ranking $\succsim$ (that is, $A \succsim B$ if and only if $\sum_{\{C \in F: A \subseteq C\}} \phi(C) \geq \sum_{\{C \in F: B \subseteq C\}} \phi(C)$) in F. Without loss of generality, I assume that there are n $\sim$-equivalence class in F and name them as the first class, second class, on so on, in the order of ranking from the most preferred to the least preferred.

Still let $w(A) = \sum_{\{C \in F: A \subseteq C\}} \phi(C)$, and $w^*(A) = \sum_{\{C \in F: A \subsetneq C\}} \phi(C)$ and $w(A) = \phi(A) + w^*(A)$. Start with the first class, notice that the set of elements X must be the only subset in the first class because of the definition of F, and $\phi(X)$ will not appear in the definition of the conjugate Moebius inverse g, moreover, X contains its all subsets. These observations indicate that the choice of $\phi(X)$ has no impact on the cardinality of the reference set $\mathcal{R}$. Consequently, one can assign $\phi(X)$ arbitrarily.

For all subsets C in the second class, $\phi(C)$ must be a negative integer as large as possible and makes subset C worse than the subset in the first class and makes itself equivalent to other subsets in the class, that is, $w(C) = w^*(C) + \phi(C) = \phi(X) + \phi(C) < \phi(X) = w(X)$ for all C in this class, then one should choose $\phi(C) = -1$ for all C in the second class.

For the k-th class, where $k = 3, 4, ...n$. For all A in k-th class, the values of $\phi(C)$ for $C \in F : A \subsetneq C$ have been determined in the previous $(k-1)$ steps. As a result,

doi:10.6342/NTU202303269

the corresponding $w^*(A)$ have also been determined. Then one can focus on the subset $\underline{A}$ in the k-th class such that $w^*(\underline{A})$ is the minimum among all subsets in this class. If $w^*(\underline{A}) < w(C)$ for C in the $(k-1)$-th class, then one can assign $\phi(\underline{A}) = 0$; if $w^*(\underline{A}) \geq w(C)$ for C in the $(k-1)$-th class, then one can assign $\phi(\underline{A})$ be a negative integer, as large as possible, and $w^*(\underline{A}) + \phi(\underline{A}) < w(C)$ for C in the $(k-1)$-th class, that is, $\phi(\underline{A}) = w(C) - w^*(\underline{A}) - 1$. In both scenarios, assign $\phi(A) = w(\underline{A}) - w^*(A)$ for other A in the class.

Finally, the non-negative integers $\phi(C)$ satisfy that for all $A, B \in F$, $A \succsim B$ if and only if

$$\sum_{\{C \in F : A \subseteq C\}} \phi(C) \geq \sum_{\{C \in F : B \subseteq C\}} \phi(C).$$

Moreover, since $\phi(C)$ is defined to be as large as possible for each subset C within each $\sim$-equivalence class, the corresponding values of g and their sum, which is the cardinalty of reference set, are minimized. □

Furthermore, one can also obtain the second proposition from the previous proof.

Propositin 2. *Let $\succsim$ be a transitive and complete ranking on 2^X satisfying (i) $B \subseteq A$ implies that $A \succsim B$, and (ii) $A \cup B \sim A$ implies that $A \cup B \cup C \sim A \cup C$ for all $C \in 2^X$. And let m be the minimal cardinality of reference set $\mathcal{R}$ representing $\succsim$ in formula (3.1).*

Suppose that

1. *both $\mathcal{R}$ and $\mathcal{R}'$ can represent $\succsim$ in formula (3.1), and*

2. *$|\mathcal{R}| = |\mathcal{R}'| = m$.*

doi:10.6342/NTU202303269

Then for all $A \in 2^X$,

$$|\{R \in \mathcal{R} : A\, R\, r\, R\, A^c\}| = |\{R' \in \mathcal{R}' : A\, R'\, r\, R'\, A^c\}|,$$

where $A\, R\, r\, R\, A^c$ means that all elements in set A is preferred to reference point r, and r is preferred to all elements in set A^c for the preference R.

Proof. From the preceding proof process, one can conclude that the minimal cardinatily m and the corresponding $\mathcal{G} = \{(G_i, g_i)\}_{i=1}^{k}$ is uniquely determined, where $m = \sum_{i=1}^{k} g_i$. Therefore, the reference sets $\mathcal{R}$ and $\mathcal{R}'$ have the same upper contour collection $\mathcal{G}$.

For $A \in 2^X$, if $A \neq G_i$ for $i = 1, ..., k$, then

$$|\{R \in \mathcal{R} : A\, R\, r\, R\, A^c\}| = |\{R' \in \mathcal{R}' : A\, R'\, r\, R'\, A^c\}| = 0;$$

if $A = G_i$ for $i = 1, ..., k$, then

$$|\{R \in \mathcal{R} : A\, R\, r\, R\, A^c\}| = |\{R' \in \mathcal{R}' : A\, R'\, r\, R'\, A^c\}| = g_i.$$

□

Proposition 2 indicates that, if two reference sets $\mathcal{R}$ and $\mathcal{R}'$ can give the representation of a ranking $\succsim$ in the formula (3.1) and they both have the minimal cardinality, then $\mathcal{R}'$ can be generated by rearranging the order of elements above and below the reference point r, respectively, for each preference in $\mathcal{R}$, and conversely.

doi:10.6342/NTU202303269

Chapter 6 Comparison with Kreps

6.1 Different Interpretation of Representations

Recall the theorem in Kreps (1979). Kreps considers a scenario where the individual encounters a two-stage problem where he first select a set and subsequently choose an element within that set. However, the individual is uncertain about his second-stage preference when he select the sets. Kreps provides a characterization under the circumstance where where only the ranking over sets can be observed.

Kreps shows that a transitive and complete ranking $\succsim^*$ on $2^X \setminus \{\emptyset\}$ satisfies (i) $B \subseteq A$ implies that $A \succsim^* B$, and (ii) $A \cup B \sim^* A$ implies that $A \cup B \cup C \sim^* A \cup C$ for all $C \in 2^X \setminus \{\emptyset\}$ if and only if there exists a finite collection of utility functions $\{U_1, U_2, ..., U_n\}$, where $U_i : X \to \mathbb{R}$ for $i = 1, 2, ..., n$ such that

$$v(A) = \sum_{i=1}^{n} \left[\max_{a \in A} U_i(a) \right],$$

where $A \in 2^X \setminus \{\emptyset\}$, represents $\succsim^*$. Notice that the empty set is not in Kreps' original discussion.

Because this representation adds up the maximum values that a set of utility functions can achieve within a sets, it implies that this ranking over sets is easily influenced

doi:10.6342/NTU202303269

by the preference which strongly favors some elements, while downplaying or disregarding the others.

The concept of satisfaction is crucial in some context as it allows the realization of fairness among the society (equivalently, the collection of preferences).

My result provides an alternative explanation for the rankings over sets with properties (i) and (ii). In some context, for example, choosing restaurant for group dinning, the fairness within the group is crucial factor for decision. Thus, it is more reasonable that the restaurant ranking for a group is generated from the mechanism taking the fairness or satisfaction into consideration, rather than from adding up the maximal utility value realized by restaurants.

6.2 Demonstration of the Equivalence between Representations

To illustrate how this type of set ranking can yield both my representation and Kreps' representation, I utilize an example and derive his representation following his proof, the n convert his representation into mine.

Consider $X = \{x, y, z\}$, the desirable ranking $\succsim$ on $2^X\{\emptyset\}$ such that

$$\{x, y, z\} \sim \{x, y\} \succ \{x, z\} \succ \{x\} \sim \{y, z\} \sim \{y\} \succ \{z\} \succ .$$

Following Kreps' proof, one can obtain four utility functions U_1, U_2, U_3, U_4 such

doi:10.6342/NTU202303269

that

$$U_1(a) = \begin{cases} 0 & \text{if } a = y \\ -1 & \text{otherwise,} \end{cases}$$

$$U_2(a) = \begin{cases} 0 & \text{if } a = x \\ -3 & \text{otherwise,} \end{cases}$$

$$U_3(a) = \begin{cases} 0 & \text{if } a = y \text{ or } z \\ -2 & \text{otherwise,} \end{cases}$$

$$U_4(a) = \begin{cases} 0 & \text{if } a = x \text{ or } y \\ -1 & \text{otherwise} \end{cases}$$

which provide Kreps' representation.

One can shift each utility function such that the utility value is non-negative. The shifted utility functions

$$U_1'(a) = \begin{cases} 1 & \text{if } a = y \\ 0 & \text{otherwise,} \end{cases}$$

$$U_2'(a) = \begin{cases} 3 & \text{if } a = x \\ 0 & \text{otherwise,} \end{cases}$$

$$U_3'(a) = \begin{cases} 2 & \text{if } a = y \text{ or } z \\ 0 & \text{otherwise,} \end{cases}$$

doi:10.6342/NTU202303269

$$U'_4(a) = \begin{cases} 1 & \text{if } a = x \text{ or } y \\ \\ 0 & \text{otherwise} \end{cases}$$

can still give the representation. Furthermore, for all $A \in 2^X \setminus \{\emptyset\}$,

$$v(A) = \sum_{i=1}^{n} \left[\max_{a \in A} U'_i(a) \right], = \sum_{\{i: A \cap G_i \neq \emptyset\}} g_i = n_{\mathcal{G}}(A),$$

where $\{(G_1, g_1), (G_2, g_2), (G_3, g_3)\} = \{(\{x\}, 3), (\{y\}, 1), (\{x, y\}, 1), (\{y, z\}, 2)\} = \mathcal{G}$. With Lemma 1, we can generate the reference set $\mathcal{R} = \{R_i\}_{i=1}^{7}$ such that

$$\begin{array}{lll} x\, R_1\, r\, R_1\, y\, R_1\, z, & x\, R_2\, r\, R_2\, y\, R_2\, z, & x\, R_3\, r\, R_3\, z\, R_3\, y, \\ y\, R_4\, r\, R_4\, x\, R_4\, z, & y\, R_5\, x\, R_5\, r\, R_5\, z, & y\, R_6\, z\, R_6\, r\, R_6\, x, \\ z\, R_7\, y\, R_7\, r\, R_7\, x. & & \end{array}$$

represent $\succsim$ in formula (3.1).

Intuitively, this disparity between these representations lies in the differing interpretations of the scores obtained to the sets:: Kreps regards them as the sum of maximum of utility value of utility function, while I interpret them as the numbers of preferences which is satisfied.

doi:10.6342/NTU202303269

Chapter 7 Conclusion

In this paper, I explore a new model which ranks sets based on a collection of preferences. This model stands apart from others for various reasons. Firstly, it does not rigorously assume that these preferences must be maximized by the elements within the sets; rather, they simply need to be satisfied. I employ a reference point as a benchmark or threshold to elaborate the concept of satisfaction.

Secondly, in contrast to the rankings characterized in the literature, which assess sets based on the quantity of elements possessing a specific property stemming from multiple preferences, my model further considers not only the existence of this property but also its magnitude. I refer to this the magnitude of property, resulting from multiple preferences, as the attached value derived from multiple preferences.

Finally, I utilize the same axioms as Kreps to characterize this model in a scenario where one can merely observe the ranking of sets rather than the collection of preferences. This result shows that a set ranking, which satisfies both (i) $B \subseteq A$ implies $A \succsim B$, and (ii) $A \cup B \sim A$ implies $A \cup B \cup C \sim A \cup C$ for all $C \in 2^X$, can be explained not only by the existence of a collection of preferences (or, utility functions) from the maximization viewpoint, but also the perspective of satisfaction.

doi:10.6342/NTU202303269

References

Bossert, W., Pattanaik, P. K., Xu, Y., et al. (2001). The measurement of diversity. Centre de recherche et développement en économique, Université de Montréal.

Chateauneuf, A., & Jaffray, J.-Y. (1989). Some characterizations of lower probabilities and other monotone capacities through the use of möbius inversion. Mathematical social sciences, 17(3), 263–283.

Kreps, D. M. (1979). A representation theorem for" preference for flexibility". Econometrica: Journal of the Econometric Society, 565–577.

Nehring, K. (1999). Preference for flexibility in a savage framework. Econometrica, 67(1), 101–119.

Nehring, K., & Puppe, C. (1999). On the multi-preference approach to evaluating opportunities. Social Choice and Welfare, 16(1), 41–63.

Nehring, K., & Puppe, C. (2002). A theory of diversity. Econometrica, 70(3), 1155–1198.

Pattanaik, P. K., & Xu, Y. (1990). On ranking opportunity sets in terms of freedom of choice. Recherches Économiques de Louvain/Louvain Economic Review, 56(3-4), 383–390.

Pattanaik, P. K., & Xu, Y. (1998). On preference and freedom. Theory and Decision, 44, 173–198.

Pattanaik, P. K., & Xu, Y. (2000). On diversity and freedom of choice. Mathematical Social Sciences, 40(2), 123–130.

Peragine, V., & Romero-Medina, A. (2006). On preference, freedom and diversity. Social Choice and Welfare, 27, 29–40.

doi:10.6342/NTU202303269

Puppe, C. (1996). An axiomatic approach to “preference for freedom of choice”. Journal of Economic Theory, 68(1), 174–199.

Puppe, C. (2002). Individual freedom and social choice. In Freedom in economics (pp. 59–78). Routledge.

Puppe, C., & Xu, Y. (2010). Essential alternatives and freedom rankings. Social Choice and Welfare, 35, 669–685.

doi:10.6342/NTU202303269