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在殼方法應用於黑洞準正規模式之散射振幅研究

On-shell Amplitude of Black Hole Quasinormal Modes

謝天

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本論文係 謝天 (姓名) R10222016 （學號）在國立臺灣大學 物理 (系/所/學位學程)完成之碩士學位論文，於民國 112 年 7 月 18 日承下列考試委員審查通過及口試及格，特此證明。

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摘要

黑洞的準正規模式來自於黑洞微擾理論的角度方向和徑向的解。這些模式代表了重力波在黑洞背景下傳遞的獨特頻率和衰減速率，例如雙黑洞系統中的重力波發射。

本論文探討了在使用自旋-螺旋形式時，利用涉及重力子發射的在殼三點樹級散射振幅，來描述黑洞準正規模式的角度向特殊函數。受到主導黑洞準正規模式的微擾度量的洛倫茲不變性的啓發，我們利用自旋-螺旋形式來表示史瓦西黑洞準正規模式的角度向特殊函數，即自旋加權球諧函數，並以不同自旋配置的不等質量振幅來表示。

接著，通過結合可用於描述古典自旋的自旋相干態和來自不等質量散射過程的在殼方法，這個被建構的張量，即具有自旋配置的在殼相干張量，可以重現克爾黑洞準正規模式的角度向特殊函數，即自旋加權橢球諧函數。

總體而言，本研究提供了一個框架，利用具有$SU(2)$自旋配置的在殼自旋-螺旋形式，來理解$SO(3)$表示的球對稱和旋轉黑洞準正規模式的角度向特殊函數。

關鍵字：散射振幅、自旋螺旋形式、在殼方法、史瓦西黑洞、克爾黑洞、黑洞微擾理論、準正規模式

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Abstract

Quasinormal Modes (QNMs) of black holes are from the angular and radial solution of the Black Hole Perturbation Theory (BHPT). These modes represent the unique transmitted frequencies and decay rates of the gravitational waves under a background of the black hole metric, such as the emission of gravitational waves in binary black hole system.

This thesis explores the application of the spinor-helicity formalism in using on-shell 3pt tree-level scattering amplitudes involving graviton emissions to describe the angular dependence of the black hole QNMs. The Lorentz invariance of the perturbed metrics, which govern the black hole QNMs, motivate us to represent the angular dependence of the Schwarzschild QNMs, the spin-weighted spherical harmonics, by the unequal masses amplitudes using the spinor-helicity formalism with different spin configurations.

Then, by combining the coherent spin state which can be used to describe a classical spin and the on-shell elements from unequal masses scattering process, the constructed tensors, which are called the on-shell coherent tensors with spin configurations, can reproduce the angular dependence of the of Kerr QNMs, the spin-weighted spheroidal harmonics.

Overall, this research provides a framework for understanding the angular dependence of spherically symmetric and rotating black hole QNMs which are $SO(3)$ representation by using the on-shell spinor-helicity formalism with the $SU(2)$ spin configurations.

Keywords: Scattering amplitude, Spinor-helicity formalism, On-shell methods, Schwarzschild black holes, Kerr black holes, Black hole perturbation theory, Quasinormal modes

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Chapter 1

Introduction

History

In classical gravity theory, as everyone knows, Einstein predicted the existence of the gravitational wave by his General Relativity. In 2016, this prediction was first confirmed experimentally by a gravitational-wave observatory at the Laser Interferometer Gravitational Wave Observatory (LIGO) and the Virgo interferometer (Virgo) [1], since they directly measured the gravitational wave signal GW150914 from a binary black hole system that contains the inspiral, merger, and ring-down phase. Again, people verified the prediction of General Relativity.

In constructing theoretical models for gravitational waves, commonly used approaches include post-Newtonian gravity, numerical relativity, and perturbation theory [2]. When considering the inspiral phase of binary black hole system, people prefer employing analytic treatments of the dynamics, rather than iterative numerical methods [3].

An analytic approach involves determining the effective Hamiltonian of the system, which describes the inspiral phase. Typically, the above effective Hamiltonian can be obtained through calculations based on the post-Newtonian approximation (PN approximation), an effective theory for describing slow-moving objects in weak gravitational fields. Since objects move slowly, that is, $v^2/c^2 \ll 1$, expansions involving v^2/c^2 and GM/r with the same scales are performed. This expansion method, known as n-PN, expands the Hamiltonian up to order $\mathcal{O}(1/c^2)^n$, and these correc-

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tion terms are commonly employed for calculating linear classical gravity theories. Lorentz and Dorste [4] in 1917 and Einstein, Infeld and Hoffmann [5] later obtained the 1PN calculation. Today, the calculations have been extended up to 4PN, and the higher PN calculations are currently being pursued to compare with gravitational wave detectors. To this end, the on-shell method offers convenient calculations for higher PN approximations.

In addition, there is another method to expand the Hamiltonian for binary black hole systems, which is known as the post-Minkowskian (PM) expansion. This method is expanded in terms of the Newtonian constant G, rather than the velocity. It is worth noting that, this expansion, involving terms like G, G^2, and so on, is similar to the scattering amplitudes of gravity, since the Lorentz invariant scattering amplitudes are also perturbations in terms of G. One can say that scattering amplitudes provide a useful method to describe the classical gravitational potential.

For example, in 1985, the PM calculations were carried out up to $\mathcal{O}(G)^2$ to describe scattering angles [6], followed by predictions of quantum corrections to the classical gravitational potential [7, 8]. Today, people employ the on-shell approach, spinor-helicity formalism, and previous scattering amplitudes to calculate the higher order PM Hamiltonians for binary black hole systems, where the spinor-helicity formalism gives us easy and clean calculations of scattering amplitudes involving momenta.

Recent years, some modern techniques just like unitarity, on-shell recursion relations, and double copy relations enable the construction of one-loop amplitudes, and through this way the classical effects can be identified. In recent years, conservative potential up to 4PM for non-spinning objects [9–11] and the Hamiltonian up to 2PM at quartic order spin for spinning objects [12] have been computed. Besides computing higher order corrections of G, this on-shell formalism can also be used to describe other physical processes, such as tidal effects, radiation effects, and so on.

These successful applications of on-shell amplitudes in gravity motivate us to utilize these amplitudes to describe the angular dependence of the Lorentz invariant

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black hole quasinormal modes.

Quasinormal Modes of Black Holes

The quasinormal modes (QNMs) of black holes [13–17] are a dissipative system in which a black hole emits gravitational waves that perturb the surrounding spacetime. These waves decay and disappear at spatial infinity, rather than continuing to propagate indefinitely. The QNMs of a black hole consist of angular and radial components, with their frequencies depending on the mass and spin of the black hole. The most famous example of the QNMs in black holes is the ring-down phases in the binary black hole system, which consists of inspiral, merger, and the final ring-down phase. The resulting black hole will emit gravitational waves which gradually dissipate during the ring-down phase [14].

The black hole QNMs are from the Black Hole Perturbation Theory and some boundary conditions. In the context of the Black Hole Perturbation Theory (BHPT), the metric of a black hole

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

is perturbed by a linearly perturbed term $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ satisfies the linearly perturbed Einstein equation and describes the gravitational waves emission. The BHPT can describe various systems involving the field fluctuations under a black hole background, and the QNMs is just one of applications. By BHPT, given the boundary conditions at the horizon and spatial infinity, the waves should be only incoming waves at the horizon $\psi \sim e^{-i\omega(t+r_*)}$ and only outgoing waves at spatial infinity $\psi \sim e^{-i\omega(t-r_*)}$, the boundary conditions connect the BHPT and the QNMs.

First, for spherically symmetric black holes, the Schwarzschild QNMs are derived by Regge and Wheeler in [13], and Zerilli in [18], respectively. They take linear perturbed metric $g_{\mu\nu} + h_{\mu\nu}$ into the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0, \tag{1.2}$$

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by using the perturbed connection

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} (\nabla_\gamma h_{\beta\nu} + \nabla_\beta h_{\gamma\nu} - \nabla_\nu h_{\beta\gamma}) \tag{1.3}$$

and the perturbed Ricci tensor condition

$$\delta R_{\mu\nu} = -\nabla_\beta \delta\Gamma^{\beta}_{\mu\nu} + \nabla_\nu \delta\Gamma^{\beta}_{\mu\beta} = 0 \tag{1.4}$$

which is a second order differential equation of $h_{\mu\nu}$. The angular dependence of the solutions can be separated as the following form

$$\begin{aligned} h_{\mu\nu} &\sim Y_{lm}(\theta) \quad \text{for a perturbed scalar field;} \\ h_{\mu\nu} &\sim {}_{\pm1}Y_{lm}(\theta) \quad \text{for a perturbed photon field;} \\ h_{\mu\nu} &\sim {}_{\pm2}Y_{lm}(\theta) \quad \text{for a perturbed graviton field,} \end{aligned} \tag{1.5}$$

where the angular special functions are the spin-weighted spherical harmonics ${}_sY_{lm}(\theta)$ with the spin weight s, orbital angular momentum l, and projection m, which we want to reproduce by the on-shell scattering amplitudes.

Then, consider the nonspherically symmetric black holes, the gravitational waves from Kerr black holes can be described by the Teukolsky equation using Kerr's four tetrads l_μ, n_μ, m_μ, and $\bar{m}_\mu$ in [19, 20] and the Weyl scalar with conditions. By exploiting the symmetry and separability of the Teukolsky equation with respect to the t and ϕ directions,

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta), \tag{1.6}$$

where ω is the propagating frequency and m is the eigenvalue of the z-axis orbital angular momentum operator, and the Teukolsky equation can be separated into two single-variable differential equations, the angular equation and radial equation. The

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angular differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)+\left((a\omega)^2\cos^2\theta-2(a\omega)s\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+s+A\right)S=0, \tag{1.7}$$

where the spin weight s means a spin-s perturbed field and the eigenvalue A is a separation constant, can be solved by two methods. One is the perturbation theory in [21–24], people obtain the eigenvalue

$$_sA_{lm}=l(l+1)-\langle slm|\mathcal{H}_1|slm\rangle-\sum_{l'\neq l}\frac{|\langle sl'm|\mathcal{H}_1|slm\rangle|^2}{l(l+1)-l'(l'+1)}+... \tag{1.8}$$

and the eigenfunction

$$\begin{aligned}{}_sS_{lm}=&{}_sY_{lm}+\sum_{l'\neq l}\frac{\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1)-l'(l'+1)}{}_sY_{l'm}\\&+\sum_{l'\neq l}\frac{1}{l(l+1)-l'(l'+1)}\left[\sum_{l''\neq l}\frac{\langle sl'm|\mathcal{H}_1|sl''m\rangle\langle sl''m|\mathcal{H}_1|slm\rangle}{l(l+1)-l''(l''+1)}-\frac{{}_sA^{(1)}_{lm}\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1)-l'(l'+1)}\right]{}_sY_{l'm}\\&-\frac{1}{2}{}_sY_{lm}\sum_{l'\neq l}\left|\frac{\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1)-l'(l'+1)}\right|^2+...,\end{aligned} \tag{1.9}$$

of the angular equation order by order with the small parameter $a\omega$, which they can solve by perturbation around the spin-weighted spherical harmonics ${}_sY_{lm}$ with the quantum number l, that is why ${}_sA_{lm}$ and ${}_sS_{lm}$ have the label l which doesn't appear in the angular differential equation. The angular solutions ${}_sS_{lm}(a\omega,\theta)$ are known as the spin-weighted spheroidal harmonics [25], and this is the function which we try to reproduce in this thesis. Another way is Leaver's method in [26] with the ansatz of the angular solution

$${}_sS_{lm}(X)=e^{a\omega X}(1+X)^{\frac{1}{2}|m-s|}(1-X)^{\frac{1}{2}|m+s|}\sum_{n=0}^{\infty}a_n(1+X)^n, \tag{1.10}$$

where $X=\cos\theta$. By the ansatz, people numerically solve a continuous fraction equation and then can obtain the unknown separation constant $A(a\omega)$ for fixed s, m.

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With the eigenvalue of the angular equation and appropriate boundary conditions (an incoming mode at the horizon and an outgoing mode at spatial infinity), the radial solution and a series of complex, discrete mode frequencies $\omega_n = \omega_{R,n} + i\omega_{I,n}$ can be obtained by solving the radial equation from the Teukolsky equation. The real part of the mode frequency represents the frequency of the transmitted gravitational wave, while the imaginary part represents its dissipation rate. Note that in BHPT, the eigenfunction of the angular equation is always a spin-weighted spheroidal harmonic, regardless of boundary conditions, such as the angular solutions of the QNMs that we have mentioned and the scattering waves by a black hole in [27, 28] are the same thing.

Organization

In this thesis, we focus on the solution of the angular differential equation, namely the spin-weighted spherical harmonics and spheroidal harmonics, since our technology, scattering amplitudes, represent the observable of a transition process on an asymptotically flat background, and people successfully use the 4pt scattering amplitudes to extract classical observable of inspiralling Kerr black holes previously. Based on the special angular functions, ${}_hY_{lm}(\theta)$ and ${}_hS_{lm}(a\omega, \theta)$, of the black hole QNMs involving the quantum numbers, spin weight h, orbital angular momentum l, and the projection m, and the fact that BHPT satisfies Lorentz invariance, we describe the QNMs of a black hole by using the on-shell 3pt tree-level scattering amplitude $\mathcal{M}^{h_2,(J_1 \ldots J_{2l})}$ (two unequal massive spinors and one massless graviton $\mathbf{3}^l \to \mathbf{1} + 2^{h=-2}$):

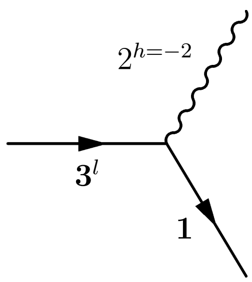

Figure 1.1: A hypothesis about the 3pt tree-level scattering amplitude

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A massive state 3 with mass m_3 and spin-l emits a graviton with energy E_2 and helicity $h = -2$ and transitions to another spinless state 1 with mass m_1, which carries the spin weight h and the orbital angular momentum l in the spin-weighted spherical harmonics. The leg 3 carries the $SU(2)$ little group indices $(J_1...J_{2l})$ which are fully symmetric. First, by the spinor-helicity formalism

$$\mathcal{M}^{h_2,(J_1...J_{2l})} = \lambda_{3,\beta_1}^{(J_1}...\lambda_{3,\beta_{2l}}^{J_{2l})}\mathcal{M}^{h_2,\{\beta_1...\beta_{2l}\}} \tag{1.11}$$

in [29–31], we can compute the 3pt scattering amplitude easily by the spin and helicity counting, and use the spin configurations of the fully symmetric $SU(2)$ little group indices to represent the $2l+1$ different m in the spin-weighted spherical harmonics

$$\boxed{\mathcal{M}^{-2,(J_1...J_{2l})} = g_l^{-2}\left(\frac{2E_2}{m_1}\right)^l \frac{1}{(2l)!}\sqrt{\frac{4\pi}{2l+1}}\sqrt{(l+2)!(l-2)!(l+m)!(l-m)!}\,{}_{-2}Y_{lm}(\theta).} \tag{1.12}$$

Moreover, we also use the on-shell amplitudes which emit a spin-h massless particle to reproduce the spin-weighted spherical harmonics with different spin weights h, which is an one-to-one correspondence.

Next, we describe the angular dependence of the Kerr QNMs by a scattering process of two spinning states 1 and 3 with different masses from an on-shell perspective ($\mathbf{3}^{l\oplus s} \to \mathbf{1}^s + 2^{h=-2}$), combining the coherent spin state

$$|\alpha\rangle = e^{-\tilde{\alpha}_J\alpha^J/2}\sum_{2s=0}^{\infty}\sum_{I_1,...,I_{2s}=\uparrow,\downarrow}\frac{(\alpha^{I_i})^{2s}}{\sqrt{(2s)!}}|s,(I_1...I_{2s})\rangle. \tag{1.13}$$

which can describe the classical spin of Kerr black holes in [32], then we establish the on-shell coherent elements and on-shell coherent tensors to reproduce the Kerr QNMs. The coherent tensors satisfy the helicity counting, $2l$ free little group indices, and coherent spin states contraction rule for the $SU(2)$ indices of the leg 1, leg 3.

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Therefore, such a coherent tensor

$$
\begin{aligned}
\mathcal{A}^{-2,(J_1\dots J_{2l})} = e^{-a} e^{\tilde{\alpha}_K \langle 3^K 1_I \rangle \alpha^I} \\
\times g_l^{-2} \sum_{n=0}^{l} \sum_{i,j=0}^{\infty} \Big\{ & c_{n,1,i,j} \langle 23^J \rangle^{l+2-n} [23^J]^{l-2} \left(\tilde{\alpha}_K \langle 23^K \rangle \right)^n \left(\langle 3^J 1_I \rangle \alpha^I \right)^n (k_2 \cdot p_3)^n \\
& + c_{n,2,i,j} \langle 23^J \rangle^{l+2} [23^J]^{l-2-n} \left(\tilde{\alpha}_K [23^K] \right)^n \left(\langle 3^J 1_I \rangle \alpha^I \right)^n (k_2 \cdot p_3)^n \Big\} \\
& \times (\tilde{\alpha}_K [23^K] \langle 21_I \rangle \alpha^I)^i (\tilde{\alpha}_K \langle 23^K \rangle [21_I] \alpha^I)^j
\end{aligned}
\tag{1.14}
$$

by summing all possible on-shell elements, can reproduce the spin-weighted spheroidal harmonics by the relationship

$$
\boxed{\mathcal{A}^{-2,(J_1\dots J_{2l})} = g_l^{-2} \left(\frac{2E_2}{m_1} \right)^l \frac{1}{(2l)!} \sqrt{\frac{4\pi}{2l+1}} \sqrt{(l+2)!(l-2)!(l+m)!(l-m)!}\, {}_{-2}S_{lm}(a\omega, \theta)}
\tag{1.15}
$$

with the coefficients $c_{n,1,i,j}$ and $c_{n,2,i,j}$ which are independent of the quantum number m, or independent of the spin configurations. Although we find the correspondence, there are some redundant structures when the order of expansion is large, that is to say, the expression is not an one-to-one correspondence.

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Chapter 2

Background on Black Hole Perturbation Theory

In this chapter, we will review the early approaches of Black Hole Perturbation Theory (BHPT) and the form of the angular differential equation, but we especially focus on the latter. Starting with the simple spherically symmetric case, the Schwarzschild black hole. The first quasinormal modes (QNMs) under the Schwarzschild background were obtained by Regge, Wheeler, and Zerilli.

We then introduce the non-spherically symmetric case, Kerr black hole with spin. If we were to use perturbed metrics and solve the Einstein field equation, as in the Schwarzschild case, it would become highly complicated. Therefore, Teukolsky employed a perturbed null tetrad and derived the famous Teukolsky equation, which is a wave equation in the Kerr background. They focused on the eigenvalues of the angular equation, and then used these eigenvalues along with boundary conditions to solve for the radial equation's angular frequencies which are a series of complex numbers representing different modes of gravitational waves.

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2.1 Regge-Wheeler–Zerilli equation for Schwarzschild Black Hole QNMs

At the begging, recall the spherically symmetric geometry, the Schwarzschild metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{2.1}$$

with $f(r) = 1 - \frac{2M}{r}$. In [13, 17], the linear perturbed metric $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ substitute into the vacuum Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{2.2}$$

where the Ricci tensor $R_{\mu\nu} = 0$ under the spherically symmetric background, and then take the linear order of $h_{\mu\nu}$ from the linear perturbed Ricci tensor, such that

$$\delta R_{\mu\nu} = -\nabla_\beta \delta\Gamma^\beta_{\mu\nu} + \nabla_\nu \delta\Gamma^\beta_{\mu\beta} \tag{2.3}$$

where the covariant derivative is from unperturbed connection, and the linear perturbed connection is given by

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\nu}(\nabla_\gamma h_{\beta\nu} + \nabla_\beta h_{\gamma\nu} - \nabla_\nu h_{\beta\gamma}) \tag{2.4}$$

with the condition $\delta R_{\mu\nu} = 0$ for each component. That is a second order differential equation of the metric $h_{\mu\nu}$. The solutions have the following matrix forms

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -h_0(t,r)\frac{1}{\sin\theta}\partial_\phi & h_0(t,r)\sin\theta\partial_\theta \\ 0 & 0 & -h_1(t,r)\frac{1}{\sin\theta}\partial_\phi & h_1(t,r)\sin\theta\partial_\theta \\ sym & sym & h_2(\frac{1}{\sin\theta}\partial_\theta\partial_\phi - \frac{\cos\theta}{\sin^2\theta}\partial_\phi) & sym \\ sym & sym & \frac{1}{2}h_2(\frac{1}{\sin\theta}{\partial_\phi}^2 + \cos\theta\partial_\theta - \sin\theta{\partial_\theta}^2) & -h_2(\sin\theta\partial_\theta\partial_\phi - \cos\theta\partial_\phi) \end{pmatrix} Y_{lm}(\theta)e^{im\phi} \tag{2.5}$$

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where the sym means $h_{\mu\nu} = h_{\nu\mu}$; h_2 is a function of t, r, and

$$h_{\mu\nu} = \begin{pmatrix} f(r)H_0(t,r) & H_1(t,r) & h_0(t,r)\partial_\theta & h_0(t,r)\partial_\phi \\ H_1(t,r) & \frac{H_2(t,r)}{f(r)} & h_1(t,r)\partial_\theta & h_1(t,r)\partial_\phi \\ sym & sym & r^2[K + G{\partial_\theta}^2] & sym \\ sym & sym & r^2G(\partial_\theta\partial_\phi - \frac{\cos\theta}{\sin\theta}\partial_\phi) & r^2[\sin^2\theta K + G({\partial_\phi}^2 + \cos\theta\sin\theta\partial_\theta)] \end{pmatrix} Y_{lm}(\theta)e^{im\phi} \tag{2.6}$$

where K, G are functions of t, r.

Because of the gauge symmetry, they consider an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ where ξ^μ is a gauge parameter, therefore the metric perturbation becomes

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\nu\xi_\mu + \nabla_\mu\xi_\nu. \tag{2.7}$$

Under the Regge-Wheeler gauge, they choose the gauge parameter

$$\xi^\mu = -\frac{1}{2}h_2(t,r)\begin{pmatrix} 0 & 0 & -\frac{1}{\sin\theta}\partial_\phi(Y_{lm}e^{im\phi}) & \sin\theta\partial_\theta(Y_{lm}e^{im\phi}) \end{pmatrix} \tag{2.8}$$

for the first one, hence the $h_2(t,r)$ terms in $h_{\mu\nu}$ are removed, such that the perturbed metric has the odd-parity $(-1)^{l+1}$ under the parity transformation $(\theta,\phi) \to (\theta+\pi,\phi)$, and then the new matrix form reduces to

$$h_{\mu\nu} = e^{-i\omega t}\begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix}\left(\sin\theta\frac{\partial}{\partial\theta}\right)Y_{l0}(\theta) \tag{2.9}$$

for the odd wave perturbation from Regge and Wheeler. As for the other one perturbation, they choose the gauge parameter

$$\xi^\mu = -\frac{1}{2}h_2(t,r)\begin{pmatrix} M_0(Y_{lm}e^{im\phi}) & M_1(Y_{lm}e^{im\phi}) & M_2\partial_\theta(Y_{lm}e^{im\phi}) & M_2\frac{1}{\sin^2\theta}\partial_\phi(Y_{lm}e^{im\phi}) \end{pmatrix} \tag{2.10}$$

where $M_0(t,r)$, $M_1(t,r)$, and $M_2(t,r)$ is used to cancel $G(t,r)$, $h_0(t,r)$, and $h_1(t,r)$,

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such that the perturbed metric has the even parity $(-1)^l$ under the parity transformation, and then the even-perturbed metric reduces to

$$h_{\mu\nu} = e^{-i\omega t}\begin{pmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & \frac{H_2(r)}{f(r)} & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta K(r) \end{pmatrix} Y_{l0}(\theta) \tag{2.11}$$

derived by Zerilli for the different partial wave number l. Due to the spherical symmetry, there are the same radial equations that people care about for all m. Note that the angular dependence $(\sin\theta\partial_\theta)\,Y_{l0}(\theta)$ in the odd-parity perturbation is proportional to the spin-weighted spherical harmonics with spin weight $s = \pm 1$ in [25].

The perturbed metric involves the frequency of gravitational waves from the black hole. To obtain the modes ω, they substitute the perturbed metric into each component of the perturbed Ricci tensor $\delta R_{\mu\nu} = 0$. Then, they solve the radial equation and obtain a series of modes which have the form $\omega = \omega_R + i\omega_I$.

We can discover that the angular dependence of the BHPT are related to the spherical harmonics. In general, the angular dependence of the BHPT are the spin-weighted spherical harmonics ${}_sY_{lm}(\theta)$, where the label s means what kind of field is, a scalar field for $s = 0$, a photon field for $s = \pm 1$, or a graviton field $s = \pm 2$. That is what we want to explore in this thesis.

2.2 Teukolsky equation for Kerr Black Hole QNMs

The Teukolsky equation is a four variables partial differential equation which describes the linear perturbations in the spacetime of a rotating (nonsymmetric)

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black hole. The Kerr metric is given by

$$\begin{aligned} ds^2 =& \frac{\left(\Delta - a^2\sin^2\theta\right)}{\Sigma}dt^2 + \frac{a\sin^2\theta\,(2Mr)}{\Sigma}(dtd\phi + d\phi dt) \\ & - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right)d\phi^2, \end{aligned} \tag{2.12}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 + a^2 - 2Mr$.

In [19], Teukolsky use the Newman–Penrose (NP) formalism to construct the null tetrad in the Kerr background as

$$\begin{aligned} l_\mu &= \begin{pmatrix} \frac{(r^2+a^2)}{\Delta} & 1 & 0 & \frac{a}{\Delta} \end{pmatrix}, \\ n_\mu &= \begin{pmatrix} \frac{(r^2+a^2)}{2\Sigma} & -\frac{\Delta}{2\Sigma} & 0 & \frac{a}{2\Sigma} \end{pmatrix}, \\ m_\mu &= \frac{1}{\sqrt{2}(r+ia\cos\theta)}\begin{pmatrix} ia\sin\theta & 0 & 1 & \frac{i}{\sin\theta} \end{pmatrix}, \end{aligned} \tag{2.13}$$

which satisfy $\boldsymbol{l}\cdot\boldsymbol{n} = 1$, $\boldsymbol{m}\cdot\bar{\boldsymbol{m}} = -1$, $\boldsymbol{l}\cdot\boldsymbol{l} = \boldsymbol{n}\cdot\boldsymbol{l} = \boldsymbol{m}\cdot\boldsymbol{m} = 0$, and $\boldsymbol{l}\cdot\boldsymbol{m} = \boldsymbol{n}\cdot\boldsymbol{m} = 0$ in the convention $\mathrm{diag}(\eta_{\mu\nu}) = (+,-,-,-)$, and they are preserved under the Lorentz transformation. The Kerr metric that the tetrads constitute can be written as $g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu\bar{m}_\nu + \bar{m}_\mu m_\nu$.

In the linear perturbations of the Kerr metric, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, where

$$h_{\mu\nu} = 2l^{(1)}_{(\mu}n^{(0)}_{\nu)} + 2l^{(0)}_{(\mu}n^{(1)}_{\nu)} - 2m^{(1)}_{(\mu}\bar{m}^{(0)}_{\nu)} - 2m^{(0)}_{(\mu}\bar{m}^{(1)}_{\nu)} \tag{2.14}$$

with the first order perturbed tetrads

$$\begin{aligned} l^{(1)}_\mu &= \frac{1}{2}h_{ll}n^{(0)}_\mu \\ n^{(1)}_\mu &= \frac{1}{2}h_{nn}l^{(0)}_\mu + h_{ln}n^{(0)}_\mu \\ m^{(1)}_\mu &= h_{nm}l^{(0)}_\mu + h_{lm}n^{(0)}_\mu - \frac{1}{2}h_{m\bar{m}}m^{(0)}_\mu - \frac{1}{2}h_{mm}\bar{m}^{(0)}_\mu, \end{aligned} \tag{2.15}$$

and h_{ll}, h_{nn}, $h_{m\bar{m}}$... are $h_{\mu\nu}$ in the tetrad representation. Then, they use the Weyl

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scalar in [20, 21]

$$
\begin{aligned}
\Psi_0 &:= -W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \\
\Psi_1 &:= -W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta \\
\Psi_2 &:= -W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta \\
\Psi_3 &:= -W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta \\
\Psi_4 &:= -W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta
\end{aligned}
\tag{2.16}
$$

which are Lorentz invariant, and the Weyl tensor

$$
W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}(R_{\mu\sigma}g_{\nu\rho} - R_{\mu\rho}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\rho} + R_{\nu\rho}g_{\mu\sigma}) + \frac{1}{6}R(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})
\tag{2.17}
$$

under the Petrov Type D background metric, which is used to solve the problem about the gravitational field from a source which only involves the mass and angular momentum, that means

$$
\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0.
\tag{2.18}
$$

The Type D forces the perturbed tetrads $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ to satisfy some differential equations, and then they combine the equations as a single master equation which is valid for a spin-s field in the Kerr background, that is the Teukolsky equation in [20, 21, 24],

$$
\begin{aligned}
&\left\{ \left[\frac{(r^2+a^2)}{\Delta} - a^2 \sin^2\theta \right] \frac{\partial^2}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t \partial\phi} + 2s\left[r + ia\cos\theta - \frac{M(r^2+a^2)}{\Delta}\right]\frac{\partial}{\partial t} \right. \\
&- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\frac{\partial^2}{\partial\phi^2} \\
&\left. - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi} + s\left(s\cot^2\theta - 1\right) \right\}\psi = 0\,.
\end{aligned}
\tag{2.19}
$$

where the wave function ψ means the Weyl scalars Ψ_0 for $s = 2$ (incoming wave), Ψ_4 for $s = -2$ (outgoing wave), and $\Phi_0 = F_{\mu\nu} l^\mu m^\nu$ for $s = 1$ (incoming wave), $\Phi_2 = F_{\mu\nu}\bar{m}^\mu n^\nu$ for $s = -1$ (outgoing wave) in [14] where $F_{\mu\nu}$ is the electromagnetic

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tensor, and the solution should be labeled by $\psi = {}_s\psi(t, r, \theta, \phi)$ with the spin weight s corresponding to a spin-s perturbed field, such as scalar, photon, or graviton fields, just like in the Schwarzschild case. This equation is a second order partial differential equation which involves angular and radial parts. It describes the evolution of fields with spin-s in the Kerr background. By solving the Teukolsky equation, we can obtain information about the perturbations around a Kerr black hole. For example, solutions of the Teukolsky equation can describe the wave function associated with the emission of gravitational waves.

To solve this equation, they separate the variables,

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta), \tag{2.20}$$

where the frequency ω is the "modes" people care about; m is the eigenvalue of the z-axis orbital angular momentum operator. Therefore, the four variables partial differential equation become the angular equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left((a\omega)^2\cos^2\theta - 2(a\omega)s\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + s + A\right) S = 0, \tag{2.21}$$

where $A = {}_sA_{lm}(a\omega)$ is a separation constant as well as the eigenvalue which is a function of $a\omega$ and $S = {}_sS_{lm}(a\omega, \theta)$ is the eigenfunction which is called the spin-weighted spheroidal harmonics. The label l is determined by the Schwarzschild case, because the spin-weighted spheroidal harmonics can be obtained by perturbation around the spin-weighted spherical harmonics with angular momentum l. And the radial equation

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r - M)K}{\Delta} + 4is\omega r - B\right) R = 0, \tag{2.22}$$

where $K = (r^2 + a^2)\omega - am$ and $B = A + (a\omega)^2 - 2m(a\omega)$.

Note that, they usually set $\cos\theta = X$, such that the angular differential equation

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becomes

$$\left[\frac{\partial}{\partial X}(1-X^2)\frac{\partial}{\partial X}\right]S+\left[(a\omega)^2X^2-2(a\omega)sX-\frac{(m+sX)^2}{1-X^2}+s+A\right]S=0 \quad (2.23)$$

Here, we focus on the angular equation. There are two most common methods which physicists use to solve the angular equation of the Teukolsky equation. One is perturbatively solving each order of $a\omega$, and the other is to use continuous fractions, which is called the Leaver method to solve the eigenvalues.

The first way is in [21], Press and Teukolsky separate the angular equation into two parts,

$$(\mathcal{H}_0+\mathcal{H}_1)\,S=-AS \quad (2.24)$$

where $\mathcal{H}_0$ is independent of the spin a of Kerr black hole, and $\mathcal{H}_1$ is spin dependent,

$$\begin{aligned}\mathcal{H}_0 =&\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\left(-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+s\right)\\ \mathcal{H}_1 =&(a\omega)^2\cos^2\theta-2(a\omega)s\cos\theta.\end{aligned} \quad (2.25)$$

which involve the quantum number s, m, and the parameter $a\omega$. The non-perturbed operator has the eigenvalue and eigenfunction,

$$\mathcal{H}_0S_0=-l(l+1)S_0 \quad (2.26)$$

where the eigenfunction is the spin-weighted spherical harmonics $S_0={}_sY_{lm}(\theta)$ for $l\geq|s|$ and $-l\leq m\leq l$. Since then, the angular momentum l has appeared.

To find the eigenvalue A of the angular equation, suppose we can expand eigenvalue around $l(l+1)$ which is the eigenvalue of the non-perturbed operator $\mathcal{H}_0$, therefore

$$A={}_sA_{lm}=l(l+1)+{}_sA^{(1)}_{lm}+{}_sA^{(2)}_{lm}+..., \quad (2.27)$$

similarly, the eigenfunction can be assumed as

$$S={}_sS_{lm}={}_sY_{lm}+{}_sS^{(1)}_{lm}+{}_sS^{(2)}_{lm}+..., \quad (2.28)$$

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where the upper indices $^{(i)}$ mean the expansion is accurate to the i-th order of the small parameter $a\omega$. Then, the angular equation with the first order perturbation

$$(\mathcal{H}_0+\mathcal{H}_1)\left({}_sY_{lm}+{}_sS^{(1)}_{lm}\right)=-\left(l(l+1)+{}_sA^{(1)}_{lm}\right)\left({}_sY_{lm}+{}_sS^{(1)}_{lm}\right) \tag{2.29}$$

can be reduced to

$$\mathcal{H}_{1s}Y_{lm}+\mathcal{H}_{0s}S^{(1)}_{lm}=-{}_sA^{(1)}_{lm\,s}Y_{lm}-l(l+1)_sS^{(1)}_{lm} \tag{2.30}$$

by using the non-perturbed eigenvalue equation. The standard method in perturbation theory is multiplying ${}_sY^*_{lm}$ from the left side and integrating the equation. Since $\mathcal{H}_0$ is Hermitian, we will obtain

$${}_sA^{(1)}_{lm}=-\langle slm|\mathcal{H}_1|slm\rangle \tag{2.31}$$

where the sandwich is

$$\langle sl'm|\mathcal{H}_1|slm\rangle=\int d\Omega_s Y^*_{l'm}(\theta)\mathcal{H}_{1s}Y_{lm}(\theta) \tag{2.32}$$

and the precise forms are

$$\begin{aligned}\langle sl'm|\cos\theta|slm\rangle=&\sqrt{\frac{2l+1}{2l'+1}}\langle l,1,m,0|l',m\rangle\langle l,1,-s,0|l',-s\rangle,\\ \langle sl'm|\cos^2\theta|slm\rangle=&\frac{1}{3}\delta_{ll'}+\frac{2}{3}\sqrt{\frac{2l+1}{2l'+1}}\langle l,2,m,0|l',m\rangle\langle l,2,-s,0|l',-s\rangle\end{aligned} \tag{2.33}$$

with the Clebsch-Gordan coefficients $\langle j_1,j_2,m_1,m_2|J,M\rangle$, or says the Wigner $3j$-symbols

$$\langle j_1,j_2,m_1,m_2|J,M\rangle=(-1)^{-j_1+j_2-M}\sqrt{2J+1}\begin{pmatrix}j_1 & j_2 & J\\ m_1 & m_2 & -M\end{pmatrix}. \tag{2.34}$$

Similarly, we can obtain the eigenfunction by multiplying ${}_sY^*_{l'm}$ from the left-hand-

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side and integrating the equation, that is

$$\int d\Omega_s Y^*_{l'm}(\theta)_s S^{(1)}_{lm}(\theta) = \frac{\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1) - l'(l'+1)} \tag{2.35}$$

To explicitly construct ${}_sS^{(1)}_{lm}$, let's assume

$$ {}_sS^{(1)}_{lm} = \sum_{l'} c_{ll'}\,{}_sY_{l'm} \tag{2.36}$$

by the orthogonal and complete basis $\{{}_sY_{lm}|l \in \mathbb{N}, l \geq s, l \geq |m|\}$, such that the first order eigenfunction is

$$ {}_sS^{(1)}_{lm} = \sum_{l'\neq l} \frac{\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1) - l'(l'+1)}\,{}_sY_{l'm}. \tag{2.37}$$

So far, we have finished the first order perturbation.

As with the above results, we can proceed to compute the second order perturbation,

$$\mathcal{H}_0\,{}_sS^{(2)}_{lm} + \mathcal{H}_1\,{}_sS^{(1)}_{lm} = -l(l+1)\,{}_sS^{(2)}_{lm} - {}_sA^{(1)}_{lm}\,{}_sS^{(1)}_{lm} - {}_sA^{(2)}_{lm}\,{}_sY_{lm}. \tag{2.38}$$

After a similar calculation, we collect the zeroth order, first order, and second order results and show them below. The eigenvalue is

$$ {}_sA_{lm} = l(l+1) - \langle slm|\mathcal{H}_1|slm\rangle - \sum_{l'\neq l} \frac{|\langle sl'm|\mathcal{H}_1|slm\rangle|^2}{l(l+1) - l'(l'+1)} + ... \tag{2.39}$$

and the eigenfunction is

$$\begin{aligned} {}_sS_{lm} =& {}_sY_{lm} + \sum_{l'\neq l} \frac{\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1) - l'(l'+1)}\,{}_sY_{l'm} \\ &+ \sum_{l'\neq l} \frac{1}{l(l+1) - l'(l'+1)} \left[\sum_{l''\neq l} \frac{\langle sl'm|\mathcal{H}_1|sl''m\rangle\langle sl''m|\mathcal{H}_1|slm\rangle}{l(l+1)-l''(l''+1)} - \frac{{}_sA^{(1)}_{lm}\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1)-l'(l'+1)} \right] {}_sY_{l'm} \\ &- \frac{1}{2}\,{}_sY_{lm} \sum_{l'\neq l} \left| \frac{\langle sl'm|\mathcal{H}_1|slm\rangle}{l(l+1) - l'(l'+1)} \right|^2 + ..., \end{aligned} \tag{2.40}$$

which are accurate to the second order of the small parameter $a\omega$. Therefore, we

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can perturbatively obtain the spin-weighted spheroidal harmonics ${}_sS_{lm}(\theta)$, and we observe that

$$ {}_sS_{lm} = {}_sY_{lm} + \sum_{l'=l-1\neq l}^{l+1} a_{sl'ms}Y_{l'm}(a\omega) + \sum_{l'=l-2}^{l+2} b_{sl'ms}Y_{l'm}(a\omega)^2 + \mathcal{O}(a\omega)^3. \tag{2.41} $$

with coefficients $a_{sl'm}$, $b_{sl'm}$ are pure numbers.

The second way is called Leaver's method in [26,33]. It is mainly used to solve the eigenvalues, rather than the spin-weighted spheroidal harmonics, they suppose this special function may be written as the following ansatz,

$$ {}_sS_{lm}(a\omega, X) = e^{a\omega X}(1+X)^{\frac{1}{2}|m-s|}(1-X)^{\frac{1}{2}|m+s|}\sum_{n=0}^{\infty} a_n(1+X)^n \tag{2.42} $$

around the regular singular point at $X = \cos\theta = \pm 1$ in (2.23) and with a nontrivial exponential $a\omega X$, where the label l is from comparing the roots A with the eigenvalues from above perturbation theory. Usually, we expect a series expansion to converge; otherwise, the higher order terms contribute more than lower order terms, such that we cannot truncate it at some order to be a valid approximation of the spin-weighted spheroidal harmonics.

To find the separation constant A in angular equation (2.23), which are unfixed parameters in coefficients a_n, they plug the ansatz into the angular equation, which involves the differential with respect to X at most twice. After differentiation, we can separate each order of X, collect the corresponding coefficients of each order, and require them to vanish. It is similar to power series, such that the coefficients a_n satisfy the recursive relation

$$ \begin{aligned} \alpha_0^\theta a_1 + \beta_0^\theta a_0 &= 0 \\ \alpha_n^\theta a_{n+1} + \beta_n^\theta a_n + \gamma_n^\theta a_{n-1} &= 0 \end{aligned} \tag{2.43} $$

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for $n = 1, 2, ...$, where

$$\alpha_n^\theta = -2(n+1)(n+2k_1+1)$$
$$\beta_n^\theta = n(n-1) + 2n(k_1+k_2+1-2a\omega) - [2a\omega(2k_1+s+1) - (k_1+k_2)(k_1+k_2+1)] - [(a\omega)^2 + s(s+1) + A(a\omega)]$$
$$\gamma_n^\theta = 2a\omega(n+k_1+k_2+s) \tag{2.44}$$

containing an unknown parameter $A = A(a\omega)$ which is a function of $a\omega$, and the parameters

$$k_1 = \frac{1}{2}|m-s|$$
$$k_2 = \frac{1}{2}|m+s| \tag{2.45}$$

for fixed s and m.

From the recursive relation (2.43), we can derive the ratio of the coefficients with the continued fraction form,

$$\frac{a_{n+1}}{a_n} = \frac{-\gamma_{n+1}^\theta}{\beta_{n+1}^\theta - \frac{\alpha_{n+1}^\theta\gamma_{n+2}^\theta}{\beta_{n+2}^\theta - \frac{\alpha_{n+2}^\theta\gamma_{n+3}^\theta}{\beta_{n+3}^\theta - \cdots}}} = \frac{-\gamma_{n+1}^\theta}{\beta_{n+1}^\theta-}\frac{\alpha_{n+1}^\theta\gamma_{n+2}^\theta}{\beta_{n+2}^\theta-}\frac{\alpha_{n+2}^\theta\gamma_{n+3}^\theta}{\beta_{n+3}^\theta-}\cdots\frac{\alpha_{n+i}^\theta\gamma_{n+i+1}^\theta}{\beta_{n+i+1}^\theta} \tag{2.46}$$

for $i \to \infty$ in related notations form. From the initial one, we combine the two equations

$$\frac{a_1}{a_0} = -\frac{\beta_0^\theta}{\alpha_0^\theta}$$
$$\frac{a_1}{a_0} = \frac{-\gamma_1^\theta}{\beta_1^\theta-}\frac{\alpha_1^\theta\gamma_2^\theta}{\beta_2^\theta-}\frac{\alpha_2^\theta\gamma_3^\theta}{\beta_3^\theta-}\cdots\frac{\alpha_n^\theta\gamma_{n+1}^\theta}{\beta_{n+1}^\theta} \tag{2.47}$$

for $n \to \infty$, such that

$$0 = \beta_0^\theta - \frac{\alpha_0^\theta\gamma_1^\theta}{\beta_1^\theta-}\frac{\alpha_1^\theta\gamma_2^\theta}{\beta_2^\theta-}\frac{\alpha_2^\theta\gamma_3^\theta}{\beta_3^\theta-}\cdots\frac{\alpha_n^\theta\gamma_{n+1}^\theta}{\beta_{n+1}^\theta} \tag{2.48}$$

with the undetermined $A(a\omega)$ in all β_n^θ.

Similarly, the radial differential equation from Teukolsky equation has the same form, that is to say, the radial solution can be written as a series expansion whose coefficients satisfy some relation, such as (2.48). The two continued fraction equations

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from the radial equation with unknown ω and the angular equation with unknown $A(a\omega)$ are coupled, so Leaver needs to numerically solve them, simultaneously.

To find the roots ω and $A(a\omega)$, given the parameters s, m, and a, they truncate the continued fractions to some order n, and then use the root finding algorithm to find the roots ω and $A(a\omega)$. Then, they can repeat the step at a higher order $n' > n$, such that accuracy is enough. By truncating to higher order n, this not only gives us greater accuracy but also allows us to find more roots that correspond to larger l. Roughly speaking, there will be infinitely many roots $A(a\omega)$ corresponding to l with the constraint $l \geq |m|$ and $l \geq |s|$ in the spin-weighted spheroidal harmonics ${}_sS_{lm}$.

For example, given $s = -2$, $m = 5$, and fixed $a\omega = 0.1$ (but ω could be complex in general), we define a function $f(A)$ as

$$f(A) = \beta_0^\theta - \frac{\alpha_0^\theta \gamma_1^\theta}{\beta_1^\theta -}\frac{\alpha_1^\theta \gamma_2^\theta}{\beta_2^\theta} \approx \frac{-A^3 + 117.17A^2 - 4465.98A + 55100.}{A^2 - 90.38A + 2031.9} \tag{2.49}$$

according to (2.48), then we can plot $f(A)$ and solve $f(A) = 0$ numerically. There exist three real roots in $A = A(0.1) \approx 27.85424, 40.63636$, and 48.67940, respectively. The roots can correspond to the eigenvalues ${}_{-2}A_{5,5}(0.1) \approx 27.86393$, ${}_{-2}A_{6,5}(0.1) \approx 39.90200$, and ${}_{-2}A_{7,5}(0.1) \approx 53.92555$ from perturbation theory. Moreover, we can ask $f(A)$ to include more terms

$$\begin{aligned} f(A) &= \beta_0^\theta - \frac{\alpha_0^\theta \gamma_1^\theta}{\beta_1^\theta -}\frac{\alpha_1^\theta \gamma_2^\theta}{\beta_2^\theta -}\frac{\alpha_2^\theta \gamma_3^\theta}{\beta_3^\theta}\frac{\alpha_3^\theta \gamma_4^\theta}{\beta_4^\theta}\frac{\alpha_4^\theta \gamma_5^\theta}{\beta_5^\theta}\frac{\alpha_5^\theta \gamma_6^\theta}{\beta_6^\theta} \\ &\approx \frac{-A^7+501.13A^6-104445.A^5+1.1\times10^7A^4-7.5\times10^8A^3+2.8\times10^{10}A^2-5.6\times10^{11}A+4.5\times10^{12}}{A^6-474.34A^5+91724.3A^4-9.2\times10^6A^3+5.0\times10^8A^2-1.4\times10^{10}A+1.6\times10^{11}} \end{aligned} \tag{2.50}$$

and we will find five real roots at $A \approx 27.86394, 39.90200, 53.92506$, and so on, which are closer to to the eigenvalues ${}_{-2}A_{5,5}(0.1)$, ${}_{-2}A_{6,5}(0.1)$, ${}_{-2}A_{7,5}(0.1)$, and so on. Let's say, we can draw the following conclusions, the n-th root is corresponding to the eigenvalue ${}_{-2}A_{(n+5),5}(0.1)$.

So far, we do not know how the roots A depend on l. Through the Schwarzschild QNMs whose l is well-defined, we can figure out the relationship between A and l.

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When they take spin a to zero, the roots A will reduce to

$$A = (n + k_1 + k_2)(n + k_1 + k_2 + 1) - s(s + 1) \tag{2.51}$$

which is the Schwarzschild case, because of $\gamma_n^\theta = 0$ for all n as $a = 0$, this leads to the continued fraction equation become

$$\begin{aligned} 0 = \beta_n^\theta = & n(n-1) + 2n(k_1 + k_2 + 1) \\ & + (k_1 + k_2)(k_1 + k_2 + 1) - s(s+1) - A. \end{aligned} \tag{2.52}$$

Hence, they identified $n + k_1 + k_2 = l$, as well as the roots are exactly the eigenvalues ${}_sA_{lm}(a\omega)$ from perturbation theory.

By the previous example, given $s = -2$, $m = 5$, but $a = 0$, for $\beta_0^\theta = 0$, the first root is corresponding to the eigenvalue ${}_{-2}A_{5,5}(0)$; for $\beta_1^\theta = 0$, the second root is corresponding to the eigenvalue ${}_{-2}A_{6,5}(0)$; for $\beta_2^\theta = 0$, the corresponding eigenvalue is ${}_{-2}A_{7,5}(0)$. Let's say, the n-th root is from the condition $\beta_n^\theta = 0$ and is corresponding to the eigenvalue ${}_{-2}A_{(n+5),5}(0)$. Note that, in some paper, people prefer to factor out the $s(s+1)$ part from the non-perturbed operator $\mathcal{H}_0$, so their non-perturbed eigenvalue will be $l(l+1) - s(s+1)$ with a different convention.

Note that, the coefficient a_n can be expanded with respect to $a\omega$ as

$$a_n = (a\omega)^n[\text{polynomial of } (a\omega)] \tag{2.53}$$

if $a\omega$ is small enough. This patterns give us that the series expansion is exactly convergent, since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1. \tag{2.54}$$

by Ratio test for small $a\omega$.

For example, for $s = -2$, $l = 2$, $|m| \leq 2$, the coefficients have the patterns,

$$a_n = \sum_{i=0}^{\infty} c_{n,i}(a\omega)^{n+i}. \tag{2.55}$$

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with $c_{n,i}$ as the coefficient of the $(a\omega)^{n+i}$ term in a_n. Note that these coefficients are m dependent so they differ for each m. For another example, for $s = -2$, $l = 3$, $|m| \leq 2$, the patterns of the coefficients will become

$$\begin{aligned} a_0 &= \sum_{i=0}^{\infty} c_{n,i}(a\omega)^{n+i} \\ a_n &= \sum_{i=0}^{\infty} c_{n,i}(a\omega)^{n+i-1}. \end{aligned} \tag{2.56}$$

The coefficient a_n will be shifted by $(a\omega)$ when $n \geq 1$ in this case. For different l and $|m| < l$, the rules of these shifts are slightly different, but can still be found. Hence, we can truncate the Leaver's ansatz at a specific order of $a\omega$ and then compare it with the result from the perturbation theory.

Both the perturbation theory and Leaver's method give us some ideas about how to construct a new ansatz of the spin-weighted spheroidal harmonics.

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Chapter 3

Spin-weighted Spherical Harmonics from On-shell Kinematics (Schwarzschild)

In this chapter, we employ the spinor-helicity formalism to obtain an on-shell expression that describes the angular dependence of Schwarzschild black hole QNMs, specifically the spin-weighted spherical harmonics.

To achieve this, we first introduce the spinor-helicity formalism, a powerful framework extensively utilized in particle scattering studies. Drawing insights from the physical picture of the gravitational wave emission by black holes, we conjecture a 3pt tree-level scattering amplitude which aligns with this emission process. Subsequently, we set the momenta of the particles involved in the scattering for both the initial and the final states of the system. Finally, we verify that the scattering amplitudes which satisfy the on-shell spinor-helicity formalism are indeed associated with the spin-weighted spherical harmonics by relating the spin configurations and the quantum number m which is the eigenvalue of the third component of the angular momentum operators.

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3.1 Spinor-Helicity Formalism and Unequal Mass 3pt Amplitude

The Spinor-Helicity Formalism is a powerful technology for computing the scattering amplitudes easily and efficiently. The spinor-helicity formalism obeys Lorentz invariance and gauge invariance in the Feynman rules, hence the scattering amplitude can be written in the simpler form than traditional QFT. The amplitude is composed of the massive and massless 2-component Weyl spinors which are from the Weyl equation being invariant under the Lorentz transformation. The Weyl spinors are more fundamental than the 4-vectors in $SO(1,3)$, because the spinors form a irreducible representation of the Lorentz group. This method provides a invariant and efficient framework for describing particle kinematics and calculating scattering amplitudes.

Before we introduce the spinor-helicity formalism, we have to know that the $SL(2,\mathbb{C})$ momentum of a massless particle can be decomposed of two massless spinors

$$k_{\alpha\dot{\alpha}} = |k\rangle_\alpha [k|_{\dot{\alpha}} \tag{3.1}$$

and the momentum of a massive particle can be decomposed of four massive spinors

$$p_{\alpha\dot{\alpha}} = |p^I\rangle_\alpha [p_I|_{\dot{\alpha}} \tag{3.2}$$

with the $SU(2)$ little group index $I = 1, 2 = \uparrow, \downarrow$.

The point of the spinor-helicity formalism is base on the angle and square spinors which satisfy the Weyl equation

$$\begin{aligned} p_{\alpha\dot{\beta}}|p^I]^{\dot{\beta}} &= +m|p^I\rangle_\alpha \\ p^{\dot{\alpha}\beta}|p^I\rangle_\beta &= +m|p^I]^{\dot{\alpha}} \\ [p^I|_{\dot{\beta}} p^{\dot{\beta}\alpha} &= -m\langle p^I|^\alpha \\ \langle p^I|^\beta p_{\beta\dot{\alpha}} &= -m[p^I|_{\dot{\alpha}} \end{aligned} \tag{3.3}$$

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for massive spinors, and satisfy

$$\begin{aligned} k_{\alpha\dot{\beta}}|k]^{\dot{\beta}} &= 0 \\ k^{\dot{\alpha}\beta}|k\rangle_{\beta} &= 0 \\ [k|_{\dot{\beta}}k^{\dot{\beta}\alpha} &= 0 \\ \langle k|^{\beta}k_{\beta\dot{\alpha}} &= 0 \end{aligned} \tag{3.4}$$

for massless spinors.

We will use the unequal masses 3pt amplitude in the following section to reproduce the angular dependence of black hole QNMs, since the gravitational waves in a QNMs system carry energy and slowly dissipate it, so that the black hole mass reduces.

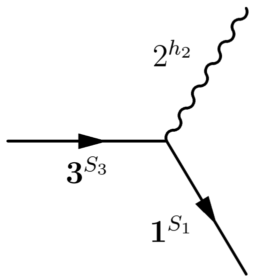

Figure 3.1: 3pt scattering process from spin-S_3 to spin-S_1 and helicity h_2

Mathematically, if we try to use the 3pt amplitude with equal masses m, then momentum conservation will give us $p_1 \cdot k_2 = p_3 \cdot k_2 = 0$, and k_2^{μ} will be a complex massless 4-momentum, such that some components of $|2\rangle$ and $|2]$ are imaginary, and some spin configurations of the amplitude cannot reproduce the spin-weighted spherical harmonics, which is a real function.

Therefore, we assume an unequal masses process for the 3pt amplitude, rather than equal masses, and then review the amplitude using the spinor-helicity formalism in [29–31]. For the spin-S_1, spin-S_3 representation of the $SU(2)$ little group indices $(I_1...I_{2S_1})$, $(J_1...J_{2S_3})$ for two massive leg 1 and leg 3 with masses m_1 and m_3, and one massless leg 2 with helicity h_2, the amplitude must contain the momenta of p_1, p_3, and k_2, and the momenta can be decomposed of the spinors, so the amplitude

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can be written as

$$\mathcal{M}^{h_2,(I_1...I_{2S_1}),(J_1...J_{2S_3})} = \lambda_{1,\alpha_1}^{(I_1}...\lambda_{1,\alpha_{2S_1}}^{I_{2S_1})}\lambda_{3,\beta_1}^{(J_1}...\lambda_{3,\beta_{2S_3}}^{J_{2S_3})}\mathcal{M}^{h_2,\{\alpha_1...\alpha_{2S_1}\},\{\beta_1...\beta_{2S_3}\}} \tag{3.5}$$

in terms of the spinors contractions $\lambda^I_{1,\alpha} = |1^I\rangle_\alpha$, $\tilde{\lambda}_1^{I\dot{\alpha}} = |1^I]^{\dot{\alpha}}$, $\lambda^J_{3,\alpha} = |3^J\rangle_\alpha$, $\tilde{\lambda}_3^{J\dot{\alpha}} = |3^J]^{\dot{\alpha}}$, $\lambda_{2,\alpha} = |2\rangle_\alpha$, and $\tilde{\lambda}_2^{\dot{\alpha}} = |2]^{\dot{\alpha}}$ where

$$\mathcal{M}^{h_2}_{\{\alpha_1...\alpha_{2S_1}\},\{\beta_1...\beta_{2S_3}\}} = g^{h_2}_{S_1,S_3}(\underbrace{u...u}_{N_u}\underbrace{v...v}_{N_v})_{\{\alpha_1...\alpha_{2S_1}\},\{\beta_1...\beta_{2S_3}\}} \tag{3.6}$$

where $g^{h_2}_{S_1,S_3}$ is a coupling constant; the $SL(2,\mathbb{C})$ Lorentz indices $\alpha_1...\alpha_{2S_1}$, $\beta_1...\beta_{2S_3}$ are carried by the basis spinors $u_\alpha = |2\rangle_\alpha$ with helicity $-\frac{1}{2}$ and $v_\alpha = \frac{p_{1\alpha\dot{\beta}}}{m_1}|2]^{\dot{\beta}}$ with helicity $+\frac{1}{2}$. Here, N_u means the number of u and N_v means the number of v, similarly, the number of $\lambda^I_{1,\alpha}$ is $2S_1$ and the number of $\lambda^J_{3,\beta}$ is $2S_3$.

The spinor-helicity formalism give the amplitudes two constraints, one is for helicity

$$-\frac{N_u}{2} + \frac{N_v}{2} = h_2, \tag{3.7}$$

and the other one is for spin

$$N_u + N_v = 2S_1 + 2S_3, \tag{3.8}$$

therefore, the number of u is determined by $N_u = S_1 + S_3 - h_2$ and the number of v is determined by $N_v = S_1 + S_3 + h_2$.

Here, the spin-weighted spherical harmonics ${}_{-2}Y_{lm}(\theta)$ carry the quantum numbers, spin weight $h = -2$, orbital angular momentum l, and projection m, therefore we conjecture the scattering amplitude $\mathcal{M}(\mathbf{1}, 2^{h=-2}, \mathbf{3}^l) = \mathcal{M}^{-2,(J_1...J_{2l})}$ which is the overlap of the quantum numbers, namely, the amplitude has the massive scalar leg 1 and spin-l leg 3 particles whose masses are m_1 and m_3, respectively, and massless leg 2 being a graviton with helicity $h = -2$. Such that the amplitude carries h and l, and the projection m from $-l$ to l is represented by spin configurations $(J_1...J_{2l})$ carried by leg 3.

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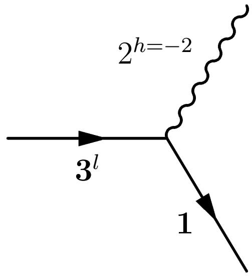

Figure 3.2: 3pt scattering process from spin-l to spinless and helicity -2

From above spinor-helicity formalism, we have the amplitude which describes that a spinning black hole emits a graviton with $h = -2$ and develops into a non-spinning black hole,

$$\mathcal{M}^{-2}_{\{\alpha_1 \dots \alpha_{2l}\}} = g_l^{-2} uuuu(uv)^{l-2}. \tag{3.9}$$

for $l \leq |h| = 2$, where the indices $\alpha_1 \dots \alpha_{2l}$ are carried by u and v, such that the amplitude satisfies the spin, helicity counting, and the Lorentz invariance under the Lorentz transformation of spinors, $|k\rangle_\alpha = \mathcal{L}(k;j)_\alpha^\beta |j\rangle_\beta$ and $[k|_{\dot{\alpha}} = [j|_{\dot{\beta}} \tilde{\mathcal{L}}(k;j)^{\dot{\beta}}_{\dot{\alpha}}$ for massless spinors; $|p^I\rangle_\alpha = W^I_J \mathcal{L}(p;q)_\alpha^\beta |q^J\rangle_\beta$ and $[p_I|_{\dot{\alpha}} = [q_J|_{\dot{\beta}} \tilde{\mathcal{L}}(p;q)^{\dot{\beta}}_{\dot{\alpha}} (W^{-1})^J_I$ for massive spinors. Then, we will use this assumption and the setup in the next section to reproduce the spin-weighted spherical harmonics.

Naively, the angular momentum in Fig.3.2 which we conjecture does not appear to be conserved. Actually, we should look at the angular momentum conservation from another point of view, since the angular momentum is coordinate dependence. As we know, the generator is the operator associated with conserved quantity, and the angular momentum tensor is the generator of the rotation and the Lorentz boost for the Lorentz group; that is to say, the Lorentz invariance guarantees the angular momentum conservation, therefore the amplitude which is a Lorentz invariant function satisfies the conservation of the angular momentum, when the angular momentum l is larger than the helicity $|h| = 2$.

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3.2 Setup of On-shell Spinors

According to the physical picture of the QNMs, we assume that the gravitational waves are emitted in the direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Furthermore, we can always set a massive particle in the rest frame. If we stay in the rest frame of particle 1, then we set the on-shell 4-momenta in $SO(1,3)$ as

$$\begin{aligned} p_1^\mu =& \begin{pmatrix} m_1 & 0 & 0 & 0 \end{pmatrix}, \\ k_2^\mu =& E_2 \begin{pmatrix} 1 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}, \end{aligned} \tag{3.10}$$

with real components which satisfy the on-shell conditions $|p_1^\mu|^2 = m_1^2$ for a massive particle and $|k_2^\mu|^2 = 0$ for massless one. Or, the momenta in $SL(2,\mathbb{C})$ by using $p_{\alpha\dot{\alpha}} = p^\mu \sigma_{\mu,\alpha\dot{\alpha}}$, such that we obtain

$$\begin{aligned} p_{1\alpha\dot{\alpha}} =& \begin{pmatrix} m_1 & 0 \\ 0 & m_1 \end{pmatrix}, \\ k_{2\alpha\dot{\alpha}} =& \begin{pmatrix} 2E_2\sin^2\left(\frac{\theta}{2}\right) & -e^{-i\phi}E_2\sin\theta \\ -e^{i\phi}E_2\sin\theta & 2E_2\cos^2\left(\frac{\theta}{2}\right) \end{pmatrix}, \end{aligned} \tag{3.11}$$

where $E_2 = \omega$ of massless particle 2 is the angular frequency (or energy) that people care about in QNMs; the determinant $\det(p_{1\alpha\dot{\alpha}}) = m_1^2$ and $\det(k_{2\alpha\dot{\alpha}}) = 0$ are Lorentz invariant. Next, we decompose the momenta in $SL(2,\mathbb{C})$ by using $p_{1\alpha\dot{\alpha}} = |1^I\rangle[1_I|$, $k_{2\alpha\dot{\alpha}} = |2\rangle[2|$ and obtain the following spinor variables, the on-shell kinematics of

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the massive spinor 1

$$\begin{aligned}|1^{\uparrow}\rangle_{\alpha} =&\sqrt{m_1}\begin{pmatrix}1\\0\end{pmatrix},\\ |1^{\uparrow}]^{\dot\alpha} =&\sqrt{m_1}\begin{pmatrix}1\\0\end{pmatrix},\\ |1^{\downarrow}\rangle_{\alpha} =&\sqrt{m_1}\begin{pmatrix}0\\1\end{pmatrix},\\ |1^{\downarrow}]^{\dot\alpha} =&\sqrt{m_1}\begin{pmatrix}0\\1\end{pmatrix},\end{aligned} \tag{3.12}$$

and the massless spinor 2

$$\begin{aligned}|2\rangle_{\alpha} =&\sqrt{2E_2}\begin{pmatrix}-e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)\end{pmatrix},\\ |2]^{\dot\alpha} =&\sqrt{2E_2}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{pmatrix},\end{aligned} \tag{3.13}$$

where the $SU(2)$ little group indices I for massive leg 1 can be chosen as $\uparrow$ and $\downarrow$; massless one $|2\rangle$ has helicity $-\frac{1}{2}$ and $|2]$ has helicity $+\frac{1}{2}$, respectively.

Then, in order to construct the 4-momentum of particle 3, the momentum conservation gives us

$$p_3^{\mu} = p_1^{\mu} + k_2^{\mu} = \begin{pmatrix} m_1 + E_2 & E_2\sin\theta\cos\phi & E_2\sin\theta\sin\phi & E_2\cos\theta \end{pmatrix}, \tag{3.14}$$

and by the on-shell condition $m_3^2 = |p_3^{\mu}|^2 = 2m_1E_2 + m_1^2$, we can first identify the rest frame momentum of leg 3 to

$$p^{\mu}_{3,rest} = \begin{pmatrix} \sqrt{2m_1E_2 + m_1^2} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} m_3 & 0 & 0 & 0 \end{pmatrix} \tag{3.15}$$

with its mass $m_3 = \sqrt{2E_2m_1 + m_1^2} = m_1 + E_2 + \mathcal{O}(E_2^2)$. And then, we also decompose the momentum $[p_{3,rest}]_{\alpha\dot\alpha}$ in $SL(2,\mathbb{C})$ into spinors which have a similar form with

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the spinor 1, $|1^I\rangle$ and $|1^I]$, but we are already in the rest frame of particle 1, so we have to boost $p^\mu_{3,rest}$ along the direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in order to set up the right coordinates, in other words, we want the spin-up spinor 3 in the rest frame to correspond to the spin-up spinor 3 in the other frame; the spin-down in the rest frame to correspond to the spin-down in the other frame. Precisely, that is

$$\begin{aligned}
|3^\uparrow\rangle_\alpha &= \begin{pmatrix} \cosh\frac{\lambda}{2} - \sinh\frac{\lambda}{2}\cos\theta & -\sinh\frac{\lambda}{2}e^{-i\phi}\sin\theta \\ -\sinh\frac{\lambda}{2}e^{i\phi}\sin\theta & \cosh\frac{\lambda}{2} + \sinh\frac{\lambda}{2}\cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{m_3} \\ 0 \end{pmatrix}, \\
|3^\uparrow]^{\dot\alpha} &= \begin{pmatrix} \cosh\frac{\lambda}{2} + \sinh\frac{\lambda}{2}\cos\theta & \sinh\frac{\lambda}{2}e^{-i\phi}\sin\theta \\ \sinh\frac{\lambda}{2}e^{i\phi}\sin\theta & \cosh\frac{\lambda}{2} - \sinh\frac{\lambda}{2}\cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{m_3} \\ 0 \end{pmatrix}, \\
|3^\downarrow\rangle_\alpha &= \begin{pmatrix} \cosh\frac{\lambda}{2} - \sinh\frac{\lambda}{2}\cos\theta & -\sinh\frac{\lambda}{2}e^{-i\phi}\sin\theta \\ -\sinh\frac{\lambda}{2}e^{i\phi}\sin\theta & \cosh\frac{\lambda}{2} + \sinh\frac{\lambda}{2}\cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{m_3} \end{pmatrix}, \\
|3^\downarrow]^{\dot\alpha} &= \begin{pmatrix} \cosh\frac{\lambda}{2} + \sinh\frac{\lambda}{2}\cos\theta & \sinh\frac{\lambda}{2}e^{-i\phi}\sin\theta \\ \sinh\frac{\lambda}{2}e^{i\phi}\sin\theta & \cosh\frac{\lambda}{2} - \sinh\frac{\lambda}{2}\cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{m_3} \end{pmatrix},
\end{aligned} \tag{3.16}$$

where the boost matrix in $SL(2,\mathbb{C})$ has the form $e^{\pm\frac{\lambda}{2}(\hat{n}\cdot\vec{\sigma})} = \cosh\frac{\lambda}{2}\mathbb{I} \pm \sinh\frac{\lambda}{2}(\hat{n}\cdot\vec{\sigma})$ from a rest frame with the rapidity $\lambda = \log\left(\frac{(E_2+m_1)+E_2}{m_3}\right)$, and the plus sign are for angle spinors (chiral spinors); the minus sign are for square spinors (anti-chiral spinors). We can easily verify the momentum conservation $p_{1\alpha\dot\alpha} + k_{2\alpha\dot\alpha} = p_{3\alpha\dot\alpha}$,

$$p_{3\alpha\dot\alpha} = \begin{pmatrix} m_1 + 2E_2\sin^2\left(\frac{\theta}{2}\right) & -e^{-i\phi}E_2\sin\theta \\ -e^{i\phi}E_2\sin\theta & m_1 + 2E_2\cos^2\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{3.17}$$

by combining the spinors $p_{3\alpha\dot\alpha} = |3^I\rangle[3_I|$ from the above.

Then, with the above settings, we can compute the spinor products (on-shell

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elements) with the angular dependence

$$\begin{aligned}
[21^{\uparrow}] =&[23^{\uparrow}] = -\sqrt{2E_2m_1}e^{i\phi}\sin\left(\frac{\theta}{2}\right),\\
[21^{\downarrow}] =&[23^{\downarrow}] = \sqrt{2E_2m_1}\cos\left(\frac{\theta}{2}\right),\\
\langle 21^{\uparrow}\rangle =&\langle 23^{\uparrow}\rangle = \sqrt{2E_2m_1}\cos\left(\frac{\theta}{2}\right),\\
\langle 21^{\downarrow}\rangle =&\langle 23^{\downarrow}\rangle = \sqrt{2E_2m_1}e^{-i\phi}\sin\left(\frac{\theta}{2}\right).
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
[3^{\uparrow}1^{\uparrow}] =&\langle 3^{\uparrow}1^{\uparrow}\rangle = \frac{\sqrt{m_1}e^{i\phi}\sin\theta(2E_2+m_1-m_3)}{2\sqrt{2E_2+m_1}},\\
[3^{\uparrow}1^{\downarrow}] =&\langle 3^{\downarrow}1^{\uparrow}\rangle = -\frac{\sqrt{m_1}[\cos\theta(2E_2+m_1-m_3)+2E_2+m_1+m_3]}{2\sqrt{2E_2+m_1}},\\
[3^{\downarrow}1^{\uparrow}] =&\langle 3^{\uparrow}1^{\downarrow}\rangle = \frac{\sqrt{m_1}[\cos\theta(-2E_2-m_1+m_3)+2E_2+m_1+m_3]}{2\sqrt{2E_2+m_1}},\\
[3^{\downarrow}1^{\downarrow}] =&\langle 3^{\downarrow}1^{\downarrow}\rangle = -\frac{\sqrt{m_1}e^{-i\phi}\sin\theta(2E_2+m_1-m_3)}{2\sqrt{2E_2+m_1}},
\end{aligned} \tag{3.19}$$

under our setup, some products are the same, since the rapidity contains the m_3 and m_3 can be replaced by m_1 and E_2. In the next section, we will use these brackets which is set to $\phi = 0$ with only angular dependence θ to plug into the scattering amplitudes, and to compare with the spin-weighted spherical harmonics, because we observed the following relationships,

$$\begin{aligned}
&\langle 23^{\uparrow}\rangle\langle 23^{\uparrow}\rangle\langle 23^{\uparrow}\rangle\langle 23^{\uparrow}\rangle \sim \cos^4\left(\frac{\theta}{2}\right) && \Rightarrow {}_{-2}Y_{2,2}(\theta),\\
&\langle 23^{\uparrow}\rangle\langle 23^{\uparrow}\rangle\langle 23^{\uparrow}\rangle\langle 23^{\downarrow}\rangle \sim \cos^3\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) && \Rightarrow {}_{-2}Y_{2,1}(\theta),\\
&\langle 23^{\uparrow}\rangle\langle 23^{\uparrow}\rangle\langle 23^{\downarrow}\rangle\langle 23^{\downarrow}\rangle \sim \cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right) && \Rightarrow {}_{-2}Y_{2,0}(\theta),\\
&\langle 23^{\uparrow}\rangle\langle 23^{\downarrow}\rangle\langle 23^{\downarrow}\rangle\langle 23^{\downarrow}\rangle \sim \cos\left(\frac{\theta}{2}\right)\sin^3\left(\frac{\theta}{2}\right) && \Rightarrow {}_{-2}Y_{2,-1}(\theta),\\
&\langle 23^{\downarrow}\rangle\langle 23^{\downarrow}\rangle\langle 23^{\downarrow}\rangle\langle 23^{\downarrow}\rangle \sim \sin^4\left(\frac{\theta}{2}\right) && \Rightarrow {}_{-2}Y_{2,-2}(\theta).
\end{aligned} \tag{3.20}$$

In addition, the above angular dependence of the spin configuration have the same

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form as $(1+X)^{\frac{1}{2}|m-s|}(1-X)^{\frac{1}{2}|m+s|}$ for $s=-2$ in Leaver's ansatz (2.42). This sparks our interest and leads us to believe that we may be able to express Leaver's ansatz by using on-shell elements in the next chapter's Kerr case.

Then, we can redefine the above spinors by dividing by a square root mass dimension

$$\begin{aligned}\frac{1}{\sqrt{m_1}}|1^I\rangle_\alpha &= |1'^I\rangle_\alpha \\ \frac{1}{\sqrt{m_1}}|2\rangle_\alpha &= |2'\rangle_\alpha \\ \frac{1}{\sqrt{m_3}}|3^I\rangle_\alpha &= |3'^I\rangle_\alpha\end{aligned} \tag{3.21}$$

such that the brackets become dimensionless. To compare with the angular equation (2.23) in BHPT, both of a and ω are dimensionless, hence we redefine the dimensionless angular frequency ω in our setup,

$$\frac{\omega}{2m_1} \to \omega, \tag{3.22}$$

such that the series expansion parameter $a\omega = \frac{aE_2}{2m_1}$ in the next chapter is also dimensionless.

3.3 Spin-weighted Spherical Harmonics

The spin-weighted spherical harmonics (or just spherical harmonics in QM) are an analytic function describing a spherically symmetric system in $SO(3)$ representation. The quantum number m represents the projection value of the angular momentum on the z-axis, and there are $2l+1$ possible projection states for $m=-l,...,0,...l$ to fix the total angular momentum l.

On the one hand, we know that a $(2l+1)$-dimension representation is corresponding to a rank-l symmetric traceless tensor in the group $SO(3)$. On the other hand, the irreducible representation of $SU(2)$ is a subgroup of the rotation group $SO(3)$.

Because spherical harmonics are representations of the $SO(3)$ group and their

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form depends on the quantum number of projection of angular momentum m, and because all representations under $SO(3)$ can be rewritten as representations under $SU(2)$ which is the more fundamental representation of rotation, we can consider the $SU(2)$ little group indices as part of the quantum number m, which constitutes spherical harmonics.

According to the last two sections, our 3pt amplitude (3.9) need to contract with the external leg 3, $\lambda_{3,\beta_1}^{(J_1}...\lambda_{3,\beta_{2l}}^{J_{2l})}$, and without leg 1, then we have

$$\mathcal{M}^{-2,(J_1...J_{2l})} = g_l^{-2}\langle 2'3'^{(J}\rangle^{l+2}[2'3'^{J)}]^{l-2}, \tag{3.23}$$

with the $SU(2)$ little group indices $(J_1...J_{2l})$ which are free and fully symmetric, and the symmetric symbol (abc) means $(abc+acb+bac+bca+cab+cba)/(3!)$. Actually, the above (3.20) is exactly the amplitude for $l=2$ with different spin configurations.

Based on the above arguments, we try to figure out how the scattering amplitudes $\mathcal{M}^{-2,(J_1...J_{2l})}$ relate to the spin-weighted spherical harmonics

$$\mathcal{M}^{-2,(J_1...J_{2l})} \sim {}_{-2}Y_{lm}(\theta). \tag{3.24}$$

We have observed that they have the same angular dependence as fixing m and spin configuration, therefore there exists an one-to-one correspondence and the amplitude is proportional to the spin-weighted spherical harmonics, which is the angular solution of black hole QNMs in the Schwarzschild case. To be explicit, the relation is

$$\boxed{\mathcal{M}^{-2,(J_1...J_{2l})} = g_l^{-2}\left(\frac{2E_2}{m_1}\right)^l \frac{1}{(2l)!}\sqrt{\frac{4\pi}{2l+1}}\sqrt{(l+2)!(l-2)!(l+m)!(l-m)!}\,{}_{-2}Y_{lm}(\theta)} \tag{3.25}$$

with a overall factor $(E_2/m_1)^l$. There are $2l$ free little group indices, that is, the spin configuration is corresponding to $2l+1$ different m.

Given l, the range of m is from $-l$ to l. For $m=0$ in the spherical harmonics, the number of spin up and spin down will be equal; for $m=\pm l$, the little group indices

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are all up or down, respectively; for arbitrarily m, there are $(l+m)$ spin up and $(l-m)$ spin down. In other words, the $SU(2)$ spin configurations $(J_1, J_2, ...J_{2l-1}, J_{2l})$ which are carried by the leg 3, as well as fully symmetric reproduce the each m by

$$(\underbrace{\uparrow, ..., \uparrow}_{l+m}, \underbrace{\downarrow, ..., \downarrow}_{l-m}) \quad \Longleftrightarrow \quad m \tag{3.26}$$

Especially, if we choose the helicity of gravitons as $h=+2$, then the amplitude will be

$$\mathcal{M}^{+2,(J_1...J_{2l})} = g_l^{+2} \langle 2'3'^{(J}\rangle^{l-2} [2'3'^{J)}]^{l+2}, \tag{3.27}$$

and will match the spin-weighted spherical harmonics with the spin weight $s=+2$, that is ${}_2Y_{lm}(\theta)$. Moreover, if we consider a photon with helicity $h=\pm1$ or a scalar particle with helicity $h=0$, then the amplitude will match the spin-weighted spherical harmonics ${}_{\pm1}Y_{lm}(\theta)$, or ${}_0Y_{lm}(\theta) = Y_{lm}(\theta)$, respectively. To be explicit, the 3pt scattering amplitudes

$$\mathcal{M}^{h,(J_1...J_{2l})} = g_l^{h} \langle 2'3'^{(J}\rangle^{l-h} [2'3'^{J)}]^{l+h} \tag{3.28}$$

are related to the spin-weighted spherical harmonics

$$\boxed{\mathcal{M}^{h,(J_1...J_{2l})} = g_l^h (-1)^h \left(\frac{2E_2}{m_1}\right)^l \frac{1}{(2l)!} \sqrt{\frac{4\pi}{2l+1}} \sqrt{(l+h)!(l-h)!(l+m)!(l-m)!}\,{}_hY_{lm}(\theta)} \tag{3.29}$$

for the helicity (spin weight) $h = 0, \pm1, \pm2$, spin $l \geq |h|$, and $-l \leq m \leq l$.

So far, the punchline is that, by establishing a unique matching of scattering amplitudes $\mathcal{M}^{h,(J_1...J_{2l})}$ and the spin-weighted spherical harmonics ${}_hY_{lm}(\theta)$, we have successfully connected the spin configurations and quantum number m, and reproduced the Schwarzschild black hole QNMs which emit a massless particle with helicity h, or says a perturbed spin-h field (spin weight h).

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Chapter 4

Spin-weighted Spheroidal Harmonics from On-shell Kinematics (Kerr)

In the previous chapter, we described spherically symmetric black holes QNMs by using the spinor-helicity formalism and the spin configurations. In this chapter, we discuss the angular dependence of Kerr black hole QNMs which involves the rotation, or says the classical spin, that is the spin-weighted spheroidal harmonics (2.41) we mentioned. Since the classical spin effect is not included in the on-shell spinor-helicity formalism kinematics which involves the quantum spin, we introduce the coherent spin state which can describe the classical behavior, then the expectation value of spin operators by the coherent spin states can represent the classical spin vector.

First, we introduce the coherent spin state as presented in [32] and its application to the minimal-coupling coherent amplitude. Next, in order to construct a on-shell basis to produce the each order of the spin-weighted spheroidal harmonics, we expand the application of the coherent spin state to the unequal masses on-shell elements which is a process from spin-$(l \oplus s)$ to spin-s and satisfy the spinor-helicity formalism. In the classical limit, we can use these elements to establish a on-shell

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coherent tensor and describe the spin-weighted spheroidal harmonics by truncating them at some order of the small parameter $a\omega$.

4.1 Review of Coherent Spin State

To obtain the spin vector of the Kerr black hole, we need to introduce the coherent spin states, since the spinor-helicity formalism does not involve the classical spin. The coherent spin state, which is an eigenstate of the annihilation operator and is composed of a series of quantum states, approximatively describes a dynamic state as the classical behavior and minimizes the uncertainty.

To find a spin operator acting on the irreducible representation of $SU(2)$, let us review the N-dimensional harmonic oscillator with $SU(N)$ symmetry in Appendix A. Now, we consider the 2-dimensional harmonic oscillator, and then there are two creation operators $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$ and two annihilation operators $\hat{a}_1$, $\hat{a}_2$ acting on $SU(2)$ representation in the system. The creation and annihilation operators satisfy the algebra

$$[\hat{a}^I, \hat{a}_J^\dagger] = \delta^I_J \tag{4.1}$$

where the $SU(2)$ indices for $I, J = 1, 2$, or says up and down. So far, they can construct an operator

$$S^i = \frac{\hbar}{2} \hat{a}_I^\dagger [\sigma^i]^I{}_J \hat{a}^J \tag{4.2}$$

acting on the $SU(2)$ spin state in [32] by above creation, annihilation operators, and the Pauli matrices which are defined by

$$[\sigma_{p\mu}]^I{}_J = \frac{1}{2m} \left(\langle p^I|\sigma_\mu|p_J] + [p^I|\bar{\sigma}_\mu|p_J\rangle \right), \tag{4.3}$$

where $|p_J]$ and $|p_J\rangle$ are decomposed from a momentum p, then we will set p at the rest frame. It is worth noting that, the operator defined in this way satisfies the angular momentum algebra

$$[S^i, S^j] = i\hbar\epsilon^{ijk} S^k, \tag{4.4}$$

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or more generally

$$[S^\mu, S^\nu] = \frac{i\hbar}{m}\epsilon^{\mu\nu\rho\sigma}p_\rho S_\sigma \tag{4.5}$$

in $SO(1,3)$ which depends on a frame p_ρ. In $SU(2)$ representation, we can compare the observable in the 2-dimensional harmonic oscillator with the spin states

$$\begin{aligned} [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \quad &\Longrightarrow \quad [x_i, p_j] = i\hbar\delta_{ij} \qquad i,j = 1,2 \\ [\hat{a}^I, \hat{a}_J^\dagger] = \delta^I_J \quad &\Longrightarrow \quad [S_i, S_j] = i\hbar\epsilon_{ijk}S_k \quad i,j,k = 1,2,3 \end{aligned} \tag{4.6}$$

for $I, J = 1, 2$. Because all of the operators x_i, p_i, and S_i are defined by the same creation and annihilation operators which satisfy the same commutation relation, we then use the same form of the coherent state to minimize the uncertainty of these observable which are in terms of the creation and annihilation operators.

This is what we want, and we will use the angular momentum operators later. Note that, the $SU(2)$ operator which is Lorentz covariant is dependent on a frame of a momentum p which we choose.

Now, let us find the coherent spin states that we will use, the coherent spin states are the coherent states for the rotation group of the 3-dimensional space $SO(3)$. In [32,34], the coherent spin states are $SU(2)$ representation, involving two creation and two annihilation operators, and satisfies the same eigenvalue equations of the annihilation operators as the 2-dimensional harmonic oscillator with complex eigenvalues α^I for $I = 1, 2$. But here the coherent spin states are used to minimize the uncertainty of the expectation value of the angular momentum operator.

Here, we briefly introduce the uncertainty of the expectation value of the angular momentum operator. Hence, first we need to know the expectation value of the angular momentum operator

$$\langle s, (I_1 ... I_{2s})|S_i|s, (J_1 ... J_{2s})\rangle = \frac{\hbar}{2} 2s[\sigma_i]^{(I_1}{}_{(J_1}\delta^{I_2}_{J_2} ... \delta^{I_{2s-1})}_{J_{2s-1})} \tag{4.7}$$

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and

$$\langle s,(I_1...I_{2s})|{S_i}^2|s,(J_1...J_{2s})\rangle = \frac{\hbar^2}{4}2s(2s-1)[\sigma^i]^{(I_1}{}_{(J_1}[\sigma_i]^{I_2}{}_{J_2}\delta^{I_3}_{J_3}...\delta^{I_{2s-2})}_{J_{2s-2})}. \tag{4.8}$$

by an arbitrary spin state in $SU(2)$ representation, where the spin state

$$|s,(I_1...I_{2s})\rangle = \frac{1}{\sqrt{(2s)!}}\hat{a}^\dagger_{I_1}...\hat{a}^\dagger_{I_{2s}}|0\rangle \tag{4.9}$$

satisfies the normalization condition

$$\langle s,(I_1...I_{2s})|s',(J_1...J_{2s'})\rangle = \delta^s_{s'}\delta^{(I_1}_{(J_1}...\delta^{I_{2s})}_{J_{2s})} \tag{4.10}$$

where $(\alpha^{I_i})^{2s} = \alpha^{I_1}\alpha^{I_2}...\alpha^{I_{2s-1}}\alpha^{I_{2s}}$. Note that the expectation value of S_i vanish as $i = 1,2$, since the two Pauli matrices σ^1 and σ^2 have no diagonal elements. After having the expectation values, moreover, we can compute the uncertainty of the angular momenta which satisfies

$$\Delta S_1 \Delta S_2 \geq \frac{\hbar}{2}\,|\langle s,(I_1...I_{2s})|S_3|s,(I_1...I_{2s})\rangle| \tag{4.11}$$

where the standard deviation of an observable is

$$\Delta\mathcal{O}_i = \sqrt{\langle s,(I_1...I_{2s})|\mathcal{O}_i^2|s,(I_1...I_{2s})\rangle - \langle s,(I_1...I_{2s})|\mathcal{O}_i|s,(I_1...I_{2s})\rangle^2} \tag{4.12}$$

by an arbitrary spin state $|s,(I_1...I_{2s})\rangle$. Take $s = 1$ spin states as examples. On the one hand, we can use the state $|1,(\uparrow,\uparrow)\rangle$ to compute the standard deviation of angular momentum

$$\Delta S_1 = \Delta S_2 = \sqrt{\frac{\hbar^2}{2}} \tag{4.13}$$

where

$$\langle 1,(\uparrow,\uparrow)|{S_1}^2|1,(\uparrow,\uparrow)\rangle = \frac{\hbar^2}{2}[\sigma_1]^\uparrow{}_\uparrow[\sigma_1]^\uparrow{}_\uparrow = \frac{\hbar^2}{2}\left(\delta^\uparrow_\uparrow\delta^\uparrow_\uparrow + \epsilon_{\uparrow\uparrow}\epsilon^{\uparrow\uparrow}\right) = \frac{\hbar^2}{2} \tag{4.14}$$

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by some properties of the Pauli matrices in [32], and then the expectation value of the third component

$$\langle 1,(\uparrow,\uparrow)|S_3|1,(\uparrow,\uparrow)\rangle = \hbar[\sigma_3]^{\uparrow}{}_{\uparrow}\delta^{\uparrow}_{\uparrow} = \hbar \tag{4.15}$$

where

$$[\sigma_3]^{\uparrow}{}_{\uparrow} = \frac{1}{2m}\left(\langle p^{\uparrow}|\sigma_3|p_{\uparrow}] + [p^{\uparrow}|\bar{\sigma}_3|p_{\uparrow}\rangle\right) = 1 \tag{4.16}$$

by the rest frame spinors $|p_I\rangle$, $|p_I]$, we can verify that this spin state makes the uncertainty principle hold; on the other hand, we can also use the other spin state $|1,(\uparrow,\downarrow)\rangle$ to compute

$$\Delta S_1 = \Delta S_2 = \hbar \tag{4.17}$$

where

$$\langle 1,(\uparrow,\downarrow)|{S_1}^2|1,(\uparrow,\downarrow)\rangle = \frac{\hbar^2}{2}\frac{1}{2}\left([\sigma_1]^{\uparrow}{}_{\uparrow}[\sigma_1]^{\downarrow}{}_{\downarrow} + [\sigma_1]^{\uparrow}{}_{\downarrow}[\sigma_1]^{\uparrow}{}_{\downarrow} + [\sigma_1]^{\downarrow}{}_{\uparrow}[\sigma_1]^{\downarrow}{}_{\uparrow} + [\sigma_1]^{\downarrow}{}_{\downarrow}[\sigma_1]^{\uparrow}{}_{\uparrow}\right) = \hbar^2 \tag{4.18}$$

and

$$\langle 1,(\uparrow,\downarrow)|S_3|1,(\uparrow,\downarrow)\rangle = \hbar[\sigma_3]^{(\uparrow}{}_{(\uparrow}\delta^{\downarrow)}_{\downarrow)} = 0 \tag{4.19}$$

where

$$[\sigma_3]^{(\uparrow}{}_{(\uparrow}\delta^{\downarrow)}_{\downarrow)} = \frac{1}{2}\left([\sigma_3]^{\uparrow}{}_{\uparrow} + [\sigma_3]^{\downarrow}{}_{\downarrow}\right) = 0 \tag{4.20}$$

by some properties of Pauli matrices, the uncertainty principle still hold by this spin state. The uncertainty of the angular momenta in $SU(2)$ representation is the same as in $SO(3)$ representation, that is to say, the standard deviation

$$\Delta S_1 = \Delta S_2 = \sqrt{\frac{\hbar^2}{2}[l(l+1) - m^2]} \tag{4.21}$$

and the expectation value

$$\langle l,m|S_3|l,m\rangle = \hbar m \tag{4.22}$$

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lead to the following inequality

$$\Delta S_1 \Delta S_2 = \frac{\hbar^2}{2}[l(l+1) - m^2] \geq \frac{\hbar}{2}\langle l, m|S_3|l, m\rangle \tag{4.23}$$

by an arbitrary state $|l, m\rangle$ in $SO(3)$ representation.

Next, in order to minimize the uncertainty of the expectation value of the angular momentum operator, we need the coherent spin states which are expressed as

$$|\alpha\rangle = e^{-\frac{1}{2}\tilde{\alpha}_J \alpha^J} e^{\alpha^I \hat{a}_I^\dagger}|0\rangle = e^{-\frac{1}{2}||\alpha||^2} \sum_{2s=0}^{\infty} \sum_{I_1,...,I_{2s}=\uparrow,\downarrow} \frac{(\alpha^I)^{2s}}{\sqrt{(2s)!}}|s, (I_1 ... I_{2s})\rangle. \tag{4.24}$$

in terms of the spin state. Naively, we can verify the eigenvalue equations of the annihilation operator by expanding that

$$\begin{aligned} \hat{a}^\uparrow|\alpha\rangle =& e^{-\frac{1}{2}||\alpha||^2} \sum_{2s=0}^{\infty} \sum_{I_1,...,I_{2s}=\uparrow,\downarrow} \frac{(\alpha^I)^{2s}}{(2s)!} \hat{a}^\uparrow \hat{a}_{I_1}^\dagger ... \hat{a}_{I_{2s}}^\dagger|0\rangle \\ =& e^{-\frac{1}{2}||\alpha||^2} \left\{ 0 + (\alpha^\uparrow)|0\rangle + 0 + \frac{2(\alpha^\uparrow\alpha^\uparrow)}{2}\hat{a}_\uparrow^\dagger|0\rangle + \frac{2(\alpha^\uparrow\alpha^\downarrow)}{2}\hat{a}_\downarrow^\dagger|0\rangle + 0 + ... \right\} \\ =& \alpha^\uparrow|\alpha\rangle, \end{aligned} \tag{4.25}$$

and $|\alpha\rangle$ is exactly a eigenstate.

To describe the behavior of the classical angular momentum, or says the classical spin, they sandwich the angular momentum operator (4.2) by the coherent spin state $|\alpha\rangle$, that is the expectation value

$$\langle\alpha|S_i S_j|\alpha\rangle = \langle\alpha|S_i|\alpha\rangle\langle\alpha|S_j|\alpha\rangle + \frac{\hbar^2}{4}\left[\delta_{ij}(\tilde{\alpha}_I \alpha^I) + i\epsilon_{ijk}(\tilde{\alpha}_I [\sigma_k]^I{}_J \alpha^J)\right], \tag{4.26}$$

where

$$\langle\alpha|S_i|\alpha\rangle = \frac{\hbar}{2}\left[\tilde{\alpha}_I [\sigma_i]^I{}_J \alpha^J\right]. \tag{4.27}$$

by $\hat{a}^I|\alpha\rangle = \alpha^I|\alpha\rangle$ and the normalization condition is the classical spin with the order $\hbar^0$, since the $SU(2)$ spinors $\tilde{\alpha}_I$, α^I have the order $\hbar^{-1/2}$ for each one. Now, we can

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verify the uncertainty of the expectation value of the angular momentum operator,

$$\Delta S_1 = \Delta S_2 = \sqrt{\frac{\hbar^2}{4}a} \tag{4.28}$$

and

$$\langle\alpha|S_3|\alpha\rangle = \frac{\hbar}{2}a \tag{4.29}$$

when the spin $a^i = (0,0,a)$ has only z component, so the coherent spin states exactly minimize the uncertainty of the expectation value of the angular momentum operator $\Delta S_1 \Delta S_2 = \frac{\hbar}{2}|\langle\alpha|S_3|\alpha\rangle|$. That is to say, the coherent spin state leads the behavior of the expectation value of the angular momentum operator to be classical.

Therefore, they can use the form in (4.24) to reproduce the classical spin a^μ. Precisely, the spin vector of a Kerr black hole is defined by

$$a_p^\mu = \frac{1}{m_p}\langle\alpha|S_p^\mu|\alpha\rangle \tag{4.30}$$

which is inspired by the expectation value of angular momentum operators (4.27) at the p frame, let's say the $SU(2)$ spinors are dependent on the frame $\alpha^I = \alpha^I(p)$.

Then, let's see the application to the equal masses on-shell scattering amplitude, which is called the classical spinning amplitudes. They start from the 3pt amplitude, the massive leg 1 and leg 2 particles which have the same masses m but spin-s_1, spin-s_2, respectively, and the massless k being a graviton with helicity $h = \pm 2$ emits.

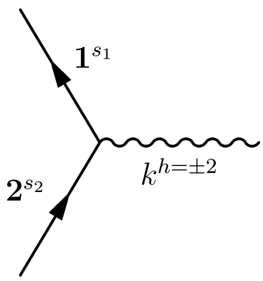

Figure 4.1: 3pt scattering process from spin-S_2 to spin-S_1 and helicity ± 2

Previously, we mention the 3pt amplitude with different masses. Now, the amplitude is with the equal masses, people cannot use the spinors $u_\alpha = |k\rangle_\alpha$ and

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$v_\alpha = \frac{p_{1\alpha\dot\beta}}{m}|k]^{\dot\beta}$ as a basis, since $v^\alpha u_\alpha = 2p_1 \cdot k/m = 0$ by momentum conservation and the same masses m, that means they are parallel. In [29,31,32], they construct a basis for the equal masses amplitudes,

$$\begin{aligned} x|k\rangle_\alpha &= \frac{p_{1\alpha\dot\beta}}{m}|k]^{\dot\beta}, \\ \frac{1}{x}|k]^{\dot\alpha} &= \frac{p_1^{\dot\alpha\beta}}{m}|k\rangle_\beta \end{aligned} \tag{4.31}$$

where the x-factor is $x = \frac{\langle\zeta|p_1|k]}{m\langle\zeta k\rangle}$ with a reference spinor ζ. Here, they start with the minimal coupling amplitudes which is the leading order of the amplitude, denoted as $\mathcal{M}_{min}(\mathbf{1}^s, \mathbf{2}^s, k^h) = \mathcal{M}_{min}^{h,\{I\}}{}_{\{J\}}$, with equal masses and equal spin-s,

$$\begin{aligned} \mathcal{M}_{min}^{+2,\{I\}}{}_{\{J\}} &= -\frac{\kappa}{2}\frac{\langle 2^I 1_J\rangle^{2s}}{m^{2s-2}}x^2, \\ \mathcal{M}_{min}^{-2,\{I\}}{}_{\{J\}} &= (-1)^{2s+1}\frac{\kappa}{2}\frac{[2^I 1_J]^{2s}}{m^{2s-2}}\frac{1}{x^2}. \end{aligned} \tag{4.32}$$

Next, to connecting to Kerr black holes, by combining the coherent spin states (4.24) and the minimal coupling amplitudes, the coherent spin amplitudes is

$$\mathcal{A}_{min}^{+2} = -\frac{\kappa}{2}x^2 e^{-\frac{1}{2}(||\alpha||^2+||\beta||^2)}\sum_{2s=0}^{\infty}\frac{1}{(2s)!}(\tilde\beta_I)^{2s}\frac{\langle 2^I 1_J\rangle^{2S}}{m^{2S-2}}(\alpha^J)^{2s}, \tag{4.33}$$

and it is obviously the series expansion of exponential, hence we can rewrite it as

$$\mathcal{A}_{min}^{+2} = -\frac{\kappa}{2}m^2x^2e^{-\frac{1}{2}(||\alpha||^2+||\beta||^2)}\exp\left\{\frac{\tilde\beta_I\langle 2^I 1_J\rangle\alpha^J}{m}\right\}. \tag{4.34}$$

According to the definition of the spin vector, we know this is dependent on the frame. Therefore, they choose an average momentum frame, defined by $p_a = \frac{p_1+p_2}{2}$, such that the spinors $|1_I\rangle$ and $|2^I\rangle$ can be boosted from $|a^I\rangle$ which is decomposed with $p_a = |a^I\rangle[a_I|$,

$$\begin{aligned} |1_I\rangle &= \exp\left\{-\frac{ip_a^\mu p_1^\nu}{m^2}\sigma_{\mu\nu}\right\}|a_I\rangle \\ |2^I\rangle &= \exp\left\{\frac{ip_a^\mu p_2^\nu}{m^2}\sigma_{\mu\nu}\right\}|a^I\rangle \end{aligned} \tag{4.35}$$

with the Lorentz generators $\sigma^{\mu\nu} = \frac{i}{2}\sigma^{[\mu}\bar\sigma^{\nu]}$ in $SL(2,\mathbb{C})$. Due to the on-shell formal-

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ism with equal masses, we know that $p_1 \cdot k = p_2 \cdot k = p_a \cdot k = 0$. Then we can rewrite the boost matrix as $\exp\left\{\pm \frac{ip_a^\mu k^\nu}{2m^2}\sigma_{\mu\nu}\right\}$ by momentum conservation $p_1 + k = p_2$, and expand the boost up to linear in k, since the higher order vanish $k^2 = 0$, such that the piece in the coherent spin amplitudes becomes

$$\tilde{\beta}_I(p_2)\langle 2^I 1_J\rangle \alpha^J(p_1) = \tilde{\beta}_I(p_a)\left(\langle a^I a_J\rangle - \frac{1}{4m}\left(\langle a^I|k|a_J] + [a^I|k|a_J\rangle\right)\right)\alpha^J(p_a), \quad (4.36)$$

where $\langle a^I a_J\rangle = m\delta^I_J$, and that is exactly corresponding to the spin vector $a^\mu_{p_a}$ as taking $\tilde{\beta}_I = \tilde{\alpha}_I$, and the coherent spin amplitudes become

$$\mathcal{A}^{\pm 2}_{min} = -\frac{\kappa}{2}m^2 x^{\pm 2} e^{\mp \bar{k}\cdot a_{p_a}}, \quad (4.37)$$

which contain the exponential form. This 3pt result with the exponential spin-multipole can be used to connect to the gravitational scattering of Kerr black holes [32], such as 4pt amplitude gluing by 3pt amplitudes is used to connect the impulse and the geodesic equation.

By observing the application, we find that the piece $k \cdot a$ is involved in their results, which is from the contraction of $SO(1,3)$ indices

$$\tilde{\alpha}_I(p)[k^\mu \sigma_{p\mu}]^I{}_J \alpha^J(p) \sim k \cdot a, \quad (4.38)$$

where $k \sim \omega$ in our language, and this give us the spin multiplied by the angular frequency, that is exactly the parameter $a\omega$ which is needed to expand the spin-weighted spheroidal harmonics. Consequently, we except the coherent spin states will give our scattering amplitudes a classical spin of Kerr black hole, such that we can solve the problem about the spinor-helicity formalism not involving the classical spin.

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4.2 Spin-weighted Spheroidal Harmonics

After we introduce the coherent spin state and the spin vector in the 3pt equal masses amplitudes, now we try to establish a on-shell basis for the each order of the spin-weighted spheroidal harmonics. The basis is composed of various on-shell elements of the 3pt process that contract with $SU(2)$ spinors $\tilde{\alpha}_I$ and α^I from coherent spin states.

First, we can find the $SU(2)$ spinors in a specific form which satisfy the following conditions. As we know from the previous section, the $SU(2)$ spinors can be combined with the spin operators to obtain the classical spin at a frame p, and now we ask the generated spin vector a^μ has only the t and z components, such that if we set p at the rest frame, then the spin vector will reduce to only the z direction, since people usually set the spin along the z direction in Kerr metric (2.12). Also, we want the length of $SU(2)$ spinors to be

$$||\alpha||^2 = \tilde{\alpha}_I(p)\alpha^I(p) = a. \tag{4.39}$$

Since the $SU(2)$ spinors depend on a reference frame, we need to make $\tilde{\alpha}_I(p)$, $\alpha^I(p)$, and the on-shell spinors $|1_I\rangle$, $|3^I\rangle$ at the same reference frame. However, our setup is not equal masses, so we have to rescale the momenta (3.21) by their masses

$$\begin{aligned} {p'_1}^\mu &= \frac{p_1^\mu}{m_1} \Rightarrow p'_{1\alpha\dot{\alpha}} = |1'^I\rangle[1'_I| \\ {p'_3}^\mu &= \frac{p_3^\mu}{m_3} \Rightarrow p'_{3\alpha\dot{\alpha}} = |3'^I\rangle[3'_I| \end{aligned} \tag{4.40}$$

where $m_3 = \sqrt{2E_2 m_1 + m_1^2}$ and define a new momentum

$${p'_a}^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.41}$$

at the rest frame, such that they are unit length $|p'_1| = |p'_3| = |p'_a| = 1$. So far, by taking ${p'_a}^\mu$ as the reference frame, we rewrite the $|1'^I\rangle$ in terms of $|a'^I\rangle$, and rewrite

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$|3'^I\rangle$ in terms of the same spinor $|a'^I\rangle$ by boost

$$\begin{aligned} |1'_I\rangle_\alpha &= |a'_I\rangle_\alpha \\ |3'^I\rangle_\alpha &= \exp\left\{i\lambda(p'_3, p'_a){p'_a}^\mu {p'_3}^\nu \sigma_{\mu\nu}\right\}_\alpha{}^\beta |a'^I\rangle_\beta \end{aligned} \tag{4.42}$$

with the Lorentz generators $\sigma^{\mu\nu}$ in $SL(2,\mathbb{C})$, where the rapidity in [35] is

$$\lambda(p_A, p_B) = \frac{\log\left[\frac{1}{m^2}\left(p_A\cdot p_B + \sqrt{(p_A\cdot p_B)^2 - m^4}\right)\right]}{\sqrt{(p_A\cdot p_B)^2 - m^4}}. \tag{4.43}$$

So far, all objects are related to ${p'_a}^\mu$ at the reference frame, such that $\alpha^I(p'_a)$ contracts with $|1'_I\rangle$ and $\tilde{\alpha}_K(p'_a)$ contracts with $|3'^K\rangle$.

Then, we can start to find the $SU(2)$ spinors which satisfy the above conditions. Assume the $SU(2)$ spinors with the form

$$\begin{aligned} \alpha^I(p'_a) &= \begin{pmatrix} \alpha^1 & \alpha^2 \end{pmatrix}, \\ \tilde{\alpha}_I(p'_a) &= \begin{pmatrix} (\alpha^1)^* & (\alpha^2)^* \end{pmatrix}, \end{aligned} \tag{4.44}$$

where * means the complex conjugate, and then we plug $SU(2)$ spinors and the reference frame ${p'_a}^\mu$ into the definition of the spin vector

$$a^\mu_{p'_a} = \frac{\hbar}{2}\left[\tilde{\alpha}_I(p'_a)[\sigma^\mu_{p'_a}]^I{}_J\alpha^J(p'_a)\right]. \tag{4.45}$$

where we have put $m_{a'} = 1$; the Pauli matrices with $SU(2)$ indices are dependent on the frame p'_a, namely,

$$[\sigma_{p'_a\mu}]^I{}_J = \frac{1}{2}\left(\langle a'^I|\sigma_\mu|a'_J] + [a'^I|\bar{\sigma}_\mu|a'_J\rangle\right). \tag{4.46}$$

Then, we force the spin vector $a^\mu_{p'_a}$ without the components x and y. Moreover, since we set ${p'_a}^\mu$ at the rest frame, the spin vector will be along the z direction.

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Therefore, we can obtain the $SU(2)$ spinors

$$\begin{aligned}\alpha^I(p'_a) &= \sqrt{a}\begin{pmatrix}0 & \kappa + i\sqrt{1-\kappa^2}\end{pmatrix},\\ \tilde{\alpha}_I(p'_a) &= \sqrt{a}\begin{pmatrix}0 & \kappa - i\sqrt{1-\kappa^2}\end{pmatrix},\end{aligned} \tag{4.47}$$

for some $|\kappa| \leq 1$.

Note that in the following discussion we will use the dimensionless spinors, so we rename $|1'^I\rangle$ to $|1^I\rangle$, $|3'^I\rangle$ to $|3^I\rangle$, $\frac{1}{\sqrt{m_1}}|2\rangle = |2'\rangle$ to $|2\rangle$, and so do the square brackets, so that their dimensionless momenta p_1, p_3 satisfy the momentum conservation $m_1 p_1 + m_1 k_2 = m_3 p_3$, which is different from the case of equal masses. And the contractions between the reference frame $SU(2)$ spinors and $|1^I\rangle$, $|3^I\rangle$ are valid, since $|1^I\rangle$, $|3^I\rangle$ can be in terms of $|a'^I\rangle$ from the Lorentz boost.

With the specific form of the $SU(2)$ spinors which are denoted $\tilde{\alpha}_I$ and α^I at the reference frame, we can construct the dimensionless on-shell elements as a basis of the spin-weighted spheroidal harmonics.

According to the previous scattering diagram of Schwarzschild case, now we consider the following process. A massive spin-$(l \oplus s)$ state 3 with mass m_3 emits a graviton with helicity $h = -2$ and reduces to a massive spin-s state 1 with a different mass m_1.

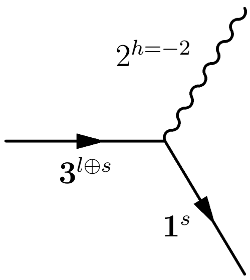

Figure 4.2: 3pt scattering process from spin-$(l \oplus s)$ to spin-s and helicity -2

Unlike the previous Schwarzschild case, leg 1 has no spin, so we only use $\langle 23^I\rangle$ and $[23^I]$ in the scattering amplitude. In this process from Fig.4.2, we now have the

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following dimensionless on-shell elements from this 3pt diagram,

$$\langle 23^I\rangle, [23^I], \langle 21_I\rangle, [21_I] \tag{4.48}$$

where the exchanging terms $[23^J] = -[3^J 2]$ for square and angle brackets, and they can be linearly combined to make

$$\begin{aligned} \langle 3^J 1_I\rangle =& \frac{-m_1}{m_3^2 - m_1^2}\left(m_3[23^J]\langle 21_I\rangle - m_1\langle 23^J\rangle[21_I]\right) \\ [3^J 1_I] =& \frac{-m_1}{m_3^2 - m_1^2}\left(m_3\langle 23^J\rangle[21_I] - m_1[23^J]\langle 21_I\rangle\right) \end{aligned} \tag{4.49}$$

with some coefficients containing m_1, m_3.

To reproduce the spin-weighted spheroidal harmonics with $a\omega$, we then build a basis by the contraction of the $SU(2)$ spinors and the angle, square brackets, so we have the on-shell coherent elements and expand them to the first order of E_2

$$\begin{aligned} \tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I &= -\frac{aE_2(1+X)}{\sqrt{m_1 m_3}} = -(1+X)\frac{aE_2}{m_1} + \mathcal{O}(aE_2^2), \\ \tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I &= -\frac{aE_2(-1+X)}{\sqrt{m_1 m_3}} = (1-X)\frac{aE_2}{m_1} + \mathcal{O}(aE_2^2). \end{aligned} \tag{4.50}$$

and

$$\begin{aligned} \tilde{\alpha}_K\langle 3^K 1_I\rangle\alpha^I &= \frac{a\left[(1+X)m_3 + (1-X)m_1\right]}{2\sqrt{m_1 m_3}} = a + \frac{X}{2}\frac{aE_2}{m_1} + \mathcal{O}(aE_2^2), \\ \tilde{\alpha}_K[3^K 1_I]\alpha^I &= \frac{a\left[(-1+X)m_3 - (1+X)m_1\right]}{2\sqrt{m_1 m_3}} = -a + \frac{X}{2}\frac{aE_2}{m_1} + \mathcal{O}(aE_2^2), \end{aligned} \tag{4.51}$$

where the $\tilde{\alpha}_K$ contract with spinor 3; the α^I contract with spinor 1. Roughly speaking, we can use $\tilde{\alpha}_K\langle 3^K 1_I\rangle\alpha^I$ and coherent spin state to obtain $e^{a\omega X}$,

$$e^{-||\alpha||^2} e^{\tilde{\alpha}_K\langle 3^K 1_I\rangle\alpha^I} \to e^{\frac{X}{2}\frac{aE_2}{m_1}} \tag{4.52}$$

in classical limit, which is inspired by the nontrivial exponential of Leaver's ansatz in (2.42), hence we can identify $\frac{aE_2}{2m_1}$ as $a\omega$. So far, we obtain the objects with the pure a, and can be used them to cancel the $e^{-||\alpha||^2}$ in the coherent spin states, such that

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a must appear together with E_2, just like the angular differential equation (2.23) in terms of $a\omega$.

Note that the classical spin parameter $||\alpha||^2 = a$ is dimensionless because we set $\hbar$ (angular momentum) to be dimensionless. Actually, the spin parameter has the following $\hbar$ counting

$$a \to \frac{a}{\hbar} \tag{4.53}$$

and the angular frequency of graviton is

$$\frac{E_2}{2m_1} \to \hbar\omega. \tag{4.54}$$

Therefore, taking the classical limit $\hbar \to 0$, we know that $a \to \infty$ and $E_2 \to 0$, respectively, but the product aE_2 is fixed.

Next, by above on-shell coherent elements, we establish an on-shell coherent tensor with the following form from (4.24) and with all possible combination of the elements

$$\mathcal{A}^{-2,(J_1\ldots J_{2l})} = e^{-||\alpha||^2} \sum_{n=0}^{\infty} c_n(\tilde{\alpha}_K)^n \mathcal{M}^{-2,(J_1\ldots J_{2l}),K_1\ldots K_n}{}_{I_1\ldots I_n}(\alpha^I)^n \tag{4.55}$$

with $2l$ fully symmetric free little group indices carried by leg 3, where the piece $\mathcal{M}^{-2,(J_1\ldots J_{2l}),K_1\ldots K_n}{}_{I_1\ldots I_n}$ in the sandwich is composed of $\langle 23^I\rangle$, $[23^I]$, $\langle 21_I\rangle$, and $[21_I]$ and satisfies the helicity counting $h = -2$. We expect the on-shell coherent tensor can reproduce the spin-weighted spheroidal harmonics

$$\mathcal{A}^{-2,(J_1\ldots J_{2l})} \sim {}_{-2}S_{lm}(a\omega, \theta). \tag{4.56}$$

Moreover, we find undetermined coefficients by truncating the coherent tensor $\mathcal{A}^{-2,(J_1\ldots J_{2l})}$ at a certain order of $a\omega$ and comparing with the spin-weighted spheroidal harmonics of perturbation theory (2.41).

For general l, there are $2l$ free little group indices in each order of $a\omega$, that is, the spin configurations correspond to $2l+1$ different m. To reproduce the spin-weighted

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spheroidal harmonics for each order, we need the coherent tensor from Fig.4.2 to satisfy

(1) helicity $h = -2$,

(2) $2l$ free $SU(2)$ indices in leg 3, and

(3) the number of α^I, $\tilde{\alpha}_I$ are the same and the α^I, $\tilde{\alpha}_I$ contraction rule for the leg 1, leg 3 indices, respectively.

Therefore, we truncate the spin-weighted spheroidal harmonics expansion at some order, and then construct a basis order by order to match the function. For the zeroth order $(a\omega)^0$, there is no spin parameter, so there is also no any α^I and $\tilde{\alpha}_I$, that means we only have one structure at this level,

$$\langle 23^{(J}\rangle^{l+2}[23^{J)}]^{l-2} \sim \left(\frac{E_2}{m_1}\right)^l {}_{-2}Y_{lm}(\theta) \tag{4.57}$$

which is the same as the Schwarzschild case (3.25). Note that the classical limit $\hbar \to 0$ means that we keep the order

$$\left(\frac{E_2}{m_1}\right)^l \left(\frac{aE_2}{2m_1}\right)^n \tag{4.58}$$

for $n = 0, 1, 2, ...$ with some fixed l, where the piece $\frac{aE_2}{2m_1}$ is the series expansion parameter $a\omega$. If we turn off the spin of the black hole, then the higher order will go to vanish, such that only the zeroth order term survives and reduces to Schwarzschild QNMs. Take $l = 3$ as an example, the zeroth order $(a\omega)^0$ of on-shell coherent tensor can be written as

$$S^{(0)}_{3m} = c_{0,1,0,0}\langle 23^{(J}\rangle^5[23^{J)}] \tag{4.59}$$

where the spin-weighted spheroidal harmonics expansion $Ne^{-a\omega\cos\theta}{}_{-2}S_{lm}(a\omega, \theta) = g_l^{-2}\sum_{n=0} S^{(n)}_{lm}(a\omega, \theta)$ where n means the order of $a\omega$ and N is a proportional constant of the spin-weighted spherical harmonics and the amplitude in Schwarzschild case in (3.25), such that $c_{0,1,0,0} = 1$ for each m. Notice that, the notation $S^{(n)}_{lm}$ and (2.28) are a little bit different.

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Therefore, the correspondence between the on-shell coherent tensor and the spin-weighted spheroidal harmonics is

$$\boxed{\mathcal{A}^{-2,(J_1\dots J_{2l})} = g_l^{-2}\left(\frac{2E_2}{m_1}\right)^l \frac{1}{(2l)!}\sqrt{\frac{4\pi}{2l+1}}\sqrt{(l+2)!(l-2)!(l+m)!(l-m)!}\,{}_{-2}S_{lm}(a\omega,\theta)} \tag{4.60}$$

from (3.25) in the previous section.

Next, before we construct the basis which we need for the next order, we check that

$$\cos\theta\,{}_{-2}Y_{lm}(\theta) = \#\,{}_{-2}Y_{l-1,m}(\theta) + \#\,{}_{-2}Y_{lm}(\theta) + \#\,{}_{-2}Y_{l+1,m}(\theta) \tag{4.61}$$

can give us the spin-weighted spherical harmonics which involve $l-1$, l, and $l+1$ as a basis, and that is enough to describe the first order spheroidal harmonics, because the first order only involves ${}_{-2}Y_{l-1,m}$, ${}_{-2}Y_{lm}$, and ${}_{-2}Y_{l+1,m}$ in (2.41).

Therefore, for the first order $(a\omega)^1$, we can use the "old structures" from the zeroth order multiplied by $(1+X)a\omega$ and $(1-X)a\omega$ in (4.50) in the classical limit, and there are two "new structures" which follow the above requirements, and the "new structures" means that the $SU(2)$ spinor α^I only contracts with the single side of $\langle 3^J 1_I\rangle$ with free little group index J,

$$\begin{aligned}&\langle 23^J\rangle^{l+1}[23^J]^{l-2}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\langle 3^J 1_I\rangle\alpha^I\right)(k_2\cdot p_3)\\&\langle 23^J\rangle^{l+2}[23^J]^{l-3}\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^J 1_I\rangle\alpha^I\right)(k_2\cdot p_3)\end{aligned} \tag{4.62}$$

with $2l$ fully symmetric free $SU(2)$ indices J on leg 3, where $\left(\langle 3^J 1_I\rangle\alpha^I\right)$ and $\left([3^J 1_I]\alpha^I\right)$ will give us the same contribution, so we just need one of them; the piece $(k_2\cdot p_3) = \langle 23_J\rangle[3^J 2]$ is used to make up for the power of E_2. Therefore, at this level, there are four possible structures as a basis for the first order spheroidal harmonics. For

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example, the first order $(a\omega)$ of on-shell coherent tensor is

$$
\begin{aligned}
S_{3m}^{(1)} =& c_{0,1,1,0}\langle 23^{(J}\rangle^5[23^{J)}](\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I) \\
&+ c_{0,1,0,1}\langle 23^{(J}\rangle^5[23^{J)}](\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I) \\
&+ c_{1,1,0,0}\langle 23^{(J}\rangle^4[23^J]\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\langle 3^{J)}1_I\rangle\alpha^I\right)(k_2\cdot p_3) \\
&+ c_{1,2,0,0}\langle 23^{(J}\rangle^5\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^{J)}1_I\rangle\alpha^I\right)(k_2\cdot p_3)
\end{aligned}
\tag{4.63}
$$

which is composed of the two old structures (black word) and two new structures (blue word) with the coefficients $c_{0,1,0,1} = \frac{1}{16}$, $c_{0,1,1,0} = \frac{3}{16}$, $c_{1,1,0,0} = \frac{25}{144}$, and $c_{1,2,0,0} = -\frac{7}{144}$ for matching each m.

For the second order $(a\omega)^2$, again, we use the "old structures" from the first order multiplied by $\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I$ and $\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I$ and there are three "new structures",

$$
\begin{aligned}
&\langle 23^J\rangle^l[23^J]^{l-2}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)^2\left(\langle 3^J1_I\rangle\alpha^I\right)^2(k_2\cdot p_3)^2 \\
&\langle 23^J\rangle^{l+1}[23^J]^{l-3}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^J1_I\rangle\alpha^I\right)^2(k_2\cdot p_3)^2 \\
&\langle 23^J\rangle^{l+2}[23^J]^{l-4}\left(\tilde{\alpha}_K[23^K]\right)^2\left(\langle 3^J1_I\rangle\alpha^I\right)^2(k_2\cdot p_3)^2
\end{aligned}
\tag{4.64}
$$

Note that, if l is not large enough, for example $l = 3$ here, there will be only two, rather than three new structures. In this order, some redundant structures start to appear, that is to say, some of which are linearly dependent in the classical limit. For instance, the second new term in (4.64) is related to

$$
\langle 23^J\rangle^{l+1}[23^J]^{l-2}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\langle 3^J1_I\rangle\alpha^I\right)(k_2\cdot p_3)\times\left(\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I\right) \tag{4.65}
$$

and

$$
\langle 23^J\rangle^{l+2}[23^J]^{l-3}\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^J1_I\rangle\alpha^I\right)(k_2\cdot p_3)\times\left(\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I\right) \tag{4.66}
$$

constructed from the previous order. In the next order, this kind of redundant structures will be more. The $l = 3$ example with higher order is placed in the

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Appendix B.

For the third order $(a\omega)^3$, the "new structures" include the following four

$$\begin{aligned}
&\langle 23^J\rangle^{l-1}[23^J]^{l-2}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)^3\left(\langle 3^J 1_I\rangle\alpha^I\right)^3(k_2\cdot p_3)^3\\
&\langle 23^J\rangle^{l}[23^J]^{l-3}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)^2\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^J 1_I\rangle\alpha^I\right)^3(k_2\cdot p_3)^3\\
&\langle 23^J\rangle^{l+1}[23^J]^{l-4}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\tilde{\alpha}_K[23^K]\right)^2\left(\langle 3^J 1_I\rangle\alpha^I\right)^3(k_2\cdot p_3)^3\\
&\langle 23^J\rangle^{l+2}[23^J]^{l-5}\left(\tilde{\alpha}_K[23^K]\right)^3\left(\langle 3^J 1_I\rangle\alpha^I\right)^3(k_2\cdot p_3)^3,
\end{aligned}\tag{4.67}$$

but there also exist the redundant structures. Here, the second and the third structures are linearly dependent on different previous structures, respectively.

Up to an arbitrary n-th order $(a\omega)^n$, similarly, we can construct some "new structures", but we only keep the first and the last new structures,

$$\begin{aligned}
&\langle 23^J\rangle^{l+2-n}[23^J]^{l-2}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)^n\left(\langle 3^J 1_I\rangle\alpha^I\right)^n(k_2\cdot p_3)^n\\
&\langle 23^J\rangle^{l+2}[23^J]^{l-2-n}\left(\tilde{\alpha}_K[23^K]\right)^n\left(\langle 3^J 1_I\rangle\alpha^I\right)^n(k_2\cdot p_3)^n,
\end{aligned}\tag{4.68}$$

since those middle new structures can always be linearly combined by the old structures which multiplied by $(\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I)^i$ and $(\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I)^j$. Furthermore, if the power is $l-2-n<0$, then this kind of new structure will be not allowed, so there will be only one new structure at this level.

So, the ansatz of the on-shell coherent tensor which describes the spin-weighted spheroidal harmonics can be expressed as the following form,

$$\begin{aligned}
\mathcal{A}^{-2,(J_1\ldots J_{2l})} = {}& e^{-a}e^{\tilde{\alpha}_K\langle 3^K 1_I\rangle\alpha^I}\\
&\times g_l^{-2}\sum_{n=0}^{\infty}\sum_{i,j=0}^{\infty}\Big\{c_{n,1,i,j}\langle 23^J\rangle^{l+2-n}[23^J]^{l-2}\left(\tilde{\alpha}_K\langle 23^K\rangle\right)^n\left(\langle 3^J 1_I\rangle\alpha^I\right)^n(k_2\cdot p_3)^n\\
&+c_{n,2,i,j}\langle 23^J\rangle^{l+2}[23^J]^{l-2-n}\left(\tilde{\alpha}_K[23^K]\right)^n\left(\langle 3^J 1_I\rangle\alpha^I\right)^n(k_2\cdot p_3)^n\Big\}\\
&\times(\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I)^i(\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I)^j
\end{aligned}\tag{4.69}$$

where the point is that a series of coefficients $c_{n,1,i,j}$ and $c_{n,2,i,j}$ for the first and the last new structures are independent of the spin configurations which are only reflected

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on the little group indices $(J_1...J_{2l})$, otherwise, the coefficients corresponding to one spin configuration will become meaningless in another spin configuration, that does not make sense. Here, we use our ansatz to match the spheroidal harmonics expansion $g_l^{-2}\sum_{n=0} S_{lm}^{(n)}(a\omega,\theta)$ order by order, where n means the order of $a\omega$, and the zeroth order coefficient $c_{0,1,0,0}=1$ for all different m. Note that, when $n=0$, the two terms with $c_{n,1,i,j}$ and $c_{n,2,i,j}$ are the same, so we just use $c_{0,1,i,j}$ term.

So far, by the coherent spin states sandwiching the on-shell elements with unequal masses from the spinor-helicity formalism and our setup of the on-shell spinors, we construct a set of bases to reproduce the Kerr black hole QNMs with spin a in the classical limit, since the spin of a Kerr black hole that is a classical spin can be approximatively described by the coherent spin states, as we explained in Section 4.1. However, unlike the Schwarzschild case with unique match between on-shell amplitudes and the angular dependence of QNMs, there are now some redundant structures for Kerr QNMs on this basis from the on-shell coherent tensors, when the order of $a\omega$ in the ${}_{-2}S_{lm}(a\omega,\theta)$ expansion is large. Finally, we give an example in Appendix B about how to specifically match the on-shell coherent tensors $\mathcal{A}^{-2,(J_1...J_{2l})}$ and the spin-weighted spheroidal harmonics ${}_{-2}S_{lm}(a\omega,\theta)$ order by order.

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Chapter 5

Discussion and Conclusion

In summary, we study the application of the 3pt unequal masses scattering process with the on-shell spinor-helicity formalism [29–31] in describing the angular dependence of the black hole quasinormal modes. Not only the Schwarzschild metric perturbation but also the Teukolsky equation from the Kerr tetrads perturbation, which involve the black hole QNMs, are Lorentz invariant. Therefore, the angular dependence must be able to be expressed by the on-shell amplitude with the spinor-helicity formalism under a suitable setup of the on-shell spinors. Based on the quantum numbers in the angular function, we conjecture a 3pt tree-level Feynman diagram in Fig.1.1, which corresponds to the scattering process with unequal masses particles and a graviton emission. We set up the on-shell momenta, where the angular dependence corresponds to the angular coordinates in the spherically and non-spherically symmetric black hole metric. By decomposing the 2 by 2 momenta of $SL(2,\mathbb{C})$ of the on-shell spinors, we can use the spin configurations of the $SU(2)$ fully symmetric little group indices associated with massive spinors to describe the spin-weighted spherical harmonics and spin-weighted spheroidal harmonics, which belong to the $2l+1$-dimensional $SO(3)$ representation.

In Chapter 3, by the spinor-helicity formalism, we compute the on-shell scattering amplitudes $\mathcal{M}^{\pm 2,(J_1 \ldots J_{2l})}$ in (3.23), (3.27) of the transition process from a spin-l black hole to a spinless black hole, accompanied by the emission of a graviton with helicity $h = \pm 2$. Then, we observe that the angular part of the amplitudes are

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exactly the same as the angular dependence of the Schwarzschild QNMs, known as the spin-weighted spherical harmonics with the spin weight ± 2. By using the spin configurations of $SU(2)$ fully symmetric little group indices $(J_1 ... J_{2l})$, we can represent all different quantum numbers m from $-l$ to l, and this is an one-to-one correspondence. Additionally, if the massless particle emitted in this transition process is a photon or scalar, the scattering amplitudes $\mathcal{M}^{h,(J_1 ... J_{2l})}$ in (3.28) will correspond to the spin-weighted spherical harmonics with the spin weight $h = \pm 1$ or the common spherical harmonics in (3.29), respectively.

In Chapter 4, our research progresses to include two spinning black holes, and we successfully utilize the spinor-helicity formalism to construct some possible on-shell elements, as well as combine the coherent spin states to describe the angular dependence of the Kerr black hole QNMs, that is, the spin-weighted spheroidal harmonics. Initially, we employ the coherent spin state which can reproduce the classical spin of the Kerr black holes from [32], and then contract the $SU(2)$ spinors with the on-shell elements. The on-shell elements are from the unequal masses 3pt scattering process in Fig.4.2 involving a transition from a spin-$(l \oplus s)$ black hole to a spin-s black hole and a helicity $h = -2$ graviton emission. Then, by the on-shell elements and the $SU(2)$ spinors, we construct the on-shell coherent elements which satisfy the helicity counting $h = -2$, furthermore, we construct an on-shell coherent tensor $\mathcal{A}^{-2,(J_1 ... J_{2l})}$ in (4.69) with $2l$ fully symmetric little group indices $(J_1 ... J_{2l})$. Consequently, in the classical limit, by the $2l + 1$ spin configurations, we can use the on-shell coherent tensors and a series of coefficients which are independent of the spin configurations to match different m in the spin-weighted spheroidal harmonics. But there exist some redundant structures, so it is not a unique mapping.

Our next work will be to derive the recursive relations for these coefficients and further extend them to represent the angular differential equation by using the on-shell formalism. This involves rewriting the differential operators with respect to $\cos\theta$ in terms of differentials with respect to spinors, aiming to obtain a new expression in the on-shell spinor-helicity formalism. We also expect to obtain some new

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insights from this new expression which differ from the previously obtained physical information about QNMs from General Relativity. For example, we hope to understand the significance of the quantum number l in the oblate spheroidal coordinates, as it does not correspond to the eigenvalue of the total angular momentum operator as it does in the spherical coordinates.

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Appendix A

Review of Coherent State

We need an operator acting on the $SU(2)$ irreducible representation, hence, we start from the N-dimensional harmonic oscillator in QM, the Hamiltonian is

$$\hat{H} = \hbar\omega \sum_{i=1}^{N} \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right) \tag{A.1}$$

where $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the creation and annihilation operators

$$\begin{aligned} \hat{a}_i^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(x_i - \frac{i}{m\omega} p_i \right) \\ \hat{a}_i &= \sqrt{\frac{m\omega}{2\hbar}} \left(x_i + \frac{i}{m\omega} p_i \right) \end{aligned} \tag{A.2}$$

for $i = 1, 2, ..., N$ and satisfy the commutation relation $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, since the Hamiltonian has $SU(N)$ symmetry, namely, the N-dimensional harmonic oscillator Hamiltonian is invariant under $SU(N)$ transformation. And we will use the $SU(2)$ representation in Section 4.1.

Based on the angular momentum operators (4.2), they try to find a coherent spin state $|\alpha\rangle$ expression, such that the expectation value $\langle\alpha|S^i|\alpha\rangle$ can represent the classical spin. Here, before talking about the coherent spin state, let us review the coherent state in 2-dimensional harmonic oscillator. We can see the eigenstate of

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the Hamiltonian first,

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!}}\frac{1}{\sqrt{n_2!}}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}|0,0\rangle \tag{A.3}$$

which satisfies the eigenvalue equation

$$\hat{H}|n_1, n_2\rangle = \hbar\omega\left(n_1 + n_2 + 1\right)|n_1, n_2\rangle. \tag{A.4}$$

Then, in general, the uncertainty between position and momentum is given by

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2} \tag{A.5}$$

where the definition of the standard deviations is

$$\Delta\mathcal{O}_i = \sqrt{\langle n_1, n_2|\mathcal{O}_i^2|n_1, n_2\rangle - \langle n_1, n_2|\mathcal{O}_i|n_1, n_2\rangle^2} \tag{A.6}$$

for $i = 1, 2$ by an arbitrary state $|n_1, n_2\rangle$, because we can compute the expectation values of position and momentum

$$\begin{aligned}
\langle n_1, n_2|x_i|n_1, n_2\rangle &= 0 \\
\langle n_1, n_2|x_i^2|n_1, n_2\rangle &= \frac{\hbar}{m\omega}\left(n_i + \frac{1}{2}\right) \\
\langle n_1, n_2|p_i|n_1, n_2\rangle &= 0 \\
\langle n_1, n_2|p_i^2|n_1, n_2\rangle &= \hbar m\omega\left(n_i + \frac{1}{2}\right),
\end{aligned} \tag{A.7}$$

and then compute the standard deviation of them

$$\begin{aligned}
\Delta x_i &= \sqrt{\frac{\hbar}{m\omega}\left(n_i + \frac{1}{2}\right)} \\
\Delta p_i &= \sqrt{\hbar m\omega\left(n_i + \frac{1}{2}\right)}
\end{aligned} \tag{A.8}$$

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Therefore, the uncertainty of position and momentum is

$$\Delta x_i \Delta p_i = \frac{\hbar}{2}(2n_i + 1) \geq \frac{\hbar}{2} \tag{A.9}$$

for $n_i = 0, 1, 2, ...$

Next, in order to minimize the uncertainty, the coherent state of the 2-dimensional harmonics oscillator is expressed as

$$|\alpha\rangle = e^{-\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)} \sum_{n_1,n_2=0}^{\infty} \frac{(\alpha_1)^{n_1}(\alpha_2)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}} |n_1, n_2\rangle = e^{-|\alpha|^2/2} e^{\alpha_1 \hat{a}_1^\dagger} e^{\alpha_2 \hat{a}_2^\dagger} |0, 0\rangle \tag{A.10}$$

which satisfies the eigenvalue equations of the annihilation operators

$$\begin{aligned} \hat{a}_1|\alpha\rangle &= e^{-\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)} \sum_{n_1,n_2=0}^{\infty} \frac{(\alpha_1)^{n_1}(\alpha_2)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}} \sqrt{n_1} |n_1 - 1, n_2\rangle = \alpha_1 |\alpha\rangle \\ \hat{a}_2|\alpha\rangle &= e^{-\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)} \sum_{n_1,n_2=0}^{\infty} \frac{(\alpha_1)^{n_1}(\alpha_2)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}} \sqrt{n_1} |n_1, n_2 - 1\rangle = \alpha_2 |\alpha\rangle, \end{aligned} \tag{A.11}$$

in other words, that is $\hat{a}_i|\alpha\rangle = \alpha_i|\alpha\rangle$ for $i = 1, 2$.

After having this coherent state, we can verify that it minimizes the uncertainty of position and momentum. We also start by calculating the expectation values of position and momentum

$$\begin{aligned} \langle\alpha|x_i|\alpha\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(\alpha_i^* + \alpha_i) \\ \langle\alpha|x_i^2|\alpha\rangle &= \frac{\hbar}{2m\omega}(\alpha_i^{*2} + \alpha_i^2 + 2\alpha_i^*\alpha_i + 1) \\ \langle\alpha|p_i|\alpha\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}(\alpha_i^* - \alpha_i) \\ \langle\alpha|p_i^2|\alpha\rangle &= -\frac{\hbar m\omega}{2}(\alpha_i^{*2} + \alpha_i^2 - 2\alpha_i^*\alpha_i - 1) \end{aligned} \tag{A.12}$$

by the coherent state and the eigenvalue equation of the annihilation operators.

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Then, we can discover the coherent state leads to the standard deviations

$$\begin{aligned}\Delta x_i =& \sqrt{\langle\alpha|x_i^2|\alpha\rangle - \langle\alpha|x_i|\alpha\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \\ \Delta p_i =& \sqrt{\langle\alpha|p_i^2|\alpha\rangle - \langle\alpha|p_i|\alpha\rangle^2} = \sqrt{\frac{\hbar m\omega}{2}}\end{aligned} \tag{A.13}$$

be some constants, and the product exactly satisfies the equation $\Delta x_i \Delta p_i = \frac{\hbar}{2}$ from the uncertainty of position and momentum.

So far, we verify the coherent state minimizes the uncertainty of position and momentum, which are expressed as the creation and annihilation operators acting on $SU(2)$ representation. Hence, in Section 4.1, we use the coherent spin state to minimize the uncertainty of the expectation values of angular momentum operators, which are also expressed as the creation and annihilation operators acting on $SU(2)$ spin states.

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Appendix B

Example of Coherent Tensors

Give a non-trivial concrete example. For $l = 3$, there are 6 free and fully symmetric little group indices which represent m from -3 to 3. The $SU(2)$ spin configurations $(J_1, J_2, J_3, J_4, J_5, J_6)$ which are symmetric reproduce the 7 different m by

$$\begin{aligned}
(\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow) \quad &\Rightarrow \quad m = 3 \\
(\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow) \quad &\Rightarrow \quad m = 2 \\
(\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow) \quad &\Rightarrow \quad m = 1 \\
(\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow) \quad &\Rightarrow \quad m = 0 \\
(\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow) \quad &\Rightarrow \quad m = -1 \\
(\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow) \quad &\Rightarrow \quad m = -2 \\
(\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow) \quad &\Rightarrow \quad m = -3
\end{aligned} \tag{B.1}$$

carried by the leg 3. Then, we check the on-shell coherent tensor (4.69) can reproduce the spin-weighted spheroidal harmonics in (2.41)

$$\mathcal{A}^{-2,(J_1,J_2,J_3,J_4,J_5,J_6)} = N\,{}_{-2}S_{3m}(a\omega, \theta) \tag{B.2}$$

where

$$N = g_l^{-2} \left(\frac{2E_2}{m_1}\right)^l \frac{1}{(2l)!} \sqrt{\frac{4\pi}{2l+1}} \sqrt{(l+2)!(l-2)!(l+m)!(l-m)!}\Bigg|_{l=3} \tag{B.3}$$

by expanding both sides and truncating them at some order.

For the zeroth order $(a\omega)^0$, on the one hand, the spin-weighted spheroidal harmonics are shown as

$$\begin{aligned}
S^{(0)}_{3,3} =& -\left(\frac{E_2}{m_1}\right)^3 \sqrt{1-X}(1+X)^{5/2} \\
S^{(0)}_{3,2} =& \frac{1}{3}\left(\frac{E_2}{m_1}\right)^3 (1+X)^2(3X-2) \\
S^{(0)}_{3,1} =& \frac{1}{3}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1-X^2}\left(3X^2+2X-1\right) \\
S^{(0)}_{3,0} =& \left(\frac{E_2}{m_1}\right)^3 X\left(1-X^2\right) \\
S^{(0)}_{3,-1} =& \frac{1}{3}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1-X^2}\left(-3X^2+2X+1\right) \\
S^{(0)}_{3,-2} =& \frac{1}{3}\left(\frac{E_2}{m_1}\right)^3 (1-X)^2(3X+2) \\
S^{(0)}_{3,-3} =& \left(\frac{E_2}{m_1}\right)^3 \sqrt{1+X}(1-X)^{5/2},
\end{aligned} \tag{B.4}$$

for different m, where $S^{(0)}_{3m}$ is from $g_l^{-2}\sum_{n=0} S^{(n)}_{3m}(a\omega,\theta)$, and the variable X means $\cos\theta$. On the other hand, the zeroth order of the on-shell coherent tensor in (3.25) or (4.69) is shown as

$$g_l^{-2} S^{(0)}_{3m} = N_{-2}Y_{3m} = g_l^{-2} c_{0,1,0,0}\langle 23^J\rangle^5[23^J] \tag{B.5}$$

in right hand side with the coefficient $c_{0,1,0,0} = 1$ is actually the Schwarzschild case.

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For the first order $(a\omega)^1$, the spin-weighted spheroidal harmonics expansion $S^{(1)}_{3m}$ which contain the spin of the black hole are shown as

$$\begin{aligned}
S^{(1)}_{3,3} =& \frac{1}{4}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1-X}(1+X)^{5/2}(2X+1)(a\omega)\\
S^{(1)}_{3,2} =& \frac{1}{54}\left(\frac{E_2}{m_1}\right)^3 (1+X)^2\left(-27X^2+9X+1\right)(a\omega)\\
S^{(1)}_{3,1} =& -\frac{1}{108}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1-X}(1+X)^{3/2}\left(54X^2-9X+13\right)(a\omega)\\
S^{(1)}_{3,0} =& \frac{1}{6}\left(\frac{E_2}{m_1}\right)^3 \left(3X^4-2X^2-1\right)(a\omega)\\
S^{(1)}_{3,-1} =& \frac{1}{108}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1+X}(1-X)^{3/2}\left(54X^2+9X+13\right)(a\omega)\\
S^{(1)}_{3,-2} =& -\frac{1}{54}\left(\frac{E_2}{m_1}\right)^3 (1-X)^2\left(27X^2+9X-1\right)(a\omega)\\
S^{(1)}_{3,-3} =& -\frac{1}{4}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1+X}(1-X)^{5/2}(2X-1)(a\omega)
\end{aligned} \tag{B.6}$$

whose $a\omega$ means $\frac{aE_2}{2m_1}$ in our language; the 7 projection states with different m can be obtained by our ansatz (4.69) expansion involving one $\tilde{\alpha}_K$ and one α^I, shown as

$$\begin{aligned}
S^{(1)}_{3m} =& c_{0,1,1,0}\langle 23^J\rangle^5[23^J](\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I)\\
&+ c_{0,1,0,1}\langle 23^J\rangle^5[23^J](\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I)\\
&+ c_{1,1,0,0}\langle 23^J\rangle^4[23^J]\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\langle 3^J 1_I\rangle\alpha^I\right)(k_2\cdot p_3)\\
&+ c_{1,2,0,0}\langle 23^J\rangle^5\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^J 1_I\rangle\alpha^I\right)(k_2\cdot p_3).
\end{aligned} \tag{B.7}$$

which is composed of the two old structures (black word) and two new structures (blue word) with the coefficients $c_{0,1,0,1} = \frac{1}{16}$, $c_{0,1,1,0} = \frac{3}{16}$, $c_{1,1,0,0} = \frac{25}{144}$, and $c_{1,2,0,0} = -\frac{7}{144}$ for matching all spin configurations. If we only consider one of the spin configurations, then there will be some free coefficients, such as we only fixed two or three of them

$c_{0,1,0,1} = \frac{1}{16}$, and $c_{0,1,1,0} = \frac{3}{16}$ for $m = 3$;

$c_{0,1,1,0} = \frac{1}{4} - c_{0,1,0,1}$, $c_{1,1,0,0} = \frac{35}{72} - 5c_{0,1,0,1}$, and $c_{1,2,0,0} = \frac{1}{72} - c_{0,1,0,1}$ for $m = 2$;

$c_{0,1,1,0} = \frac{1}{4} - c_{0,1,0,1}$, $c_{1,1,0,0} = \frac{95}{288} - \frac{5}{2}c_{0,1,0,1}$, and $c_{1,2,0,0} = -\frac{1}{2}c_{0,1,0,1} - \frac{5}{288}$ for $m = 1$;

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$c_{0,1,1,0} = \frac{1}{4} - c_{0,1,0,1}$, $c_{1,1,0,0} = \frac{5}{18} - \frac{5}{3}c_{0,1,0,1}$, and $c_{1,2,0,0} = -\frac{1}{3}c_{0,1,0,1} - \frac{1}{36}$ for $m = 0$;

$c_{0,1,1,0} = \frac{1}{4} - c_{0,1,0,1}$, $c_{1,1,0,0} = \frac{145}{576} - \frac{5}{4}c_{0,1,0,1}$, and $c_{1,2,0,0} = -\frac{1}{4}c_{0,1,0,1} - \frac{19}{576}$ for $m = -1$;

$c_{0,1,1,0} = \frac{1}{4} - c_{0,1,0,1}$, $c_{1,1,0,0} = \frac{17}{72} - c_{0,1,0,1}$, and $c_{1,2,0,0} = -\frac{1}{5}c_{0,1,0,1} - \frac{13}{360}$ for $m = -2$;

$c_{0,1,1,0} = \frac{1}{4} - c_{0,1,0,1}$, and $c_{1,2,0,0} = \frac{3}{16} - c_{0,1,0,1} - c_{1,1,0,0}$ for $m = -3$,

and their intersection is as we mentioned above.

Next, for the second order $(a\omega)^2$, the expansion $S^{(2)}_{3m}$ are shown as

$$\begin{aligned}
S^{(2)}_{3,3} &= -\frac{1}{288}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1-X}(1+X)^{5/2}\left(48X^2+48X-7\right)(a\omega)^2 \\
S^{(2)}_{3,2} &= \frac{1}{1944}\left(\frac{E_2}{m_1}\right)^3 (1+X)^2\left(324X^3-105X+74\right)(a\omega)^2 \\
S^{(2)}_{3,1} &= \frac{1}{7776}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1-X}(1+X)^{3/2}\left(1296X^3+219X+199\right)(a\omega)^2 \\
S^{(2)}_{3,0} &= \frac{1}{18}\left(\frac{E_2}{m_1}\right)^3 X\left(-3X^4+2X^2+1\right)(a\omega)^2 \\
S^{(2)}_{3,-1} &= \frac{1}{7776}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1+X}(1-X)^{3/2}\left(1296X^3+219X-199\right)(a\omega)^2 \\
S^{(2)}_{3,-2} &= \frac{1}{1944}\left(\frac{E_2}{m_1}\right)^3 (1-X)^2\left(324X^3-105X-74\right)(a\omega)^2 \\
S^{(2)}_{3,-3} &= -\frac{1}{288}\left(\frac{E_2}{m_1}\right)^3 \sqrt{1+X}(1-X)^{5/2}\left(48X^2-48X-7\right)(a\omega)^2
\end{aligned} \tag{B.8}$$

correspond to

$$\begin{aligned}
S^{(2)}_{3m} =& c_{0,1,2,0}\langle 23^J\rangle^5[23^J](\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I)^2 \\
&+ c_{0,1,1,1}\langle 23^J\rangle^5[23^J](\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I)(\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I) \\
&+ c_{0,1,0,2}\langle 23^J\rangle^5[23^J](\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I)^2 \\
&+ c_{1,1,1,0}\langle 23^J\rangle^4[23^J]\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\langle 3^J1_I\rangle\alpha^I\right)(k_2\cdot p_3)(\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I) \\
&+ c_{1,1,0,1}\langle 23^J\rangle^4[23^J]\left(\tilde{\alpha}_K\langle 23^K\rangle\right)\left(\langle 3^J1_I\rangle\alpha^I\right)(k_2\cdot p_3)(\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I) \\
&+ c_{1,2,1,0}\langle 23^J\rangle^5\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^J1_I\rangle\alpha^I\right)(k_2\cdot p_3)(\tilde{\alpha}_K[23^K]\langle 21_I\rangle\alpha^I) \\
&+ c_{1,2,0,1}\langle 23^J\rangle^5\left(\tilde{\alpha}_K[23^K]\right)\left(\langle 3^J1_I\rangle\alpha^I\right)(k_2\cdot p_3)(\tilde{\alpha}_K\langle 23^K\rangle[21_I]\alpha^I) \\
&+ c_{2,1,0,0}\langle 23^J\rangle^3[23^J]\left(\tilde{\alpha}_K\langle 23^K\rangle\right)^2\left(\langle 3^J1_I\rangle\alpha^I\right)^2(k_2\cdot p_3)^2.
\end{aligned} \tag{B.9}$$

which is composed of the seven old structures (black and blue word) and one new

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structure (red word) with $c_{0,1,0,2} = -\frac{7}{4608}$, $c_{0,1,1,1} = \frac{55}{2304}$, $c_{0,1,2,0} = \frac{89}{1608}$, $c_{1,1,0,1} = \frac{895}{41472}$, $c_{1,2,0,1} = -\frac{371}{41472}$, $c_{1,2,1,0} = -\frac{371}{41472}$, $c_{2,1,0,0} = \frac{85}{10368}$, and $c_{1,1,1,0} = \frac{175}{4608}$ for matching all spin configurations. Similarly, there are some undetermined coefficients in each spin configuration, but there exists a unique solution by solving all m together.

As usual, we compare the on-shell coherent tensors with the expansion of the spin-weighted spheroidal harmonics, so that we can determine the coefficients $c_{n,1,i,j}$ and $c_{n,2,i,j}$ for the third, fourth, and fifth order uniquely. However, up to the sixth order $(a\omega)^6$, we discover the corresponding coefficients
$c_{0,1,0,6} = \frac{33074183081}{500206688796672000}$, $c_{0,1,1,5} = -\frac{13718649689}{83367781466112000}$, $c_{0,1,2,4} = -\frac{180237118483}{166735562932224000}$, $c_{0,1,3,3} = \frac{236937040931}{125051672199168000}$, $c_{0,1,4,2} = \frac{486925201549}{166735562932224000}$, $c_{0,1,5,1} = \frac{143690607719}{83367781466112000}$, $c_{0,1,6,0} = \frac{1266677777}{4133939576832000}$, $c_{1,1,0,5} = -\frac{7834430112683}{24310045075518259200}$, $c_{1,1,2,3} = \frac{15445406428537}{2431004507551825920}$, $c_{1,1,3,2} = \frac{127184727812761}{12155022537759129600}$, $c_{1,1,4,1} = \frac{5726939656829}{1870003467347558400}$, $c_{1,1,5,0} = \frac{16928742866813}{24310045075518259200}$, $c_{1,2,0,5} = -c_{1,1,1,4} - \frac{15868912353539}{15193778172198912000}$, $c_{1,2,1,4} = -\frac{12168990668251}{40516741792530432000}$, $c_{1,2,2,3} = -\frac{627153474019}{810334835850608640}$, $c_{1,2,3,2} = -\frac{627153474019}{810334835850608640}$, $c_{1,2,4,1} = -\frac{12168990668251}{40516741792530432000}$, $c_{1,2,5,0} = -\frac{2584301011559}{40516741792530432000}$, $c_{2,1,0,4} = \frac{123508268749}{552501024443596800}$, $c_{2,1,1,3} = c_{1,1,1,4} + \frac{49301910994957}{8103348358506086400}$, $c_{2,1,2,2} = \frac{1913966673461}{202583708962652160}$, $c_{2,1,3,1} = -\frac{10131370876243}{3038755634439782400}$, $c_{2,1,4,0} = -\frac{4706245322851}{6077511268879564800}$, $c_{3,1,0,3} = \frac{753480953701}{1012918544813260800}$, $c_{3,1,1,2} = c_{1,1,1,4} + \frac{1410117472721}{442000819554877440}$, $c_{3,1,2,1} = -\frac{761111503219}{101291854481326080}$, $c_{3,1,3,0} = -\frac{808667194487}{337639514937753600}$, $c_{4,1,0,2} = -\frac{1274532923}{28136626244812800}$, $c_{4,1,1,1} = c_{1,1,1,4} - \frac{54333101717617}{24310045075518259200}$, $c_{4,1,2,0} = -\frac{1543858083353}{1012918544813260800}$, $c_{5,1,0,1} = -\frac{11983799185}{40516741792530432}$ $c_{5,1,1,0} = c_{1,1,1,4} + \frac{1513581786157}{2210004097774387200}$ and $c_{6,1,0,0} = c_{1,1,1,4} + \frac{23839679158727}{24310045075518259200}$ start to emerge some redundant structures in this level, such as the terms with $c_{1,1,1,4}$, $c_{1,2,0,5}$, $c_{2,1,1,3}$, $c_{3,1,1,2}$, $c_{4,1,1,1}$, $c_{5,1,1,0}$, and $c_{6,1,0,0}$ are related to each other, that means the structures in the sixth order of our ansatz (4.69) are linear dependence in the classical limit.

For higher order, there are similar redundant structures, so the expression is not a one-to-one correspondence. In future work, we should find out what the patterns for these redundant coefficients are.

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