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On the Sequential Price of Anarchy in Network Formation Game

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本論文係王怡堯（R10921104）在國立臺灣大學電機工程學系完成之碩士學位論文，於民國 112 年 7 月 26 日承下列考試委員審查通過及口試及格，特此證明。

The undersigned, appointed by the Department of Electrical Engineering on 26/07/2023 have examined a Master's thesis entitled above presented by Yi-Yao Wang (R10921104) candidate and hereby certify that it is worthy of acceptance.

口試委員 Oral examination committee:

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再來是研究室的夥伴們，雖然大家在研究的主題都不太一樣，但是我們還是常常討論和分享自己的狀況，最後大家共同努力和互相幫助，一起準備了學位口試的安排，也非常感謝學弟妹們的幫忙。最後要感謝的是家人以及朋友們，這一路走來時常有懷疑自己，不知所措的時候，總是能相信我，給我支持和鼓勵。雖然現在未來還是很不確定的，但是我相信未來我們的發展一定能找到自己的方向，謝謝。

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中文摘要

隨著網路的廣泛應用以及各種不同利益的實體參與，對於網路形成和效率的理解變得越來越重要。本研究探討網路形成遊戲，以分析個體利益和整體網路效能之間的關係。在本篇論文中，我們專注於分析網路形成賽局的序列賽局版本。該賽局模型基於一個圖形結構，其中每條邊都有固定的建立成本，並且在某些節點上有獨立的玩家。每個玩家都有自己的起點和目標終點。在賽局進行過程中，玩家從所有可能的路徑中選擇成本最低的路徑，並且每條邊的成本由使用該邊的玩家平均分擔。

我們關注的是賽局系統的效率，即序列賽局中子網路完美納許均衡（SPNE）與最佳解之間的成本比例（SPoA）。我們觀察到原始賽局模型存在一些問題，例如玩家可能選擇非簡單路徑，這在同時做決策的賽局中是不會發生的情況。這些情況引發了對公平性和合理性的疑慮。因此，我們引入了一個新的限制，限制玩家只能選擇簡單路徑，以解決這些問題。

此外，我們指出先前針對二人單一終點賽局提出的性質在序列賽局中不再適用，這對 SPoA 的上界分析帶來了挑戰。最後，我們構建了一個特定的實例，展示了在特定情況下的下界，即 $\Omega(\log n)$。這個下界可以應用於環狀網路和廣播網路等情境中，並且在這些網路的非序列賽局中，PoS 已被證明為常數 [7, 16]。

關鍵字：賽局演算法、網路設計、賽局理論、序列賽局、子賽局完美均衡

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Abstract

The widespread use of networks and the involvement of diverse entities with varying interests highlight the need to understand network formation and efficiency. This study examines network formation games to analyze the relationship between individual benefits and overall network performance. In this thesis, we focus on analyzing the sequential version of the network formation game. The game model is based on a graph structure, where each edge has a fixed construction cost, and there are independent players at certain nodes. Each player has their own starting point and target destination. During the game, players choose a lowest-cost path from all possible paths, and the cost of each edge is shared equally among the players using that edge.

Our main focus is on the efficiency of the game system, specifically the Subgame Perfect Nash Equilibrium (SPNE) and the cost ratio between SPNE and the optimal solution, known as the Sequential Price of Anarchy (SPoA). We observe that the original game model exhibits some issues, such as players potentially choosing non-simple paths, which

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would not occur in simultaneous decision-making games. These concerns raise questions about fairness and rationality. To address these issues, we introduce a new restriction where players are limited to choosing simple paths.

Furthermore, we point out that properties formulated for two-player single-sink games in previous studies are no longer applicable in the sequential version, posing challenges in analyzing the upper bounds of SPoA. Finally, we construct a specific instance that demonstrates a lower bound of $\Omega(\log n)$ in a particular case. This lower bound can be applied to ring networks and broadcast networks, where the Price of Stability PoS in non-sequential games has been proven to be constant [7, 16].

Keywords: Algorithmic Game Theory, Network Design, Game Theory, Sequential Game, Subgame Prefect Nash Equilibrium

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Chapter 1 Introduction

1.1 Motivation

In the past decades, with the emergence and widespread use of the Internet and various networks, significant impacts have been observed across various domains, including theoretical computer science, game theory, and economics. The construction and utilization of networks involve numerous independent entities with diverse economic interests. Therefore, understanding networks and analysis of efficiency has become a crucial area of research.

The formation of networks , such as social networks, communication networks, and transportation networks, often involves multiple entities with their own interests and objectives. These entities may be individuals, organizations, or even countries, each seeking to optimize their benefits while considering the costs and benefits associated with network formation.

Network Formation Game (also known as network cost-charing game or network design game) [3] is a game-theoretic model that describe the challenges encountered by multiple participants in establishing network connections. In this game, each participant aims to establish connections with other participants at the lowest cost while satisfying

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their own needs. However, due to the independence and selfishness of the participants, their decisions may lead to the instability or inefficiency of the network structure.

Numerous studies have been conducted on regular network formation games [1, 3, 4, 10, 14, 23, 24]. In certain networks, decision-making does not occur simultaneously among players. Instead, players make decisions in a sequential order. We assume that players are aware of the decision order, and while the players cannot modify their strategies once they have made their choices, all players possess knowledge of the choices made by earlier players. Additionally, players are equipped with the infinity computing resources to anticipate the decisions of subsequent players, considering different conditions that may arise due to their own path choices driven by their self-interest in minimizing their individual costs. As the game progresses from player 1 to player n, each player can accurately compute the costs associated with different paths from their source to sink and sequentially select the path with the minimum cost. This leads to a network structure that corresponds to the concept of subgame perfect Nash equilibrium (SPNE) within the sequential game.

In Network Formation Games, decisions are essentially one-time events and have a fixed decision order. For example, each company involved in constructing the network has specific time constraints, requiring them to determine their routes before certain time points. This motivation emphasizes the significance of specific decision order rather than random sequencing of decisions.

In addition to the natural considerations, another significant reason for studying the concept of SPNE is the observation that in a simple network involving arbitrarily many players, a single source, a single sink, and two edges connecting them, the cost of Nash

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equilibrium solution can be arbitary larger than the optimal solution. In contrast, the SPNE provides an optimal solution in such cases. This raises the question of whether the SPNE concept might lead to better outcomes in network scenarios. This highlights the importance of analyzing and understanding SPNE, as it offers a possibility for achieving more desirable outcomes in network formation games.

The efficiency of a network is analyzed by comparing the total cost of the equilibrium solution to the total cost of the optimal solution (i.e., the smallest cost or the largest utility). Previous studies have analyzed the efficiency of network formation games by considering the cost of the best Nash equilibrium (PoS) or the worst Nash equilibrium (PoA). In this paper, we adopt a different approach and use the concept of the Sequential Price of Anarchy (SPoA), which is the ratio of the cost of the subgame perfect Nash equilibrium (SPNE) to the cost of the optimal solution. By employing SPoA, we aim to analyze the efficiency of the sequential game in our study.

In the original Network Formation Game, mixed strategy equilibrium may lead to significantly higher social costs. For example, when there are k players with k paths having equal costs, in a pure Nash equilibrium, all players choose the same path (any one of the k paths). However, in a mixed Nash equilibrium, players may choose separated paths, resulting in a more dispersed and less efficient outcome. To address this issue and find more practically implementable assumptions, correlated equilibrium are proposed as a more suitable candidate for efficiency analyses. (see also the discussion in [8])

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1.2 Related works

The Network Formation Game was first proposed by [3]. The pioneering work by [4] introduced the original form of the game, where players move simultaneously and converge to a pure Nash equilibrium. However, the Price of Anarchy (PoA) of the Network Formation Game can be as large as the number of players, because of the presence of undesirable equilibria. To address this, [4] defined a more reasonable efficiency measure called the Price of Stability (PoS) and provided a tight bound $\Theta(\mathcal{H}_n)$ for directed Network Formation Games, where n is the number of the players and $\mathcal{H}_n$ is the n-th harmonic number. Since then, considerable research efforts have been devoted to analyzing and improving the bounds of PoS in various settings of the Network Formation Game.

For undirected networks, the upper bound $\mathcal{H}_n$ in Anshelevich et al. [4] still holds, but the lower bound is now a constant that does not match the upper bound. Disser et al. [14] and Mamageishvili et al. [23] have made some improvements to the upper bound of the PoS, but still an asymptotic bound of $O(\log n)$. However, the best-known lower bound so far is a constant value of approximately ≈ 2.245 as shown by Bilo et al. [5].

For the generalized version of the Network Formation Game with weighted players, Chen and Roughgarden [8] established an upper bound of $O(\log W)$, where W represents the sum of the weights of all players. On the other hand, a lower bound of $\Omega\left(\frac{\log W}{\log \log W}\right)$ was provided by Albers [1]. These bounds indicate the scaling behavior of the game's efficiency with respect to the total weight of the players involved.

In the subclass of Network Formation Games, specific attention has been given to multicast games, where there is only one sink. Li [22] established an upper bound of

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$O\left(\frac{\log n}{\log\log n}\right)$ for the PoS of multicast games, subsequent works [11, 18] has further improved upon this bound and proved it to be constant in specific graph scenarios. Similarly, in the context of broadcast games, where there is only one sink and every node except the sink has at least one player, several works have contributed to understanding their PoS. Fiat et al. [17] and Lee and Ligett [20] have provided upper bounds for the PoS of broadcast games and finally achieved $O(1)$ by Bilò et al. [7]. On the ring-structured network, Fanelli et al. [16] proved the tight bound $\frac{3}{2}$ for PoS. These results highlight the improved efficiency achieved in these specific subclasses of Network Formation Games. There are also other works in Strong Nash equilibrium Epstein et al. [15] and bounds on three players [6].

Indeed, most of the previous research in the field of network formation games has primarily focused on the PoS, which examines the efficiency of the best pure Nash equilibrium. However, more recently, there has been a growing interest in algorithmic game theory, where researchers have started exploring the cost of SPNE. This line of research recognizes that players in a sequential game may strategically deviate from their initial decisions to achieve better outcomes, leading to the notion of SPNE.

Sequential Price of Anarchy (SPoA) was first proposed by Leme et al. [21]. SPoA measures the ratio between the cost of a SPNE and the optimal cost in a sequential game. It quantifies the inefficiency that arises from players making sequential decisions rather than simultaneous decisions. In their work, Leme et al. [21] focused on a specific subset of network formation games called machine cost sharing games. They provided a tight bound of $O(\log n)$ for the SPoA in these games. Correa et al. [12] proved that SPoA has an non-constant lower bound in routing game, and gave a tight bound $\frac{7}{5}$ for 2-players. There also some works on Atomic Congestion Games [13] and Isolation Games [2]. In

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network formation game, Chen et al. [9] give a $O(n)$ lower bound for SPoA on directed networks, Xiao [25] provided a lower bound $O(\frac{\log n}{\log \log n})$ for SPoA on undirected networks and an upper bound $O(\log n)$ for SPoA on regular ring game. These results shed light on the performance guarantees and limitations of sequential decision-making in different types of network formation game.

In previous works on the sequential version of the network formation game [9, 25], the original game model was utilized. However, we have identified several issues when applying the original model to the sequential game setting. Specifically, the strategy set in the original model consists of all paths from the source to the sink. While in Nash Equilibrium, players never choose non-simple paths if there are no zero-cost cycles. However, we have discovered examples where players might choose non-simple paths in the SPNE. In order to address these concerns, we propose a new restriction where we limit the strategy set to only include simple paths.

1.3 Our results

In this thesis, our main focus is on establishing a lower bound for the SPoA in the network formation game. We initially observed the unfairness and unrealistic outcomes in the original game model used in the sequential version. To address these issues, we introduce a new restriction where players are only allowed to choose simple paths.

For the lower bound analysis of SPoA, we introduce a specific network game called a clockwise ring game. This game has a ring structure with a single sink, and the decision order follows a clockwise direction. We conduct a detailed analysis to determine the players' behaviors under certain conditions. To quantify the SPoA, we utilize the comple-

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mentary slackness condition of a specific linear program and its dual to compare feasible solutions. This allows us to derive the lower bound for SPoA, providing insights into the inefficiency of the network formation game.

The reason we are interested in researching the ring structure is that it provides a relatively simpler network scenario in the context of non-tree structures, where each player has multiple strategy choices. This simplicity allows us to gain valuable insights and build a foundation for understanding more complex network configurations in the future. Previous works have already explored this specific ring structure [16, 25], which serves as a starting point for our analysis. However, our ultimate goal is to extend our research towards more general network formations.

Table 1.1 provides a comparison of the PoA, PoS, and SPoA in previous work. The PoA has a tight bound of n, while the PoS lower bound remains constant for undirected cases. However, in the sequential version of the game, the SPoA lower bound reaches $O\left(\frac{\log n}{\log\log n}\right)$. The previous work on SPoA focuses on upper bounds for the regular ring game [25], where the behavior in the game is fixed.

Table 1.2 presents our work, which establishes an $\Omega(\log n)$ lower bound for SPoA in the clockwise ring game. Our lower bound construction considers both the ring and broadcast scenarios. Notably, our lower bound even exceeds the upper bound of PoS. These findings highlight the limitations of the sequential decision-making framework in the network formation game.

In chapter 2, we focus on modeling a sequential game in network formation. We consider a specific order of decision-making and assume players have perfect information about others' strategies. To illustrate this game dynamics, we provide a concise exam-

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Measure	Network		Lower Bound	Upper Bound
PoA	Directed	General	n [24]	n [24]
	Undirected	General		
		Multicast		
		Broadcast		
		Ring		
PoS	Directed	General	$\mathcal{H}_n$ [4]	$\mathcal{H}_n$ [4]
	Undirected	General	2.245 [5]	$O(\log n)$ [14, 23]
		Multicast	1.826 [5]	$O\left(\frac{\log n}{\log\log n}\right)$ [22]
		Broadcast	1.818 [5]	$O(1)$ [7]
		Ring	1.5 [16]	1.5 [16]
SPoA	Directed	General	n [9]	n [9]
	Undirected	General	$O\left(\frac{\log n}{\log\log n}\right)$ [25]	
		Multicast		
		Broadcast		
		Ring	NA	
		Regular Ring	NA	$\mathcal{H}_n$ [25]

Table 1.1: Related Works

Measure	Network		Lower Bound	Upper Bound
SPoA	Directed	General	n [9]	n [9]
	Undirected	General	$\boldsymbol{\Omega}(\log \boldsymbol{n})$ (by Theorem 3)	
		Multicast		
		Broadcast		
		Ring		
		Regular Ring		$\mathcal{H}_n$ [25]

Table 1.2: Our Results

ple highlighting the strategic interactions between players. In chapter 3, we will present examples that demonstrate peculiar behavior and potential issues in the original model. These examples highlight the need for a new restriction to address these concerns. We will discuss the challenges faced in proving upper bounds for the network formation game, specifically in scenarios involving three or more players. The propositions formulated for two-player games in previous work are ineffective in these cases. Finally, the main result of our work is presented in Chapter 4, where we establish a tight lower bound of $\Omega(\log n)$ for SPoA of clockwise ring game.

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Chapter 2 Preliminaries

2.1 Shapley Network Cost-Sharing Games

In Shapley network cost-sharing games [3, 4], we consider a graph $G = (V, E)$, either directed or undirected, and a set of players $N = \{1, 2, \ldots, n\}$. Each player $i \in N$ is associated with a source-sink pair $(s_i, t_i) \in V \times V$ and has a strategy set Γ_i containing all possible paths from s_i to t_i. In the game, each player i chooses a path $P_i \in \Gamma_i$ as their strategy.

The edges $e \in E$ are assigned costs c_e, and each edge e is used by a set of players $S_e = \{i : e \in P_i\}$. A cost sharing method ξ_e is employed to allocate nonnegative costs to the players in S_e. Specifically, we adopt the Shapley protocol [4], where the cost paid by player i for using edge e, denoted as $\xi_e(i, S_e)$, is equal to $\frac{c_e}{|S_e|}$ for every edge e. The cost of a strategy profile $(P_1, P_2, \ldots, P_n)$ is defined as follows:

$$C(P_1, P_2, \ldots, P_n) = \sum_{e \in \bigcup_i P_i} c_e \tag{2.1}$$

and in the Shapley protocol, the cost is shared evenly among the players. Thus, the cost incurred by player i for the given strategy profile is given by:

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$$c_i(P_1, P_2, \ldots, P_n) = \sum_{e \in P_i} \frac{c_e}{|S_e|}. \tag{2.2}$$

2.2 Sequential Games

In this thesis, we focus on a specific type of extensive form game with perfect information[19] known as sequential games. In sequential games, players make their decisions in a specific order, typically numbered from 1 to n. At each stage of the game, player i observes the paths taken by players 1 to $i-1$ and then selects a path $P_i \in \Gamma_i$. A history $h_i \in \times_{j=1,2,\ldots,i-1}\Gamma_j$ denoted as the sequence of strategies chosen by previous players up to stage $i-1$. A strategy profile is considered a Subgame Perfect Nash Equilibrium (SPNE) if it serves as an equilibrium for all subgames defined by the histories. In other words, for each player i and given a history h_i, their strategy at stage i must be a best response to the equilibrium of the subgame rooted at player $i+1$, which is the path P_i that minimizes their costs at that particular history.

We define the sequential price of anarchy (SPoA) of the game as the ratio between the optimal solution and the worse SPNE. Formally:

$$SPoA = \frac{\max_{s \in \text{SPNE}} C(s)}{\min_{s \in \times_i P_i} C(s)}. \tag{2.3}$$

Note that while SPNE exists in sequential games, its uniqueness may be subject to tie-break situations. In such cases, where multiple equilibria are possible, we introduce small epsilon (ϵ) terms to resolve the ties, which allow for a consistent comparison with the Price of Stability (PoS) in regular (non-sequential) games.

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2.3 An Example

In a sequential version of Shapley network cost-sharing game, as shown in Figure 2.1, player 1, starting from source s1, makes the first move, and then player 2 observes player 1's move and makes their decision. Figure 2.2 illustrates the game tree of this sequential game, where the labels top and bottom represent the two choices of the players. The tuple on each leaf node represents the costs incurred by player 1 and player 2 when they follow the actions from the root to that leaf. We will now demonstrate the process of backward induction to determine the SPNE in Figure 2.1.

Starting with the last player, let's consider their optimal strategy. When it is their turn to make a decision, they choose the action that minimizes their cost. In this case, the last player (player 2) chooses the top edge if player 1 takes the top edge, and chooses the bottom edge if player 1 takes the bottom edge. This is because, in both cases, the cost for the last player is lower when they follow these strategies ($\frac{1+\epsilon}{2} < 2$ and $1+\epsilon > 1$). Moving on to player 1, since player 1 is aware of the optimal strategy of the last player. Player 1 evaluates their own costs for choosing the top edge and the bottom edge. They find that their cost for choosing the top edge is $\frac{1+\epsilon}{2}$, while their cost for choosing the bottom edge is 1. Since $\frac{1+\epsilon}{2} < 1$, player 1 prefers the top edge. Therefore, in the SPNE, both player 1 and player 2 choose the top path.

Note that in the concept of Nash equilibrium, it is also an equilibrium for both players to choose the bottom edge in Figure 2.1. However, this equilibrium may result in a large PoA, reaching as high as the number of players, compared to the optimal solution. This observation suggests that exploring SPNE might provide better outcomes in terms of efficiency. Furthermore, it is important to note that the SPNE is unique if there is no ties

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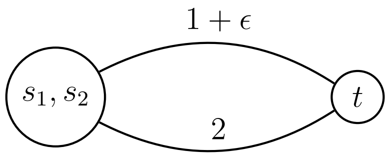

Figure 2.1: single-source single-sink example

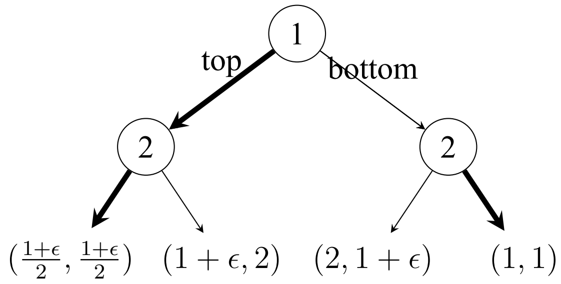

Figure 2.2: game tree of the example

in the backward induction process.

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Chapter 3 An New Restriction

3.1 The Simple Path Restriction

In the original game model and the concept of Nash Equilibrium, players always choose a simple path if there are no zero-cost cycles. This is because removing a cycle always decreases the cost for the specific player. However, we have found that this proposition no longer holds true in the sequential version of the game. In this section, we will present examples, both directed and undirected, to illustrate this observation. Subsequently, we will propose a restriction to the game model to address this issue and make it more reasonable.

Figure 3.1 illustrates an example where player 1 deviates from choosing a simple path. Instead of following the direct path from s_1 to t, player 1 selects a non-simple path $s_1 \rightarrow a \rightarrow b \rightarrow c \rightarrow a \rightarrow t$, which includes a cycle of length three. The reason for this decision is that player 1 aims to have player 2 share the cost of the edge from a to t. By traversing the cycle, player 1 creates an opportunity for player 2 to join them at node a and collectively bear the cost of the edge from a to t.

In undirected networks, there can be situations where players employ strategies that are not considered reasonable, such as using an edge multiple times by walking back and

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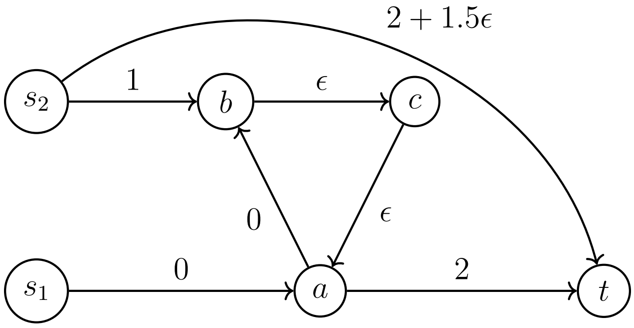

Figure 3.1: Non-Simple Path Example (Directed)

forth. Figure 3.2 presents an example where player 1 chooses the path $s_1 \to a \to s_1 \to t$, which involves traversing an edge twice and intentionally inducing player 2 to share the cost of the edge (s_1, t). In such cases, there may be some ambiguities regarding the definitions, such as whether there are penalties for using a single edge multiple times, or if the cost is shared equally among the players who have used the edge. Regardless of the specific definition chosen, the behavior exhibited by player 1 in Figure 3.2 remains the same.

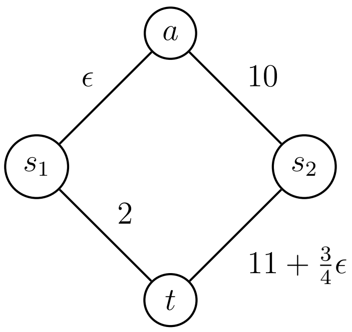

Figure 3.2: Non-Simple Path Example (Undirected)

From the two situations mentioned above, we have identified two issues with the original model in sequential games. The first issue is the rationality of players choosing non-simple paths, and in some cases even engaging in back and forth movements. The second issue relates to the definition problem we mentioned earlier, which involves determining whether there are penalties for using an edge multiple times or simply sharing the cost among players who have used it. Addressing these two issues is important for

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improving the realism and fairness of the game model in sequential games.

In the subsequent chapters of this thesis, we introduce a new model that involves a small restriction to the model defined in Chapter 2. Specifically, we adjust the strategy set Γ_i for player i to include all possible simple paths from s_i to t_i, rather than containing all possible paths. This modification restricts the strategies to simple paths only. For the remaining part of this thesis, we will use this updated model as our default model.

3.2 Challenges in Upper Bounds

After modifying the model, our focus shifts towards proving the upper bound of SPoA in the network formation game, with an initial emphasis on scenarios involving a small number of players. In the work presented by [25], they conducted an analysis of the 2-player single-sink upper bound. Their analysis began by establishing a crucial lemma that demonstrates the non-separation of two merged players, implying the formation of a tree-like structure in the SPNE. Here, we cite the lemma statement proposed by the author.

Lemma 1 (Lemma 1 [25]). *In SPNE of a 2-player single-sink network design game, once the paths of the two players merge, they do not separate again.*

The proof of this lemma does not consider the non-simple path situation mentioned in the previous section and may be affected by the definition issue. However, it holds true in our modified model. They continued to prove the upper bound for the 2-player single-sink case, which is $\frac{7}{5}$. Unfortunately, we discovered examples that shows the lemma does not hold true when there are more than two players. We will provide both directed and undirected examples to illustrate this.

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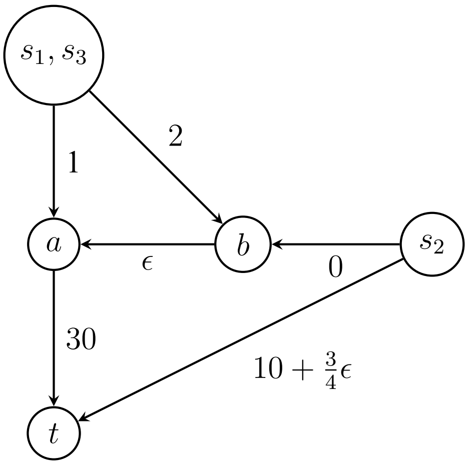

Figure 3.3: Merged and Separated Example (Directed)

In Figure 3.3, we present an example where player 1 and player 3 merge and separate in the SPNE, despite sharing the same source and sink. An intuitive explanation for this behavior is that they prefer to have player 2 share the cost of the edge from a to t with them since they must use that edge. For player 2, there are two strategies: a direct path from s_2 to t or sharing the edge with player 1 and 3, which is $s_t \to b \to a \to t$. The cost of these two strategies for player 2 is $10 + \frac{3}{4}\epsilon$ and $10 + \epsilon$, respectively, if no one shares the edge $b \to a$. Therefore, in order to incentivize player 2 to share the edge $a \to t$, someone must go to b and share the ϵ cost on the edge $b \to a$. By considering the decision order, we can see that player 1 must take the responsibility since player 3 makes the decision after player 2, and $s_3 \to a \to t$ is always better than $s_3 \to b \to a \to t$ when fixing the decisions of the other players. This results in the situation where player 1 and player 3 merge and separate even in a single-sink scenario where they have the same source.

In the case of undirected networks, we have a similar network depicted in Figure 3.4, which illustrates the same situation mentioned earlier. However, in this scenario, Lemma 1(cited in the previous paragraph) does not hold true. This negative observation undermines our initial hope that the SPNE necessarily forms a tree. Consequently, it becomes more challenging to prove positive results for the upper bound, as we may have to

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consider more complicated cases. Even after modifying the strategy set to simple paths, there still might be some unusual behavior. It is important to note that such behavior does not occur in the Nash equilibrium of the original game.

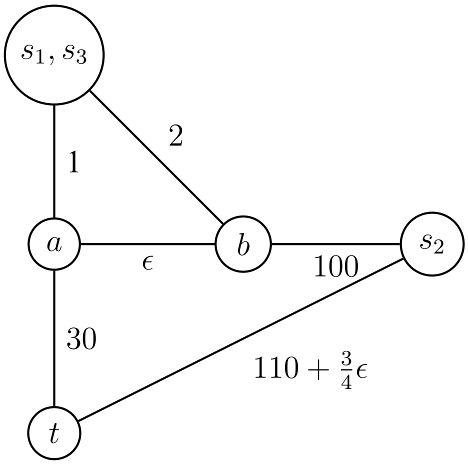

Figure 3.4: Merged and Separated Example (Undirected)

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Chapter 4 Lower Bound of SPoA

In this chapter, we will provide a non-trivial lower bound of $\Omega(\log n)$ for SPoA by construct an instance on a ring-structured network. It should be noted that this lower bound also applies to the general undirected network formation game. The analysis of the costs and SPoA is quite complicated and requires several steps. We will determine the type of SNPE that can lead to a large SPoA, as well as how the edge's cost can cause such an SPNE. Next, to analyze the asymptotic value of SPoA, we introduce a special linear program and use the complementary slackness condition of the linear program and its dual to compare the value of a feasible solution with the costs we have constructed. We then obtain the lower bound of $\Omega(\log n)$.

4.1 Clockwise Ring Game

To provide a instance of the lower bound, we introduce a specific network structure known as a clockwise ring game. A clockwise ring game consists of a single-sink network with a ring structure, where the decision order follows a clockwise direction. Figure 4.1 illustrates an example of a clockwise ring game structure.

Definition 1 (Clockwise ring game). *In a clockwise ring game, we are given a cost vector $d \in \mathbb{R}^{n+1}$ of length $n + 1$, and this network formation game can be represented by an*

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undirected graph $G = (V, E)$, where the vertex set V consists of elements from 1 to n with player on it and an additional vertex t. The edge set E includes edges $(i, i+1)$ for $1 \leq i \leq n-1$, and it also includes edges $(t, 1)$ and (t, n). Each edge $(i, i+1)$ has a cost d_{i+1}, and the edges $(t, 1)$ and (t, n) have costs d_1 and d_{n+1}, respectively.

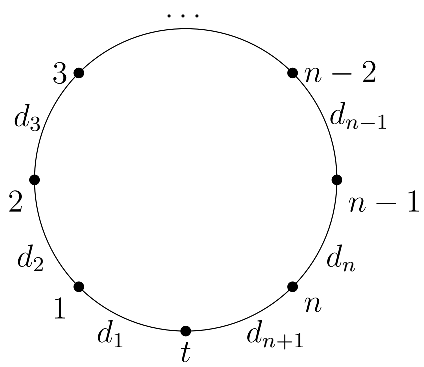

Figure 4.1: Clockwise Ring Game Structure

In Lemma 2 and Theorem 1, we prove that when the inequalities in eq. (4.1) and eq. (4.4) are satisfied, the SPNE will take on a desired form in which all players choose the counter-clockwise direction, as demonstrated in Lemma 2. In order to construct a lower bound, we let the equalities in inequalities eq. (4.4) hold, which also satisfy the inequalities in eq. (4.1), as demonstrated in eq. (4.10). In this case, the optimal solution is for all players to choose the clockwise direction, as compared to the SPNE.

Lemma 2. *In n-players clockwise ring games, if the costs satisfy following $n-1$ inequality, the subgame with player 1 to $k-1$ choose counter-clockwise direction and player k chooses clockwise direction has an SPNE with player $k+1$ to n chooses clockwise direction for $1 \leq k \leq n-1$.*

$$\sum_{i=1}^{k} \frac{d_i}{n-i} + \frac{d_{k+1}}{n-k+1} > \sum_{i=k+2}^{n+1} \frac{d_i}{i-k}, \text{for} \quad 1 \leq k \leq n-1. \tag{4.1}$$

Proof. In the subgame where players 1 to $k-1$ choose the counter-clockwise direction

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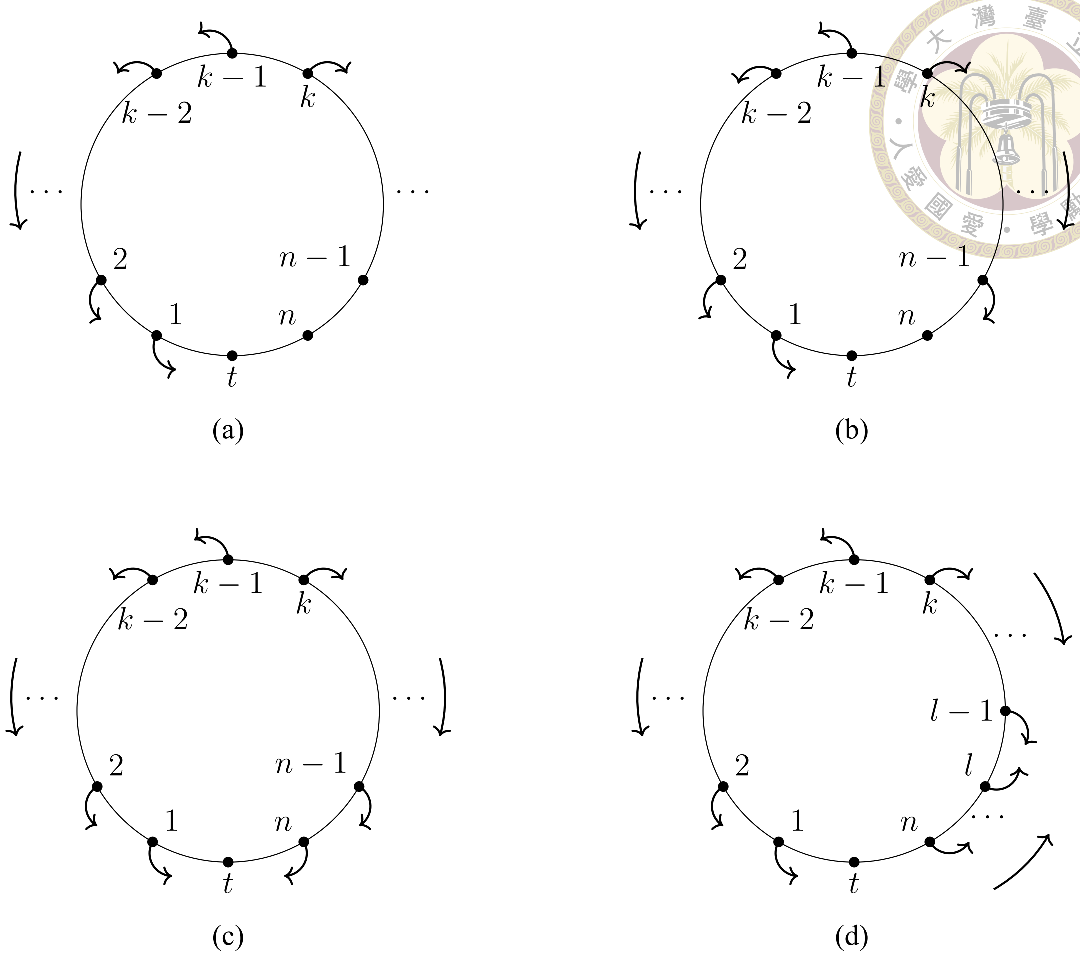

Figure 4.2: Steps of induction

and player k chooses the clockwise direction (see Figure 4.2a), let's consider the scenario where players $k+1$ to $n-1$ also choose the clockwise direction (see Figure 4.2b). In this case, if player n chooses the clockwise direction, it will incur a cost of $\frac{d_{n+1}}{n-k+1}$. On the other hand, if player n chooses the counter-clockwise direction, it will incur a cost of at least $\sum_{i=1}^{k} \frac{d_i}{k-i+1} + \frac{d_{k+1}}{2}$. By eq. (4.1) we have

$$
\begin{aligned}
\sum_{i=1}^{k} \frac{d_i}{k-i+1} + \frac{d_{k+1}}{2} &\geq \sum_{i=1}^{k} \frac{d_i}{n-i} + \frac{d_{k+1}}{n-k+1} \\
&> \sum_{i=k+2}^{n+1} \frac{d_i}{i-k} \\
&\geq \frac{d_{n+1}}{n-k+1}.
\end{aligned} \tag{4.2}
$$

Thus, player n will chooses the clockwise direction (see Figure 4.2c).

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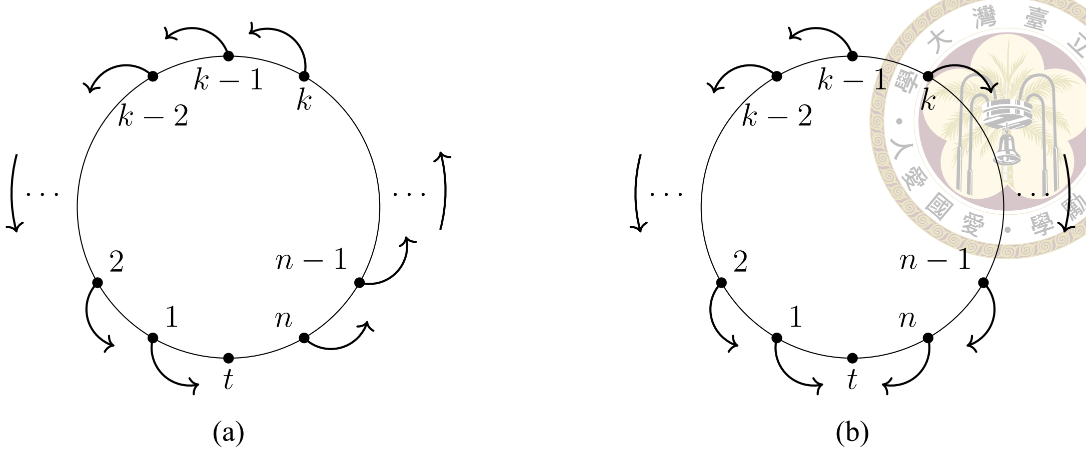

Figure 4.3: Two strategies for player k

By repeating the process, let's consider player l from $k+1$ to $n-1$. In the subgame where players $k+1$ to $l-1$ choose the clockwise direction (see Figure 4.2d), if player l chooses the clockwise direction, it will incur a cost of $\sum_{i=l+1}^{n+1} \frac{d_i}{i-k}$. On the other hand, if player l chooses the counter-clockwise direction, it will incur a cost of at least $\sum_{i=1}^{k} \frac{d_i}{k-i+n-l+1} + \frac{d_{k+1}}{n-l+2}$. By eq. (4.1) we have

$$\begin{aligned}\sum_{i=1}^{k} \frac{d_i}{k-i+n-l+1} + \frac{d_{k+1}}{n-l+2} &\geq \sum_{i=1}^{k} \frac{d_i}{n-i} + \frac{d_{k+1}}{n-k+1} \\ &> \sum_{i=k+2}^{n+1} \frac{d_i}{i-k} \\ &\geq \sum_{i=l+1}^{n+1} \frac{d_i}{i-k}.\end{aligned} \tag{4.3}$$

Therefore, it follows that player l will choose the clockwise direction (see Figure 4.2c), for $k+1 \leq l \leq n$. $\square$

Theorem 1. *In n-players clockwise ring games, if the costs satisfy both eq.* (4.1) *and the following* n *inequalities, the configuration in which all player choosing the counter-clockwise direction is an SPNE.*

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$$\sum_{i=1}^{k} \frac{d_i}{n-i+1} \leq \sum_{i=k+1}^{n+1} \frac{d_i}{i-k}, \text{for} \quad 1 \leq k \leq n. \tag{4.4}$$

Proof. Prove by backward induction. Consider the subgame where players 1 to $n-1$ choose the counter-clockwise direction. In this case, if player n chooses the clockwise direction, it will incur a cost of d_{n+1}, while choosing the counter-clockwise direction will incur a cost of $\sum_{i=1}^{n} \frac{d_i}{n-i+1}$. By eq. (4.4), we can conclude that player n will choose the counter-clockwise direction.

By repeating the process, let's consider the subgame where players 1 to $k-1$ choose the counter-clockwise direction. In this scenario, if player k chooses the clockwise direction, it will result in a cost of $\sum_{i=k+1}^{n+1} \frac{d_i}{i-k}$ (as shown in 4.3b), by Lemma 2. On the other hand, if player k chooses the counter-clockwise direction, the cost will be $\sum_{i=1}^{k} \frac{d_i}{n-i+1}$ (as shown in 4.3a), based on the given premise. By eq. (4.4), we can conclude that player k will choose the counter-clockwise direction, for $1 \leq k \leq n$. □

In the remaining part of this chapter, we will make the assumption that n is sufficiently large $(n \geq 75)$ such that $\ln(n) - 2 \cdot \ln(2 \cdot \ln n) > 0$. For convenience, we organize the original inequalities eq. (4.4) with an $n \times n$ matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and two n-vectors $x, b \in \mathbb{R}^n$:

$$Ax \succeq b. \tag{4.5}$$

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Where

$$a_{ij} = \begin{cases} 1, & \text{if } i = j, \\ \frac{-1}{n-i}, & \text{if } i > j, \\ \frac{1}{j-i+1}, & \text{if } i < j. \end{cases} \tag{4.6}$$

$$x_i = d_{i+1}, \qquad \text{for } 1 \le i \le n,$$

$$b_i = \frac{d_1}{n}, \qquad \text{for } 1 \le i \le n.$$

4.2 The Lower bound Instance

Our construction for the lower bound is a specific case of the clockwise ring game, where the edge costs are selected in a way that satisfies the equalities in the inequalities eq. (4.4). Please refer to Figure 4.4 for a visual representation. in eq. (4.10), In eq. (4.10), we will demonstrate that the costs satisfies both the inequalities eq. (4.4) and eq. (4.1), and it can be shown that $\hat{d}_1$ is the largest edges cost. This implies that in the SPNE, all players choose the counter-clockwise direction, while in the optimal solution, all players choose the clockwise direction. Additionally, we will demonstrate that the solution is component-wise positive, ensuring that it represents feasible costs for the game. We denote these solutions as $\hat{d} \in \mathbb{R}^{n+1}$ with a fixed $\hat{d}_1 = n \ln n - 2n \ln(2 \ln n)$ and

$$\begin{bmatrix} \hat{d}_2 \\ \hat{d}_3 \\ \vdots \\ \hat{d}_{n+1} \end{bmatrix}^T = A^{-1} \begin{bmatrix} \frac{\hat{d}_1}{n} \\ \frac{\hat{d}_1}{n} \\ \vdots \\ \frac{\hat{d}_1}{n} \end{bmatrix}. \tag{4.7}$$

Lemma 3. *If $\hat{d}_1$ is positive, $\hat{d}_i$ is also positive for $2 \le i \le n+1$.*

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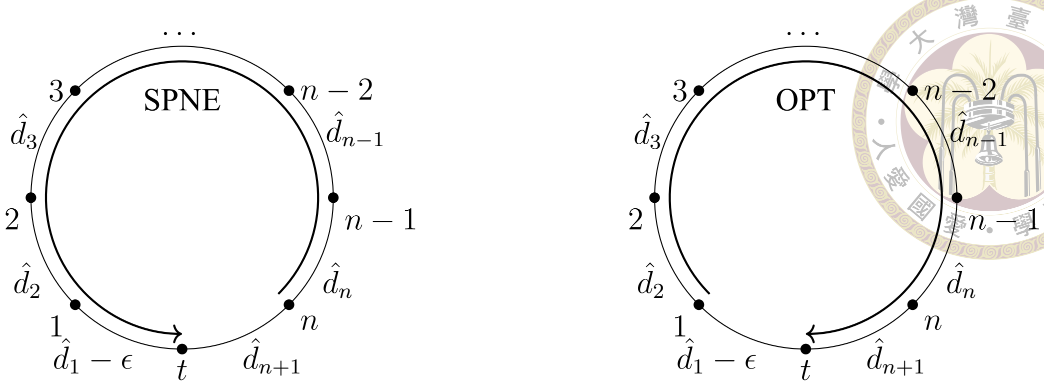

Figure 4.4: Lower Bound Instance Construction

Proof. Prove by solving the linear system $Ax = b$, we can select two adjacent equations from the given system and subtract them. we obtain:

$$x_i = \frac{n-i}{n-i+1} \cdot \sum_{k=1}^{n-i} \frac{x_{i+k}}{k(k+1)}, \text{for } 1 \le i \le n-1. \tag{4.8}$$

We can infer that x_i will be a linear combination of x_n with positive coefficients for all i. Then we take the first equation:

$$\sum_{k=1}^{n} \frac{x_i}{n-k+1} = \frac{\hat{d}_1}{n}. \tag{4.9}$$

Since $\hat{d}_1 > 0$ and $\sum_{k=1}^{n} \frac{x_i}{n-k+1}$ can be written as $c \cdot x_n$ with a positive c, we have $x_n > 0$ and therefore, x_i is positive for all i. □

It can be easily seen that $\hat{d}$ satisfies both eq. (4.1) and eq. (4.4), since it holds the equalities in eq. (4.4).

$$\begin{aligned}
\sum_{i=1}^{k} \frac{\hat{d}_i}{n-i} + \frac{\hat{d}_{k+1}}{n-k+1} &> \sum_{i=1}^{k} \frac{\hat{d}_i}{n-i+1} \\
&= \sum_{i=k+1}^{n+1} \frac{\hat{d}_i}{i-k} \\
&> \sum_{i=k+2}^{n+1} \frac{\hat{d}_i}{i-k}, \text{for} \quad 1 \le k \le n-1.
\end{aligned} \tag{4.10}$$

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Thus, clockwise ring game with costs $\hat{d}$ has SPNE that every player choose counter-clockwise direction. However, based on equation eq. (4.9), which indicates that $\hat{d}_1$ is the maximum cost, the optimal solution would be for every player to choose the clock-wise direction. This implies that the SPoA $= \frac{\hat{d}_1+\sum_{i=2}^{n}\hat{d}_i}{\sum_{i=2}^{n+1}\hat{d}_i}$. If $\frac{\hat{d}_1+\sum_{i=2}^{n}\hat{d}_i}{\sum_{i=2}^{n+1}\hat{d}_i} = \Omega(\log n)$, then we are done.

4.3 The Value of SPoA

To proceed, it is necessary to analyze the values of $\hat{d}$. Specifically, we aim to prove that $\sum_{i=2}^{n+1}\hat{d}_i$ is sufficiently small while keeping $\hat{d}_1$ fixed. However, directly analyzing this sum can be complex and intricate. Therefore, we will employ alternative methods and techniques to analyze it. Specifically, we will consider a primal-dual linear program with the inequalities given in eq. (4.4). The high-level idea of the proof is to construct a feasible solution $\hat{y}$ for the dual program and demonstrate that $\hat{d}_{2:n+1}$ (the vector of $\hat{d}_2$ to $\hat{d}_{n+1}$) and $\hat{y}$ satisfy the relaxed complementary slackness condition. This will allow us to conclude that $\hat{d}_{2:n+1}$ is not far from the optimal solution in the primal program. We will then construct a feasible solution in the primal program that can be easily calculated, allowing us to compare its value with our construction $\hat{d}$.

Primal:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} x_i \\ \text{subject to} \quad & Ax \succeq \ln(n) - 2\cdot\ln(2\cdot\ln n), \\ & x_i \geq 0, \qquad\qquad \text{for } 1 \leq i \leq n. \end{aligned} \tag{4.11}$$

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Dual:

$$\begin{aligned}
\text{maximize} \quad & (\ln(n) - 2 \cdot \ln(2 \cdot \ln n)) \cdot \sum_{i=1}^{n} y_i \\
\text{subject to} \quad & A^T y \preceq 1, \\
& y_i \geq 0, \qquad \text{for } 1 \leq i \leq n.
\end{aligned} \tag{4.12}$$

and let $x^\star$ be the optimal solution of the primal program. Our objective now is to compare this optimal solution with $\hat{d}_{2:n+1}$, which represents the denominator part of the SPoA ratio.

Theorem 2. *The objective value of $\hat{d}_{2:n+1}$ is at most 5 times the objective value of $x_i^\star$ in the primal linear program eq.* (4.11).

Proof. We will construct a valid dual solution $\hat{y}$ and show that $(\hat{d}_{2:n+1}, \hat{y})$ satisfied a set of relaxed complementary slackness conditions in eq. (4.13) and eq. (4.15). Based on our construction eq. (4.7), we can observe that $\hat{d}_{2:n+1}$ satisfies the the primal complementary slackness conditions that make the inequalities in the primal problem hold as equalities.

$$\sum_{j=1}^{n} a_{ij} \hat{d}_{j+1} = \ln(n) - 2 \cdot \ln(2 \cdot \ln n), \quad \text{for } 1 \leq i \leq n. \tag{4.13}$$

Then we construct a feasible solution of the dual program $\hat{y}$:

$$\hat{y}_i = \begin{cases} \frac{1}{5} \cdot \left(\frac{1}{\mathcal{H}_2 - 1} + 1 \right), & \text{if } i = 1, \\ \frac{1}{5 \cdot (\mathcal{H}_{i+1} - 1)}, & \text{if } 1 \leq i \leq n. \end{cases} \tag{4.14}$$

Where $\mathcal{H}_n = \sum_{k=1}^{n} \frac{1}{k}$. We will show that $\hat{y}$ satisfy the relaxed dual complementary slackness conditions:

$$\frac{1}{5} \leq \sum_{i=1}^{n} a_{ij} \hat{y}_i \leq 1, \quad \text{for } 1 \leq j \leq n. \tag{4.15}$$

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let's examine the k-th row in $A^T\hat{y}$ and split it into three cases:

$$\sum_{i=1}^{n} a_{ik}\hat{y}_i. \tag{4.16}$$

Case 1: If $k=1$:

$$\begin{aligned}5\cdot\sum_{i=1}^{n} a_{i1}\hat{y}_i &= 5\cdot\left(\hat{y}_1-\sum_{i=2}^{n}\frac{\hat{y}_i}{n-1}\right)\\ &= 3-\sum_{i=2}^{n}\frac{1}{(n-1)(\mathcal{H}_{i+1}-1)}\\ &\geq 3-\sum_{i=2}^{n}\frac{1}{(n-1)(\mathcal{H}_2-1)}=1.\end{aligned} \tag{4.17}$$

and for upper bound

$$\begin{aligned}5\cdot\sum_{i=1}^{n} a_{i1}\hat{y}_i &= 3-\sum_{i=2}^{n}\frac{1}{(n-1)(\mathcal{H}_{i+1}-1)}\\ &\leq 5.\end{aligned} \tag{4.18}$$

By inequalities eq. (4.17) and eq. (4.18), we can see that the 1-st row of the matrix $A^T\hat{y}$ satisfies the condition in equation eq. (4.15).

Case 2: If $k=2$:

$$5\cdot\sum_{i=1}^{n} a_{i2}\hat{y}_i = \frac{3}{2}+\left(\frac{6}{5}-\sum_{i=3}^{n}\frac{1}{(n-2)(\mathcal{H}_{i+1}-1)}\right)$$

by replacing $\mathcal{H}_{i+1}$ with $\mathcal{H}_3$, we have

$$\begin{aligned}&\geq \frac{3}{2}+\left(\frac{6}{5}-\sum_{i=3}^{n}\frac{1}{(n-2)(\mathcal{H}_3-1)}\right)\\ &\geq 1,\end{aligned} \tag{4.19}$$

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and for upper bound

$$
\begin{aligned}
5 \cdot \sum_{i=1}^{n} a_{i2}\hat{y}_i &= 5 \cdot \left(\frac{\hat{y}_1}{2} + \hat{y}_2 - \sum_{i=3}^{n} \frac{\hat{y}_i}{n-2} \right) \\
&\leq 5 \cdot \left(\frac{\hat{y}_1}{2} + \hat{y}_2 \right) \\
&= \frac{3}{2} + \frac{6}{5} \leq 5. \qquad (4.20)
\end{aligned}
$$

By inequalities eq. (4.19) and eq. (4.20), we can see that the 2-st row of the matrix $A^T\hat{y}$ also satisfies the condition in equation eq. (4.15).

Case 3: For $k \geq 3$:

$$
\begin{aligned}
5 \cdot \sum_{i=1}^{n} a_{ik}\hat{y}_i &= 5 \cdot \left(\sum_{i=1}^{k} \frac{\hat{y}_i}{k-i+1} - \sum_{i=k+1}^{n} \frac{\hat{y}_i}{n-k} \right) \\
&= \frac{3}{k} + \sum_{i=2}^{k-1} \frac{1}{(k-i+1)(\mathcal{H}_{i+1}-1)} + \left(\frac{1}{\mathcal{H}_{k+1}-1} - \sum_{i=k+1}^{n} \frac{1}{(n-k)(\mathcal{H}_{i+1}-1)} \right)
\end{aligned}
$$

by replacing $\mathcal{H}_{i+1}$ with $\mathcal{H}_{k+1}$ and simplify the above, we have

$$
\begin{aligned}
&= \frac{3}{k} + \sum_{i=2}^{k-1} \frac{1}{(k-i+1)(\mathcal{H}_{i+1}-1)} \\
&\geq \sum_{i=1}^{k-1} \frac{1}{(k-i+1)(\mathcal{H}_{i+1}-1)}
\end{aligned}
$$

by replacing $\mathcal{H}_{i+1}$ with $\mathcal{H}_k$, we have

$$
\geq \sum_{i=1}^{k-1} \frac{1}{(k-i+1)(\mathcal{H}_k-1)} = 1 \qquad (4.21)
$$

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and for upper bound

$$
\begin{aligned}
5 \cdot \sum_{i=1}^{n} a_{ik}\hat{y}_i &= 5 \cdot \left(\sum_{i=1}^{k} \frac{\hat{y}_i}{k-i+1} - \sum_{i=k+1}^{n} \frac{\hat{y}_i}{n-k} \right) \\
&\leq 5 \cdot \sum_{i=1}^{k} \frac{\hat{y}_i}{k-i+1}
\end{aligned}
$$

split the analysis into two segments and we have

$$
= 5 \cdot \sum_{i=1}^{\lceil \frac{k}{2} \rceil} \frac{\hat{y}_i}{k-i+1} + \sum_{i=\lceil \frac{k}{2} \rceil + 1}^{k-1} \frac{1}{(k-i+1)(\mathcal{H}_{i+1}-1)} + \frac{1}{\mathcal{H}_{k+1}-1}
$$

by replacing $\mathcal{H}_{i+1}$ with $\mathcal{H}_{\lceil \frac{k}{2} \rceil + 2}$, we have

$$
\begin{aligned}
&\leq 5 \cdot \sum_{i=1}^{\lceil \frac{k}{2} \rceil} \frac{\hat{y}_i}{k-i+1} + \sum_{i=\lceil \frac{k}{2} \rceil + 1}^{k-1} \frac{1}{(k-i+1)(\mathcal{H}_{\lceil \frac{k}{2} \rceil + 2}-1)} + 1 \\
&= 5 \cdot \sum_{i=1}^{\lceil \frac{k}{2} \rceil} \frac{\hat{y}_i}{k-i+1} + \frac{\mathcal{H}_{k-\lceil \frac{k}{2} \rceil} - 1}{\mathcal{H}_{\lceil \frac{k}{2} \rceil + 2} - 1} + 1 \\
&\leq 5 \cdot \sum_{i=1}^{\lceil \frac{k}{2} \rceil} \frac{\hat{y}_i}{k-i+1} + 2
\end{aligned}
$$

by substituting $\hat{y}$ into the expression, we have

$$
= \frac{3}{k} + \sum_{i=2}^{\lceil \frac{k}{2} \rceil} \frac{1}{(k-i+1)(\mathcal{H}_{i+1}-1)} + 2
$$

by replacing $\mathcal{H}_{i+1}$ with $\mathcal{H}_2$, we have

$$
\begin{aligned}
&\leq \frac{3}{k} + \sum_{i=2}^{\lceil \frac{k}{2} \rceil} \frac{1}{(k-\lceil \frac{k}{2} \rceil + 1)(\mathcal{H}_2 - 1)} + 2 \\
&= \frac{3}{k} + 2 \cdot \frac{\lceil \frac{k}{2} \rceil - 1}{k - \lceil \frac{k}{2} \rceil + 1} + 2 \\
&\leq 1 + 2 + 2 \leq 5. \qquad (4.22)
\end{aligned}
$$

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By inequalities eq. (4.21) and eq. (4.22), we can observe that the k-th row of the matrix $A^T\hat{y}$ satisfies the condition in equation eq. (4.15).

Thus, we have shown that dual complementary slackness conditions eq. (4.15) hold. By examining equations eq. (4.13) and inequalities eq. (4.15), we can observe that $\hat{d}_{2:n+1}$ and $\hat{y}$ obey the relaxed complementary conditions. Based on Lemma 3 and eq. (4.14), both $\hat{d}_{2:n+1}$ and $\hat{y}$ are component-wise positive, making them feasible solutions in both the primal and dual programs. Consequently, the objective values of $\hat{d}_{2:n+1}$ and $\hat{y}$ for the primal and dual programs are bounded by at most five times their respective values.

$$\sum_{i=1}^{n} \hat{d}_{i+1} \leq 5 \cdot (\ln n - 2\ln(2\ln n)) \cdot \sum_{i=1}^{n} \hat{y}_i \leq 5 \cdot \sum_{i=1}^{n} x_i^\star. \tag{4.23}$$

$\square$

Now, we are ready to proceed with the final step of the analysis, which is to demonstrate that the SPoA of $\hat{d}$ is asymptotically $\log n$.

Theorem 3. *The worst-case SPoA of n-players clockwise ring games has a lower bound* $\Omega(\log n)$.

Proof. Let us define an n-vector $\bar{d} \in \mathbb{R}^n$:

$$\bar{d} = \begin{cases} 1, & \text{if } i < n, \\ n, & \text{if } i = n. \end{cases} \tag{4.24}$$

Next, since the sum of the values in $\bar{d}$ is easy to determine, if we can prove that $\bar{d}$ is a feasible solution for the primal program eq. (4.11), we can bound the value of $\hat{d}_{2:n+1}$ linearly while fixing the value of $\hat{d}_1$ to $n\ln n - 2n\ln(2\ln n)$. This implies that the ratio

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between SPNE and OPT is at least a $\log n$ factor, since the SPNE utilizes $\hat{d}_1$ while the OPT does not. Additionally, we will demonstrate that the ratio is at most $\log n$ in our construction.

Let's examine k-th row from the bottom in the primal. We split into three cases: (Note that $\ln(i+1) \leq \mathcal{H}_i \leq \ln(i) + 1 \leq \ln(i+1) + 1$)

$$\sum_{j=1}^{n} a_{kj}\bar{d}_j. \tag{4.25}$$

Case 1: For $2 \leq k \leq \frac{n}{2 \cdot \ln n}$:

$$\sum_{j=1}^{n} a_{kj}\bar{d}_j = \frac{n}{k} + \sum_{i=1}^{k-1} \frac{1}{i} - \sum_{i=k}^{n-1} \frac{1}{i}$$

by $k \leq \frac{n}{2 \cdot \ln n}$, we have

$$\geq 2 \cdot \ln n + \sum_{i=1}^{k-1} \frac{1}{i} - \sum_{i=k}^{n-1} \frac{1}{i}$$

using the inequality $\ln n \geq \mathcal{H}_n - 1$, we can simplify the above, and we have

$$\geq \ln n + \sum_{i=1}^{k-1} \frac{1}{i} \geq \ln(n) - 2 \cdot \ln(2 \cdot \ln n). \tag{4.26}$$

Case 2: For $k > \frac{n}{2 \cdot \ln n}$:

$$\sum_{j=1}^{n} a_{kj}\bar{d}_j = \frac{n}{k} + \sum_{i=1}^{k-1} \frac{1}{i} - \sum_{i=k}^{n-1} \frac{1}{i} \tag{4.27}$$

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by adding $\sum_{i=1}^{k-1} \frac{1}{i}$ to the last two terms, we have

$$
\begin{aligned}
&= \frac{n}{k} + \left(\sum_{i=1}^{k-1} \frac{1}{i} + \sum_{i=1}^{k-1} \frac{1}{i} \right) - \left(\sum_{i=1}^{k-1} \frac{1}{i} + \sum_{i=k}^{n-1} \frac{1}{i} \right) \\
&= \frac{n}{k} + 2 \cdot \mathcal{H}_{k-1} - \mathcal{H}_{n-1}
\end{aligned}
$$

by $k > \frac{n}{2 \cdot \ln n}$, we have

$$
\begin{aligned}
&\geq \frac{n}{k} + 2 \cdot (\ln(n) - \ln(2 \cdot \ln n)) - \ln(n) - 1 \\
&= \frac{n}{k} - 1 + \ln(n) - 2 \cdot \ln(2 \cdot \ln n) \\
&\geq \ln(n) - 2 \cdot \ln(2 \cdot \ln n). \qquad (4.28)
\end{aligned}
$$

Case 3: For the case where $k = 1$, the value of the first row from the bottom is equal to the value of the second row from the bottom plus $\frac{n}{2} - 2$, which is positive for $n \geq 2$.

Finally, we are ready to bound the SPoA of $\hat{d}$. By summing up all the equations obtained from the construction of $\hat{d}$, we have:

$$
\hat{d}_1 = \sum_{i=2}^{n+1} \left(\sum_{k=1}^{i-1} \frac{1}{k} \right) \cdot \hat{d}_i. \qquad (4.29)
$$

Then for the upper bound on the SPoA, we have

$$
\frac{\hat{d}_1 + \sum_{i=2}^{n} \hat{d}_i}{\sum_{i=2}^{n+1} \hat{d}_i} \leq \frac{\hat{d}_1}{\sum_{i=2}^{n+1} \hat{d}_i} + 1
$$

by eq. (4.29), we have

$$
= \frac{\sum_{i=2}^{n+1} \left(\sum_{k=1}^{i-1} \frac{1}{k} \right) \cdot \hat{d}_i}{\sum_{i=2}^{n+1} \hat{d}_i} + 1
$$

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by replacing $\sum_{k=1}^{i-1} \frac{1}{k}$ with $\sum_{k=1}^{n} \frac{1}{k}$, we have

$$
\begin{aligned}
&\leq \frac{\sum_{i=2}^{n+1} \left(\sum_{k=1}^{n} \frac{1}{k}\right) \cdot \hat{d}_i}{\sum_{i=2}^{n+1} \hat{d}_i} + 1 \\
&= \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{\sum_{i=2}^{n+1} \hat{d}_i}{\sum_{i=2}^{n+1} \hat{d}_i} + 1 \\
&= \mathcal{H}_n + 1 = O(\log n). \qquad (4.30)
\end{aligned}
$$

For the lower bound on the SPoA, we have

$$
\frac{\hat{d}_1 + \sum_{i=2}^{n} \hat{d}_i}{\sum_{i=2}^{n+1} \hat{d}_i} \geq \frac{\hat{d}_1}{\sum_{i=2}^{n+1} \hat{d}_i}
$$

by Theorem 2, we have

$$
\geq \frac{\hat{d}_1}{5 \cdot \sum_{i=1}^{n} x_i^{\star}}
$$

since $x_i^{\star}$ is the optimal solution for linear program eq. (4.11), we have

$$
\begin{aligned}
&\geq \frac{\hat{d}_1}{5 \cdot \sum_{i=1}^{n} \bar{d}_i} \\
&= \frac{\hat{d}_1}{5 \cdot (2n-1)} \\
&= \frac{n \ln n - 2n \ln(2 \ln n)}{10n - 5} = \Omega(\log n) \qquad (4.31)
\end{aligned}
$$

In summary, we have constructed a specific instance of a clockwise ring game with edge costs denoted as $\hat{d}$. According to Theorem 1, we have established that in the SPNE, every player chooses the counter-clockwise direction, while the optimal solution requires all players to choose the clockwise direction. As a result, the SPoA for this instance is given by the expression $\frac{\hat{d}1+\sum i=2^n \hat{d}i}{\sum i=2^{n+1} \hat{d}_i}$. To analyze this SPoA value, we introduced the com-

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plementary slackness condition and conducted a thorough analysis. Ultimately, we proved that the value of the SPoA is $\Theta(\log n)$, which provides a lower bound for the worst-case sequential price of anarchy in the network formation game. $\square$

Corollary 1. *The worst-case Sequential Price of Anarchy of n-player undirected network formation games has a lower bound* $\Omega(\log n)$.

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Chapter 5 Conclusion and Future Study

In conclusion, our study highlights the limitations of sequential decision-making in network formation games. We propose a new restriction on the game model to address these limitations and improve the realism of outcomes in sequential games. By applying this restriction, we construct an instance that establishes a lower bound of $\Omega(\log n)$ that can applied in various undirected network scenarios. This lower bound reveals the trade-offs and inefficiencies inherent in sequential decision-making, as quantified by the SPoA. Our findings indicate that while sequential decision-making can avoid the unrealistic worst Nash Equilibrium in the example with two parallel edges, it may not always result in networks that closely approach the most efficient or optimal structures. In fact, in certain network configurations, it may even yield worse outcomes than the best Nash Equilibrium.

For future work, one direction is to explore upper bounds for undirected networks in the network formation game. Another avenue is to investigate alternative modifications to the game model and analyze their effects. In our study, we introduced the simple path restriction, but there may be other potential modifications worth exploring.

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