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碩士論文

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歐氏空間之傅立葉變換限制問題的探討

A Survey of The Restriction Problems in Euclidean Spaces

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摘要

本論文的目的是研究歐幾里得空間中的傅立葉限制問題及其在法爾科納距離集猜想中的應用。限制問題是調和分析領域中最知名的研究問題之一，且與其他研究領域（如偏微分方程和幾何測度論）有著重要的聯繫。在本論文中，我們主要詳細介紹已知的 Tomas-Stein 成果、雙線性限制估計以及 Bourgain 對法爾科納距離猜想的結果。

關鍵字：傅立葉限制問題，福爾科納猜想，調和分析，幾何測度論，Tomas-Stein 定理

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Abstract

The purpose of this dissertation is to study the Fourier restriction problems in Euclidean spaces and their applications to Falconer's distance set conjecture. Restriction problems are one of the most known research problems in the area of Harmonic analysis and have been found to have important connections to other research fields such as partial differential equations and geometric measure theory. In this dissertation, we mainly introduce in details the known Tomas-Stein results, bilinear restriction estimates and Bourgain's work on Falconer distance conjecture.

keywords:Fourier restriction problems, Falconer's conjecture, harmonic analysis, geometric measure theory, Tomas-Stein theorem

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Contents

致謝 **i**

摘要 **ii**

Abstract **iii**

List of Figures **v**

1 Introduction **1**

2 Introduction of restriction problems and the results of Tomas and Stein **4**

2.1 Some Known Results of Restriction Problem 4

2.2 Proof of Tomas-Stein Restriction Theorem 9

2.3 Conclusion of Restriction Conjecture 19

3 Connections between restriction problems and geometric measure theory **21**

3.1 The case for $n = 2$ 21

3.2 The case for $n \geq 3$ 35

References **52**

doi:10.6342/NTU202301952

List of Figures

Figure 1 19
Figure 2 20

doi:10.6342/NTU202301952

1 Introduction

The restriction problems of Fourier transform was first purposed by C. Fefferman in [5] in 1970. Given an appropriate function f on $\mathbb{R}^n$, its Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx \tag{1}$$

or the alternative form

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i x\cdot\xi} dx$$

(we usually use the form **(1)**). And we give the definition of restriction estimate on hypersurface.

Definition 1.1. *Let S be a hypersurface with boundary, we denote the estimate*

$$\left(\int_S \left|\widehat{f}(\xi)\right|^q d\sigma(\xi)\right)^{\frac{1}{q}} = \left\|\widehat{f}\right\|_{L^q(S)} \lesssim \|f\|_p \tag{2}$$

holds for all $f \in S(\mathbb{R}^n)$ by $R_S(p \to q)$ for $1 \leq p, q \leq \infty$, where $d\sigma$ is the surface measure of S.

Our question is when does $R_S(p \to q)$ holds for S be the unit sphere S^{n-1} in $\mathbb{R}^n$? Of course there are some variations of this problem. First we may think that the norm is of whole $\mathbb{R}^n$, which is

$$\left\|\widehat{f}\right\|_q \lesssim \|f\|_p \tag{3}$$

for all function f, we may assume f is a test function since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. There are now many well-known results of this question. We can also consider S be some specific surfaces, for example, paraboloid, or the bilinear form of the inequality, that is

$$\left\|\widehat{f_1 d\sigma_1}\widehat{f_2 d\sigma_2}\right\|_q \lesssim \|f_1\|_p \|f_2\|_p\,,$$

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where f_i is supported on the surface Σ_i with the measure σ_i for $i = 1, 2$.

In this paper we will introduce the famous result of the standard restriction problem, which is Tomas-Stein theorem.

Theorem 1.1 (Tomas-Stein restriction theorem). *$R_S(p \to 2)$ holds for $1 \le p \le \dfrac{2(n+1)}{n+3}$.*

It is proved by Peter A. Tomas [14] for $1 \le p < \dfrac{2(n+1)}{n+3}$, and the end point case is proved by Elias M. Stein [12].

We also introduce a application of Fourier restriction, K.J. Falconer given a conjecture of geometric measure theory in [4]:

Conjecture 1.1 (Falconer's conjecture). *For $n \ge 2$ and a compact subset E of $\mathbb{R}^n$, define*

$$\Delta(E) = \{|x - y| : x, y \in E\}.$$

Then $\dim_H(E) > \dfrac{n}{2} \Rightarrow |\Delta(E)| > 0$. This is called the Falconer's conjecture. We say Falconer's conjecture holds for constant C if $\dim_H(E) > C \Rightarrow |\Delta(E)| > 0$.

This problem is open in every dimension, Falconer proved $\dfrac{n}{2}$ is optimal and showed this conjecture holds for $\dfrac{n+1}{2}$ in the same literature. In [1], Bourgain improved the result in all n, especially in $n = 2$, it holds for $\dfrac{13}{9}$. Later, Wolff showed a better result in [15], he showed it holds for $\dfrac{4}{3}$ in $\mathbb{R}^n$. Erdgan [3] use the bilinear restriction estimate to improve the bound to $\dfrac{n}{2} + \dfrac{1}{3}$ for $n \ge 3$. Recently, the bound of $\mathbb{R}^2$ is improved to $\dfrac{5}{4}$ in [6], and the authors of [2] proved that $\dim(E) > \dfrac{n}{2} + \dfrac{1}{4}$ suffices if $n \ge 4$ is an even integer.

In **Section 2**, we will give some known results of restriction problem, and a detail of proof of **Theorem 1.1**. In **Section 3**, we will go through the proof of [1] for $\mathbb{R}^2$ and [3].

The following is the list of notation.

1. $X \lesssim Y$: $X \le CY$ for some constant C.

2. C_T : the constant depends on T.

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3. $X \approx Y$: $X \lesssim Y$ and $Y \lesssim X$.

4. $\dim_H(E)$: the Hausdorff dimension of E.

5. $|X|$: the Lebesgue measure of X.

6. χ_E : the characteristic function of E.

7. $B(x,r) := \{y : |x-y| < r\}$.

8. $A_R(L) := \{x \in \mathbb{R}^n : ||x| - R| < L,\ \text{where } R, L \in \mathbb{R}\}$.

9. $H^s(E)$: the s-dimensional Hausdorff measure of E.

10. $M(E) := \{\mu : \text{measure } \mu \text{ satisfies } 0 < \mu(E) < \infty\}$.

11. $S(E)$: the set of all Schwartz function on E.

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2 Introduction of restriction problems and the results of Tomas and Stein

2.1 Some Known Results of Restriction Problem

First, we consider the problem in whole $\mathbb{R}^n$. By Plancherel's theorem, we get

$$\left\|\widehat{f}\right\|_2 = \|f\|_2 ,$$

and by the triangle inequality, we have

$$\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$$

immediately. Using these two estimates with the interpolation, we obtain

$$\left\|\widehat{f}\right\|_{p'} \leq \|f\|_p \tag{4}$$

for all $1 \leq p \leq 2$, and p' is the Hölder conjugate exponent of p, which means it satisfies

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ for } 1 < p \leq 2,$$

and define $p' = \infty$ if $p = 1$.

Moreover, the estimate (4) is the best possible.

Theorem 2.1. *If* (3) *holds, then* $q = p'$ *and* $1 \leq p \leq 2$.

Proof. First, we prove that q needs to equal to p'. Consider

$$f(x) = \phi\left(\frac{x}{t}\right)$$

for some Schwartz function ϕ and $t > 0$ can be varied. We can see that

$$\widehat{f}(\xi) = t^n \widehat{\phi}(t\xi) ,$$

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so

$$\left\|\widehat{f}\right\|_q = t^n \left(\int_{\mathbb{R}^n} \left|\widehat{\phi}\,(t\xi)\right|^q d\xi\right)^{\frac{1}{q}} = t^{n-\frac{n}{q}} \left(\int_{\mathbb{R}^n} \left|\widehat{\phi}\,(\xi)\right|^q d\xi\right)^{\frac{1}{q}} \approx t^{n-\frac{n}{q}},$$
$$\|f\|_p \approx t^{\frac{n}{p}}.$$

Then **(3)** becomes

$$t^{n-\frac{n}{q}} \lesssim t^{\frac{n}{p}}.$$

It is only true if $n - \dfrac{n}{q} = \dfrac{n}{p} \Rightarrow q = p'$, since we can let $t > 1$ or $0 < t < 1$.

For $1 \leq p \leq 2$, since we assumed $1 \leq p$ at first, we only need prove $p \leq 2$. Given a Schwartz function ψ supported on $[0,1]^n$ and $\{a_k\}_{k=1}^N$ be i.i.d. random variables with

$$\mathbb{P}\,(a_k = 1) = \mathbb{P}\,(a_k = -1) = \frac{1}{2}$$

for all $1 \leq k \leq N$. Then we choose function f be

$$f(x) = \sum_{k=1}^N a_k \psi\,(x - ke_1)\,,$$

where e_1 is the first vector of standard basis of $\mathbb{R}^n$. We can easily see that

$$\widehat{f}\,(\xi) = \sum_{k=1}^N a_k \widehat{\psi}\,(\xi)\, e^{2\pi i k \xi_1}$$

and

$$\|f\|_p = \left(\int \left|\sum_{k=1}^N a_k \psi\,(x - ke_1)\right|^p dx\right)^{\frac{1}{p}} \approx \left(N \int |\psi(x)|^p\, dx\right)^{\frac{1}{p}} \approx N^{\frac{1}{p}}. \qquad (5)$$

Next, by Khinchin's inequality, we obtain

$$\mathbb{E}\left(\left|\widehat{f}(\xi)\right|^p\right)^{\frac{1}{p}} = \mathbb{E}\left(\left|\sum_{k=1}^N a_k \widehat{\psi}\,(\xi)\, e^{2\pi i k \xi_1}\right|^p\right)^{\frac{1}{p}}$$

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$$\approx \left(\sum_{k=1}^{N} \left| \widehat{\psi}\,(\xi)\, e^{2\pi i k \xi_1} \right|^2 \right)^{\frac{1}{2}} = \left(\sum_{k=1}^{N} \left| \widehat{\psi}(\xi) \right|^2 \right)^{\frac{1}{2}}$$

since $\left|e^{2\pi i k\xi_1}\right| = 1$. Then raise the both side to the power of p, integrate with respect to ξ and interchange the expectation and integral, we will get

$$\mathbb{E}\left(\left\| \widehat{f} \right\|_q^q \right) \approx \left\| \left(\sum_{k=1}^{N} \left| \widehat{\psi} \right|^2 \right)^{\frac{1}{2}} \right\|_q^q \approx N^{\frac{q}{2}}.$$

Thus, by the definition of expectation, there is a choice of $\{a_k\}_{k=1}^N$ such that

$$N^{\frac{q}{2}} \lesssim \left\| \widehat{f} \right\|_q^q \Rightarrow N^{\frac{1}{2}} \lesssim \left\| \widehat{f} \right\|_q . \tag{6}$$

According to the assumption, **(5)** and **(6)**, we obtain

$$N^{\frac{1}{2}} \lesssim \left\| \widehat{f} \right\|_q \lesssim \|f\|_p \approx N^{\frac{1}{p}},$$

so we need $p \leq 2$ if we let N tends to infinity. □

Motivated by these elementary inequalities, the restriction problems are to study the same kind of inequalities by replacing the left hand side by restricting $\widehat{f}$ on some subset of $\mathbb{R}^n$. These problems also play a very important role in many different problems in PDE and geometric measure theory e.t.c.

Theorem 2.2. *$R_S(p \to q)$ is equivalent to the following estimate:*

$$\left\| \widehat{f d\sigma} \right\|_{p'} \lesssim \|f\|_{L^{q'}(S)} \tag{7}$$

if $d\sigma \in M(\mathbb{R}^n)$.

Proof. Suppose that $R_S(p \to q)$ holds, by the Riesz representation formula, we have

$$\left\| \widehat{f d\sigma} \right\|_{p'} = \sup_{\|g\|_p = 1} \left| \int_{\mathbb{R}^n} \widehat{f d\sigma}(x) g(x) dx \right| = \sup_{\|g\|_p = 1} \left| \int_{\mathbb{R}^n} f(x) \widehat{g}(x) d\sigma(x) \right| \tag{8}$$

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for g a Schwartz function. Then

$$\left|\int_{\mathbb{R}^n} f(x)\widehat{g}(x)d\sigma(x)\right| \leq \|\widehat{g}\|_{L^q(S)} \|f\|_{L^{q'}(S)} \lesssim \|g\|_p \|f\|_{L^{q'}(S)}$$

by Hölder's inequality and our assumption. Putting this estimate back to **(8)**, we obtain

$$\sup_{\|g\|_p=1} \left|\int_{\mathbb{R}^n} f(x)\widehat{g}(x)d\sigma(x)\right| \lesssim \sup_{\|g\|_p=1} \|g\|_p \|f\|_{L^{q'}(S)} \leq \|f\|_{L^{q'}(S)}.$$

For the other side, we can use the same method to get

$$\left|\int_{\mathbb{R}^n} \widehat{f}(x)g(x)d\sigma(x)\right| = \left|\int_{\mathbb{R}^n} f(x)\widehat{gd\sigma}(x)dx\right| \leq \|f\|_p \left\|\widehat{gd\sigma}\right\|_{p'} \lesssim \|f\|_p \|g\|_{L^{q'}(S)},$$

so

$$\|f\|_{L^q(S)} = \sup_{\|g\|_{L^{q'}(S)}=1} \left|\int_{\mathbb{R}^n} \widehat{f}(x)g(x)d\sigma(x)\right| \lesssim \sup_{\|g\|_{L^{q'}(S)}=1} \|f\|_p \|g\|_{L^{q'}(S)} \leq \|f\|_p.$$

Thus, we have shown these two statements are equivalent. □

The Fourier transform of f restricted on S^{n-1} may not always make sense for arbitrary $f \in L^p$. First, for $p = 1$, we have $\widehat{f}$ is continuous and decays to zero as it goes to infinity. Thus, it is meaningful of $\widehat{f}$ restricted on S^{n-1}, we can also see that $R_S(1 \to q)$ holds for $q = \infty$. From the Hölder's inequality, we have

$$\left\|\widehat{f}\right\|_{L^q(S^{n-1})} \leq \left\|\widehat{f}\right\|_{L^\infty(S^{n-1})} \cdot \|1\|_{L^q(S^{n-1})} \lesssim \left\|\widehat{f}\right\|_{L^\infty(S^{n-1})} \lesssim \|f\|_1$$

for all $1 \leq q < \infty$ since S^{n-1} is compact. Hence, $R_S(1 \to q)$ holds for all $1 \leq q \leq \infty$. But for $p = 2$, let f be a L^2 function, $\widehat{f}$ is also a L^2 function by Plancherel's theorem. But $\widehat{f}$ is not meaningful on S^{n-1} in general since it is a measure zero set.

Now we introduce the restriction conjecture.

Conjecture 2.1 (restriction conjecture)**.** *Given $S = S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and $1 \leq p, q < \infty$, then $R_S(p \to q)$ holds if and only if the following two inequalities*

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hold.

$$p' \geq \frac{(n+1)q}{n-1} \tag{9}$$

$$p < \frac{2n}{n+1} \tag{10}$$

Why do we have this conjecture? Consider Σ be a surface with the form

$$\Sigma = \left\{ (\underline{x}, \phi(\underline{x})) : \underline{x} \in \mathbb{R}^{n-1} \right\},$$

where ϕ is a smooth function maps from $\mathbb{R}^{n-1}$ to $\mathbb{R}$. Without loss of generality, we may assume $\phi(0) = \nabla\phi(0) = 0$. Then we have the following property.

Property 2.1. *Suppose $\phi(x) = O\left(|x|^a\right)$ for $a \geq 2$, then $R_S(p \to q)$ holds only possible if*

$$p' \geq \frac{n+a-1}{n-1} q.$$

The proof is similar with part one of **theorem 2.1**. Thus, if $\Sigma = S$, we have $\phi(x) = O\left(|x|^2\right)$, so the constraint of property above becomes **(9)**. On the other hand, if $R_S(p \to q)$ holds, by the duality (**theorem 2.2**), we have

$$\left\| \widehat{f d\sigma} \right\|_{p'} \lesssim \|f\|_{L^{q'}(S)}$$

for all $f \in L^{q'}(S)$. In particular, choose f be the constant function 1, the this inequality becomes

$$\left\| \widehat{d\sigma} \right\|_{p'} \lesssim 1. \tag{11}$$

Using the fact that

$$\left| \widehat{d\sigma}(\xi) \right| \lesssim |1+\xi|^{\frac{1-n}{2}} \tag{12}$$

(see **chapter 14.2** of [9]), we obtain

$$\left\| \widehat{d\sigma} \right\|_{p'} \lesssim \int_{\mathbb{R}^n} |x|^{\frac{1-n}{2}} \, dx = \int_0^\infty (1+u)^{\frac{1-n}{2}p'+n-1} du.$$

If we consider u is far from the origin, the term inside the integral is comparable

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with $u^{\frac{1-n}{2}p'+n-1}$, so if we want it bounded by a constant, we need

$$\frac{1-n}{2}p' + n - 1 < -1 \Rightarrow p < \frac{2n}{n+1}. \tag{13}$$

Consequently, these two inequalities are proved to be necessary conditions, and the restriction conjecture says that they are also sufficient conditions.

The following theorem is an important result that proved by Peter A. Tomas and Elias M. Stein. Their proofs are often called Tomas-Stein methods.

2.2 Proof of Tomas-Stein Restriction Theorem

Proof. First, if **conjecture 2.1** is true, take $q = 2$, we have

$$p' \geq \frac{2n+2}{n-1} \Rightarrow 1 - \frac{1}{p} \leq \frac{n-1}{2n+2} \Rightarrow \frac{1}{p} \geq \frac{n+3}{2(n+1)} \Rightarrow p \leq \frac{2(n+1)}{2n+3}.$$

It seems that the conjecture is somewhat believable.

We say that $R_S(p \to 2)$ holds means

$$\begin{aligned} \left\|\widehat{f}\right\|_{L^2(S)} \lesssim \|f\|_p \Rightarrow \int \left|\widehat{f}(\xi)\right|^2 d\sigma(\xi) &\lesssim \|f\|_p^2 \\ \Rightarrow \int \widehat{f}(\xi) \cdot \overline{\widehat{f}}(\xi) d\sigma(\xi) &\lesssim \|f\|_p^2 \\ \Rightarrow \int \widehat{f}(\xi) \cdot \overline{\widehat{f * \widehat{d\sigma}}}(\xi) d\xi &\lesssim \|f\|_p^2 \\ \Rightarrow \int f(\xi) \cdot \overline{f * \widehat{d\sigma}}(\xi) d\xi &\lesssim \|f\|_p^2 \end{aligned} \tag{14}$$

by the Plancherel theorem. Next, by the Hölder's inequality, we can see if

$$\left\|f * \widehat{d\sigma}\right\|_{p'} \lesssim \|f\|_p \tag{15}$$

holds, then **(14)** holds. Therefore, we transform the $R_S(p \to 2)$ problem to the some $R(p \to p')$ type problem, and this transformation can be achieved because of $q = 2$.

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To approach the estimation **(15)**, we consider a radial bump function ϕ satisfies

$$\phi = 1 \text{ for } |x| \leq 1,$$
$$\phi = 0 \text{ for } |x| \geq 2$$

and has compact support. Then we define

$$\psi_k(x) = \phi\left(2^{-k}x\right) - \phi\left(2^{1-k}x\right), \tag{16}$$

ψ_k is supported on the annulus of $|x| \approx 2^k$ and is of size ≈ 1. Next, by the definition of ϕ and ψ, we have the following two equations:

$$\psi_k(x) = \psi_0\left(2^{-k}x\right) \tag{17}$$

and

$$1 - \phi(x) = \sum_{k=1}^{\infty} \psi_k(x). \tag{18}$$

(18) implies that

$$f * \widehat{d\sigma} = f * \left(\phi\widehat{d\sigma}\right) + f * \sum_{k=1}^{\infty} \psi_k \widehat{d\sigma}$$
$$\Rightarrow \left\|f * \widehat{d\sigma}\right\|_{p'} \leq \left\|f * \left(\phi\widehat{d\sigma}\right)\right\|_{p'} + \left\|\sum_{k=1}^{\infty} f * \left(\psi_k\widehat{d\sigma}\right)\right\|_{p'}.$$

Consequently, we can estimate $\left\|f * \left(\phi\widehat{d\sigma}\right)\right\|_{p'}$ and $\left\|\sum_{k=1}^{\infty} f * \left(\psi_k\widehat{d\sigma}\right)\right\|_{p'}$ separately instead of $\left\|f * \widehat{d\sigma}\right\|_{p'}$.

For the first one, by the Young's convolution inequality,

$$\left\|f * \left(\phi\widehat{d\sigma}\right)\right\|_{p'} \leq \|f\|_p \left\|\phi\widehat{d\sigma}\right\|_{\frac{p'}{2}} \lesssim \|f\|_p \tag{19}$$

since $\phi\widehat{d\sigma}$ is a bump function in C_0^∞.

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For the other one, first we observe that

$$\left\|\sum_{k=1}^{\infty} f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{p'} \leq \sum_{k=1}^{\infty} \left\|f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{p'},$$

so if we can show $\left\|f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{p'} \lesssim 2^{-\varepsilon k} \|f\|_p$ for some $\varepsilon > 0$, we may easily conclude that

$$\left\|\sum_{k=1}^{\infty} f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{p'} \lesssim \|f\|_p .$$

To show $\left\|f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{p'} \lesssim 2^{-\varepsilon k} \|f\|_p$, we use the interpolation of $R_S(1 \to \infty)$ and $R_S(2 \to 2)$ with the restricted constants depend on some order of 2^k.

First, for $R_S(\infty \to 1)$, since ψ_k is supported on the annulus of $|x| \approx 2^k$, we get

$$\left\|f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{\infty} \lesssim \left\|\psi_k \widehat{d\sigma}\right\|_{\infty} \|f\|_1 \lesssim 2^{-\frac{n-1}{2}k} \|f\|_1 \tag{20}$$

by the Young's convolution inequality and **(12)**

Next, for $R_S(2 \to 2)$, we can see that

$$\left\|f * \left(\psi_k \widehat{d\sigma}\right)\right\|_2 = \left\|\widehat{f} \cdot \left(\widehat{\psi_k} * d\sigma\right)\right\|_2 \leq \left\|\widehat{\psi_k} * d\sigma\right\|_{\infty} \left\|\widehat{f}\right\|_2 \tag{21}$$

by Plancherel's theorem and Hölder's inequality. Compute $\widehat{\psi_k}$ directly by its definition, we obtain

$$\widehat{\psi_k}(x) = 2^{nk} \widehat{\psi_0}\left(2^k x\right),$$

and we have $\widehat{\psi_0}$ is a Schwartz function since ψ_0 is also a Schwartz function. Thus, $\widehat{\psi_k}(x)$ has the Schwartz decay:

$$\left|\widehat{\psi_k}(x)\right| = \left|2^{nk} \widehat{\psi_0}\left(2^k x\right)\right| \lesssim \frac{2^{nk}}{\left(1 + 2^k |x|\right)^N} \tag{22}$$

for all $N > 0$. Take $N = n$, we obtain the estimate

$$\left|\widehat{\psi_k} * d\sigma\right| \lesssim \int_{S^{n-1}} \frac{2^{nk}}{\left(1 + 2^k |x-y|\right)^n} d\sigma(y)$$

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$$
\begin{aligned}
&= \int_{\{y:|x-y|\le 2^{-k}\}} \frac{2^{nk}}{(1+2^k|x-y|)^n} d\sigma(y) \\
&\quad + \sum_{i=-k}^{\infty} \int_{\{y:2^i<|x-y|\le 2^{i+1}\}} \frac{2^{nk}}{(1+2^k|x-y|)^n} d\sigma(y) \\
&\le 2^{nk} \int_{\{y:|x-y|\le 2^{-k}\}} 1\cdot d\sigma(y) + \sum_{i=-k}^{\infty} \int_{\{y:2^i<|x-y|\le 2^{i+1}\}} \frac{2^{nk}}{(1+2^{k+i})^n} d\sigma(y) \\
&\lesssim 2^{nk}\cdot 2^{-k(n-1)} + \sum_{i=-k}^{\infty} 2^{nk}\cdot 2^{-(k+i)n}\cdot 2^{(i+1)(n-1)} \\
&= 2^k + 2^{n-1}\sum_{i=-k}^{\infty} 2^{-i} = 2^k + 2^{n-1}\cdot 2^{k+1} \lesssim 2^k. \qquad (23)
\end{aligned}
$$

Hence, we have

$$
\left\| \widehat{\psi_k} * d\sigma \right\|_\infty \lesssim 2^k,
$$

and **(21)** becomes

$$
\left\| f * \left(\psi_k \widehat{d\sigma} \right) \right\|_2 \lesssim 2^k \|f\|_2 . \qquad (24)
$$

As we mention above, using the interpolation by **(20)** and **(24)**, we obtain

$$
\left\| f * \left(\psi_k \widehat{d\sigma} \right) \right\|_{p'} \lesssim 2^{(1-\theta)\left(\frac{k(1-n)}{2}\right)} \cdot 2^{\theta k} \|f\|_p = 2^{k\left(\theta + \frac{(1-n)(1-\theta)}{2}\right)} \|f\|_p
$$

for

$$
\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{1} \Rightarrow p = \frac{2}{2-\theta} \qquad (25)
$$

with $0 < \theta < 1$. As our desire, we want

$$
2^{k\left(\theta + \frac{(1-n)(1-\theta)}{2}\right)} \le 2^{-\varepsilon k} \Rightarrow \theta + \frac{(1-n)(1-\theta)}{2} \le -\varepsilon < 0 \qquad (26)
$$

$$
\Rightarrow 2\theta + (n-1)\theta + 1 - n < 0 \Rightarrow \theta < \frac{n-1}{n+1}. \qquad (27)
$$

Putting this back to **(25)**, we have the range of p is

$$
p = \frac{2}{2-\theta} < \frac{2}{2-\frac{n-1}{n+1}} = \frac{2(n+1)}{n+3}. \qquad (28)
$$

Therefore, we conclude that $R_S(p \to 2)$ if $1 \le p < \dfrac{2(n+1)}{n+3}$.

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For the endpoint $p = \dfrac{2(n+1)}{n+3}$, Stein proved it by the similar method of Tomas in 1975, he used the complex interpolation instead of the real interpolation.

As the proof above, it suffices to show that

$$\left\|\sum_{k=1}^{\infty} f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{p'} \lesssim \|f\|_p$$

for $p = \dfrac{2(n+1)}{n+3}$. If we use the same method of Tomas, we have $\varepsilon = 0$ and then the sum will go to infinity. Thus, we prove the following two inequalities instead of **(20)** and **(24)**:

$$\left\|\sum_{k=1}^{\infty} 2^{\left(\frac{n-1}{2}+iy\right)k} f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{\infty} \lesssim \|f\|_1 \tag{29}$$

and

$$\left\|\sum_{k=1}^{\infty} 2^{(-1+iy)k} f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{2} \lesssim \|f\|_2 \tag{30}$$

for all $y \in \mathbb{R}$.

Similar with **(20)**, we use the Young's convolution inequality to get

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} 2^{\left(\frac{n-1}{2}+iy\right)k} f * \left(\psi_k \widehat{d\sigma}\right)\right\|_{\infty} &= \left\|f * \sum_{k=1}^{\infty} 2^{\left(\frac{n-1}{2}+iy\right)k} \left(\psi_k \widehat{d\sigma}\right)\right\|_{\infty} \\ &\leq \left\|\sum_{k=1}^{\infty} 2^{\left(\frac{n-1}{2}+iy\right)k} \left(\psi_k \widehat{d\sigma}\right)\right\|_{\infty} \|f\|_1 .\end{aligned}$$

If we can prove

$$\left\|\sum_{k=1}^{\infty} 2^{\left(\frac{n-1}{2}+iy\right)k} \left(\psi_k \widehat{d\sigma}\right)\right\|_{\infty} \lesssim 1,$$

then **(29)** holds immediately. But this is easy because of **(12)** and

$$\sum_{k=1}^{\infty} 2^{\left(\frac{n-1}{2}+iy\right)k} \psi_k(x) = O\left(|x|^{\frac{n-1}{2}}\right),$$

since $|x| \approx 2^k$.

For **(30)**, we need to use some estimate which is different from Tomas'. First,

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by the Young's convolution inequality, it is sufficient to show

$$\left\| \sum_{k=1}^{\infty} 2^{(-1+iy)k} \left(\widehat{\psi_k} * d\sigma \right) \right\|_{\infty} \lesssim 1. \tag{31}$$

We may ignore the imaginary term, so it is only need to prove

$$\sum_{k=1}^{\infty} 2^{-k} \left| \widehat{\psi_k} * d\sigma \right| \lesssim 1. \tag{32}$$

First, we show that

$$\left| \widehat{\psi_k} * d\sigma \right| \lesssim \begin{cases} 1 + 2^{2k} d(x,S), & \text{if } d(x,S) \leq 2^{(-1-\varepsilon)k}, \\ \dfrac{2^{nk}}{(2^k d(x,S))^M}, & \text{if } d(x,S) \geq 2^{(-1+\varepsilon)k} \end{cases} \tag{33}$$

for every $\varepsilon > 0$ and $M > 0$, where $d(x,S) = ||x| - 1|$, it means the distance between x and the unit sphere. For $d(x,S) \geq 2^{(-1+\varepsilon)k}$, just use the same method of estimate **(23)** and the fact that $|x-y| \geq ||x| - |y|| = d(x,S)$. For the other side, we can not just use the same estimate, so we need to find another way to approach $|\psi_k * d\sigma|$. Consider

$$\nabla \widehat{\psi_k}(x) = \nabla 2^{nk} \widehat{\psi_0}(2^k x) = 2^{(n+1)k} \nabla \widehat{\psi_0}(2^k x) \tag{34}$$

and $\nabla \widehat{\psi_0}$ is also a Schwartz function, so it has the Schwartz decay, too. Therefore, we have

$$2^{-k} \left| \left(\nabla \widehat{\psi_k} \right) * d\sigma \right| \lesssim 2^k \Rightarrow \left| \nabla \left(\widehat{\psi_k} * d\sigma \right) \right| = \left| \left(\nabla \widehat{\psi_k} \right) * d\sigma \right| \lesssim 2^{2k}$$

by **(23)**. Write $F_k(x) = \widehat{\psi_k} * d\sigma(x)$, we may assume $|x| < 1$, then

$$\left| F_k(x) - F_k\left(\frac{x}{|x|}\right) \right| \leq \sup_{t \in \left[x, \frac{x}{|x|}\right]} |\nabla F_k(t)| \cdot \left| x - \frac{x}{|x|} \right| \lesssim 2^{2k} d(x,S).$$

$$\Rightarrow |F_k(x)| \lesssim \left| F_k\left(\frac{x}{|x|}\right) \right| + 2^{2k} d(x,S).$$

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If we can show $\left|F_k\left(\frac{x}{|x|}\right)\right|$ is bounded, then we are done. By the rotational invariance of $d\sigma$, we suppose $\frac{x}{|x|} = e_n$ be the last vector of basis, so

$$|F_k(e_n)| \leq \int_{|e_n-y|\leq 100} \left|\widehat{\psi_k}(e_n-y)\right| d\sigma(y) + \int_{|e_n-y|>100} \left|\widehat{\psi_k}(e_n-y)\right| d\sigma(y).$$

The second part is finite by the similar method of **(23)**, For the first part, we write

$$y = \left(\underline{y}, \left(1-|\underline{y}|^2\right)^{\frac{1}{2}}\right),$$

where $\underline{y} \in \mathbb{R}^{n-1}$. Thus, the integral becomes

$$\int_{|\underline{y}|\leq R} \left|\widehat{\psi_k}\left(\underline{y}, 1-\left(1-|\underline{y}|^2\right)^{\frac{1}{2}}\right)\right| \cdot \frac{1}{\left(1-|\underline{y}|^2\right)^{\frac{1}{2}}} d\underline{y} \lesssim \int_{\mathbb{R}^{n-1}} \left|\widehat{\psi_k}\left(\underline{y}, O\left(|\underline{y}|^2\right)\right)\right| d\underline{y}$$

for some constant R. We now to show that this integral is comparable with

$$\int_{\mathbb{R}^{n-1}} \left|\widehat{\psi_k}\left(\underline{y}, 0\right)\right| d\underline{y} + O(1).$$

By the Schwartz decay, we obtain

$$\left|\widehat{\psi_k}\left(\underline{y}, O\left(|\underline{y}|^2\right)\right) - \widehat{\psi_k}\left(\underline{y}, 0\right)\right| \lesssim \frac{2^{(n+1)k}}{\left(1+2^k|\underline{y}|\right)^N} \cdot O\left(|\underline{y}|^2\right)$$

for all $N > 0$, and then

$$\left|\widehat{\psi_k}\left(\underline{y}, O\left(|\underline{y}|^2\right)\right)\right| \lesssim \left|\widehat{\psi_k}\left(\underline{y}, 0\right)\right| + \frac{2^{(n+1)k}}{\left(1+2^k|\underline{y}|\right)^N} \cdot O\left(|\underline{y}|^2\right)$$
$$\Rightarrow \int_{\mathbb{R}^{n-1}} \left|\widehat{\psi_k}\left(\underline{y}, O\left(|\underline{y}|^2\right)\right)\right| d\underline{y} \lesssim \int_{\mathbb{R}^{n-1}} \left|\widehat{\psi_k}\left(\underline{y}, 0\right)\right| d\underline{y} + \int_{\mathbb{R}^{n-1}} \frac{2^{(n+1)k} O\left(|\underline{y}|^2\right)}{\left(1+2^k|\underline{y}|\right)^N} d\underline{y}.$$

Consequently, we need to show the last integral is $O(1)$. We use the method of

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estimate **(23)** again, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \frac{2^{(n+1)k}\left|\underline{y}\right|^2}{\left(1+2^k\left|\underline{y}\right|\right)^N} d\underline{y} \\
=&\int_{\left\{\underline{y}:\left|\underline{y}\right|\le 2^{-k}\right\}} \frac{2^{(n+1)k}\left|\underline{y}\right|^2}{\left(1+2^k\left|\underline{y}\right|\right)^N} d\underline{y} + \sum_{i=1}^{\infty}\int_{\left\{\underline{y}:2^{-k+i-1}<\left|\underline{y}\right|\le 2^{-k+i}\right\}} \frac{2^{(n+1)k}\left|\underline{y}\right|^2}{\left(1+2^k\left|\underline{y}\right|\right)^N} d\underline{y} \\
\lesssim& 2^{(n+1)k}\int_{\left\{\underline{y}:\left|\underline{y}\right|\le 2^{-k}\right\}} 2^{-2k} d\underline{y} + 2^{(n+1)k}\sum_{i=1}^{\infty}\int_{\left\{\underline{y}:2^{-k+i-1}<\left|\underline{y}\right|\le 2^{-k+i}\right\}} 2^{2(-k+i)}\cdot 2^{-N(i-i)} d\underline{y} \\
\lesssim& 2^{(n-1)k}\cdot 2^{-k(n-1)} + \sum_{i=1}^{\infty} 2^{(n-N+1)i+N} = 1 + \sum_{i=1}^{\infty} 2^{(n-N+1)i+N}.
\end{aligned}$$

Taking $N = n+2$, the summation converges, then this integral is $O(1)$. Next, we may choose suitable ϕ such that the integral

$$\int_{\mathbb{R}^{n-1}} \left|\widehat{\psi_k}\left(\underline{y},0\right)\right| d\underline{y} = 0.$$

Thus, we prove that $\left|F_k\left(\frac{x}{|x|}\right)\right|$ is $O(1)$, so **(23)** is true. According to this inequality, we have

$$\sum_{k=1}^{\infty} 2^{-k}\left|\widehat{\psi_k} * d\sigma\right| \lesssim \begin{cases} \sum_{k=1}^{\infty} 2^{-k} + 2^{-\varepsilon k}, & \text{if } d(x,S) \le 2^{(-1-\varepsilon)k}, \\ \sum_{k=1}^{\infty} 2^{(n-1-M\varepsilon)k}, & \text{if } d(x,S) \ge 2^{(-1+\varepsilon)k}. \end{cases} \tag{35}$$

The above one converges, the under one also converges if we choose M such that $n-1-M\varepsilon < 0$. Therefore, by the limit argument, the sum is convergent and bounded by some constant for all $d(x,S)$. Then we prove that **(32)** holds. Finally, using the Stein complex interpolation, we can finish the end point case of Tomas-Stein theorem.

Theorem 2.3 (Stein complex interpolation theorem)**.** *Let T_z be an analytic family of linear operators of admissible growth defined in the strip $\{z : 0 \le R(z) \le 1\}$.*

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Suppose that $1 \le p_1, p_2, q_1, q_2 \le \infty$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$ *with* $0 \le \theta \le 1$ *and*

$$\|T_{iy}(f)\|_{q_1} \le C_1(y)\|f\|_{p_1}, \ \|T_{1+iy}(f)\|_{q_2} \le C_2(y)\|f\|_{p_2}$$

with $\log|C_i(y)| \le Ce^{a|y|}$, $i = 1, 2$ *and* $a < \pi$, *then*

$$\|T_\theta(f)\|_{q_\theta} \lesssim \|f\|_{p_\theta}.$$

The proof and detailed definition is in [11]. To use this interpolation, we set $T_{iy}(f) = \sum_{k=1}^{\infty} 2^{(iy)k} f * \left(\psi_k \widehat{d\sigma}\right)$, we have $p_1 = 2$, $p_2 = 1$, normalize the interval $\left[-1, \frac{n-1}{2}\right]$ to have the length 1, the interval becomes $\left[\frac{n-1}{n+1}, \frac{-2}{n-1}\right]$. And then

$$\|T_0(f)\|_{p_0'} = \left\| \sum_{k=1}^{\infty} f * \left(\psi_k \widehat{d\sigma}\right) \right\|_{p_0'} \lesssim \|f\|_{p_0},$$

where P_0 satisfies

$$\frac{1}{p_0} = \frac{\frac{n-1}{n+1} - \theta}{2} + \frac{\theta - \frac{-2}{n-1}}{1} \Rightarrow p_0 = \frac{2(n+1)}{n+3}.$$

So we finish the part of endpoint $\frac{2(n+1)}{n+3}$, and then completes the whole proof. □

The theorem says that $R_S(p \to 2)$ is valid for $1 \le p \le \frac{2(n+1)}{2n+3}$, next we use the Knapp example to show the bound $\frac{2(n+1)}{2n+3}$ is sharp. Let $e_n = (0, 0, \cdots, 1)$ be the last unit vector of $\mathbb{R}^n$, define

$$A_\delta = \left\{ t \in S^{n-1} : 1 - e_n \cdot t \le \delta^2 \right\}$$

with $0 < \delta < 1$. Consider $f = \chi_{A_\delta}$, we have

$$\|f\|_{L^2(S^{n-1})} \approx \delta^{\frac{n-1}{2}}. \tag{36}$$

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Next, using **lemma 3.18** of [9] with $c = \dfrac{1}{12n}$, we have

$$\left|\widehat{f}(\xi)\right| \gtrsim \delta^{n-1}$$

for $\xi \in B_\delta$, where

$$B_\delta = \left\{\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n : |\xi_i| \le \frac{c}{\delta} \text{ for } i = 1, \cdots, n-1,\ |\xi_n| \le \frac{c}{\delta^2}\right\}.$$

Then we can estimate the L^q norm of $\widehat{f}$,

$$\left\|\widehat{f}\right\|_{q'} \gtrsim \delta^{n-1} |B_\delta|^{\frac{1}{q'}} = \delta^{n-1} \left(2^n c^n \delta^{-n-1}\right)^{\frac{1}{q'}} \gtrsim \delta^{n-1-\frac{n+1}{q'}}. \tag{37}$$

Thus, combine **(36)**, **(37)** and the dual form of restriction inequality, we get

$$\delta^{n-1-\frac{n+1}{q'}} \lesssim \left\|\widehat{f}\right\|_{q'} \lesssim \|f\|_{L^2(S^{n-1})} \approx \delta^{\frac{n-1}{2}},$$

which implies that $n - 1 - \dfrac{n+1}{q'} \ge \dfrac{n-1}{2} \Rightarrow q' \ge \dfrac{2(n+1)}{n-1}$ since $0 < \delta < 1$. Hence the bound of q is $q \le \dfrac{2(n+1)}{2n+3}$, and it can not greater than $\dfrac{2(n+1)}{2n+3}$. That shows the bound is sharp.

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2.3 Conclusion of Restriction Conjecture

In the **Section 2.1**, we give two necessary conditions of **(7)** for $S = S^{n-1}$. By Tomas-Stein theorem, we have $R_S(p \to q)$ holds for $(p', q') = \left(\frac{2(n+1)}{n-1}, 2\right)$, and by a simple estimate

$$\left|\widehat{fd\sigma}(\xi)\right| = \left|\int_{S^{n-1}} f(x)e^{-2\pi i x\cdot\xi} d\sigma(x)\right| \leq \int_{S^{n-1}} |f(x)|\, d\sigma(x) = \|f\|_{L^1(S)},$$

we have $R_S(p \to q)$ holds for $(p', q') = (\infty, 1)$. Thus, we have $R_S(p \to q)$ holds for $\left(\frac{1}{p'}, \frac{1}{q'}\right)$ on the line between $\left(\frac{1}{\infty}, 1\right)$ and $\left(\frac{n-1}{2(n+1)}, \frac{1}{2}\right)$ using the interpolation Next, if **(7)** holds for p' and q', then it also holds for $\tilde{p'} \geq p'$ and $\tilde{q'} \geq q'$ by Hölder's inequality. Combining all these conditions, we can draw a region of validity on $\left(\frac{1}{q'}, \frac{1}{p'}\right)$-diagram. On the other hand, by the argument of **(13)**, we know that $\widehat{d\sigma}$ is

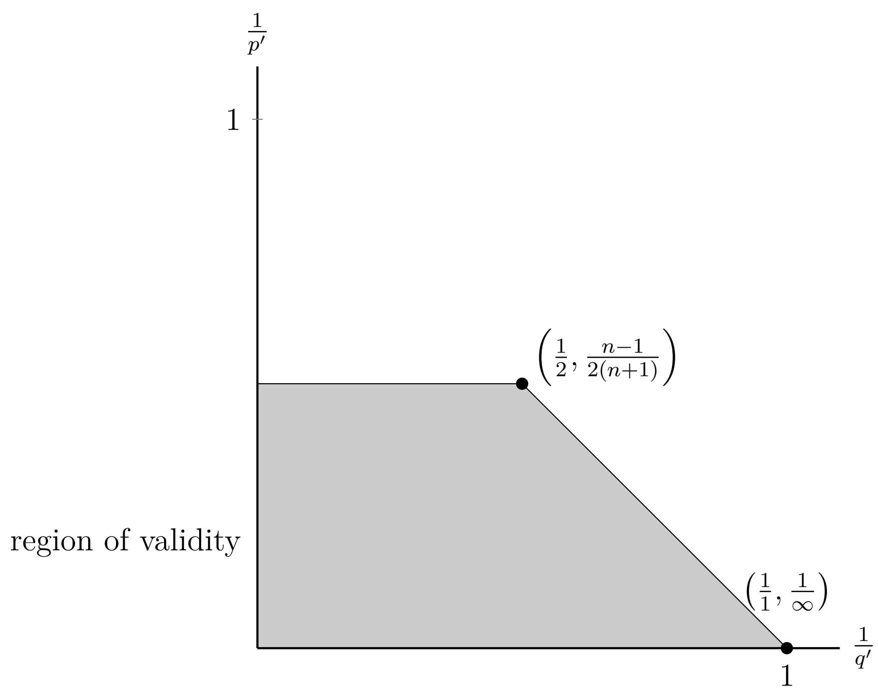

Figure 1

not a $L^{p'}$ function for $p' \leq \frac{2n}{n-1}$. Because of the sharpness of Tomas-Stein theorem, the best point can not exceed the line $y = \frac{n-1}{n+1} \cdot (-x+1)$. Thus, we guess the best possible point will be $\left(\frac{n-1}{2n}, \frac{n-1}{2n}\right)$. Also, we may impose a weaker conjecture that to be $\left(\frac{1}{q}, \frac{1}{p'}\right) = \left(\frac{1}{\infty}, \frac{n-1}{2n}\right)$ i.e. $p' = \frac{2n}{n-1}$ and $q = \infty$. However, it has been proven

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by Bourgain that they are equivalent.

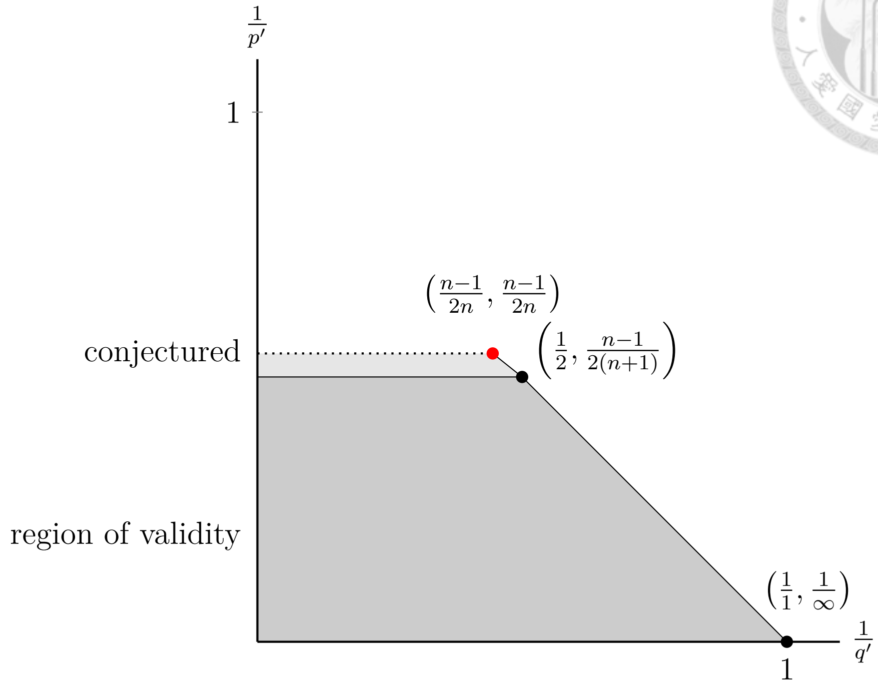

Figure 2

There is a large gap from $\left(\frac{1}{2}, \frac{n-1}{2(n+1)}\right)$ to $\left(\frac{n-1}{2n}, \frac{n-1}{2n}\right)$. Back to the proof of Tomas-Stein theorem, Due to the case we deal with is $q = 2$, we can easily transform $\left\|\widehat{f}\right\|_{L^2(S)} \lesssim \|f\|_p$ into $\left\|f * \widehat{d\sigma}\right\|_{p'} \lesssim \|f\|_p$. In other cases, it would not be such easy to get a clear form to estimate. Also we know that $\widehat{d\sigma}$ will affect the bound directly, so for the other surfaces, the best point will be different. That is why restriction problem is a vast and fascinating field which is still activating nowadays.

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3 Connections between restriction problems and geometric measure theory

3.1 The case for $n = 2$

Before going through the detail of [1], we need some lemmas first.

Lemma 3.1 (Frostman lemma). *Given a Borel set $E \subset \mathbb{R}^n$ and $0 \le s \le n$, then $H^s(E) > 0$ if and only if there exists $\mu \in M(E)$ satisfies*

$$\mu\left(B\left(x, r\right)\right) \le r^s, \forall x \in \mathbb{R}^n,\ r > 0. \tag{38}$$

Proof. ($\Longleftarrow$)

Given a covering of balls $\{B_i\}$ with radius r cover E and $\mu \in M(E)$ satisfies **(38)**, then

$$\sum_i \left(diam\left(B_i\right)\right)^s = \sum_i (r)^s \ge \sum_i \mu\left(B_i\right)^s \ge \mu(A) > 0. \tag{39}$$

So by the definition of $H^s(E)$, we obtain $H^s(E) > 0$.

($\Rightarrow$)

We only prove the version that E is compact. Assume E is compact and is contained in a dyadic cube. Since $H^s(E) > 0$, there is a constant $c > 0$ such that

$$\sum_j diam(E_j)^s \ge c > 0$$

for all covering $\{E_j\}$ of E. Now consider a dyadic system, let

$$D_m = \left\{\text{all dyadic cubes of length } 2^{-m}\right\} = \left\{\prod_{i=1}^n \left[\frac{k_i}{2^m}, \frac{k_i + 1}{2^m}\right) \middle| (k_1, \cdots, k_n) \in \mathbb{Z}^n\right\}.$$

For all $m > 0$, define a measure $\mu_{m,m}$ on $\mathbb{R}^n$ by

$$\mu_{m,m}\Big|_Q = \begin{cases} \frac{1}{2^{ms}} \frac{|\cdot|_Q}{|Q|}, & \text{if } E \cap Q \ne \phi, \\ 0, & \text{if } E \cap Q = \phi \end{cases} \tag{40}$$

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for all $Q \in D_m$. Since E is compact, there exists a smallest $k_m \in \mathbb{R}$ such that $E \subseteq Q'$ for some $Q' \in D_{m-k_m}$. So for $0 \leq k \leq k_m - 1$, we define

$$\mu_{m,m-k-1}\Big|_P = \begin{cases} \mu_{m,m-k}\Big|_P, & \text{if } \mu_{m,m-k}(P) \leq \frac{1}{2^{(m-k-1)s}}, \\ \frac{1}{2^{(m-k-1)s}} \frac{\mu_{m,m-k}\big|_P}{\mu_{m,m-k}(P)}, & \text{if } \mu_{m,m-k}(P) > \frac{1}{2^{(m-k-1)s}} \end{cases} \tag{41}$$

for all $P \in D_{m-k-1}$. Let $\mu_m = \mu_{m,m-k_m}$, we can see that for each stages, the measure of dyadic cubes is non-increasing since $\mu_{m,m-k}(P) > \dfrac{1}{2^{(m-k-1)s}} \Rightarrow \dfrac{1}{2^{(m-k-1)s}} \dfrac{1}{\mu_{m,m-k}(P)} < 1$. So we have for $0 \leq k \leq k_m$,

$$\mu_m(Q) \leq \frac{1}{2^{(m-k)s}} \tag{42}$$

for all $Q \in D_{m-k}$. Next, for all $x \in E$ and $m \geq 0$, we can find a maximal dyadic cube $Q \in D_{m-k}$ for some k satisfies

$$\mu_m(Q) = \frac{1}{2^{(m-k)s}} = \frac{diam(Q)^s}{n^{\frac{s}{2}}}. \tag{43}$$

Picking for each of $x \in E$ the largest such Q, we obtain disjoint cubes $Q_1, \cdots, Q_l$ with $E \subseteq \bigcup_{i=1}^{l} Q_i$ since E is compact. Then

$$\mu_m(\mathbb{R}^n) = \sum_{i=1}^{l} \mu_m(Q_i) = \sum_{i=1}^{l} \frac{diam(Q_i)^s}{n^{\frac{s}{2}}} \geq \frac{c}{n^{\frac{s}{2}}}. \tag{44}$$

Consider $\nu_m = \dfrac{\mu_m}{\mu_m(\mathbb{R}^n)}$, then $\nu_m(\mathbb{R}^n) = 1$ and

$$\nu_m(Q) = \frac{\mu_m(Q)}{c \cdot n^{-\frac{s}{2}}} \leq \frac{n^{\frac{s}{2}}}{c \cdot 2^{(m-k)s}} < \infty \tag{45}$$

for $Q \in D_{m-k}$ by **(42)** and **(44)**. So by the theorem 1.23 in [8], $\{\nu_m\}_{m=1}^{\infty}$ possesses a weakly convergent subsequence $\{\nu_{m_l}\}_{l=1}^{\infty}$ with $\lim_{l\to\infty} \nu_{m_l} = \nu$ and $\nu \in M(E)$ with

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$\nu(E) \leq \nu(\mathbb{R}^n) = 1$. Finally, for $x \in \mathbb{R}^n,\ 0 < r < 1$, we have

$$B(x,r) \subseteq int\left(\bigcup_{i=1}^{2^n} Q_i\right)$$

with $Q_i \in D_p$ for some p such that $diam(Q_i) = \dfrac{n^{\frac{1}{2}}}{2^p} \leq 4n^{\frac{1}{2}}r$. Then for $m \geq p$, we obtain

$$\nu_m\left(\bigcup_{i=1}^{2^n} Q_i\right) \leq \sum_{i=1}^{2^n} 2^{-ps} \cdot \frac{n^{\frac{s}{2}}}{c} = 2^n \cdot 2^{-ps} \cdot \frac{n^{\frac{s}{2}}}{c} \leq 2^n\,(4r)^s\,\frac{2^{\frac{s}{2}}}{c} = \left(\frac{2^{n+2s} \cdot n^{\frac{s}{2}}}{c}\right) r^s$$

by **(45)**. So

$$\nu\left(B\left(x,r\right)\right) \leq \nu\left(\bigcup_{i=1}^{2^n} Q_i\right) \leq \liminf_{l\to\infty} \nu_{m_l}\left(\bigcup_{i=1}^{2^n} Q_i\right) \leq \left(\frac{2^{n+2s} \cdot n^{\frac{s}{2}}}{c}\right) r^s. \tag{46}$$

Then the measure $\mu = \left(\dfrac{2^{n+2s} \cdot n^{\frac{s}{2}}}{c}\right)^{-1} \nu$ is what we want. □

Now given $s > 0$ and a Borel measure μ, we define the s-energy be

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu(x)d\mu(y) = \int \left(k * \mu\right)(x) d\mu(x), \tag{47}$$

where $k_s(x)$ is the Riesz kernel:

$$k_s\left(x\right) = |x|^{-s},\ x \in \mathbb{R}^n. \tag{48}$$

We have the following proposition immediately.

Property 3.1. *If μ has compact support and satisfies* (38), *then*

$$I_s(\mu) < \infty \Rightarrow I_t(\mu) < \infty$$

for $0 < t < s$.

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Proof. First we have

$$\int |x-y|^{-s}\, d\mu(x) = \int_0^\infty \mu\left(\left\{x : |x-y|^{-s} \geq u\right\}\right) du = \int_0^\infty \mu\left(B(y, u^{-\frac{1}{s}})\right) du$$
$$= s\int_0^\infty r^{-s-1}\mu(B(y,r))dr,$$

and then

$$I_t(\mu) \leq t \iint_0^{diam(supp(\mu))} \frac{\mu\left(B\left(y,r\right)\right)}{r^{t+1}} dr d\mu(y) \leq t \iint_0^{diam(supp(\mu))} \frac{r^s}{r^{t+1}} dr d\mu(y)$$
$$= t\mu(\mathbb{R}^n)\int_0^{diam(supp(\mu))} r^{s-t-1} dr < \infty$$

since $t,\ \mu(\mathbb{R}^n)\ diam(supp\,(\mu))$ are finite and $s-t-1 > -1$. $\square$

If μ has compact support, then $I_s(\mu) < \infty \Rightarrow \int |x-y|^{-s}\, d\mu(x) < \infty$ for μ almost $y \in \mathbb{R}^n$. Thus, we can find $0 < M < \infty$ such that

$$A := \left\{ y : \int |x-y|^{-s}\, d\mu(x) < M \right\}$$

has positive measure for μ. It means that for all $x \in \mathbb{R}^n$ and $r > 0$, we have

$$\mu\left(A \cap B(x,r)\right) = \int_{A\cap B(x,r)} d\mu(y) = \int_{A\cap B(x,r)} |z-y|^s\, |z-y|^{-s}\, d\mu(y)$$
$$\leq (2r)^s \int_{A\cap B(x,r)} |z-y|^{-s}\, d\mu(y) < M(2r)^s,$$

where $z \in A\cap B(x,r)$. Then by Frostman's lemma, if $\mu \in M(A)$, we have $H^s(A) > 0$. Moreover, we have

$$dim_H(E) = \sup\left\{s : \text{there is a } \mu \in M(E) \text{ such that } I_s(\mu) < \infty\right\}.$$

The two theorems show that $I_s(\mu)$ can be expressed as some integral form of Fourier transform of μ.

Theorem 3.1. *For $0 < s < n$, there is a constant $0 < C_{n,s} < \infty$ such that*

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$\widehat{k_s} = C_{n,s} k_{n-s}$ *as tempered distribution, which means*

$$\int k_s \widehat{\phi} = \int C_{n,s} k_{n-s} \phi \tag{49}$$

for all $\phi \in S(\mathbb{R}^n)$.

Proof. First assume $\frac{n}{2} < s < n$, then $k_s \in L^1 + L^2 := \left\{ f_1 + f_2 : f_1 \in L^1,\ f_2 \in L^2 \right\}$ since 7

$$\int_{B(0,1)} k_s < \infty \text{ and } \int_{\mathbb{R}^n \setminus B(0,1)} k_s^2 < \infty.$$

So we can define the Fourier transform of such $k_s = f_1 + f_2 \in L^1 + L^2$ as

$$\widehat{k_s} = \widehat{f_1} + \widehat{f_2} \in L^\infty + L^2$$

by the duality. We can see that k_s is radial, so $\widehat{k_s}$ is also radial. Also we have

$$k_{n-s}(r\xi) = k_{n-s}(\xi) r^{s-n}$$

and

$$\begin{aligned} \widehat{k_s}(r\xi) &= \int_{\mathbb{R}^n} |x|^{-s} e^{-2\pi i x \cdot r\xi} dx = \int_{\mathbb{R}^n} \left| \frac{u}{r} \right|^{-s} e^{-2\pi i u \cdot \xi} du \cdot r^{-n} \\ &= r^{s-n} \int_{\mathbb{R}^n} |u|^{-s} e^{-2\pi i u \cdot \xi} du = r^{s-n} \widehat{k_s}(\xi) \end{aligned}$$

for $r > 0$. Fix $\xi = \xi_0$, view these as functions of r, we obtain

$$\widehat{k_s}(\xi_0 r) = \widehat{k_s}(\xi_0) r^{s-n}$$

and

$$k_{n-s}(r\xi_0) = k_{n-s}(\xi_0) r^{s-n} = \frac{k_{n-s}(\xi_0)}{\widehat{k_s}(\xi_0)} \widehat{k_s}(\xi_0 r),$$

doi:10.6342/NTU202301952

so $\widehat{k_s} = C_{n,s}k_{n-s}$ for $\frac{n}{2} < s < n$. Then for any $\phi \in S(\mathbb{R}^n)$,

$$\int k_s\widehat{\phi} = \int \widehat{k_s}\phi = \int C_{n,s}k_s\phi$$

by the Fubini's theorem because $k_s \in L^1(B(0,1))$, $k_s\phi \in L^1(\mathbb{R}^n \setminus B(0,1))$.
For $0 < s < \frac{n}{2}$, we have $\widehat{\widehat{f}}(x) = f(-x)$ and k_s is radial, so

$$\widehat{\widehat{k_s}}(x) = k_s(x).$$

And it is also true for tempered distribution by the Fubini's theorem. Therefore,

$$\widehat{k_{n-s}}(x) = C_{n,s}k_s(x) \Rightarrow k_{n-s}(x) = C_{n,s}\widehat{k_s}(x) \Rightarrow \widehat{k_s}(x) = C_{n,s}^{-1}k_{n-s}(x).$$

The first equality follows from the result above since $\frac{n}{2} < n-s < n$.
Now for $s = \frac{n}{2}$, by the Lebesgue dominated convergence theorem, we obtain

$$\int k_{\frac{n}{2}}\widehat{\phi} = \lim_{s\to\frac{n}{2}}\int k_s\widehat{\phi} = \lim_{s\to\frac{n}{2}}\int C_{n,s}k_{n-s}\phi = \lim_{s\to\frac{n}{2}}C_{n,s}\int k_{\frac{n}{2}}\phi. \tag{50}$$

Then if we show $\lim_{s\to\frac{n}{2}} C_{n,s}$ is finite, we complete the proof. Actually, we can find the exact form of $C_{n,s}$. Consider $\phi(x) = e^{-\pi|x|^2}$, we have

$$\begin{aligned}\int |x|^{-s} e^{-\pi|x|^2}dx &= \int_0^\infty u^{-s}e^{-\pi u^2}\cdot u^{n-1}du = \int_0^\infty u^{n-s-1}e^{-\pi u^2}du \\ &= \int_0^\infty \left(\frac{t}{\pi}\right)^{\frac{n-s-1}{2}} e^{-t}\cdot\frac{1}{2\pi}\cdot\left(\frac{t}{\pi}\right)^{-1}dt \\ &= \frac{\pi^{\frac{s-n}{2}}}{2}\int_0^\infty t^{\frac{n-s}{2}-1}e^{-t}dt = \frac{\pi^{\frac{s-n}{2}}}{2}\Gamma\left(\frac{n-s}{2}\right).\end{aligned} \tag{51}$$

Similarly, by replacing s with $n-s$, we obtain

$$\int |x|^{-(n-s)} e^{-\pi|x|^2}dx = \frac{\pi^{-\frac{s}{2}}}{2}\Gamma\left(\frac{s}{2}\right) \tag{52}$$

doi:10.6342/NTU202301952

we know that $\widehat{\phi} = \phi$. Then by **(51)** and **(52)**,

$$\int |x|^{-s} e^{-\pi|x|^2} dx = C_{n,s} \int |x|^{s-n} e^{-\pi|x|^2} dx$$

for $s \neq \dfrac{n}{2}$, which implies

$$C_{n,s} = \pi^{s-\frac{n}{2}} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Since Gamma function is continuous,

$$\lim_{s \to \frac{n}{2}} \pi^{s-\frac{n}{2}} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \pi^0 \frac{\Gamma\left(\frac{n}{4}\right)}{\Gamma\left(\frac{n}{4}\right)} = 1$$

which is finite. Hence we complete the proof of this theorem. □

Theorem 3.2. *Let $\mu \in M(\mathbb{R}^n)$ and $0 < s < n$, then*

$$I_s(\mu) = \iint |x-y|^{-s} \, d\mu(x) d\mu(y) = C_{n,s} \int |\widehat{\mu}(x)|^2 \, |x|^{s-n} \, dx \tag{53}$$

Proof. For the common function sense, we have

$$\begin{aligned} I_s(\mu) &= \int k_s * \mu d\mu = \int \widehat{k_s * \mu} \cdot \bar{\widehat{\mu}} = \int \widehat{k_s} \cdot \widehat{\mu} \cdot \bar{\widehat{\mu}} \\ &= \int \widehat{k_s} \, |\widehat{\mu}|^2 = C_{n,s} \int k_{n-s} \, |\widehat{\mu}|^2 = C_{n,s} \int |x|^{s-n} \, |\widehat{\mu}(x)|^2 \, dx \end{aligned}$$

by some basic formulas, Plancherel's theorem and **theorem 3.1**. But $\widehat{k_s}$ only exists in distribution sense. So we first give $\psi \in S(\mathbb{R}^n)$ be real valued, we can get

$$\begin{aligned} I_s(\psi) &= \int k_s * \psi d\psi = \iint k_s\,(x-y)\,\psi(y)\psi(x) dy dx \\ &= \iint k_s(z)\psi(x-z)\psi(x) dz dx = \int k_s\left(\tilde{\psi} * \psi\right), \end{aligned}$$

where $\tilde{\psi}(x) = \psi(-x)$. Moreover, we can see that

$$\widehat{\left|\widehat{\psi}\right|^2} = \widehat{\left(\widehat{\psi} \cdot \bar{\widehat{\psi}}\right)} = \widehat{\widehat{\psi}} * \widehat{\bar{\widehat{\psi}}} = \tilde{\psi} * \psi \tag{54}$$

doi:10.6342/NTU202301952

since ψ is real valued. So $\tilde{\psi} * \psi$ is the Fourier transform of $\left|\widehat{\psi}\right|^2$, thus

$$\begin{aligned} I_s(\psi) &= \int k_s \left(\tilde{\psi} * \psi\right) = \int k_s \widehat{\left|\widehat{\psi}\right|^2} \\ &= \int \widehat{k_s} \left|\widehat{\psi}\right|^2 = C_{n,s} \int |x|^{s-n} \left|\widehat{\psi}(x)\right|^2 dx. \end{aligned}$$

Now for $\mu \in M(\mathbb{R}^n)$, we use

$$\mu_\varepsilon = \phi_\varepsilon * \mu$$

to approximate μ, where

$$\phi_\varepsilon(x) = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right) \tag{55}$$

with $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\int \phi = 1$, then $\mu_\varepsilon \in S(\mathbb{R}^n)$ and $\mu_\varepsilon \to \mu$ weakly. Next, we see that

$$\iint |x-y|^{-s} \phi_\varepsilon(x-z)\phi_\varepsilon(y-w) dx dy = \iint |\varepsilon(u-v)+z-w|^{-s} \phi(u)\phi(v) du dv$$

by changing variable with $u = \frac{x-z}{\varepsilon}$, $v = \frac{y-w}{\varepsilon}$. Then since ϕ has compact support and $\int \phi = 1$,

$$|\varepsilon(u-v)+z-w|^{-s} \lesssim |z-w|^{-s} < \infty.$$

Therefore, by Lebesgue dominated convergence theorem, we have

$$\iint |\varepsilon(u-v)+z-w|^{-s} \phi(u)\phi(v) du dv \to |z-w|^{-s}$$

as $\varepsilon \to 0$ if $z \neq w$. Then we get

$$\iint |\varepsilon(u-v)+z-w|^{-s} \phi(u)\phi(v) du dv \lesssim |z-w|^{-s} < \infty \tag{56}$$

hence $\int \phi = 1$. So

$$\iint \left(\iint |x-y|^{-s} \phi_\varepsilon(x-z)\phi_\varepsilon(y-w) dx dy\right) \mu(z)\mu(w) dz dw$$

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$$
\begin{aligned}
&= \iint |x-y|^{-s} \left(\int \phi_\varepsilon(x-z)\mu(z)dz \right) \left(\int \phi_\varepsilon(y-w)\mu(w)dw \right) dxdy \\
&= \iint |x-y|^{-s} \mu_\varepsilon(x)\mu_\varepsilon(y)dxdy \\
&= I_s(\mu_\varepsilon) = C_{n,s} \int |x|^{s-n} \left| \widehat{\mu}(x)\widehat{\phi_\varepsilon}(x) \right|^2 dx = C_{n,s} \int |x|^{s-n} |\widehat{\mu}(x)|^2 \left| \widehat{\phi}(\varepsilon x) \right|^2 dx. \quad (57)
\end{aligned}
$$

The first equality because of Fubini's theorem, which can be used since **(56)**. The last line follows from the conclusion above with $\psi = \mu_\varepsilon$ and $\widehat{\phi_\varepsilon}(x) = \widehat{\phi}(\varepsilon x)$. Now if $I_s(\mu) = \int |z-w|^{-s} \mu(z)\mu(w)dzdw < \infty$,we can use the Lebesgue dominated convergence theorem on the last term of **(57)** by **(56)**, then it tends to $C_{n,s} \int |x|^{s-n} |\widehat{\mu}(x)|^2 dx$ because $\widehat{\phi}(0) = \int \phi = 1$. On the other hand, if $I_s(\mu) = \infty$, by Fatou's lemma, we obtain

$$
\begin{aligned}
\infty = I_s(\mu) &\le \liminf_{\varepsilon \to 0} \iint \left(\iint |x-y|^{-s} \phi_\varepsilon(x-z)\phi_\varepsilon(y-w)dxdy \right) \mu(z)\mu(w)dzdw \\
&= \liminf_{\varepsilon \to 0} C_{n,s} \int |x|^{s-n} |\widehat{\mu}(x)|^2 \left| \widehat{\phi}(\varepsilon x) \right|^2 dx = C_{n,s} \int |x|^{s-n} |\widehat{\mu}(x)|^2 dx.
\end{aligned}
$$

Then we complete the proof. □

Now we can present the result of Bourgain.

Theorem 3.3. *Given a Borel compact set $E \subset \mathbb{R}^2$, we have*

$$
dim_H(E) > \frac{13}{9} \Rightarrow |\Delta E| > 0.
$$

Proof. Given a set $E \subset \mathbb{R}^2$ and take $\alpha < dim_H(E)$, by **property 3.1**, we know there is a probability measure μ (since we may assume $\mu(E) = 1$) on E which satisfies

$$
I_\alpha(\mu) \iint |x-y|^{-\alpha} d\mu(x)d\mu(y) < \infty.
$$

Then by **theorem 3.2**, it becomes

$$
\int |\widehat{\mu}(x)|^2 |x|^{\alpha-2} dx = \int \frac{|\widehat{\mu}(x)|^2}{|x|^{2-\alpha}} dx < \infty \quad (58)
$$

doi:10.6342/NTU202301952

and implies that

$$\int_{|x|>1} \frac{|\widehat{\mu}(x)|^2}{|x|^{2-\alpha}} dx < \infty. \tag{59}$$

Consider a ball B with radius R in $\mathbb{R}^2$, then **(59)** implies

$$\int_B |\widehat{\mu}(x)|^2 \, dx \lesssim R^{2-\alpha}. \tag{60}$$

This is trivial if B is contained in $B(0, 100R)$, so if not, assume $B = B(x_0, R)$ and consider a function ϕ with its Fourier transform is a bump function and satisfies

$$\begin{aligned} \phi, \widehat{\phi} &\geq 0, \\ \widehat{\phi} &\approx 1 \ \text{ on } B(0,R), \\ \widehat{\phi} &= 0 \ \text{ out of } B(0,2R). \end{aligned}$$

Hence, we can see that $f(x) = (1 - \cos < x, x_0 >)\, \phi(x) \geq 0$ and

$$\widehat{f}(\xi) = \widehat{\phi}(\xi) - \frac{1}{2}\left(\widehat{\phi}(\xi + x_0) + \widehat{\phi}(\xi - x_0)\right)$$

since $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and the Fourier transform of $f(x)$ here is $\int f(x) e^{ix\cdot\xi} dx$. Then

$$\begin{aligned} 0 &\leq \int f(x) \mu * \mu(x) dx = \int \widehat{f}(\xi) \widehat{\mu * \mu}(\xi) d\xi = \int \widehat{f}(\xi) \, |\widehat{\mu}(\xi)|^2 \, d\xi \\ &= \int \widehat{\phi}(\xi) \, |\widehat{\mu}(\xi)|^2 \, d\xi - \frac{1}{2} \int \left(\widehat{\phi}(\xi + x_0) + \widehat{\phi}(\xi - x_0)\right) |\widehat{\mu}(\xi)|^2 \, d\xi \\ &\lesssim \int_{B(0,2R)} |\widehat{\mu}(\xi)|^2 \, d\xi - \int_B |\widehat{\mu}(\xi)|^2 \, d\xi. \end{aligned} \tag{61}$$

The last line is because of $\widehat{\phi}(\xi + x_0)$ is almost supported on B and $\widehat{\phi}$ is symmetric w.r.t. the origin. Therefore, we get

$$\int_B |\widehat{\mu}(\xi)|^2 \, d\xi \lesssim \int_{B(0,2R)} |\widehat{\mu}(\xi)|^2 \, d\xi \lesssim R^{2-\alpha}. \tag{62}$$

Let σ_s be the arc length measure of $S^1(s)$. If we want to show that $\Delta(E)$ has positive

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measure, we need

$$\int_a^b |< \mu, \mu * \sigma_s >|^2 \, ds < \infty \tag{63}$$

for some $0 < a < b < \infty$. This is called Mattila's method. As its name, it is given by Mattila in [7]. If the inequality above holds, it means the Fourier transform of the push-forward measure $\mu \times \mu$ under the distance map is a L^2 function. Hence it is absolutely continuous with L^2 density. Also we can see that the support of $\mu \times \mu$ is contained in $\Delta(E)$. Combine these two results, we get $|\Delta(E)| \geq |\mathrm{supp}\,(\mu \times \mu)| > 0$.

Then we first see that

$$\begin{aligned} < \mu, \mu * \sigma > &= \int \mu(x) \overline{\mu * \sigma}(x) dx = \int \widehat{\mu}(\xi) \overline{\widehat{\mu}}(\xi) \overline{\widehat{\sigma}}(\xi) d\xi \\ &\approx \int |\widehat{\mu}(\xi)|^2 \frac{e^{is|\xi|}}{|\xi|^{\frac{1}{2}}} d\xi = \iint \left|\widehat{\mu}\left(re^{i\theta}\right)\right| r^{\frac{1}{2}} e^{isr} dr d\theta, \end{aligned}$$

so if

$$\int \left(\int \left|\widehat{\mu}\left(re^{i\theta}\right)\right|^2 d\theta \right)^2 r dr < \infty, \tag{64}$$

then **(63)** holds. Now let

$$g_\mu(r) = \int \left|\widehat{\mu}\left(re^{i\theta}\right)\right|^2 d\theta \tag{65}$$

as a function of r as it tends to ∞, and we want to show that it can be bounded by some order of r.

Consider $F \geq 0$ be a convolution of μ which only depends on r and satisfying

$$\left|\widehat{F}(\xi)\right| \leq |\widehat{\mu}(\xi)|$$

and by **(59)** we have

$$\int |F|^2 = \int \left|\widehat{F}\right|^2 \leq \int |\widehat{\mu}|^2 \leq r^{2-\alpha}. \tag{66}$$

Given a annulus $A_r(1)$ on $\mathbb{R}^2$, we can consider $\{R_i\}$ be a set of rectangles with

doi:10.6342/NTU202301952

dimensions $1 \times r^{\frac{1}{2}}$ to be almost fill up $A_r(1)$, and we have the amount of R_i is $\approx r^{\frac{1}{2}}$. Thus,

$$g_\mu(r) \approx \int \left|\widehat{F}\left(re^{i\theta}\right)\right|^2 d\theta \approx \int_r^{r+1} \int \left|\widehat{F}\left(\rho e^{i\theta}\right)\right|^2 d\theta d\rho, \tag{67}$$

and by our construction above, this integral becomes

$$\frac{1}{r}\int_A \left|\widehat{F}(x)\right|^2 dx = \frac{1}{r}\int \widehat{F}(x)\cdot \widehat{F}(x)\chi_A(x)dx = \frac{1}{r}\int F(x)\cdot F(x) * \widehat{\chi_A}(x)dx \tag{68}$$

$$\leq \frac{1}{r}\|F\|_{\frac{4}{3}} \left\|\sum_i \left(\widehat{F}\cdot \chi_{R_i}\right)^\vee\right\|_4, \tag{69}$$

by Hölder's inequality. To estimate $\|F\|_{\frac{4}{3}}$, since

$$\|F\|_1 \lesssim \|\mu\|_1 = 1 \text{ and } \|F\|_2 \leq r^{1-\frac{\alpha}{2}},$$

the interpolation formula tells us

$$\|F\|_{\frac{4}{3}} \leq \|F\|_1^{\frac{1}{2}} \|F\|_2^{\frac{1}{2}} \lesssim r^{\frac{1}{2}-\frac{\alpha}{4}}. \tag{70}$$

For the other one, by the square function equivalence,

$$\left\|\sum_i \left(\widehat{F}\chi_{R_i}\right)^\vee\right\|_4 \lesssim \left\|\left(\sum_i \left|\left(\widehat{F}\chi_{R_i}\right)^\vee\right|^2\right)^{\frac{1}{2}}\right\|_4 = \left\|\sum_i \left|\left(\widehat{F}\chi_{R_i}\right)^\vee\right|^2\right\|_2^{\frac{1}{2}}. \tag{71}$$

Then define

$$a_i = \int \left|\left(\widehat{F}\chi_{R_i}\right)^\vee\right|^2 = \int \left|\widehat{F}\chi_{R_i}\right|^2 = \int_{R_i} \left|\widehat{F}\right|^2 \tag{72}$$

and

$$b_i = \frac{\left|\left(\widehat{F}\chi_{R_i}\right)^\vee\right|^2}{a_i}. \tag{73}$$

since R_i is with dimensions $1 \times r^{\frac{1}{2}}$, it is contained in a ball Q_i of radius $r^{\frac{1}{2}}$. Then we have

$$a_i = \int_{R_i} \left|\widehat{F}\right|^2 \leq \int_{B_i} \left|\widehat{F}\right|^2 \leq \int_{B_i} |\widehat{\mu}|^2 \lesssim r^{1-\frac{\alpha}{2}} \tag{74}$$

doi:10.6342/NTU202301952

by **(60)** and $\left|\widehat{F}\right| \leq |\widehat{\mu}|$. Also, we can observe that

$$\sum_i a_i = \int_{\cup R_i} \left|\widehat{F}\right|^2 \leq \int_A \left|\widehat{F}\right|^2 = r \cdot g_\mu(r) \tag{75}$$

Next, given $\omega \in S^1$, consider the maximal function

$$M_\delta f(\omega) = \sup_{\mathcal{T}} \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} f, \tag{76}$$

where $\mathcal{T}$ is a rectangle with dimensions $1 \times \delta$ and the length 1 side is along the direction ω. Given a function f with $\|f\|_2 \leq 1$, then we have

$$\left\langle \sum_i a_i b_i, f \right\rangle \leq \sum_i a_i M_{\frac{1}{\sqrt{r}}} f(\omega_i) \leq \left(\sum_i a_i^2\right)^{\frac{1}{2}} \left(\sum_i \left|M_{\frac{1}{\sqrt{r}}} f(\omega_i)\right|^2\right)^{\frac{1}{2}} \tag{77}$$

$$= r^{\frac{1}{4}} \left(\sum_i a_i^2\right)^{\frac{1}{2}} \left(\sum_i r^{-\frac{1}{2}} \left|M_{\frac{1}{\sqrt{r}}} f(\omega_i)\right|^2\right)^{\frac{1}{2}} \tag{78}$$

where $\{\omega_i\}$ $i = 1, ..., \lfloor r^{\frac{1}{2}} \rfloor$ is a τ separated set on S^1 with $\tau \approx r^{-\frac{1}{2}}$. Thus,

$$r^{\frac{1}{4}} \left(\sum_i a_i^2\right)^{\frac{1}{2}} \left(\sum_i r^{-\frac{1}{2}} \left|M_{\frac{1}{\sqrt{r}}} f(\omega_i)\right|^2\right)^{\frac{1}{2}} \approx r^{\frac{1}{4}} \left(\sum_i a_i^2\right)^{\frac{1}{2}} \left\|M_{\frac{1}{\sqrt{r}}} f\right\|_2. \tag{79}$$

Then use the result of Kakeya-type maximal operator (see [10], Theorem 9.1.2), we get the inequality

$$\left\|M_{\frac{1}{\sqrt{r}}} f\right\|_{L^2(S^1)} \lesssim \left(\log \frac{1}{\frac{1}{\sqrt{r}}}\right)^{\frac{1}{2}} \|f\|_2, \tag{80}$$

and then

$$\textbf{(79)} \lesssim r^{\frac{1}{4}} \left(\sum_i a_i^2\right)^{\frac{1}{2}} (\log r)^{\frac{1}{2}} \leq r^{\frac{1}{4}} (\log r)^{\frac{1}{2}} \left(\max_i a_i\right)^{\frac{1}{2}} \left(\sum_i a_i\right)^{\frac{1}{2}}. \tag{81}$$

Finally, by **(67)**, **(69)**, **(70)**, **(71)**, **(74)**, **(75)** and **(81)**, we obtain

$$g_\mu(r) \lesssim \frac{1}{r} \cdot r^{\frac{1}{2} - \frac{\alpha}{4}} \cdot r^{\frac{1}{8}} (\log r)^{\frac{1}{4}} r^{\frac{1}{4} - \frac{\alpha}{8}} (r \cdot g_\mu(r))^{\frac{1}{4}}, \tag{82}$$

doi:10.6342/NTU202301952

and we may conclude that

$$g_\mu(r) \lesssim r^{-\frac{\alpha}{2}+\frac{1}{6}+\varepsilon} \tag{83}$$

for any $\varepsilon > 0$. Back to our main goal, we want **(64)** to be true. Since $g_\mu(r) \lesssim r^{\frac{1}{6}-\frac{\alpha}{2}}$, we have

$$\int \left(\int \left| \widehat{\mu}\left(re^{i\theta} \right) \right|^2 d\theta \right)^2 rdr \lesssim \int r^{-\frac{\alpha}{2}+\frac{1}{6}} \int r \left| \widehat{\mu}(re^{i\theta}) \right|^2 d\theta dr \quad = \int \left| \widehat{\mu}(\xi) \right|^2 |\xi|^{\frac{1}{6}-\frac{\alpha}{2}} d\xi. \tag{84}$$

Then by **(59)**, if $\dfrac{1}{6} - \dfrac{\alpha}{2} > 2 - \alpha \Rightarrow \alpha > \dfrac{13}{9}$, **(84)** will be finite. Hence we conclude that if $\dfrac{13}{9} < \alpha < dim_H(E)$, the distance set of E has the positive measure. □

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3.2 The case for $n \geq 3$

First we give a bilinear restriction proved by Tao in [13].

Theorem 3.4. *Let $n \geq 2$, Σ_1, Σ_2 be the compact subsets of Σ with $d(\Sigma_1, \Sigma_2) > 1$, where $\Sigma = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_n = x_1^2 + \cdots + x_{n-1}^2\}$, and $d\sigma$ be the surface measure on Σ. Then for $q > \dfrac{n+2}{n}$, we have*

$$\left\| \widehat{f_1 d\sigma} \widehat{f_2 d\sigma} \right\|_{L^q(\mathbb{R}^n)} \lesssim_{q,n} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)}$$

for all f_1 is supported in Σ_1, f_2 is supported in Σ_2.

Basic on this theorem, we can consider a weighted type.

Theorem 3.5. *For $n \geq 3$, and $\alpha \in (0, n)$, given a function $W : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\begin{cases} \|W\|_\infty \lesssim 1, \\ \displaystyle\int_{B(x,r)} |W(y)|\, dy \lesssim r^\alpha, \forall x \in \mathbb{R}^n, r > 0. \end{cases} \tag{85}$$

Then under the hypothesis of ***Theorem 3.4****, we have the inequality*

$$\left\| \widehat{f_1 d\sigma} \widehat{f_2 d\sigma} \right\|_{L^q(dW)} \lesssim_{\alpha,q,n} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)} \tag{86}$$

holds for $q > q_0(\alpha, n) := \max\left(1, \min\left(\dfrac{4\alpha}{n+2\alpha-2}, \dfrac{n+2}{n}\right)\right)$.

We can first see that **(86)** holds for $q > \dfrac{n+2}{n}$ by **Theorem 3.4** and **(85)**, and we will improve the lower bound of q from $\dfrac{n+2}{n}$ to q_0 for $\alpha \leq \dfrac{n+2}{2}$.

Proof. To prove this theorem, we need the following epsilon-removed lemma:

Lemma 3.2. *Let $n \geq 2$, and let Σ_1 and Σ_2 be compact hypersurfaces with boundary in $\mathbb{R}^n$, denote $d\sigma_1$, $d\sigma_2$ be their surface measure respectively. If*

$$\left|\widehat{d\sigma_i}\right| \lesssim (1 + |x|)^{-N} \tag{87}$$

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for some $M > 0$*, then for all* $1 < q < 1 + \frac{1}{N}$*,* $R > 1$*, we have*

$$\left\| \widehat{f_1 d\sigma_1 f_2 d\sigma_2} \right\|_{L^q(B(0,R),Wd\xi)} \leq C_{\eta,q,n} R^{\eta} \|f_1\|_{L^2(d\sigma_1)} \|f_2\|_{L^2(d\sigma_2)} \tag{88}$$

holds implies (86) *if* $\frac{1}{p}\left(1 + \frac{2\eta}{N}\right) < \frac{1}{q} + \frac{\eta}{1+N}$.

By this lemma, it is sufficient to show that

$$\left\| \widehat{f_1 d\sigma} \widehat{f_1 d\sigma} \right\|_{L^{q_0}(B(0,R),Wd\xi)} \leq C_{n,\alpha,\eta} R^{\eta} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)} \tag{89}$$

for all $\eta > 0$ and $R > 1$ with the assumption in the **Theorem 3.5**. We prove this by induction. First, using Young's convolution inequality, we have

$$\left\| \widehat{f_1 d\sigma} \right\|_{\infty} = \left\| \widehat{f_1} * \widehat{d\sigma} \right\|_{\infty} \lesssim \left\| \widehat{f_1} \right\|_2 \left\| \widehat{d\sigma} \right\|_2 = \|d\sigma\|_2 \|f_1\|_2 \lesssim \|f_1\|_2 ,$$

and similarly, we have $\left\| \widehat{f_2 d\sigma} \right\|_{\infty} \lesssim \|f_2\|_2$. Then

$$\begin{aligned} \left\| \widehat{f_1 d\sigma} \widehat{f_1 d\sigma} \right\|_{L^{q_0}(B(0,R),Wd\xi)} &\lesssim \|f_1\|_2 \|f_2\|_2 \int_{B(0,R)} |W(\xi)| \, d\xi \\ &\lesssim R^{\alpha} \|f_1\|_2 \|f_2\|_2 \lesssim R^{\eta_0} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)} \end{aligned}$$

for some η_0 large enough, so **(89)** holds for $\eta \geq \eta_0$.

Next, we show that **(89)** holds if $\eta = \max((1-\delta)\eta_0, C\delta) + C\varepsilon$ for all $0 < \delta, \varepsilon < 1$ and C is a constant which is independent of δ and ε. If this is true, we may choose suitable δ and ε such that $\max((1-\delta)\eta_0, C\delta) + C\varepsilon = \eta_0 - C_0\eta_0^2 < \eta_0$ for some small constant C_0, and then iterating this process, we will see that **(89)** holds for all $\eta > 0$. Therefore, our goal becomes to prove

$$\left\| \widehat{f_1 d\sigma} \widehat{f_1 d\sigma} \right\|_{L^{q_0}(B(0,R),Wd\xi)} \leq C_{n,\alpha,\eta} R^{\max((1-\delta)\eta_0, C\delta) + C\varepsilon} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)} . \tag{90}$$

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We use the wave packet decomposition, then

$$\widehat{f_i d\sigma}(\xi) = \sum_{T_i} C_{T_i} \phi_{T_i}(\xi)$$

for $\xi \in B(0,R)$ and $i = 1,2$. T_i are $R^{\frac{1}{2}}$-separated tubes with dimensions $R \times R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}}$, C_{T_i} are constants, ϕ_{T_i} are the functions satisfy $\phi_{T_i}^{\vee}$ is supported in the dual tube of T_i which contained in a $O(R^{-1})$ neighborhood of S_i and ϕ_{T_i} is essentially supported on T_i with the Schwartz decay away from T_i. And we have the following properties:

$$\|\phi_{T_i}\|_2 \approx R^{\frac{1}{2}}, \tag{91}$$

$$\sum_{T_i} |C_{T_i}|^2 \lesssim \|f_i\|^2_{L^2(d\sigma)}, \tag{92}$$

$$\left\| \sum_{T_i} \phi_{T_i} \right\|_2^2 \lesssim \sum_{T_i} \|\phi_{T_i}\|_2^2. \tag{93}$$

Moreover, we may assume C_{T_i} be either 0 or 1 by the pigeonholing. Hence, **(90)** becomes

$$\left\| \sum_{T_1 \in T} \sum_{T_2 \in T'} \phi_{T_1} \phi_{T_2} \right\|_{L^{q_0}(B(0,R), W d\xi)} \lesssim R^{\max((1-\delta)\eta_0, C\delta) + C\varepsilon} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)}, \tag{94}$$

Let T be the collection of T_1 and T' be the collection of T_2, since by **(92)**, we have $(\#T)^{\frac{1}{2}} \lesssim \|f_1\|_{L^2(d\sigma)}$ and $(\#T')^{\frac{1}{2}} \lesssim \|f_2\|_{L^2(d\sigma)}$. Thus, to prove **(94)**, it is sufficient to show

$$\left\| \sum_{T_1 \in T} \sum_{T_2 \in T'} \phi_{T_1} \phi_{T_2} \right\|_{L^{q_0}(B(0,R), W d\xi)} \lesssim R^{\max((1-\delta)\eta_0, C\delta) + C\varepsilon} (\#T)^{\frac{1}{2}} (\#T')^{\frac{1}{2}}. \tag{95}$$

Cover $B(0,R)$ by a collection $\mathcal{B}$, which is of $\approx O(R^{C\delta})$ many finitely overlapping balls with radius $R^{1-\delta}$, then we have the following lemma:

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Lemma 3.3. *There exists a relation* $\sim$ *between* $B \in \mathcal{B}$ *and* $\tilde{T} \in T \cup T'$ *such that*

$$\#\left\{B \in \mathcal{B} : \tilde{T} \sim B\right\} \lesssim R^{\varepsilon} \tag{96}$$

for each $\tilde{T} \in T \cup T'$, *and*

$$\|f_B\|_{L^2(B)} \lesssim R^{C\varepsilon + C\delta - \frac{n-2}{4}} (\#T)^{\frac{1}{2}} (\#T')^{\frac{1}{2}}, \tag{97}$$

where $\tilde{B} = \{(T_1, T_2) \in T \times T' : T_1$ *is not* $\sim B$ *or* T_2 *is not* $\sim B\}$, *and*

$$f_B(\xi) := \sum_{(T_1,T_2)\in\tilde{B}} \phi_{T_1}(\xi)\phi_{T_2}(\xi).$$

Now we can do the main part of the proof, using the rough estimate, we obtain

$$\begin{aligned}\left\|\sum_{T_1\in T}\sum_{T_2\in T'}\phi_{T_1}\phi_{T_2}\right\|_{L^{q_0}(B(0,R),Wd\xi)} &\le \sum_{B\in\mathcal{B}}\left\|\sum_{T_1\in T}\sum_{T_2\in T'}\phi_{T_1}\phi_{T_2}\right\|_{L^{q_0}(B,Wd\xi)}\\ \le \sum_{B\in\mathcal{B}}\left\|\sum_{T_1\in T}\sum_{T_2\in T'}\phi_{T_1}\phi_{T_2}\right\|_{L^{q_0}(B,Wd\xi)} &+ \sum_{B\in\mathcal{B}}\left\|\sum_{(T_1,T_2)\in\tilde{B}}\phi_{T_1}\phi_{T_2}\right\|_{L^{q_0}(B,Wd\xi)}.\end{aligned}$$

For the first part, since B has radius $R^{1-\delta}$ and ϕ_{T_1}, ϕ_{T_2} are supported on $O\left(R^{-1}\right)$-neighborhood of S_1, S_2, by the induction hypothesis, we have

$$\begin{aligned}&\sum_{B\in\mathcal{B}}\left\|\sum_{T_1\in T}\sum_{T_2\in T'}\phi_{T_1}\phi_{T_2}\right\|_{L^{q_0}(B,Wd\xi)}\\ \lesssim& R^{-1}R^{(1-\delta)\eta_0}\sum_{B\in\mathcal{B}}\left\|\sum_{T_1\sim B}\phi_{T_1}\right\|_2\left\|\sum_{T_2\sim B}\phi_{T_2}\right\|_2\\ \lesssim& R^{-1}R^{(1-\delta)\eta_0}\sum_{B\in\mathcal{B}}\left(\sum_{T_1\sim B}\|\phi_{T_1}\|_2^2\sum_{T_2\sim B}\|\phi_{T_2}\|_2^2\right)^{\frac{1}{2}}\\ \approx& R^{-1}R^{(1-\delta)\eta_0}\sum_{B\in\mathcal{B}}\left(\sum_{T_1\sim B}R\sum_{T_2\sim B}R\right)^{\frac{1}{2}}\end{aligned}$$

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$$
\begin{aligned}
&= R^{(1-\delta)\eta_0} \sum_{B\in\mathcal{B}} \left(\#\left\{T_1 \in T : T_1 \sim B\right\} \#\left\{T_2 \in T' : T_2 \sim B\right\}\right)^{\frac{1}{2}} \\
&\lesssim R^{(1-\delta)\eta_0} \left(\sum_{B\in\mathcal{B}} \#\left\{T_1 \in T : T_1 \sim B\right\}\right)^{\frac{1}{2}} \left(\sum_{B\in\mathcal{B}} \#\left\{T_1 \in T : T_1 \sim B\right\}\right)^{\frac{1}{2}} \\
&= R^{(1-\delta)\eta_0} \left(\sum_{T_1\in T} \#\left\{B \in \mathcal{B} : T_1 \sim B\right\}\right)^{\frac{1}{2}} \left(\sum_{T_2\in T'} \#\left\{B \in \mathcal{B} : T_2 \sim B\right\}\right)^{\frac{1}{2}} \\
&\lesssim R^{(1-\delta)\eta_0+\varepsilon} \left(\#T\right)^{\frac{1}{2}} \left(\#T'\right)^{\frac{1}{2}}.
\end{aligned}
$$

The third and fourth line follows from **(91)** and **(93)**, and the second last line follows from **(96)**.

For the second part, since $q_0 < 2$ and

$$
\frac{1}{\frac{2}{2-q_0}} + \frac{1}{\frac{2}{q_0}} = 1,
$$

using the Hölder's inequality, we have

$$
\begin{aligned}
\|f_B\|_{L^{q_0}(B, W d\xi)} &= \left(\int_B |f_B(\xi)|^{q_0} W(\xi) d\xi\right)^{\frac{1}{q_0}} \\
&\lesssim \left(\left(\int_B |f_B(\xi)|^{q_0\cdot\frac{2}{q_0}} d\xi\right)^{\frac{q_0}{2}} \left(\int_B |W(\xi)|^{\frac{2}{2-q_0}} d\xi\right)^{\frac{2-q_0}{2}}\right)^{\frac{1}{q_0}} \\
&= \left(\int_B |f_B(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_B |W(\xi)| d\xi\right)^{\frac{1}{q_0}-\frac{1}{2}} \\
&\lesssim \|f_B\|_{L^2(B)} \cdot R^{(1-\delta)\alpha\cdot\left(\frac{1}{q_0}-\frac{1}{2}\right)} \leq R^{\frac{\alpha}{q_0}-\frac{\alpha}{2}} \|f_B\|_{L^2(B)}
\end{aligned}
$$

because $\dfrac{2}{2-q_0} > 1$ and W satisfies **(85)**. Therefore, by **(97)**, we obtain

$$
\begin{aligned}
\sum_{B\in\mathcal{B}} \left\| \sum_{(T_1,T_2)\in\tilde{B}} \phi_{T_1}\phi_{T_2} \right\|_{L^{q_0}(B, W d\xi)} &\lesssim \sum_{B\in\mathcal{B}} R^{\frac{\alpha}{q_0}-\frac{\alpha}{2}} \|f_B\|_{L^2(B)} \\
&\lesssim R^{\frac{\alpha}{q_0}-\frac{\alpha}{2}} \sum_{B\in\mathcal{B}} R^{C\delta+C\varepsilon-\frac{n-2}{4}} \left(\#T\right)^{\frac{1}{2}} \left(\#T'\right)^{\frac{1}{2}} \\
&\lesssim R^{C\delta+C\varepsilon} \cdot R^{\frac{\alpha}{q_0}-\frac{\alpha}{2}-\frac{n}{4}+\frac{1}{2}} \cdot R^{C\delta} \left(\#T\right)^{\frac{1}{2}} \left(\#T'\right)^{\frac{1}{2}} \\
&\lesssim R^{C\delta+C\varepsilon} \left(\#T\right)^{\frac{1}{2}} \left(\#T'\right)^{\frac{1}{2}}.
\end{aligned}
$$

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The last line follows from the definition of q_0, since $1 < \frac{n+2}{n}$, we have three conditions,

$$\begin{aligned}
&\frac{4\alpha}{n+2\alpha-2} \leq 1 < \frac{n+2}{n} \Rightarrow q_0 = 1, \\
&1 \leq \frac{4\alpha}{n+2\alpha-2} < \frac{n+2}{n} \Rightarrow q_0 = \frac{4\alpha}{n+2\alpha-2}, \\
&1 < \frac{n+2}{n} \leq \frac{4\alpha}{n+2\alpha-2} \Rightarrow q_0 = \frac{n+2}{n}.
\end{aligned}$$

For the first case,

$$\frac{\alpha}{q_0} - \frac{\alpha}{2} - \frac{n}{4} + \frac{1}{2} = \frac{2\alpha - (n-2)}{4} < 0$$

since $1 > \frac{4\alpha}{n+2\alpha-2} \Rightarrow n-2 > 2\alpha$. For the second case,

$$\frac{\alpha}{q_0} - \frac{\alpha}{2} - \frac{n}{4} + \frac{1}{2} = 0.$$

For the last case, because we fix $\alpha \leq \frac{n+2}{2}$, $\frac{n+2}{n} \leq \frac{4\alpha}{n+2\alpha-2}$ is only true for the equality holds, so it is same as the case two. Then we have

$$R^{\frac{\alpha}{q_0} - \frac{\alpha}{2} - \frac{n}{4} + \frac{1}{2}} \leq R^0 = 1.$$

Combine these two results, we may see that **(95)** holds, and then complete the proof. $\square$

Definition 3.1. *We say $\varphi : B(0,1) \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is a (N, δ_0)-elliptic phase if it satisfies*

(1) $\|\varphi\|_\infty \leq N$,

(2) $\varphi(0) = \nabla\varphi(0) = 0$,

(3) *all eigenvalues of the Heissan matrix $H_{i,j}(x)$ lie in the interval* $[1-\varepsilon_0, 1+\varepsilon_0]$.

And S is said to be a (N, δ_0)-elliptic surface if

$$S = \{(x, \varphi(x)) : (x, \varphi(x)) \in B(0,1) \times \mathbb{R} \subset \mathbb{R}^n\},$$

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where φ is a (N,δ_0)-elliptic phase.

There are two properties of (M,ε_0)-elliptic phase.

Property 3.2. (1) *If φ is a (N,δ_0)-elliptic phase, then for a ball $B(x_0,r_0) \subset B(0,1)$, let*

$$\tilde{\varphi}(x) := \frac{1}{r_0^2}(\varphi(xr_0+x_0) - \varphi(x_0) - r_0 x \cdot \nabla\varphi(x_0)),\ x \in B(0,1), \tag{98}$$

it's a $(C_n N,\delta_0)$-elliptic phase.

(2) *Let S be a smooth compact submanifold of $\mathbb{R}^n$ with strictly positive principal curvatures. Then for any $\delta_0 > 0$ and for any $s \in S$, exists a neighborhood U_s of s and an affine bijection a_s of $\mathbb{R}^n$ s.t. $a_s(U_s)$ is an (N,δ_0)-elliptic surface, with M depends on $n, \|\varphi\|_{C^\infty}$ and the principal curvature at s. Furthermore, we can use the partition of unity to write S as a union of affine images of finitely many (N,δ_0)-elliptic surfaces.*

Now we can get the following theorem be the generalization of Theorem **3.5**.

Theorem 3.6. *Let $n \geq 3, \alpha \in (0,n)$, and W satisfies* (85)*. Then for any $N > 0$, there exists $\delta_0 > 0$ such that the following holds.*
Let Σ_1, Σ_2 be two compact subsets of diameter ≈ 1 of (N,δ_0)-elliptic surface in $\mathbb{R}^n$ with $d(\Sigma_1,\Sigma_2) > \frac{1}{100}$, σ_i be the Lebesgue measure on Σ_i, $i = 1,2$. Then $\forall q > q_0(\alpha,n)$, we have

$$\left\|\widehat{f_1 d\sigma_1}\widehat{f_2 d\sigma_2}\right\|_{L^q(dW)} \lesssim \|f_1\|_{L^2(\Sigma_1,d\sigma_1)}\|f_2\|_{L^2(\Sigma_2,d\sigma_2)} \tag{99}$$

for all $f_i \in L^2(d\sigma)$ is supported in Σ_i, $i = 1,2$.

Definition 3.2. *Let μ be a compactly supported probability measure, we say μ is an α- dimensional measure if*

$$\mu(B(x,r)) \lesssim_\mu r^\alpha,\ \forall x \in \mathbb{R}^n,\ \forall r > 0.$$

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Consider ψ be a Schwartz bump function satisfies $\psi(x) = 1$ for $|x| < 2$, $\psi(x) = 0$ for $|x| > 4$. For each ball $B \subset \mathbb{R}^n$, take an affine bijection a_B of $\mathbb{R}^n$ maps B to $B(0,1)$, and define $\psi_B(x) = \psi(a_B(x))$. Then we have the following lemma.

Lemma 3.4. *Let μ be an α-dimensional measure in $\mathbb{R}^n$, and B be a ball of radius r in $\mathbb{R}^n$. Define a function $\mu_B := |\psi_B^\vee| * \mu$, we have*

(1) $\|\mu_B\|_\infty \lesssim r^{n-\alpha}$,

(2) $\|\mu_B\|_1 \lesssim 1$,

(3) $\int_D \mu_B(y)dy \lesssim s^\alpha$ *for any ball D of radius $s \geq \frac{1}{r}$.*

Proof. (1) Since ψ is a Schwartz function, by the decay of Schwartz function, for $M \in \mathbb{N}$, we can write

$$|\psi_B^\vee(x)| = r^n\, |\psi^\vee(rx)| \lesssim_{M,n} r^n \sum_{j=1}^{\infty} 2^{-Mj} \chi_{B(0,2^j r^{-1})}(x).$$

So we obtain

$$\begin{aligned} 0 \leq \mu_B(x) &\lesssim r^n \sum_{j=1}^{\infty} 2^{-Mj} \int \chi_{B(0,2^j r^{-1})}(x) d\mu(y) \\ &\lesssim r^n \sum_{j=1}^{\infty} 2^{-Mj} (2^j r^{-1})^\alpha \lesssim r^{n-\alpha}, \end{aligned}$$

by using the definition of α-dimensional measure and taking $M \geq n$.

(2) since ψ is Schwartz function, $\psi_B^\vee$ is also a Schwartz function. Then we know $\|\psi_B^\vee\|_1 \lesssim 1$, so by Young's convolution inequality, $\|\mu_B\| \lesssim \|\psi_B^\vee\|_1 \|\mu\|_1 \lesssim 1$ since μ is compactly supported probability measure.

(3)

$$\begin{aligned} \int_D \mu_B(y)dy &\lesssim r^n \sum_{j=1}^{\infty} 2^{-Mj} \iint \chi_D(y) \chi_{B(0,2^j r^{-1})}(y-u) d\mu(u) dy \\ &= r^n \sum_{j=1}^{\infty} 2^{-Mj} \iint \chi_{D+B(0,2^j r^{-1})}(u) \chi_{B(0,2^j r^{-1})}(y-u) dy d\mu(u) \end{aligned}$$

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$$\lesssim r^n \sum_{j=1}^{\infty} 2^{-Mj} \left(2^j r^{-1}\right)^n \left(s + 2^j r^{-1}\right)^{\alpha}$$
$$\lesssim \sum_{j=1}^{\infty} 2^{-Mj/2} \left(s + 2^j r^{-1}\right)^{\alpha} \lesssim s^{\alpha}$$

The second line follows from the Fubini's theorem and for $y \in D, y - u \in B(0, 2^j r^{-1})$, we have $u \in D + B(0, 2^j r^{-1})$, and the last inequality because of taking M large enough.

□

Corollary 3.1. *Let μ be an α-dimensional measure, $L > 0$ and R, η satisfies $LR^{-\frac{1}{2}} \lesssim \eta \lesssim 1$. Consider I_1, I_2 be the subsets of $A_R(L)$ satisfies diam(I_1), diam$(I_2) \approx R\eta$, $d(I_1, I_2) \approx R\eta$. Then for all $q > q_0(\alpha, n)$, we have*

$$\left\|\widehat{f_1}\widehat{f_2}\right\|_{L^q(d\mu)} \lesssim L\,(R\eta)^{n-1-\frac{\alpha}{q}}\, \eta^{-\frac{1}{q}} \left\|f_1\right\|_2 \left\|f_2\right\|_2. \tag{100}$$

for all functions f_1, f_2 is supported on I_1, I_2 separately.

Proof. diam(I_1), diam$(I_2) \approx R\eta$, so $f_1 * f_2$ is contained in a ball B of radius $r \approx R\eta$. Hence we have

$$\left\|\widehat{f_1}\widehat{f_2}\right\|_{L^q(d\mu)} = \left\|\left(\widehat{f_1}\widehat{f_2}\right) * \psi_B^{\vee}\right\|_{L^q(d\mu)} = \left(\int \left|\int \widehat{f_1}(y)\widehat{f_2}(y)\psi_B^{\vee}(x-y)dy\right|^q \mu(x)dx\right)^{\frac{1}{q}}$$
$$\lesssim \left\{\int \left[\left(\int \left|\widehat{f_1}(y)\widehat{f_2}(y)\psi_B^{\vee}(x-y)^{\frac{1}{q}}\right|^q dy\right)^{\frac{1}{q}} \cdot \left(\int \left|\psi_B^{\vee}(y)^{\frac{1}{q'}}\right|^{q'} dy\right)^{\frac{1}{q'}}\right]^q \mu(x)dx\right\}^{\frac{1}{q}}$$
$$= \left[\int \left(\int \left|\widehat{f_1}(y)^q\widehat{f_2}(y)^q\psi_B^{\vee}(x-y)\right| dy\right) \cdot \left(\int |\psi_B^{\vee}(y)dy|^{\frac{q}{q'}}\right) \mu(x)dx\right]^{\frac{1}{q}}$$
$$= \left(\iint \left|\widehat{f_1}(y)^q\widehat{f_2}(y)^q\psi_B^{\vee}(x-y)\right| dy\mu(x)dx\right)^{\frac{1}{q}} \cdot \|\psi_B^{\vee}\|_1^{\frac{1}{q'}}$$
$$= \left(\int \left|\widehat{f_1}(y)^q\widehat{f_2}(y)^q\right| \int |\psi_B^{\vee}(x-y)|\, \mu(x)dxdy\right)^{\frac{1}{q}} \cdot \|\psi_B^{\vee}\|_1^{\frac{1}{q'}}$$
$$= \left(\int \left|\widehat{f_1}(y)^q\widehat{f_2}(y)^q\right| |\psi_B^{\vee}| * \mu(y)dy\right)^{\frac{1}{q}} \cdot \|\psi_B^{\vee}\|_1^{\frac{1}{q}}$$
$$= \left\|\widehat{f_1}\widehat{f_2}\right\|_{L^q\left(|\psi_B^{\vee}|*\mu\right)} \|\psi_B^{\vee}\|_1^{\frac{1}{q'}} \lesssim \left\|\widehat{f_1}\widehat{f_2}\right\|_{L^q\left(|\psi_B^{\vee}|*\mu\right)},$$

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where q' is the Hölder conjugate exponent. The first equality because of that $\psi_B = 1$, the second last line because of ψ_D is radial, others follows from Hölder's inequality and Fubini's theorem.

Next step we consider e be the unit vector in the direction of the center of mass of $I_1 \cup I_2$ and a orthonormal basis $\{e_1 = e, e_2, \cdots, e_n\}$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with

$$T(e_1) = \frac{1}{R\eta^2} e_1, \ T(e_i) = \frac{1}{R\eta} e_i, \text{ for } 2 \leq i \leq n.$$

Let $C_1 = T(I_1), C_2 = T(I_2)$, we can see that C_i is contained in $\approx \dfrac{L}{R\eta^2}$-neighborhood of an affine image of the surface S_i that satisfies the assumption of **Theorem 3.6** for $i = 1, 2$.

Consider $g_i(x) = f_i(T^{-1}x)$, then g_i is supported in C_i and $\widehat{f_i}(\xi) = \dfrac{1}{\det(T)} \widehat{g_i}(T^{-1}\xi) = (R\eta)^n \, \eta \widehat{g_i}(T^{-1}\xi)$. Then we obtain

$$\left\| \widehat{f_1} \widehat{f_2} \right\|_{L^q(\mu_D)} = (R\eta)^{2n} \eta^2 \left(\int \left| \widehat{g_1}\left(T^{-1}x\right) \widehat{g_2}\left(T^{-1}x\right) \right|^q \mu_D(x) dx \right)^{\frac{1}{q}} \tag{101}$$

$$= (R\eta)^{2n - \frac{n}{q}} \eta^{2 - \frac{1}{q}} \left(\int \left| \widehat{g_1}\left(x\right) \widehat{g_2}\left(x\right) \right|^q \mu_D(Tx) dx \right)^{\frac{1}{q}} \tag{102}$$

$$= (R\eta)^{2n - \frac{n}{q}} \eta^{2 - \frac{1}{q}} (R\eta)^{\frac{n-\alpha}{q}} \left\| \widehat{g_1} \widehat{g_2} \right\|_{L^q(dW)}, \tag{103}$$

where $W(x) = (R\eta)^{\alpha - n} \mu_D(Tx)$. If we can say W satisfies **(85)**, then by **Theorem 3.6**, we have

$$\left\| \widehat{g_1} \widehat{g_2} \right\|_{L^q(dW)} \lesssim \frac{L}{R\eta^2} \left\| g_1 \right\|_2 \left\| g_2 \right\|_2. \tag{104}$$

The constant $\dfrac{L}{R\eta^2}$ because C_i is contained in $\approx \dfrac{L}{R\eta^2}$-neighborhood of affine image of S_i.

Now we check these conditions. For the infinity norm,

$$\left\| W \right\|_\infty = (R\eta)^{\alpha - n} \left\| \mu_D \right\|_\infty \lesssim (R\eta)^{\alpha - n} (R\eta)^{n - \alpha} = 1$$

by (1) of **Lemma 3.4** and D is a ball of radius $\approx R\eta$.

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For the other one, given $B(x,r)$,

$$\begin{aligned}\int_{B(x,r)} |W(y)|\,dy &= \int_{B(x,r)} (R\eta)^{\alpha-n} |\mu_D(Ty)|\,dy \\ &= (R\eta)^{\alpha-n} \int_{T(B(x,r))} (R\eta)^n \eta\, |\mu_D(t)|\,dt \\ &= (R\eta)^{\alpha} \eta \int_{T(B(x,r))} |\mu_D(t)|\,dt \\ &\lesssim (R\eta)^{\alpha} \eta \left(\frac{R\eta}{R\eta^2}\right) \int_{D'} |\mu_D(t)|\,dt = (R\eta)^{\alpha} \int_{D'} |\mu_D(t)|\,dt,\end{aligned}$$

where D' is a ball or radius $\approx \frac{r}{R\eta}$. T scales one axis into $\frac{1}{R\eta^2}$ long, others are $\frac{1}{R\eta}$ long, so $T(B(x,r))$ is covered by at most $\frac{R\eta}{R\eta^2}$ many balls with radius $\frac{r}{R\eta}$. Then using (3) of **Lemma 3.4**, if $\frac{r}{R\eta} \geq \frac{1}{R\eta} \Rightarrow r \geq 1$, we have $\int_{D'} |\mu_D(t)|\,dt \lesssim \left(\frac{r}{R\eta}\right)^{\alpha}$. So $\int_{B(x,r)} |W(y)|\,dy \lesssim (R\eta)^{\alpha} \cdot \left(\frac{r}{R\eta}\right)^{\alpha} = r^{\alpha}$. If $r < 1$, $\int_{B(x,r)} |W(y)|\,dy \lesssim \|W\|_\infty r^n \lesssim r^{\alpha}$ by (2) of **Lemma 3.4**.

Thus we have $\|\widehat{g_1}\widehat{g_2}\|_{L^q(dW)} \lesssim \frac{L}{R\eta^2} \|g_1\|_2 \|g_2\|_2$, and since $g_i(x) = f_i(T^{-1}x)$,

$$\|g_i\|_2 = (R\eta)^{-\frac{n}{2}} \eta^{-\frac{1}{2}} \|f_i\|_2 \tag{105}$$

for $i = 1, 2$. Finally, combine **(103)**, **(104)**, **(105)**, we obtain

$$\begin{aligned}\left\|\widehat{f_1}\widehat{f_2}\right\|_{L^q(\mu_D)} &\lesssim (R\eta)^{2n-\frac{n}{q}} \eta^{2-\frac{1}{q}} (R\eta)^{\frac{n-\alpha}{q}} \left(\frac{L}{R\eta^2}\right) (R\eta)^{-n} \eta^{-1} \|f_1\|_2 \|f_2\|_2 \\ &= L\,(R\eta)^{n-1-\frac{\alpha}{q}} \eta^{-\frac{1}{q}} \|f_1\|_2 \|f_2\|_2 .\end{aligned}$$

□

Theorem 3.7. *Let $\alpha \in (0,n)$, $q > q_0(\alpha,n)$, $\forall\alpha$-dimensional measure μ, $R > 1$, f is supported in $A_R(1)$. Then*

$$\left|\int f^{\vee}(x)\,d\mu(x)\right| \leq C_{q,\mu} R^{\frac{n-1}{2}-\frac{\alpha}{2q}} \|f\|_2 \tag{106}$$

Proof. W.L.O.G., we may assume $\|f\|_2 = 1$. Since $\left(\int |f^{\vee}|\,d\mu\right)^2 \leq \int |f^{\vee}|^2\,d\mu$ by

doi:10.6342/NTU202301952

Cauchy-Schwartz inequality and μ has compactly support, we only need to prove

$$\|f^\vee\|_{L^2(d\mu)} \lesssim R^{\frac{n-1}{2}-\frac{\alpha}{2q}}.$$

Consider a dyadic decomposition of $A_R(1)$ into spherical caps I with dimensions $2 \times 2^k \times \cdots \times 2^k$ for $R^{\frac{1}{2}} \leq 2^k \leq R$, we denote $l(I) = 2^k$ and define the **parent** of I be the unique cap contained I with length 2^{k+1}. Given two spherical caps I, J with $l(I) = 2^k$, we say $I \sim J$ if

1. $l(I) = l(J)$,
2. I, J are not adjacent,
3. the parent of I and the parent of J ar adjacent.

Then let $f_I = f \cdot \chi_I$, we have

$$\left(f^\vee\right)^2(x) = \sum_{\sqrt{R} \ll 2^k \ll R} \sum_{l(I)=2^k, I\sim J} f_I^\vee(x) f_J^\vee(x) + error, \tag{107}$$

where the term has the bound

$$|error| \lesssim \sum_{I_E} \left|f_I^\vee(x)\right|^2. \tag{108}$$

$I \in I_E$ is the diagonal dyadic caps term with $I \approx \sqrt{R}$ satisfies $\left\| \sum_{I\in I_E} \chi_I \right\|_\infty \lesssim 1$. Combine **(107)** and **(108)**, we obtain

$$\|f^\vee\|^2_{L^2(d\mu)} \lesssim \sum_{\sqrt{R} \ll 2^k \ll R} \sum_{l(I)=2^k, I\sim J} \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} + \sum_{I\in I_E} \|f_I^\vee\|^2_{L^2(d\mu)}. \tag{109}$$

First, we show that $\sum_{I\in I_E} \|f_I^\vee\|^2_{L^2(d\mu)} \lesssim R^{n-1-\frac{\alpha}{q}}$. For $I \in I_E$, since $l(I) \approx \sqrt{R}$, I is contained in a ball D of radius $C\sqrt{R}$ for some constant C. Then

$$|f_I^\vee(x)| = |f_I^\vee * \psi_D^\vee(x)| = \left| \int f_I^\vee(x-y) \psi_D^\vee(y) dy \right|$$

doi:10.6342/NTU202301952

$$= \left| \int f_I^\vee(x-y) \left(\psi_D^\vee\right)^{\frac{1}{2}}(y) \left(\psi_D^\vee\right)^{\frac{1}{2}}(y)dy \right|$$

$$\lesssim \left(|f_I^\vee|^2 * |\psi_D^\vee| \right)^{\frac{1}{2}} \|\psi_D^\vee\|_1^{\frac{1}{2}} \lesssim \left(|f_I^\vee|^2 * |\psi_D^\vee| \right)^{\frac{1}{2}},$$

where ψ_D has defined above the **Lemma 3.4**, and the last line follows from Hölder's inequality. Using this, we get

$$\sum_{I \in I_E} \|f_I^\vee\|_{L^2(d\mu)}^2 \lesssim \int \left(|f_I^\vee|^2 * |\psi_D^\vee| \right) d\mu = \int |f_I^\vee(x)|^2 \left(\mu * |\psi_D^\vee|\right)(x)dx$$

$$\lesssim \|f_I^\vee\|_2^2 \, \|\mu * |\psi_D^\vee|\|_\infty \lesssim \|f_I\|_2^2 \cdot \left(\sqrt{R}\right)^{n-\alpha} = R^{\frac{n-\alpha}{2}} \|f_I\|_2^2$$

by (2) of **Lemma 3.4**. Then

$$\sum_{I \in I_E} \|f_I^\vee\|_{L^2(d\mu)}^2 \lesssim R^{\frac{n-\alpha}{2}} \sum_{I \in I_E} \|f_I\|_2^2 \lesssim R^{\frac{n-\alpha}{2}} \|f\|_2^2 = R^{\frac{n-\alpha}{2}} \lesssim R^{n-1-\frac{\alpha}{q}}. \tag{110}$$

The second inequality because of $\left\| \sum_{I \in I_E} \chi_I \right\|_\infty \lesssim 1$. And the last inequality follows from $\alpha < n$, $q > q_0 \geq \dfrac{4\alpha}{n+2\alpha-2}, n \geq 3$.

For $\sum_{\sqrt{R} \ll 2^k \ll R} \sum_{l(I)=2^k, I \sim J} \|f_I^\vee f_J^\vee\|_{L^1(d\mu)}$, we prove

$$\|f_I^\vee f_J^\vee\|_{L^1(d\mu)} \leq C_{\alpha,q,n} R^{n-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2 \tag{111}$$

for $I \sim J$, $l(I) = 2^k$. Now by the similar method in the proof of **Corollary 3.1**, let e be the unit vector in the direction of center of mass of $I \cup J$, consider a rectangle H with side lengths $100 \times 100 \cdot \dfrac{2^k}{R} \times \cdots \times 100 \cdot \dfrac{2^k}{R}$, where the axis has side length 100 is in the direction e. We use H to tile $\mathbb{R}^n$ and assume a_H be the affine bijection maps H to the unit cube $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]^n$. Let ψ be a Schwartz function satisfies

$$\psi(x) \geq \chi_{B(0,1)}(x) \text{ for } x \in \mathbb{R}^n \text{ , } supp\left(\widehat{\psi}\right) \subset B(0,1).$$

doi:10.6342/NTU202301952

Let $\psi_H = \psi \circ a_H$ and $f_{I,H} = \widehat{f_I^\vee \psi_H}$, then we have

$$\begin{aligned}\|f_I^\vee f_J^\vee\|_{L^1(d\mu)} &= \int |f_I^\vee f_J^\vee|\, d\mu = \sum_H \int_H |f_I^\vee f_J^\vee|\, d\mu \\ &\lesssim \sum_H \int \left|f_{I,H}^\vee f_{J,H}^\vee\right| (\psi_H)^{\frac{1}{q'}}\, d\mu \lesssim \sum_H \left\|f_{I,H}^\vee f_{J,H}^\vee\right\|_{L^q(d\mu)} \|\psi_H\|_{L^1(d\mu)}^{\frac{1}{q'}} \quad (112)\end{aligned}$$

since $\psi_H \geq 1$ and q' is the Hölder conjugate exponent of q. Let I_H be the support of $f_{I,H}$, we have $I_H \subseteq I + supp\left(\widehat{\psi_H}\right) = I + H_{dual}$ where H_{dual} is a rectangle with side lengths $\dfrac{1}{100} \times \dfrac{R}{100 \cdot 2^k} \times \cdots \dfrac{R}{100 \cdot 2^k}$ centered at the origin. Then $I + H_{dual}$ is contained in a sperical cap of $10 \times \dfrac{11}{10} 2^k \times \cdots \times \dfrac{11}{10} 2^k$ in $A_R(10)$ which contains I. We prove this later. By this fact, we have the diameter of $I_H, J_H \approx 2^k$ and they are contained in $A_R(10)$ with $d\left(I_H, J_H\right) \approx 2^k$. So by**Corollary 3.1** with $R\eta = 2^k$, we get

$$\begin{aligned}\left\|f_{I,H}^\vee f_{J,H}^\vee\right\|_{L^q(d\mu)} &\lesssim 10 \cdot 2^{k\left(n-1-\frac{\alpha}{q}\right)} \left(\frac{2^k}{R}\right)^{-\frac{1}{q}} \|f_{I,H}\|_2 \|f_{J,H}\|_2 \\ &\lesssim R^{\frac{1}{q}} 2^{k\left(n-1-\frac{1}{q}-\frac{\alpha}{q}\right)} \|f_{I,H}\|_2 \|f_{J,H}\|_2 . \quad (113)\end{aligned}$$

since $R^{\frac{1}{2}} \ll 2^k \ll R \Rightarrow R^{-\frac{1}{2}} \ll \dfrac{2^k}{R} \ll 1$.

For $\|\psi_H\|_{L^1(d\mu)}$, since $2^j H$ is of dimensions $100 \cdot 2^j \times 100 \cdot \dfrac{2^j 2^k}{R} \times \cdots \times 100 \cdot \dfrac{2^j 2^k}{R}$, it can be covered by $\approx 100 \cdot 2^j \Big/ \dfrac{100 \cdot 2^j 2^k}{R} = \dfrac{R}{2^k}$ many balls of radius $\approx \dfrac{2^j 2^k}{R}$ and ψ_H is Schwartz function, given M large enough, we obtain

$$\begin{aligned}\|\psi_H\|_{L^1(d\mu)} &\leq \sum_{j=1}^{\infty} 2^{-Mj} \int \chi_{2^j H}(x) d\mu(x) \\ &\lesssim \sum_{j=1}^{\infty} 2^{-Mj} \frac{R}{2^k} \left(\frac{2^j 2^k}{R}\right)^{\alpha} \\ &\leq \sum_{j=1}^{\infty} 2^{-\frac{Mj}{2}} 2^{k(\alpha-1)} R^{1-\alpha} \lesssim 2^{k(\alpha-1)} R^{1-\alpha}. \quad (114)\end{aligned}$$

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Combine **(112)**, **(113)** and **(114)**, we get

$$
\begin{aligned}
\|f_I^\vee f_J^\vee\|_{L^1(d\mu)} &\lesssim \sum_H \left\|f_{I,H}^\vee f_{J,H}^\vee\right\|_{L^q(d\mu)} \|\psi_H\|_{L^1(d\mu)}^{\frac{1}{q'}} \\
&\lesssim \sum_H \left(R^{\frac{1}{q}} 2^{k\left(n-1-\frac{1}{q}-\frac{\alpha}{q}\right)} \|f_{I,H}\|_2 \|f_{I,H}\|_2\right) \left(2^{k\alpha-k} R^{1-\alpha}\right)^{\frac{1}{q'}} \\
&= R^{\frac{1}{q}+\frac{1}{q'}-\frac{\alpha}{q'}} 2^{k\left(n-1-\frac{1}{q}-\frac{\alpha}{q}+\frac{\alpha}{q'}-\frac{1}{q'}\right)} \sum_H \|f_{I,H}\|_2 \|f_{I,H}\|_2 \\
&= R^{\frac{1}{q}+\frac{1}{q'}-\frac{\alpha}{q'}} 2^{k\left(n-2+\alpha-\frac{2\alpha}{q}\right)} \|f_{I,H}\|_2 \|f_{I,H}\|_2 \\
&\lesssim R^{\frac{1}{q}+\frac{1}{q'}-\frac{\alpha}{q'}} R^{n-2+\alpha-\frac{2\alpha}{q}} \sum_H \|f_{I,H}\|_2 \|f_{I,H}\|_2 \\
&= R^{n-1-\frac{\alpha}{q}} \sum_H \|f_{I,H}\|_2 \|f_{I,H}\| \\
&\lesssim R^{n-1-\frac{\alpha}{q}} \left(\sum_H \|f_{I,H}\|_2^2\right)^{\frac{1}{2}} \left(\sum_H \|f_{J,H}\|_2^2\right)^{\frac{1}{2}}, \qquad (115)
\end{aligned}
$$

since $2^k \lesssim R$ and $q > q_0 \geq \dfrac{4\alpha}{n+2\alpha-2} \Rightarrow n-2+\alpha-\dfrac{2\alpha}{q} > \dfrac{n}{2}-1 > 0$.
For $\|f_{I,H}\|_2$, by Plancherel theorem and Schwartz decay of ψ_H, we have

$$
\begin{aligned}
\|f_{I,H}\|_2^2 = \int_H |f_{I,H}|^2 &= \int_H \left|\widehat{f_I^\vee \psi_H}\right|^2 \\
&= \int_H |f_I^\vee|^2 \psi_H^2 \lesssim \int_H |f_I^\vee|^2 = \int_H |f_I|^2 .
\end{aligned}
$$

So

$$
\left(\sum_H \|f_{I,H}\|_2^2\right)^{\frac{1}{2}} \lesssim \left(\sum_H \int_H |f_I|^2\right)^{\frac{1}{2}} = \left(\int |f_I|^2\right)^{\frac{1}{2}} = \|f_I\|_2 .
$$

Similarly, we have $\left(\sum_H \|f_{J,H}\|_2^2\right)^{\frac{1}{2}} \lesssim \|f_J\|_2$.
Then **(111)** follows from this and **(115)**. Now by **(109)**, **(110)** and **(111)**, we have

$$
\|f^\vee\|_{L^2(d\mu)}^2 \lesssim \sum_{\sqrt{R} \ll 2^k \ll R} \sum_{l(I)=2^k, I\sim J} R^{n-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2 + R^{n-1-\frac{\alpha}{q}}. \qquad (116)
$$

For each I with $l(I) = 2^k$, there are finitely many J such that $I \sim J$. So there is a

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spherical cap I' with $l(I') = C2^k$ for some constant C contain I such that

$$\sum_{l(I)=2^k, I\sim J} \|f_J\|_2 \lesssim \|f_{I'}\|_2 . \tag{117}$$

And for each k, such I' are finitely overlapping. Therefore,

$$\sum_{l(I)=2^k} \|f_I\|_2^2 \lesssim \sum_{l(I)=2^k} \|f_{I'}\|_2^2 \approx \|f\|_2^2 . \tag{118}$$

Then by **(117)**, **(118)** and Cauchy-Schwartz inequality,

$$\begin{aligned}\sum_{l(I)=2^k, I\sim J} \|f_I\|_2 \|f_J\|_2 &\le \left(\sum_{l(I)=2^k} \|f_I\|_2^2\right)^{\frac{1}{2}} \left(\sum_{l(I)=2^k, I\sim J} \|f_J\|_2^2\right)^{\frac{1}{2}} \\ &\lesssim \|f\|_2 \cdot \|f\|_2 = \|f\|_2^2 = 1.\end{aligned}$$

So **(116)** becomes

$$\begin{aligned}\|f^\vee\|_{L^2(d\mu)}^2 &\lesssim \sum_{\sqrt{R}\ll 2^k \ll R} R^{n-1-\frac{\alpha}{q}} + R^{n-1-\frac{\alpha}{q}} \\ &\lesssim (\log R)\, R^{n-1-\frac{\alpha}{q}} . \end{aligned} \tag{119}$$

Since we can find another $\tilde{q}$ satisfies $q > \tilde{q} > q_0$ and all statements above, the inequality becomes

$$\|f^\vee\|_{L^2(d\mu)}^2 \lesssim (\log R)\, R^{n-1-\frac{\alpha}{\tilde{q}}} = (\log R)\, R^{\alpha\left(\frac{1}{q}-\frac{1}{\tilde{q}}\right)} R^{n-1-\frac{\alpha}{q}} \lesssim R^{n-1-\frac{\alpha}{q}} .$$

This follows from $\log R \lesssim R^l$ for all $l > 0$ and $\dfrac{1}{q} < \dfrac{1}{\tilde{q}}$. So we complete the proof. □

Lemma 3.5. *$I + H_{dual}$ in proof of* ***theorem 3.7*** *is contained in a spherical cap of $10 \times \frac{11}{10}2^k \times \cdots \frac{11}{10}2^k$ in $A_R(10)$ which contains I.*

Proof. First we have H_{dual} is of dimensions $\dfrac{1}{100} \times \dfrac{1}{100} \cdot \dfrac{R}{2^k} \times \cdots \times \dfrac{1}{100} \cdot \dfrac{R}{2^k}$. Let the short axis being in the direction e, then for $h \in H_{dual}$, $x \in I$, the angle between $p - \langle p, e\rangle e$ and the hyperplane E_x which goes through the origin with normal vector x is less

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than $10 \cdot \dfrac{2^k}{R}$. Hence, H_{dual} is contained in the $\dfrac{1}{10}$-neighborhood of $E_x \cap B(0, \frac{R}{100 \cdot 2^k})$. Furthermore, we can see that $x + (E_x \cap B(0,r))$ is contained in a spherical cap containing x of dimensions $\approx 1 \times r \times \cdots \times r$ in $A_{|x|}(1)$ if $|x| \approx R$ and $r \lesssim R^{\frac{1}{2}}$. Then the statement we want is holds since $\dfrac{R}{100 \cdot 2^k} \lesssim 2^k$.

□

Finally, we use a variation of Mattila's theorem(see [7]) to obtain the result.

Theorem 3.8 (Mattila's theorem)**.** *Fix* $\alpha \in \left[\dfrac{n}{2}, \dfrac{n+1}{2}\right]$ *and* $q_0 \in [1,2]$ *such that* $\alpha\left(1 + \dfrac{1}{q_0}\right) \geq n$*. If for all* α*-dimensional measure* μ*,* $R > 1$*, and* f *is supported in* $A_R(1)$*, we have*

$$\left|\int f^{\vee}(x) d\mu(x)\right| \leq C_{q,\mu} R^{\frac{n-1}{2} - \frac{\alpha}{2q}} \left\|f\right\|_2 \tag{120}$$

holds for all $q > q_0$*, then Falconer's conjecture holds for* α*.*

Using **Theorem 3.7**, we can give a bound of α in **Theorem 3.8**. Since we need $\alpha \in \left[\dfrac{n}{2}, \dfrac{n+1}{2}\right]$ and

$$\frac{4\alpha}{n + 2\alpha - 2} = \frac{n+2}{n} \Leftrightarrow \alpha = \frac{n+2}{2},$$

we have $q_0 = \min\left(1, \dfrac{4\alpha}{n+2\alpha-2}\right)$. Also, we can see that $q_0 \in [1,2]$ because $n \geq 2$. Therefore, to accord the inequality, if $q_0 = 1$, it is trivial. If $q_0 = \dfrac{4\alpha}{n+2\alpha-2}$,

$$\begin{aligned} \alpha\left(1 + \frac{1}{q_0}\right) \geq n \Leftrightarrow \frac{n+2\alpha-2}{4\alpha} &\geq \frac{3n-2}{3n+2} \\ \Leftrightarrow 3n^2 - 4n - 4 &\geq 6n\alpha - 12\alpha \Leftrightarrow \alpha \leq \frac{n}{2} + \frac{1}{3}. \end{aligned}$$

So we obtain Falconer's conjecture holds for $\dfrac{n}{2} + \dfrac{1}{3}$.

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