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碩士論文

Department of Mathematics
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Master Thesis

高維度初抵滲透模型的時間常數與極限形狀

Time Constant and Limit Shape of High-Dimensional First-Passage Percolation

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摘要

初抵滲透模型是一個由 Hammersley 跟 Welsh 在 1965 年提出的廣爲人知的統計物理模型。模型的定義如下。令 $\mathbb{Z}^d$ 爲 d 維的整數網格、$\mathcal{E}^d$ 爲收集所有在 $\mathbb{Z}^d$ 上的（無向）最近鄰邊的集合，也就是

$$\mathcal{E}^d = \{\{\mathbf{x}, \mathbf{y}\} \mid \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d, \|\mathbf{x} - \mathbf{y}\|_1 = 1\}。$$

對於每條邊 $e \in \mathcal{E}^d$，我們指定一個非負的隨機變數 τ_e, 稱爲 e 的**權重（穿越時間）**。對於在 $\mathbb{Z}^d$ 上任意的網格路徑 γ，γ **的穿越時間**有如下的定義

$$\tau(\gamma) = \sum_{e \in \gamma} \tau_e。$$

對於兩個整數點 $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$，我們定義 $\Gamma(\mathbf{x}, \mathbf{y})$ 爲在 $\mathbb{Z}^d$ 上所有連接 $\mathbf{x}$ 與 $\mathbf{y}$ 的網格路徑。對於一個點 $\mathbf{x} \in \mathbb{R}^d$，我們定義 $\lfloor \mathbf{x} \rfloor \in \mathbb{Z}^d$ 爲唯一使得 $\mathbf{x} \in \lfloor \mathbf{x} \rfloor + [0, 1)^d$ 的整數點。現在，我們定義**從 $\mathbf{x}$ 到 $\mathbf{y}$ 的初抵時間**爲

$$\tau(\mathbf{x}, \mathbf{y}) = \inf\{\tau(\gamma) \mid \gamma \in \Gamma(\lfloor \mathbf{x} \rfloor, \lfloor \mathbf{y} \rfloor)\}。$$

儘管模型本身很易於理解，但時至今日，仍有許多關於這個模型的問題懸而未決。其中一個就是關於時間常數與極限形狀的決定。在這篇碩士論文中，我們會考慮時間常數在維度趨向無限時的漸進行爲，並以此在維度夠高時去否決部分的極限形狀候選。

關鍵字：初抵滲透模型、時間常數、極限形狀。

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Abstract

First-passage percolation (FPP) is a widely recognized statistical physics model that was originally proposed by Hammersley and Welsh in 1965. The model is defined as follows. Let $\mathbb{Z}^d$ be the d-dimensional integer lattice and $\mathcal{E}^d$ be the collection of (non-oriented) nearest-neighbour edges in $\mathbb{Z}^d$. That is,

$$\mathcal{E}^d = \{\{\mathbf{x}, \mathbf{y}\} \mid \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d, \|\mathbf{x} - \mathbf{y}\|_1 = 1\}.$$

We assign to each edge $e \in \mathcal{E}^d$ a non-negative random variable τ_e, called the **weight (passage time)** of e. For any lattice path γ on $\mathbb{Z}^d$, the **passage time of** γ is defined by

$$\tau(\gamma) = \sum_{e \in \gamma} \tau_e.$$

For two points $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, we define $\Gamma(\mathbf{x}, \mathbf{y})$ to be all lattice paths on $\mathbb{Z}^d$ connecting $\mathbf{x}$ to $\mathbf{y}$. For a point $\mathbf{x} \in \mathbb{R}^d$, we define the "floor" of $\mathbf{x}$, $\lfloor \mathbf{x} \rfloor \in \mathbb{Z}^d$, to be the unique vertex in $\mathbb{Z}^d$ such that $\mathbf{x} \in \lfloor \mathbf{x} \rfloor + [0, 1)^d$. Now, we define the **first-passage time from x to y** by

$$\tau(\mathbf{x}, \mathbf{y}) = \inf\{\tau(\gamma) \mid \gamma \in \Gamma(\lfloor \mathbf{x} \rfloor, \lfloor \mathbf{y} \rfloor)\}.$$

Despite its apparent simplicity, numerous unresolved problems persist within this model. Among these, determining the time constant and the limit shape are particularly noteworthy. In this master thesis, we aim to investigate the asymptotic behavior of the time constant as the dimension tends to infinity and subsequently reject several potential

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limit shapes in sufficiently high dimensions.

Keywords: First-passage percolation, time constant, limit shape.

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Chapter 1

Introduction

First-passage percolation (FPP) is a model of fluid flow through a random medium introduced by Hammersley and Welsh (see [9]) in 1965. The model is defined as follows. Let $\mathbb{Z}^d$ be the d-dimensional integer lattice and $\mathcal{E}^d$ be the collection of (non-oriented) nearest-neighbour edges in $\mathbb{Z}^d$. That is,

$$\mathcal{E}^d = \{\{\mathbf{x}, \mathbf{y}\} \mid \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d, \|\mathbf{x} - \mathbf{y}\|_1 = 1\}.$$

We assign to each edge $e \in \mathcal{E}^d$ a non-negative random variable τ_e, called the **weight (passage time)** of e. We will assume throughout this thesis that the collection $(\tau_e)_{e \in \mathcal{E}^d}$ is independent and identically distributed (i.i.d.) and denote the distribution function

$$F_{\tau_e}(t) = \mathbb{P}(\tau_e \leqslant x).$$

For any lattice path γ on $\mathbb{Z}^d$, the **passage time of** γ is defined by

$$\tau(\gamma) = \sum_{e \in \gamma} \tau_e.$$

For two points $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, we define $\Gamma(\mathbf{x}, \mathbf{y})$ to be all lattice paths on $\mathbb{Z}^d$ connecting $\mathbf{x}$ to $\mathbf{y}$. For a point $\mathbf{x} \in \mathbb{R}^d$, we define the "floor" of $\mathbf{x}$, $\lfloor \mathbf{x} \rfloor \in \mathbb{Z}^d$, to be the unique vertex in $\mathbb{Z}^d$

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such that $\mathbf{x} \in \lfloor \mathbf{x} \rfloor + [0,1)^d$. Now, we define the **first-passage time from x to y** by

$$\tau(\mathbf{x},\mathbf{y}) = \inf\{\tau(\gamma) \mid \gamma \in \Gamma(\lfloor \mathbf{x} \rfloor, \lfloor \mathbf{y} \rfloor)\}.$$

For $t \geqslant 0$, define

$$B_d(t) = \{\mathbf{x} \in \mathbb{R}^d \mid \tau(\mathbf{0},\mathbf{x}) \leqslant t\}. \tag{1.1}$$

$B_d(t)$ is the set of vertices that can be reached from **0** by time t. We are interested in studying the geometry of the random ball $B_d(t)$.

1.1 Known results

One approach studying the ball $B_d(t)$ is to ask whether it "converges" to some deterministic set (under a proper normalization). Therefore, it will be convenient to study the asymptotic behavior of passage times between points becoming gradually distant.

Theorem 1.1 ([9], Theorem 4.3.2)**.** If τ_e satisfies

$$\mathbb{E}\left[\min\{\tau_1,\ldots,\tau_{2d}\}\right] < \infty, \tag{$\clubsuit$}$$

where $\tau_1,\ldots,\tau_{2d}$ are independent copies of τ_e, then there exists a constant $\mu_d(\mathbf{e}_1) \in [0,\infty)$ (called the **time constant**) such that

$$\frac{\tau(\mathbf{0}, n\mathbf{e}_1)}{n} \to \mu_d(\mathbf{e}_1) \quad \text{almost surely and in } \mathcal{L}^1. \tag{1.2}$$

The theorem above indicates that $\tau(\mathbf{0}, n\mathbf{e}_1) \approx n \cdot \mu_d(\mathbf{e}_1)$ when n is large. We can in fact extend $\mu_d \colon \mathbb{R}^d \to \mathbb{R}$ to a semi-norm on $\mathbb{R}^d$; the details will be discussed in Section 3.1 later.

Note that if $\mathbb{P}(\tau_e = 0) > 0$, there are cost-free edges to cross. It is intuitive that as $\mathbb{P}(\tau_e = 0)$ increases, the probability for **0** to reach ∞ within zero time also increases. Let

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$\{0 \leftrightarrow \infty\}$ be the event that there exists an infinite self-avoiding path γ starting from $\mathbf{0}$ with $\tau_e = 0$ for all $e \in \gamma$, we define the critical threshold

$$p_c(d) = \sup\{p \in [0,1] \mid \mathbb{P}_p(\{0 \leftrightarrow \infty\}) = 0\},$$

where $\mathbb{P}_p$ is the measure of the FPP model with weight τ_e satisfying $\mathbb{P}(\tau_e = 0) = p$. It is known that $0 < p_c(d) < 1$ for all $d \geqslant 2$ (see the standard text [7] for more details on Bernoulli percolation) and we actually have the following theorem determining whether $\mu_d(\mathbf{e}_1)$ is positive or not.

Theorem 1.2 ([10], Theorem 6.1)**.** If τ_e satisfies the moment assumption (♣), then $\mu_d(\mathbf{e}_1) > 0$ if and only if $\mathbb{P}(\tau_e = 0) < p_c(d)$.

Under the condition $\mathbb{P}(\tau_e = 0) < p_c(d)$, there is (almost surely) no cost-free path for $\mathbf{0}$ to reach ∞, we may then expect that $B_d(t)$ won't grow too fast on $\mathbb{R}^d$. The following shape theorem indicates that $B_d(t)$ (under a proper normalization) does "converge" to a deterministic set under this assumption.

Theorem 1.3 ([4], Theorem 4)**.** If τ_e satisfies

(S1) $\mathbb{E}\left[\min\{\tau_1, \ldots, \tau_{2d}\}^d\right] < \infty$ (where $\tau_1, \ldots, \tau_{2d}$ are independent copies of τ_e) and

(S2) $\mathbb{P}(\tau_e = 0) < p_c(d)$,

then there is a deterministic, convex, compact set $\mathcal{B}_d \subseteq \mathbb{R}^d$ such that for all $\epsilon > 0$,

$$\mathbb{P}\left((1-\epsilon)\mathcal{B}_d \subseteq \frac{B_d(t)}{t} \subseteq (1+\epsilon)\mathcal{B}_d \text{ for all } t \text{ large}\right) = 1.$$

Moreover, the **limit shape** given by $\mathcal{B}_d = \left\{\mathbf{x} \in \mathbb{R}^d \mid \mu_d(\mathbf{x}) \leqslant 1\right\}$ has non-empty interior and is symmetric about the axes of $\mathbb{R}^d$.

Observe that we may determine $\mathcal{B}_d$ in Theorem 1.3 if we are able to compute the time constant μ_d. Unfortunately, the exact value of μ_d for non-degenerate edge weight distributions remains an open problem. However, one can determine the asymptotic be-

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havior of μ_d as $d \to \infty$ and therefore investigate the geometry of $\mathcal{B}_d$ when d is large. This problem was first considered by Kesten in [10]. To state his result, we define, for $p \geqslant 1$ and $a \in [0, \infty]$, the class of distributions

$$\mathcal{M}_p(a) = \left\{ F\colon \text{ distribution on } [0, \infty) \;\middle|\; \int_0^\infty x \, \mathrm{d}F(x) < \infty \text{ and } \lim_{t \to 0^+} \frac{F(t)}{t^p} = a \right\}. \quad (\spadesuit)$$

Kesten showed that if $F_{\tau_e} \in \mathcal{M}_1(a)$ with $a \in (0, \infty)$, then there is a universal constant $\epsilon > 0$ (independent of the distribution of τ_e) such that

$$\frac{\epsilon}{a} \leqslant \liminf_{d \to \infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} \leqslant \limsup_{d \to \infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} \leqslant \frac{11}{a}. \tag{1.3}$$

In fact, Kesten assumed a stronger condition ($\heartsuit$) with $C = o(1)$ (defined in Section 4.2), but this can be overcome by stochastically dominance, which will be shown in Corollary 4.10. Under the same condition ($p = 1$, $a \in (0, \infty)$) and by using a similar argument (of the lower bound), he also showed that the **diagonal time constant** $\mu_d(\mathbf{1})$ (which will be defined in Section 3.1 and $\mathbf{1} = \mathbf{e}_1 + \cdots + \mathbf{e}_d$) satisfies

$$\frac{1}{6ea} \leqslant \liminf_{d \to \infty} \mu_d(\mathbf{1}) \leqslant \limsup_{d \to \infty} \mu_d(\mathbf{1}) \leqslant \frac{1}{2a}. \tag{1.4}$$

The upper bound in (1.4) was due to Cox and Durrett [5], who actually showed that the limit of $\mu_d(\mathbf{1})$ as $d \to \infty$ exists if the edge weight distribution is exponential. With (1.3) and (1.4), Kesten showed that $\mathcal{B}_d$ is not a Euclidean ball if d is sufficiently large.

Following Kesten, Dhar showed in [6] that if $\tau_e \sim \texttt{Exponential}(a)$, then we have

$$\lim_{d \to \infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} = \frac{1}{2a}. \tag{1.5}$$

However, his proof cannot be generalized to general distribution since it relies on the Eden model, a random growth model related to FPP when $\tau_e \sim \texttt{Exponential}(a)$.

For the diagonal time constant, Couronné, Enriquez and Gerin showed in [3] that if $\tau_e \sim \texttt{Exponential}(a)$, then one can improve the lower bound in (1.4) to $\frac{\sqrt{\alpha_1^2 - 1}}{2a}$, where

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$\alpha_1 \approx 1.19968$ is the positive solution of $\coth \alpha = \alpha$.

Recently, Auffinger and Tang showed in [2] that if τ_e satisfies ($\heartsuit$) with parameter $p = 1$, $a \in [0, 1]$, then (1.5) still holds. However, we believe that some of the arguments are incorrect in the paper [2], which will be discussed in Section 4.3. Nevertheless, their methods (originated from [10]) remain valid to prove our main result (1.6) for the case $p > 1$. They also proved (1.8) for $p = 1$ under the same assumption.

Last, Martinsson showed in [13] (using a completely different argument) that the constant given in [3] is sharp for (1.4). That is,

$$\lim_{d\to\infty} \mu_d(\mathbf{1}) = \frac{\sqrt{\alpha_1^2 - 1}}{2a}$$

for all distributions $F_{\tau_e} \in \mathcal{M}_1(a)$ with parameter $a \in [0, \infty]$.

1.2 Main results

The main objective of this thesis is to study the limit shape $\mathcal{B}_d$ in high-dimensional integer lattice by deriving the asymptotic behavior of $\mu_d(\mathbf{e}_1)$ as $d \to \infty$. The first main result is stated as follows.

Theorem 1.4 (Asymptotic behavior of high-dimensional time constant $\mu_d(\mathbf{e}_1)$)**.** If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p > 1$ and $a \in [0, \infty]$, then the time constant satisfies

$$\lim_{d\to\infty} \frac{\mu_d(\mathbf{e}_1) d^{\frac{1}{p}}}{\log d} = \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}}, \tag{1.6}$$

where Γ is the gamma function. For the case $p = 1$ and $a \in [0, \infty]$, one has

$$\frac{1}{2a} \leqslant \liminf_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} \leqslant \limsup_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} \leqslant \frac{6}{a}. \tag{1.7}$$

Remark 1.5. Note that the distributions in the class $\mathcal{M}_p(a)$ are Gamma-like (or in the case $p = 1$, exponential-like) near the passage time at 0. In fact, it contains a large class

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of distributions.

We also consider the asymptotic behavior of the diagonal time constant (recall $\mathbf{1} = \mathbf{e}_1 + \cdots + \mathbf{e}_d$). The result is stated as follows.

Theorem 1.6 (Asymptotic behavior of high-dimensional time constant $\mu_d(\mathbf{1})$)**.** If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p \geqslant 1$ and $a \in [0, \infty]$, then the time constant satisfies

$$p\left(\frac{\alpha_p^{p-1}\sqrt{\alpha_p^2-1}}{2a\Gamma(p+1)}\right)^{\frac{1}{p}} \leqslant \liminf_{d\to\infty} \frac{\mu_d(\mathbf{1})}{d^{1-\frac{1}{p}}} \leqslant \limsup_{d\to\infty} \frac{\mu_d(\mathbf{1})}{d^{1-\frac{1}{p}}} \leqslant \frac{1}{a^{\frac{1}{p}}}, \tag{1.8}$$

where $\alpha_p > 0$ is the positive solution of $\coth(p\alpha) = \alpha$.

Last, note that by the convexity of $\mathcal{B}_d$, one always has $\ell^1 \subseteq \mathcal{B}_d \subseteq \ell^\infty$, where

$$\ell^q = \left\{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\|_q \leqslant \mu_d(\mathbf{e}_1)^{-1}\right\}$$

for $q \in [1, \infty]$. It has been conjectured that $\mathcal{B}_2 \neq \ell^2$ if $\tau_e \sim \texttt{Exponential}(1)$. This leads to our second main result.

Theorem 1.7 (Limit shape)**.** If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p \geqslant 1$ and $a \in (0, \infty)$, then there exists $d_0 \in \mathbb{N}$ such that for any $d \geqslant d_0$,

$$\ell^1 \subsetneq \mathcal{B}_d \subsetneq \ell^\infty \quad \text{and} \quad \mathcal{B}_d \neq \ell^q \quad \text{for all} \quad 1 < q < \infty.$$

1.3 Outline of the thesis

In Chapter 2, we will introduce some useful bounds of the numbers of various lattice paths on $\mathbb{Z}^d$. This part is mainly based on [2, 3, 8, 10].

In Chapter 3, we gather and prove some classical results in FPP, such as the existence of the time constant and the shape theorem. This part is mainly based on [1, 4, 9, 10].

In Chapter 4, we make use of the tools in the previous two chapters and prove our

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main results (serving as a generalization of the results in [2]).

The elementary calculation in Appendix A will be used in Section 2.3 and the ergodic theorem introduced in Appendix B will be used in Section 3.1. In appendix C, we briefly introduce the sharp upper bound of the diagonal time constant in the particular case $p = 1$ derived in [13].

1.4 Notations

The following notations will be utilized throughout the thesis; some of which will be defined more rigorously in the content.

- Ordering. Let $f, g\colon [0,\infty) \to [0,\infty)$.

  1. $f(x) = O(g(x))$ as $x \to \infty$ if and only if there are constants $C_1 > 0, C_2 > 0$ independent of x such that $f(x) \leqslant C_1 g(x)$ for all $x \geqslant C_2$.
  2. $f(x) = o(g(x))$ as $x \to \infty$ if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
  3. $f(x) \asymp g(x))$ as $x \to \infty$ if and only if $f(x) = O(g(x))$ and $g(x) = O(f(x))$.
  4. $f(x) \sim g(x)$ as $x \to \infty$ if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

  Most of the asymptotic behavior we are interested is as $d \to \infty$, and we will omit the "$d \to \infty$" part in the definition if there is no ambiguity.

- Sets and Coordinates.

  1. $\mathbb{N}$: set of positive integers; $\mathbb{N}_0$: set of non-negative integers; $\mathbb{Z}$: set of integers.
  2. For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, the 1-norm is defined by

  $$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_d|.$$

  3. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ with $\|\mathbf{x} - \mathbf{y}\|_1 = 1$, we denote by $\{\mathbf{x}, \mathbf{y}\}$ the unique edge in $\mathcal{E}^d$ connecting $\mathbf{x}$ and $\mathbf{y}$.

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4. $\mathbf{e}_1, \ldots, \mathbf{e}_d$: standard basis of $\mathbb{R}^d$.
5. $\mathbf{0} \in \mathbb{Z}^d$ is the origin and $\mathbf{1} = \mathbf{e}_1 + \cdots + \mathbf{e}_d \in \mathbb{Z}^d$.
6. For all $\mathbf{x} \in \mathbb{R}^d$ and $d \in \mathbb{N}$, the floor $\lfloor \mathbf{x} \rfloor \in \mathbb{Z}^d$ is the unique vertex on $\mathbb{Z}^d$ such that $\mathbf{x} \in \lfloor \mathbf{x} \rfloor + [0,1)^d$.
7. $H_n = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 = n\right\}$.
8. $J_n^{(\nu)} = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d = \lfloor nd^\nu \rfloor\right\}$ where $\nu \geqslant 0$.
9. $L_1 = \{t\mathbf{e}_1 \mid t \in \mathbb{R}\}$ is the first coordinate axis.
10. $L_{\text{diag}} = \{t\mathbf{1} \mid t \in \mathbb{R}\}$ is the diagonal line.

- Paths.

1. $\Gamma(\mathbf{x}, \mathbf{y})$: lattice paths from $\mathbf{x} \in \mathbb{Z}^d$ to $\mathbf{y} \in \mathbb{Z}^d$. We define in a similar way the set of point-to-plane paths by $\Gamma(\mathbf{x}, H_n) = \bigcup_{\mathbf{y} \in H_n} \Gamma(\mathbf{x}, \mathbf{y})$.
2. $\Lambda_{d,k}$: the set of self-avoiding paths of length k starting from $\mathbf{0}$ on $\mathbb{Z}^d$.
3. $\lambda_{d,k}$: number of self-avoiding paths of length k starting from $\mathbf{0}$ on $\mathbb{Z}^d$.
4. ξ_d: connective constant of $\mathbb{Z}^d$ (see Equation (2.4)).
5. $\mathcal{P}_{d,k,r}$: the set of self-avoiding paths of length k from $\mathbf{0}$ to H_1 on $\mathbb{Z}^d$ whose
   - first $k-1$ steps use only directions $\pm\mathbf{e}_i$, $2 \leqslant i \leqslant r+1$ and
   - the last step is $\mathbf{e}_1$.

   In practice, the constants $k, r \in \mathbb{N}$ will be given in (4.15).
6. $\mathcal{D}_{d,k,n}^{(\nu)}$: the set of self-avoiding paths of length k starting from $\mathbf{0}$ on $\mathbb{Z}^d$ and arrive $J_n^{(\nu)}$ for the first time at the k-th edge (see Section 2.3).

- First-Passage Percolation.

1. τ_e^F: the weight of edge e where F is the distribution function, that is,

$$F(t) = \mathbb{P}\left(\tau_e^F \leqslant t\right).$$

   We will omit F if there is no ambiguity on the distribution.

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2. $\tau^F(\mathbf{x}, \mathbf{y})$: passage time from **x** to **y** on $\mathbb{R}^d$ if the edge weight have distribution F. We will omit F if there is no ambiguity on the distribution.

3. $\mu_d^F(\mathbf{x})$: time constant in direction **x** where $\mathbf{x} \in \mathbb{R}^d$ if the edge weight have distribution F. We will omit F if there is no ambiguity on the distribution.

4. $\mathcal{B}_d$: limit shape in $\mathbb{R}^d$.

5. $N_{d,k,r,t} = \#\{\gamma \in \mathcal{P}_{d,k,r} \mid \tau(\gamma) \leqslant t\}$ (see Section 4.2).

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Chapter 2

Preliminaries: lattice paths

In this chapter, we will introduce some estimates on the numbers of various types of lattice paths. These upper and lower bounds will be utilized in the proof in Chapter 4. Most of these results can be found in [2, 3, 8, 10].

Definition 2.1 (Lattice path)**.** A **lattice path (Pólya walk)** of length $k \in \mathbb{N}$ on $\mathbb{Z}^d$ is an ordered sequence of $k+1$ points in $\mathbb{Z}^d$,

$$\gamma = (\mathbf{s}_0, \dots, \mathbf{s}_k),$$

where $\mathbf{s}_{i-1}$ and $\mathbf{s}_i$ are neighbors (that is, $\|\mathbf{s}_i - \mathbf{s}_{i-1}\|_1 = 1$) for all $1 \leqslant i \leqslant k$. We denote by $\#\gamma$ the length of γ and by Δ_i the i-th ($1 \leqslant i \leqslant k$) increments

$$\Delta_i = \mathbf{s}_i - \mathbf{s}_{i-1} \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \dots, \pm\mathbf{e}_d\}.$$

We will abbreviate "lattice path" by "path" throughout the thesis.

The following result proven by Kesten in [10] gives an upper bound on the number of paths connecting $\mathbf{0}$ to H_n, that is,

$$\Gamma(\mathbf{0}, H_n) = \bigcup_{\mathbf{x} \in H_n} \Gamma(\mathbf{0}, \mathbf{x}) \quad \text{where} \quad H_n = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 = n\}.$$

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This result will be useful when we estimate the lower bound of (axis-direction) time constant in Section 4.1.

Lemma 2.2 ([10], Equation (6.20))**.** The number of paths of length k on $\mathbb{Z}^d$ from $\mathbf{0}$ to H_n is bounded above by

$$\#\{\gamma \in \Gamma(\mathbf{0}, H_n) \mid \#\gamma = k\} \leqslant (2d)^k \min\left\{1, \exp\left(-n\rho + \frac{k}{d}(\cosh\rho - 1)\right)\right\}$$

for any $\rho > 0$.

Proof. Note that there are exactly $(2d)^k$ paths of length k starting from $\mathbf{0}$, hence the number is bounded above by $(2d)^k$. Moreover, we can bound the number of paths of length k on $\mathbb{Z}^d$ from $\mathbf{0}$ to H_n by

$$\frac{\#\{\gamma \in \Gamma(\mathbf{0}, H_n) \mid \#\gamma = k\}}{(2d)^k} \leqslant \mathbb{P}(U_1 + \cdots + U_k = n),$$

where $\{U_i\}_{i=1}^k$ are independent random variables with distribution

$$\mathbb{P}(U_i = 1) = \mathbb{P}(U_i = -1) = \frac{1}{2d} \quad \text{and} \quad \mathbb{P}(U_i = 0) = 1 - \frac{1}{d}.$$

The collection $\{U_i\}_{i=1}^k$ records the change of the first coordinates of the vertices of the path (if the i-th increment of γ is $\pm\mathbf{e}_1$, then $U_i = \pm 1$; otherwise, the first coordinate remains unchanged and $U_i = 0$). Therefore, for any $\rho > 0$,

$$\begin{aligned}
\frac{\#\{\gamma \in \Gamma(\mathbf{0}, H_n) \mid \#\gamma = k\}}{(2d)^k} &\leqslant \mathbb{P}(U_1 + \cdots + U_k \geqslant n) \\
&\leqslant e^{-\rho n}\mathbb{E}\left[e^{\rho U_1}\right]^k \qquad (2.1)\\
&= e^{-\rho n}\left[1 + \frac{1}{d}(\cosh\rho - 1)\right]^k \\
&\leqslant \exp\left(-\rho n + \frac{k}{d}(\cosh\rho - 1)\right), \qquad (2.2)
\end{aligned}$$

where (2.1) holds by Markov's inequality and independence and (2.2) holds by the elementary inequality $\log(1 + x) \leqslant x$ for $x \geqslant 0$. □

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2.1 Self-avoiding paths

Definition 2.3 (Self-avoiding path)**.** A path $\gamma = (\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_k)$ is said to be **self-avoiding** if $\mathbf{s}_i \neq \mathbf{s}_j$ for all $i \neq j$. We denote the set of all self-avoiding paths of length $k \in \mathbb{N}$ starting from $\mathbf{0}$ on $\mathbb{Z}^d$ by $\Lambda_{d,k}$ and denote

$$\lambda_{d,k} = \#\Lambda_{d,k}.$$

For convenience, let $\Lambda_{d,0} = \{\varnothing\}$ and $\lambda_{d,0} = 1$.

In this section, we will prove the following bounds for $\lambda_{d,k}$:

$$[2d - 1 - \log(2d-1)]^k \leqslant \lambda_{d,k} \leqslant 2d(2d-1)^{k-1} \tag{2.3}$$

given by Hammersley in [8]. The upper bound is direct. The lower bound will be crucial when it comes to estimating the passage time afterwards. In particular, we can treat the passage time of a self-avoiding path as a sum of i.i.d. random variables.

Note that for any $n, m \in \mathbb{N}$, any path in $\Lambda_{d,n+m}$ can be decomposed into a path in $\Lambda_{d,n}$ concatenating a path in $\Lambda_{d,m}$. Therefore, we have $\lambda_{d,n+m} \leqslant \lambda_{d,n} \cdot \lambda_{d,m}$ for all $d, n, m \in \mathbb{N}$. That is, $\lambda_{d,k}$ is submultiplicative (in k) for all $d \in \mathbb{N}$. By Fekete's lemma, there is a constant $\xi_d \in [0, \infty)$ (called **connective constant**) such that

$$\xi_d = \lim_{k\to\infty} \lambda_{d,k}^{\frac{1}{k}} = \inf_{k\in\mathbb{N}} \lambda_{d,k}^{\frac{1}{k}}. \tag{2.4}$$

Observe that by the upper bound of $\lambda_{d,k}$ in (2.3), we have $\xi_d \leqslant 2d-1$ for all $d \in \mathbb{N}$. Also, since $\lambda_{d,k} \geqslant 1$, we have $\xi_d \geqslant 1$ for all $d \in \mathbb{N}$. By (2.4), we can bound $\lambda_{d,k}$ by deriving a lower bound for ξ_d. First, we introduce a special type of restricted self-avoiding paths.

Definition 2.4. Fix $1 \leqslant r \leqslant d-1$. For a self-avoiding path $\gamma \in \Lambda_{d,k}$, let

$$I_\gamma = \{1 \leqslant i \leqslant k \mid \Delta_i \in \{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_r\}\} \tag{2.5}$$

be the indices of increments in γ that lie in the first r coordinates (possibly empty). If

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$\#I_\gamma = m \in \mathbb{N}_0$, order the set I_γ by $I_\gamma = \{i_1 < i_2 < \cdots < i_m\}$. Define, for $1 \leqslant r \leqslant d-1$, the set $\Lambda_{d,k}^{(r)} \subseteq \Lambda_{d,k}$ to be the collection of self-avoiding paths of length $k \in \mathbb{N}$ starting from $\mathbf{0}$ on $\mathbb{Z}^d$ such that $\#I_\gamma = m$ and

$$\gamma_r = (\mathbf{0}, \Delta_{i_1}, \Delta_{i_1} + \Delta_{i_2}, \dots, \Delta_{i_1} + \cdots + \Delta_{i_m}) \in \Lambda_{r,m},$$

where we treat γ_r as a path in $\mathbb{Z}^r$. Define, for all $1 \leqslant r \leqslant d-1$,

$$\lambda_{d,k}^{(r)} = \#\Lambda_{d,k}^{(r)}.$$

For convenience, let $\Lambda_{d,0}^{(r)} = \{\varnothing\}$, $\lambda_{d,0}^{(r)} = 1$.

Define, for $d \in \mathbb{N}$, the generating function

$$\Phi_d(x) = \sum_{k=0}^{\infty} \lambda_{d,k} x^k = 1 + 2dx + 2d(2d-1)x^2 + \sum_{k=2}^{\infty} \lambda_{d,k} x^k,$$

which has radius of convergence ξ_d^{-1} by (2.4). Similarly, for $1 \leqslant r \leqslant d-1$, let

$$\Phi_d^{(r)}(x) = \sum_{k=0}^{\infty} \lambda_{d,k}^{(r)} x^k,$$

which has radius of convergence no smaller than ξ_d^{-1}, since we have the obvious bound $\lambda_{d,k}^{(r)} \leqslant \lambda_{d,k}$ for all $1 \leqslant r \leqslant d-1$. The following proposition establishes the connection between these generating functions.

Proposition 2.5 ([8], Theorem 11)**.** For $x \geqslant 0$ and $1 \leqslant r \leqslant d-1$, one has

$$\Phi_{d-r}(x)\Phi_r\big(x\Phi_{d-r}(x)\big) = \Phi_d^{(r)}(x) \leqslant \Phi_d(x). \tag{2.6}$$

Proof. The inequality is obvious since $\lambda_{d,k}^{(r)} \leqslant \lambda_{d,k}$ for all $1 \leqslant r \leqslant d-1$. For the equality, we claim that for all $d \in \mathbb{N}$, $1 \leqslant r \leqslant d-1$, one has

$$\lambda_{d,k}^{(r)} = \sum_{m=0}^{k} \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} \lambda_{r,m} \prod_{j=1}^{m+1} \lambda_{d-r, i_j - i_{j-1} - 1}, \tag{2.7}$$

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where we treat $i_0 = 1$ and $i_{m+1} = k+1$. Given a path $\gamma \in \Lambda_{d,k}^{(r)}$ and define $\Delta_{i_1}, \ldots, \Delta_{i_m}$ as in Definition 2.4, we have the decomposition

$$\gamma = \gamma_1 \cup \pi_1 \cup \gamma_2 \cup \pi_2 \cup \cdots \cup \gamma_m \cup \pi_m \cup \gamma_{m+1},$$

where π_j ($1 \leqslant j \leqslant m$) are paths of length 1 with direction Δ_{i_j} and γ_j ($1 \leqslant j \leqslant m+1$) are paths of length $i_j - i_{j-1} - 1$ within them (located in the last $d-r$ coordinates) which is possibly empty. We may then count $\Lambda_{d,k}^{(r)}$ by considering the path

$$\pi = \pi_{i_1} \cup \cdots \cup \pi_{i_m} \in \Lambda_{r,m}$$

and the $m+1$ paths $\gamma_{i_1} \in \Lambda_{d-r, i_j - i_{j-1} - 1}$ ($1 \leqslant j \leqslant m+1$). Such enumeration gives us exactly the right hand side of (2.7). By expanding the left hand side of (2.6) in the power series, we see that it coincides with the right hand side of (2.7) and therefore the identity holds as asserted. □

Proposition 2.6 ([8], Theorem 12)**.** For all $a, b \in \mathbb{N}$,

$$\xi_{a+b} \geqslant \xi_a + \xi_b.$$

Proof. Note that for all $|x| < (\xi_a + \xi_b)^{-1}$, by (2.6) with $d = a+b$, $s = a$,

$$\begin{aligned}
\Phi_{a+b}(x) &\geqslant \Phi_b(x)\Phi_a(x\Phi_b(x)) && (2.8)\\
&\geqslant \left(\sum_{k=0}^{\infty} \xi_b^k x^k\right)\left(\sum_{k=0}^{\infty} \xi_a^k (x\Phi_b(x))^k\right) && \text{(subadditivity (2.4))}\\
&= \frac{1}{1-\xi_b x} \cdot \frac{1}{1-\xi_a \Phi_b(x) x}\\
&\geqslant \frac{1}{1-\xi_b x} \cdot \frac{1}{1-\xi_a x \cdot \frac{1}{1-\xi_b x}} = \frac{1}{1-(\xi_a+\xi_b)x}. && (2.9)
\end{aligned}$$

The radius of convergence of the right hand side of (2.9) is $(\xi_a + \xi_b)^{-1}$, which must be greater than the one of the left hand side of (2.8), ξ_{a+b}^{-1}. Thus, $\xi_{a+b} \geqslant \xi_a + \xi_b$. □

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Proposition 2.7 ([8], Theorem 13)**.** The function Ψ_d defined by

$$\Psi_d(x) = \begin{cases} \dfrac{x}{\Phi_d\left(\frac{1}{x}\right)} & , x > \xi_d \\ 0 & , x \leqslant \xi_d \end{cases}$$

satisfies the functional inequality

$$\Psi_{a+b}(x) \leqslant \Psi_a(\Psi_b(x)) \quad \text{for all} \quad x \geqslant \xi_{a+b}, a, b \in \mathbb{N}. \tag{2.10}$$

Moreover, Ψ_d is differentiable on (ξ_d, ∞) with

$$0 < \Psi_d'(x) < 1 \quad \text{for all} \quad x > \xi_d. \tag{2.11}$$

Proof. (2.10) follows from (2.6) with $d = a + b$, $s = a$:

$$\Psi_{a+b}(x) = \frac{x}{\Phi_{a+b}\left(\frac{1}{x}\right)} \leqslant \frac{x}{\Phi_b\left(\frac{1}{x}\right)\Phi_a\left(\frac{1}{x}\Phi_b\left(\frac{1}{x}\right)\right)} = \Psi_a\left(\frac{x}{\Phi_b\left(\frac{1}{x}\right)}\right) = \Psi_a(\Psi_b(x))$$

for all $x \geqslant \xi_{a+b}$. Moreover, $\Psi_d(x)$ is differentiable on (ξ_d, ∞) since $\Phi_d(x)$ is analytic on $\left(0, \xi_d^{-1}\right)$. For the lower bound in (2.11), observe that $\Psi_d(x)$ is a strictly increasing function on $\left(0, \xi_d^{-1}\right)$ (since it has positive integer coefficients). Therefore, the function $\Psi_d(x)$ is also strictly increasing on (ξ_d, ∞) and we have $\Psi_d'(x) > 0$. For the upper bound, note that

$$\begin{aligned}
\Psi_d'(x) < 1 &\iff \frac{1 \cdot \Phi_d\left(\frac{1}{x}\right) - x \cdot \Phi_d'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\Phi_d\left(\frac{1}{x}\right)^2} < 1 \\
&\iff \Phi_d\left(\frac{1}{x}\right) + \frac{1}{x}\Phi_d'\left(\frac{1}{x}\right) < \Phi_d\left(\frac{1}{x}\right)^2 \\
&\iff \sum_{k=0}^{\infty} \lambda_{d,k}\left(\frac{1}{x}\right)^k + \frac{1}{x}\sum_{k=1}^{\infty} k\lambda_{d,k}\left(\frac{1}{x}\right)^{k-1} < \left(\sum_{k=0}^{\infty} \lambda_{d,k}\left(\frac{1}{x}\right)^k\right)^2 \\
&\iff \sum_{k=0}^{\infty}(k+1)\lambda_{d,k}\left(\frac{1}{x}\right)^k < \sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{k} \lambda_{d,\ell}\lambda_{d,k-\ell}\right)\left(\frac{1}{x}\right)^k
\end{aligned}$$

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and the conclusion follows from $\lambda_{d,k} \leqslant \lambda_{d,\ell}\lambda_{d,k-\ell}$ for all $0 \leqslant \ell \leqslant k$ and

$$\begin{aligned}(2+1)\lambda_{d,2} &= 3 \cdot 2d(2d-1) \\ &< 2d(2d-1) + (2d)^2 + 2d(2d-1) = \lambda_{d,0}\lambda_{d,2} + \lambda_{d,1}^2 + \lambda_{d,2}\lambda_{d,0}\end{aligned}$$

which gives the strict inequality (2.11) for $x > \xi_d$. □

Remark 2.8. We believe that the derivative of $\Psi_d(x)$ in [8] is incorrect and therefore we cannot deduce $\Psi_d'(x) < 1$ directly. However, the inequality (2.11) still holds by the argument presented above.

Theorem 2.9 ([8], Theorem 14)**.** For $a, d \in \mathbb{N}$,

$$\int_{\xi_a}^{\xi_{ad}} \frac{\mathrm{d}x}{x - \Psi_a(x)} \geqslant d - 1. \tag{2.12}$$

Proof. Observe that $\Phi_a(x) \geqslant (1 - \xi_a x)^{-1} \to \infty$ as $x \to (\xi_a^{-1})^-$ and therefore $\Psi_a(x)$ is continuous at $x = \xi_d$. Moreover, by integrating (2.11) from $x = \xi_a$ to $x = y$, we obtain

$$\Psi_a(y) - \Psi_a(\xi_a) \leqslant y - \xi_a$$

giving

$$y - \Psi_a(y) \geqslant \xi_a \geqslant 1.$$

Hence, the integrand in (2.12) is bounded, continuous and positive. Define, for $x \geqslant \xi_a$, the indefinite integral

$$I(x) = \int_{\xi_a}^{x} \frac{\mathrm{d}y}{y - \Psi_a(y)},$$

which is a strictly increasing function of x. Observe that $I(x) \to \infty$ as $x \to \infty$ and hence there exists an inverse function $\Xi = I^{-1} \colon [0, \infty) \to [\xi_a, \infty)$ such that

$$x = \int_{\xi_a}^{\Xi(x)} \frac{\mathrm{d}y}{y - \Psi_a(y)} \quad \text{for all} \quad x \in [0, \infty). \tag{2.13}$$

Since I is differentiable on (ξ_a, ∞), Ξ is differentiable on $(0, \infty)$. By differentiating (2.13),

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we derive the identity

$$\Xi'(x) = \Xi(x) - \Psi_a(\Xi(x)), \tag{2.14}$$

where the right hand side is differetiable on $(0, \infty)$ since $\Xi(x) > \xi_a$ for $x > 0$. Differentiating both sides of (2.14), we obtain

$$\Xi''(x) = \Xi'(x)(1 - \Psi_a'(\Xi(x))),$$

which is strictly positive by (2.11) and the fact that Ξ is strictly increasing. Therefore, for all $x \geqslant 1$, there is $x - 1 \leqslant \theta_x \leqslant x$ such that

$$\begin{aligned}\Xi(x-1) &= \Xi(x) - \Xi'(x) + \frac{\Xi''(\theta_x)}{2} \\ &> \Xi(x) - \Xi'(x) = \Psi_a(\Xi(x))\end{aligned} \tag{2.15}$$

by the mean value theorem and (2.14). Note that for $1 \leqslant k \leqslant d-1$, by (2.10) (with $a(d-k+1) = a(d-k) + a$) and (2.15) (with $x = d-k \geqslant 1$), we have

$$\Psi_{a(d-k+1)}(\Xi(d-k)) \leqslant \Psi_{a(d-k)}(\Psi_a(\Xi(d-k))) < \Psi_{a(d-k)}(\Xi(d-k-1)). \tag{2.16}$$

Hence,

$$\Psi_{ad}(\Xi(d-1)) \leqslant \Psi_a(\Xi(0)) = \Psi_a(\xi_a) = 0.$$

Therefore, we must have $\Xi(d-1) \leqslant \xi_{ad}$, which is exactly (2.12). □

Corollary 2.10. For all $d \in \mathbb{N}$,

(i) $2d - 1 - \log(2d-1) \leqslant \xi_d \leqslant 2d - 1$, and

(ii) $[2d - 1 - \log(2d-1)]^k \leqslant \lambda_{d,k} \leqslant 2d(2d-1)^{k-1}$.

Proof. For (i), we see when the dimension is 1 (that is, on $\mathbb{Z}$),

$$\xi_1 = \lim_{k\to\infty} \lambda_{1,k}^{\frac{1}{k}} = \lim_{k\to\infty} 2^{\frac{1}{k}} = 0 \quad \text{and} \quad \Phi_1(x) = \sum_{k=1}^{\infty} \lambda_{1,k} x^k = 1 + \sum_{k=1}^{\infty} 2x^k = \frac{1+x}{1-x}$$

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and therefore $\Psi_1(x) = \frac{x}{\Phi_1\left(\frac{1}{x}\right)} = \frac{x(x-1)}{x+1}$. Plugging these in (2.12) with $a = 1$, we have

$$\int_1^{\xi_d} \frac{\mathrm{d}x}{1 - \frac{x(x-1)}{x+1}} \geqslant d - 1 \iff \frac{\xi_d}{2} + \frac{1}{2}\log(\xi_d) - \frac{1}{2} \geqslant d - 1$$

and hence one may deduce by $\xi_d \leqslant 2d - 1$ that

$$\xi_d \geqslant 2d - 1 - \log(\xi_d) \geqslant 2d - 1 - \log(2d - 1)$$

as asserted. (ii) follows directly from (i) and subadditivity of $\lambda_{d,k}$. $\square$

2.2 Pairs of self-avoiding paths

In this section, we are going to estimate the number of pairs of self-avoiding paths. This result will be crucial when we are deriving an upper bound for $\mu_d(\mathbf{e}_1)$ in Section 4.2.

Definition 2.11. For $d, k, r \in \mathbb{N}$ ($r \leqslant d - 1$), define $\mathcal{P}_{d,k,r}$ to be the set of self-avoiding paths of length k from $\mathbf{0}$ to H_1 on $\mathbb{Z}^d$ whose

- first $k - 1$ steps use only directions $\pm\mathbf{e}_i$, $2 \leqslant i \leqslant r + 1$ and
- the last step is $\mathbf{e}_1$.

We may then define, for all $1 \leqslant \ell \leqslant k$ (the number of intersections),

$$\mathcal{P}_{d,k,r}^{(\ell)} = \{(\gamma, \gamma') \mid \gamma, \gamma' \in \mathcal{P}_{d,k,r}, \#(\gamma \cap \gamma') = \ell\} \tag{2.17}$$

The following estimate can be trace back to Eq (8.14) - (8.17) in [10].

Proposition 2.12 ([2], Proposition 3.3)**.** If $k, r \in \mathbb{N}$ ($r \leqslant d - 1$), $1 \leqslant \ell \leqslant k - 1$ satisfy $r = r(d) \sim d$ and $k = k(d) = O(\log d)$ as $d \to \infty$, then we have

$$\frac{\#\mathcal{P}_{d,k,r}^{(\ell)}}{(2r)^{2(k-1)}} \leqslant \left(\frac{1}{2r}\right)^{\ell} \left(1 + o\left(\frac{1}{\sqrt{r}}\right)\right) \quad \text{as} \quad d \to \infty.$$

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Proof. Since last step of all paths in $\mathcal{P}_{d,k,r}$ is $\mathbf{e}_1$, we may deduce that for all pairs $(\gamma,\gamma') \in \mathcal{P}^{(\ell)}_{d,k,r}$ where $\gamma = (\mathbf{s}_0,\ldots,\mathbf{s}_k)$ and $\gamma' = (\mathbf{s}'_0,\ldots,\mathbf{s}'_k)$, there are two possible intersection behaviors:

(i) $\mathbf{s}_{k-1} \neq \mathbf{s}'_{k-1}$ and all of the ℓ intersections occur in the first $k-1$ steps.

(ii) $\mathbf{s}_{k-1} = \mathbf{s}'_{k-1}$ and there are exactly $\ell - 1$ intersections in the first $k-1$ steps.

Therefore, by setting $\mathbb{P}$ to be the uniform measure on all pairs of paths of length $k-1$ starting from $\mathbf{0}$ on $\mathbb{Z}^r$, we may bound the desired quantity by

$$\frac{\mathcal{P}^{(\ell)}_{d,k,r}}{(2r)^{2(k-1)}} \leqslant \mathbb{P}\left(\gamma,\gamma' \in \Lambda_{r,k-1} \text{ and } \#(\gamma\cap\gamma') = \ell\right)$$
$$+\,\mathbb{P}\left(\gamma,\gamma' \in \Lambda_{d,k-1}, \mathbf{s}_{k-1} = \mathbf{s}'_{k-1} \text{ and } \#(\gamma\cap\gamma') = \ell - 1\right), \qquad (2.18)$$

where (γ,γ') is sampled from $\mathbb{P}$. To estimate (2.18), we introduce the following notations:

- On the event $\{\#(\gamma\cap\gamma') = \ell\}$, let $\mathfrak{C}$ be the number of pieces where the ℓ overlapping edges of γ and γ' are clustered in ($1 \leqslant \mathfrak{C} \leqslant \ell$).
- On the event $\{\mathfrak{C} = C\}$, we denote $m_1,\ldots,m_C$ (resp. $m'_1,\ldots,m'_C$) to be the indices of their starting points in γ (resp. γ').

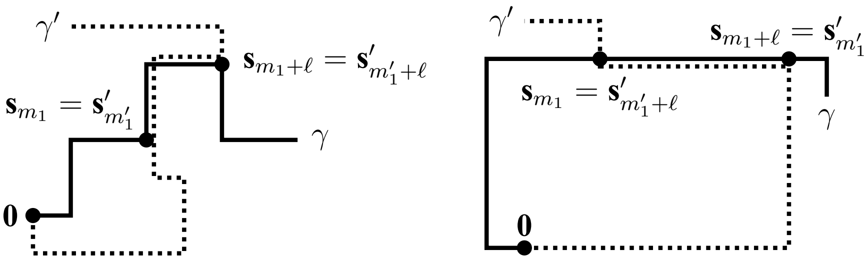

Figure 2.1: Pairs $(\gamma,\gamma') \in \mathcal{P}^{(\ell)}_{d,k,r}$ such that $\mathfrak{C} = 1$ (The left hand side is a positively oriented match and the right hand side is a negatively oriented match)

For the first term in (2.18), we see that if $\mathfrak{C} = 1$, there are two possible matching behaviors: one positively oriented (that is, $\mathbf{s}_{m_1+i} = \mathbf{s}'_{m'_1+i}$ for all $0 \leqslant i \leqslant \ell$) and one negatively oriented (that is, $\mathbf{s}_{m_1+i} = \mathbf{s}'_{m'_1+\ell-i}$ for all $0 \leqslant i \leqslant \ell$), see Figure 2.1. The

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probability of the occurrence of the positively oriented case is bounded by

$$
\begin{aligned}
&\sum_{m_1,m_1'=0}^{k-1-\ell} \mathbb{P}\left(\bigcap_{i=0}^{\ell-1}\left\{\mathbf{s}_{m_1+i}=\mathbf{s}'_{m_1'+i}\right\}\right) \\
&\qquad=\left(\frac{1}{2r}\right)^{\ell}\sum_{m_1,m_1'=0}^{k-1-\ell}\mathbb{P}\left(\mathbf{s}_{m_1}=\mathbf{s}'_{m_1'}\right) \\
&\qquad=\left(\frac{1}{2r}\right)^{\ell}\left[1+2\sum_{m_1=0}^{k-1-\ell}\mathbb{P}\left(\mathbf{S}_{m_1}=\mathbf{0}\right)+\sum_{m_1,m_1'=1}^{k-1-\ell}\mathbb{P}\left(\mathbf{S}_{m_1+m_1'}=\mathbf{0}\right)\right] \qquad (2.19)\\
&\qquad\leqslant\left(\frac{1}{2r}\right)^{\ell}\left(1+\frac{2k+k^2}{r}\right) \\
&\qquad=\left(\frac{1}{2r}\right)^{\ell}\left(1+o\left(\frac{1}{\sqrt{r}}\right)\right), \qquad (2.20)
\end{aligned}
$$

where $(\mathbf{S}_m)_{m\in\mathbb{N}_0}$ in (2.19) is a standard symmetric random walk on $\mathbb{Z}^r$ and (2.20) holds by the fact that $r\sim d$ and $k=O(\log d)$ as $d\to\infty$. For the negatively oriented case, by a similar calculation we have,

$$
\begin{aligned}
&\sum_{m_1,m_1'=0}^{k-1-\ell} \mathbb{P}\left(\bigcap_{i=0}^{\ell-1}\left\{\mathbf{s}_{m_1+i}=\mathbf{s}'_{m_1'+\ell-i}\right\}\right) \\
&\qquad=\left(\frac{1}{2r}\right)^{\ell}\sum_{m_1,m_1'=0}^{k-\ell-1}\mathbb{P}\left(\mathbf{s}_{m_1}=\mathbf{s}'_{m_1'+\ell}\right) \\
&\qquad=\left(\frac{1}{2r}\right)^{\ell}\left[2\sum_{m_1=0}^{k-1-\ell}\mathbb{P}\left(\mathbf{S}_{m_1}=\mathbf{0}\right)+\sum_{m_1,m_1'=1}^{k-1-\ell}\mathbb{P}\left(\mathbf{S}_{m_1+m_1'+\ell}=\mathbf{0}\right)\right] \qquad (2.21)\\
&\qquad\leqslant\left(\frac{1}{2r}\right)^{\ell}\left(\frac{2k+k^2}{r}\right) \\
&\qquad=\left(\frac{1}{2r}\right)^{\ell}o\left(\frac{1}{\sqrt{r}}\right),
\end{aligned}
$$

where (2.21) holds since in the negatively oriented case, it is not possible that $m_1=m_1'=0$. Hence, we have now derived for the first term in (2.18),

$$
\mathbb{P}(\gamma,\gamma\in\Lambda_{r,k-1},\#(\gamma\cap\gamma')=\ell\text{ and }\mathfrak{C}=1)\leqslant\left(\frac{1}{2r}\right)^{\ell}\left(1+o\left(\frac{1}{\sqrt{r}}\right)\right). \qquad (2.22)
$$

For the case $2\leqslant\mathfrak{C}\leqslant\ell$, we count first the total number of ways to matching $\mathfrak{C}$

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segments in γ with $\mathfrak{C}$ segments in γ'.

1° First, we divide the ℓ overlapping edges into $\mathfrak{C}$ (nonempty) clusters, which gives a combinatorial factor

$$\binom{(\ell-\mathfrak{C})+(\mathfrak{C}-1)}{\mathfrak{C}-1} = \binom{\ell-1}{\mathfrak{C}-1} \leqslant \ell^{\mathfrak{C}-1}.$$

2° Next, we determine the positions of each clusters on γ and γ'. Note that there are at most $\binom{k-1}{\mathfrak{C}}$ ways to choose the starting points $m_1 < \cdots < m_{\mathfrak{C}}$ of these segments, and once the starting points are chosen, there are at most $\mathfrak{C}!$ ways to associate a running length chosen in step 1°. The same idea works for γ'.

3° Last, there are at most two possible ways to pair each of these clusters: positively or negatively oriented (as demonstrated in the $\mathfrak{C} = 1$ case). Therefore, there is an additional $2^{\mathfrak{C}}$ factor in the counting of matching segments.

By our observations above, we see that the total number of ways to match $\mathfrak{C}$ segments in γ with $\mathfrak{C}$ segments in γ' is bounded above by

$$\ell^{\mathfrak{C}-1} \cdot \left[\binom{k-1}{\mathfrak{C}} \cdot \mathfrak{C}!\right]^2 \cdot 2^{\mathfrak{C}} \leqslant \frac{(2\ell k^2)^{\mathfrak{C}}}{\ell}. \tag{2.23}$$

Now, we will derive an upper bound for the probability of the occurrence of each matching pattern discussed above. Fix $2 \leqslant \mathfrak{C} \leqslant \ell$ and a matching pattern. That is, we fix the cluster lengths $n_1, \ldots, n_{\mathfrak{C}}$ (enumerated by the appearance in γ starting from $\mathbf{0}$ for convenience and set $n_0 = 0$), the starting points of the clusters $m_1, \ldots, m_{\mathfrak{C}}$, $m_1', \cdots, m_{\mathfrak{C}}'$ (set $m_0 = 0$) and the orientation of each matching. For all $\mathbf{x}_1, \ldots, \mathbf{x}_\ell \in \{\pm\mathbf{e}_i\}_{i=1}^r$ and $\mathbf{y}_1, \ldots, \mathbf{y}_{\mathfrak{C}} \in \mathbb{Z}^r$, let $\overline{\gamma} = (\bar{\mathbf{s}}_0, \ldots, \bar{\mathbf{s}}_{k-1})$ be a fixed path of length $k-1$ starting from $\mathbf{0}$ on $\mathbb{Z}^r$ such that

- for all $1 \leqslant i \leqslant \mathfrak{C}$, $\bar{\mathbf{s}}_{m_i+j} - \bar{\mathbf{s}}_{m_i+j-1} = \mathbf{x}_{n_0+\cdots+n_{i-1}+j}$ for each $1 \leqslant j \leqslant n_i$ and
- $\bar{\mathbf{s}}_{m_i} - \bar{\mathbf{s}}_{m_{i-1}+n_{i-1}} = \mathbf{y}_i$.

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That is, we fix the intersecting paths and the increments between them. Let $n'_1, \ldots, n'_{\mathfrak{C}}$ be the cluster lengths enumerated by the appearance in γ' starting from $\mathbf{0}$ (hence, a permutation of $n_1, \ldots, n_{\mathfrak{C}}$) and $n'_0 = m'_0 = 0$. On the event $\gamma = \overline{\gamma}$, we see that there is at most one admissible $\mathbf{x}'_1, \ldots, \mathbf{x}'_\ell \in \{\mathbf{e}_i\}_{i=1}^r$ (a permutation of $\mathbf{x}_1, \ldots, \mathbf{x}_\ell$) and $\mathbf{y}'_1, \ldots, \mathbf{y}'_{\mathfrak{C}} \in \mathbb{Z}^r$ (a permutation of $\mathbf{y}_1, \ldots, \mathbf{y}_{\mathfrak{C}}$) such that for all $1 \leqslant i \leqslant \mathfrak{C}$,

- $\mathbf{s}'_{m'_i+j} - \mathbf{s}'_{m'_i+j-1} = \mathbf{x}'_{n'_0+\cdots+n'_{i-1}+j}$ for each $1 \leqslant j \leqslant n'_i$,
- $\mathbf{s}'_{m'_i} - \mathbf{s}_{m'_{i-1}+n'_{i-1}} = \mathbf{y}'_i$ and
- γ' intersects γ in the given pattern.

We may then derive

$$\begin{aligned}
&\mathbb{P}(\gamma' \text{ intersects } \gamma \text{ in the given pattern} \mid \gamma = \overline{\gamma}) \\
&\leqslant \left(\prod_{i=1}^{\mathfrak{C}} \prod_{j=1}^{n'_i} \mathbb{P}\left(\mathbf{s}'_{m'_i+j} - \mathbf{s}'_{m'_i+j-1} = \mathbf{x}'_{n'_0+\cdots+n'_{i-1}+j} \right) \right) \\
&\qquad\qquad \cdot \left(\prod_{i=1}^{\mathfrak{C}} \mathbb{P}\left(\mathbf{s}'_{m'_i} - \mathbf{s}_{m'_{i-1}+n'_{i-1}} = \mathbf{y}'_i \right) \right) \\
&\leqslant \left(\frac{1}{2r} \right)^{\ell} \cdot \left(\frac{1}{2r} \right)^{\mathfrak{C}-1},
\end{aligned} \tag{2.24}$$

where the second factor is $\mathfrak{C} - 1$ but not $\mathfrak{C}$ since there might only be $\mathfrak{C} - 1$ increments within these $\mathfrak{C}$ intersections. Combining (2.23) and (2.24), we then see

$$\begin{aligned}
&\sum_{C=2}^{\ell} \mathbb{P}\left(\gamma, \gamma \in \Lambda_{r,k-1}, \#(\gamma \cap \gamma') = \ell \text{ and } \mathfrak{C} = C \right) \\
&\leqslant \sum_{C=2}^{\ell} \frac{(2\ell k^2)^C}{\ell} \cdot \left(\frac{1}{2r} \right)^{\ell} \left(\frac{1}{2r} \right)^{C-1} \\
&= \left(\frac{1}{2r} \right)^{\ell} \frac{4\ell^2 k^4}{2r\ell} \sum_{C=2}^{\ell} \left(\frac{2\ell k^2}{2r} \right)^{C-2} \\
&\leqslant \left(\frac{1}{2r} \right)^{\ell} \frac{2\ell k^4}{r} \frac{1}{1 - \frac{\ell k^2}{r}} \\
&= \left(\frac{1}{2r} \right)^{\ell} o\left(\frac{1}{\sqrt{r}} \right).
\end{aligned} \tag{2.25}$$

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With (2.22) and (2.25), we may then bound the first term in (2.18) by

$$\mathbb{P}\left(\gamma, \gamma' \in \Lambda_{r,k-1} \text{ and } \#(\gamma \cap \gamma') = \ell\right) \leqslant \left(\frac{1}{2r}\right)^{\ell} \left(1 + o\left(\frac{1}{\sqrt{r}}\right)\right).$$

The second term in (2.18) can be controlled by a similar argument, giving

$$\mathbb{P}\left(\gamma, \gamma' \in \Lambda_{d,k-1}, \mathbf{s}_{k-1} = \mathbf{s}'_{k-1} \text{ and } \#(\gamma \cap \gamma') = \ell - 1\right) = \left(\frac{1}{2r}\right)^{\ell} o\left(\frac{1}{\sqrt{r}}\right)$$

More details can be found in Lemma 3.4 of [2]. □

2.3 Diagonal paths

In this section, we will consider the following type of paths.

Definition 2.13 (Diagonal Paths). For $d, n \in \mathbb{N}$, $\nu \geqslant 0$, we define

$$J_n^{(\nu)} = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d = \lfloor nd^{\nu} \rfloor\right\}.$$

For $k \in \mathbb{N}$, let $\mathcal{D}_{d,k,n}^{(\nu)}$ be the set of self-avoiding paths of length k starting from $\mathbf{0}$ on $\mathbb{Z}^d$ and arrive $J_n^{(\nu)}$ for the first time at the k-th edge.

We will estimate the number of these diagonal paths on $\mathbb{Z}^d$ by treating them as simple symmetric random walk on $\mathbb{Z}$ (see Appendix A). The following proposition (which is essentially a modification of Lemma 3 of [3]) will be used later in Section 4.6 to bound the passage time of diagonal direction.

Proposition 2.14. For all $\nu \geqslant 0$ and $k = \alpha \lfloor nd^{\nu} \rfloor \in \mathbb{N}$ for some $\alpha \geqslant 1$,

$$\#\mathcal{D}_{d,k,n}^{(\nu)} \leqslant \left(\frac{2d\alpha}{(\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}}\right)^{k},$$

where we set the right hand side to be $d^{-\lfloor nd^{\nu} \rfloor}$ when $\alpha = 1$ so that it is continuous at $\alpha = 1$.

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Proof. Let $(\mathbf{S}_m)_{m\in\mathbb{N}_0}$ in (2.19) be the simple symmetric random walk on $\mathbb{Z}^d$ with $\mathbf{S}_0 = \mathbf{0}$. Then

$$\frac{\#\mathcal{D}^{(\nu)}_{d,k,n}}{(2d)^k} = \mathbb{P}\left((\mathbf{S}_m)_{m\in\mathbb{N}_0} \text{ hits } J^{(\nu)}_n \text{ for the first time at } m = k\right).$$

Observe the orthogonal projection $(\mathbf{T}_m)_{m\in\mathbb{N}_0}$ of $(\mathbf{S}_m)_{m\in\mathbb{N}_0}$ (see Figure 2.2) on the diagonal line $L_{\text{diag}} = \{t\mathbf{1} \mid t \in \mathbb{R}\}$ is a one-dimensional symmetric random walk with increment $d^{-\frac{1}{2}}$ and therefore

$$\frac{\#\mathcal{D}^{(\nu)}_{d,k,n}}{(2d)^k} = \mathbb{P}\left((\mathbf{T}_m)_{m\in\mathbb{N}_0} \text{ hits } \lfloor nd^{\nu}\rfloor \text{ for the first time at } m = k\right).$$

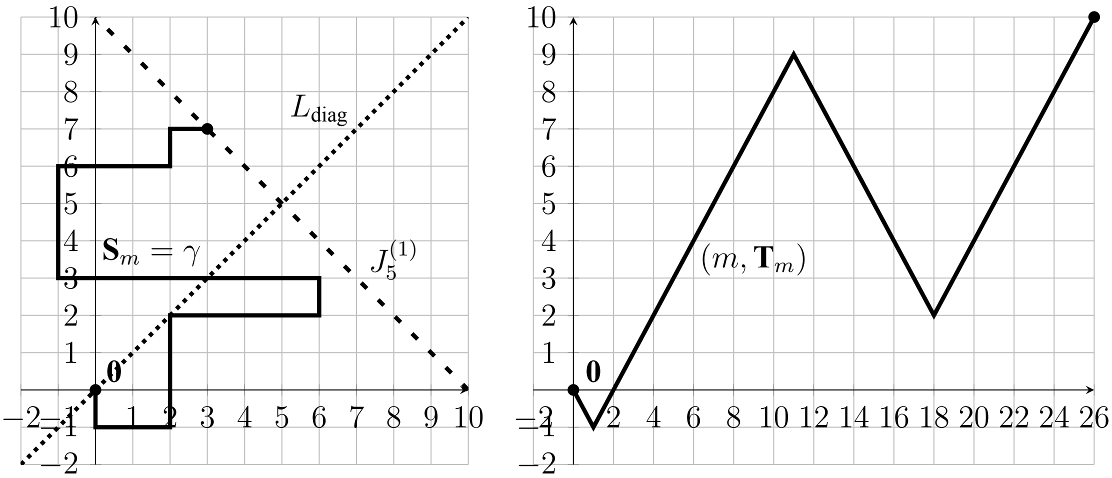

Figure 2.2: $\gamma \in D^{(1)}_{2,26,5}$ and the corresponding one-dimensional random walk

Obviously, if $\alpha = 1$, we have

$$\frac{\#\mathcal{D}^{(\nu)}_{d,k,n}}{(2d)^k} = \frac{1}{2^k}, \quad \text{giving} \quad \#\mathcal{D}^{(\nu)}_{d,k,n} = d^k = d^{\lfloor nd^{\nu}\rfloor},$$

since the only possibility is $\mathbf{T}_m = md^{-\frac{1}{2}}$ for each $0 \leqslant m \leqslant \lfloor nd^{\nu}\rfloor$. For $\alpha > 1$, by the distribution of first hitting time (see Lemma A.3)

$$(2d)^{-k}\#\mathcal{D}^{(\nu)}_{d,k,n} = \frac{k}{\lfloor nd^{\nu}\rfloor}\binom{k}{\frac{k+\lfloor nd^{\nu}\rfloor}{2}}\frac{1}{2^k} = \frac{k}{\lfloor nd^{\nu}\rfloor}\cdot\frac{k!}{\left(\frac{k+\lfloor nd^{\nu}\rfloor}{2}\right)!\left(\frac{k-\lfloor nd^{\nu}\rfloor}{2}\right)!}\cdot\frac{1}{2^k}.$$

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Applying the non-asymptotic version of Stirling's approximation

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leqslant n! \leqslant e^{\frac{1}{12}}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

we may derive

$$\begin{aligned}
&(2d)^{-k}\#\mathcal{D}_{d,k,n}^{(\nu)} \\
&\quad\leqslant \frac{k}{\lfloor nd^\nu\rfloor}\cdot\frac{e^{\frac{1}{12}}\cdot\sqrt{2\pi k}\left(\frac{k}{e}\right)^k}{\sqrt{2\pi\cdot\frac{k+\lfloor nd^\nu\rfloor}{2}}\left(\frac{k+\lfloor nd^\nu\rfloor}{2e}\right)^{\frac{k+\lfloor nd^\nu\rfloor}{2}}\cdot\sqrt{2\pi\cdot\frac{k-\lfloor nd^\nu\rfloor}{2}}\left(\frac{k-\lfloor nd^\nu\rfloor}{2e}\right)^{\frac{k-\lfloor nd^\nu\rfloor}{2}}}\cdot\frac{1}{2^k} \\
&\quad= \sqrt{\frac{2e^{\frac{1}{6}}}{\pi\left(\alpha^2-1\right)k}}\left(\frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}}\right)^k \\
&\quad\leqslant \left(\frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}}\right)^k,
\end{aligned}$$

where the last inequality holds by $e^{\frac{1}{6}} \leqslant \pi$ and

$$\left(\alpha^2-1\right)k \geqslant \left[\left(\frac{\lfloor nd^\nu\rfloor+1}{\lfloor nd^\nu\rfloor}\right)^2-1\right]\left(\lfloor nd^\nu\rfloor+1\right) \geqslant 2.$$

Hence, the proposition holds as asserted. □

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Chapter 3

Preliminaries: first-passage percolation

In this chapter, we will introduce some known results in first-passage percolation. Most of these classical results can be found in [1, 4, 9, 10].

3.1 Time constant and its properties

In this section, we will prove Theorem 1.1 and introduce some results and extensions of the time constant μ_d.

Proof. (of Theorem 1.1) We apply Kingman's subadditive ergodic theorem (see Theorem B.1). We need to show the random sequence $X_{m,n} = \tau(m\mathbf{e}_1, n\mathbf{e}_1)$ satisfies conditions (K1) - (K4).

- Note that for all paths $\gamma_1 \in \Gamma(\mathbf{0}, m\mathbf{e}_1)$ and $\gamma_2 \in \Gamma(m\mathbf{e}_1, n\mathbf{e}_1)$, their concatenation $\gamma = \gamma_1 \cup \gamma_2 \in \Gamma(\mathbf{0}, n\mathbf{e}_1)$. Therefore,

$$X_{0,n} = \inf_{\gamma \in \Gamma(\mathbf{0}, n\mathbf{e}_1)} \tau(\gamma) \leqslant \inf_{\substack{\gamma_1 \in \Gamma(\mathbf{0}, m\mathbf{e}_1) \\ \gamma_2 \in \Gamma(m\mathbf{e}_1, n\mathbf{e}_1)}} \tau(\gamma_1 \cup \gamma_2)$$
$$= \inf_{\gamma_1 \in \Gamma(\mathbf{0}, m\mathbf{e}_1)} \tau(\gamma_1) + \inf_{\gamma_2 \in \Gamma(m\mathbf{e}_1, n\mathbf{e}_1)} \tau(\gamma_2) = X_{0,m} + X_{m,n}$$

  which shows subadditivity (K1).

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- The conditions (K2), (K3) and ergodicity follows directly from that $(\tau_e)_{e\in\mathcal{E}^d}$ is i.i.d..
- Since $X_{m,n} \geqslant 0$, we have $\mathbb{E}[X_{0,n}] > -cn$ for $c = 1$. The only remaining condition to show is $\mathbb{E}[X_{0,1}] < \infty$. Build $2d$ disjoint self-avoiding paths $\{\gamma_i\}_{i=1}^{2d}$ from $\mathbf{0}$ to $\mathbf{e}_1$ in $\mathbb{Z}^d$ (see Figure 3.1). If we order these paths so that γ_1 is the longest path with length L, then for all $t \geqslant 0$,

$$\begin{aligned}
\mathbb{P}(X_{0,1} > t) &\leqslant \prod_{i=1}^{2d} \mathbb{P}(\tau(\gamma_i) > t) && \text{(independence)} \\
&\leqslant \mathbb{P}(\tau(\gamma_1) > t)^{2d} \\
&= \mathbb{P}\left(\bigcup_{e\in\gamma_1} \left\{\tau_e > \frac{t}{L}\right\}\right)^{2d} && \left(\tau(\gamma_1) = \textstyle\sum_{e\in\gamma_1} \tau_e\right) \\
&\leqslant \left[L\mathbb{P}\left(\tau_e > \frac{t}{L}\right)\right]^{2d} = L^{2d}\mathbb{P}\left(\min\{\tau_1, \dots, \tau_{2d}\} > \frac{t}{L}\right) && (3.1)
\end{aligned}$$

and $\mathbb{E}[X_{0,1}] < \infty$ holds by integrating both sides of (3.1) together with the moment assumption (♣).

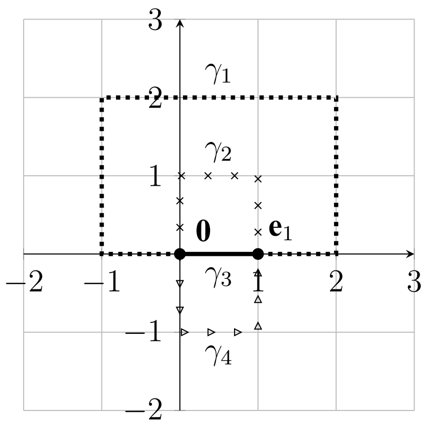

Figure 3.1: $2d$ paths from $\mathbf{0}$ to $\mathbf{e}_1$ $(d = 2)$

Since the conditions (K1) - (K4) of Theorem B.1 are satisfied, we may deduce by (B.1) and (B.2) that there exists a constant $\mu_d(\mathbf{e}_1)$ such that

$$\lim_{n\to\infty} \frac{\tau(\mathbf{0}, n\mathbf{e}_1)}{n} = \mu_d(\mathbf{e}_1) = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}[\tau(\mathbf{0}, n\mathbf{e}_1)]}{n} \quad \text{almost surely and in } \mathcal{L}^1.$$

as asserted. □

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Remark 3.1.

1. One may apply the same argument in the proof above and show that for all $k \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^d$, we have

$$\mathbb{E}\left[\{\tau_1, \ldots, \tau_{2d}\}^k\right] < \infty \iff \mathbb{E}\left[\tau(\mathbf{0}, \mathbf{x})^k\right] < \infty.$$

2. Note that having finite mean $\mathbb{E}[\tau_e] < \infty$ (as in condition (♠)) is a stronger assumption in comparison to (♣).

In fact, one may extend the argument given in Theorem 1.1 to show that under the assumption (♣), we have for all $\mathbf{z} \in \mathbb{Z}^d$ a constant $\mu_d(\mathbf{z})$ such that

$$\lim_{n\to\infty} \frac{\tau(\mathbf{0}, n\mathbf{z})}{n} = \mu_d(\mathbf{z}) = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}[\tau(\mathbf{0}, n\mathbf{z})]}{n} \quad \text{almost surely and in } \mathcal{L}^1.$$

Now, we extend the function $\mu_d \colon \mathbb{Z}^d \to [0, \infty)$ to $\mathbb{Q}^d$. Let $\mathbf{q} \in \mathbb{Q}^d$ and $M \in \mathbb{N}$ be such that $M\mathbf{q} \in \mathbb{Z}^d$, then we define

$$\mu_d(\mathbf{q}) = \lim_{n\to\infty} \frac{\tau(\mathbf{0}, nM\mathbf{q})}{nM} = \frac{1}{M} \lim_{n\to\infty} \frac{\tau(\mathbf{0}, n(M\mathbf{q}))}{n} = \frac{\mu_d(M\mathbf{q})}{M},$$

which is independent of the choice of M, since one clearly sees $\mu_d(k\mathbf{z}) = |k|\mu_d(\mathbf{z})$ if $k \in \mathbb{Z}$ and $\mathbf{z} \in \mathbb{Z}^d$. Note that (1.2) also holds if we replace $\mathbf{e}_1$ by $\mathbf{q} \in \mathbb{Q}^d$. Indeed, let $M \in \mathbb{N}$ be as before. For $nM \leqslant k < (n+1)M$, we see that

$$\mathbb{E}[|\tau(\mathbf{0}, k\mathbf{q}) - \tau(\mathbf{0}, nM\mathbf{q})|] \leqslant \mathbb{E}[\tau(nM\mathbf{q}, k\mathbf{q})] \leqslant \|(k - nM)\mathbf{q}\|_1 \mathbb{E}[\tau(\mathbf{0}, \mathbf{e}_1)].$$

We may then derive for $0 \leqslant i \leqslant M - 1$ that

$$\begin{aligned}\sum_{n=0}^{\infty} &\mathbb{P}\left(|\tau(\mathbf{0}, (nM+i)\mathbf{q}) - \tau(\mathbf{0}, nM\mathbf{q})| \geqslant n\epsilon\right) \\ &\leqslant \frac{\mathbb{E}\left[|\tau(\mathbf{0}, (nM+i)\mathbf{q}) - \tau(\mathbf{0}, nM\mathbf{q})|\right]}{\epsilon} \leqslant \frac{\|i\mathbf{q}\|_1 \mathbb{E}[\tau(\mathbf{0}, \mathbf{e}_1)]}{\epsilon} < \infty.\end{aligned}$$

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Hence, by the (first) Borel-Cantelli lemma, we have

$$\lim_{n\to\infty} \frac{\tau(\mathbf{0}, n\mathbf{q})}{n} = \lim_{n\to\infty} \frac{\tau(\mathbf{0}, nM\mathbf{q})}{nM} = \mu_d(\mathbf{q})$$

as asserted. One may verify from the definition of the generalized time constant $\mu_d \colon \mathbb{Q}^d \to [0,\infty)$ that for all $r \in \mathbb{Q}$, $\mathbf{q}, \mathbf{p} \in \mathbb{Q}^d$,

(i) $\mu_d(\mathbf{q}+\mathbf{p}) \leqslant \mu_d(\mathbf{q}) + \mu_d(\mathbf{p})$.

(ii) $\mu_d(r\mathbf{q}) = |r|\mu_d(\mathbf{q})$.

(iii) μ_d is invariant under symmetries of $\mathbb{Z}^d$ that fix the origin.

(iv) μ_d is Lipschitz on $\mathbb{Q}^d$. Indeed, by subadditivity,

$$|\mu_d(\mathbf{q}) - \mu_d(\mathbf{p})| \leqslant \mu_d(\mathbf{q}-\mathbf{p}) \leqslant \sum_{i=1}^{d} \mu_d((q_i - p_i)\mathbf{e}_i) = \|\mathbf{q}-\mathbf{p}\|_1 \mu_d(\mathbf{e}_1).$$

Therefore, μ_d admits a unique continuous extension to $\mathbb{R}^d$.

We have now the time constant $\mu_d \colon \mathbb{R}^d \to [0,\infty)$ where the properties above can be extended to real arguments, meaning that μ_d is a semi-norm on $\mathbb{R}^d$. In fact, by Theorem 1.2 and the following proposition, it is a norm if and only if $\mathbb{P}(\tau_e = 0) < p_c(d)$.

Proposition 3.2 ([10], Equation (3.15))**.** If $\mu_d(\mathbf{x}) = 0$ for some $\mathbf{x} \neq \mathbf{0}$, then $\mu_d(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathbb{R}^d$.

Proof. Without loss of generality, assume $\mu_d(\mathbf{x}) = 0$ for some $\mathbf{x} \neq \mathbf{0}$ with $x_1 \neq 0$. We have by symmetry $\mu_d(\mathbf{x}') = 0$, where $\mathbf{x}' = -\mathbf{x} + 2x_1\mathbf{e}_1$ is the reflection of $\mathbf{x}$ with respect to the first coordinate axis $L_1 = \{t\mathbf{e}_1 \mid t \in \mathbb{R}\}$. Then,

$$2|x_1|\mu_d(\mathbf{e}_1) = \mu_d(2x_1\mathbf{e}_1) = \mu_d(\mathbf{x}+\mathbf{x}') \leqslant \mu_d(\mathbf{x}) + \mu_d(\mathbf{x}') = 0$$

and therefore $\mu_d(\mathbf{e}_1) = 0$. By symmetry, $\mu_d(\mathbf{e}_j) = 0$ for all $1 \leqslant j \leqslant d$ and hence $\mu_d(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathbb{R}^d$ by subadditivity. $\square$

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Now, we consider some other types of passage times which are mentioned in [9] and [10]. Some of these will be used in the argument later.

Definition 3.3 (Other passage times)**.** For all $m < n$, we define:

$$a_{m,n} = \inf\{\tau(\gamma) \mid \gamma \in \Gamma(m\mathbf{e}_1, n\mathbf{e}_2)\} \quad \text{and} \quad b_{m,n} = \inf\{\tau(\gamma) \mid \gamma \in \Gamma(m\mathbf{e}_1, H_n)\}.$$

For $\mathbf{x} \in H_m$, $\mathbf{y} \in H_n$, we define the **cylinder path**

$$\Gamma_{\text{cyl}}(\mathbf{x}, \mathbf{y}) = \{\gamma \in \Gamma(\mathbf{x}, \mathbf{y}), \gamma \subseteq [m, n-1) \times \mathbb{R}^{d-1} \text{ except for its final point}\}$$

and $\Gamma_{\text{cyl}}(\mathbf{x}, H_n) = \bigcup_{\mathbf{y} \in H_n} \Gamma_{\text{cyl}}(\mathbf{x}, \mathbf{y})$. The **cylinder passage times** is defined by

$$t_{m,n} = \inf\{\tau(\gamma) \mid \gamma \in \Gamma_{\text{cyl}}(m\mathbf{e}_1, n\mathbf{e}_1)\} \quad \text{and} \quad s_{m,n} = \inf\{\tau(\gamma) \mid \gamma \in \Gamma_{\text{cyl}}(m\mathbf{e}_1, H_n)\}.$$

We have the obvious inequalities

$$b_{m,n} \leqslant a_{m,n} \leqslant t_{m,n} \quad \text{and} \quad b_{m,n} \leqslant s_{m,n} \leqslant t_{m,n} \tag{3.2}$$

for all $m < n$. Note that $a_{m,n}$ and $s_{m,n}$ are not comparable.

Theorem 3.4 ([9], Theorem 4.3.7)**.** If τ_e satisfies $\mathbb{E}[\tau_e] < \infty$, then

$$\lim_{n \to \infty} \frac{t_{0,n}}{n} = \mu_d(\mathbf{e}_1).$$

Proof. Similar to the idea in the proof of Theorem 1.1, we show that the random sequence $X_{m,n} = t_{m,n}$ satisfies conditions (K1) - (K4). The subadditivity condition (K1), stationary conditions (K2), (K3) and ergodicity follows directly from definition. We also have $\mathbb{E}[X_{0,n}] > -cn$ for $c = 1$. Last, $\mathbb{E}[X_{0,1}] \leqslant \mathbb{E}[\tau_e] < \infty$ gives the condition (K4). Hence, we may deduce from Theorem B.1 that there exists some constant ν_d such that $\frac{t_{0,n}}{n} \to \nu_d$ almost surely and in $\mathcal{L}^1$. It remains to show $\nu_d = \mu_d(\mathbf{e}_1)$.

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By (3.2), it suffices to show $\nu_d \leqslant \mu_d(\mathbf{e}_1)$. Define $q^k_{m,n}$ ($m < n$ and $k \in \mathbb{N}$) to be the first-passage time from $m\mathbf{e}_1$ to $n\mathbf{e}_1$, over paths $\gamma \subseteq [m-k, n+k) \times \mathbb{R}^{d-1}$ except for its final point. One can also apply Theorem B.1 (by checking (K1) - (K4) immediately) and deduce that there exists a constant ν^k_d such that

$$\lim_{n\to\infty} \frac{q^k_{0,n}}{n} = \nu^k_d = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}\left[q^k_{0,n}\right]}{n} \quad \text{almost surely and in } \mathcal{L}^1. \tag{3.3}$$

By construction, one sees that for $k \geqslant 2$, $a_{0,n} \leqslant q^k_{0,n} \leqslant q^{k-1}_{0,n} \leqslant t_{0,n}$. Therefore, we have

$$\mu_d(\mathbf{e}_1) \leqslant \nu^k_d \leqslant \mu^{k-1}_d \leqslant \nu_d \quad \text{for all} \quad k \geqslant 2. \tag{3.4}$$

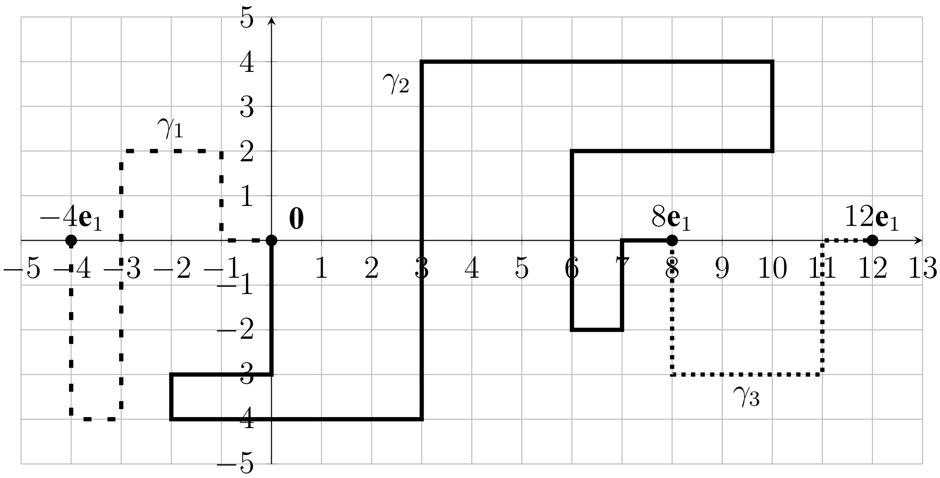

Figure 3.2: A depiction of $\gamma_1 \cup \gamma_2 \cup \gamma_3$ where $k = 4$, $n = 8$

Observe that for all paths $\gamma_1 \in \Gamma_{\text{cyl}}(-k\mathbf{e}_1, \mathbf{0})$, $\gamma_2 \in \Gamma(\mathbf{0}, n\mathbf{e}_1)$ with the property $\gamma_2 \subseteq [-k, n+k) \times \mathbb{R}^{d-1}$ except for its final point and $\gamma_3 \in \Gamma_{\text{cyl}}(n\mathbf{e}_1, (n+k)\mathbf{e}_1)$, we must have $\gamma_1 \cup \gamma_2 \cup \gamma_3 \in \Gamma_{\text{cyl}}(-k\mathbf{e}_1, (n+k)\mathbf{e}_1)$ (see Figure 3.2). Therefore, we have

$$\begin{aligned} t_{-k,n+k} &\leqslant \inf_{\gamma_1,\gamma_2,\gamma_3} \tau(\gamma_1 \cup \gamma_2 \cup \gamma_3) \\ &= \inf_{\gamma_1} \tau(\gamma_1) + \inf_{\gamma_2} \tau(\gamma_2) + \inf_{\gamma_3} \tau(\gamma_3) = t_{-k,0} + q^k_{0,n} + t_{n,n+k}. \end{aligned}$$

Taking expectation on both sides, by stationarity we have that

$$\mathbb{E}[t_{0,n+2k}] \leqslant 2\mathbb{E}[t_{0,k}] + \mathbb{E}\left[q^k_{0,n}\right]. \tag{3.5}$$

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Dividing (3.5) by n and taking $n \to \infty$, we may then deduce then $\nu_d \leqslant \nu_d^k$. Combining (3.4), we derive

$$\nu_d = \nu_d^k \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.6}$$

Observe that $q_{0,n}^k \nearrow a_{0,n}$ as $k \to \infty$ and hence by the monotone convergence theorem, $\mathbb{E}\left[q_{0,n}^k\right] \nearrow \mathbb{E}[a_{0,n}]$. Therefore, by (3.3) and (3.6), we have

$$\mu_d(\mathbf{e}_1) = \lim_{n\to\infty} \frac{\mathbb{E}[a_{0,n}]}{n} = \lim_{n\to\infty}\left(\lim_{k\to\infty} \frac{\mathbb{E}\left[q_{0,n}^k\right]}{n}\right) \geqslant \lim_{n\to\infty} \nu_d^k = \nu_d^k = \nu_d$$

and our conclusion holds by (3.4). □

Remark 3.5. In fact, $\frac{b_{0,n}}{n}$ and $\frac{s_{0,n}}{n}$ also converge to $\mu_d(\mathbf{e}_1)$. However, we cannot apply Theorem B.1 directly since these sequences do not satisfy the subadditivity condition (K1). We will postpone the proof (of Corollary 3.9) after we introduce the shape theorem.

3.2 Cox-Durret shape theorem

Before proving Theorem 1.3, we need a lemma to estimate the difference of passage times between different directions. The following lemma essentially comes from Lemma 3.6 of [10].

Lemma 3.6 ([1], Lemma 2.22)**.** If the edge weight τ_e satisfies (S1), then there is $\kappa < \infty$ such that for all $\mathbf{x} \in \mathbb{Z}^d$, one has

$$\mathbb{P}\left(\sup_{\mathbf{z}\in\mathbb{Z}^d, \mathbf{z}\neq\mathbf{x}} \frac{\tau(\mathbf{x},\mathbf{z})}{\|\mathbf{x}-\mathbf{z}\|_1} < \kappa\right) > 0. \tag{3.7}$$

Proof. We will prove the lemma under the stronger assumption $\mathbb{E}\left[\tau_e^2\right] < \infty$ which carries the main idea. Details on weakening the condition to (S1) can be found in [4].

By translation invariance, we may assume $\mathbf{x} = \mathbf{0}$. As in the proof of Theorem 1.1, we may build $2d$ disjoint self-avoiding paths $\{\gamma_i\}_{i=1}^{2d}$ from $\mathbf{0}$ to $\mathbf{z}$ (see Figure 3.3) with length

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at most $\|\mathbf{z}\|_1 + 2d$ (with γ_1 being the longest one). We have as usual

$$
\begin{aligned}
\mathbb{P}\big(\tau(\mathbf{0},\mathbf{z}) \geqslant 2\mathbb{E}[\tau_e](\|\mathbf{z}\|_1 + 2d)\big) &\leqslant \prod_{i=1}^{2d} \mathbb{P}\big(\tau(\gamma_i) \geqslant 2\mathbb{E}[\tau_e](\|\mathbf{z}\|_1 + 2d)\big) \\
&\leqslant \mathbb{P}\big(\tau(\gamma_1) \geqslant 2\mathbb{E}[\tau_e](\|\mathbf{z}\|_1 + 2d)\big)^{2d} \\
&\leqslant \left(\frac{\operatorname{Var}(\tau(\gamma_1))}{(2\mathbb{E}[\tau_e]\|(\|\mathbf{z}\|_1 + 2d) - \mathbb{E}[\tau(\gamma_1)])^2}\right)^{2d} \qquad (3.8) \\
&\leqslant \left(\frac{\mathbb{E}\left[\tau_e^2\right] \cdot \#\gamma_1}{(2\mathbb{E}[\tau_e](\|\mathbf{z}\|_1 + 2d) - \mathbb{E}[\tau_e] \cdot \#\gamma_1)^2}\right)^{2d} \\
&\leqslant \left(\frac{\mathbb{E}\left[\tau_e^2\right]}{\mathbb{E}\left[\tau_e\right]^2}\right)^{2d} \cdot \frac{1}{(\|\mathbf{z}\|_1 + 2d)^{2d}},
\end{aligned}
$$

where we apply Chebyshev's inequality in (3.8). Therefore, we see that

$$
\sum_{\mathbf{0}\neq\mathbf{z}\in\mathbb{Z}^d} \mathbb{P}\left(\frac{\tau(\mathbf{0},\mathbf{z})}{\|\mathbf{z}\|_1} \geqslant 2\mathbb{E}[\tau_e]\right) \leqslant \left(\frac{\mathbb{E}\left[\tau_e^2\right]}{\mathbb{E}\left[\tau_e\right]^2}\right)^{2d} \sum_{\mathbf{0}\neq\mathbf{z}\in\mathbb{Z}^d} \frac{1}{(\|\mathbf{z}\|_1 + 2d)^{2d}} < \infty.
$$

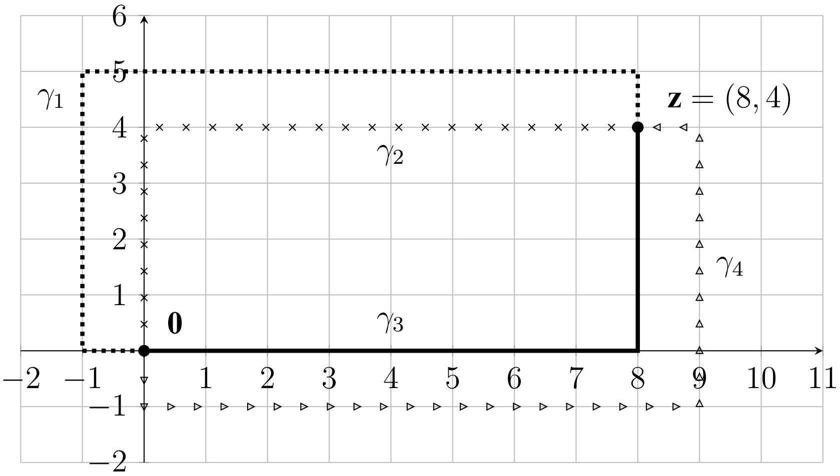

Figure 3.3: $2d$ paths from $\mathbf{0}$ to $\mathbf{z} = (8,4)$ ($d = 2$ with $\#\gamma_1 = \|\mathbf{z}\|_1 + 4$)

Hence, by the (first) Borel-Cantelli lemma, we have

$$
\mathbb{P}\left(\frac{\tau(\mathbf{0},\mathbf{z})}{\|\mathbf{z}\|_1} < 2\mathbb{E}[\tau_e] \text{ for all } \mathbf{z} \text{ with } \|\mathbf{z}\|_1 \text{ large enough}\right) = 1
$$

and by enlarging $\kappa > 2\mathbb{E}[\tau_e]$ if necessary, the lemma holds as asserted. $\square$

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If the event in (3.7) occurs for some $\mathbf{x} \in \mathbb{Z}^d$, we will call $\mathbf{x}$ a good vertex (note that being "good" is a random property).

Lemma 3.7 ([1], Proof of Theorem 2.17)**.** Let $\mathbf{0} \neq \mathbf{z} \in \mathbb{Z}^d$, if $\{n_k\}_{k=1}^{\infty}$ is the (random) sequence of natural numbers such that $n_k\mathbf{x}$ is a good vertex, then $\{n_k\}_{k=1}^{\infty}$ is an infinite set and

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1 \quad \text{almost surely.}$$

Proof. By Birkhoff's ergodic theorem and Lemma 3.6, we have almost surely

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{\{j\mathbf{z} \text{ is good}\}} \to \mathbb{E}\left[\mathbb{1}_{\{\mathbf{z} \text{ is good}\}}\right] > 0.$$

Therefore, the set $\{n_k\}_{k=1}^{\infty}$ must be infinite almost surely. Moreover,

$$\frac{k}{n_k} = \frac{1}{n_k}\sum_{j=1}^{n_k} \mathbb{1}_{\{j\mathbf{z} \text{ is good}\}} \to \mathbb{E}\left[\mathbb{1}_{\{\mathbf{z} \text{ is good}\}}\right] > 0 \quad \text{almost surely}$$

by Birkhoff's ergodic theorem again. Therefore, $\frac{k}{n_k}$ is positive almost surely for large k and

$$\frac{n_{k+1}}{n_k} = \frac{n_{k+1}}{k+1}\cdot\frac{k+1}{k}\cdot\frac{k}{n_k} \to 1 \quad \text{almost surely}$$

as asserted. $\square$

Proof. (of Theorem 1.3) We will prove the theorem by claiming

$$\mathcal{B}_d = \left\{\mathbf{x} \in \mathbb{R}^d \mid \mu_d(\mathbf{x}) \leqslant 1\right\},$$

which is convex (by convexity of μ_d), compact (by continuity of μ_d and the fact that it is bounded away from 0 on $\|\mathbf{x}\|_1 = 1$ by Theorem 1.2 and Proposition 3.2). Hence, it suffices to show

$$\limsup_{\|\mathbf{x}\|_1\to\infty} \frac{|\tau(\mathbf{0},\mathbf{x}) - \mu_d(\mathbf{x})|}{\|\mathbf{x}\|_1} = 0 \quad \text{almost surely.} \tag{3.9}$$

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Suppose now there is $\epsilon > 0$ and an event D_ϵ with $\mathbb{P}(D_\epsilon) > 0$ such that for any outcome in D_ϵ, there are infinitely many vertices $\mathbf{x} \in \mathbb{Z}^d$ such that

$$|\tau(\mathbf{0}, \mathbf{x}) - \mu_d(\mathbf{x})| \geqslant 2\epsilon \|\mathbf{x}\|_1. \tag{3.10}$$

Set the following events

$$\Pi_1 = \left\{ \lim_{n\to\infty} \frac{\tau(\mathbf{0}, n\mathbf{q})}{n} = \mu_d(\mathbf{q}) \text{ almost surely for all } \mathbf{q} \in \mathbb{Q}^d \right\} \quad \text{and}$$

$$\Pi_2 = \left\{ \text{the event in Lemma 3.7 holds for all } \mathbf{z} \in \mathbb{Z}^d \right\}.$$

Since both $\mathbb{Q}^d$ and $\mathbb{Z}^d$ are countable, we have by the discussion in Section 3.1 and Lemma 3.7 that $\mathbb{P}(\Pi_1) = \mathbb{P}(\Pi_2) = 1$. Therefore, there is some outcome $\omega \in \Pi_1 \cap \Pi_2 \cap D_\epsilon$. For such an outcome, there is a sequence $\{\mathbf{x}_n\}_{n=1}^\infty \subseteq \mathbb{Z}^d$ such that (3.10) holds with $\mathbf{x} = \mathbf{x}_n$. Moreover, by compactness of the ℓ^1-sphere, we may further assume that $\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|_1} \to \mathbf{y}$ as $n \to \infty$ for some $\mathbf{y}$ with $\|\mathbf{y}\|_1 = 1$. Let $0 < \delta < 1$ be arbitrary and choose N large enough such that

$$\left\| \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|_1} - \mathbf{y} \right\|_1 < \delta \quad \text{and} \quad \left| \frac{\mu_d(\mathbf{x}_n)}{\|\mathbf{x}_n\|_1} - \mu_d(\mathbf{y}) \right| < \epsilon \tag{3.11}$$

for $n \geqslant N$ (this is valid by the continuity of μ_d). Together with (3.10), we see that the second inequality in (3.11) implies

$$\left| \frac{\tau(\mathbf{0}, \mathbf{x}_n)}{\|\mathbf{x}_n\|_1} - \mu_d(\mathbf{y}) \right| \geqslant \epsilon \quad \text{for all} \quad n \geqslant N. \tag{3.12}$$

Choose $\mathbf{z} \in \mathbb{Q}^d$ with $\|\mathbf{z}\|_1 = 1$ such that

$$\|\mathbf{z} - \mathbf{y}\|_1 < \delta \tag{3.13}$$

and let $M \in \mathbb{N}$ be the smallest positive integer such that $M\mathbf{z} \in \mathbb{Z}^d$. Let $\{n_k\}_{k=1}^\infty$ be the increasing sequence of integers such that $n_k M\mathbf{z}$ is good. We choose K large enough (by

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$\omega \in \Pi_1 \cap \Pi_2$) such that

$$\frac{n_{k+1}}{n_k} < 1+\delta \quad \text{and} \quad \left|\frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{n_k M} - \mu_d(\mathbf{z})\right| < \delta \quad \text{for all} \quad k \geqslant K. \tag{3.14}$$

Note that for any $n \in \mathbb{N}$, there is $k = k(n)$ such that

$$n_k M < \|\mathbf{x}_n\|_1 \leqslant n_{k+1} M. \tag{3.15}$$

We choose $n \geqslant N$ large enough such that $k(n) \geqslant K$. Now, we will derive a contradiction by bounding the left hand side of (3.12) with a small number. Observe that

$$\begin{aligned}\left|\frac{\tau(\mathbf{0}, \mathbf{x}_n)}{\|\mathbf{x}_n\|_1} - \mu_d(\mathbf{y})\right| \leqslant \left|\frac{\tau(\mathbf{0}, \mathbf{x}_n)}{\|\mathbf{x}_n\|_1} - \frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{\|\mathbf{x}_n\|_1}\right| + \left|\frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{\|\mathbf{x}_n\|_1} - \frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{n_k M}\right| \\ + \left|\frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{n_k M} - \mu_d(\mathbf{z})\right| + |\mu_d(\mathbf{z}) - \mu_d(\mathbf{y})|.\end{aligned} \tag{3.16}$$

First, we see that when $n \geqslant N$ and $k = k(n) \geqslant K$,

$$\begin{aligned}\|\mathbf{x}_n - n_k M\mathbf{z}\|_1 &\leqslant \|\mathbf{x}\|_1 \left(\left\|\frac{\mathbf{x}_n}{\|\mathbf{x}\|_1} - \mathbf{y}\right\|_1 + \|\mathbf{y} - \mathbf{z}\|_1\right) + (\|\mathbf{x}_n\|_1 - n_k M)\|\mathbf{z}\|_1 \\ &< 2\delta\|\mathbf{x}_n\|_1 + (n_{k+1} - n_k)M\|\mathbf{z}\|_1 \qquad (3.17) \\ &< 2\delta\|\mathbf{x}_n\| + \delta n_k M \cdot 1 \qquad (3.18) \\ &\leqslant 3\delta\|\mathbf{x}_n\|_1, \qquad (3.19)\end{aligned}$$

where (3.17) holds by (3.11), (3.13), $\|\mathbf{x}_n\|_1 \leqslant n_{k+1}M$; (3.18) holds by (3.14), $\|\mathbf{z}\|_1 = 1$ and (3.19) holds by $n_k M < \|\mathbf{x}_n\|_1$. Then, we may bound the first term in (3.16) by

$$\left|\frac{\tau(\mathbf{0}, \mathbf{x}_n)}{\|\mathbf{x}_n\|_1} - \frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{\|\mathbf{x}_n\|_1}\right| \leqslant \frac{\tau(\mathbf{x}_n, n_k M\mathbf{z})}{\|\mathbf{x}_n\|_1} \leqslant \frac{\kappa\|\mathbf{x}_n - n_k M\mathbf{z}\|}{\|\mathbf{x}_n\|_1} \leqslant 3\kappa\delta, \tag{3.20}$$

where the first inequality holds by subadditivity, the second holds since $n_k M\mathbf{z}$ is good and the last holds by (3.19). For the second term in (3.16),

$$\left|\frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{\|\mathbf{x}_n\|_1} - \frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{n_k M}\right| = \frac{\tau(\mathbf{0}, n_k M\mathbf{z})}{n_k M}\left(1 - \frac{n_k M}{\|\mathbf{x}_n\|_1}\right)$$

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$$\leqslant (\mu_d(\mathbf{z}) + \delta) \left(1 - \frac{1}{1+\delta}\right), \tag{3.21}$$

where the inequality holds by (3.14) and $\|\mathbf{x}_n\|_1 \leqslant n_{k+1}M$. Moreover, we see directly from (3.14) that the third term in (3.16) is bounded by δ. Last, since μ_d is a semi-norm on $\mathbb{R}^d$, the fourth term in (3.16) is bounded by $c\|\mathbf{z} - \mathbf{y}\|_1 < c\delta$ where $c > 0$ is a constant depending only on μ_d. Combining these with (3.12), (3.16), (3.20) and (3.21), we obtain

$$\epsilon \leqslant 3\kappa\delta + (\mu_d(\mathbf{z}) + \delta) \left(1 - \frac{1}{1+\delta}\right) + \delta + c\delta.$$

Since the choice of $\delta > 0$ is arbitrary, we derive a contradiction and hence the intersection $\Pi_1 \cap \Pi_2 \cap D_\epsilon = \varnothing$. Therefore, (3.9) holds and so does the shape theorem. □

Remark 3.8. Note that having finite mean $\mathbb{E}[\tau_e] < \infty$ (as in condition (♠)) is a stronger assumption in comparison to (S1).

The uniform convergence characterized in the shape theorem allows us to find the asymptotic behavior of passage time between two sets more easily. In fact, the following corollary shows that the point-to-point passage time and the point-to-plane passage time share the same asymptotics.

Corollary 3.9 ([10], Equation (1.13))**.** If (S1) and (S2) hold, then

$$\lim_{n\to\infty} \frac{b_{0,n}}{n} = \mu_d(\mathbf{e}_1) \quad \text{and} \quad \lim_{n\to\infty} \frac{s_{0,n}}{n} = \mu_d(\mathbf{e}_1) \quad \text{almost surely and in } \mathcal{L}^1. \tag{3.22}$$

Proof. Since $b_{0,n} \leqslant a_{0,n}$, it suffices to show

$$\liminf_{n\to\infty} \frac{b_{0,n}}{n} \geqslant \mu_d(\mathbf{e}_1).$$

Fix $\epsilon > 0$. Suppose on the contrary that there exists a configuration (in the intersection of almost sure events $\left\{\frac{\tau(0,n\mathbf{e}_1)}{n} \to \mu_d(\mathbf{e}_1)\right\}$ and $\left\{(1-\epsilon)\mathcal{B}_d \subseteq \frac{B_d(t)}{t} \subseteq (1+\epsilon)\mathcal{B}_d \text{ eventually}\right\}$) such that

$$\liminf_{n\to\infty} \frac{b_{0,n}}{n} \geqslant \mu_d(\mathbf{e}_1) - 3\delta$$

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for some $\delta > 0$. We may pick a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\frac{b_{0,n_k}}{n_k} \to \mu_d(\mathbf{e}_1) - 3\delta$ as $k \to \infty$. By the definition of $b_{0,n}$, there exists a sequence of vertices $\mathbf{x}_k \in \mathbb{Z}^d$ where the first coordinate of each $\mathbf{x}_k$ is n_k and $\tau(\mathbf{0}, \mathbf{x}_k) < b_{0,n_k} + \delta$. Hence, for k large enough,

$$\tau(\mathbf{0}, \mathbf{x}_k) < b_{0,n_k} + \delta \leqslant n_k(\mu_d(\mathbf{e}_1) - 2\delta) + \delta \leqslant n_k(\mu_d(\mathbf{e}_1) - \delta)$$

and hence $\mathbf{x}_k \in B(n_k(\mu_d(\mathbf{e}_1) - \delta))$. Fix $\epsilon > 0$. By Theorem 1.3,

$$\frac{\mathbf{x}_k}{n_k(\mu_d(\mathbf{e}_1) - \delta)} \in (1+\epsilon)\mathcal{B}_d \quad \text{for all } k \text{ large enough.}$$

Therefore, there is $\mathbf{y} = (y_1, \ldots, y_d) \in \mathcal{B}_d$ with first coordinate satisfying

$$(1+\epsilon)y_1 = \frac{n_k}{n_k(\mu_d(\mathbf{e}_1) - \delta)}, \quad \text{which gives} \quad y_1 = \frac{1}{(1+\epsilon)(\mu_d(\mathbf{e}_1) - \delta)}.$$

By symmetry and convexity of $\mathcal{B}_d$, we have

$$y_1\mathbf{e}_1 = (y_1, 0, \ldots, 0) = \frac{1}{2}(y_1, y_2, \ldots, y_d) + \frac{1}{2}(y_1, -y_2, \ldots, -y_d) \in \mathcal{B}_d.$$

Again by Theorem 1.3, we have $(1-\epsilon)ty_1\mathbf{e}_1 \in B_d(t)$ for all t large enough. Therefore,

$$\begin{aligned}
\mu_d(\mathbf{e}_1) = \lim_{n\to\infty} \frac{\tau(0, n\mathbf{e}_1)}{n} &\leqslant \lim_{t\to\infty} \frac{\tau(\mathbf{0}, (1-\epsilon)ty_1\mathbf{e}_1)}{\lfloor (1-\epsilon)ty_1 \rfloor} \\
&\leqslant \lim_{t\to\infty} \frac{t}{\lfloor (1-\epsilon)ty_1 \rfloor} \\
&= \frac{1}{(1-\epsilon)y_1} = \frac{1+\epsilon}{1-\epsilon} \cdot (\mu_d(\mathbf{e}_1) - \delta).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we may take $\epsilon \to 0^+$ and derive a contradiction. The convergence for $\frac{s_{0,n}}{n}$ follows from (3.2). Their $\mathcal{L}^1$-convergences simply follow from the dominated convergence theorem. □

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3.3 Stochastic dominance

In this section, we will briefly compare different time constants $\mu_d^F(\mathbf{e}_1)$ and $\mu_d^G(\mathbf{e}_1)$ of different edge weight distributions satisfying a certain stochastic order.

Definition 3.10 (Stochastic dominance)**.** Let F, G be two distributions on $\mathbb{R}$.

1. We say F **stochastically dominates** G if $F(t) \leqslant G(t)$ for all $t \in \mathbb{R}$.

2. We say F is **more variable** than G if

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}F(x) \leqslant \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}G(x)$$

   for every concave non-decreasing function $\varphi\colon \mathbb{R} \to \mathbb{R}$ provided the integrals exist.

Lemma 3.11 (Characterization of stochastic dominance)**.** Let F, G be two distributions on $\mathbb{R}$. Then F stochastically dominates G if and only if

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}F(x) \leqslant \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}G(x)$$

for all non-increasing functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ provided the integrals exist.

Proof. Sufficiency follows by taking $\varphi(x) = \mathbb{1}_{\{x \leqslant t\}}$. For necessity, we have

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}F(x) &= [\varphi(x)F(x)]_{x\to-\infty}^{x\to\infty} - \int_{\mathbb{R}} F(x)\,\mathrm{d}\varphi(x) \qquad (3.23)\\ &\leqslant \varphi(\infty) - \int_{-\infty}^{\infty} G(x)\,\mathrm{d}\varphi(x) = \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}G(x),\end{aligned}$$

where (3.23) holds since φ is non-increasing and $F(x) \leqslant G(x)$ for all $x \in \mathbb{R}$. □

Corollary 3.12. If F stochastically dominates G, then G is more variable than F.

Lemma 3.13 ([15], Theorem 2.13 (a))**.** If $d \geqslant 2$ and F, G are distributions on $[0, \infty)$ such that F is more variable than G, then $\mu_d^F(\mathbf{e}_1) \leqslant \mu_d^G(\mathbf{e}_1)$.

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Proof. Note that by conditioning and induction, for all increasing concave function $\varphi\colon \mathbb{R}^k \to \mathbb{R}$ (meaning $\varphi(\mathbf{x}) \geqslant \varphi(\mathbf{x}+\delta\mathbf{e}_j)$ for all $\mathbf{x} \in \mathbb{R}^k$, $\delta > 0$ and $1 \leqslant j \leqslant d$), we have

$$\mathbb{E}[\varphi(X_1, \ldots, X_k)] \leqslant \mathbb{E}[\varphi(Y_1, \ldots, Y_k)], \tag{3.24}$$

where $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ are i.i.d. copies of random variables with distribution F and G, respectively. Denote the set of edges within the box $[-N, N]^d$ by $\mathcal{E}\left([-N, N]^d\right)$ and define the function $\varphi_{n,N}\colon \mathbb{R}^{\mathcal{E}\left([-N,N]^d\right)} \to \mathbb{R}$ by

$$\varphi_{n,N}\left((\tau_e)_{e\in\mathcal{E}\left([-N,N]^d\right)}\right) = \min\left\{T(\gamma) \mid \gamma \text{ is a path from } \mathbf{0} \text{ to } n\mathbf{e}_1 \text{ in } [-N,N]^d\right\}.$$

Clearly, $\varphi_{n,N}$ is increasing on its domain. Moreover, it is concave since it is the minimum of linear functions $\varphi_\gamma\left((\tau_e)_{e\in\mathcal{E}\left([-N,N]^d\right)}\right) = \sum_{e\in\gamma}\tau_e$. Therefore, by (3.24),

$$\mathbb{E}\left[\varphi_{n,N}\left(\left(\tau_e^F\right)_{e\in\mathcal{E}\left([-N,N]^d\right)}\right)\right] \leqslant \mathbb{E}\left[\varphi_{n,N}\left(\left(\tau_e^G\right)_{e\in\mathcal{E}\left([-N,N]^d\right)}\right)\right].$$

Taking $N \to \infty$, we have $\mathbb{E}\left[\tau^F(0, n\mathbf{e}_1)\right] \leqslant \mathbb{E}\left[\tau^G(0, n\mathbf{e}_1)\right]$. Dividing by n and taking $n \to \infty$, we obtain $\mu_d^F(\mathbf{e}_1) \leqslant \mu_d^G(\mathbf{e}_1)$ from Theorem 1.1. □

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Chapter 4

Proof of Theorems 1.4, 1.6 and 1.7

First, we will prove Theorem 1.4 in Section 4.1 (treating the lower bound) and Section 4.2 (treating the upper bound) with parameter $p \geqslant 1$ and $a \in (0, \infty)$ (for the upper bound, we leave the discussion for $p = 1$ to Section 4.3). The "boundary cases" $a = \infty$ and $a = 0$ in Theorem 1.4 will be proved in Section 4.4. We will also demonstrate an interesting asymptotic scaling relationship between time constants in Section 4.5.

Next, we will prove Theorem 1.6 in Section 4.6 (treating the lower bound) and Section 4.7 (treating the upper bound).

Last, we combine the results above and prove Theorem 1.7 in Section 4.8.

4.1 Proof of the lower bound

In this section, we will prove the lower bound in Theorem 1.4 holds for $p \geqslant 1$ and $a \in (0, \infty)$. To be precise, we will prove the following proposition.

Proposition 4.1. If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p \geqslant 1$ and $a \in (0, \infty)$, then the time constant satisfies

$$\liminf_{d\to\infty} \frac{\mu_d(\mathbf{e}_1) d^{\frac{1}{p}}}{\log d} \geqslant \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}}.$$

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Recall the point-to-plane passage time $b_{0,n}$ in Definition 3.3. Since $b_{0,n} \leqslant \tau(\mathbf{0}, n\mathbf{e}_1)$ for all $n \in \mathbb{N}$, we may derive a lower bound for $\mu_d(\mathbf{e}_1)$ by bounding the passage time $b_{0,n}$ below.

Lemma 4.2. If (♣) holds and there exists $t > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}(b_{0,n} \leqslant nt) < \infty, \tag{4.1}$$

then $\mu_d(\mathbf{e}_1) \geqslant t$.

Proof. By the (first) Borel-Cantelli lemma, (4.1) implies

$$\mathbb{P}(b_{0,n} \leqslant nt \text{ i.o.}) = 0.$$

Therefore, we have

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{b_{0,n}}{n} > t\right) = 1.$$

Since $b_{0,n} \leqslant \tau(\mathbf{0}, n\mathbf{e}_1)$ for all $n \in \mathbb{N}$, we have $\mu_d(\mathbf{e}_1) \geqslant t$. □

By Lemma 4.2, it suffices to bound $\mathbb{P}(b_{0,n} \leqslant nt)$ in order to derive a lower bound for $\mu_d(\mathbf{e}_1)$. Using a union bound, we see that

$$\begin{aligned}
\mathbb{P}(b_{0,n} \leqslant nt) &= \mathbb{P}\left(\text{there exist a self-avoiding path } \gamma \in \Gamma(\mathbf{0}, H_n) \text{ with } \tau(\gamma) \leqslant nt\right) \\
&= \mathbb{P}\left(\bigcup_{\gamma \in \Gamma(\mathbf{0}, H_n),\ \text{self-avoiding}} \{\tau(\gamma) \leqslant nt\}\right) \\
&\leqslant \sum_{\gamma \in \Gamma(\mathbf{0}, H_n),\ \text{self-avoiding}} \mathbb{P}(\tau(\gamma) \leqslant nt) \\
&\leqslant \sum_{k=n}^{\infty} \#\{\gamma \in \Gamma(\mathbf{0}, H_n) \mid \#\gamma = k\} \cdot \mathbb{P}(S_k \leqslant nt),
\end{aligned} \tag{4.2}$$

where S_k is the sum of k i.i.d. random variables with distribution τ_e. Since we already derive an upper bound the number of paths of length k on $\mathbb{Z}^d$ from $\mathbf{0}$ to H_n in Lemma 2.2, it suffices to bound $\mathbb{P}(S_k \leqslant nt)$ from above.

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Lemma 4.3. If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p \geqslant 1$ and $a \in (0, \infty)$, then for any $\delta > 0$, there exists $t_0 > 0$ small enough so that for all $n \in \mathbb{N}$ and $k \geqslant n$,

$$\mathbb{P}(S_k \leqslant nt) \leqslant \left[(1+\delta)a\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k \quad \text{for all} \quad 0 \leqslant t < t_0.$$

Proof. By condition (♠), for any fixed $\delta > 0$, we may pick t_1 small enough such that

$$\mathbb{P}(\tau_e \leqslant t) \leqslant \left(1 + \frac{\delta}{2}\right) at^p < 1 \quad \text{for all} \quad 0 \leqslant t < t_1.$$

Let Y be any non-negative random variable with density $f_Y(y) = \left(1 + \frac{\delta}{2}\right) pay^{p-1}$ on $[0, t_1)$ (such a random variable exists since $\left(1 + \frac{\delta}{2}\right) at_1^p < 1$). Fix $\epsilon \in [0, t_1)$. Define

$$X_1 = \tau_e \mathbb{1}_{\{\tau_e < \epsilon\}} + \epsilon \mathbb{1}_{\{\tau_e \geqslant \epsilon\}} \quad \text{and} \quad X_2 = Y \mathbb{1}_{\{Y < \epsilon\}} + \epsilon \mathbb{1}_{\{Y \geqslant \epsilon\}}.$$

By construction, X_1 stochastically dominates X_2:

$$\mathbb{P}(X_1 \leqslant t) = \mathbb{P}(\tau_e \leqslant t) \leqslant \left(1 + \frac{\delta}{2}\right) at^p = \mathbb{P}(Y \leqslant t) = \mathbb{P}(X_2 \leqslant t)$$

if $t \in [0, \epsilon)$ and $\mathbb{P}(X_1 \leqslant t) = \mathbb{P}(X_2 \leqslant t) = 1$ if $t \geqslant \epsilon$. By taking $\varphi(t) = e^{-\gamma t}$ (where $\gamma > 0$ is a constant that will be chosen later) and applying Lemma 3.11, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-\gamma \tau_e} \mathbb{1}_{\{\tau_e < \epsilon\}}\right] + e^{-\gamma\epsilon}\mathbb{P}(\tau_e \geqslant \epsilon) \\
&\quad = \mathbb{E}[\varphi(X_1)] \\
&\quad \leqslant \mathbb{E}[\varphi(X_2)] \\
&\quad = \mathbb{E}\left[e^{-\gamma Y} \mathbb{1}_{\{Y < \epsilon\}}\right] + e^{-\gamma\epsilon}\mathbb{P}(Y \geqslant \epsilon) \\
&\quad \leqslant \int_0^\epsilon e^{-\gamma y} \cdot \left(1 + \frac{\delta}{2}\right) pay^{p-1} \, \mathrm{d}y + e^{-\gamma\epsilon} \\
&\quad \leqslant \frac{\left(1 + \frac{\delta}{2}\right) pa \cdot \Gamma(p)}{\gamma^p} + e^{-\gamma\epsilon} = \frac{\left(1 + \frac{\delta}{2}\right) a\Gamma(p+1)}{\gamma^p} + e^{-\gamma\epsilon}.
\end{aligned} \tag{4.3}$$

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Note that

$$\begin{aligned}
\mathbb{P}(S_k \leqslant nt) &= \mathbb{P}\left(e^{-\gamma S_k} \geqslant e^{-n\gamma t}\right) \\
&\leqslant e^{n\gamma t}\mathbb{E}\left[e^{-\gamma S_k}\right] \qquad (4.4)\\
&\leqslant e^{n\gamma t}\left(\mathbb{E}\left[e^{-\gamma\tau_e}\mathbb{1}_{\{\tau_e<\epsilon\}}\right] + e^{-\gamma\epsilon}\mathbb{P}(\tau_e \geqslant \epsilon)\right)^k \qquad (4.5)\\
&\leqslant e^{n\gamma t}\left(\frac{\left(1+\frac{\delta}{2}\right)a\Gamma(p+1)}{\gamma^p} + e^{-\gamma\epsilon}\right)^k, \qquad (4.6)
\end{aligned}$$

where (4.4) holds by Markov's inequality, (4.5) holds since $\tau_e \geqslant X_1$ and (4.6) holds by (4.3). Pick t small enough such that $\sqrt{t} < \min\{t_1, 1\}$ and such that for all $y \geqslant 1$,

$$y \exp\left(-\frac{y}{\sqrt{t}}\right) \leqslant \frac{1}{p}\left(\frac{\delta}{2}\cdot a\Gamma(p+1)\right)^{\frac{1}{p}} t. \qquad (4.7)$$

(4.7) is possible by considering $f(y) = y\exp\left(-\frac{y}{\sqrt{t}}\right)$ for $y \in [1,\infty)$. Note that

$$f'(y) = \exp\left(-\frac{y}{\sqrt{t}}\right)\left(1-\frac{y}{\sqrt{t}}\right) \leqslant \exp\left(-\frac{y}{\sqrt{x}}\right)\left(1-\frac{1}{\sqrt{t}}\right) < 0$$

if $\sqrt{t} < 1$. Therefore, $f(y)$ is decreasing on $y \in [1,\infty)$. It suffices to find t small enough such that $1\cdot\exp\left(-\frac{1}{\sqrt{t}}\right) \leqslant ct$, which can be simplified to

$$\sqrt{t}\log c + 2\sqrt{t}\log\sqrt{t} \geqslant -1, \qquad (4.8)$$

where $c = \frac{1}{p}\left(\frac{\delta}{2}\cdot a\Gamma(p+1)\right)^{\frac{1}{p}}$. This is valid since the left hand side of (4.8) goes to 0 as $t \to 0^+$. Set $t_0 < t_1$ small enough such that (4.8) holds for all $t \in [0, t_0)$ and pick $\gamma = \frac{pk}{nt}$, $\epsilon = \sqrt{t}$ for $t \in [0, t_0)$. Then

$$\begin{aligned}
\mathbb{P}(S_k \leqslant nt) &\leqslant e^{n\cdot\frac{pk}{nt}\cdot t}\left[\left(1+\frac{\delta}{2}\right)a\Gamma(p+1)\left(\frac{nt}{pk}\right)^p + e^{-\frac{pk}{nt}\cdot\sqrt{t}}\right]^k \\
&\leqslant e^{pk}\left[\left(1+\frac{\delta}{2}\right)a\Gamma(p+1)\left(\frac{nt}{pk}\right)^p + \frac{\delta}{2}\cdot a\Gamma(p+1)\left(\frac{nt}{pk}\right)^p\right]^k \\
&= \left[(1+\delta)a\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k,
\end{aligned}$$

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where the second inequality holds by (4.7) with $y = \frac{k}{n} \geqslant 1$. $\square$

We can now prove Proposition 4.1 by combining Lemma 2.2, Lemma 4.3 and (4.2).

Proof. (of Proposition 4.1) Fix $\delta > 0$ and set

$$t = (1-\delta) \cdot \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}} \cdot \frac{\log d}{d^{\frac{1}{p}}}, \quad M = \frac{(4ad\Gamma(p+1))^{\frac{1}{p}} ent}{p} = \frac{(1-\delta)2^{\frac{1}{p}} en \log d}{p},$$

where we choose d large enough such that $0 \leqslant t < t_0$ as in Lemma 2.2. By Lemma 2.2 (with $\rho = \log\left(\frac{2nd}{k}\right)$ and $n \leqslant k \leqslant M$) and Lemma 4.3, we have

$$\begin{aligned}
&\sum_{n \leqslant k \leqslant M} \#\{\gamma\colon \text{from } \mathbf{0} \text{ to } H_n \mid \#\gamma = k\} \cdot \mathbb{P}(S_k \leqslant nt) \\
&\leqslant \sum_{n \leqslant k \leqslant M} (2d)^k \exp\left(-n\rho + \frac{k}{d}(\cosh \rho - 1)\right) \left[(1+\delta) a \Gamma(p+1) \left(\frac{ent}{pk}\right)^p\right]^k \\
&= \sum_{n \leqslant k \leqslant M} \left(\frac{k}{2nd}\right)^n \exp\left(\frac{k}{d}\left(\frac{2nd}{2k} + \frac{k}{4nd} - 1\right)\right) \left[2ad(1+\delta)\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k \\
&= \sum_{n \leqslant k \leqslant M} \left(\frac{ek}{2nd}\right)^n \exp\left(\frac{k}{d}\left(\frac{k}{4nd} - 1\right)\right) \left[2ad(1+\delta)\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k \\
&\leqslant \sum_{n \leqslant k \leqslant M} \left(\frac{ek}{2nd}\right)^n \left[2ad(1+\delta)\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k,
\end{aligned}$$

where the last inequality holds since $\frac{k}{4nd} \leqslant \frac{M}{4nd} \to 0$ as $d \to \infty$. Therefore, by (4.2), we have

$$\begin{aligned}
\mathbb{P}(b_{0,n} \leqslant nt) \leqslant &\sum_{n \leqslant k \leqslant M} \left(\frac{ek}{2nd}\right)^n \left[2ad(1+\delta)\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k \\
&+ \sum_{k > M} \left[2ad(1+\delta)\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k,
\end{aligned} \tag{4.9}$$

where we used the trivial bound $(2d)^k$ in Lemma 2.2 for $k > M$. By choosing $\delta \leqslant \frac{1}{2}$, the second term on the right hand side of (4.9) is bounded by

$$\sum_{k > M} \left[2ad(1+\delta)\Gamma(p+1)\left(\frac{net}{pk}\right)^p\right]^k$$

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$$\leqslant \sum_{k>M} \left(\frac{1+\delta}{2}\right)^k \leqslant \left(\frac{1+\delta}{2}\right)^M \frac{1}{1-\frac{1+\delta}{2}} \leqslant 4\left[\left(\frac{3}{4}\right)^{O(1)\log d}\right]^n, \tag{4.10}$$

where the first inequality holds since $k > M$. Note that the last term in (4.10) is indeed summable in n. On the other hand, by taking $z = \frac{k}{n} \geqslant 1$, we see that the first term on the right hand side of (4.9) is bounded above by

$$\sum_{n\leqslant k\leqslant M} \left(\frac{ek}{2nd}\right)^n \left[2ad(1+\delta)\Gamma(p+1)\left(\frac{ent}{pk}\right)^p\right]^k \tag{4.11}$$

$$= \sum_{n\leqslant k\leqslant M} \left(\frac{ek}{2nd}\right)^n \left[(1+\delta)\left(\frac{en}{pk}\right)^p (1-\delta)^p(\log d)^p\right]^k \qquad \text{(definition of } t\text{)}$$

$$\leqslant \sum_{n\leqslant k\leqslant M} \left(\frac{ek}{2nd}\right)^n \left[(1-\delta^2)\left(\frac{en}{pk}\right)^p (\log d)^p\right]^k$$

$$\leqslant \left(O(1)\cdot\frac{\log d}{d}\right)^n \sum_{n\leqslant k\leqslant M} \left[\left((1-\delta^2)\left(\frac{en}{pk}\right)^p (\log d)^p\right)^{\frac{k}{n}}\right]^n \qquad (k \leqslant M)$$

$$\leqslant \left(O(1)\cdot\frac{\log d}{d}\right)^n \cdot M \cdot \sup_{z\geqslant 1}\left(\frac{D}{z^p}\right)^z,$$

where $D = (1-\delta^2)\left(\frac{e}{p}\right)^p(\log d)^p$. Consider the function $f(z) = \left(\frac{D}{z^p}\right)^z$ on $z \in [0,\infty)$ (where we extend $f(z)$ to $z = 0$ continuously). Observe that

$$f'(z) = \left(\frac{D}{z^p}\right)^z \log\left(\frac{D}{(ez)^p}\right)$$

and hence $f'(z) = 0$ if $z = \frac{D^{\frac{1}{p}}}{e}$. The maximum of $f(z)$ on $z \in [0,\infty)$ is then achieved by

$$\max\{f(z) \mid z \geqslant 0\} = f\left(\frac{D^{\frac{1}{p}}}{e}\right) = \exp\left(\frac{pD^{\frac{1}{p}}}{e}\right). \tag{4.12}$$

Therefore, we have by (4.12) that

$$(4.11) \leqslant \left(O(1)\frac{\log d}{d}\right)^n \cdot M \cdot \exp\left[\frac{p}{e}(1-\delta^2)^{\frac{1}{p}}\frac{e}{p}\log d\right]^n$$

$$= \left(O(1)\cdot\frac{\log d}{d}\right)^n \cdot M \cdot d^{(1-\delta^2)^{\frac{1}{p}} n}$$

$$= \left[O(1)\cdot d^{(1-\delta^2)^{\frac{1}{p}}-1}\log d\right]^n \cdot O(1) n\log d,$$

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which is summable in n for d large enough. Hence, by Lemma 4.2, we have

$$\mu_d(\mathbf{e}_1) \geqslant t = (1-\delta) \cdot \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}} \cdot \frac{\log d}{d^{\frac{1}{p}}} \quad \text{for } d \text{ large enough.}$$

By dividing $\frac{\log d}{d^{\frac{1}{p}}}$, taking $\liminf_{d\to\infty}$ and letting $\delta \to 0^+$, we obtain the desired inequality in Proposition 4.1. $\square$

4.2 Proof of the upper bound

In this section, we will prove the upper bound in Theorem 1.4 holds for $p > 1$ and $a \in (0, \infty)$. In order to do that, we will first consider the theorem under a slightly stronger assumption, then overcome the restriction by approximation. To be precise, we will prove the following proposition.

Proposition 4.4. If τ_e satisfies

$$\mathbb{E}[\tau_e] < \infty \quad \text{and} \quad \left| \frac{\mathbb{P}(\tau_e \leqslant t)}{t^p} - a \right| \leqslant \frac{C}{|\log t|} \tag{$\heartsuit$}$$

for some $p > 1$, $C > 0$ and $a \in (0, \infty)$ for $t \in [0, t_*)$ (for convenience, we assume $0 < t_* < 1$), then

$$\limsup_{d\to\infty} \frac{\mu_d(e_1) d^{\frac{1}{p}}}{\log d} \leqslant \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}}. \tag{4.13}$$

Remark 4.5. Note that the condition ($\heartsuit$) is a stronger condition in comparison to the conditions in ($\spadesuit$) since we require the convergent rate to be at most $|\log t|^{-1}$. However, we can show that (4.13) still holds under ($\spadesuit$) by approximating distributions satisfying ($\spadesuit$) with the one satisfying ($\heartsuit$); see Corollary 4.10. Also, the cases $a = 0$ and $a = \infty$ in Proposition 4.4 will be proved later in Section 4.4.

The idea of the proof of Proposition 4.4 essentially follows from Chapter 8 of [10]. First, we bound the time constant above by $\mathbb{E}[s_{0,1}]$ (introduced in Definition 3.2).

Lemma 4.6. If ($\clubsuit$) holds, then $\mu_d(\mathbf{e}_1) \leqslant \mathbb{E}[s_{0,1}]$.

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Proof. Observe that for all $n \geqslant 2$, if $\gamma_1 \in \Gamma_{\text{cyl}}(\mathbf{0}, \mathbf{x})$ (recall the notation in Definition 3.3) for some $\mathbf{x} \in H_{n-1}$, we can construct $\gamma_2 \in \Gamma_{\text{cyl}}(\mathbf{x}, H_n)$ giving $\gamma = \gamma_1 \cup \gamma_2 \in \Gamma_{\text{cyl}}(\mathbf{0}, H_n)$. The infimum of $\tau(\gamma_2)$ over $\Gamma_{\text{cyl}}(\mathbf{x}, H_n)$ is independent of γ_1 and has the same distribution as $s_{0,1}$ by translation invariance. Consequently, we have

$$\mathbb{E}[s_{0,n}] \leqslant \mathbb{E}[s_{0,n-1}] + \mathbb{E}[s_{0,1}].$$

By induction, we have $\mathbb{E}\left[s_{0,n}\right] \leqslant n\mathbb{E}\left[s_{0,1}\right]$ for all $n \in \mathbb{N}$ and we may deduce by Corollary 3.9 that

$$\mu_d(\mathbf{e}_1) = \lim_{n\to\infty} \frac{\mathbb{E}\left[s_{0,n}\right]}{n} \leqslant \mathbb{E}\left[s_{0,1}\right]$$

as asserted. □

We are then motivated to bound the passage time $s_{0,1}$. Precisely, we will show that there is a path (which is admissible for $s_{0,1}$) with sufficiently small passage time.

Recall the sub-collection of paths $\mathcal{P}_{d,k,r} \subseteq \Gamma_{\text{cyl}}(\mathbf{0}, H_1)$ given in Definition 2.11 (where $d, k, r \in \mathbb{N}$). We define, for $t \geqslant 0$,

$$N_{d,k,r,t} = \#\{\gamma \in \mathcal{P}_{d,k,r} \mid \tau(\gamma) \leqslant t\}.$$

Throughout the remaining section, we fix $0 < \eta < 1$ small enough such that

$$p - 1 > 3\eta, \tag{4.14}$$

which is possible since $p > 1$. Fix

$$k = \left\lceil \frac{\log d}{p} \right\rceil, \quad \delta = \frac{1}{(\log d)^{\eta}}, \quad r = d - \left\lfloor \frac{d}{(\log d)^{2\eta}} \right\rfloor \tag{4.15}$$

and

$$t = \frac{1}{(1-\delta)^{\frac{1}{p}}} \cdot \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}} \cdot \frac{\log d}{d^{\frac{1}{p}}}. \tag{4.16}$$

Note that $\delta, t = o(1)$ and $r \sim d$ as $d \to \infty$. In the remaining section, we will show that

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for the parameters (4.14), (4.15) and (4.16), one has

$$\mathbb{P}\left(N_{d,k,r,t} \geqslant 1\right) \geqslant \frac{1}{2} \quad \text{if } d \text{ is sufficiently large.} \tag{4.17}$$

We will derive this by showing

$$\mathbb{E}\left[N_{d,k,r,t}^2\right] \leqslant 2 \cdot \mathbb{E}\left[N_{d,k,r,t}\right]^2, \tag{4.18}$$

where (4.17) then follows directly from the Cauchy-Schwarz inequality.

First, we show how (4.18) implies Proposition 4.4.

Proof. (of Proposition 4.4) Suppose (4.18) holds (which, will be proved in Lemma 4.9), then (4.17) holds consequently. Let E_j (where $r+2 \leqslant j \leqslant d$) be the event $\{N_{d,k,r,t} \geqslant 1\}$ translated by $\mathbf{e}_j$. That is, E_j is the event such that there is a self-avoiding path γ of length k from $\mathbf{e}_j$ to H_1 such that

- the first $k-1$ steps of γ use only directions $\pm\mathbf{e}_i$, $2 \leqslant i \leqslant r+1$,
- the last step of γ is $\mathbf{e}_1$ and
- the passage time of γ is less than t.

We have $\mathbb{P}(E_j) \geqslant \frac{1}{2}$ for all $r+2 \leqslant j \leqslant d$ by (4.17) and translation invariance. Let

$$y = \left(\frac{\delta}{a}\right)^{\frac{1}{p}} \frac{\log d}{d^{\frac{1}{p}}} \quad \text{and} \quad F_j = E_j \cap \left\{\tau_{\{\mathbf{0},\mathbf{e}_j\}} \leqslant y\right\} \quad (r+2 \leqslant j \leqslant d).$$

First, observe that for all $r+2 \leqslant j \leqslant d$, the event E_j and $\left\{\tau_{\{\mathbf{0},\mathbf{e}_j\}} \leqslant y\right\}$ are independent, since E_j is measurable relative to the σ-algebra

$$\sigma\left(\left\{\tau_{\{\mathbf{e}_j+\mathbf{x},\mathbf{e}_j+\mathbf{x}+\mathbf{e}_i\}} \,\middle|\, 1 \leqslant i \leqslant r+1, \mathbf{x} \in \operatorname{span}\{\mathbf{e}_1,\ldots,\mathbf{e}_r\}\right\}\right)$$

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whereas $\tau_{\{\mathbf{0},\mathbf{e}_j\}}$ is independent of it. Hence, by condition ($\heartsuit$), for all $r+2 \leqslant j \leqslant d$,

$$\mathbb{P}(F_j) = \mathbb{P}(E_j) \cdot \mathbb{P}\left(\tau_{\{\mathbf{0},\mathbf{e}_j\}} \geqslant y\right) \geqslant \frac{1}{2} \cdot ay^p \left(1 - \frac{C}{a|\log t|}\right) \geqslant \frac{\delta(\log d)^p}{4d}$$

if d is large enough so that $1 - \frac{C}{a|\log t|} \geqslant \frac{1}{2}$ (recall $t = o(1)$). Next, note that on $\bigcup_{j=r+2}^{d} F_j$, the passage time satisfies

$$s_{0,1} \leqslant y + t = \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}} \cdot \frac{\log d}{d^{\frac{1}{p}}} \cdot \left[(2\delta\Gamma(p+1))^{\frac{1}{p}} + \frac{1}{1-\delta}\right].$$

Therefore, we have by Lemma 4.6,

$$\begin{aligned}
\mu_d(\mathbf{e}_1) &\leqslant \mathbb{E}\left[s_{0,1}\right] && (4.19)\\
&\leqslant (y+t) \cdot \mathbb{P}\left(\bigcup_{j=r+2}^{d} F_j\right) + \mathbb{E}\left[\tau_{\{\mathbf{0},\mathbf{e}_1\}} \mathbb{1}_{\bigcap_{j=r+2}^{d} F_j^{\complement}}\right] \\
&\leqslant y + t + \mathbb{E}[\tau_e] \prod_{j=r+2}^{d} \mathbb{P}\left(F_j^{\complement}\right) \\
&\leqslant \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}} \cdot \frac{\log d}{d^{\frac{1}{p}}} \cdot (1+o(1)) + \mathbb{E}[\tau_e] \left(1 - \frac{\delta(\log d)^p}{4d}\right)^{d-r-1}. && (4.20)
\end{aligned}$$

The third inequality holds by independence. Note that

$$\begin{aligned}
\left(1 - \frac{\delta(\log d)^p}{4d}\right)^{d-r-1} &= \left(1 - \frac{\delta(\log d)^p}{4d}\right)^{\left\lfloor \frac{d}{(\log d)^{2\eta}} \right\rfloor - 1} \\
&\leqslant 2\left(1 - \frac{\delta(\log d)^p}{4d}\right)^{\frac{d}{(\log d)^{2\eta}}} \leqslant 2\exp\left(-\frac{1}{4}(\log d)^{p-3\eta}\right),
\end{aligned}$$

where the factor 2 appears to overcome the modification of exponent. Observe that

$$\frac{2\exp\left(-\frac{1}{4}(\log d)^{p-3\eta}\right)}{\frac{\log d}{d^{\frac{1}{p}}}} = 2\exp\left(-\frac{1}{4}(\log d)^{p-3\eta} + \frac{\log d}{p} - \log\log d\right) \to 0$$

since $p - 3\eta > 1$ by assumption (4.14). Hence, by dividing $\frac{\log d}{d^{\frac{1}{p}}}$ and taking $\limsup_{d\to\infty}$ on the left hand side of (4.19) and (4.20), we derive the the upper bound (4.13). $\square$

Now, we start the proof of the inequality (4.18). Since the paths in $\mathcal{P}_{d,k,r}$ are self-

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avoiding by construction, the first step of estimating moments of $N_{d,k,r,t}$ is to estimate the sum of i.i.d. copies of τ_e that satisfies the condition ($\heartsuit$).

Lemma 4.7. Let $\{\tau_i\}_{i=1}^\infty$ be i.i.d. copies of τ_e satisfying ($\heartsuit$) for some $p \geqslant 1$, $C > 0$ and $a \in (0, \infty)$ for $t \in [0, t_*)$ and let $S_k = \sum_{i=1}^k \tau_i$ be the partial sum. Then, for $n \geqslant 1$ and $t \in [0, t_*)$,

$$A_{k,p}\left[at^p\left(1 - \frac{C'}{|\log t|}\right)\right]^k \leqslant \mathbb{P}(S_k \leqslant t) \leqslant A_{k,p}\left[at^p\left(1 + \frac{C'}{|\log t|}\right)\right]^k,$$

where the constants $A_{k,p}$ and C' are defined by

$$A_{k,p} = \frac{\Gamma(p+1)^k}{\Gamma(kp+1)} \quad \text{and} \quad C' = \frac{C}{a}.$$

Proof. We will only prove the upper bound (the case for lower bound is similar). We apply induction on $k \geqslant 1$. The case $k = 1$ holds trivially by ($\heartsuit$). Suppose $F(t) = \mathbb{P}(\tau_e \leqslant t)$ is the distribution function for τ_e (note that $F(0) = 0$ since $a < \infty$) and the statement holds for $k - 1$ for some $k \geqslant 2$. Then

$$\begin{aligned}
\mathbb{P}(S_k \leqslant t) &= \int_{\{t_1+\cdots+t_k \leqslant t\}} \mathrm{d}F(t_1)\cdots \mathrm{d}F(t_k) \\
&= \int_0^t \left(\int_{\{t_2+\cdots+t_k \leqslant t-t_1\}} \mathrm{d}F(t_2)\cdots \mathrm{d}F(t_{k-1})\right) \mathrm{d}F(t_1) \\
&\leqslant \int_0^t A_{k-1,p}\left[a(t-t_1)^p\left(1 + \frac{C'}{|\log(t-t_1)|}\right)\right]^{k-1} \mathrm{d}F(t_1) \qquad (4.21)\\
&\leqslant A_{k-1,p}\left[a\left(1 + \frac{C'}{|\log t|}\right)\right]^{k-1} \int_0^t (t-t_1)^{p(k-1)}\,\mathrm{d}F(t_1), \qquad (4.22)
\end{aligned}$$

where (4.21) follows from the induction hypothesis ($t - t_1 \in [0, t_*)$) and (4.22) follows from the inequality $|\log(t - t_1)| \geqslant |\log t|$ (recall $t < 1$). Observe that

$$\begin{aligned}
&\int_0^t (t-t_1)^{p(k-1)}\,\mathrm{d}F(t_1) \\
&\quad= \left[(t-t_1)^{p(k-1)}F(t_1)\right]_{t_1=0}^{t_1=t} - \int_0^t -p(k-1)(t-t_1)^{p(k-1)-1}F(t_1)\,\mathrm{d}t_1 \\
&\quad= p(k-1)\int_0^t (t-t_1)^{p(k-1)-1}F(t_1)\,\mathrm{d}t_1 \qquad (F(0) = 0)
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant (k-1)\int_0^t (t-t_1)^{p(k-1)-1}\cdot A_{1,p}\left[at_1^p\left(1+\frac{C'}{|\log t_1|}\right)\right]\mathrm{d}t_1\\
&\leqslant A_{1,p}\left[a\left(1+\frac{C'}{|\log t|}\right)\right]\int_0^t t_1^p(t-t_1)^{p(k-1)-1}\,\mathrm{d}t_1\\
&= A_{1,p}\left[a\left(1+\frac{C'}{|\log t|}\right)\right]t^{pk}B\big(p+1,(k-1)p\big),
\end{aligned}
$$

where B is the beta function. Combine these with (4.22), we see that the desired inequality follows by taking $A_{k,p} = A_{1,p}A_{k-1,p}B\big(p+1,(k-1)p\big)$, which gives $A_{k,p} = \frac{\Gamma(p+1)^k}{\Gamma(kp+1)}$. □

Next, we show $\mathbb{E}\left[N_{d,k,r,t}\right]^{-1} = o(1)$ as $d\to\infty$, which will be useful on the estimation of the second moment of $N_{d,k,r,t}$.

Lemma 4.8. The first moment of $N_{d,k,r,t}$ satisfies $\mathbb{E}\left[N_{d,k,r,t}\right]\to\infty$ as $d\to\infty$.

Proof. By the definition of $N_{d,k,r,t}$, the expectation equals

$$
\mathbb{E}\left[N_{d,k,r,t}\right] = \sum_{\gamma\in\mathcal{P}_{d,k,r}}\mathbb{P}(\tau(\gamma)\leqslant t) = \#\mathcal{P}_{d,k,r}\cdot\mathbb{P}(S_k\leqslant t) = \lambda_{r,k-1}\cdot\mathbb{P}(S_k\leqslant t),
$$

where S_k is the sum of k i.i.d. random variable with distribution τ_e. Applying Lemma 4.7 (applicable since $t\to 0^+$ as $d\to\infty$) and Corollary 2.10, we have

$$
\mathbb{E}\left[N_{d,k,r,t}\right] \geqslant (2r-1-\log(2r-1))^{k-1}\cdot A_{k,p}\left[at^p\left(1-\frac{C'}{|\log t|}\right)\right]^k. \tag{4.23}
$$

We will estimate the right hand side of (4.23) term by term. First, note that we may choose d large enough such that $\delta\leqslant\frac{1}{2}$. Then, for d large enough,

$$
|\log t| = \frac{\log d}{p} - \log\log d + \frac{\log((1-\delta)\cdot 2a\Gamma(p+1))}{p} \geqslant \frac{\log d}{2p}. \tag{4.24}
$$

Therefore, we have

$$
\left(1-\frac{C'}{|\log t|}\right)^n \leqslant \exp\left(-\frac{C'}{|\log t|}\cdot\left\lceil\frac{\log d}{p}\right\rceil\right) \leqslant e^{-2C'}, \tag{4.25}
$$

where the first inequality holds by the elementary inequality $(1-x)^y\leqslant\exp(-xy)$ $(x,y>$

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0), (4.24) and the second holds by $\left\lceil \frac{\log d}{p} \right\rceil \geqslant \frac{\log d}{p}$. Therefore, by (4.23) and (4.25),

$$\begin{aligned}
\mathbb{E}\left[N_{d,k,r,t}\right] &\geqslant (2r-1-\log(2r-1))^{k-1} \cdot A_{k,p}\left(at^p\right)^k e^{-2C'} \\
&\geqslant e^{-2C'} \cdot \frac{\left(at^p(2r-1-\log(2r-1))\right)^n}{2r-1-\log(2r-1)} \cdot \frac{\Gamma(p+1)^k}{\Gamma(kp+1)} \\
&= e^{-2C'} \cdot \frac{\left[a \cdot \frac{1}{1-\delta} \cdot \frac{1}{2a\Gamma(p+1)} \cdot \frac{(\log d)^p}{d} \cdot (2r-1-\log(2r-1))\right]^k}{2r-1-\log(2r-1)} \cdot \frac{\Gamma(p+1)^k}{\Gamma(kp+1)} \\
&\geqslant \frac{e^{-2C'}}{2d} \cdot \left[\frac{(\log d)^p(2r-1-\log(2r-1))}{2(1-\delta)d}\right]^{\frac{\log d}{p}} \cdot \frac{1}{2\sqrt{2\pi kp}\left(\frac{kp}{e}\right)^{kp}} &(4.26)\\
&\geqslant \frac{e^{-2C'}}{2d} \cdot \left[\frac{2r-1-\log(2r-1)}{2(1-\delta)d}\right]^{\frac{\log d}{p}} \\
&\qquad \cdot (\log d)^{\log d} \cdot \frac{d}{2\sqrt{2\pi \cdot 2\log d \cdot (kp)^{kp}}} &(4.27)\\
&= \frac{e^{-2C'}}{8\sqrt{\pi \log d}} \cdot \left[\frac{2r-1-\log(2r-1)}{2(1-\delta)d}\right]^{\frac{\log d}{p}} \cdot \frac{(\log d)^{\log d}}{(kp)^{kp}} &(4.28)
\end{aligned}$$

where (4.26) holds since the denominator $(2r-1-\log(2r-1)) \leqslant 2d$, the exponent $k = \left\lceil \frac{\log d}{p} \right\rceil \geqslant \frac{\log d}{p}$ and we set d large enough so that we have $\Gamma(kp+1) \leqslant 2\sqrt{2\pi kp}\left(\frac{kp}{e}\right)^{kp}$; (4.27) holds since $e^{kp} = e^{\left\lceil \frac{\log d}{p} \right\rceil p} \geqslant e^{\frac{\log d}{p} \cdot p} = e^{\log d} = d$ and we set d large enough so that $\sqrt{kp} = \sqrt{\left\lceil \frac{\log d}{p} \right\rceil p} \leqslant \sqrt{2 \cdot \frac{\log d}{p} \cdot p} = \sqrt{2\log d}$.

Now, we will bound the two latter terms of (4.28). First,

$$\begin{aligned}
&\left[\frac{2r-1-\log(2r-1)}{2(1-\delta)d}\right]^{\frac{\log d}{p}} &(4.29)\\
&\quad = \exp\left[\frac{\log d}{p} \cdot \log\left(\frac{2r-1-\log(2r-1)}{2(1-\delta)d}\right)\right] \\
&\quad \geqslant \exp\left[\frac{\log d}{p} \cdot \log\left(\frac{1-\frac{1}{d}\left\lfloor \frac{d}{(\log d)^{2\eta}} \right\rfloor - \frac{1+\log(2d)}{2d}}{1-\delta}\right)\right] &(4.30)\\
&\quad \geqslant \exp\left[\frac{\log d}{p} \cdot \log\left(\frac{1-\frac{\delta}{2}}{1-\delta}\right)\right], &(4.31)
\end{aligned}$$

where (4.30) holds since $r \leqslant d$ and (4.31) holds since we can set d large enough such that

$$\delta = \frac{1}{(\log d)^\eta} \geqslant \frac{4}{(\log d)^{2\eta}} \geqslant 2\left(\frac{1}{d}\left\lfloor \frac{d}{(\log d)^{2\eta}} \right\rfloor + \frac{1+\log(2d)}{2d}\right). \tag{4.32}$$

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Observe the exponent in (4.31) satisfies

$$\begin{aligned}
\log\left(\frac{1-\frac{\delta}{2}}{1-\delta}\right) &= \log\left(1+\frac{\delta}{2(1-\delta)}\right) \\
&\geqslant \frac{\delta}{2(1-\delta)} - \frac{1}{2}\left(\frac{\delta}{2(1-\delta)}\right)^2 \\
&\geqslant \frac{(\log d)^{-\eta}}{2} - \frac{1}{2}\left(\frac{(\log d)^{-\eta}}{2\left(1-\frac{1}{2}\right)}\right)^2 && (0<\delta\leqslant\tfrac{1}{2}) \\
&= \frac{1}{2}\left[(\log d)^{-\eta} - (\log d)^{-2\eta}\right] \\
&\geqslant (\log d)^{-2\eta}, && (4.33)
\end{aligned}$$

where the last inequality holds because we may set d large enough so that $(\log d)^{-\eta} \geqslant 3(\log d)^{-2\eta}$. Therefore, the left hand side of (4.29) satisfies

$$\left[\frac{2r-1-\log(2r-1)}{2(1-\delta)d}\right]^{\frac{\log d}{p}} \geqslant \exp\left(\frac{(\log d)^{1-2\eta}}{p}\right). \tag{4.34}$$

Now, for the last term in (4.28),

$$\begin{aligned}
\frac{(\log d)^{\log d}}{(np)^{np}} &\geqslant \frac{(\log d)^{\log d}}{\left[\left(\frac{\log d}{p}+1\right)p\right]^{\left(\frac{\log d}{p}+1\right)p}} \\
&= \frac{(\log d)^{\log d}}{(\log d+p)^{\log d+p}} \\
&= \left(1-\frac{p}{\log d+p}\right)^{\log d+p} \cdot \frac{1}{(\log d)^p} \\
&\geqslant \frac{e^{-(p+1)}}{(\log d)^p}, && (4.35)
\end{aligned}$$

where the last inequality holds for all d large enough since $\left(1-\frac{p}{\log d+p}\right)^{\log d+p} \to e^{-p}$ as $d \to \infty$.

Combining (4.28), (4.34) and (4.35), we then see that $\mathbb{E}\left[N_{d,k,r,t}\right] \to \infty$ as $d \to \infty$, since the right hand side of (4.34) obviously dominates the $\sqrt{\log d}$ term in (4.28) and the $(\log d)^p$ term in (4.35). □

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Finally, we are able to prove (4.18) by bounding the second moment of $N_{d,k,r,t}$ from above. The main estimation will be on the numbers of pairs of paths in $\mathcal{P}_{d,k,r}$, which we have already done in Section 2.2.

Lemma 4.9. For d large enough, the following holds.

$$\mathbb{E}\left[N_{d,k,r,t}^2\right] \leqslant 2 \cdot \mathbb{E}\left[N_{d,k,r,t}\right]^2.$$

Proof. By the definition of $N_{d,k,r,t}$,

$$\begin{aligned}\mathbb{E}\left[N_{d,k,r,t}^2\right] &= \sum_{\gamma,\gamma'\in\mathcal{P}_{d,k,r}} \mathbb{P}(\tau(\gamma) \leqslant t, \tau(\gamma') \leqslant t) \\ &= \sum_{\ell=0}^{n} \sum_{(\gamma,\gamma')\in\mathcal{P}_{d,k,r}^{(\ell)}} \mathbb{P}(\tau(\gamma) \leqslant t, \tau(\gamma') \leqslant t) \end{aligned} \tag{4.36}$$

(recall the definition of $\mathcal{P}_{d,k,r}^{(\ell)}$ from Section 2.2). For the case $\ell = 0$ (that is, γ and γ' are disjoint), $\tau(\gamma)$ and $\tau(\gamma')$ are independent, so

$$\begin{aligned}\sum_{(\gamma,\gamma')\in\mathcal{P}_{d,k,r}^{(0)}} \mathbb{P}(\tau(\gamma) \leqslant t, \tau(\gamma') \leqslant t) &= \sum_{(\gamma,\gamma')\in\mathcal{P}_{d,k,r}^{(0)}} \mathbb{P}(\tau(\gamma) \leqslant t)\mathbb{P}(\tau(\gamma') \leqslant t) \\ &= \sum_{\gamma,\gamma'\in\mathcal{P}_{d,k,r}^{(0)}} \mathbb{P}(S_k \leqslant t)^2 \\ &\leqslant (\#\mathcal{P}_{d,k,r})^2 \cdot \mathbb{P}(S_k \leqslant t)^2 = \mathbb{E}\left[N_{d,k,r,t}\right]^2, \end{aligned} \tag{4.37}$$

where S_k is the sum of n i.i.d. random variables with distribution τ_e. For the case $\ell = k$ (that is, γ and γ' are the same path),

$$\sum_{(\gamma,\gamma')\in\mathcal{P}_{d,k,r}^{(k)}} \mathbb{P}(\tau(\gamma) \leqslant t, \tau(\gamma') \leqslant t) = \sum_{\gamma\in\mathcal{P}_{d,k,r}} \mathbb{P}(\tau(\gamma) \leqslant t) = \mathbb{E}\left[N_{d,k,r,t}\right]. \tag{4.38}$$

For the rest of the cases, $1 \leqslant \ell \leqslant k-1$, if $(\gamma,\gamma') \in \mathcal{P}_{d,k,r}^{(\ell)}$,

$$\mathbb{P}(\tau(\gamma) \leqslant t, \tau(\gamma') \leqslant t)$$

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$$\leqslant \mathbb{P}\big(\tau(\gamma \setminus (\gamma \cap \gamma')) \leqslant t, \tau(\gamma') \leqslant t\big) = \mathbb{P}(S_{k-\ell} \leqslant t)\mathbb{P}(S_k \leqslant t), \tag{4.39}$$

where the equality holds by independence. By Lemma 4.7, we have

$$\begin{aligned}
\frac{\mathbb{P}(S_{k-\ell} \leqslant t)}{\mathbb{P}(S_k \leqslant t)} &\leqslant \frac{A_{k-\ell,p}\left[at^p\left(1+\frac{C'}{|\log t|}\right)\right]^{k-\ell}}{A_{k,p}\left[at^p\left(1-\frac{C'}{|\log t|}\right)\right]^{k}} \\
&\leqslant \frac{\frac{\Gamma(p+1)^{k-\ell}}{\Gamma((k-\ell)p+1)}}{\frac{\Gamma(p+1)^k}{\Gamma(kp+1)}} \cdot \frac{1}{(at^p)^\ell} \cdot \left(1 + \frac{2C'|\log t|^{-1}}{1-C'|\log t|^{-1}}\right)^k \\
&= \frac{\Gamma(kp+1)}{\Gamma((k-\ell)p+1)} \cdot \frac{1}{(\Gamma(p+1)at^p)^\ell} \cdot \exp\left[k\log\left(1+\frac{2C'}{|\log t| - C'}\right)\right] \\
&\leqslant \left(\prod_{i=1}^{\ell p}((k-\ell)p+i)\right) \cdot \frac{1}{(\Gamma(p+1)at^p)^\ell} \cdot \exp\left(k \cdot \frac{2C'}{|\log t| - C'}\right) \\
&\leqslant \left(\frac{(kp)^p}{\Gamma(p+1)at^p}\right)^\ell \cdot \exp\left(\frac{2kC'}{|\log t| - C'}\right).
\end{aligned} \tag{4.40}$$

Note that for d large enough,

$$\frac{2kC'}{|\log t| - C'} = \frac{2 \cdot \frac{2\log d}{p} \cdot C'}{\frac{\log d}{p} + \frac{\log((1-\delta)\cdot 2a\Gamma(p+1))}{p} - C'} = 2C'(1+o(1)) \quad \text{as} \quad d \to \infty. \tag{4.41}$$

Combining (4.39), (4.40), and (4.41), we have the upper bound

$$\mathbb{P}(\tau(\gamma) \leqslant t, \tau(\gamma') \leqslant t)\mathbb{1}_{\{\#(\gamma\cap\gamma')=\ell\}} \leqslant e^{2C'(1+o(1))}\mathbb{P}(S_k \leqslant t)^2 \left(\frac{(kp)^p}{\Gamma(p+1)at^p}\right)^\ell. \tag{4.42}$$

To bound (4.42) by the first moment of $N_{d,k,r,t}$, we apply Corollary 2.10 and observe

$$\begin{aligned}
\frac{(2r)^{k-1}\mathbb{P}(S_k \leqslant t)}{\mathbb{E}\left[N_{d,k,r,t}\right]} = \frac{(2r)^{k-1}}{\#\mathcal{P}_{d,k,r}} &\leqslant \left(\frac{2r}{2r-1-\log(2r-1)}\right)^{k-1} \\
&= \left(1 + \frac{1+\log(2r-1)}{2r-1-\log(2r-1)}\right)^{k-1} \\
&\leqslant \exp\left(\frac{(k-1)(1+\log(2r-1))}{2r-1-\log(2r-1)}\right).
\end{aligned} \tag{4.43}$$

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Recall the definitions of k, δ, r in (4.15), we see

$$\frac{(k-1)(1+\log(2r-1))}{2r-1-\log(2r-1)} \leqslant \frac{\frac{\log d}{p}\cdot 2\log(2d)}{2d\left(1-(\log d)^{-2\eta}\right)-1-\log(2d)} = o(1) \quad \text{as} \quad d\to\infty.$$

Together with (4.36), (4.37) and (4.38), we have

$$\begin{aligned}
\frac{\mathbb{E}\left[N_{d,k,r,t}^2\right]}{\mathbb{E}\left[N_{d,k,r,t}\right]^2} &\leqslant \frac{1}{\mathbb{E}\left[N_{d,k,r,t}\right]} + 1 + \frac{(1+o(1))e^{2C'}}{(2r)^{2(k-1)}} \sum_{\ell=1}^{k-1} \sum_{(\gamma,\gamma')\in\mathcal{P}_{d,k,r}^{(\ell)}} \left(\frac{(kp)^p}{\Gamma(p+1)at^p}\right)^{\ell} \\
&= 1+o(1)+(1+o(1))e^{2C'}\sum_{\ell=1}^{k-1}\left(\frac{(kp)^p}{\Gamma(p+1)at^p}\right)^{\ell}\frac{\#\mathcal{P}_{d,k,r}^{(\ell)}}{(2r)^{2(k-1)}} \qquad (4.44)\\
&\leqslant 1+o(1)+(1+o(1))e^{2C'}\sum_{\ell=1}^{k-1}\left(\frac{(kp)^p}{\Gamma(p+1)at^p}\right)^{\ell}\left(\frac{1}{2r}\right)^{\ell}. \qquad (4.45)
\end{aligned}$$

Note that (4.44) holds because $\mathbb{E}\left[N_{d,k,r,t}\right]^{-1} = o(1)$ as $d\to\infty$ by Lemma 4.8. Also, by the definition of $\mathcal{P}_{d,k,r}^{(\ell)}$ (recall from (2.17)), (4.45) holds by Proposition 2.12 (applicable since $r\sim d$ and $k=O(\log r)$) and combining the $\left(1+o\left(r^{-\frac{1}{2}}\right)\right)$ term with $(1+o(1))$. Recalling the definitions of k, δ, r in (4.15) and t in (4.16), we have

$$\begin{aligned}
\frac{\mathbb{E}\left[N_{d,k,r,t}^2\right]}{\mathbb{E}\left[N_{d,k,r,t}\right]^2} &\leqslant 1+o(1)+(1+o(1))e^{2C'}\sum_{\ell=1}^{k-1}\left(\frac{\left(\left\lceil\frac{\log d}{p}\right\rceil\cdot p\right)^p}{\left(d-\left\lfloor\frac{d}{(\log d)^{2\eta}}\right\rfloor\right)\cdot\frac{1}{1-\delta}\cdot\frac{(\log d)^p}{d}}\right)^{\ell}\\
&\leqslant 1+o(1)+(1+o(1))e^{2C'}\sum_{\ell=1}^{\infty}\left[\frac{1-\delta}{1-(\log d)^{-2\eta}}\cdot\left(\frac{\left\lceil\frac{\log d}{p}\right\rceil}{\frac{\log d}{p}}\right)^p\right]^{\ell}. \qquad (4.46)
\end{aligned}$$

Note that by (4.32) and (4.33), we have, for d large enough,

$$\frac{1-\delta}{1-(\log d)^{-2\eta}} \leqslant \frac{1-\delta}{1-\frac{\delta}{2}} = \exp\left[\log\left(\frac{1-\delta}{1-\frac{\delta}{2}}\right)\right] \leqslant \exp\left(-(\log d)^{-2\eta}\right)$$

and hence the summand in (4.46) satisfies

$$\begin{aligned}
\frac{1-\delta}{1-(\log d)^{-2\eta}}\cdot\left(\frac{\left\lceil\frac{\log d}{p}\right\rceil}{\frac{\log d}{p}}\right)^p &\leqslant \exp\left(-(\log d)^{-2\eta}\right)\cdot\left(1+\frac{p}{\log d}\right)^p\\
&\leqslant \exp\left(\frac{p^2}{\log d}-\frac{1}{(\log d)^{2\eta}}\right),
\end{aligned}$$

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which is less than 1 for all d large enough. Hence, the sum in (4.46) converges for d large enough, giving

$$\frac{\mathbb{E}\left[(N_{d,k,r,t})^2\right]}{\mathbb{E}\left[N_{d,k,r,t}\right]^2} \leqslant 1 + o(1) + (1+o(1))e^{2C'} \sum_{\ell=1}^{\infty} \left(e^{\frac{p^2}{\log d} - \frac{1}{(\log d)^\eta}} \right)^\ell = 1 + o(1).$$

Choosing d large enough such that the $o(1)$ term in right hand side of the equation above is less than 1, the lemma then holds as asserted. $\square$

Lemma 4.9 concludes the proof of (4.18). Now, we will prove the upper bound holds under the weaker condition (♠).

Corollary 4.10. If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p > 1$ and $a \in (0, \infty)$. Then, the time constant satisfies

$$\limsup_{d\to\infty} \frac{\mu_d(\mathbf{e}_1) d^{\frac{1}{p}}}{\log d} \leqslant \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}}.$$

Proof. Since the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$, for all $\epsilon > 0$, there is $\delta > 0$ such that

$$\mathbb{P}(\tau_e \leqslant t) \geqslant (a-\epsilon)t^p \quad \text{for all} \quad 0 \leqslant t < \delta.$$

Fix $\epsilon > 0$ and let $\delta > 0$ be given as above. Let F_ϵ be a distribution on $[0, \infty)$ such that

(i) $F_\epsilon(t) = (a-\epsilon)t^p$ for $y \in [0, \delta)$,

(ii) $F_\epsilon(t) \leqslant \mathbb{P}(\tau_e \leqslant t)$ for all $t \in \mathbb{R}$.

We may construct such a distribution F_ϵ by choosing a non-decreasing function F on $[\delta, \infty)$ with $F(t) \to 1$ as $t \to \infty$ and $F(t) \leqslant \mathbb{P}(\tau_e \leqslant t)$. Hence, using Lemma 3.13, we have the dominance

$$\mu_d(\mathbf{e}_1) \leqslant \mu_d^{F_\epsilon}(\mathbf{e}_1) \quad \text{for all} \quad d \geqslant 2.$$

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Since F_ϵ satisfies (♡) with parameter $a-\epsilon$, we have by Proposition 4.4 that

$$\limsup_{d\to\infty}\frac{\mu_d(\mathbf{e}_1)d^{\frac{1}{p}}}{\log d}\leqslant\limsup_{d\to\infty}\frac{\mu_d^{F_\epsilon}(\mathbf{e}_1)d^{\frac{1}{p}}}{\log d}\leqslant\frac{1}{(2(a-\epsilon)\Gamma(p+1))^{\frac{1}{p}}}$$

Taking $\epsilon\to 0^+$ gives us the desired result. □

One can then derive Theorem 1.4 directly by Proposition 4.1 and Proposition 4.4 for the case $a\in(0,\infty)$ and $p>1$.

4.3 A discussion of the upper bound for $p=1$

Unfortunately, the proof in Section 4.2 does not work for $p=1$ due to the restriction (4.14), which is introduced to balance the growth between δ and r. This is the problem in [2] we mentioned briefly in the introduction (to be precise, it is not possible for both (3.2) in [2] tending to ∞ and the last term in (3.4) tending to 0 as $d\to\infty$, $\delta\to 0^+$). One might argue non-rigorously that if the distribution $F_{\tau_e}\in\mathcal{M}_p(a)$ for $p=1$ and some $a\in(0,\infty)$, then we have

$$\begin{aligned}\limsup_{d\to\infty}\frac{\mu_d(\mathbf{e}_1)d}{\log d}&=\limsup_{d\to\infty}\lim_{p\to1^+}\frac{\mu_d^{F_p}d^{\frac{1}{p}}}{\log d}\\&=\lim_{p\to1^+}\limsup_{d\to\infty}\frac{\mu_d^{F_p}d^{\frac{1}{p}}}{\log d}\leqslant\lim_{p\to1^+}\frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}}=\frac{1}{2a},\end{aligned}\tag{4.47}$$

where $F_p\in\mathcal{M}_p(a)$ with parameter $p>1$ are distributions such that $F_p\Rightarrow F_{\tau_e}$ as $p\to1^+$. However, (4.47) does holds for the special case $\tau_e\sim\texttt{Exponential}(a)$ as mentioned in the introduction; see [6].

One can modify the proof in Section 4.2 to derive the bound given in (1.7). We will sketch the critical steps in the following. Let

$$k=\lceil\log d\rceil\,,\quad\delta=\frac{3}{4},\quad r=\left\lceil\frac{d}{2}\right\rceil\quad\text{and}\quad t=\frac{1}{1-\delta}\cdot\frac{1}{2a}\cdot\frac{\log d}{d}.$$

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First, note that the middle term in (4.27) satisfies

$$\left[\frac{2r-1-\log(2r-1)}{2(1-\delta)d}\right]^{\log d} \geqslant d^{\log\left(\frac{2\left\lceil\frac{d}{2}\right\rceil-(1+\log(2d))}{2\left(1-\frac{3}{4}\right)d}\right)} = d^{\log(2-o(1))},$$

which dominates the other terms in (4.27). Hence, we have $\mathbb{E}\left[N_{k,d,r,t}\right]^{-1} = o(1)$ as $d \to \infty$. For the second moment, we can still apply Proposition 2.12 since $k = O(\log r)$. Moreover, the summand in (4.46) is bounded by

$$\frac{1-\frac{3}{4}}{1-\frac{1}{2}} \cdot \frac{\lceil \log d \rceil}{\log d} = \frac{1}{2}(1+o(1))$$

and hence the sum converges. In particular, the right hand side of (4.46) can then be bounded above by the constant

$$1+o(1)+(1+o(1))e^{2C'} \cdot \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1+e^{2C'}(1+v(d)),$$

where $v(d) = o(1)$. Therefore, we have

$$\mathbb{P}\left(N_{k,d,r,t} \geqslant 1\right) \geqslant \frac{\mathbb{E}\left[N_{k,d,r,t}\right]^2}{\mathbb{E}\left[N_{k,d,r,t}^2\right]} \geqslant \frac{1}{1+e^{2C'}(1+v(d))}.$$

Similar to the proof of Proposition 4.4, we choose

$$y = \frac{2\left(1+e^{2C'}(1+v(d))\right)}{a\left(1-\frac{C'}{|\log t|}\right)} \cdot \frac{\log d}{d}.$$

Then, by the same argument, we have

$$\begin{aligned}
\mathbb{E}[s_{0,1}] &\leqslant y + t + \mathbb{E}[\tau_e] \cdot 2\left[1-\mathbb{P}\left(E_j \cap \{\tau_{\{\mathbf{0},\mathbf{e}_j\}} \leqslant y\}\right)\right]^{\frac{d}{2}} \\
&= \frac{\log d}{2ad}\left(\frac{4\left(1+e^{2C'}(1+v(d))\right)}{1-\frac{C'}{|\log t|}} + \frac{1}{1-\delta}\right) + 2\mathbb{E}[\tau_e]\left(1-\frac{2\log d}{d}\right)^{\frac{d}{2}} \\
&\leqslant \frac{\log d}{2ad}\left(\frac{4\left(1+e^{2C'}(1+v(d))\right)}{1-\frac{C'}{|\log t|}} + \frac{1}{1-\delta}\right) + 2\mathbb{E}[\tau_e]\exp\left(-\frac{2\log d}{d}\cdot\frac{d}{2}\right) \\
&= \frac{\log d}{2ad}\left(\frac{4\left(1+e^{2C'}(1+v(d))\right)}{1-\frac{C'}{|\log t|}} + \frac{1}{1-\delta} + \frac{4a\mathbb{E}[\tau_e]}{\log d}\right).
\end{aligned}$$

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Dividing $\frac{\log d}{d}$ and taking $\limsup_{d\to\infty}$, we obtain

$$\limsup_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} \leqslant \frac{4\left(1+e^{2C'}\right)+4}{2a}$$

provided τ_e satisfies ($\heartsuit$). Following the proof of Corollary 4.10, if τ_e satisfies ($\spadesuit$) with parameters $p=1$ and a, we can construct distributions F_ϵ satisfying ($\heartsuit$) with parameters $p=1$ and $a-\epsilon$ (and for all $C>0$) and dominating F_{τ_e} for all $\epsilon>0$. Therefore,

$$\limsup_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} \leqslant \limsup_{d\to\infty} \frac{\mu_d^{F_\epsilon}(\mathbf{e}_1)d}{\log d} \leqslant \frac{4\left(1+e^{2C'}\right)+4}{2(a-\epsilon)}.$$

Taking $C'\to 0^+$ (that is, $C\to 0^+$) and $\epsilon\to 0^+$ shows that

$$\limsup_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d}{\log d} \leqslant \frac{6}{a}.$$

4.4 The cases $a=\infty$ and $a=0$

In this section, we will prove Theorem 1.4 for the cases $a=\infty$ or $a=0$. First, we consider the case $a=\infty$.

Proof. Note that if $\mathbb{P}(\tau_e=0)\geqslant p_c(d)>0$, then ($\spadesuit$) holds with $a=\infty$. By Theorem 1.2, we have $\mu_d(\mathbf{e}_1)=0$ in this case. Therefore,

$$\limsup_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d^{\frac{1}{p}}}{\log d} = 0 = \frac{1}{(2a\Gamma(p+1))^{\frac{1}{p}}}$$

as asserted. Suppose now $\mathbb{P}(\tau_e=0)<p_c$. Note that $\frac{\mathbb{P}(\tau_e\leqslant t)}{t^p}\to\infty$ as $t\to 0^+$ implies that for any $M>0$, there exists $\delta>0$ such that

$$\mathbb{P}(\tau_e\leqslant t)\geqslant Mt^p \quad \text{for all} \quad 0\leqslant t<\delta.$$

Fix $M>0$ and let $\delta>0$ be given as above. Let F_M be a distribution on $[0,\infty)$ such that

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(i) $F_M(t) = Mt^{p+1}$ for $t \in [0, \delta)$,

(ii) $F_M(t) \leqslant \mathbb{P}(\tau_e \leqslant t)$ for all $t \in \mathbb{R}$.

Hence, using Lemma 3.13, we have

$$\mu_d(\mathbf{e}_1) \leqslant \mu_d^{F_M}(\mathbf{e}_1) \quad \text{for all} \quad d \geqslant 2.$$

Since $F_M \in \mathcal{M}_p(M)$, by Theorem 1.4,

$$\limsup_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d^{\frac{1}{p}}}{\log d} \leqslant \lim_{d\to\infty} \frac{\mu_d^{F_M}(\mathbf{e}_1)d^{\frac{1}{p}}}{\log d} = \frac{1}{(2M\Gamma(p+1))^{\frac{1}{p}}}.$$

Taking $M \to \infty$ gives us the desired result. $\square$

We now consider the case $a = 0$.

Proof. We must have $\mathbb{P}(\tau_e = 0) < p_c(d)$ (otherwise $a = \infty$). Note that $\frac{\mathbb{P}(\tau_e \leqslant t)}{t^p} \to 0$ as $t \to 0^+$ implies that for all $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ such that

$$\mathbb{P}(\tau_e \leqslant t) \leqslant \epsilon t^p \quad \text{for all} \quad 0 \leqslant t < \delta.$$

Fix $\epsilon > 0$ and let $\delta > 0$ be given as above. Let F_ϵ be a distribution on $[0, \infty)$ such that

(i) $F_\epsilon(t) = \epsilon t^p$ for $t \in [0, \delta)$,

(ii) $F_\epsilon(t) \geqslant \mathbb{P}(\tau_e \leqslant t)$ for all $t \in \mathbb{R}$.

Again, by Lemma 3.13, $\mu_d(\mathbf{e}_1) \geqslant \mu_d^{F_\epsilon}(\mathbf{e}_1)$ for $d \geqslant 2$. Since $F_\epsilon \in \mathcal{M}_p(\epsilon)$, by Theorem 1.4,

$$\liminf_{d\to\infty} \frac{\mu_d(\mathbf{e}_1)d^{\frac{1}{p}}}{\log d} \geqslant \lim_{d\to\infty} \frac{\mu_d^{F_\epsilon}(\mathbf{e}_1)d^{\frac{1}{p}}}{\log d} = \frac{1}{(2\epsilon\Gamma(p+1))^{\frac{1}{p}}}.$$

Taking $\epsilon \to 0^+$ gives us the desired result. $\square$

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4.5 A scaling relation

In this section, we will derive an interesting asymptotic relationship between time constants and the convolution of the underlying edge weight distribution. For two distributions F, G on $\mathbb{R}$, we define

$$(F * G)(t) = \int_{\mathbb{R}} F(t-x)\,\mathrm{d}G(x) = \int_{\mathbb{R}} G(t-x)\,\mathrm{d}F(x)$$

as usual. The main result follows from the following simple observation.

Lemma 4.11. If $F \in \mathcal{M}_p(a)$, $G \in \mathcal{M}_q(b)$ for some $p, q \geqslant 1$ and $a, b \in (0, \infty)$, then

$$F * G \in \mathcal{M}_{p+q}\left(ab \cdot \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)}\right).$$

Proof. This follows from direct computation. Fix $\epsilon > 0$, and choose $\delta > 0$ such that

$$|F(t) - at^p| < \epsilon t^p \quad \text{and} \quad |G(t) - bt^q| < \epsilon t^q$$

for all $0 < t < \delta$. Fix $0 < t < \delta$, then since F and G are supported on $[0, \infty)$,

$$\begin{aligned}
(F * G)(t) &= \int_0^t F(t-x)\,\mathrm{d}G(x) \\
&\leqslant \int_0^t (a+\epsilon)(t-x)^p\,\mathrm{d}(b+\epsilon)x^q \\
&= (a+\epsilon)(b+\epsilon)q \int_0^t (t-x)^p x^{q-1}\,\mathrm{d}x \\
&= (a+\epsilon)(b+\epsilon)qB(p+1, q) = (a+\epsilon)(b+\epsilon) \cdot \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)}.
\end{aligned}$$

One can derive a similar lower bound by the same method. Therefore,

$$\lim_{t \to 0^+} \frac{(F * G)(t)}{t^{p+q}} = ab \cdot \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)}$$

as asserted in the lemma. □

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Corollary 4.12. If $F_i \in \mathcal{M}_{p_i}(a_i)$ for some $p_i > 1$ and $a_i \in (0, \infty)$ for all $1 \leqslant i \leqslant n$, then

$$\mu_d^{F_1 * \cdots * F_n}(\mathbf{e}_1) \sim \left[(2d)^{n-1} \prod_{i=1}^{n} \mu_d^{F_i}(\mathbf{e}_1)^{p_i} \right]^{\frac{1}{p_1 + \cdots + p_n}} \quad \text{as} \quad d \to \infty.$$

In particular, if $F \in \mathcal{M}_p(a)$ for some $a \in (0, \infty)$ and $p > 1$, then

$$\mu_d^{F^{*n}}(\mathbf{e}_1) \sim (2d)^{\frac{1}{p}\left(1 - \frac{1}{n}\right)} \mu_d^F(\mathbf{e}_1) \quad \text{as} \quad d \to \infty.$$

Proof. By Theorem 1.4, we have for all $1 \leqslant i \leqslant n$,

$$\mu_d^{F_i}(\mathbf{e}_1) \sim \frac{1}{(2a_i \Gamma(p_i + 1))^{\frac{1}{p_i}}} \cdot \frac{\log d}{d^{\frac{1}{p_i}}} \quad \text{or that} \quad \mu_d^{F_i}(\mathbf{e}_1)^{p_i} \sim \frac{1}{2a_i \Gamma(p_i + 1)} \cdot \frac{(\log d)^{p_i}}{d}$$

as $d \to \infty$. By applying Lemma 4.11 iteratively, we obtain

$$F_1 * \cdots * F_n \in \mathcal{M}_{p_1 + \cdots + p_n}\left(a_1 \cdots a_n \cdot \frac{\Gamma(p_1 + 1) \cdots \Gamma(p_n + 1)}{\Gamma(p_1 + \cdots + p_n + 1)} \right).$$

Again, by Theorem 1.4, we then have

$$\begin{aligned}
&\mu_d^{F_1 * \cdots * F_n}(\mathbf{e}_1) \\
&\sim \left(2a_1 \cdots a_n \cdot \frac{\Gamma(p_1 + 1) \cdots \Gamma(p_n + 1)}{\Gamma(p_1 + \cdots + p_n + 1)} \cdot \Gamma(p_1 + \cdots + p_n + 1) \right)^{-\frac{1}{p_1 + \cdots + p_n}} \cdot \frac{\log d}{d^{\frac{1}{p_1 + \cdots + p_n}}} \\
&= \left(\frac{1}{2a_1 \cdots a_n \cdot \Gamma(p_1 + 1) \cdots \Gamma(p_n + 1)} \cdot \frac{(\log d)^{p_1 + \cdots + p_n}}{d} \right)^{\frac{1}{p_1 + \cdots + p_n}} \\
&\sim \left[(2d)^{n-1} \prod_{i=1}^{n} \mu_d^{F_i}(\mathbf{e}_1)^{p_i} \right]^{\frac{1}{p_1 + \cdots + p_n}}
\end{aligned}$$

as asserted. The second assertion can be shown by taking $F_1 = \cdots = F_n$. $\square$

4.6 Lower bound in the diagonal direction

In this section, we will show an asymptotic lower bound for $\mu_d(\mathbf{1})$. The method is quite similar to the one in Section 4.1. However, instead of considering passage time from

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$\mathbf{0}$ to some fixed diagonal plane, we "rescale" the plane a bit to derive a better bound.

Recall $J_n^{(\nu)}$ from Section 2.3. Observe that $d^{\nu-1}\mathbf{1} \in J_1^{(\nu)}$, we may show as in the proof of Corollary 3.9 that[1]

$$\frac{\mu_d(\mathbf{1})}{d^{1-\nu}} = \lim_{n\to\infty} \frac{\tau(\mathbf{0}, d^{\nu-1}\mathbf{1})}{n} = \lim_{n\to\infty} \frac{\tau\left(\mathbf{0}, J_n^{(\nu)}\right)}{n} \quad \text{almost surely and in } \mathcal{L}^1. \tag{4.48}$$

Proposition 4.13. If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p \geqslant 1$ and $a \in (0,\infty)$, then

$$\liminf_{d\to\infty} \frac{\mu_d(\mathbf{1})}{d^{1-\frac{1}{p}}} \geqslant p\left(\frac{\alpha_p^{p-1}\sqrt{\alpha_p^2-1}}{2a\Gamma(p+1)}\right)^{\frac{1}{p}},$$

where $\alpha_p > 0$ is the positive solution of $\coth(p\alpha) = \alpha$.

Proof. Fix $0 < \nu < \frac{1}{p}$ and $\delta = d^{-\frac{\nu}{2}}$, set

$$t = (1-\delta)\cdot p\left(\frac{\alpha_p^{p-1}\sqrt{\alpha_p^2-1}}{2a\Gamma(p+1)}\right)^{\frac{1}{p}} \cdot d^{\nu-\frac{1}{p}}.$$

Note that $t = o(1)$ as $d \to \infty$. By Proposition 2.14 and Lemma 4.3 (with $k \geqslant \lfloor nd^\nu \rfloor \geqslant n$ and choosing d large enough such that $t < t_0$), with $\alpha = \frac{k}{\lfloor nd^\nu \rfloor} \geqslant 1$,

$$\begin{aligned}
&\mathbb{P}\left(\tau\left(\mathbf{0}, J_n^{(\nu)}\right) \leqslant nt\right) \\
&\quad\leqslant \mathbb{P}\left(\text{there exist } k \geqslant \lfloor nd^\nu \rfloor \text{ and } \gamma \in \mathcal{D}_{d,k,n}^{(\nu)} \text{ such that } \tau(\gamma) \leqslant nt\right) \\
&\quad= \sum_{k=\lfloor nd^\nu \rfloor}^{\infty} \#\mathcal{D}_{d,k,n}^{(\nu)} \cdot \mathbb{P}(S_k \leqslant nt) \\
&\quad\leqslant \sum_{k=\lfloor nd^\nu \rfloor}^{\infty} \left(\frac{2d\alpha}{(\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}}\right)^k \cdot \left[(1+\delta)\Gamma(p+1)a\left(\frac{ent}{pk}\right)^p\right]^k \\
&\quad= \sum_{k=\lfloor nd^\nu \rfloor}^{\infty} \left[\frac{(1+\delta)(1-\delta)^p e^p \alpha_p^{p-1}\sqrt{\alpha_p^2-1}}{(\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}\alpha^{p-1}} \cdot \frac{nd^\nu}{\lfloor nd^\nu \rfloor}\right]^k.
\end{aligned} \tag{4.49}$$

[1]The definition of diagonal constant μ^* in [10] is defined to be the right hand side of (4.48) for $\nu = 0$; μ^* in [2] and [3] is defined to be the right hand side of (4.48) for $\nu = \frac{1}{2}$ and μ^* in [13] is defined to be the right hand side of (4.48) for $\nu = 1$.

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Consider the function $f_p(\alpha) = (\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}\alpha^{p-1}$ defined on $\alpha \in [1,\infty)$ (where we extend $f_p(\alpha)$ to $\alpha = 1$ continuously). Observe that

$$f_p'(\alpha) = \frac{f_p(\alpha)}{2\alpha^2}\left[2\alpha p - \log\left(\frac{\alpha+1}{\alpha-1}\right)\right] \overset{\text{set}}{=} 0 \qquad \Longleftrightarrow \qquad \alpha = \coth(p\alpha).$$

Let $\alpha_p > 0$ be the positive solution of $\coth(p\alpha) = \alpha$, we see that $f_p(\alpha)$ achieves its minimum at $\alpha = \alpha_p$ with

$$f_p(\alpha_p) = e^p \alpha_p^{p-1}\sqrt{\alpha_p^2 - 1}.$$

Therefore, (4.49) is bounded by

$$\begin{aligned}
\mathbb{P}\left(\tau\left(0, J_n^{(\nu)}\right) \leqslant nt\right) &\leqslant \sum_{k=\lfloor nd^\nu \rfloor}^{\infty}\left[\frac{(1-\delta)(1+\delta)^p e^p \alpha_p^{p-1}\sqrt{\alpha_p^2-1}}{(\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}\alpha^p} \cdot \frac{nd^\nu}{\lfloor nd^\nu \rfloor}\right]^k \\
&\leqslant \sum_{k=\lfloor nd^\nu \rfloor}^{\infty}\left(\frac{1-\delta^2}{1-\frac{1}{nd^\nu}}\right)^k \\
&= \sum_{k=\lfloor nd^\nu \rfloor}^{\infty}\left(\frac{1-\delta^2}{1-\frac{\delta^2}{n}}\right)^k = \frac{\delta^2(n-1)}{n-\delta^2}\left(\frac{1-\delta^2}{1-\frac{\delta^2}{n}}\right)^{\lfloor nd^\nu \rfloor},
\end{aligned}$$

which is summable in n, and hence

$$\liminf_{n\to\infty}\frac{\tau\left(\mathbf{0}, J_n^{(\nu)}\right)}{n} \geqslant t = (1-\delta)\cdot p\left(\frac{\alpha_p^{p-1}\sqrt{\alpha_p^2-1}}{2a\Gamma(p+1)}\right)^{\frac{1}{p}} \cdot d^{\nu-\frac{1}{p}} \quad \text{almost surely}$$

by the (first) Borel-Cantelli lemma. Thus, by (4.48),

$$\frac{\mu_d(\mathbf{1})}{d^{1-\nu}} \geqslant (1-\delta)\cdot p\left(\frac{\alpha_p^{p-1}\sqrt{\alpha_p^2-1}}{2a\Gamma(p+1)}\right)^{\frac{1}{p}} \cdot d^{\nu-\frac{1}{p}}.$$

By dividing $d^{\nu-\frac{1}{p}}$, taking $\liminf_{d\to\infty}$, we derive the desired inequality in Proposition 4.13. $\square$

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4.7 Upper bound in the diagonal direction

In the this section, we will prove a loose upper bound for the diagonal time constant. The order for the lower bound actually matches the one for the upper bound, but finding the exact limit seems to be challenging, and we will leave it to our future work.

In Appendix C, we will present the result from [13], which shows that the lower bound given in Proposition 4.13 is in fact sharp for $p = 1$.

Proposition 4.14. If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p \geqslant 1$ and $a \in (0, \infty)$, then

$$\limsup_{d\to\infty} \frac{\mu_d(\mathbf{1})}{d^{1-\frac{1}{p}}} \leqslant \frac{1}{a^{\frac{1}{p}}}.$$

Proof. Recall the definition of $J_n^{(\nu)}$ from the previous section, we will choose $\nu = 1$ in this proof. Let γ be a (random) path of length $\lfloor nd^{\nu} \rfloor = nd$ from $\mathbf{0}$ to $J_n^{(1)}$, defined by the following greedy algorithm:

- $\gamma = (\mathbf{s}_0, \dots, \mathbf{s}_{nd})$, where $\mathbf{s}_0 = \mathbf{0}$ and $\mathbf{s}_{i+1} - \mathbf{s}_i \in \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$, and

- the passage time after each $\mathbf{s}_i$ is chosen to be the minimum one among the d directions in $\{\mathbf{e}_1, \dots, \mathbf{e}_j\}$, that is,

$$\tau_{\{\mathbf{s}_i, \mathbf{s}_{i+1}\}} = \min_{1\leqslant j\leqslant d} \tau_{\{\mathbf{s}_i, \mathbf{s}_i + \mathbf{e}_j\}} \quad \text{for all} \quad 0 \leqslant i \leqslant nd - 1.$$

Define $Y = \min\{\tau_1, \dots, \tau_d\}$, where $\tau_1, \dots, \tau_d$ are independent copies of τ_e. We then have

$$\mathbb{E}[\tau(\gamma)] = nd \cdot \mathbb{E}[Y]. \tag{4.50}$$

By our assumption (♠), there are $\epsilon, \delta > 0$ such that

$$\mathbb{P}\left(\tau_e > t^{\frac{1}{p}}\right) = 1 - \mathbb{P}\left(\tau_e \leqslant t^{\frac{1}{p}}\right) \leqslant 1 - at(1 - \epsilon) \quad \text{for all} \quad 0 \leqslant t < \delta$$

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and $\mathbb{P}\left(\tau_e > \delta^{\frac{1}{p}}\right) < 1$. Choose $M > 0$ such that $\mathbb{E}[Y] < M^{\frac{1}{p}}$. Then

$$\begin{aligned}
\mathbb{E}\left[Y^p\right] &= \int_0^\infty \mathbb{P}\left(Y > t^{\frac{1}{p}}\right) \mathrm{d}t \\
&\leqslant \int_0^\delta [1 - at(1-\epsilon)]^d \, \mathrm{d}t + \int_\delta^M \mathbb{P}\left(Y > \delta^{\frac{1}{p}}\right)^d \mathrm{d}t + \int_M^\infty \left(\frac{\mathbb{E}[Y]}{t^{\frac{1}{p}}}\right)^d \mathrm{d}t \\
&\leqslant \frac{1}{d+1} \cdot \frac{1}{a(1-\epsilon)} + (M-\delta)\mathbb{P}\left(Y > \delta^{\frac{1}{p}}\right)^d + \left(\frac{\mathbb{E}[Y]}{M^{\frac{1}{p}}}\right)^d \frac{pM}{d-p} \\
&\leqslant \frac{1}{ad}\left(\frac{1}{1-\epsilon} + \epsilon\right)
\end{aligned}$$

for d large enough. Using Hölder's inequality, (4.48) and (4.50), we can then deduce

$$\begin{aligned}
\mu_d(\mathbf{1}) &= \lim_{n\to\infty} \frac{\mathbb{E}\left[\tau\left(\mathbf{0}, J_n^{(1)}\right)\right]}{n} \\
&\leqslant \lim_{n\to\infty} \frac{\mathbb{E}\left[\tau(\gamma)\right]}{n} = d \cdot \mathbb{E}[Y] \leqslant d \cdot \mathbb{E}\left[Y^p\right]^{\frac{1}{p}} \leqslant d^{1-\frac{1}{p}} \left[\frac{1}{a}\left(\frac{1}{1-\epsilon} + \epsilon\right)\right]^{\frac{1}{p}}.
\end{aligned}$$

We can obtain the desired upper bound in Proposition 4.14 by dividing $d^{1-\frac{1}{p}}$, taking $\limsup_{d\to\infty}$ and letting $\epsilon \to 0^+$. $\square$

One can then derive Theorem 1.6 directly by Proposition 4.13 and Proposition 4.14. The cases for $a = 0$ and $a = \infty$ can be argued similarly as in Section 4.4.

4.8 Application to limit shape

By Theorem 1.4 (and the discussion in Section 4.3) and Theorem 1.6, we see that

$$\mu_d(\mathbf{e}_1) \asymp \frac{\log d}{d^{\frac{1}{p}}} \quad \text{and} \quad \mu_d(\mathbf{1}) \asymp d^{1-\frac{1}{p}} \tag{4.51}$$

if $F_{\tau_e} \in \mathcal{M}_p(a)$ for some $p \geqslant 1$ and $a \in (0, \infty)$. One can then use (4.51) to reject several possible candidates for $\mathcal{B}_d$ when d is sufficiently large.

Proof. (of Theorem 1.7) By convexity of $\mathcal{B}_d$, we have $\ell^1 \subseteq \mathcal{B}_d \subseteq \ell^\infty$. First, we show

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$\mathcal{B}_d \neq \ell^q$ for $1 < q < \infty$ if d is sufficiently large. Observe that the limit shape $\mathcal{B}_d$ intersects the first coordinate axis $L_1 = \{t\mathbf{e}_1 \mid t \in \mathbb{R}\}$ at $\mu_d(\mathbf{e}_1)^{-1}\mathbf{e}_1$, which has ℓ^q-norm

$$\left\| \frac{\mathbf{e}_1}{\mu_d(\mathbf{e}_1)} \right\|_q = \frac{1}{\mu_d(\mathbf{e}_1)} \asymp \frac{d^{\frac{1}{p}}}{\log d} \tag{4.52}$$

for d large enough. Similarly, $\mathcal{B}_d$ intersects the diagonal line $L_{\text{diag}} = \{t\mathbf{1} \mid t \in \mathbb{R}\}$ at $\mu_d(\mathbf{1})^{-1}\mathbf{1}$, which has ℓ^q-norm

$$\left\| \frac{\mathbf{1}}{\mu_d(\mathbf{1})} \right\|_q = \frac{1}{\mu_d(\mathbf{1})} \left(\sum_{i=1}^{d} 1^q \right)^{\frac{1}{q}} = \frac{d^{\frac{1}{q}}}{\mu_d(\mathbf{1})} \asymp d^{\frac{1}{p}+\frac{1}{q}-1} \tag{4.53}$$

for d large enough. Comparing (4.52) and (4.53), we see that

$$\left\| \frac{\mathbf{e}_1}{\mu_d(\mathbf{e}_1)} \right\|_q > \left\| \frac{\mathbf{1}}{\mu_d(\mathbf{1})} \right\|_q \tag{4.54}$$

for d large enough and therefore $\mathcal{B}_d \neq \ell^q$ for $1 < q < \infty$ (otherwise the quantities in (4.54) should coincide).

For the case $q = \infty$, we have

$$\left\| \frac{\mathbf{e}_1}{\mu_d(\mathbf{e}_1)} \right\|_\infty = \frac{1}{\mu_d(\mathbf{e}_1)} \asymp \frac{d^{\frac{1}{p}}}{\log d} > d^{\frac{1}{p}-1} \asymp \frac{1}{\mu_d(\mathbf{1})} = \left\| \frac{\mathbf{1}}{\mu_d(\mathbf{1})} \right\|_\infty$$

for d large enough. Therefore, $\mathcal{B}_d \subsetneq \ell^\infty$.

Finally, for the case $q = 1$, we have

$$\left\| \frac{\mathbf{e}_1}{\mu_d(\mathbf{e}_1)} \right\|_1 = \frac{1}{\mu_d(\mathbf{e}_1)} \asymp \frac{d^{\frac{1}{p}}}{\log d} < d \cdot d^{\frac{1}{p}-1} \asymp \frac{d}{\mu_d(\mathbf{1})} = \left\| \frac{\mathbf{1}}{\mu_d(\mathbf{1})} \right\|_1$$

for d large enough. Therefore, $\ell^1 \subsetneq \mathcal{B}_d$ and our conclusion holds. □

Figure 4.1 shows the relationship between $\mathcal{B}_d$ and ℓ^q for d large enough. The shape of $\mathcal{B}_d$ is probably not accurate but one should focus on the difference between the $\mathbf{e}_1$- and $\mathbf{1}$-direction.

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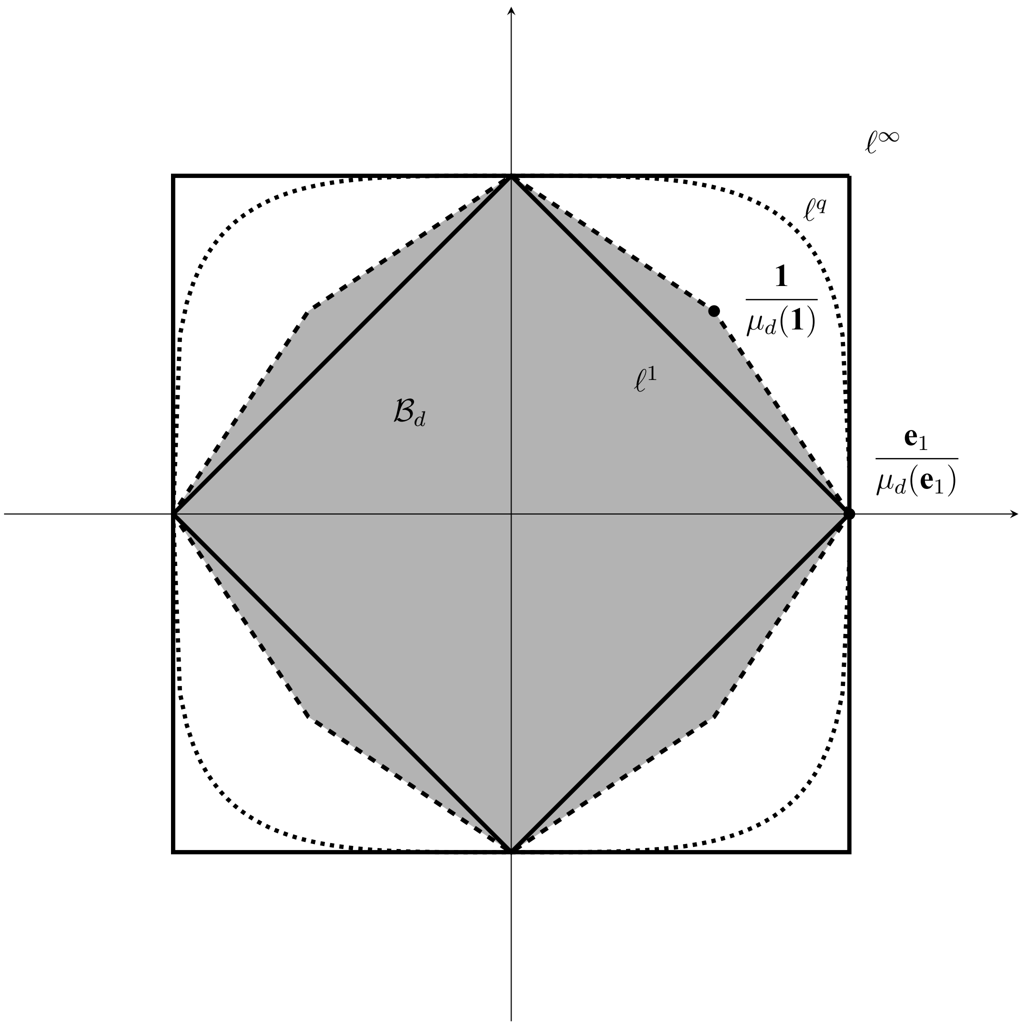

Figure 4.1: Relationship between limit shape $\mathcal{B}_d$ (the colored one) and ℓ^q for d large enough

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Appendix A

Simple symmetric random walk

Let $X_0 = 0$ and $\{X_i\}_{i=1}^{\infty}$ be i.i.d. $\texttt{Bernoulli}\left(\frac{1}{2}\right)$ random variables. Define the random walk

$$S_k = \sum_{i=0}^{k} X_i \quad \text{for all} \quad k \in \mathbb{N}_0.$$

First, we consider the distribution of the maximum of the random walk.

Lemma A.1 ([14], Theorem 2.4)**.** For $k, n \in \mathbb{N}_0$ and $n \leqslant k$,

$$\mathbb{P}(M_k = n) = \frac{1}{2^k}\binom{k}{\lfloor \frac{k-n}{2} \rfloor} \quad \text{where} \quad M_k = \max_{0 \leqslant i \leqslant k} S_i.$$

Proof. Define $a_{k,n} = \mathbb{P}(M_k = n)$ for all $k, n \in \mathbb{N} \cup \{0\}$. Then, for $k, n \geqslant 1$,

$$\begin{aligned} a_{k,n} &= \mathbb{E}[\mathbb{P}(M_k = n \mid S_1)] \\ &= \frac{\mathbb{P}(M_k = n \mid S_1 = 1) + \mathbb{P}(M_k = n \mid S_1 = -1)}{2} \\ &= \frac{\mathbb{P}(M_{k-1} = n-1) + \mathbb{P}(M_{k-1} = n+1)}{2} = \frac{1}{2}(a_{k-1,n-1} + a_{k-1,n+1}). \end{aligned}$$

Similarly, for $n = 0$, we may compute

$$\begin{aligned} a_{k,0} &= \frac{1}{2}\mathbb{P}(M_k = 0 \mid S_1 = -1) \\ &= \frac{\mathbb{P}(M_{k-1} = 0) + \mathbb{P}(M_{k-1} = 1)}{2} = \frac{1}{2}(p_{k-1,0} + p_{k-1,1}). \end{aligned}$$

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Together with the initial condition $a_{k,k} = 2^{-k}$ for all $k \in \mathbb{N}_0$ and induction, we can prove the desired formula. $\square$

Definition A.2 (First hitting time). The **first hitting time** of $n \in \mathbb{N}$ by S_k is

$$\zeta_n = \inf\{k \in \mathbb{N} \mid S_k = n\},$$

where we use the convention $\inf \varnothing = \infty$.

Lemma A.3 ([14], Theorem 9.1). For $k, n \in \mathbb{N}$ and $n \leqslant k$,

$$\mathbb{P}(\zeta_n = k) = \frac{n}{k}\binom{k}{\frac{k+n}{2}}\frac{1}{2^k}$$

if $k \equiv n \pmod 2$ and $\mathbb{P}(\tau_n = k) = 0$ if $k \not\equiv n \pmod 2$.

Proof. Note that for all $n \in \mathbb{N}$ and $n \leqslant k$,

$$\mathbb{P}(\zeta_n \geqslant k) = \mathbb{P}(M_{k-1} \leqslant n-1) = \sum_{j=0}^{n-1} \frac{1}{2^{k-1}}\binom{k-1}{\lfloor \frac{k-1-j}{2} \rfloor}$$

by Lemma A.1. Suppose $k \equiv n \pmod 2$. Then

$$\begin{aligned}
\mathbb{P}(\zeta_n = k) &= \mathbb{P}(\zeta_n \geqslant k) - \mathbb{P}(\zeta_n \geqslant k+1) \\
&= \sum_{j=0}^{n-1} \frac{1}{2^{k-1}}\binom{k-1}{\lfloor \frac{k-1-j}{2} \rfloor} - \sum_{j=0}^{n-1} \frac{1}{2^k}\binom{k}{\lfloor \frac{k-j}{2} \rfloor} \\
&= \frac{1}{2^k} \sum_{j \equiv k \pmod 2} \left[\frac{2(k-1)!}{\left(\frac{k-j}{2}-1\right)!\left(\frac{k+j}{2}\right)!} - \frac{k!}{\left(\frac{k-j}{2}\right)!\left(\frac{k+j}{2}\right)!} \right] \\
&\quad + \frac{1}{2^k} \sum_{j \not\equiv k \pmod 2} \left[\frac{2(k-1)!}{\left(\frac{k-j-1}{2}\right)!\left(\frac{k+j-1}{2}\right)!} - \frac{k!}{\left(\frac{k-j-1}{2}\right)!\left(\frac{k+j+1}{2}\right)!} \right] \\
&= \frac{1}{k2^k}\left[- \sum_{j \equiv k \pmod 2} j\binom{k}{\frac{k+j}{2}} + \sum_{j \not\equiv k \pmod 2} (j+1)\binom{k}{\frac{k+j+1}{2}} \right] \\
&= \frac{n}{k}\binom{k}{\frac{k+n}{2}}\frac{1}{2^k},
\end{aligned}$$

since $k \equiv n \pmod 2$. For $k \not\equiv n \pmod 2$, the probability is obviously 0. $\square$

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Appendix B

Kingman's subadditve ergodic theorem

Theorem B.1 (Kingman's subadditive ergodic theorem; [11])**.** Let $(X_{m,n})_{0\leqslant m<n}$ be a family of random variables that satisfies the following conditions:

(K1) $X_{0,n} \leqslant X_{0,m} + X_{m,n}$ for all $0 < m < n$.

(K2) For all $m \in \mathbb{N}_0$, we have

$$(X_{m,m+k})_{k\in\mathbb{N}} \stackrel{\mathrm{d}}{=} (X_{m+1,m+k+1})_{k\in\mathbb{N}}.$$

(K3) For each $k \in \mathbb{N}$, the sequence $(X_{nk,(n+1)k})_{n\in\mathbb{N}}$ is stationary.

(K4) $\mathbb{E}\left[X_{0,1}^{+}\right] < \infty$ and $\mathbb{E}[X_{0,n}] > -cn$ for some finite constant c.

Then

$$\lim_{n\to\infty} \frac{X_{0,n}}{n} \quad \text{exists} \quad \text{almost surely and in } \mathcal{L}^1. \tag{B.1}$$

Furthermore, if the stationary sequences in (K3) are also ergodic, then the limit in (B.1) is constant almost surely and is equal to

$$\lim_{n\to\infty} \frac{X_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}[X_{0,n}]}{n} = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}[X_{0,n}]}{n}. \tag{B.2}$$

Proof. The proof provided here is modified from the one in [12]. Observe that by condi-

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tions (K1) and (K2), we have for all $m \in \mathbb{N}$,

$$\mathbb{E}[X_{0,n}^{+}] \leqslant \sum_{i=1}^{n} \mathbb{E}[X_{i-1,i}^{+}] = \sum_{i=1}^{n} \mathbb{E}[X_{0,1}^{+}] = n\mathbb{E}[X_{0,1}^{+}]. \tag{B.3}$$

Therefore, for all $n \in \mathbb{N}$,

$$\mathbb{E}[|X_{0,n}|] = 2\mathbb{E}[X_{0,n}^{+}] - \mathbb{E}[X_{0,n}] \leqslant 2n\mathbb{E}[X_{0,1}^{+}] + cn < \infty.$$

Hence, $X_{m,n} \in \mathcal{L}^1$ for all $0 \leqslant m < n$.

First, we show that

$$\lim_{n\to\infty} \frac{\mathbb{E}[X_{0,n}]}{n} = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}[X_{0,n}]}{n} = \gamma \in (-\infty, \infty). \tag{B.4}$$

Note that by conditions (K1) and (K2), we have

$$\mathbb{E}[X_{0,m+n}] \leqslant \mathbb{E}[X_{0,m}] + \mathbb{E}[X_{m,m+n}] = \mathbb{E}[X_{0,m}] + \mathbb{E}[X_{0,n}]$$

and hence the convergence in (B.4) follows directly from Fekete's lemma. Moreover, $\gamma \leqslant \mathbb{E}[X_{0,1}] < \infty$ and $\gamma \geqslant -c > -\infty$ by condition (K4).

Next, we show that if $\overline{X} = \limsup_{n\to\infty} \frac{X_{0,n}}{n}$, then $\mathbb{E}\left[\overline{X}\right] \leqslant \gamma$ and that if the sequence in (K3) is ergodic, then $\overline{X} \leqslant \gamma$ almost surely. For all $m \in \mathbb{N}$, and $n \in \mathbb{N}$, there is a unique $q_n \in \mathbb{N}_0$ and $0 \leqslant r_n < m$ such that $n = mq_n + r_n$. By applying (K1) iteratively, we have $X_{0,n} \leqslant X_{0,m} + X_{m,2m} + \cdots + X_{q_n m,n}$ (if $q_n m = n$, then we omit the last term). Dividing by n and taking lim sup, we obtain

$$\overline{X} \leqslant \limsup_{n\to\infty} \frac{q_n}{n} \cdot \frac{1}{q_n} \sum_{i=1}^{q_n} X_{(i-1)m,im} + \limsup_{n\to\infty} \frac{X_{q_n m,n}}{n} \tag{B.5}$$

Note that by (K3) and Birkoff's ergodic theorem, we have

$$\limsup_{n\to\infty} \frac{q_n}{n} \cdot \frac{1}{q_n} \sum_{i=1}^{q_n} X_{(i-1)m,im} = \frac{\mathbb{E}\left[X_{0,m} \mid \mathcal{I}_m\right]}{m} \quad \text{almost surely,}$$

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where $\mathcal{I}_m$ is the (shift-) invariant σ-algebra for the sequence $(X_{nm,(n+1)m})_{n\in\mathbb{N}}$ (if the sequence is ergodic, then $\mathcal{I}_m$ is trivial and $\mathbb{E}\left[X_{0,m} \mid \mathcal{I}_m\right] = \mathbb{E}[X_{0,m}]$). By (K1) and (K2), we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(X_{q_n m,n} > n\epsilon\right) = \sum_{n=1}^{\infty} \mathbb{P}(X_{0,r_n} > n\epsilon) \leqslant \sum_{n=1}^{\infty} \left(\sum_{r=1}^{m-1} \mathbb{P}(X_{0,r} > n\epsilon)\right) < \infty$$

since $\mathbb{E}\left[X_{0,r}^+\right] < \infty$ for all $1 \leqslant r \leqslant m-1$. We may then deduce from (first-) Borel-Cantelli lemma that the second term in (B.5) vanishes almost surely. Therefore, for all $m \in \mathbb{N}$,

$$\mathbb{E}[\overline{X}] \leqslant \mathbb{E}\left[\frac{\mathbb{E}\left[X_{0,m} \mid \mathcal{I}_m\right]}{m}\right] = \frac{\mathbb{E}[X_{0,m}]}{m}$$

and our assertion follows by taking infimum over all $m \in \mathbb{N}$.

Thirdly, we show that if $\underline{X} = \liminf_{n\to\infty} \frac{X_{0,n}}{n}$, then $\mathbb{E}\left[\underline{X}\right] \geqslant \gamma$. Let U_n be a random variable that is uniformly distributed over $\{1, \dots, n\}$ for each $n \in \mathbb{N}$ and independent of the sequence $(X_{m,n})_{0\leqslant m<n}$. Define, for all $k \in \mathbb{N}$, $Y_k^n = X_{0,k+U_n} - X_{0,k+U_n-1}$. Observe that by (K2), we have $\left(Y_n^k\right)_{n\in\mathbb{N}} \overset{\mathrm{d}}{=} (X_{0,n})_{n\in\mathbb{N}}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[Y_k^n\right] &= \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \mathbb{E}\left[X_{0,k+\ell} - X_{0,k+\ell-1}\right] \\ &= \lim_{n\to\infty} \frac{\mathbb{E}[X_{0,k+n}] - \mathbb{E}[X_{0,k}]}{n} = \gamma \end{aligned} \tag{B.6}$$

by the conclusion in 1°. Moreover, by (K2) again,

$$\mathbb{E}\left[(Y_k^n)^+\right] \leqslant \mathbb{E}\left[X_{U_n+k-1,U_n+k}^+\right] = \mathbb{E}[X_{0,1}^+] \tag{B.7}$$

for all $k \in \mathbb{N}$. Therefore, by (B.6) and (B.7), we have

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left[|Y_k^n|\right] = \sup_{k\in\mathbb{N}} \left(2\mathbb{E}\left[(Y_k^n)^+\right] - \mathbb{E}[Y_k^n]\right) < \infty$$

and that the sequence $\{(Y_k^n)_{k\in\mathbb{N}}\}_{n\in\mathbb{N}}$ is tight. We may extract a subsequence $\{(Y_k^{n_i})_{k\in\mathbb{N}}\}_{n_i\in\mathbb{N}}$ with weak limit $(Y_k)_{k\in\mathbb{N}}$. That is, for all bounded continuous function f defined on $\mathbb{R}^{\mathbb{N}}$

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which depends on only finitely many coordinates,

$$\mathbb{E}\left[f\left((Y_k)_{k\in\mathbb{N}}\right)\right] = \lim_{i\to\infty} \frac{1}{n_i} \sum_{\ell=1}^{n_i} \mathbb{E}\left[f\left((X_{0,k+\ell} - X_{0,k+\ell-1})_{k\in\mathbb{N}}\right)\right]. \tag{B.8}$$

By (B.8), we see that $(Y_k)_{k\in\mathbb{N}}$ is stationary. Moreover, by Fatou's lemma, the expectation $\mathbb{E}[Y_1] \geqslant \limsup_{n\to\infty} \mathbb{E}[Y_1^n] = \gamma$. By Birkoff's ergodic theorem, we have $Y = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k$ exists almost surely and that $\mathbb{E}[Y] = \mathbb{E}[Y_1] \geqslant \gamma$. It remains to prove that $\mathbb{E}\left[\underline{X}\right] \geqslant \mathbb{E}[Y]$, which can be shown by stochastic dominance (recall from Section 3.3). Observe that if f is a non-increasing bounded continuous function function defined on $\mathbb{R}^{\mathbb{N}}$ which depends on only finitely many coordinates, then

$$\mathbb{E}\left[f\left(X_{0,k}\right)_{k\in\mathbb{N}}\right] = \lim_{i\to\infty} \frac{1}{n_i} \sum_{\ell=1}^{n_i} \mathbb{E}\left[f\left((X_{\ell,k+\ell})_{k\in\mathbb{N}}\right)\right] \tag{B.9}$$

$$\leqslant \lim_{i\to\infty} \frac{1}{n_i} \sum_{\ell=1}^{n_i} \mathbb{E}\left[f\left((X_{0,k+\ell} - X_{0,\ell})_{k\in\mathbb{N}}\right)\right] \tag{B.10}$$

$$= \mathbb{E}\left[f\left(Y_1 + \cdots + Y_k\right)_{k\in\mathbb{N}}\right], \tag{B.11}$$

where (B.9) holds by condition (K2), (B.10) holds by condition (K1) and (B.11) holds by (B.8). Hence, $(X_{0,k})_{k\in\mathbb{N}}$ dominates $(Y_1 + \cdots + Y_k)_{k\in\mathbb{N}}$ and we have

$$\mathbb{E}\left[\underline{X}\right] = \mathbb{E}\left[\liminf_{n\to\infty} \frac{X_{0,n}}{n}\right] \geqslant \mathbb{E}\left[\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k\right] = \mathbb{E}[Y] \geqslant \gamma.$$

Last, utilizing the results in 2°, 3° and the fact that γ is a finite constant, we have, we have $\overline{X} = \underline{X}$ almost surely (say, X) and hence (B.1) holds almost surely. For $\mathcal{L}^1$-convergence, note that by (B.3), the sequence $\left(\frac{X_{0,n}^+}{n}\right)_{n\in\mathbb{N}}$ is uniformly integrable. Therefore,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{X_{0,n}}{n} - X\right]^+ = 0$$

and hence (B.1) holds in $\mathcal{L}^1$. □

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Appendix C

Diagonal direction for $p = 1$

Define the generating function of lattice paths connecting $x, y \in \mathbb{Z}$ by

$$m_{\mathbb{Z}}(x, y, t) = \sum_{\gamma \in \Gamma(x,y)} \frac{t^{\#\gamma}}{(\#\gamma)!}$$

and set $t^*(x, y)$ to be the critical value such that $m_{\mathbb{Z}}\left(x, y, t^*(x, y)\right) = 1$. Martinsson proved the following result in [13].

Theorem C.1 ([13], Theorem 1.4)**.** If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for $p = 1$ and $a \in (0, \infty)$, then for all $n \in \mathbb{N}$,

$$\tau(\mathbf{0}, n\mathbf{1}) \to \frac{t^*(0, n)}{a} \quad \text{in probability and in } \mathcal{L}^1$$

as $d \to \infty$.

We will now apply Theorem C.1 to argue that the lower bound of the diagonal constant given in Section 4.6 is sharp when $p = 1$.

Corollary C.2 ([13], Theorem 1.7)**.** If the distribution $F_{\tau_e} \in \mathcal{M}_p(a)$ for $p = 1$ and $a \in (0, \infty)$, then the time constant satisfies

$$\limsup_{d \to \infty} \mu_d(\mathbf{1}) \leqslant \frac{\sqrt{\alpha_1^2 - 1}}{2a}, \tag{C.1}$$

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where $\alpha_1 > 0$ (as before) is the positive solution of $\coth(\alpha) = \alpha$.

Proof. Observe that for all $n \in \mathbb{N}$, there are $\binom{n+2k}{k}$ paths from 0 to n that take k steps to the left ($k \in \mathbb{N}_0$). Therefore, the generating function satisfies

$$m_{\mathbb{Z}}(0, n, t) = \sum_{k=0}^{\infty} \binom{n+2k}{k} \frac{t^{n+2k}}{(n+2k)!} = \sum_{k=0}^{\infty} \frac{t^{n+2k}}{k!(n+k)!}.$$

Let $f_1(\alpha) = (\alpha+1)^{\frac{\alpha+1}{2\alpha}}(\alpha-1)^{\frac{\alpha-1}{2\alpha}}$ (as in Section 4.6) and $x_k = \frac{n+2k}{k}$. Applying Stirling's approximation, we have

$$\frac{t^{n+2k}}{k!(n+k)!} \asymp \frac{1}{\sqrt{k(n+k)}} \left(\frac{2et}{nf_1(x_k)}\right)^{n+2k} \quad \text{as} \quad k \to \infty.$$

As discussed in Section 4.6, $f_1(\alpha)$ is defined on $\alpha \in [1, \infty)$ (where we extend $f_1(\alpha)$ to $\alpha = 1$ continuously) and it achieves its minimum at $\alpha = \alpha_1$ with $f_1(\alpha_1) = e\sqrt{\alpha_1^2 - 1}$. Hence we have

$$m_{\mathbb{Z}}(0, n, nt) \to \begin{cases} 0 & , \text{ if } t < \frac{1}{2}\sqrt{\alpha_1^2 - 1} \\ \infty & , \text{ if } t > \frac{1}{2}\sqrt{\alpha_1^2 - 1} \end{cases} \quad \text{as} \quad n \to \infty,$$

which gives $t^*(0, n) \sim \frac{1}{2}\sqrt{\alpha_1^2 - 1} \cdot n$ as $n \to \infty$. Therefore, by subadditivity and the convergence in Theorem C.1, for all $n \in \mathbb{N}$,

$$\mu_d(\mathbf{1}) \leqslant \frac{\mathbb{E}[\tau(\mathbf{0}, n\mathbf{1})]}{n} = \frac{t^*(0, n)}{an} + o(1) \quad \text{as} \quad d \to \infty.$$

Taking $\limsup_{d \to \infty}$ on both sides and letting $n \to \infty$, we see that

$$\limsup_{d \to \infty} \mu_d(\mathbf{1}) \leqslant \lim_{n \to \infty} \frac{t^*(0, n)}{an} = \frac{\sqrt{\alpha_1^2 - 1}}{2a}$$

as asserted in the corollary. $\square$

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