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在殼方法與磁單極

On-shell Method and Monopole

賴允忠

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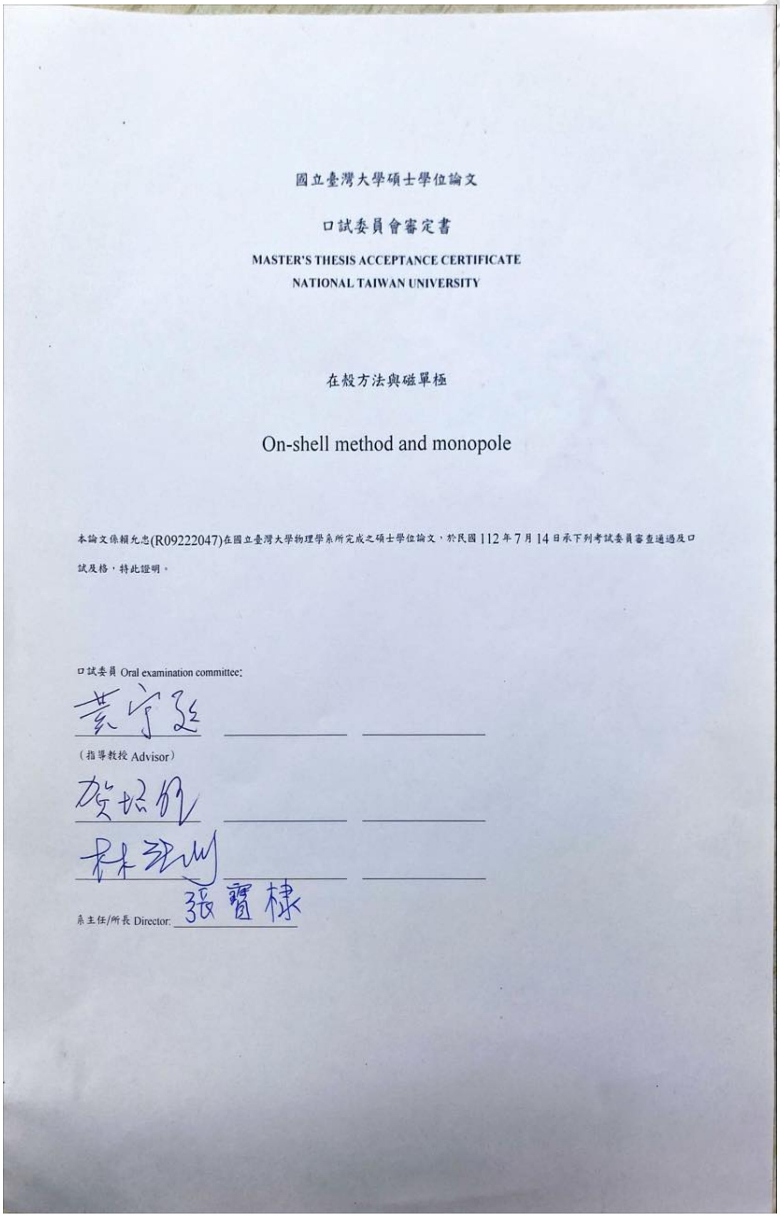

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摘要

在這篇碩論中我們討論了電子與磁單極之間的交互作用並從古典的角度出發去計算衝量跟散射角度. 同時我們也透過解薛丁格方程在磁單極的位能下得到波函數, 進一步取它的漸近形式來獲取散射幅度. 以及我們也透過在殼旋轉方法得到電子與磁單極的傳播子, 其中也包含了狄拉克弦的部分，但我們透過計算可以發現在古典極限下狄拉克弦並不會出現在最後觀測量, 包括衝量, 散射角度以及散射幅度.

關鍵字：磁單極、散射幅度、本徵近似

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Abstract

In this thesis, we consider the interaction between an electron and a monopole. We calculate the impulse in the classical picture and the cross section in quantum mechanics. Additionally, we employ on-shell phase rotation to reproduce the same result and demonstrate that the observables are independent of the Dirac string in the eikonal limit.

Keywords: Monopole, scattering amplitude, eikonal approximation

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Chapter 1 Introduction

Since Dirac's first paper on the appearance of monopoles, researchers have been actively searching for experimental evidence and examining its consistency with contemporary theories. The concept of a monopole can be naturally extended from Maxwell's equations, implying a non-vanishing Bianchi identity

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= J_e^\nu \\ \partial_\mu \widetilde{F}^{\mu\nu} &= J_g^\nu \end{aligned} \tag{1.1}$$

If we focus on the U(1) gauge symmetry for the monopole, the field strength automatically satisfies the Bianchi identity and possesses zero magnetic charge

$$\begin{aligned} \widetilde{F}^{\mu\nu} &\equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \\ \partial_\mu \widetilde{F}^{\mu\nu} &= \epsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\alpha A_\beta = 0 \end{aligned}$$

To address this issue, Zwanziger proposed the two-potential formalism [37], which involves non-local variables. Similarly, in order to construct the classical monopole, Dirac introduced a magnetic point source with an attached infinite string. Undoubtedly, the string breaks Lorentz symmetry and encodes its information in both the theory and observables.

There is another problematic issue known as Dirac quantization

$$eg = 2\pi n, \quad n \in \mathbb{Z} \tag{1.2}$$

In [12], Dirac combined the gauge symmetry for the monopole with the single-valued constraint on the wave function to argue for nontrivial quantization results. Furthermore, according to Dirac quantization and electromagnetic duality, electric and magnetic charges

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must be quantized simultaneously. The electric part of this quantization was indicated by Millikan's oil experiment. This quantization also poses a challenge to perturbation theory. In classical results for gravitational or electric forces, the leading order is based on $\mathcal{O}(g^2)$, and higher-order effects depend on loop corrections, which can be calculated in quantum field theory (QFT). However, due to quantization, perturbing both types of coupling constants is prohibited, rendering renormalization meaningless. In this thesis, we aim to investigate whether physical observables depend on strings or not.

In our research, we follow the approach outlined in [18] to derive the tree-level amplitude for the $2 \to 2$ charge-monopole system. We calculate the impulse using the KMOC formalism [23]. Additionally, we extend our analysis to the 1-loop amplitude using the unitary cut method and demonstrate that the results are consistent with classical computations. Furthermore, we provide a comprehensive review of calculations already known in classical mechanics, quantum mechanics, and the eikonal approximation for charge-monopole interactions.

This thesis is organized as follows. Chapter 2 provides a review of the classical point source for the monopole. We combine the Lorentz force to calculate classical observables such as impulse and deflection angle. Additionally, we explore the two vector potentials for the monopole and examine gauge transformations.Chapter 3, we solve the Schrödinger equation in the presence of a monopole background to obtain the wave function. We then use this wave function to calculate the scattering amplitude. Chapter 4 focuses on using the on-shell phase rotation technique to recover the propagator between the electron and the monopole. From this propagator, we deduce the deflection angle and impulse, which are consistent with classical results. We also explore the summation of all-order Feynman diagrams in the eikonal limit, even considering Dirac quantization, to obtain the scattering amplitude. The results are consistent with those obtained from quantum mechanics in the small-angle limit. Finally, in Chapter 5, we conclude our findings and provide some outlook for future research directions.

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Chapter 2 Classical Dynamics in monopole background

In this chapter, we provide a brief review of the monopole in the abelian U(1) gauge group. We solve the equation of motion for the electron-monopole system and examine its dynamics.

Next, we introduce the vector potential for the monopole and demonstrate that the transformation involving the string is actually a gauge transformation. Additionally, we touch upon the duality between the electron and the monopole.

The content of this chapter is largely adapted from the works of [31] and [7].

2.1 Monopole as a point source

To construct the monopole, we can start from generalized Maxwell equation (1.1) and static point source with magnetic charge at r'. The magnetic field can be solved by

$$\begin{aligned} F^{ij} &\equiv \varepsilon_{ijk} B_k, \quad F^{i0} \equiv E_i \\ \Rightarrow &\partial_i \tilde{F}^{i0} = \nabla \cdot \mathbf{B}(\mathbf{r}) = g\,\delta(\mathbf{r'} - \mathbf{r}) \end{aligned} \tag{2.1}$$

The solution can be extended from the electric case by replacing the electric charge e with the magnetic charge g.

$$\mathbf{B}(\mathbf{r}) = \frac{g}{4\pi} \frac{\mathbf{r} - \mathbf{r'}}{(\mathbf{r} - \mathbf{r'})^3} \tag{2.2}$$

To detect the magnetic field, we can consider a test particle that carries an electric charge.

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In this case, the equation of motion for the electrically charged particle in an electromagnetic background is described by the Lorentz force equation.

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = e\,F^{\mu\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}\tau} \Longleftrightarrow \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{2.3}$$

The force experienced by a particle with an electric charge in a monopole background is given by

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = e[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{B_g}] \tag{2.4}$$

$$= \frac{eg}{4\pi r^3}[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{r}] \tag{2.5}$$

with the monopole located at the origin and the position of the electric charge particle denoted by the vector **r**

$$\frac{m}{2}\frac{\mathrm{d}v^2}{\mathrm{d}t} = \frac{1}{m}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = 0$$

$$\frac{mv^2}{2} \equiv E \tag{2.6}$$

To determine the magnitude of radius, we observe that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}r^2 = 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right) = 2v^2 + 2\mathbf{r}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = 2v^2 \tag{2.7}$$

Taking into account time reversal symmetry, we can express

$$r = \sqrt{v^2t^2 + b^2}$$

$$\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{r}\cdot\mathbf{v} = v^2t \tag{2.8}$$

Here, we introduce the impact parameter b as the minimum distance between the electron and the monopole. Before calculating the equation of motion for the electron-monopole system, we need to consider some time-independent constants, in addition to the kinematic energy, that describe the system's properties. Furthermore, we are interested in studying the orbital angular momentum associated with the motion of the particles.

$$\widetilde{\mathbf{L}} = m[\mathbf{r}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}] \tag{2.9}$$

In an electron-electron system, the orbital angular momentum is a constant of motion

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{e^2}{4\pi}\frac{\mathbf{r}}{r^3} \tag{2.10}$$

$$\frac{\mathrm{d}\widetilde{\mathbf{L}}}{\mathrm{d}t} = m[\mathbf{r}\times\ddot{\mathbf{r}}] = \frac{e^2}{4\pi r^3}[\mathbf{r}\times\mathbf{r}] = 0 \tag{2.11}$$

but in the background field of monopole (4.1), the orbital angular momentum is no longer constant

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\mathbf{L}}}{\mathrm{d}t} = m[\mathbf{r}\times\ddot{\mathbf{r}}] &= \frac{eg}{4\pi r^3}[\mathbf{r}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{r}] \\ &= \frac{eg}{4\pi}[\frac{1}{r}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} - (\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t})\frac{\mathbf{r}}{r^3}] \\ &= \frac{eg}{4\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right)\end{aligned} \tag{2.12}$$

However, we can define angular momentum as a new constant of motion

$$\mathbf{L} \equiv \widetilde{\mathbf{L}} - \frac{eg}{4\pi}\frac{\mathbf{r}}{r} \tag{2.13}$$

which satisfy

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = 0$$

$$L^2 \equiv \mathbf{L}^2 = \widetilde{L}^2 + \frac{e^2g^2}{(4\pi)^2} = (mvb)^2 + \frac{e^2g^2}{(4\pi)^2}$$

The physical meaning of the additional angular momentum in the presence of a monopole background field arises from the electromagnetic field itself. The angular momentum of the electromagnetic field can be described by

$$\begin{aligned}\mathbf{L}_{em} &= \int \mathrm{d}^3r'\,[\mathbf{r}'\times(\mathbf{E}\times\mathbf{B})] = \frac{g}{4\pi}\int \mathrm{d}^3r'\,[\mathbf{r}'\times(\mathbf{E}\times\frac{\mathbf{r}'}{r'^3})] \\ &= \frac{g}{4\pi}\int \mathrm{d}^3r'\,[\mathbf{E}\frac{1}{r'} - \frac{\mathbf{r}'}{r'^3}(\mathbf{E}\cdot\mathbf{r}')] = \frac{g}{4\pi}\int \mathrm{d}^3r'\,(\mathbf{E}\cdot\boldsymbol{\nabla}')\frac{\mathbf{r}'}{r'} \\ &= -\frac{g}{4\pi}\int \mathrm{d}^3r'\,(\boldsymbol{\nabla}\cdot\mathbf{E})\frac{\mathbf{r}'}{r'} = -\frac{eg}{4\pi}\frac{\mathbf{r}}{r}\end{aligned} \tag{2.14}$$

In last equation, we use integration by part and Gauss law for electric point source

$$\boldsymbol{\nabla}'\cdot\mathbf{E} = e\,\delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{2.15}$$

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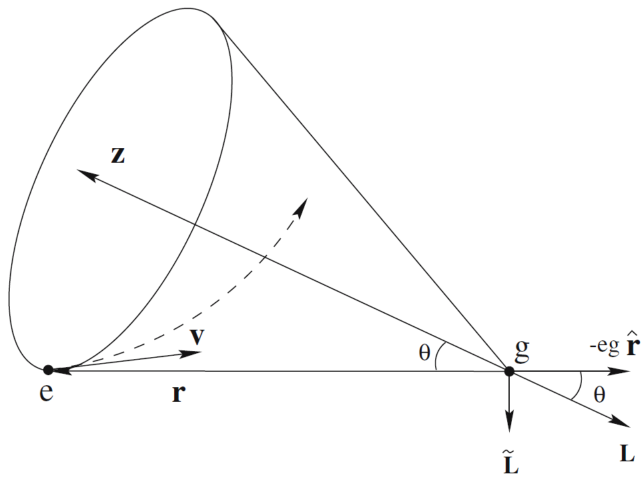

Figure 2.1: geometry of scattering of an electron by a monopole at the origin from [31]

2.2 Deflection angle and Impulse

In this section, we calculate the deflection angle for the electron-monopole system. We consider an electron approaching the monopole at the origin from the asymptotic region, and the deflection angle is measured by the inner product of the unit vectors of velocity at $t = \pm\infty$. The trajectory of the probe particle in the background field of the monopole can be observed by

$$\mathbf{L} \cdot \hat{\mathbf{r}} = -\frac{eg}{4\pi} \tag{2.16}$$

Thus, the trajectory form a cone (Fig:2.1). The corresponding cone angle is

$$\tan\theta = \frac{\widetilde{L}}{\mu} = \frac{mvb}{\mu} \tag{2.17}$$

where $\mu \equiv \frac{eg}{4\pi}$. Since the trajectory lies on a cone, the system possesses only two degrees of freedom. We have already established that the magnitude of velocity v and the angular momentum **L** are constants of motion. Therefore, the equation of motion can be completely determined by these two variables. We can observe

$$\begin{aligned} \mathbf{L} \times \mathbf{r} &= m\mathbf{r} \times \mathbf{v} \times \mathbf{r} = mr^2\mathbf{v} - m(\mathbf{r} \cdot \mathbf{v})\mathbf{r} \\ &= mr^2\mathbf{v} - mv^2t\mathbf{r} \\ \Rightarrow \mathbf{v} &= \frac{\mathbf{L} \times \mathbf{r}}{mr^2} + \frac{v^2t}{r}\hat{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} + v_r\hat{\mathbf{r}} \end{aligned} \tag{2.18}$$

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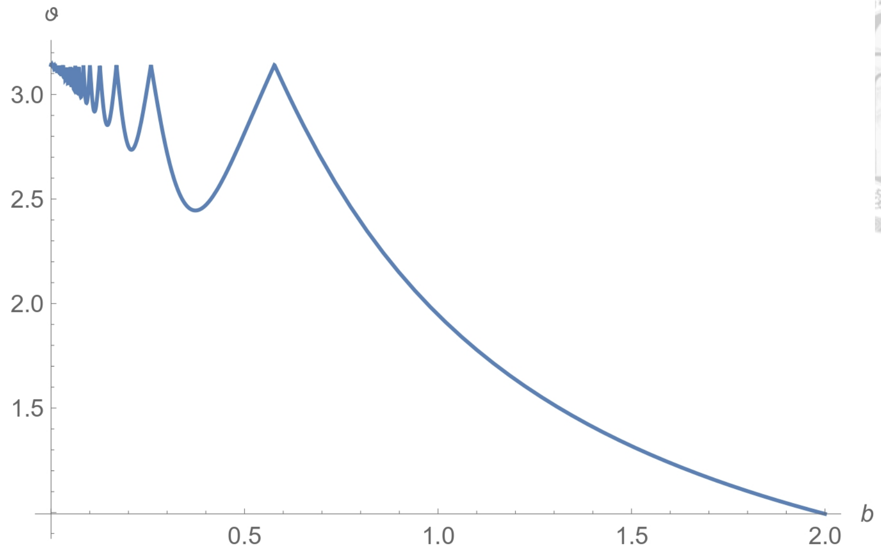

Figure 2.2: Deflection angle $\vartheta(b)$ with rescaling $eg/mv \equiv 1$

where the angular and radial velocity are

$$\boldsymbol{\omega} \equiv \frac{\mathbf{L}}{mr^2}, \quad v_r \equiv \frac{v}{\sqrt{1+(b/vt)^2}}$$

Then boundary condition will be

$$\lim_{t\to\pm\infty} \omega \equiv \lim_{t\to\pm\infty} |\boldsymbol{\omega}| = 0, \quad \lim_{t\to\pm\infty} v_r = \pm v$$

Because the angular velocity $\omega = d\phi/dt$, the azimuthal angle ϕ will be

$$\phi(t) = \frac{|\mathbf{L}|}{m} \int \frac{\mathrm{d}t}{v^2t^2+b^2} = \frac{1}{\sin\theta} \arctan\frac{vt}{b} \tag{2.19}$$

Due to the boundary condition, we can parameterize the velocity in asymptotic form

$$\mathbf{v}|_{t=\pm\infty} = v\left(\pm\sin\theta\cos\frac{\Delta\phi}{2}, \sin\theta\sin\frac{\Delta\phi}{2}, \pm\cos\theta\right) \tag{2.20}$$

where $\Delta\phi = \phi(\infty) - \phi(-\infty) = \pi/\sin\theta$. The deflection angle ϑ is defined as

$$\begin{aligned} \cos\vartheta &= \frac{\mathbf{v}|_{t=\infty}\cdot\mathbf{v}|_{t=-\infty}}{v^2} = 2\sin^2\theta\sin^2\left(\frac{\pi}{2\sin\theta}\right) - 1 \\ \Longleftrightarrow \cos\frac{\vartheta}{2} &= \sin\theta\left|\sin\left(\frac{\pi}{2\sin\theta}\right)\right| \end{aligned} \tag{2.21}$$

The deflection angle of electron-monopole scattering is not a monotonous function of impact parameter b.The function $\vartheta(b)$ is ploted in Fig 2.2. In large impact parameter, We

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can consider $\mathcal{O}\big(\frac{1}{b}\big)$ of deflection angle

$$\vartheta = \pi - 2\theta \approx \frac{2\mu}{mvb} \tag{2.22}$$

Another classical observable closely related to scattering is impulse. Impulse is a classical observable from Newtonian mechanics that measures the difference in momentum between the incoming and outgoing particles. In order to obtain a non-perturbative result for impulse, we follow the approach outlined in [24]. In three dimensions, we can expand the outgoing momentum in terms of a basis consisting of the incoming momentum and the impact parameter.

$$\mathbf{k}_{out} = c_1\,\mathbf{k}_{in} + c_2|\mathbf{k}_{in}|\,\hat{\mathbf{b}} + c_3\,(\mathbf{k}_{in}\times\hat{\mathbf{b}}) \tag{2.23}$$

The coefficient can be determined by

$$\begin{aligned}&\hat{\mathbf{k}}_{in}\cdot\hat{\mathbf{k}}_{out} = \cos\vartheta, \quad (\mathbf{k}_{in}+\mathbf{k}_{out})\cdot\mathbf{L} = 0, \quad |\mathbf{k}_{in}| = |\mathbf{k}_{out}|\\ &\Rightarrow c_1 = \cos\vartheta, \quad c_2 = \cos\theta\sin\theta\Big[1-\cos\Big(\frac{\pi}{\sin\theta}\Big)\Big], \quad c_3 = \sin\theta\sin\Big(\frac{\pi}{\sin\theta}\Big)\end{aligned} \tag{2.24}$$

Then the impulse of electron-monopole scattering can be read as

$$\Delta\mathbf{k} = \mathbf{k}_{out} - \mathbf{k}_{in} \tag{2.25}$$

Perturbatively, We can expand the impulse order by order in $\frac{eg}{4\pi L}$.

$$\begin{aligned}&\mathcal{O}(eg): \quad 2\frac{eg}{4\pi L}(\mathbf{k}_{in}\times\hat{\mathbf{b}})\\ &\mathcal{O}\big(e^2g^2\big): \quad (\frac{eg}{4\pi L})^2\Big(-2\mathbf{k}_{in} - \frac{\pi}{2}|\mathbf{k}_{in}|\hat{\mathbf{b}}\Big)\end{aligned} \tag{2.26}$$

Another way to derive the impulse of order eg can be found in [17]. This approach involves considering a probe charged particle with a velocity of $(0,0,v)$ being deflected by a monopole located at the origin, as depicted in Figure 2.3. In large impact parameter, we can consider $\mathcal{O}\big(\frac{1}{b}\big)$ of impulse, which can be easily calculate as

$$\Delta\mathbf{k}^{(1)} = \int_{-\infty}^{\infty}\mathbf{F}\,dt \approx \frac{egbv}{4\pi}\int_{-\infty}^{\infty}\frac{dt}{(b^2+v^2t^2)^{3/2}}\,\hat{\mathbf{y}} = 2\frac{eg}{4\pi b}\,\hat{\mathbf{y}} \tag{2.27}$$

the same as (2.26) with identify $\hat{\mathbf{b}} \equiv \hat{x}$ and $mvb \equiv L$

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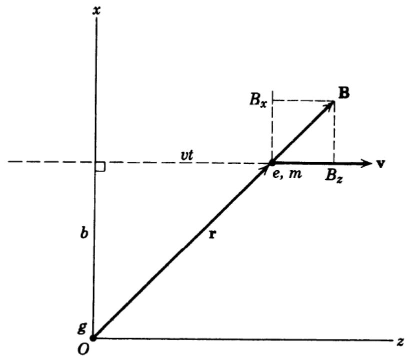

Figure 2.3: Charged particle pass a monopole at large impact parameter from [19]

2.3 Vector Potential of Monopole

In the previous section, we discussed classical observables for electron-monopole scattering. However, to extend the calculation to quantum physics, we need to consider the Lagrangian that describes the dynamics of both electric and magnetic charges. The standard Lagrangian for an electrically charged particle in the background of a vector potential is given by

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^2 + e\mathbf{v}\cdot\mathbf{A} \tag{2.28}$$

Then equation of motion can be solved by Lagrangian equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right) &= \frac{\partial\mathcal{L}}{\partial\mathbf{r}} \\ \Rightarrow \frac{\partial\mathcal{L}}{\partial\mathbf{r}} &= \nabla\mathcal{L} = e\nabla(\mathbf{A}\cdot\mathbf{v}) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}+e\mathbf{A}) \end{aligned} \tag{2.29}$$

with the identity of vector analysis

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}) \tag{2.30}$$

then Lagrangian equation will be

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}+e\mathbf{A}) = e\,(\mathbf{v}\cdot\nabla)\mathbf{A} + e\,\mathbf{v}\times(\nabla\times\mathbf{A}) \tag{2.31}$$

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The total differential, $\mathrm{d}A$, contains two parts: the change with respect to time, $\mathrm{d}t$, and the change with respect to displacement.$\mathrm{d}\mathbf{r}$.

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}$$

Then the equation of motion is just

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\partial \mathbf{A}}{\partial t} + e\,\mathbf{v}\times(\nabla\times\mathbf{A}) \tag{2.32}$$

If vector potential is time-independent, the EOM is nothing but Lorentz fore equation (2.4) with identify

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{2.33}$$

However, there is a disaster for monopole. Since

$$\mathbf{B}_g = g\frac{\hat{\mathbf{r}}}{4\pi r^2} \Rightarrow \nabla\cdot\mathbf{B} = g\,\delta^{(3)}(\mathbf{r}) \tag{2.34}$$

To resolve the contradiction in equation (2.33) which implies $\nabla\cdot\mathbf{B} = 0$, we introduce a vector potential that does not cover $\mathbb{R}^3$ globally but contains a singularity region representing the Dirac string. To achieve this, we assume the existence of two types of vector potentials, $\mathbf{A}_N$ and $\mathbf{A}_S$, as described in [36]. These vector potentials satisfy(2.33)

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{r}} = \mathbf{B}_g = g\frac{\hat{\mathbf{r}}}{4\pi r^2} \tag{2.35}$$

To process azimuthal symmetry, we take A_θ is zero [1]. The simplest solution is

$$A_\phi = g\frac{-\cos\theta + c}{4\pi r\sin\theta} \tag{2.36}$$

c is constant.To determine the constant c for the two vector potentials $\mathbf{A}_N$ and $\mathbf{A}_S$, we assume that they coincide on the equatorial plane (or any other chosen closed curve). By making this assumption, we can integrate the surface

$$\oint \mathrm{d}\mathbf{S}\cdot\mathbf{B}_g = \oint(\mathbf{A}_N - \mathbf{A}_S)\cdot \mathrm{d}\boldsymbol{\ell} = \frac{1}{2}g(c_N - c_S) = g \tag{2.37}$$

[1] Since r.h.s only contain $f(r)$,the solution for non-zero A_θ is $\phi + g(r,\theta)$, which means $A(\phi+2\pi) \neq A(\phi)$

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We can trivially choose $c_N = 1$ and $c_S = -1$. Then final result would be

$$\mathbf{A}_N = \frac{g}{4\pi}\frac{-\cos\theta + 1}{r\sin\theta}\hat{\boldsymbol{\phi}}$$
$$\mathbf{A}_S = \frac{g}{4\pi}\frac{-\cos\theta - 1}{r\sin\theta}\hat{\boldsymbol{\phi}} \tag{2.38}$$

We combine these two expression into

$$\mathbf{A}(\mathbf{r}) = \frac{g}{4\pi r}\frac{\mathbf{r}\times\mathbf{n}}{r - (\mathbf{r}\cdot\mathbf{n})} \tag{2.39}$$

For the vector potential $\mathbf{A}_N$, the normal vector is $\mathbf{n} = (0,0,-1)$, while for $\mathbf{A}_S$, the normal vector is $\mathbf{n} = (0,0,1)$. It is evident that each vector potential has its own singularity: $\mathbf{A}_N$ diverges at the south pole, and $\mathbf{A}_S$ diverges at the north pole. These singularities correspond to the existence of the Dirac string. In classical observables, the singularity can be avoided due to the presence of the cone, on which the trajectory of the probe particle lies. Since we have two vector potentials to cover each singularity, the next issue to consider is their overlapping region, where the vector potential is not uniquely defined. However, the curl of A remains the same in this region, and therefore the difference between the two vector potentials must be a U(1) gauge transformation. This gauge transformation ensures that the physical properties of the system are preserved despite the non-uniqueness in the overlapping region

$$\mathbf{A}_N - \mathbf{A}_S = \frac{g}{2\pi r\sin\theta} = \nabla\lambda(\mathbf{r})$$
$$\Rightarrow \lambda(\mathbf{r}) = \frac{g}{2\pi}\phi \tag{2.40}$$

The general gauge transformation for arbitrary string is

$$\mathbf{A}_{\mathbf{n}'}(\mathbf{r}) = \mathbf{A}_{\mathbf{n}'}(\mathbf{r}) + \nabla\Omega_{n',n}(\mathbf{r}) \tag{2.41}$$

where $\Omega_{n',n}(r)$ is solid angle under the surface between new string, old string and position vector. We leave it to the reader to prove this nontrivial result. However, in the next chapter, we will focus solely on equation (2.38) and consider the vector potentials $\mathbf{A}_N$ and $\mathbf{A}_S$ derived from it. The significance of gauge transformations in quantum mechanics will be explored and discussed in the subsequent chapter, where we will examine their implications and the resulting effects.

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Chapter 3 Quantum Daynamics in monopole background

In this chapter, we provide a comprehensive review of gauge symmetry in quantum mechanics, emphasizing its significance and implications. We particularly focus on one of its most intriguing consequences, known as Dirac quantization. This phenomenon highlights the constraint placed on the allowed values of electric and magnetic charges in a gauge theory.

Furthermore, we delve into solving the Schrödinger equation in the background of a monopole, aiming to derive the scattering amplitude and cross section in the framework of quantum mechanics. By considering the interaction between particles and the monopole, we explore the quantum mechanical aspects of the scattering process.

The content of this chapter is primarily based on the research and findings presented in the works of [7] and [31].

3.1 Quantum Mechanics in Monopole Background

Recall the Lagrangian with external vector potential

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^2 + e\,\mathbf{A}\cdot\mathbf{v}$$

Then the generalized momentum is defined as

$$\mathbf{\Pi} \equiv \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = m\mathbf{v} + e\,\mathbf{A}$$

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The corresponding Hamiltonian is

$$\mathcal{H} \equiv \mathbf{\Pi} \cdot \mathbf{v} - \mathcal{L} = \frac{1}{2} m \mathbf{v}^2 = \frac{1}{2m} (\mathbf{\Pi} - e\,\mathbf{A})^2$$

In quantum mechanics, generalized momentum is replace by operator

$$\mathbf{\Pi} \rightarrow -i \boldsymbol{\nabla}$$

The time-independent Schrödinger equation with vector potential is

$$-\frac{1}{2m} (\boldsymbol{\nabla} - ie\mathbf{A})^2 \psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{3.1}$$

If we recall the gauge transformation

$$\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}(\mathbf{r}) + \nabla \lambda(\mathbf{r})$$

To guarantee the gauge invariant of Schrödinger equation, wave function must transform as

$$\psi(\mathbf{r}) \rightarrow e^{ie\lambda(\mathbf{r})} \psi(\mathbf{r})$$

Since wave function must be single-valued, the gauge transformation for monopole (2.40) lead to **dirac quantization**

$$e^{ieg} = 1 \Rightarrow eg = 2\pi n, \quad n \in \mathbb{Z} \tag{3.2}$$

If the wave function is symmetric under rotation $\phi \rightarrow \phi + 2\pi$, then $eg = 2\pi n,\ n \in \mathbb{Z}$. Or it is anti-symmetric , then $eg = (2n+1)\pi$.

3.2 Schrödinger equation in the monopole background

The wave function in the background of a monopole can be solved using a similar procedure as that for the hydrogen atom. We begin by substituting the vector potential $\mathbf{A}_N$, as given in equation (2.38), into the Schrödinger equation. This substitution allows

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us to express the Hamiltonian as follows

$$\mathcal{H} = -\frac{1}{2mr^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \mathbf{L}^2 - \mu^2\right\} \tag{3.3}$$

where $\mu \equiv \frac{eg}{4\pi}$ and angular momentum in quantum mechanics can be derived from (2.13)

$$\begin{aligned}\mathbf{L} &= \mathbf{r}\times\mathbf{p} - \mu\hat{\mathbf{r}} = \mathbf{r}\times(\mathbf{\Pi} - e\mathbf{A}) - \mu\,\hat{\mathbf{r}} = \widetilde{\mathbf{L}} - e(\mathbf{r}\times\mathbf{A}) - \mu\,\hat{\mathbf{r}} \\ &= \frac{1}{\sin\theta}\left(i\frac{\partial}{\partial\phi} + \mu(1-\cos\theta)\right)\hat{\boldsymbol{\theta}} - i\frac{\partial}{\partial\theta}\,\hat{\boldsymbol{\phi}} - \mu\,\hat{\mathbf{r}}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathbf{L}^2 &= -\frac{1}{\sin\theta^2}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \left(\frac{\partial}{\partial\phi} - i\mu(1-\cos\theta)\right)^2\right] + \mu^2 \\ L_3 &= -i\frac{\partial}{\partial\phi} - \mu\end{aligned}$$

with separation of variables

$$\psi(\mathbf{r}) = F_{k\tilde{\ell}}(r)Y_{\mu lm}(\theta,\phi) \tag{3.5}$$

where

$$\begin{aligned}&\mathbf{L}^2Y_{\mu lm}(\theta,\phi) = l(l+1)Y_{\mu lm}, \quad L_3Y_{\mu lm} = mY_{\mu lm}(\theta,\phi) \\ &l = \mu,\mu+1,\ldots \quad m = -l,-l+1,\ldots,+l\end{aligned} \tag{3.6}$$

$Y_{\mu,lm}(\theta.\phi)$ is generalized spherical harmonics [36] and some property list in appendix C. The radial part of the wave function satisfy

$$-\frac{1}{2m}\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\ell(\ell+1)-\mu^2}{r^2}\right\}F_{k\tilde{\ell}}(r) = EF_{k\tilde{\ell}}(r) \tag{3.7}$$

It can be solved by spherical Bessel functions of the order

$$\tilde{\ell} = \sqrt{\left(l+\frac{1}{2}\right)^2 - \mu^2} - \frac{1}{2} \tag{3.8}$$

The solution is

$$F_{k\tilde{\ell}}(r) = \sqrt{\frac{k}{r}}J_{\tilde{\ell}+1/2}(kr) = \frac{1}{k}\sqrt{\frac{2}{\pi}}j_{\tilde{\ell}}(kr), \quad k = \sqrt{2mE} \tag{3.9}$$

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where Bessel function is defined as

$$j_n(x) \equiv (-x)^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{\sin x}{x} \tag{3.10}$$

3.3 Quantum Scattering in Electron-Monopole System

In scattering theory [30], we assume a time-independent Hamiltonian as

$$\mathcal{H} = \mathcal{H}_0 + V \tag{3.11}$$

where $\mathcal{H}_0 \equiv \frac{\mathbf{\Pi}^2}{2m}$ is the kinetic-energy operator. In usual, we have two eigenstate

$$\mathcal{H} |\psi\rangle = E |\psi\rangle$$
$$\mathcal{H}_0 |\phi\rangle = E_0 |\phi\rangle$$

The perturbation theory can guide us in obtaining the wave function $|\psi\rangle$ from an initial state $|\phi\rangle$. However, in the electron-monopole system, there is no free Hamiltonian that can be used as a reference for perturbation theory. Consequently, alternative approaches are needed.

To analyze the wave function in the absence of a free Hamiltonian, we adopt a partial wave expansion technique. This involves expanding the wave function using a series of partial waves, which correspond to different angular momentum states. By employing this expansion, we can express the wave function as:

$$\Psi_k(\mathbf{r}) = \sum_{\ell,m} S_{\ell,m}\, j_{\tilde{\ell}}(kr) Y_{\mu\ell m}(\theta, \phi) \tag{3.12}$$

with the asymptotic form of radial Bessel function

$$j_{\tilde{\ell}}(kr) \xrightarrow{r\gg 1} \frac{1}{kr} \sin\left(kr - \frac{\pi\tilde{\ell}}{2}\right) = \frac{1}{2ikr}\left(e^{ikr - i\pi\tilde{\ell}/2} - e^{-ikr + i\pi\tilde{\ell}/2}\right) \tag{3.13}$$

The wave function will be

$$\Psi(\mathbf{r}) \xrightarrow{r\gg 1} \sum_{\ell,m} S_{\ell,m} \frac{1}{2ikr}\left(e^{ikr - i\pi\tilde{\ell}/2} - e^{-ikr + i\pi\tilde{\ell}/2}\right) Y_{\mu\ell m}(\theta, \phi) \tag{3.14}$$

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The plane wave can be expanded on ordinary spherical harmonics and Bessel function

$$e^{i\,\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{\ell,m} i^{\ell}\, j_{\ell}(kr)\, Y_{\ell m}(\hat{\mathbf{k}})\, Y^{*}_{\ell m}(\hat{\mathbf{r}}) \tag{3.15}$$

In asymptotic form with the completeness relation, plane wave can be expressed as

$$e^{i\,\mathbf{k}\cdot\mathbf{r}} \xrightarrow{r\gg 1} \frac{e^{ikr}}{2ikr}\,\delta^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) - \frac{e^{-ikr}}{2ikr}\,\delta^2(-\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) \tag{3.16}$$

The first term in equation (3.12) corresponds to the outgoing wave function, while the latter term represents the incoming wave function. Considering the principle of causality, as the scattering process is solely outgoing, the incoming part of the wave function in equation (3.12) must match with the incoming part of the plane wave function. By ensuring this match, we can determine the coefficients $S_{\ell,m}$ using the completeness of the generalized spherical harmonics.

$$\sum_{\ell,m} Y^{*}_{\mu\ell m}(\theta,\phi) Y_{\mu\ell m}(\theta',\phi') = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi')$$
$$\Rightarrow S_{\ell,m} = Y^{*}_{\mu\,\ell\,m}(-\hat{\mathbf{k}}) e^{-i\pi\tilde{\ell}/2} \tag{3.17}$$

Thus, the outgoing part of (3.12) in asymptotic limit is

$$\Psi_{out}(r) \approx \frac{e^{ikr}}{2ikr} \sum_{\ell m} Y^{*}_{\mu\ell m}(-\hat{\mathbf{k}}) Y_{\mu\ell m}(\hat{\mathbf{r}}) \tag{3.18}$$

With the relation between Wigner D matrix and spin-weighted spherical harmonics (appendix C), the additional theorem is [21]

$$\sum_b \mathcal{D}^{l}_{ab}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^{l}_{bc}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^{l}_{ac}(\alpha,\beta,\gamma) \tag{3.19}$$

The angle α, β, γ can be expressed in terms of $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$

$$\cot(\alpha - \alpha_2) = \cos\beta_2 \cot(\alpha_1+\gamma_2) + \cot\beta_1 \frac{\sin\beta_2}{\sin(\alpha_1+\gamma_2)}$$
$$\cos\beta = \cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2\cos(\alpha_1+\gamma_2)$$
$$\cot(\gamma - \gamma_2) = \cos\beta_1 \cot(\alpha_1+\gamma_2) + \cot\beta_2 \frac{\sin\beta_1}{\sin(\alpha_1+\gamma_2)} \tag{3.20}$$

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The outgoing part can be simplify as

$$
\begin{aligned}
\Psi_{out}(r) &\approx \frac{e^{ikr}}{r} f(\Theta, \Phi) \\
2ikf(\Theta, \Phi) &= \sum_{\ell} \frac{2\ell+1}{4\pi} \mathcal{D}^{\ell}_{\mu\mu}(\Phi, \pi - \Theta, \Phi) e^{-i\pi\tilde{\ell}}
\end{aligned} \tag{3.21}
$$

where $f(\Theta, \Phi)$ is defined as scattering amplitude. Θ is the angle between $\hat{\mathbf{k}}$ and $\hat{\mathbf{r}}$ and Φ can be determined by (3.20)

$$
\cos\Theta = \cos\theta_k \cos\theta_r + \sin\theta_k \sin\theta_r \cos(\phi_k - \phi_r) \tag{3.22}
$$

In eikonal limit $\Theta \ll 1$, the main contribution for amplitude is from large ℓ. Therefore, we can approximate

$$
\tilde{\ell} = \sqrt{(\ell + \frac{1}{2})^2 - \mu^2} - \frac{1}{2} \xrightarrow{\ell \gg 1} \ell
$$

Then the amplitude become

$$
2ikf(\Theta, \Phi) \approx \sum_{\ell} (-1)^{\ell} (2\ell+1) d^{\ell}_{\mu\mu}(\pi - \Theta) e^{2i\mu\Phi} \tag{3.23}
$$

To carry out this series, we utilize the generating function of characteristic function [21]

$$
\sum_{\ell} t^{2\ell} \chi^{\ell}(\kappa) = \frac{1}{1 - 2t\cos\kappa + t^2} \tag{3.24}
$$

where

$$
\begin{aligned}
\chi^{\ell}(\kappa) &= \sum_{m=-\ell}^{m=\ell} \mathcal{D}^{\ell}_{mm}(\alpha, \beta, \gamma) \\
&= \frac{\sin(\ell + 1/2)\kappa}{\sin\kappa/2}
\end{aligned} \tag{3.25}
$$

in which

$$
\cos\frac{\kappa}{2} = \cos\frac{\beta}{2} \cos\frac{\alpha + \gamma}{2} \tag{3.26}
$$

However, we only need sum of ℓ. Thus we take $\alpha = \gamma$ and take Fourier transform γ on

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both side.

$$\sum_{\ell} t^{2\ell}\mathcal{D}^{\ell}_{\mu\mu}(0,\pi-\Theta,0) = \int_0^{2\pi} \frac{\mathrm{d}\gamma}{2\pi} \frac{e^{2i\mu\gamma}}{1-t\Theta\cos\gamma+t^2} \tag{3.27}$$

And times t, differentiating respect t and substitute t as i

$$\sum_{\ell}(-1)^{\ell}(2\ell+1)d^{\ell}_{\mu\mu}(\pi-\Theta) = \int_0^{2\pi} \frac{\mathrm{d}\gamma}{2\pi} \frac{\partial}{\partial t}\left(\frac{te^{2i\mu\gamma}}{1-t\Theta\cos\gamma+t^2}\right)\Bigg|_{t=i} \tag{3.28}$$

This integral can be implemented in complex plane with $z \equiv e^{i\gamma}$. It turn out to be

$$\frac{1}{2\pi i}\oint_C \mathrm{d}z \, \frac{-2z^{1+2\mu}}{(1+z^2)^2}\frac{4}{\Theta^2} \tag{3.29}$$

The pole of integrand $\pm i$ lie on the contour which is unit circle. To prevent the divergent, we take the integral as **principal value** [1] and deform contour to

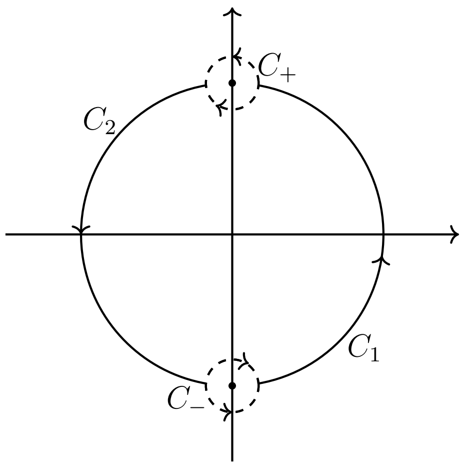

Figure 3.1: Contour of the integral

C_1,C_2 are part of unit circle and $C_\pm$ are half circle where center is $\gamma_\pm = \pm i$. Principal value of the integral is

$$\text{P.V.}\int f(z)\,dz = \int_{C_1+C_2} f(z)\,dz, \quad \text{where } f(z) \equiv \frac{z^{2\mu+1}}{(z-\gamma_+)^2(z-\gamma_-)^2}$$

There are two different contour for each $C_\pm$, include the pole or exclude the pole. If we

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choose including the pole, residue theorem imply

$$
\begin{aligned}
&\oint_C f(z)\,dz = 2\pi i \sum_k Res(f,\gamma_k) \\
&\Rightarrow \text{P.V.} \int f(z)\,dz + i\pi \sum_k Res(f,\gamma_k) = 2\pi i \sum_k Res(f,\gamma_k) \\
&= \text{P.V.} \int f(z)\,dz - i\pi \sum_k Res(f,\gamma_k) = 0
\end{aligned} \tag{3.30}
$$

The same contribution for excluding the pole. Thus, the principal value will be

$$
\begin{aligned}
&\text{P.V.} \int f(z)\,dz = i\pi \sum_k Res(f,\gamma_k) \\
&= i\pi \frac{\partial[f(z) * (z-\gamma_+)^2]}{\partial z}\bigg|_{z=\gamma_+} + i\pi \frac{\partial[f(z) * (z-\gamma_-)^2]}{\partial z}\bigg|_{z=\gamma_-} \\
&= -i\mu\pi(-1)^\mu
\end{aligned} \tag{3.31}
$$

Restoring all the factor, the eikonal limit of the amplitude is

$$
2ikf(\Theta,\Phi) \approx \sum_\ell (-1)^\ell (2\ell+1) d^\ell_{\mu\mu}(\pi-\Theta) e^{2\mu\Phi} = (-1)^\mu \frac{4\mu}{\Theta^2} e^{2i\mu\Phi} \tag{3.32}
$$

Then the differential cross section can be easily computed as

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\Theta,\Phi)|^2 \approx \frac{4\mu^2}{k^2\Theta^4}(\Theta \ll 1) \tag{3.33}
$$

3.4 Electromagnetic Duality

In this section, we discuss the duality between electric charge and monopole charge. We can start from a free Lagrangian

$$
\mathcal{L} = \frac{1}{4}(F^2 + \tilde{F}^2) \tag{3.34}
$$

where $\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. Apparently, the Lagrangian possesses an SO(2) symmetry.

$$
\begin{pmatrix} F \\ \widetilde{F} \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F \\ \widetilde{F} \end{pmatrix}
$$

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If we consider an additional current source coupled to the gauge field in the Lagrangian, the equation of motion would be

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= J_e^\nu \\ \partial_\mu \widetilde{F}^{\mu\nu} &= J_g^\nu\end{aligned} \tag{3.35}$$

J_e^ν, J_g^ν represent the source of two type of current and $\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. Then the corresponding transformation of current is

$$\begin{pmatrix} J_e \\ J_g \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} J_e \\ J_g \end{pmatrix} \tag{3.36}$$

If we consider only $F_{\mu\nu}$ without the dual part

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.37}$$

After dual transformation

$$\begin{aligned}\mathcal{L}' &= \frac{1}{4}\Big\{F_{\mu\nu}F^{\mu\nu}\cos^2\theta + \widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu}\sin^2\theta + F_{\mu\nu}\widetilde{F}^{\mu\nu}\sin 2\theta\Big\} \\ &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\sin 2\theta}{2}\partial_\mu(A_\nu \widetilde{F}^{\mu\nu})\end{aligned} \tag{3.38}$$

Since the latter term is a total derivative as a boundary condition, the equation of motion is still invariant under dual transformation. Although we won't consider the boundary term in this thesis, but the topological effect plays crucial role in SU(N) monopole and instanton[33].

To simplify the situation, we consider a point source as classical particle. Then current can be read as

$$J_e^\mu = \rho_e \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad J_g^\mu = \rho_g \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{3.39}$$

Since $Q = \int \rho dx^3$, the dual transformation can be treated as the rotation of charge

$$e + ig \to e^{i\theta}(e + ig) \tag{3.40}$$

Therefore, if the Lagrangian is dual-invariant, the charge of a single particle is irrelevant. Since we can always apply a dual transformation to obtain another charge, the only rele-

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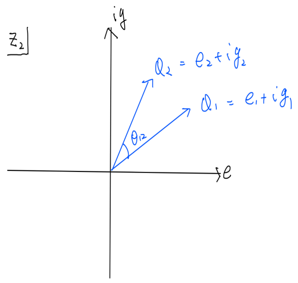
vant factor is the phase of the charge difference between any two particles.



$$|Q_1||Q_2|\cos\theta_{12} = e_1e_2 + g_1g_2$$
$$|Q_1||Q_2|\sin\theta_{12} = e_1g_2 - e_2g_1$$

For pure charge-monopole system, $\theta_{12} = \pi/2$. The electric charge and magnetic charge are exchanged, which means we have to treat the monopole and electron on equal footing. However, from experiments, we already know the value of the unit electric charge. Combined with Dirac quantization (3.2), there are two different couplings for this two-charge system. In low energy, such as our daily life, the electric coupling is weak, while the magnetic coupling is strong. This can explain why laboratories cannot find the existence of a monopole. However, in Grand Unified Theory (GUT), a solution for the monopole is needed. Therefore, in high energy, perhaps experiments can tell us whether the monopole exists or not.

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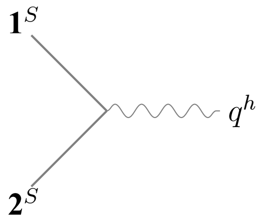
Chapter 4 On-Shell approach to monopole scattering

In this chapter, we begin with the $2 \to 2$ tree-level scattering amplitude for the electron-monopole system[1]. We then derive the impulse, scattering angle, and non-perturbative amplitude using the eikonal approximation.

4.1 The on-shell phase rotation

The minimal coupling[2] for a spin S particle to a photon is given by



$$M_3(1^S, 2^S, q^+) = \sqrt{2}Q(xm)^h \frac{\langle \mathbf{12} \rangle^{2S}}{m^{2S}} \tag{4.1}$$

The factor x is defined in (A.1.6.3), and Q represents the coupling constant, which serves as an effective charge. According to [8, 18], we can apply a complex phase shift to the x factor in order to obtain the dyon-dyon amplitude.

$$x \to xe^{i\theta} \tag{4.2}$$

This corresponds to an electromagnetic duality transformation (3.4) that rotates an electri-

[1] In this chapter, we consider the electron as an electric-charged scalar and the monopole as a magnetic-charged scalar

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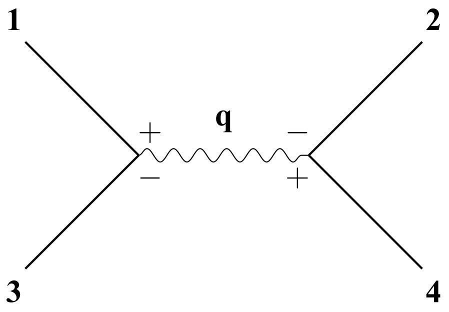

Figure 4.1: 4-point tree level in t channel

cally charged particle into a dyon, which carries both electric and magnetic charges. More specifically

$$Q \equiv e^2 + g^2, \quad \theta \equiv \arctan\left(\frac{g}{e}\right) \tag{4.3}$$

We are interested in the tree-level $2 \to 2$ scattering amplitude, whose limit as $q^2 \to 0$ is the product of the three-point amplitudes summed over all possible helicities. Combining (4.2) and (4.1),we have

$$M_3(q^{\pm}13) = \sqrt{2}Q_1 m_1 x_1^{\pm 1} e^{\pm i\theta_1}, \quad M_3(q^{\pm}24) = \sqrt{2}Q_2 m_2 x_2^{\pm 1} e^{\pm i\theta_2} \tag{4.4}$$

Therefore

$$\begin{aligned} M_4(1,2 \to 3,4)\big|_{q^2\to 0} &= \frac{M_3(q^+13)M_3(q^-24) + M_3(q^-13)M_3(q^+24)}{q^2} \\ &= \frac{2m_1 m_2 Q_1 Q_2}{q^2}\Big(\frac{x_1 e^{i\theta_1}}{x_2 e^{i\theta_2}} + \frac{x_2 e^{i\theta_2}}{x_1 e^{i\theta_1}}\Big) \end{aligned} \tag{4.5}$$

With the relation (4.3), we can write the above expression as

$$M_4(1,2 \to 3,4)\big|_{q^2\to 0} = \frac{2m_1 m_2}{q^2}\{Q_1 Q_2 \cos\theta_{12}(\frac{x_1}{x_2} + \frac{x_2}{x_1}) + iQ_1 Q_2 \sin\theta_{12}(\frac{x_1}{x_2} - \frac{x_2}{x_1})\} \tag{4.6}$$

As we explicitly show in appendix (A.1.7), the ratio of x is

$$\frac{x_1}{x_2} = \frac{p_1 \cdot p_2}{m_1 m_2} + \frac{i\epsilon(\eta, p_1, q, p_2)}{m_1 m_2 (q \cdot \eta)} \tag{4.7}$$

$$\frac{x_2}{x_1} = \frac{p_1 \cdot p_2}{m_1 m_2} - \frac{i\epsilon(\eta, p_1, q, p_2)}{m_1 m_2 (q \cdot \eta)} \tag{4.8}$$

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We have also defined

$$\epsilon(a,b,c,d) \equiv \epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma$$

Thus eq.(4.6) will be

$$M_4(1,2\to 3,4)\big|_{q^2\to 0} = \frac{4m_1m_2}{q^2}\{Q_1Q_2\cos\theta_{12}[\frac{p_1\cdot p_2}{m_1m_2}]+iQ_1Q_2\sin\theta_{12}[\frac{i\epsilon(\eta,p_1,q,p_2)}{m_1m_2(q\cdot\eta)}]\} \tag{4.9}$$

In this thesis, we focus on electron-monopole system , which means $Q_1 = e$, $Q_2 = g$, and $\theta_{12} = \pi/2$. Then

$$M_{2\to 2}(e,g\to e,g)\big|_{q^2\to 0} = 4eg\frac{\epsilon(\eta,p_1,q,p_2)}{q^2}\frac{1}{q\cdot\eta} \tag{4.10}$$

This exactly match with photon propagator[35, 37] between electric current and a magnetic current in $q^2 \to 0$.

4.2 KMOC formalism

In this section, we will provide a brief overview of the relationship between impulse and amplitude [23, 26]. We prepare the initial state as $|\psi_{\text{in}}\rangle$, which consists of two incoming particles with wave functions $\phi_i(p_i)$. Since we are interested in classical point-like particles, the wave function should have a a well-defined momentum and position. The wave packet can be defined as follows

$$|\psi\rangle_{\text{in}} \equiv \int d\Phi(p_1)d\Phi(p_2)\phi_1(p_1)\phi_2(p_2)e^{ib\cdot p_1/\hbar}|p_1p_2\rangle \tag{4.11}$$

where

$$\begin{aligned} d\Phi(p) &\equiv \hat{d}^4p\,\hat{\delta}^{(+)}(p^2-m^2) \\ \hat{d}^np &\equiv \frac{d^np}{(2\pi)^n} \\ \hat{\delta}^{(+)}(p^2-m^2) &\equiv 2\pi\Theta(p^0)\delta(p^2-m^2) \end{aligned} \tag{4.12}$$

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and b is impact parameter, which is distance between two particle. In quantum physics, the observable is the expectation value of operator. Then the outing momentum is

$$\begin{aligned}\langle p^{\mu}_{out,1}\rangle &= {}_{\text{out}}\langle\psi|\mathbb{P}^{\mu}_1|\psi\rangle_{\text{out}} \\ &= {}_{\text{in}}\langle\psi|U(\infty,-\infty)^{\dagger}\mathbb{P}^{\mu}_1 U(\infty,-\infty)|\psi\rangle_{\text{in}}\end{aligned}$$

where $U(\infty,-\infty)$ is the time operator from the past to future in asymptotic state. In quantum field theory, the time evolution operator is nothing but S-matrix. Impulse is defined as the difference between incoming momentum and outgoing momentum. Therefore

$$\begin{aligned}\langle\Delta p^{\mu}_1\rangle &= {}_{\text{out}}\langle\psi|\mathbb{P}^{\mu}_1|\psi\rangle_{\text{out}} - {}_{\text{in}}\langle\psi|\mathbb{P}^{\mu}_1|\psi\rangle_{\text{in}} \\ &= {}_{\text{in}}\langle\psi|S^{\dagger}\mathbb{P}^{\mu}_1 S|\psi\rangle_{\text{in}} - {}_{\text{in}}\langle\psi|\mathbb{P}^{\mu}_1|\psi\rangle_{\text{in}}\end{aligned} \tag{4.13}$$

We can write S-matrix in terms of transition matrix T via $S = 1 + iT$ and combine the unitary condition $S^{\dagger}S = 1$. Eq.(4.13) can reduce to

$$\begin{aligned}\langle\Delta p^{\mu}_1\rangle &= I^{\mu}_{(1)} + I^{\mu}_{(2)} \\ I^{\mu}_{(1)} &\equiv \langle\psi|i[\mathbb{P}^{\mu}_1, T]|\psi\rangle \\ I^{\mu}_{(2)} &\equiv \langle\psi|T^{\dagger}[\mathbb{P}^{\mu}_1, T]|\psi\rangle\end{aligned} \tag{4.14}$$

Insert eq.(4.11) into eq.(4.13) and first part contribute to amplitude would be

$$\begin{aligned}I^{\mu}_{(1)} = \int d\Phi(p_1)d\Phi(p_2)d\Phi(p'_1)d\Phi(p'_2)\phi_1(p_1)\phi^*_1(p'_1)\phi_2(p_2)\phi^*_2(p'_2) \\ \times e^{ib\cdot(p_1-p'_1)/\hbar} i(p'^{\mu}_1 - p^{\mu}_1)\langle p'_1 p'_2|T|p_1 p_2\rangle\end{aligned} \tag{4.15}$$

where the matrix element is

$$\langle p'_1 p'_2|T|p_1 p_2\rangle = M_4(p_1, p_2 \to p'_1, p'_2)\hat{\delta}^{(4)}(p'_1 + p'_2 - p_1 - p_2) \tag{4.16}$$

Here, we label the incoming state as $p_{1,2}$ and their conjugate as $p'_{1,2}$. And we introduce the momentum mismathch $q^{\mu}_i = p^{'\mu}_i - p^{\mu}_i$ and change the integral variable from p'_i to q_i

$$d\Phi(q_i + p_i) = \hat{d}^4 q_i \hat{\delta}((p_i + q_i)^2 - m^2_i)\Theta(p^0_i + q^0_i) \tag{4.17}$$

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Then

$$
\begin{aligned}
I^{\mu}_{(1)} = \int & d\Phi(p_1) d\Phi(p_2) \Phi(p_1+q_1) d\Phi(p_2+q_2) \\
& \times e^{-ib\cdot q_1/\hbar} \phi(p_1)\phi^*(p_1+q_1)\phi(p_2)\phi^*(p_2+q_2) \\
& \times iq_1^{\mu} M_4(p_1, p_2 \to p_1+q_1, p_2+q_2)
\end{aligned} \tag{4.18}
$$

We can integrate q_2 variable and relabel q_1 to q, then

$$
\begin{aligned}
I^{\mu}_{(1)} = \int & d\Phi(p_1) d\Phi(p_2) \hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2) \Theta(p_1^0 + q^0) \hat{\delta}(2p_2 \cdot q - q^2) \Theta(p_2^0 - q^0) \\
& \times e^{-ib\cdot q/\hbar} \phi(p_1)\phi^*(p_1+q)\phi(p_2)\phi^*(p_2-q) \\
& \times iq^{\mu} M_4(p_1, p_2 \to p_1+q, p_2-q)
\end{aligned} \tag{4.19}
$$

Now we shift our focus to the second part of the impulse and utilize the complete set of intermediate states labeled by momentum ℓ_1 and ℓ_2, along with an additional degree of freedom denoted as X.

$$
\begin{aligned}
I^{\mu}_{(2)} &= \langle \psi | T^{\dagger} [\mathbb{P}^{\mu}_1, T] | \psi \rangle \\
&= \sum_X \int \prod_{i=1,2} d\Phi(\ell_i) \langle \psi | T^{\dagger} | \ell_1 \ell_2 X \rangle \langle \ell_1 \ell_2 X | [\mathbb{P}^{\mu}_1, T] | \psi \rangle
\end{aligned} \tag{4.20}
$$

Here we adopt the normalization of momentum state as

$$
\langle p' | p \rangle \equiv 2E_p \, \hat{\delta}^{(3)}(\mathbf{p}' - \mathbf{p}) \tag{4.21}
$$

Insert the definition of incoming state and the matrix element,then

$$
\begin{aligned}
I^{\mu}_{(2)} = \sum_X \int \prod_{i=1,2} & d\Phi(\ell_i) d\Phi(p_i) d\Phi(p_i') \phi(p_i) \phi(p_i') e^{ib\cdot(p_1 - p_1')/\hbar} (\ell_1^{\mu} - p_1^{\mu}) \\
& \times \hat{\delta}^{(4)}(p_1 + p_2 - \ell_1 - \ell_2) \hat{\delta}^{(4)}(p_1' + p_2' - \ell_1 - \ell_2) \\
& \times M(p_1, p_2 \to \ell_1, \ell_2) M^*(p_1', p_2' \to \ell_1, \ell_2)
\end{aligned} \tag{4.22}
$$

We introduce the momentum mismatch $q_i^{\mu} = p_i'^{\mu} - p_i^{\mu}$ and momentum transfer $\omega_i^{\mu} =$

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$\ell_i^\mu - p_i^\mu$. Then we integrate out q_1, ω_1 and relabel $q_2 \to q$ and $\omega_2 \to \omega$

$$
\begin{aligned}
I_{(2)}^\mu = \sum_X \int \prod_{i=1,2} & d\Phi(p_i) \hat{d}^4 q \hat{d}^4 \omega \hat{\delta}(2p_1 \cdot q + q^2)\Theta(p_1^0 + q^0)\hat{\delta}(2p_2 \cdot q - q^2)\Theta(p_2^0 - q^0) \\
& \times \phi(p_1)\phi(p_2)\phi^*(p_1+q)\phi^*(p_2-q)\hat{\delta}(2p_1 \cdot \omega + \omega^2)\Theta(p_1^0 + \omega^0)\hat{\delta}(2p_2 \cdot \omega - \omega^2)\Theta(p_2^0 - \omega^0) \\
& \times e^{-ib\cdot q/\hbar}\omega^\mu M(p_1, p_2 \to p_1+\omega, p_2-\omega) M^*(p_1+q, p_2-q \to p_1+\omega, p_2-\omega)
\end{aligned}
\tag{4.23}
$$

Since our interest lies in classical observables, we need to extract the classical contribution from the impulse. In practice, we rescale the momentum transfer as $q \to \hbar q$, $\omega \to \hbar\omega$ and the coupling constant as $g_c \to g_c/\sqrt{\hbar}$, and then consider the terms up to $\mathcal{O}(\hbar^0)$ in the impulse. The remaining part involves integrating over the wave function $\phi(p_i)$. From a classical standpoint, the peak of the wave function corresponds to the classical value, and the expectation of momentum coincides with the classical momentum.

$$
\langle p^\mu \rangle = \int d\Phi(p) p^\mu |\phi(p)|^2 = p^\mu_{\text{classical}} + \mathcal{O}(\ell_c) \tag{4.24}
$$

where ℓ_c is Compton wavelength.

4.3 Impulse from tree amplitude

In this section, we derive the impulse formula for tree-level scattering. In the classical scenario, we prepare particles with certain momenta and capture them after scattering. The impulse formula can be simplified as

$$
\Delta p_1^\mu = -\frac{\partial \chi(b)}{\partial b_\mu} + i\chi^*(-b)\frac{\partial \chi(b)}{\partial b_\mu} \tag{4.25}
$$

where $\chi(b)$ is eikonal phase[25] and defined as

$$
\chi(b) = \int d^4\hat{q}\, \hat{\delta}(2q \cdot p_1)\hat{\delta}(2q \cdot p_2) \times e^{ib\cdot q} M_4(1, 2 \to 1', 2')|_{q^2 \to 0} \tag{4.26}
$$

$$
d^n\hat{q} \equiv \frac{d^n q}{(2\pi)^n}, \quad \hat{\delta}(x) \equiv 2\pi\delta(x)
$$

Since the mass of the monopole is assumed to be very heavy, we can consider particle 2 as the monopole and set it in the rest frame, where $p_2 = (m_2, 0, 0, 0)$ acts as a classical background. To simplify our calculations, we consider particle 1, the electron, moving

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along the z-axis, which gives $p_1 = m_1(\gamma, 0, 0, \gamma\beta)$. The impact parameter **b** lies in the xy plane as a two-dimensional vector, and η is represented as $(0, \mathbf{n}, 0)$. The eikonal phase is then given by

$$\chi(b) = \frac{eg}{4\pi^2} \int dq^2\, e^{i\mathbf{b}\cdot\mathbf{q}}\, \frac{n_x q_y - n_y q_x}{q^2} \frac{1}{\mathbf{q}\cdot\mathbf{n}} \tag{4.27}$$

To carry out this integral, we follow [6] to regulate the integral to

$$\chi(b) = \frac{eg}{8\pi^2} \int dq^2\, e^{i\mathbf{b}\cdot\mathbf{q}}\, \frac{n_x q_y - n_y q_x}{q^2} \left(\frac{1}{\mathbf{q}\cdot\mathbf{n} + i\varepsilon} + \frac{1}{\mathbf{q}\cdot\mathbf{n} - i\varepsilon} \right) \tag{4.28}$$

and use Schwinger parameters

$$\begin{aligned} \frac{1}{q^2} &= \int_0^\infty ds\, e^{-sq^2} \\ \frac{1}{\mathbf{q}\cdot\mathbf{n} + i\varepsilon} &= \frac{1}{i} \int_0^\infty d\lambda\, e^{i\lambda(\mathbf{q}\cdot\mathbf{n}) - \lambda\varepsilon}, \quad \mathrm{Re}\{\varepsilon\} > 0 \\ \frac{1}{\mathbf{q}\cdot\mathbf{n} - i\varepsilon} &= \frac{-1}{i} \int_0^\infty d\lambda\, e^{-i\lambda(\mathbf{q}\cdot\mathbf{n}) - \lambda\varepsilon}, \quad \mathrm{Re}\{\varepsilon\} > 0 \end{aligned}$$

Then integral become

$$\begin{aligned} \chi(b) =& \frac{eg}{8\pi^2} \int_0^\infty d\lambda \int_0^\infty ds \int d^2q\, e^{-sq^2} (n_x q_y - n_y q_x) \frac{1}{i} \\ & \left[e^{i\lambda(\mathbf{q}\cdot\mathbf{n})} - e^{-i\lambda(\mathbf{q}\cdot\mathbf{n})} \right] e^{-\lambda\epsilon} e^{i\mathbf{q}\cdot\mathbf{b}} \end{aligned}$$

$$= -\frac{eg}{4\pi}(n_x b_y - n_y b_x) \int_0^\infty ds \left(\frac{1}{(\mathbf{b} + s\mathbf{n})^2} - \frac{1}{(\mathbf{b} - s\mathbf{n})^2} \right) e^{-s\varepsilon} \tag{4.29}$$

We also know that ε as regulator have to be small. Therefore, we can perturb ε to first order

$$\chi(b) = -\frac{eg}{4\pi}(n_x b_y - n_y b_x) \int_0^\infty ds \left(\frac{1}{(\mathbf{b} + s\mathbf{n})^2} - \frac{1}{(\mathbf{b} - s\mathbf{n})^2} \right) \left(1 - s\varepsilon + \mathcal{O}(\varepsilon^2)\right) \tag{4.30}$$

and decompose $\chi = \chi_{(0)} + \varepsilon\chi_{(1)} + \mathcal{O}(\varepsilon^2)$. If we change our variable as

$$u = s + \mathbf{n}\cdot\mathbf{b} \tag{4.31}$$

$$v = s - \mathbf{n}\cdot\mathbf{b} \tag{4.32}$$

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Then $\chi_{(0)}$ can be expressed as

$$\chi_{(0)} = -\frac{eg}{4\pi}(n_x b_y - n_y b_x)\left(\int_{\mathbf{n}\cdot\mathbf{b}}^{\infty} \frac{du}{u^2+b^2-(\mathbf{n}\cdot\mathbf{b})^2} - \int_{-\mathbf{n}\cdot\mathbf{b}}^{\infty} \frac{dv}{v^2+b^2-(\mathbf{n}\cdot\mathbf{b})^2}\right)$$
$$= -\frac{eg}{4\pi}(n_x b_y - n_y b_x)\int_{\mathbf{n}\cdot\mathbf{b}}^{-\mathbf{n}\cdot\mathbf{b}} \frac{du}{u^2+b^2-(\mathbf{n}\cdot\mathbf{b})^2} \quad (4.33)$$

move to polar coordinate $\mathbf{n} = (\cos\theta, \sin\theta)$, $\mathbf{b} = b(\cos\phi, \sin\phi)$

$$\chi_{(0)} = -\frac{eg}{4\pi} b\sin(\phi-\theta)\int_{b\cos(\phi-\theta)}^{-b\cos(\phi-\theta)} \frac{du}{u^2+b^2\sin^2(\phi-\theta)}$$
$$= -\frac{eg}{4\pi}[\arctan(-\cot(\phi-\theta)) - \arctan(\cot(\phi-\theta))]$$
$$= 2\frac{eg}{4\pi}(\frac{\pi}{2}+\theta-\phi) \quad (4.34)$$

According (4.25),the leading order of impulse is

$$\Delta\mathbf{p} = -\frac{1}{b}\frac{\partial\chi^{(0)}}{\partial\phi} = 2\frac{eg}{4\pi b}\hat{\phi} \quad (4.35)$$

The result matches the classical result (2.27), and this correspondence has already been demonstrated in [18, 24]. In the non-relativistic limit, the momentum of the incoming particle is given by $\mathbf{p}_{\text{in}} = m\mathbf{v}$. The deflection angle is defined as

$$\vartheta \equiv \frac{|\Delta\mathbf{p}|}{|\mathbf{p}_{in}|} = \frac{2eg}{4\pi\tilde{L}} \quad (4.36)$$

the same formula with electron-monopole scattering (eq.2.22) in small angle.

4.4 One-loop amplitude to impulse

In this section, we consider the next leading order of the impulse. Therefore, we need to compute the one-loop amplitude using the unitarity method [3]. Since the impulse is a purely classical observable, we only consider the box and triangle diagrams. Hence, we expand the amplitude using an integral basis.

$$M_{\text{classical}}^{\text{one-loop}} = c_{\triangle}I_{\triangle} + c_{\bigtriangledown}I_{\bigtriangledown} + c_{\Box}I_{\Box} + c_{\bowtie}I_{\bowtie} \quad (4.37)$$

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where box and triangle integral are

$$I_{\square} = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2[(\ell-p_1)^2-m_a^2](\ell-q)^2[(\ell+p_2)^2-m_b^2]}$$
$$I_{\bowtie} = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2[(\ell+p_4)^2-m_a^2](\ell-q)^2[(\ell+p_2)^2-m_b^2]}$$
$$I_{\triangle} = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2(\ell-q)[(\ell-p_1)^2-m_a^2]}$$
$$I_{\bigtriangledown} = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2(\ell-q)^2[(\ell+p_2)^2-m_b^2]} \tag{4.38}$$

we set the external momentum in center of momentum (COM) frame

$$p_1 = (E_a, \vec{\mathbf{p}}),\ p_2 = (E_b, -\vec{\mathbf{p}}),\ p_3 = (E_b, -\vec{\mathbf{p}}'),\ p_4 = (E_a, \vec{\mathbf{p}}') \tag{4.39}$$

where $\vec{\mathbf{p}}' = \vec{\mathbf{p}} - \vec{\mathbf{q}}$ and $\vec{\mathbf{q}}$ is the momentum transfer between two particle. Expanding in $|\vec{\mathbf{q}}|$ variable and taking classical limit [5, 13], we have

$$I_{\bowtie}\Big|_{\text{classical}} = -\frac{i}{16\pi^2|\vec{\mathbf{q}}|^2}\left[\frac{1}{\epsilon} - \log|\vec{\mathbf{q}}|^2\right]\left[\frac{\log\left[\sigma-\sqrt{\sigma^2-1}\right]}{m_a m_b\sqrt{\sigma^2-1}} + \frac{i\pi}{|\mathbf{p}|\sqrt{s}}\right]$$
$$I_{\square}\Big|_{\text{classical}} = \frac{i}{16\pi^2|\vec{\mathbf{q}}|^2}\left[\frac{1}{\epsilon} - \log|\vec{\mathbf{q}}|^2\right]\frac{\log\left[\sigma-\sqrt{\sigma^2-1}\right]}{m_a m_b\sqrt{\sigma^2-1}}$$
$$I_{\triangle,\bigtriangledown}\Big|_{\text{classical}} = -\frac{1}{32 m_{a,b}}\frac{i}{|\vec{\mathbf{q}}|} \tag{4.40}$$

We calculate the integral coefficient in (4.37) using the unitarity cut methods provided in [4, 9]. The unitarity cut of the one-loop amplitude can be obtained by sewing together three-point and four-point tree amplitudes. For example, the box coefficient and triangle coefficient can be generated by the quadruple and triple cut,respectively. We perform the

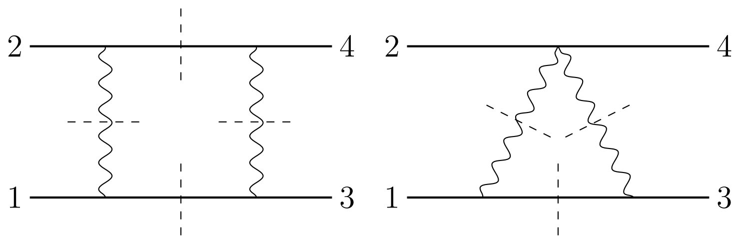

Figure 4.2: quadruple cut and triangle cut

sewing of the double cut C_2 with the Compton amplitudes [2] along with a phase rotation.

(4.2)

$$M_4(p_1, k_2^{+1}, k_3^{+1}, p_4) = -\frac{m^2[23]^2}{(s-m^2)(u-m^2)}e^{2i\phi}$$
$$M_4(p_1, k_2^{+1}, k_3^{-1}, p_4) = -\frac{[2|p_1|3\rangle^2}{(s-m^2)(u-m^2)}$$
$$M_4(p_1, k_2^{-1}, k_3^{-1}, p_4) = -\frac{m^2\langle 23\rangle^2}{(s-m^2)(u-m^2)}e^{-2i\phi} \quad (4.41)$$

The quadruple and triple cut can be generated by the basis C_2 in [9]

$$C_2 = \sum_{h_0,h_1=\pm 1} M_4(p_1, -p_4, -\ell^{h_0}, \ell_1^{h_1}) M_4(p_2, -p_3, \ell^{h_0}, -\ell_1^{h_1}) \quad (4.42)$$

Then triple cut $C_{3,\#}$ can be extracted from C_2

$$C_{3,\triangle} = (-2\ell \cdot p_1)\, C_2\Big|_{\ell\cdot p_1=0}, \quad C_{3,\bigtriangledown} = (2\ell\cdot p_2)\, C_2\Big|_{\ell\cdot p_2=0}, \quad (4.43)$$

The cut condition for triangle diagram is

$$\ell^2 = (\ell+q)^2 = (\ell-p_1)^2 - m_a^2 = 0 \quad (4.44)$$

We adopt loop parametrization with one free variable as [4, 9]

$$\ell_\pm^\mu(z) = \alpha q^\mu + \beta p_1^\mu + z u_\pm^\mu + \frac{\alpha^2\gamma_\pm}{16z} v_\pm^\mu \quad (4.45)$$

where $\pm$ label the two solution of cut solution and

$$u_\pm^\mu = \langle Q_\pm^\flat|\sigma^\mu|P_\pm^\flat], \quad v_\pm^\mu = \langle P_\pm^\flat|\sigma^\mu|Q_\pm^\flat], \quad P_\pm^{\flat\mu} = p_1^\mu + \frac{q^\mu}{\gamma_\pm}, \quad Q_\pm^{\flat\mu} = q^\mu + \frac{q^2}{m_a^2\gamma_\pm}p_1^\mu,$$
$$\alpha = \frac{2m_a}{4m_a^2 - q^2}, \quad \beta = -\frac{q^2}{4m_a^2-q^2}, \quad \gamma_\pm = -\frac{q^2 \pm \sqrt{q^2(q^2-4m_a^2)}}{2m_a^2} \quad (4.46)$$

By substituting (4.45) into (4.43), we obtain the triple cut, and the corresponding triangle coefficient can be generated as described in [14]

$$c_\triangle = \frac{1}{2}\left[\sum_{\ell=\ell_\pm(z)} \mathrm{Inf}_z C_{3,\triangle}(\ell)\right] \quad (4.47)$$

Here, $\mathrm{Inf}_z C_{3,\triangle}(\ell)$ represents the constant term in the Laurent expansion of $C_{3,\triangle}$ at $z=\infty$. Another coefficient, $c_\bigtriangledown$, can be obtained by swapping m_a and m_b. The triangle contribu-

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tion to the one-loop amplitude is

$$M_{\triangle,\bigtriangledown} = c_{\triangle} I_{\triangle} + c_{\bigtriangledown} I_{\bigtriangledown} \tag{4.48}$$

To extract the classical contribution, we need to restore factors of $\hbar$ with the appropriate scaling

$$m_{a/b} \to m_{a/b}, \quad p_{1/2} \to p_{1/2}, \quad q \to q\hbar \tag{4.49}$$

and only the terms of order $\mathcal{O}(\hbar^0)$ survive. In the classical limit, the triangle contribution is

$$M_{\triangle,\bigtriangledown} = \frac{m_a + m_b}{16|\vec{\mathbf{q}}|} \tag{4.50}$$

Since the classical contribution from same helicity Compton amplitude is zero, there is no phase rotation in triangle coefficient.

The box and crossed box coefficients can also be extracted from C_2 as

$$C_{4,\square} = (-2\ell \cdot p_1)(2\ell \cdot p_2)\, C_2\Big|_{\ell\cdot p_1 = \ell \cdot p_2 = 0}, \; C_{4,\bowtie} = (2\ell \cdot p_4)(2\ell \cdot p_2)\, C_2\Big|_{\ell\cdot p_4 = \ell \cdot p_2 = 0} \tag{4.51}$$

The relation between box coefficient and crossed box coefficient is

$$c_{\bowtie} = c_{\square}\Big|_{p_1 \to -p_4, p_4 \to -p_1} \tag{4.52}$$

The cut condition is

$$\ell^2 = (\ell + q)^2 = (\ell - p_1)^2 - m_a^2 = (\ell + p_2)^2 - m_b^2 = 0 \tag{4.53}$$

We adopt the four solutions in [4, 9]

$$\begin{aligned} \ell^\mu_{\pm,1} &= -\frac{q^2 \eta^\mu}{2q \cdot \eta_\pm}, \\ \ell^\mu_{\pm,2} &= \frac{N_1 p_1^\mu + N_2 p_2^\mu + N_q q^\mu}{\mathcal{N}} \end{aligned} \tag{4.54}$$

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where

$$N_1 = 2m_b(m_b + m_a\sigma)q^2,\ N_2 = -2m_a(m_a + m_b\sigma)q^2$$
$$N_q = 4m_a^2m_b^2(\sigma^2 - 1), \quad \mathcal{N} = N_q + \frac{N_2 - N_1}{2}$$
$$\eta_\pm^\mu = \langle k_1|\sigma^\mu|k_2]$$
$$k_{1,\pm}^\mu = p_1^\mu + m_a^2 p_2^\mu \zeta_\pm, \quad k_{2,\pm}^\mu = p_2^\mu + m_b^2 p_1^\mu \zeta_\pm, \quad \zeta_\pm = \frac{-\sigma \pm \sqrt{\sigma^2 - 1}}{m_a m_b} \tag{4.55}$$

By substituting the cut solution into C_2, we can obtain the box coefficient as

$$c_\square = \frac{1}{4}\sum_{i=\pm}\sum_{j=1,2} C_{4,\square}(\ell_{i,j}) = 4m_a^2m_b^2(2\sigma^2 + \cos 2\phi - 1)\Big|_{\phi=\frac{\pi}{2}} \tag{4.56}$$
$$= 4m_a^2m_b^2(2\sigma^2 - 2) \tag{4.57}$$

The contribution from box and cross box diagram is

$$M_{\square+\bowtie} = im_a^2m_b^2(2 - 2\sigma^2)\frac{\log q^2}{4q^2}\frac{1}{|\vec{\mathbf{p}}|\sqrt{s}} \tag{4.58}$$

The pre-eikonal phase from one-loop amplitude with coupling constant is

$$\chi_{\triangle+\triangledown} = -e^2g^2\frac{m_a + m_b}{32\pi|\vec{\mathbf{p}}|\sqrt{s}b}$$
$$\chi_{\square+\bowtie} = \frac{ie^2g^2m_a^2m_b^2(1-\sigma^2)}{|\vec{\mathbf{p}}|^2 s}\frac{\log^2 b}{8\pi^2} \tag{4.59}$$

The result is inconsistent with the eikonal approximation

$$\chi_{\square,\bowtie} \neq \frac{i}{2}\chi_{(0)}^2 \tag{4.60}$$

To bridge the gap, we can select another tree amplitude to perform an on-shell phase rotation using the Gram determinant

$$\epsilon(\eta, p_1, q, p_2)^2 = m_a^2m_b^2(q\cdot\eta)^2(1-\sigma^2), \tag{4.61}$$
$$\text{where} \quad p_1\cdot q = p_2\cdot q = q^2 = \eta^2 = 0$$

Then the tree amplitude, with the hotfix, is

$$M_{\text{tree}}^{\text{fix}} = \pm 4eg\frac{m_am_b\sqrt{1-\sigma^2}}{q^2} \tag{4.62}$$

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Then the eikonal phase is reduced to

$$\chi^{\text{fix}}_{(0)} = \mp \frac{egm_a m_b\sqrt{1-\sigma^2}}{|\vec{\mathbf{p}}|\sqrt{s}}\frac{\log b}{2\pi} \tag{4.63}$$

which satisfies the exponentiation of the hotfix eikonal phase, and the contributions to the impulse in $\mathcal{O}(1/\hbar)$ cancel each other out.

$$-\frac{\partial\chi_{\square+\bowtie}(b)}{\partial b_\mu} + i\chi^{\text{fix}}_{(0)}(-b)\frac{\partial\chi^{\text{fix}}_{(0)}(b)}{\partial b_\mu} = 0 \tag{4.64}$$

Therefore, the contribution of the impulse from the one-loop calculation only depends on the triangle part

$$\begin{aligned}\Delta\mathbf{p} &= -\frac{\partial\chi_{\triangle+\nabla}}{\partial\vec{\mathbf{b}}} = -\frac{e^2g^2(m_a+m_b)}{32\pi|\mathbf{p}|\sqrt{s}b^2}\hat{\mathbf{b}} \\ &\approx -\frac{e^2g^2}{32\pi|\mathbf{p}|b^2}\hat{\mathbf{b}} \end{aligned}\tag{4.65}$$

This satisfies the classical result in impact parameter space (2.26) with the identification $\tilde{L} \equiv |\vec{\mathbf{p}}|b$, and it is consistent with the electromagnetic case up to $\mathcal{O}(e^4)$ [29]. In the last equation, we make the approximations $m_a + m_b \approx m_a$ and $\sqrt{s} \approx m_a$ due to the heavy mass of the monopole.

4.5 From eikonal phase to all order amplitude

In standard quantum field theory, Feynman rules play a vital role in calculating amplitudes order by order. As the order of the coupling constant increases, the complexity of constructing the amplitude also increases. However, in the eikonal limit ($\frac{t}{s} \ll 1$) [1], we can obtain this infinite series in a simple formula [20, 25]

$$M_4^{eik}(s,t) = 4\sqrt{s}\,|\vec{\mathbf{p}}|\int d^2\hat{b}\,e^{-i\mathbf{b}\cdot\mathbf{q}}(e^{i\chi}-1) \tag{4.66}$$

where χ is defined in (4.26), and p_1 and p_2 are in the center-of-momentum (COM) frame (4.39). There are two different eikonal phases depending on whether strings are involved or not. We will start with the string-dependent eikonal phase. In order to ensure the convergence of the integral, we need to introduce a regulator to the eikonal phase. Naturally,

[1] Here, s and t are the Mandelstam variables, where $s = (p_1+p_2)^2$ and $t = (p_1-p_3)^2$

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we can compute the correction to the eikonal phase at $\mathcal{O}(\varepsilon)$ using a regularization scheme (4.28).

$$\chi^{(1)} = -\frac{eg}{4\pi}(n_x b_y - n_y b_x)\int_0^\infty ds\left(\frac{s}{(\mathbf{b}-s\mathbf{n})^2} - \frac{s}{(\mathbf{b}+s\mathbf{n})^2}\right) \tag{4.67}$$

By changing variables according to (4.31), the equation becomes

$$\chi^{(1)} = -\frac{eg}{4\pi}(n_x b_y - n_y b_x)\left(\int_{-\mathbf{n}\cdot\mathbf{b}}^\infty dv\frac{v+\mathbf{n}\cdot\mathbf{b}}{v^2+b^2-(\mathbf{n}\cdot\mathbf{b})^2} - \int_{\mathbf{n}\cdot\mathbf{b}}^\infty du\frac{u-\mathbf{n}\cdot\mathbf{b}}{u^2+b^2-(\mathbf{n}\cdot\mathbf{b})^2}\right) \tag{4.68}$$

We need to evaluate two integrals

$$\begin{aligned}\int \frac{u}{u^2+c} &= \frac{1}{2}\ln(1+u^2)\\ \int \frac{1}{u^2+c} &= \frac{\arctan\left(\frac{u}{\sqrt{c}}\right)}{\sqrt{c}}\end{aligned} \tag{4.69}$$

Both the first and second terms exhibit logarithmic divergence, but they can cancel each other out. This is precisely why we employ two different regularization schemes. By utilizing the same polar parameterization in $\chi^{(0)}$, the final result will be

$$\begin{aligned}\chi^{(1)} =& -\frac{egb}{4\pi}\sin(\phi-\theta)\frac{1}{2}\ln\left(b^2+u^2-b\cos(\phi-\theta)\right)\Big|_{u=-b\cos(\phi-\theta)}^{u=b\cos(\phi-\theta)}\\ &-\frac{egb}{4\pi}\cos(\phi-\theta)\left(\arctan\left(\frac{u}{b\sin(\phi-\theta)}\right)\Big|_{\mathbf{n}\cdot\mathbf{b}}^{\infty} + \arctan\left(\frac{u}{b\sin(\phi-\theta)}\right)\Big|_{-\mathbf{n}\cdot\mathbf{b}}^{\infty}\right)\\ =& -\frac{egb}{4\pi}\pi\cos(\phi-\theta)\end{aligned} \tag{4.70}$$

Combining the above calculations, the eikonal phase up to $\mathcal{O}(\varepsilon)$ is

$$\begin{aligned}\chi(b) &= -2\frac{eg}{4\pi}(\phi-\theta-\frac{\pi}{2}) - \frac{eg}{4\pi}\varepsilon\pi b\cos(\phi-\theta) + \mathcal{O}(\varepsilon^2)\\ &= 2\frac{eg}{4\pi}\arctan\left(\frac{\mathbf{n}\cdot\mathbf{b}}{\hat{\mathbf{z}}\cdot(\mathbf{b}\times\mathbf{n})}\right) - \frac{eg}{4\pi}\varepsilon\pi(\mathbf{b}\cdot\mathbf{n}) + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{4.71}$$

Since the integrand of -1 is just a two-fold delta function, the former term is our target. We can substitute χ from (4.71) with polar parametrization up to $\mathcal{O}(\varepsilon)$ and $\mathbf{q} =$

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$q(\cos\psi, \sin\psi)$. Thus, we have

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \,\exp\left(i2\frac{eg}{4\pi}(\frac{\pi}{2}+\theta-\phi)\right) \int_0^{\infty} b\,db\,\exp\left(-iqb\cos(\psi-\phi) - i\frac{eg}{4\pi}\varepsilon\pi b\cos(\phi-\theta)\right) \quad (4.72)$$

Here, we assume $\mathrm{Im}(\varepsilon) > 0$ to ensure the convergence of the integral. After integrating over the radial part, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \,\frac{-\exp\left(i2\frac{eg}{4\pi}(\frac{\pi}{2}+\theta-\phi)\right)}{\left(q\cos(\psi-\phi)+\frac{eg}{4\pi}\pi\varepsilon\cos(\phi-\theta)\right)^2} \quad (4.73)$$

To simplify integral, let $y=\psi-\phi$,$z=e^{-iy}$ and $\mu\equiv eg/(4\pi)$

$$\Rightarrow \frac{1}{2\pi i}\exp\left(i2\mu(\frac{\pi}{2}+\theta-\psi)\right)\oint_C dz\,\frac{4z^{2\mu+1}}{(q+\alpha\beta)^2(z-\gamma_+)^2(z-\gamma_-)^2} \quad (4.74)$$

where $\alpha\equiv\mu\varepsilon\pi$, $\beta\equiv e^{i(\psi-\theta)}$, and $\gamma_\pm$ are the two poles of the polynomial

$$\gamma_\pm = \pm i\sqrt{\frac{\alpha+\beta q}{\alpha\beta^2+\beta q}}$$

The contour of the integral is the unit circle, but unfortunately, both $\gamma_\pm$ lie on the boundary. Therefore, we evaluate this integral using its principal value. Due to Dirac quantization (3.2), there are no branch cuts appearing in the integral region, and the contour is

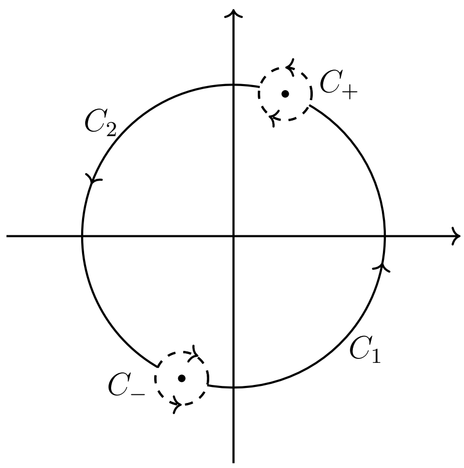

Figure 4.3: Contour of the integral

C_1,C_2 are part of unit circle and $C_\pm$ are half circle where center is $\gamma_\pm$. Principal value of

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the integral is

$$\text{P.V.}\int f(z)\,dz = \int_{C_1+C_2} f(z)\,dz, \quad \text{where } f(z) \equiv \frac{z^{2\mu+1}}{(z-\gamma_+)^2(z-\gamma_-)^2}$$

Following the same technique as in the previous chapter, the principal value of the integral is

$$\begin{aligned}\text{P.V.}\int f(z)\,dz &= i\pi\sum Res(f,\gamma_k)\\ &= i\pi\frac{\partial[f(z)*(z-\gamma_+)^2]}{\partial z}\bigg|_{z=\gamma_+} + i\pi\frac{\partial[f(z)*(z-\gamma_-)^2]}{\partial z}\bigg|_{z=\gamma_-}\\ &= -\frac{i\pi\beta\mu(\alpha\beta+q)}{2(\alpha+\beta q)}\left(\left(-\frac{\sqrt{-\alpha-\beta q}}{\sqrt{\beta(\alpha\beta+q)}}\right)^{2\mu} + \left(\frac{\sqrt{-\alpha-\beta q}}{\sqrt{\beta(\alpha\beta+q)}}\right)^{2\mu}\right)\end{aligned} \tag{4.75}$$

Restoring all the prefactors and omitting the delta function, the non-perturbative amplitude is

$$M_4^{eik} = 4\sqrt{s}\,|\vec{\mathbf{p}}|\left(-\frac{\beta\mu\left(\left(-\frac{\sqrt{-\alpha-\beta q}}{\sqrt{\beta(\alpha\beta+q)}}\right)^{2\mu}+\left(\frac{\sqrt{-\alpha-\beta q}}{\sqrt{\beta(\alpha\beta+q)}}\right)^{2\mu}\right)}{(\alpha+\beta q)(\alpha\beta+q)}e^{i2\mu\left(\frac{\pi}{2}+\theta-\psi\right)}\right) \tag{4.76}$$

But this answer may not be accurate as we have only considered χ up to $\mathcal{O}(\varepsilon)$. Contributions of $\mathcal{O}(\varepsilon^2)$ can also affect M_4^{eik}. To account for the $\mathcal{O}(\varepsilon^2)$ effects from χ, we can perform a double Taylor expansion with respect to ε.

$$\begin{aligned}e^{i\chi} &= e^{i(\chi_{(0)}+\varepsilon\chi_{(1)}+\varepsilon^2\chi_{(2)}+\cdots)}\\ &= (1+i\chi_{(0)}+\cdots)(1+i\varepsilon\chi_{(1)}-\frac{1}{2}\varepsilon^2\chi_{(1)}^2+\cdots)(1+i\varepsilon^2\chi_{(2)}-\frac{1}{2}\varepsilon^4\chi_{(2)}^2+\cdots)\end{aligned} \tag{4.77}$$

We can immediately observe that (4.76) is valid up to $\mathcal{O}(\varepsilon)$. This is because part of the $\mathcal{O}(\varepsilon^2)$ contribution from M_4^{eik} depends on $\chi_{(2)}$. Performing a Taylor expansion of (4.76) up to $\mathcal{O}(\varepsilon)$ yields

$$\begin{aligned}M_4^{eik} = 4\sqrt{s}\,|\vec{\mathbf{p}}|\big[&\frac{-2\mu(-1)^\mu}{q^2} + \frac{2(-1)^\mu(\mu)^2\varepsilon\pi(1-\mu+\beta^2+\mu\beta^2)}{q^3\beta}\\ &+\mathcal{O}\big(\varepsilon^2\big)\big]e^{i2\mu\left(\frac{\pi}{2}+\theta-\psi\right)}\end{aligned} \tag{4.78}$$

If we take $\mathcal{O}(1)$ of amplitude to obtain differential cross section in COM (4.39), where

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$q^2 = t = 4p_c^2 \sin^2 \frac{\varphi}{2}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{16s}\left|M_4^{eik(0)}\right|^2 = \left|\frac{\mu}{2p_c \sin^2 \frac{\varphi}{2}}\right|^2 \approx \frac{4\mu^2}{p_c^2\varphi^4}(\varphi \ll 1) \tag{4.79}$$

This approximation reproduces the result of quantum mechanics (3.33) and agrees with the findings in [15, 34].

If we choose the string-independent eikonal phase given by equation (4.63) for exponentiation, the amplitude would recover as the Coulomb problem [20]. In this case, the cross section of the hotfix approach coincides with the string-dependent one. Therefore, there is no difference between the two eikonal phases for the cross section in the eikonal approximation [27].

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Chapter 5 Conclusion

We have calculated the classical observables, such as the impulse and cross section, using different approaches and have shown that their results are consistent with each other. We summarize the results in Table (5.1) and Table (5.2). In this thesis, we have specifically considered the case of phase rotation given by equation (4.2), where $\theta = \pi/2$. For arbitrary values of θ, the impulse obtained from the one-loop amplitude remains the same since there is no phase factor in the triangle coefficient.Hence, the next leading order of the impulse should also remain the same for arbitrary phase differences. We have already observed the consequences for $\theta_{12} = 0,\pi/2$. In the case of gravitational dyons, the impulse only consists of the 1PM term and disagrees with the 2PM term for general θ [22].

When we extend the amplitude to include quantum corrections, the Dirac quantization condition remains a significant challenge. In Zangwill's formalism [37], fixed-order calculations can be hindered by the presence of Dirac strings. In the on-shell method, there are two different tree amplitudes that can be selected. Both of these amplitudes should be taken into account in order to map the classical result and obtain the same cross section in the eikonal limit. In the $q^2 \to 0$ limit, these two amplitudes are consistent with each other. However, for general kinematics, their behavior is completely different and can affect the quantum behavior of observables.

Recently, a new formalism [11] has been proposed to construct pairwise states that carry the phase of the string [10]. The partial wave analysis using pairwise helicity can be used to construct spin-weighted spherical harmonics [36]. However, the coefficients of these spin-weighted spherical harmonics are still unconfirmed. Additionally, there is hope that the pairwise formalism can provide insights into the Montonen-Olive conjecture. This conjecture suggests that in the Bogomol'nyi limit, all particles from the spectrum of the Georgi-Glashow model can be composed as

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particle	mass	charge (q,g)	spin
Higgs	0	(0,0)	0
γ	0	(0,0)	1
$A_{\pm}$	ve	(e,0)	1
g	vg	(0,g)	0

According to Dirac quantization, there exist two distinct coupling regimes for magnetic monopoles. Therefore, verifying the Montonen-Olive conjecture requires non-perturbative results, which present a challenge similar to the string issue.

In this thesis, we have focused on magnetic monopoles within the U(1) gauge group. To satisfy the Gauss law for magnetic charge, the field strength must include a string variable. However, there is another approach that involves the SU(2) gauge group and spontaneous symmetry breaking. Monopoles in the SU(2) group, as proposed by 't Hooft and Polyakov [28, 32], allow for the existence of magnetic solutions and automatically satisfy Dirac quantization. In the context of SU(2) symmetry, the singularity associated with the string completely disappears. Instead, the magnetic charge is encoded in the equation of motion as the winding number. The interaction between monopoles can be described by geodesic motion on the moduli space [16]. In the future, we hope to establish a connection between these results and the on-shell approach.

	Amplitude M	Eikonal phase $\chi \equiv \int dq^2 e^{-iqb} M$
Tree	$M_{(0)} = 4eg\frac{\epsilon(\eta,p_1,q,p_2)}{q^2}\frac{1}{q\cdot\eta}$ $M^{\text{fix}}_{(0)} = \pm 4eg\frac{m_a m_b\sqrt{1-\sigma^2}}{q^2}$	$\chi_{(0)} = 2\frac{eg}{4\pi}\text{arctan}\left(\frac{\hat{\mathbf{n}}}{\hat{\mathbf{z}}\cdot(\mathbf{b}\times\hat{\mathbf{n}})}\right)$ $\chi^{\text{fix}}_{(0)} = \mp\frac{eg m_a m_b\sqrt{1-\sigma^2}}{\lvert\vec{\mathbf{p}}\rvert\sqrt{s}}\frac{\log b}{2\pi}$
1-Loop	$M_{\triangle+\triangledown} = 4e^2g^2\frac{m_a+m_b}{16\lvert\vec{\mathbf{q}}\rvert}$ $M_{\square+\bowtie} = i4e^2g^2m_a^2m_b^2(2-2\sigma^2)\frac{\log q^2}{4q^2}\frac{1}{\lvert\vec{\mathbf{p}}\rvert\sqrt{s}}$	$\chi_{\triangle+\triangledown} = -e^2g^2\frac{m_a+m_b}{32\pi\lvert\vec{\mathbf{p}}\rvert\sqrt{s}b}$ $\chi_{\square+\bowtie} = \frac{ie^2g^2m_a^2m_b^2(1-\sigma^2)}{\lvert\vec{\mathbf{p}}\rvert^2 s}\frac{\log^2 b}{8\pi^2}$

Table 5.1: Amplitude and Eikonal phase

	Tree	1-Loop
Impulse $\Delta\mathbf{p} \equiv -\frac{\partial\chi(b)}{\partial\mathbf{b}} + i\chi^*(-b)\frac{\partial\chi(b)}{\partial\mathbf{b}}$	$2\frac{eg}{4\pi b}\hat{\mathbf{b}}_{\perp}$	$-\frac{e^2g^2(m_a+m_b)}{32\pi\lvert\mathbf{p}\rvert\sqrt{s}b^2}\hat{\mathbf{b}}$

Table 5.2: Impulse of charge-monopole scattering

Appendix A — Spinor Helicity Formalism

A.1 Spinor Helicity Formalism

A.1.1 Contraction and the Levi-Civita Tensor

We choose the convention of contracting the dotted and undotted spinors into square and angle brackets as :

$$\langle \lambda\mu \rangle \equiv \lambda^{\alpha}\mu_{\alpha} = \varepsilon_{\alpha\beta}\lambda^{\alpha}\mu^{\beta}, \quad [\lambda\mu] \equiv \tilde{\lambda}_{\dot{\alpha}}\tilde{\mu}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\alpha}}\mu_{\dot{\beta}}$$

Same for massive spinors that carry SU(2) indicies.Here the Levi Civita tensor in matrix form is given by:

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\alpha\beta} = -\varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

such that

$$\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

A.1.2 Pauli matrices and Gamma matrices

Space-time metric is four-dimension,Minkowski flat

$$\eta^{\mu\nu} = g^{\mu\nu} = \text{Diag}\big(+1, -1, -1, -1\big)$$

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The Paulu matrices are given by

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with

$$\sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma} = (1, -\vec{\sigma})$$

Satisfying the following identities

$$\begin{aligned}
(\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} &= \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\beta\dot{\beta}} \\
(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma_\mu)_{\beta\dot{\beta}} &= 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} \\
(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_a^{\ b} &= 2\eta^{\mu\nu}\delta_a^{\ b} \\
\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) = \mathrm{Tr}(\sigma^\nu\bar{\sigma}^\mu) &= 2\eta^{\mu\nu}
\end{aligned}$$

The gamma matrices in the Weyl representation with the $SL(2, C)$indices shown explictly are

$$\gamma^\mu = \begin{bmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\alpha}} \\ (\dot{\sigma}^\mu)^{\dot{\alpha}\alpha} & 0 \end{bmatrix}, \quad \{\gamma^\mu\gamma^\nu\} = 2\eta^{\mu\nu}$$

and γ^5 which satisfy $\{\gamma^5, \gamma^\mu\} = 0$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the chiral projectors are

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2}$$

Some useful identities are

$$\begin{aligned}
\eta^{\mu\nu}\eta_{\mu\nu} &= 4 \\
\gamma^\mu\gamma_\mu &= 4I_4 \\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho} \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu
\end{aligned}$$

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and trace identities

$$\begin{aligned}
\text{Tr}[\gamma^5] = \text{Tr}[\gamma^\mu] = \text{Tr}[\gamma^\mu\gamma^\alpha\gamma^\nu] &= \text{Tr}[\text{odd \# of } \gamma\text{-matrices}] = 0 \\
\text{Tr}[\gamma^\mu\gamma^\nu] &= 4\eta^{\mu\nu} \\
\text{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] &= 4(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}) \\
\text{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] &= -4i\varepsilon^{\mu\nu\alpha\beta}
\end{aligned}$$

A.1.3 Massive Momentum and Massless Momentum

The momentum of a massless particle can be written as as product of two two-component spnior

$$\begin{aligned}
k_{\alpha\dot{\alpha}} &\equiv k_\mu(\sigma^\mu)_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} = |k\rangle_\alpha [k|_{\dot{\alpha}} \\
k^{\alpha\dot{\alpha}} &\equiv k_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}} = |k]^{\dot{\alpha}}\langle k|^\alpha
\end{aligned}$$

and a massive momentum can be written as a product of two 2-by-2 matrices

$$p_{\alpha\dot{\alpha}} \equiv p_\mu(\sigma^\mu)_{\alpha\dot{\alpha}} = \lambda^I_\alpha\tilde{\lambda}_{I\dot{\alpha}} = |\lambda^I\rangle_\alpha [\lambda_I|_{\dot{\alpha}}$$

relation between spinor and gamma matrices

$$p_\mu\gamma^\mu = \begin{bmatrix} 0 & p_\mu(\sigma^\mu)_{\alpha\dot{\alpha}} \\ p_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} & 0 \end{bmatrix} = \begin{bmatrix} 0 & p_{\alpha\dot{\alpha}} \\ p^{\dot{\alpha}\alpha} & 0 \end{bmatrix}$$

A.1.4 Vector Inner Product in Spinor Representation

We adopt convention that the inner product between two momentum to be:

- Massless-Massless

$$2k_i \cdot k_j \equiv \langle ij\rangle[ji] = \langle ji\rangle[ij]$$

- Massless-Massive

$$2k_i \cdot p_j \equiv \langle i^I j\rangle[j i_I] = \langle j i^I\rangle[i_I j]$$

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- Massive-Massive

$$2p_i \cdot p_j \equiv \langle i^I j^J \rangle [j_J i_I] = \langle j^J i^I \rangle [i_I j_J]$$

From above,we can see that for massless momentum $k_i^2 = k_i \cdot k_i = 0$

A.1.5 Determinant of Massive Spinor

The on-shell condition for massive momentum is given by

$$p^2 = \frac{1}{2} \langle \lambda^I \lambda^J \rangle [\lambda_J \lambda_I] = m^2$$

Since $\varepsilon^{IJ} \varepsilon_{JI} = 2$,we are free to chooese

$$\langle \lambda^I \lambda^J \rangle = zm\varepsilon^{IJ}, \quad [\lambda_J \lambda_I] = z^{-1} m \varepsilon_{JI}$$

On the other hand,the on-shell condition are given bt det(p) = det(λ)det($\tilde{\lambda}$) = m^2. We choose det(λ) = m, which imply

$$\begin{aligned} \det(\lambda)\varepsilon_{\alpha\beta} &= \varepsilon_{IJ} \lambda^{IJ}_{\alpha\beta} \\ \Rightarrow -2\det(\lambda) &= \varepsilon_{IJ} \langle \lambda^J \lambda^I \rangle \\ \Rightarrow \langle \lambda^I \lambda^J \rangle &= -m\varepsilon^{IJ} \end{aligned}$$

This fix $z = 1$. Summing up we have

$$\boxed{\langle \lambda^I \lambda^J \rangle = -m\varepsilon^{IJ}, \quad [\lambda_I \lambda_J] = -m\varepsilon_{IJ}}$$

Other combinations of identical massive spinor are given by:

$$\begin{aligned} \langle \lambda_I \lambda_J \rangle &= +m\varepsilon_{IJ}, \quad [\lambda^I \lambda^J] = +m\varepsilon_{IJ} \\ \langle \lambda^I \lambda_J \rangle &= +m\delta^I{}_J, \quad [\lambda^I \lambda_J] = -m\delta^I{}_J \\ \langle \lambda_I \lambda^J \rangle &= -m\delta_I^{\ J}, \quad [\lambda_I \lambda^J] = +m\delta_I^{\ J} \\ \langle \lambda^I \lambda_I \rangle &= +2m \quad , \quad [\lambda^I \lambda_I] = -2m \\ \langle \lambda_I \lambda^I \rangle &= -2m \quad , \quad [\lambda_I \lambda^I] = +2m \end{aligned}$$

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With these Conventions, we find that momentum contracting with the massive spinors are:

$$\boxed{\begin{aligned} p_{\alpha\dot{\alpha}}\tilde{\lambda}^{I\dot{\alpha}} &= +m\lambda^I_\alpha, \quad p^{\dot{\alpha}\alpha}\lambda^I_\alpha = +m\tilde{\lambda}^{I\dot{\alpha}} \\ \lambda^{I\alpha}p_{\alpha\dot{\alpha}} &= -m\tilde{\lambda}^I_{\dot{\alpha}}, \quad \tilde{\lambda}^I_{\dot{\alpha}}p^{\alpha\dot{\alpha}} = -m\lambda^{I\alpha} \end{aligned}}$$

In the bra-ket notation:

$$m|\lambda^I\rangle = +p|\lambda^I], \quad m|\lambda^I] = +p|\lambda^I\rangle$$
$$m\langle\lambda^I| = -[\lambda^I|p, \quad m[\lambda^I| = -\langle\lambda^I|p$$

With SU(2) indices contracted:

$$\varepsilon_{IJ}\lambda^I_\alpha\lambda^J_\beta = +m\varepsilon_{\alpha\beta}, \quad \varepsilon_{IJ}\lambda^{\alpha I}\lambda^{\beta J} = -m\varepsilon^{\alpha\beta}$$
$$\varepsilon_{IJ}\tilde{\lambda}^I_{\dot{\alpha}}\tilde{\lambda}^J_{\dot{\beta}} = +m\varepsilon_{\dot{\alpha}\dot{\beta}}, \quad \varepsilon_{IJ}\lambda^{\dot{\alpha}I}\lambda^{\dot{\beta}J} = -m\varepsilon^{\dot{\alpha}\dot{\beta}}$$

A.1.6 Identities in Massive Amplitudes

A.1.6.1 Two Massless One Massive

We choose all three momenta to be incoming,so

$$k_1 + k_2 + p_3 = 0$$

3-pt kinematics give the condition:

$$\langle 12\rangle[21] = m_3^2$$

A.1.6.2 Two Massive:Unequal Mass

We choose all three momentum to be incoming,so

$$p_1 + p_2 + k_3 = 0$$

The basis of unequal 2-massive amplitude is

$$u_\alpha = \lambda_{3\alpha}, \quad v_\alpha = \frac{p_{1\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_3}{m_1}$$

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Here are few useful spinor combinations:

$$\langle \mathbf{1}u\rangle = \langle \mathbf{1}3\rangle, \quad \langle \mathbf{2}u\rangle = \langle \mathbf{2}3\rangle, \quad \langle \mathbf{1}v\rangle = [3\mathbf{1}], \quad \langle \mathbf{2}v\rangle = \frac{m_2}{m_1}[\mathbf{2}3]$$
$$\langle \mathbf{12}\rangle = \frac{1}{m_1^2 - m_2^2}(m_2[\mathbf{2}3]\langle 3\mathbf{1}\rangle + m_1[\mathbf{1}3]\langle 3\mathbf{2}\rangle)$$
$$[\mathbf{12}] = \frac{1}{m_2^2 - m_1^2}(m_2\langle \mathbf{2}3\rangle[3\mathbf{1}] + m_1\langle \mathbf{1}3\rangle[3\mathbf{2}])$$

A.1.6.3 Two Massive:Equal Mass

Definition and convention of x-factor

We choose all three momenta to be incoming, so

$$p_1 + p_2 + k_3 = 0$$

The momentum conservation and the on-shell condition yields:

$$2p_2 \cdot k_3 = \langle 3|p_1|3\rangle = \lambda_3^{\alpha} p_{1\alpha\dot{\alpha}} \tilde{\lambda}_3^{\dot{\alpha}} = 0$$

so that $\lambda_{3\alpha}$ is proportional to $p_{1\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}$.This allow us to define the x-factor

$$\boxed{x_1 \lambda_{3\alpha} = \frac{p_{1\alpha\dot{\alpha}}\tilde{\lambda}_3^{\dot{\alpha}}}{m}, \quad \frac{\tilde{\lambda}_3^{\dot{\alpha}}}{x_1} = \frac{p_1^{\dot{\alpha}\alpha}\lambda_{3\alpha}}{m}}$$

The 1 in the subscript of x denote that we define the x-factor with respect to massive leg 1. Suppose we want to define the x-factor with respect to leg 2, then

$$\boxed{x_2 \lambda_{3\alpha} = \frac{p_{2\alpha\dot{\alpha}}\tilde{\lambda}_3^{\dot{\alpha}}}{m}, \quad \frac{\tilde{\lambda}_3^{\dot{\alpha}}}{x_2} = \frac{p_2^{\dot{\alpha}\alpha}\lambda_{3\alpha}}{m}}$$

By momentum conservation, we find $x_1 = -x_2$. Also:

$$x_1 = \frac{\langle \eta|p_1|3]}{m\langle \eta 3\rangle}, \quad \frac{1}{x_1} = \frac{\langle 3|p_1|\eta]}{m[\eta 3]}$$
$$x_2 = \frac{\langle \eta|p_2|3]}{m\langle \eta 3\rangle}, \quad \frac{1}{x_2} = \frac{\langle 3|p_2|\eta]}{m[\eta 3]}$$

So,under complex conjugation, the x factor becomes

$$\bar{x}_1 = -\frac{1}{x_1}, \quad \bar{x}_2 = -\frac{1}{x_2}$$

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A.1.7 Explicit Kinematics

We can construct the corresponding spinors by finding the eigenvectors of $p_{\alpha\dot{\alpha}}$.

- For massless spinor

$$\lambda_\alpha = \sqrt{2E}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \quad \tilde{\lambda}_\alpha = \sqrt{2E}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{-i\phi} \end{pmatrix}$$

- For massive spinor

$$\lambda_\alpha^I = \begin{pmatrix} \sqrt{E+p_c}\sin\frac{\theta}{2} & \sqrt{E-p_c}\cos\frac{\theta}{2}e^{-i\phi} \\ -\sqrt{E+p_c}\cos\frac{\theta}{2}e^{i\phi} & \sqrt{E-p_c}\sin\frac{\theta}{2} \end{pmatrix}$$
$$\tilde{\lambda}_{\dot{\alpha}I} = \begin{pmatrix} \sqrt{E+p_c}\sin\frac{\theta}{2} & \sqrt{E-p_c}\cos\frac{\theta}{2}e^{i\phi} \\ -\sqrt{E+p_c}\cos\frac{\theta}{2}e^{-i\phi} & \sqrt{E-p_c}\sin\frac{\theta}{2} \end{pmatrix}$$

with E is COM energy and p_c is COM momentum. Note that for massive spinors, both λ and $\tilde{\lambda}$ have $SL(2,\mathbb{C})$ indices in the front and $SU(2)$ indices in the back.But $SU(2)$ index for λ is upstairs and $SU(2)$ index for $\tilde{\lambda}$ is downstairs.

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Appendix B — ratio of $\frac{x_1}{x_2}$

In this appendix we derive the express the little group invariant the ratio of $\frac{x_1}{x_2}$ in terms of momentum factors. From (A.1.6.3), we have

$$x_1 = \frac{\langle\eta|p_1|q]}{m_1\langle\eta q\rangle}, \quad \frac{1}{x_2} = \frac{\langle q|p_2|\tilde{\eta}]}{m_2[\tilde{\eta}q]} \tag{B.1}$$

where $\langle\eta|$ and $[\eta|$ are some auxiliary spinors such that$\langle\eta q\rangle$ and $[\eta q] \neq 0$ and we choose $\eta_{\alpha\dot{\alpha}} = |\eta\rangle[\tilde{\eta}|$

$$\frac{x_1}{x_2} = -\frac{1}{m_1 m_2}\frac{\langle\eta|p_1 q p_2|\tilde{\eta}]}{\langle\eta q\rangle[q\tilde{\eta}]} \tag{B.2}$$

With the relation between γ matrice and Pauli matrice in (A.1.2), ratio can be read as

$$\frac{x_1}{x_2} = -\frac{1}{4m_1 m_2}\frac{Tr[(1+\gamma 5)\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta]\eta_\mu p_{1\nu} q_\alpha p_{2\beta}}{q\cdot\eta} \tag{B.3}$$

$$= -\frac{(\eta\cdot p_1)(q\cdot p_2) - (\eta\cdot q)(p_2\cdot p_1) + (\eta\cdot p_2)(p_1\cdot q) - i\epsilon(\eta, p_1, q, p_2)}{m_1 m_2(q\cdot\eta)} \tag{B.4}$$

$$= \frac{(p_2\cdot p_1)}{m_1 m_2} + \frac{i\epsilon(\eta, p_1, q, p_2)}{m_1 m_2(q\cdot\eta)} \tag{B.5}$$

In last equation we used the on shell kinematics $(q\cdot p_1) = (q\cdot p_2) = 0$

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Appendix C — The Generalized Spherical Harmonics

In this appendix, we show the eigenfunction of the generalized momentum operator

$$\begin{aligned}\tilde{\mathbf{L}}^2 Y_{\mu lm}(\theta,\phi) &= -\left[(1-x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} - \frac{1}{1-x^2}\left(i\frac{\partial}{\partial\phi} + \mu(1-x)\right)^2 - \mu^2\right] Y_{\mu lm}(\theta,\phi) \\ &= \lambda Y_{\mu lm}(\theta,\phi) \qquad \text{(C.6)}\end{aligned}$$

where $x \equiv \cos\theta$. Then Separating the ϕ dependence from $Y_{\mu lm}$

$$Y_{\mu lm}(\theta,\phi) = P(\cos\theta)e^{i(\mu+m)\phi} \qquad \text{(C.7)}$$

Then the remaining part is

$$\left[-(1-x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + \frac{(m+\mu x)^2}{1-x^2} - \mu^2\right] P(x) = \lambda P(x) \qquad \text{(C.8)}$$

Following the standard procedure.Separating the singularity $x = \pm 1$, $P(x)$ can be expressed as

$$P(x) = (1-x)^{-\frac{\mu+m}{2}}(1+x)^{-\frac{\mu-m}{2}}F(x) \qquad \text{(C.9)}$$

Then $F(x)$ satisfy differential equation

$$(1-x^2)\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} + 2[m+(\mu-1)x]\frac{\mathrm{d}F}{\mathrm{d}x} + (\mu-\mu^2+\lambda)F = 0 \qquad \text{(C.10)}$$

With the new variable $z \equiv \frac{1+x}{2}$

$$\begin{aligned}&z(1-z)F'' + (m-\mu+1+2z(\mu-1))F' + (\mu-\mu^2+\lambda)F \\ &\Rightarrow z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0 \qquad \text{(C.11)}\end{aligned}$$

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$F(x)$ is just standard hypergeometric function ${}_2F_1(a,b,c,z)$,identify $c = m - \mu + 1$, $ab = \mu^2 - \mu - \lambda$ and $a + b + 1 - 2 - 2\mu$. And we fix $a = -n, n \in \mathbb{Z}$, λ can be read as

$$\lambda = n(n+1) - 2\mu n + \mu(\mu - 1) \tag{C.12}$$

identify $l = n - \mu$.Then the full solution is

$$Y_{\mu lm}(\theta,\phi) = (1-x)^{-\frac{\mu+m}{2}}(1+x)^{-\frac{\mu-m}{2}}\,{}_2F_1(-n, n+1-2\mu, m+1-\mu; z)e^{i(\mu+m)\phi}$$

There is more convenient to use another solution, which is related to **Jacobi polynomial**

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1-x)^{\beta+n}\right]$$

Then the solution and be rewrite as

$$Y_{\mu lm}(\theta,\phi) = N(1-x)^{\frac{-\mu-m}{2}}(1+x)^{-\frac{\mu-m}{2}}P_{l+m}^{(-\mu-m,\mu-m)}(x)e^{i(\mu+m)\phi} \tag{C.13}$$

where

$$N \equiv 2^m\sqrt{\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l-\mu)!(l+\mu)!}} \tag{C.14}$$

The relation between general spherical harmonics and Wigner D-matrix is

$$\mathcal{D}^l_{\mu-m}(-\phi,\theta,\phi) = (-1)^{m+\mu}\sqrt{\frac{4\pi}{2l+1}}Y_{\mu lm}(\theta,\phi) \tag{C.15}$$

where Wigner D matrix and little d matrix are defined as

$$\begin{aligned}\mathcal{D}^l_{\mu m}(\alpha,\beta,\gamma) &\equiv \langle l,\mu|\, e^{-iJ_z\alpha}e^{-iJ_y\beta}e^{-iJ_z\gamma}\,|l,m\rangle \\ &= e^{-i(\mu\alpha+m\gamma)}\,\langle l,\mu|\, e^{-iJ_y}\,|l,m\rangle \\ &= e^{-i(\mu\alpha+m\gamma)}\,d^l_{\mu m}(\beta)\end{aligned} \tag{C.16}$$

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