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量子形變模上的兩個定理

Two theorems for deformation quantization modules

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Abstract to the Dissertation

Two theorems for deformation quantization modules

The theory of deformation quantization modules have a great improvement recently. In this thesis, we prove two basic theorems about this theory.

The first theorem is a generalization of Riemann-Roch theorem for $\mathscr{D}$-modules. We generalize the (algebraic) Riemann-Roch theorem for $\mathscr{D}$-modules of [16] to (analytic) $\widehat{\mathscr{W}}$-modules.

The second theorem is a generalization of Serre's GAGA theorem [see 6]. Let X be a smooth complex projective variety with associated compact complex manifold X_{an}. If $\mathscr{A}_X$ is a DQ-algebroid on X, then there is an induced DQ-algebroid on X_{an}. We show that the natural functor from the derived category of bounded complexes of $\mathscr{A}_X$-modules with coherent cohomologies to the derived category of bounded complexes of $\mathscr{A}_{X_{\mathrm{an}}}$-modules with coherent cohomologies is an equivalence.

Introduction

The theory of deformation quantizaton modules (we usually call DQ-modules) was introduced and developed by M. Kashiwara and P. Schapira recently. In this thesis, two basic theorems are proved.

Let X be a smooth complex algebraic variety or a complex manifold. Set $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$ and $\mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar))$. In [13], M. Kashiwara and P. Schapira introduce the notion of a DQ-algebra $\mathscr{A}_X$ which is a $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$-algebra and locally isomorphic to an algebra $(\mathscr{O}_X[[\hbar]], \star)$ where $\star$ is a star product. They also consider the notion of a DQ-algebroid, that is, a $\mathbb{C}^\hbar$-algebroid (in the sense of stacks) locally equivalent to the algebroid associated with a DQ-algebra.

If $\mathscr{A}_X$ is a DQ-algebra or a DQ-algebroid on X, we denote by $\mathscr{A}_{X^a}$ the opposite algebra or algebroid $\mathscr{A}_X^{\mathrm{op}}$ and we denote by $\mathscr{A}_{X_1\times X_2}$ the external product of $\mathscr{A}_{X_i}$ $(i = 1, 2)$.

If $\mathscr{A}_X$ is a DQ-algebra or a DQ-algebroid on X, then we have the notion of $\mathscr{A}_X$-modules. We denote by $\mathrm{Mod}(\mathscr{A}_X)$ the category of $\mathscr{A}_X$-modules, by $\mathrm{D^b}(\mathscr{A}_X)$ its bounded derived category and by $\mathrm{D^b_{coh}}(\mathscr{A}_X)$ the full triangulated subcategory of the bounded derived category $\mathrm{D^b}(\mathscr{A}_X)$ with coherent cohomologies. An object of $\mathrm{D^b}(\mathscr{A}_X)$ is called a kernel.

Similarly, set $\mathscr{A}_X^{\mathrm{loc}} := \mathbb{C}^{\hbar,\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar} \mathscr{A}_X$, then we have the notion of good $\mathscr{A}_X^{\mathrm{loc}}$-modules. We denote by $\mathrm{Mod}(\mathscr{A}_X^{\mathrm{loc}})$ the category of $\mathscr{A}_X^{\mathrm{loc}}$-modules, by $\mathrm{D^b}(\mathscr{A}_X^{\mathrm{loc}})$ its bounded derived category and by $\mathrm{D^b_{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full triangulated subcategory of the bounded derived category $\mathrm{D^b}(\mathscr{A}_X^{\mathrm{loc}})$ with good cohomologies.

Let $(X, \mathscr{A}_X)$, $(Y, \mathscr{A}_Y)$ be two complex manifolds endowed with DQ-algebras $\mathscr{A}_X$ and $\mathscr{A}_Y$. Let $\mathscr{M} \in \mathrm{D^b}(\mathscr{A}_X)$ and $\mathscr{K} \in \mathrm{D^b}(\mathscr{A}_{X^a\times Y})$ be two kernels. Their convolution is defined as

$$\mathscr{M} \circ \mathscr{K} := Rq_{2!}(\mathscr{K} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} q_1^{-1}\mathscr{M})$$

here, q_i the i-th projection defined on $X \times Y$ $(i = 1, 2)$. One of the main theorems in [13] asserts that if $\mathscr{M}$ and $\mathscr{K}$ are coherent and if q_2 is proper on $q_1^{-1}\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(\mathscr{K})$, then $\mathscr{M} \circ \mathscr{K}$ is coherent.

Similarly, If $\mathscr{M}_1 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and $\mathscr{K}_1 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_{X^a \times Y})$ are good modules and if q_2 is proper on $q_1^{-1}\mathrm{supp}(\mathscr{M}_1) \cap \mathrm{supp}(\mathscr{K}_1)$, then their convolution $\mathscr{M}_1 \circ \mathscr{K}_1$ is also good.

If X and Y are compact, then a kernel $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a \times Y})$ defines a functor

$$\circ\mathscr{K} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_Y)$$

which is called the Fourier-Mukai transform induced by $\mathscr{K}$. Similarly, a kernel $\mathscr{K}^{\mathrm{loc}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_{X^a \times Y})$ defines a functor

$$\circ\mathscr{K}^{\mathrm{loc}} : \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_Y^{\mathrm{loc}})$$

which is called the Fourier-Mukai transform induced by $\mathscr{K}^{\mathrm{loc}}$.

The first main theorem of this thesis is the Riemann-Roch theorem for Fourier-Mukai transforms of $\mathscr{A}^{\mathrm{loc}}$-modules.

As an application, we recover the Riemann-Roch formula for $\mathscr{D}$-modules of [16].

If $(X,\ \mathscr{A}_X)$ is a smooth complex algebraic variety endowed with a DQ-algebroid, then there is an induced DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on the complex manifold X_{an} induced by X. Then we construct a functor $f^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$. The second main theorem of this thesis is the following:

Assume that X is projective. Then the functor f^*: $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is an equivalence.

This thesis is organized as follows: In Chapter 1, we briefly review the notions of coherent modules, derived categories and the projective limits of 2-categories which are used in the text.

In Chapter 2, we review Serre's GAGA theorem and translate this theorem to the derived version. In Chapter 3, we review the Riemann-Roch theorem for $\mathscr{D}$-modules in [16]. We prove the first main theorem in Chapter 4. In Chapter 5, we use the main results in Chapter 4 to recover the results in Chapter 3. In Chapter 6, we reveiw the notions and results in [13], in particular, Remark 6.6 and Finiteness theorem 6.11 are crucial to the proof of the second main theorem. We show how to induce an analytic DQ-algebroid from an algebraic DQ-algebroid in Chapter 7. In the final chapter, we prove the second main theorem.

Note: Throughout this thesis, all varieties (or schemes) are over $\mathbb{C}$ if not otherwise specified.

1 Preliminary

1.1 Coherency

Let $\mathscr{A}_X$ be a sheaf of rings on a topological space X.

Definition 1.1.

(1) A (left) $\mathscr{A}_X$-module $\mathscr{F}$ is said to be *locally finitely generated* if for each $x \in X$ there exist an open neighborhood U of x, an integer $N \in \mathbb{Z}_{\geq 0}$, and epimorphism of $\mathscr{A}_X|_U$-modules on U, $(\mathscr{A}_X|_U)^{\oplus N} \twoheadrightarrow \mathscr{F}|_U$.

(2) $\mathscr{F}$ is said to be *locally finitely presented* if for each $x \in X$ there exist an open neighborhood U of x, integers $N_0, N_1 \in \mathbb{Z}_{\geq 0}$, and exact sequence of $\mathscr{A}_X|_U$-modules on U,

$$(\mathscr{A}_X|_U)^{\oplus N_1} \to (\mathscr{A}_X|_U)^{\oplus N_0} \to \mathscr{F}|_U \to 0.$$

(3) $\mathscr{F}$ is said to be pseudo-coherent if for every open set U all locally finitely generated $\mathscr{A}_X|_U$-submodules of $\mathscr{F}|_U$ are locally finitely presented.

(4) A pseudo-coherent and locally finitely generated $\mathscr{A}_X$-module is said to be coherent.

We denote by

$$\mathrm{Mod}(\mathscr{A}_X)\text{: the category of } \mathscr{A}_X\text{-modules.}$$

Then $\mathrm{Mod}(\mathscr{A}_X)$ is a Grothendieck category [14, Theorem 18.1.6]. Recall that a Grothendieck category $\mathscr{C}$ is an abelian category such that $\mathscr{C}$ admits a generator and inductive limits and filtrant inductive limits are exact. In particular, $\mathrm{Mod}(\mathscr{A}_X)$ has enough injectives.

From basic properties of coherent modules (see [11, Proposition A.2 and Proposition A.6]), one can see that the category of coherent $\mathscr{A}_X$-modules denoted by

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$$

is a full abelian thick subcategory of $\mathrm{Mod}(\mathscr{A}_X)$. Recall that the term *thick* means that it is closed by kernels, cokernels and extensions.

We also denote by

$K(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X))$: the Grothendieck group of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$.

1.2 Homotopy Categories

Let $\mathscr{C} := \mathrm{Mod}(\mathscr{A}_X)$ be an abelian category. A *complex* M in $\mathscr{C}$ consists of a family $\{(M^n, d^n_M)\}_{n\in\mathbb{Z}}$ where M^n is an object of $\mathscr{C}$, and d^n_M is a morphism $\mathrm{d}^n_M : M^n \to M^{n+1}$, called a differential of M, satisfying $d^{n+1}_M \circ d^n_M = 0$. Denote by $\mathrm{C}(\mathscr{C})$ the additive category of complexes in $\mathscr{C}$. A morphism $f : M \to N$ in $\mathrm{C}(\mathscr{C})$ consists of a family $\{f^n\}_{n\in\mathbb{Z}}$ of morphisms $f^n : M^n \to N^n$ ($n \in \mathbb{Z}$) satisfying $d^n_N \circ f^n = f^{n+1} \circ d^n_M$. For a complex M and an integer k, define a complex $M[k]$ by $M[k]^n = M^{k+n}$ and $d^n_{M[k]} = (-1)^k d^{k+n}_M$. For $f : M \to N$, define $f[k] : M[k] \to N[k]$ by $f[k]^n = f^{k+n}$.

A complex M is said to be bounded below if $M^n = 0$ ($n \ll 0$), bounded above if $M^n = 0 (n \gg 0)$, and bounded if it is bounded above and below. Denote by $\mathrm{C}^+(\mathscr{C})$, $\mathrm{C}^-(\mathscr{C})$, and $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ the full subcategories of $\mathrm{C}(\mathscr{C})$ consisting of complexes of bounded below, bounded above and bounded, respectively.

To $X \in \mathscr{C}$, we associate a complex X^n defined by X if $n = 0$ and 0 if $n \neq 0$ and $d^n_X = 0$. We thus consider $\mathscr{C}$ as a full subcategory of $\mathrm{C}(\mathscr{C})$.

For complexes M and N in $\mathscr{C}$, define a complex $\mathrm{Hom}(M,N)$ of modules by

$$\mathrm{Hom}(M,N)^n = \prod_k \mathrm{Hom}_{\mathscr{C}}(M^k, N^{k+n}).$$

For $f = (f^k) \in \mathrm{Hom}(M,N)^n$ ($f^k \in \mathrm{Hom}(M^k, N^{k+n})$), we define the differential $df \in \mathrm{Hom}(M,N)^{n+1}$ by letting its k-th component $(df)^k : M^k \to N^{k+n+1}$ be the sum $M^k \xrightarrow{f^k} N^{k+n} \xrightarrow{d_N^{k+n}} N^{k+n+1}$ and $M^k \xrightarrow{(-1)^{n+1}d_M^k} M^{k+1} \xrightarrow{f^{k+1}} N^{k+n+1}$.

Then $d^2 = 0$ and $\mathrm{Hom}(M,N)$ is a complex of $\mathscr{A}_X$-modules. The 0-th cocycle

$$Z^0(\mathrm{Hom}(M,N)) := \mathrm{Ker}(\mathrm{Hom}(M,N)^0 \xrightarrow{d} \mathrm{Hom}(M,N)^1)$$

is nothing but $\mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(M,N)$. We denote by $\mathrm{Ht}(M,N)$ the 0-th coboundary

$$B^0(\mathrm{Hom}(M,N)) := \mathrm{Im}(\mathrm{Hom}(M,N)^{-1} \xrightarrow{d} \mathrm{Hom}(M,N)^0);$$

its element is called a morphism from M to N homotopic to 0. By the definition, $f = (f^n) \in \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(M,N)$ is homotopic to 0 if and only if there exists $s = (s^n : M^n \to N^{n-1})_{n\in\mathbb{Z}}$ such that

$$f^n = d_N^{n-1}s^n + s^{n+1}d_M^n.$$

Define a new additive category $\mathrm{K}(\mathscr{C})$ by

$\mathrm{Ob}(\mathrm{K}(\mathscr{C})) := \mathrm{Ob}(\mathrm{C}(\mathscr{C}))$,

$\mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(M,N) := H^0(\mathrm{Hom}(M,N)) = \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(M,N)/\mathrm{Ht}$ (M,N).

The composition of morphisms is the one induced from $\mathrm{C}(\mathscr{C})$. Then $\mathrm{K}(\mathscr{C})$ is an additive category. Similarly to the case $\mathrm{C}(\mathscr{C})$, we consider $\mathscr{C}$ as a full subcategory of $\mathrm{K}(\mathscr{C})$.

1.3 Triangulated Categories

The category $\mathrm{C}(\mathscr{C})$ is also an abelian category, but $\mathrm{K}(\mathscr{C})$ is not. Instead, $\mathrm{K}(\mathscr{C})$ has a structure called a triangulated category.

For a morphism $f : M \to N$ in $\mathrm{C}(\mathscr{C})$, define a new complex $M(f)$, called a mapping cone of f, as follows: $M(f)^n = M^{n+1} \oplus N^n$, and its differential $d^n_{M(f)} : M(f)^n \to M(f)^{n+1}$ is the map such that $M^{n+1} \longrightarrow M(f)^n \xrightarrow{d^n_{M(f)}} M(f)^{n+1}$ is the sum of $M^{n+1} \xrightarrow{f^{n+1}} N^{n+1} \longrightarrow M(f)^{n+1}$ and $M^{n+1} \xrightarrow{-d^{n+1}_M} M^{n+2} \longrightarrow M(f)^{n+1}$ and $N^n \longrightarrow M(f)^n \xrightarrow{d^n_{M(f)}} M(f)^{n+1}$ equals $N^n \xrightarrow{d^n_N} N^{n+1} \longrightarrow M(f)^{n+1}$. Then $d^{n+1}_{M(f)} \circ d^n_{M(f)} = 0$ and thus $M(f)$ is a complex.

Define morphisms $\alpha_f : N \to M(f)$ and $\beta_f : M(f) \to M[1]$ in $\mathrm{C}(\mathscr{C})$ by

$$\alpha^n_f : N^n \to M^{n+1} \oplus N^n = M(f)^n,$$

$$\beta^n_f : M(f)^n = M^{n+1} \oplus N^n \to M^{n+1} = M[1]^n.$$

A diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathrm{K}(\mathscr{C})$ is called a triangle. A morphism from a triangle $X \to Y \to Z \to X[1]$ to a triangle $X' \to Y' \to Z' \to X'[1]$ is a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\xi \big\downarrow & & \eta \big\downarrow & & \zeta \big\downarrow & & \xi[1] \big\downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1].
\end{array}$$

If ξ, η and ζ are isomorphisms, these two triangles are said to be isomorphic. A triangle $X \to Y \to Z \to X[1]$ in $\mathrm{K}(\mathscr{C})$ is said to be distinguished if there exists a morphism $X' \xrightarrow{f} Y'$ such that the triangle is isomorphic to $X' \xrightarrow{f} Y' \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X'[1]$.

With the automorphism $[1] : \mathrm{K}(\mathscr{C}) \to \mathrm{K}(\mathscr{C})$ sending M to $M[1]$ and distinguished triangles, one can see that $\mathrm{K}(\mathscr{C})$ is a triangulated category.

Similarly, one can define the distinguished categories $\mathrm{K}^+(\mathscr{C})$, $\mathrm{K}^-(\mathscr{C})$ and $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$.

1.4 Derived Categories

A morphism $f : X \to Y$ in $\mathrm{K}(\mathscr{C})$ is called a quasi-isomorphism if $H^n(X) \to H^n(Y)$ are isomorphisms for all n.

We obtain the derived category from $\mathrm{K}(\mathscr{C})$ by regarding quasi-isomorphisms are isomorphisms. More precisely, we define the derived category $\mathrm{D}(\mathscr{C})$ as follows:

Define the family $\mathrm{Ob}(\mathrm{D}(\mathscr{C}))$ of objects in $\mathrm{D}(\mathscr{C})$ to equal $\mathrm{Ob}(\mathrm{K}(\mathscr{C}))$. For $X, Y \in \mathrm{Ob}(\mathrm{D}(\mathscr{C})) = \mathrm{Ob}(\mathrm{K}(\mathscr{C}))$, define the family $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y)$ of morphisms to be $\mathcal{S}(X, Y)/\sim$, where $\mathcal{S}(X, Y)$ is a family given in the following, and $\sim$ is its equivalence relation. $\mathcal{S}(X, Y)/\sim$ denotes the family of $\sim$ equivalence classes in $\mathcal{S}(X, Y)$.

(1) $\mathcal{S}(X, Y)$ is the set of pairs (s, f) where $s : X' \to X$ is a quasi-isomorphism and $f : X' \to Y$ is a morphism in $\mathrm{K}(\mathscr{C})$.

(2) for $(s_1, f_1), (s_2, f_2) \in \mathcal{S}(X, Y)$ where $s_i : X'_i \to X$ are quasi-isomorphisms and $f_i : X'_i \to Y$ for $i = 1, 2$, we define $(s_1, f_1) \sim (s_2, f_2)$ if there exists quasi-isomorphisms $t_i : X'_0 \to X'_i$ for $i = 1, 2$ and a morphism $g : X'_0 \to Y$ such that $s_1 \circ t_1 = s_2 \circ t_2$ and $f_1 \circ t_1 = f_2 \circ t_2 = g$.

Similarly, one can define the derived categories $\mathrm{D}^+(\mathscr{C})$, $\mathrm{D}^-(\mathscr{C})$ and $\mathrm{D}^{\mathrm{b}}(\mathscr{C})$. Note that $\mathrm{D}^+(\mathscr{C})$ (resp. $\mathrm{D}^-(\mathscr{C})$, resp. $\mathrm{D}^{\mathrm{b}}(\mathscr{C})$) is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects

X such that $H^n(X) = 0$ for $n \ll 0$ (resp. $n \gg 0$, resp. $|n| \gg 0$).

We denote by

$\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$: the bounded derived category of $\mathrm{Mod}(\mathscr{A}_X)$,

$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$: the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$ consisting of complexes with cohomology sheaves belonging to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$,

$K_{\mathrm{coh}}(\mathscr{A}_X)$: the Grothendieck group of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$.

Note that by [2, P.283 Lemma 1.6], we have

$$K(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)) \simeq K_{\mathrm{coh}}(\mathscr{A}_X). \tag{1.1}$$

1.5 Projective limits of 2-cat

Recall that a presite X is nothing but a category which we denote by $\mathcal{C}_X$. If $\mathfrak{S}$ is a prestack on X, then we have the morphism $u : U_1 \to U_2$ in $\mathcal{C}_X$, and the functor $r_u : \mathfrak{S}(U_2) \to \mathfrak{S}(U_1)$ for $U_1, U_2 \in \mathcal{C}_X$.

Definition 1.2. Let $\mathfrak{S}$ be a prestack on X. We denote by $\varprojlim_{U \in \mathcal{C}_X} \mathfrak{S}(U)$ the category defined as follows.

(a) An object F of $\varprojlim_{U \in \mathcal{C}_X} \mathfrak{S}(U)$ is a family $\{(F_U, \varphi_u)\}_{U \in \mathcal{C}_X, u \in \mathrm{Mor}(\mathcal{C}_X)}$ where:

(i) for any $U \in \mathcal{C}_X$, F_U is an object of $\mathfrak{S}(U)$,

(ii) for any morphism $u : U_1 \to U_2$ in $\mathcal{C}_X$, $\varphi_u : r_u F_{U_2} \xrightarrow{\sim} F_{U_1}$

is an isomorphism such that for any sequence $U_1 \xrightarrow{u} U_2 \xrightarrow{v} U_3$ of morphisms in $\mathcal{C}_X$, the following diagram commutes (this is a so-called cocycle condition):

$$\begin{tikzcd}
r_u r_v F_{U_3} \arrow[r, "r_u(\varphi_v)"] \arrow[d, "\mathcal{C}_{u,v}"'] & r_u F_{U_2} \arrow[d, "\varphi_u"] \\
r_{v\circ u} F_{U_3} \arrow[r, "\varphi_{v\circ u}"] & F_{U_1}
\end{tikzcd}$$

(b) For two objects $F = \{(F_U, \varphi_u)\}$ and $F' = \{(F'_U, \varphi'_u)\}$ in $\varprojlim_{U\in\mathcal{C}_X} \mathfrak{S}(U)$, $\mathrm{Hom}_{\varprojlim_{U\in\mathcal{C}_X} \mathfrak{S}(U)}(F, F')$ is the set of families $f = \{f_U\}_{U\in\mathcal{C}_X}$ such that $f_U \in \mathrm{Hom}_{\mathfrak{S}(U)}(F_U, F'_U)$ and the following diagram commutes for any $u : U_1 \to U_2$

$$\begin{tikzcd}
r_u F_{U_2} \arrow[r, "\varphi_u"] \arrow[d, "r_u(f_{U_2})"'] & F_{U_1} \arrow[d, "f_{U_1}"] \\
r_u F'_{U_2} \arrow[r, "\varphi'_u"] & F'_{U_1}.
\end{tikzcd}$$

Therefore,

$$\mathrm{Hom}_{\varprojlim_{U\in\mathcal{C}_X} \mathfrak{S}(U)}(F, F') \simeq \varprojlim_{U\in\mathcal{C}_X} \mathrm{Hom}_{\mathfrak{S}|_U}(F_U, F'_U).$$

For any $A \in \mathcal{C}_X^\wedge := \mathrm{Fct}(\mathcal{C}_X^{\mathrm{op}}, \mathbf{Set})$, we set

$$\mathfrak{S}(A) = \varprojlim_{(U\to A)\in\mathcal{C}_A} (\mathfrak{S}|_A)(U) = \varprojlim_{(U\to A)\in\mathcal{C}_A} \mathfrak{S}(U).$$

Hence, $\varprojlim_{U\in\mathcal{C}_X} \mathfrak{S}(U) = \mathfrak{S}(\mathrm{pt}_X)$, where pt_X denotes as usual the terminal object of $\mathcal{C}_X^\wedge$.

We set

$$\mathfrak{S}(X) := \mathfrak{S}(\mathrm{pt_X}) = \varprojlim_{U\in\mathcal{C}_X} \mathfrak{S}(U).$$

A morphism $v : A \to A'$ in $\mathcal{C}_X^\wedge$ defines a functor

$$r_v : \mathfrak{S}(A') = \varprojlim_{U \to A'} \mathfrak{S}(U) \to \varprojlim_{U \to A} \mathfrak{S}(U) = \mathfrak{S}(A)$$

and it is easy to check that the conditions of prestack are satisfied.

Proposition 1.3. ([14, §19]) *Let $\mathfrak{S}$ be a prestack on the small presite X. Then $\mathfrak{S}$ extends naturally to a prestack on $\widehat{X}$, where $\widehat{X}$ denotes the presite associated with the category $\mathcal{C}_X^\wedge$.*

2 Review on the GAGA Theorem

Let X be a scheme of finite type and let X_{an} be the associated complex analytic space. Denote by $\mathrm{Mod}(\mathscr{O}_X)$ (resp. $\mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}})$) the category of sheaves on X (resp. X_{an}) and $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ (resp. $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$) the full subcategory of coherent sheaves. There is a continuous map $\varphi : X_{\mathrm{an}} \to X$ of the underlying topological spaces and there is also a natural map of the structure sheaves $\varphi^{-1}\mathscr{O}_X \to \mathscr{O}_{X_{\mathrm{an}}}$. To $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X)$, one associates its complex analytic sheaf $\mathscr{F}^{\mathrm{an}} := \mathscr{O}_{X_{\mathrm{an}}} \otimes_{\varphi^{-1}\mathscr{O}_X} \varphi^{-1}\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}})$. Hence we obtain a functor:

$$(*) \qquad \boldsymbol{\Upsilon}_{\mathbf{X}} : \mathrm{Mod}(\mathscr{O}_X) \to \mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}}).$$

If $\mathscr{F}$ is a coherent sheaf, then $\mathscr{F}^{\mathrm{an}}$ is also coherent.

The following theorem for a projective scheme is proved in Serre's famous paper GAGA (cf [19]) which is generalized by Grothendieck for a proper scheme (cf [8, XII]).

Theorem 2.1. *Let X be a projective scheme. Then the functor $(*)$ induces an equivalence of categories*

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}}).$$

Furthemore, for every coherent sheaf $\mathscr{F}$ on X, the natural maps

$$H^i(X; \mathscr{F}) \to H^i(X_{\mathrm{an}}; \mathscr{F}^{\mathrm{an}})$$

are isomorphisms, for all $i \geq 0$. □

The following lemma is Theorem 2.2.8 in [5].

Lemma 2.2. *Let $\mathcal{A}'$ and $\mathcal{B}'$ be thick subcategories of abelian categories $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be an exact* *functor that takes $\mathcal{A}'$ to $\mathcal{B}'$. Assume furthermore that the following properties are satisfied:*

1. *$\mathcal{A}$ and $\mathcal{B}$ have enough injectives;*

2. *Φ is an equivalence of categories when restricted to $\mathcal{A}' \to \mathcal{B}'$;*

3. *Φ induces a natural isomorphism*

$$\mathrm{Ext}^i_{\mathcal{A}}(F,G) \cong \mathrm{Ext}^i_{\mathcal{B}}(\Phi(F),\Phi(G))$$

for any $F, G \in \mathcal{A}'$ and any i.
Then the natural functor $\widetilde{\Phi} : \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ induced by Φ is an equivalence of categories.
Proof. We briefly sketch the proof here for reader's convenience.
(i) We prove that the functor $\widetilde{\Phi}$ is fully faithful, i.e. that for any $F^\bullet$, $G^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$, $\widetilde{\Phi}$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{A})}(F^\bullet, G^\bullet) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet)). \tag{2.1}$$

We'll use a technique known as *dévissage* to prove it. The *dévissage* technique is just induction on the number $n(E^\bullet)$ defined as

$$n(E^\bullet) = \max\{j - i \mid H^j(E^\bullet) \neq 0, H^i(E^\bullet) \neq 0\}.$$

Hence we shall prove (1.1) by induction on $N = n(F^\bullet) + n(G^\bullet)$. If $N = -\infty$, then one of $F^\bullet$ or $G^\bullet$ is the zero complex, so there is nothing to prove. If $N = 0$, then there exist $F \in \mathcal{A}'$, $G \in \mathcal{A}'$ such that $F^\bullet = F[a]$ and $G^\bullet = G[b]$ for some $a, b \in \mathbb{Z}$. Then

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{A})}(F^\bullet, G^\bullet) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{A})}(F[a], G[b]) = \mathrm{Ext}^{b-a}_{\mathcal{A}}(F,G)$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet)) &= \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{B})}(\widetilde{\Phi}(F[a]), \widetilde{\Phi}(G[b])) \\ &= \mathrm{Ext}^{b-a}_{\mathcal{B}}(\Phi(F), \Phi(G)).\end{aligned}$$

Hence (2.1) follows from property 3 above.

Assume that $\widetilde{\Phi}$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(F^\bullet, G^\bullet) \cong \mathrm{Hom}_{\mathrm{D^b}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet))$$

for all $F^\bullet$, $G^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ with $n(F^\bullet) + n(G^\bullet) < N$, and let $F^\bullet$, $G^\bullet$ be objects of $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ with $n(F^\bullet)+n(G^\bullet) = N > 0$. We may assume that $n(G^\bullet) = N > 0$ and that $G^i = 0$ for $i < 0$, and $H^0(G^\bullet) \neq 0$.

Let $G'^\bullet$ be the complex with single non zero object $H^0(G^\bullet)$ in degree zero. From the morphism $G'^\bullet \to G^\bullet$, there exists a distinguished triangle $G''^\bullet \to G'^\bullet \to G^\bullet \to G''^\bullet[1]$. By the long exact cohomology sequence, one deduces $n(G''^\bullet) < n(G^\bullet)$; also, from the assumption, $n(G'^\bullet) = 0 < n(G^\bullet)$. From the long exact sequence of Hom's, the five-lemma and the induction hypothesis we conclude that

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(F^\bullet, G^\bullet) \cong \mathrm{Hom}_{\mathrm{D^b}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet))$$

which is what we needed to prove that $\widetilde{\Phi}$ is fully faithful. (The case when $n(G^\bullet) = 0$ but $n(F^\bullet) > 0$ follows in a similar way.)

(ii) The functor $\widetilde{\Phi}$ is essentially surjective: any object $G^\bullet$ of $\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ is isomorphic to an object of the form $\widetilde{\Phi}(F^\bullet)$ for some $F^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$. We prove this by induction on $n = n(G^\bullet)$: the case $n = -\infty$ is trivial, and $n = 0$ follows from property 2.

So assume $n > 0$, and construct a distinguished triangle $G''^\bullet \to G'^\bullet \to G^\bullet \to G''^\bullet[1]$ where $G'^\bullet = H^0(G^\bullet) \neq 0$ and we assume $G^\bullet$ is zero in degrees < 0. Since Φ is an equivalence of categories between $\mathcal{A}'$ and $\mathcal{B}'$, we can find an $F'^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ such that $\Phi(F'^\bullet) \cong G'^\bullet$. Also, by the induction hypothesis, we can find an $F''^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ such that $\widetilde{\Phi}(F''^\bullet) \cong G''^\bullet$. Since we proved that $\widetilde{\Phi}$ is fully faithful, we can find a map $F''^\bullet \to F'^\bullet$ whose image by $\widetilde{\Phi}$ is just the side of the distinguished triangle constructed before. Again, we have the distinguished triangle

$F''^{\bullet} \to F'^{\bullet} \to F^{\bullet} \to F''^{\bullet}[1]$. Then, since $\widetilde{\Phi}$ is a ∂ functor (a functor commutes with the translation functor and sends a distinguished triangle to a distinguished triangle) because Φ is exact, we see that $\widetilde{\Phi}(F^{\bullet})$ is isomorphic to $G^{\bullet}$, as required. $\square$

Let X be a scheme of finite type, set $\mathcal{A} = \mathrm{Mod}(\mathscr{O}_X)$, $\mathcal{B} = \mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}})$, $\mathcal{A}' = \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ and $\mathcal{B}' = \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$, and denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{\mathrm{an}}) = \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$. Clearly, $\mathcal{A}'$ (resp. $\mathcal{B}'$) is a full thick subcategory of $\mathcal{A}$ (resp. $\mathcal{B}$).

As an application of Lemma 2.2, we have

Corollary 2.3. *Let X be a projective scheme, then the functor $\boldsymbol{\Upsilon}_{\mathbf{X}}$ of $(*)$ induces an equivalence (we keep the same notation)*

$$\boldsymbol{\Upsilon}_{\mathbf{X}} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{\mathrm{an}}).$$

Proof. We shall apply Lemma 2.2. First note that $\mathscr{O}_{X_{\mathrm{an}},x}$ is a flat $\mathscr{O}_{X,x}$-module for each $x \in X$, hence the functor $\boldsymbol{\Upsilon}_{\mathbf{X}} : \mathcal{A} \to \mathcal{B}$ is an exact functor. Both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives [see §1.1]. The functor $\boldsymbol{\Upsilon}_{\mathbf{X}}$ is an equivalence by Theorem 2.1. Hence it is sufficient to check condition 3 of Lemma 2.2. It is enough to prove that

$$\text{(2.2)} \qquad \mathrm{RHom}(\mathscr{F}, \mathscr{G}) \simeq \mathrm{RHom}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}}) \text{ for } \mathscr{F}, \mathscr{G} \in \mathcal{A}'.$$

Since $\mathrm{RHom}(\mathscr{F}, \mathscr{G}) \simeq \mathrm{R}\Gamma(X, \mathscr{F}^{*} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{G})$ and $\mathrm{RHom}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}})$ $\simeq \mathrm{R}\Gamma(X_{\mathrm{an}}, (\mathscr{F}^{\mathrm{an}})^{*} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X_{\mathrm{an}}}} \mathscr{G}^{\mathrm{an}})$, where $\mathscr{F}^{*} = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$ and $(\mathscr{F}^{\mathrm{an}})^{*} = \mathrm{R}\mathscr{H}om_{\mathscr{O}_{X_{\mathrm{an}}}}(\mathscr{F}^{\mathrm{an}}, \mathscr{O}_{X_{\mathrm{an}}})$, we reduce (2.2) to the following isomorphism

$$\text{(2.3)} \quad \mathrm{R}\Gamma(X, \mathscr{F}^{\bullet}) \simeq \mathrm{R}\Gamma(X_{\mathrm{an}}, (\mathscr{F}^{\bullet})^{\mathrm{an}}), \text{ where } \mathscr{F}^{\bullet} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X).$$

Assume that (2.3) holds for $\mathscr{F}^{\bullet} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ of amplitude $\leq$ N. Now let $\mathscr{F}^{\bullet}$ be of amplitude N+1. Assume for example

$H^j(\mathscr{F}^\bullet) = 0$ for $j \notin [a, a+\mathrm{N}+1]$ and consider the distinguished triangle

$$\tau^{\leq a+\mathrm{N}}\mathscr{F}^\bullet \longrightarrow \mathscr{F}^\bullet \longrightarrow H^{a+\mathrm{N}+1}(\mathscr{F}^\bullet)[-a-\mathrm{N}-1] \xrightarrow{+1} .$$

Let $b = a + \mathrm{N}$. Since $H^{b+1}(\tau^{\leq b}\mathscr{F}^\bullet) = H^{b+1}(H^{b+1}(\mathscr{F}^\bullet)[-b-1]) = 0$, one deduces that $H^j(\mathscr{F}^\bullet) = 0$ for $j \notin [a, b]$. Hence by d*évissage*, one reduces $\mathscr{F}^\bullet$ to a single sheaf $\mathscr{F}$. By Theorem 2.1, we have $\mathrm{R}\Gamma(X, \mathscr{F}) \simeq \mathrm{R}\Gamma(X_{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$ and the result follows.$\square$

3 Review on the results of Laumon

Let X, Y be smooth algebraic varieties with sheaf of differential operators $\mathscr{D}_X$ and $\mathscr{D}_Y$. For a morphism $f : X \to Y$, we have induced morphisms:

$$T^*X \xleftarrow{f_d} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y \tag{3.1}$$

where T^*X and T^*Y denote the cotangent bundles of X and Y, respectively.

If f is proper, there are well defined functors (see [16, section 5]):

$$f_d^! : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{T^*X}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X\times_Y T^*Y}) \tag{3.2}$$

defined by

$$f_d^!(-) := f_d^*(-) \otimes_{\mathscr{O}_X} \omega_{X/Y}$$

where $\omega_{X/Y} = f^{-1}(\omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathscr{O}_Y} \omega_X$ denotes relative canonical bundle of f
and

$$f_{\pi *} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X\times_Y T^*Y}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{T^*Y}). \tag{3.3}$$

Hence the functors of (3.2) and (3.3) induce group homomorphisms (see [16, 6.2]):

$$f_d^! : K_{\mathrm{coh}}(\mathscr{O}_{T^*X}) \to K_{\mathrm{coh}}(\mathscr{O}_{X\times_Y T^*Y})$$

and

$$f_{\pi *} : K_{\mathrm{coh}}(\mathscr{O}_{X\times_Y T^*Y}) \to K_{\mathrm{coh}}(\mathscr{O}_{T^*Y}).$$

Let $(\mathscr{M}, F)$ be a filtered $\mathscr{D}_X$-module with filtration $\{F_i\mathscr{M}\}$. Recall from [10, Definition 2.1.2] that F is a good filtration of $\mathscr{M}$ if the graded module $\mathrm{gr}^F\mathscr{M}$ is coherent over $\pi_*\mathscr{O}_{T^*X}$ where $\mathrm{gr}^F\mathscr{M} := \bigoplus F_i\mathscr{M}/F_{i-1}\mathscr{M}$ and $\pi : T^*X \to X$ is the projection and set

$$\widetilde{\mathrm{gr}^F\mathscr{M}} := \mathscr{O}_{T^*X} \otimes_{\pi^{-1}\pi_*\mathscr{O}_{T^*X}} \pi^{-1}\mathrm{gr}^F\mathscr{M}.$$

If $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$, then there exists globally a good filtration F on $\mathscr{M}$ (see [10, Theorem 2.1.3]). We denote by $Car(\mathscr{M})$ the element of $K_{\mathrm{coh}}(\mathscr{O}_{T^*X})$ defined by $\widetilde{\mathrm{gr}^F\mathscr{M}}$. For a proper morphism $f: X \to Y$, we also denote by $\int_f : K_{\mathrm{coh}}(\mathscr{D}_X) \to K_{\mathrm{coh}}(\mathscr{D}_Y)$ the group homomorphism induced by the direct image functor $\int_f : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{\mathrm{X}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{\mathrm{Y}})$ given by $\int_f \mathscr{M} = Rf_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})$ for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{\mathrm{X}})$.

The following theorem is the main result of [16].

Theorem 3.1. *Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties. Then the following diagram is commutative*

$$\begin{tikzcd} K_{\mathrm{coh}}(\mathscr{D}_X) \arrow[r, "\int_f"] \arrow[d, "Car"'] & K_{\mathrm{coh}}(\mathscr{D}_Y) \arrow[d, "Car"] \\ K_{\mathrm{coh}}(\mathscr{O}_{T^*X}) \arrow[r, "f_{\pi*}f_d^!"] & K_{\mathrm{coh}}(\mathscr{O}_{T^*Y}). \end{tikzcd}$$

□

Remark 3.2. Combining (algebraic) Grothendieck-Riemann-Roch formula at the level of cotangent bundles with Theorem 3.1, then we have a Riemann-Roch formula for $\mathscr{D}$-modules.

Recall that for a smooth algebraic variety X, we have the corresponding complex manifold X^{an} and morphism $\iota : X^{\mathrm{an}} \to X$. Then we have a canonical morphism

$$\iota^{-1}\mathscr{D}_X \to \mathscr{D}_{X^{\mathrm{an}}}$$

of sheaves of rings satisfying

$$\mathscr{D}_{X^{\mathrm{an}}} \simeq \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}\mathscr{D}_X.$$

Hence we obtain a functor

$$(\bullet)^{\mathrm{an}} : \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{D}_{X^{\mathrm{an}}})$$

sending $\mathscr{M}$ to $\mathscr{M}^{\mathrm{an}} := \mathscr{D}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{D}_X} \iota^{-1}\mathscr{M}$. Note that since $\mathscr{D}_{X^{\mathrm{an}}}$ is faithfully flat over $\iota^{-1}(\mathscr{D}_X)$, this functor is exact. We denote by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_{X^{\mathrm{an}}})$ the category of good $\mathscr{D}_{X^{\mathrm{an}}}$-modules (for definition and its properties, see [11, 4.7]), in particular, $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_{X^{\mathrm{an}}})$ is an abelian thick subcategory of $\mathrm{Mod}(\mathscr{D}_{X^{\mathrm{an}}})$. We also denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_{\mathrm{X}^{\mathrm{an}}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_{\mathrm{X}^{\mathrm{an}}})$ consisting of complexes with cohomology sheaves belonging to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_{X^{\mathrm{an}}})$ and $K_{\mathrm{gd}}(\mathscr{D}_{\mathrm{X}^{\mathrm{an}}})$ the Grothendieck group of $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_{\mathrm{X}^{\mathrm{an}}})$.

Recall from [10, Corollary 1.4.17] that for a coherent $\mathscr{D}_X$-module $\mathscr{M}$, $\mathscr{M}$ is generated by a coherent $\mathscr{O}_X$-module $\mathscr{F}$, i.e. $\mathscr{M} = \mathscr{D}_X \cdot \mathscr{F}$. Denote by

$$F_i\mathscr{D}_X := \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathscr{O}_X) \mid [P, f] \in F_{l-1}\mathscr{D}_X, \forall f \in \mathscr{O}_X\} \text{ and}$$
$$\mathscr{M}_i := F_i\mathscr{D}_X \cdot \mathscr{F},$$

then $\mathscr{M}_i \subset \mathscr{M}_{i+1}$ for each $i \geq 0$ and $\mathscr{M} = \bigcup_i \mathscr{M}_i = \varinjlim_i \mathscr{M}_i$. Hence we have the well defined functor:

$$\Upsilon : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X) \longrightarrow \mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_{X^{\mathrm{an}}})$$

sending $\mathscr{M} = \bigcup_i \mathscr{M}_i$ to $\mathscr{M}^{\mathrm{an}} := \bigcup_i \mathscr{M}_i^{\mathrm{an}}$. On the other hand, if X is proper, then by GAGA theorem, we have the well defined inverse funtor to the functor Υ. Hence we get the equivalence:

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_{X^{\mathrm{an}}}).$$

In particular, by [2, P.283 Lemma 1.6], we have

$$K_{\mathrm{coh}}(\mathscr{D}_X) \simeq K_{\mathrm{gd}}(\mathscr{D}_{X^{\mathrm{an}}}). \tag{3.4}$$

For a morphism $f : X \to Y$ of smooth algebraic varieties, we have the induced analytic morphism of complex manifolds $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ and the corresponding morphisms to (3.1):

$$T^*X^{\mathrm{an}} \xleftarrow{f_d^{\mathrm{an}}} X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} T^*Y^{\mathrm{an}} \xrightarrow{f_\pi^{\mathrm{an}}} T^*Y^{\mathrm{an}}.$$

If f is a proper morphism, then we have the corresponding functors to (3.2) and (3.3):

$$f_d^{\mathrm{an}!} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{T^*X^{\mathrm{an}}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} T^*Y^{\mathrm{an}}}) \tag{3.5}$$

defined by

$$f_d^{\mathrm{an}!}(-) := f_d^{\mathrm{an}*}(-) \otimes_{\mathscr{O}_{X^{\mathrm{an}}}} \omega_{X^{\mathrm{an}}/Y^{\mathrm{an}}}$$

and

$$f_{\pi\ *}^{\mathrm{an}} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} T^*Y^{\mathrm{an}}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{T^*Y^{\mathrm{an}}}). \tag{3.6}$$

Hence (3.5) and (3.6) induce group homomorphisms:

$$f_d^{\mathrm{an}!} : K_{\mathrm{coh}}(\mathscr{O}_{T^*X^{\mathrm{an}}}) \to K_{\mathrm{coh}}(\mathscr{O}_{X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} T^*Y^{\mathrm{an}}})$$

and

$$f_{\pi\ *}^{\mathrm{an}} : K_{\mathrm{coh}}(\mathscr{O}_{X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} T^*Y^{\mathrm{an}}}) \to K_{\mathrm{coh}}(\mathscr{O}_{T^*Y^{\mathrm{an}}}).$$

Hence, by Theorem 3.1 and (3.4), we get following theorem.

Theorem 3.3. *Let $f : X \to Y$ be a proper morphism of proper smooth algebraic varieties, then we have the following commutative diagram*

$$\begin{array}{ccc} K_{\mathrm{gd}}(\mathscr{D}_{X^{\mathrm{an}}}) & \xrightarrow{\int_{f^{\mathrm{an}}}} & K_{\mathrm{gd}}(\mathscr{D}_{Y^{\mathrm{an}}}) \\ {\scriptstyle Car}\downarrow & & \downarrow{\scriptstyle Car} \\ K_{\mathrm{coh}}(\mathscr{O}_{T^*X^{\mathrm{an}}}) & \xrightarrow{f_{\pi\ *}^{\mathrm{an}} f_d^{\mathrm{an}!}} & K_{\mathrm{coh}}(\mathscr{O}_{T^*Y^{\mathrm{an}}}) \end{array}$$

where the group homomorphism $\int_{f^{\mathrm{an}}}$ is induced by the direct image functor $\int_f$. □

Remark 3.4. As in Remark 3.2. By using (analytic) Riemann-Roch formula for $\mathscr{O}$-modules, we get a Riemann-Roch formula for analytic $\mathscr{D}$-modules.

4 The first main theorem

4.1 Star-products

Let X be a complex manifold (or a smooth variety). We denote by $\delta_X : X \hookrightarrow X \times X$ the diagonal embedding and we set $\triangle_X = \delta_X(X)$. We denote by $\mathscr{O}_X$ the structure sheaf on X, by Ω_X the sheaf of differential forms of maximal degree and by Θ_X the sheaf of vector fields. As usual, we denote by $\mathscr{D}_X$ the sheaf of rings of differential operators on X. Recall that a bi-differential operator P on X is a $\mathbb{C}$-bilinear morphism $\mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$ which is obtained as the composition $\delta_X^{-1} \circ \tilde{P}$ where $\tilde{P}$ is a differential operator on $X \times X$ defined on a neighborhood of the diagonal and δ^{-1} is the restriction to the diagonal:

$$P(f, g)(x) = (\tilde{P}(x_1, x_2; \partial_{x_1}, \partial_{x_2})(f(x_1)g(x_2))|_{x_1 = x_2 = x}.$$

Hence the sheaf of bi-differential operators is isomorphic to

$$\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X,$$

where the both $\mathscr{D}_X$ are regarded as $\mathscr{O}_X$-modules by the left multiplications.

Definition 4.1. A *star algebra* on $\mathscr{O}_X[[\hbar]]$ is a $\mathbb{C}^\hbar$-bilinear sheaf morphism

$$\star : \mathscr{O}_X[[\hbar]] \times \mathscr{O}_X[[\hbar]] \to \mathscr{O}_X[[\hbar]]$$

satisfies the following conditions:

(i) the star product makes $\mathscr{O}_X[[\hbar]]$ into a sheaf of associated unital $\mathbb{C}^\hbar$-algebra with unit $1 \in \mathscr{O}_X$.

(ii) there is a sequence $P_i : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$ of bi-differential operators, such that for any two local sections $f, g \in \mathscr{O}_X$ one has

$$f \star g = fg + \sum_{i=1}^{\infty} P_i(f,g)\hbar^i.$$

Note that $f \star g \equiv fg \bmod \hbar$, and $P_i(f,1) = P_i(1,f) = 0$ for all f and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a star algebra.

4.2 DQ-algebras

Definition 4.2. A DQ-algebra $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebra locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]], \star)$ as a $\mathbb{C}^\hbar$-algebra.

Clearly, a DQ-algebra is a sheaf of $\hbar$-adically complete flat $\mathbb{C}^\hbar$-algebra on X satisfying $\mathscr{A}/\hbar\mathscr{A} \simeq \mathscr{O}_X$. Note also that for an algebraic variety X, a DQ-algebra $\mathcal{A}$ is called *deformation quantization* of $\mathscr{O}_X$ in [3] and [20].

Remark 4.3. For a smooth projective variety X, there exists a DQ-algebra $\mathscr{A}_X$ on X. For details, one refers to [3].

4.3 Riemann-Roch theorem for DQ-modules

Let X be a complex manifold. To a DQ-algebra $\mathscr{A}_X$ on X, we associate its *$\hbar$-localization*, the $\mathbb{C}^{\hbar,\mathrm{loc}}$-algebra

$$\mathscr{A}_X^{\mathrm{loc}} = \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \mathscr{A}_X.$$

Hence we have an exact functor

$$\mathrm{Mod}(\mathscr{A}_X) \xrightarrow{\otimes_{\mathbb{C}^\hbar}\mathbb{C}^{\hbar,\mathrm{loc}}} \mathrm{Mod}(\mathscr{A}_X^{\mathrm{loc}}).$$

If $\mathscr{M}$ is an $\mathscr{A}_X^{\mathrm{loc}}$-module, $\mathscr{M}_0$ is an $\mathscr{A}_X$-submodule and $\mathscr{M}_0 \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \xrightarrow{\sim} \mathscr{M}$, we shall say that $\mathscr{M}_0$ generates $\mathscr{M}$.

A coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is *good* if for any open relatively compact subset U of X, there exists a coherent $\mathscr{A}_X|_U$-module which generates $\mathscr{M}|_U$.

We denote by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the thick abelian subcategory of $\mathrm{Mod}\,(\mathscr{A}_X^{\mathrm{loc}})$ consisting of good $\mathscr{A}_X^{\mathrm{loc}}$-modules. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects with cohomology sheaves belonging to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$. We also denote by $K(\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}}))$ the Grothendieck group of $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and $K_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the Grothendieck group of $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and note that

$$K(\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})) \simeq K_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}}) \tag{4.1}$$

by [2, P. 283 Lemma 1.6].

If $(Y, \mathscr{A}_Y)$ is another complex manifold endowed with a DQ-algebra $\mathscr{A}_Y$, denote by q_i the i-th projection defined on $X \times Y$ $(i = 1, 2)$. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a\times Y})$. Set

$$\mathscr{M} \circ \mathscr{K} := Rq_{2!}(\mathscr{K} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} q_1^{-1}\mathscr{M}).$$

If q_2 is proper on $q_1^{-1}\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(\mathscr{K})$, then by [13, Theorem 9.1], $\mathscr{M} \circ \mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_Y)$.

Similarly, let $\mathscr{M}_1 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and $\mathscr{K}_1 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_{X^a\times Y})$. Set

$$\mathscr{M}_1 \circ \mathscr{K}_1 := Rq_{2!}(\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X^{\mathrm{loc}}} q_1^{-1}\mathscr{M}_1).$$

If q_2 is proper on $q_1^{-1}\mathrm{supp}(\mathscr{M}_1) \cap \mathrm{supp}(\mathscr{K}_1)$, then $\mathscr{M}_1 \circ \mathscr{K}_1 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_Y^{\mathrm{loc}})$ (see [18, Corollary 3.3.5]).

If X and Y are compact, then a kernel $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a\times Y})$ defines a functor

$$\circ\mathscr{K} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_Y) \tag{4.2}$$

which is called the Fourier-Mukai transform induced by $\mathscr{K}$. Hence the Fourier-Mukai transform of (4.2) defines a group homomorphism of Grothendieck groups

$$\circ[\mathscr{K}] : K_{\mathrm{coh}}(\mathscr{A}_X) \to K_{\mathrm{coh}}(\mathscr{A}_Y)$$

where $[\mathscr{K}] \in K_{\mathrm{coh}}(\mathscr{A}_{X^a \times Y})$. Similarly, a kernel $\mathscr{K}^{\mathrm{loc}}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_{X^a \times Y})$ defines a functor

$$\circ\mathscr{K}^{\mathrm{loc}} : \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_Y) \tag{4.3}$$

which is called the Fourier-Mukai transofrm induced by $\mathscr{K}^{\mathrm{loc}}$ and the Fourier-Mukai transform of (4.3) defines a group homomorphism of Grothendieck groups

$$\circ[\mathscr{K}^{\mathrm{loc}}] : K_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_X) \to K_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_Y)$$

where $[\mathscr{K}^{\mathrm{loc}}] \in K_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_{X^a \times Y})$.

Denote by

$$\mathrm{gr} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \tag{4.4}$$

the functor $\mathscr{M} \to \mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar} \mathscr{M}$ (see [13]).

We have the following proposition.

Proposition 4.4. *Let $(X, \mathscr{A}_X)$ and $(Y, \mathscr{A}_Y)$ be two compact complex manifolds endowed with DQ-algebras $\mathscr{A}_X$ and $\mathscr{A}_Y$ and let $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a \times Y})$. Then the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) & \xrightarrow{\circ\mathscr{K}} & \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_Y) \\ {\scriptstyle \mathrm{gr}}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}} \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) & \xrightarrow{\circ(\mathrm{gr}\mathscr{K})} & \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y) \end{array} \tag{4.5}$$

which induces the following commutative diagram

$$\begin{array}{ccc} K_{\mathrm{coh}}(\mathscr{A}_X) & \xrightarrow{\circ[\mathscr{K}]} & K_{\mathrm{coh}}(\mathscr{A}_Y) \\ {\scriptstyle \mathrm{gr}}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}} \\ K_{\mathrm{coh}}(\mathscr{O}_X) & \xrightarrow{\circ[\mathrm{gr}\mathscr{K}]} & K_{\mathrm{coh}}(\mathscr{O}_Y) \end{array}$$

where the group homomorphisms gr *(4.4) are induced by (4.5).*
Proof. Using the fact that the functor gr commutes with the convolution $\circ$. $\square$

Definition 4.5. Let X be a compact complex manifold, and denote by $H^*(X, \mathbb{C}) = \bigoplus_i H^i(X, \mathbb{C})$. One defines the Mukai vector of an object $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ as the cohomology class

$$v : K_{\mathrm{coh}}(\mathscr{O}_X) \to H^*(X, \mathbb{C})$$

by the formula

$$v([E]) = \mathrm{ch}([E]).\sqrt{\mathrm{td}(X)}$$

where $\mathrm{ch}([E])$ is the Chern character of $[E]$ and $\mathrm{td}(X)$ is the Todd class of tangent bundle of X.

Let X and Y be compact complex manifolds and let $E \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}$ $(\mathscr{O}_{X\times Y})$. Define the *cohomological integral transform* associated to E

$$\Phi_{[E]} : H^*(X, \mathbb{C}) \to H^*(Y, \mathbb{C})$$

by $\Phi_{[E]}(\alpha) = q_{2*}(v([E].{q_1}^*(\alpha))$, where q_i denotes the i-th projection defined on $X \times Y$ $(i = 1, 2)$.

We have the following theorem.

Theorem 4.6.*([5, Proposition 3.1.9] or [9, Corollary 5.29]) The following diagram is commutative*

$$\begin{array}{ccc}
K_{\mathrm{coh}}(\mathscr{O}_X) & \xrightarrow{\circ[E]} & K_{\mathrm{coh}}(\mathscr{O}_Y) \\
{\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle v} \\
H^*(X, \mathbb{C}) & \xrightarrow{\Phi_{[E]}} & H^*(Y, \mathbb{C}).
\end{array}$$

□

Combining Proposition 4.4 with Theorem 4.6, we obtain the following theorem.

Theorem 4.7. *(Riemann-Roch for $\mathscr{A}$-modules) Let $(X, \mathscr{A}_X)$ and $(Y, \mathscr{A}_Y)$ be two compact complex manifolds endowed with DQ-algebras $\mathscr{A}_X$ and $\mathscr{A}_Y$ and let $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a \times Y})$. Then the following diagram is commutative*

$$\begin{tikzcd} K_{\mathrm{coh}}(\mathscr{A}_X) \arrow[r, "\circ[\mathscr{K}]"] \arrow[d, "v \circ \mathrm{gr}"'] & K_{\mathrm{coh}}(\mathscr{A}_Y) \arrow[d, "v \circ \mathrm{gr}"] \\ H^*(X, \mathbb{C}) \arrow[r, "\Phi_{[\mathrm{gr}\mathscr{K}]}"] & H^*(Y, \mathbb{C}). \end{tikzcd}$$

□

4.2 Riemann-Roch theorem for $\mathscr{A}^{\mathrm{loc}}$-modules

First, we need the following lemma.

Lemma 4.8. *Let $(X, \mathscr{A}_X)$ be a compact complex manifold endowed with a DQ-algebra $\mathscr{A}_X$. Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and let $\mathscr{M}_0 \subset \mathscr{M}$ which generates $\mathscr{M}$. Then $[\mathscr{M}_0/\hbar\mathscr{M}_0]$ belongs to $K(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X))$ depends only on $\mathscr{M}$.*

Proof. We consider another generator $\mathscr{M}_0^{'}$ of $\mathscr{M}$. Since X is compact, there exists $m, n \geq 0$ such that $\mathscr{M}_0^{'} \subset \hbar^{-n}\mathscr{M}_0$ and $\mathscr{M}_0 \subset \hbar^{-m}\mathscr{M}_0^{'}$. Hence

$$\mathscr{M}_0^{'} \subset \hbar^{-n}\mathscr{M}_0 \subset \hbar^{-m-n}\mathscr{M}_0^{'}.$$

Since our modules have no $\hbar$-torsion, we have

$$\hbar^{-n}\mathscr{M}_0/\hbar^{-n+1}\mathscr{M}_0 \simeq \mathscr{M}_0/\hbar\mathscr{M}_0.$$

Hence we may assume that for N large enough, we have

$$\mathscr{M}_0' \subset \mathscr{M}_0 \subset \hbar^{-N}\mathscr{M}_0'.$$

We shall prove $[\mathscr{M}_0/\hbar\mathscr{M}_0] = [\mathscr{M}_0'/\hbar\mathscr{M}_0']$ by induction on $N \geq 0$.
When $N = 0$, it is trivial.
When $N = 1$, we have the following exact sequences:

$$0 \to \hbar^{-1}\mathscr{M}_0'/\mathscr{M}_0 \to \hbar^{-1}\mathscr{M}_0/\mathscr{M}_0 \to \hbar^{-1}\mathscr{M}_0/\hbar^{-1}\mathscr{M}_0' \to 0$$

$$0 \to \mathscr{M}_0/\mathscr{M}_0' \to \hbar^{-1}\mathscr{M}_0'/\mathscr{M}_0' \to \hbar^{-1}\mathscr{M}_0'/\mathscr{M}_0 \to 0.$$

Since $\hbar^{-1}\mathscr{M}_0/\hbar^{-1}\mathscr{M}_0' \simeq \mathscr{M}_0/\mathscr{M}_0'$, we get $[\mathscr{M}_0/\hbar\mathscr{M}_0] = [\mathscr{M}_0'/\hbar\mathscr{M}_0']$.

For $N > 1$. Put $\mathscr{M}_0'' = \mathscr{M}_0' + \hbar\mathscr{M}_0$. Then

$$\mathscr{M}_0'' \subset \mathscr{M}_0 \subset \hbar^{-1}\mathscr{M}_0'',$$

hence, as above, $[\mathscr{M}_0''/\hbar\mathscr{M}_0''] = [\mathscr{M}_0/\hbar\mathscr{M}_0]$. On the other hand,

$$\mathscr{M}_0' \subset \mathscr{M}_0'' \subset \hbar^{-N+1}\mathscr{M}_0'$$

hence by induction hypothesis, we obtain

$$[\mathscr{M}_0''/\hbar\mathscr{M}_0''] = [\mathscr{M}_0'/\hbar\mathscr{M}_0'].$$

Hence $[\mathscr{M}_0/\hbar\mathscr{M}_0] = [\mathscr{M}_0'/\hbar\mathscr{M}_0']$, as desired. □

Theorem 4.9. *Let $(X, \mathscr{A}_X)$ be a compact complex manifold endowed with a DQ-algebra $\mathscr{A}_X$, then we have a well defined group homomorphism*

$$\mu : K(\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})) \to K(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)). \tag{4.6}$$

Proof. Define a function

$$\Gamma : \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}}) \to K(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X))$$

sending $\mathscr{M}$ to $[\mathscr{M}_0/\hbar\mathscr{M}_0]$ with $\mathscr{M}_0$ a generator of $\mathscr{M}$. We need to show that Γ is additive. For an exact sequence

$$(\star) \quad 0 \to \mathscr{N}_1 \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \xrightarrow{\alpha} \mathscr{N} \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \xrightarrow{\beta} \mathscr{N}_2 \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \to 0$$

in $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ with $\mathscr{N}_1, \mathscr{N}$ and $\mathscr{N}_2$ of no $\hbar$-torsions and set $\mathscr{M}_2 := \beta(\mathscr{N})$, $\mathscr{M} := \mathscr{N}$ and $\mathscr{M}_1 := ker(\beta|_{\mathscr{N}} : \mathscr{N} \to \beta(\mathscr{N}))$. Then we have the exact sequence

$$(*) \qquad 0 \to \mathscr{M}_1 \to \mathscr{M} \to \mathscr{M}_2 \to 0$$

in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ and $(*) \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \simeq (\star)$ with $\mathscr{M}_1, \mathscr{M}$ and $\mathscr{M}_2$ of no $\hbar$-torsions. By Lemma 4.8, we have $[\mathscr{N}_1/\hbar\mathscr{N}_1] = [\mathscr{M}_1/\hbar\mathscr{M}_1]$, $[\mathscr{N}/\hbar\mathscr{N}] = [\mathscr{M}/\hbar\mathscr{M}]$ and $[\mathscr{N}_2/\hbar\mathscr{N}_2] = [\mathscr{M}_2/\hbar\mathscr{M}_2]$. Hence from the exact sequence $(*)$, we get $[\mathscr{N}/\hbar\mathscr{N}] = [\mathscr{N}_1/\hbar\mathscr{N}_1] + [\mathscr{N}_2/\hbar\mathscr{N}_2]$. Hence Γ is additive and by universal property, and this defines a group homomorphism $\mu : K(\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})) \to K(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X))$. $\square$

From (1.1) and (4.1), the group homomorphism of (4.6) induces the group homomorphism

$$(4.7) \qquad \lambda : K_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}}) \to K_{\mathrm{coh}}(\mathscr{O}_X).$$

Now let X, Y be compact complex manifolds. From Lemma 4.8, we obtain the following theorem.

Theorem 4.10. *Let $(X, \mathscr{A}_X)$, $(Y, \mathscr{A}_Y)$ be compact complex manifolds endowed with DQ-algebras $\mathscr{A}_X$ and $\mathscr{A}_Y$ and let $\mathscr{K}^{\mathrm{loc}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}$ $(\mathscr{A}_{X^a \times Y})$. Then we have the following commutative diagram*

$$\begin{array}{ccc}
K_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}}) & \xrightarrow{\circ[\mathscr{K}^{\mathrm{loc}}]} & K_{\mathrm{gd}}(\mathscr{A}_Y^{\mathrm{loc}}) \\
\lambda \downarrow & & \downarrow \lambda \\
K_{\mathrm{coh}}(\mathscr{O}_X) & \xrightarrow{\circ\lambda([\mathscr{K}^{\mathrm{loc}}])} & K_{\mathrm{coh}}(\mathscr{O}_Y).
\end{array}$$

where λ's denote the group homomorphisms of (4.7).

Proof. Let $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a \times Y})$ which generates $\mathscr{K}^{\mathrm{loc}}$. Let $\mathscr{M}^{\mathrm{loc}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ which generates $\mathscr{M}^{\mathrm{loc}}$. Then $\mathscr{M} \circ \mathscr{K}$ generates $\mathscr{M}^{\mathrm{loc}} \circ \mathscr{K}^{\mathrm{loc}}$ and $\lambda([\mathscr{M}^{\mathrm{loc}} \circ \mathscr{K}^{\mathrm{loc}}]) =$

$[\mathrm{gr}(\mathscr{M} \circ \mathscr{K})] = [\mathrm{gr}\mathscr{M} \circ \mathrm{gr}\mathscr{K}] = [\mathrm{gr}\mathscr{M}] \circ [\mathrm{gr}\mathscr{K}] = \lambda[\mathscr{M}^{\mathrm{loc}}] \circ \lambda[\mathscr{K}^{\mathrm{loc}}]$ by Lemma 4.8, as desired. $\square$

Combining Theorem 4.6 and Theorem 4.10, we obtain the following theorem.

Theorem 4.11. *(Riemann-Roch for $\mathscr{A}^{\mathrm{loc}}$-modules) Let $(X, \mathscr{A}_X)$ and $(Y, \mathscr{A}_Y)$ be two compact complex manifolds endowed with DQ-algebras $\mathscr{A}_X$ and $\mathscr{A}_Y$ and let $\mathscr{K}^{\mathrm{loc}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_{X^a \times Y})$. Then the following diagram is commutative*

$$\begin{tikzcd} K_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}}) \arrow[r, "\circ[\mathscr{K}^{\mathrm{loc}}]"] \arrow[d, "v\circ\lambda"'] & K_{\mathrm{gd}}(\mathscr{A}_Y^{\mathrm{loc}}) \arrow[d, "v\circ\lambda"] \\ H^*(X, \mathbb{C}) \arrow[r, "\Phi_{\lambda([\mathscr{K}^{\mathrm{loc}}])}"] & H^*(Y, \mathbb{C}). \end{tikzcd}$$

$\square$

5 Applications to $\mathscr{D}$-modules

Let X be a complex manifold with sheaf of differential operators $\mathscr{D}_X$. Recall that the sheaf of $\mathbb{C}^{\hbar,\mathrm{loc}}$-algebras $\widehat{\mathscr{W}}_{T^*X}$ on the cotangent bundle T^*X has been constructed in [17], see also [18, chapter 3]. This sheaf of algebras contains a subsheaf $\widehat{\mathscr{W}}_{T^*X}(0)$ which is a DQ-algebra such that $\widehat{\mathscr{W}}_{T^*X}(0) \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \xrightarrow{\sim} \widehat{\mathscr{W}}_{T^*X}$. Hence $\widehat{\mathscr{W}}_{T^*X}$ is a *$\hbar$-localization* of $\widehat{\mathscr{W}}_{T^*X}(0)$.

Since $\widehat{\mathscr{W}}_{T^*X}|_X$ is flat over $\mathscr{D}_X$, we have an exact functor (see [18, chapter 3]):

$$(\bullet)^{\mathrm{W}} : \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\widehat{\mathscr{W}}_{T^*X})$$

which sends $\mathscr{M}$ to $\widehat{\mathscr{W}}_{T^*X} \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}$, where $\pi : T^*X \to X$ is the projection. Then $(\bullet)^{\mathrm{W}}$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}}_{T^*X})$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*X})$.

Let $f : X \to Y$ be a morphism of complex manifolds. Define $\widehat{\mathscr{W}}_{T^*X \leftarrow T^*Y}$ as $(\mathscr{D}_{X\leftarrow Y})^{\mathrm{W}}$ and we denote by $\mathscr{K} := \widehat{\mathscr{W}}_{T^*X\leftarrow T^*Y}(0) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}}_{(T^*X)^a\times T^*Y}(0))$, and $\mathscr{K}^{\mathrm{loc}} := \widehat{\mathscr{W}}_{T^*X\leftarrow T^*Y} = \mathscr{K} \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_{(T^*X)^a\times T^*Y})$. We also denote by $\circ\mathscr{K} := \widehat{\mathscr{W}}_{T^*X\leftarrow T^*Y}(0) \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{W}}(0)}$ and $\circ\mathscr{K}^{\mathrm{loc}} := \widehat{\mathscr{W}}_{T^*X\leftarrow T^*Y} \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{W}}}$. If f is proper, then $\circ\mathscr{K}$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}}_{T^*X}(0))$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}}_{T^*Y}(0))$ and $\circ\mathscr{K}^{\mathrm{loc}}$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*X})$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*Y})$ as we mentioned in chapter 4. Hence we have group homomorphisms $\circ[\mathscr{K}] : K_{\mathrm{coh}}(\widehat{\mathscr{W}}_{T^*X}(0)) \to K_{\mathrm{coh}}(\widehat{\mathscr{W}}_{T^*Y}(0))$ and $\circ[\mathscr{K}^{\mathrm{loc}}] : K_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*X}) \to K_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*Y})$.

Recall that from the proof of [18, Theorem 7.4.4], we have the following commutative diagram

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X) \arrow[r, "\int_f"] \arrow[d, "(\bullet)^{\mathrm{W}}"'] & \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y) \arrow[d, "(\bullet)^{\mathrm{W}}"] \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*X}) \arrow[r, "\circ\mathscr{K}^{\mathrm{loc}}"] & \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*Y}) \end{tikzcd} \tag{5.1}$$

where $\int_f$ is the direct image functor.

Hence we get:

Theorem 5.1. *Let $f : X \to Y$ be a proper morphism of complex manifolds and let $\mathscr{K}^{\mathrm{loc}} := \widehat{\mathscr{W}}_{T^*X \leftarrow T^*Y} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_{(T^*X)^a \times T^*Y})$. Then we have the following commutative diagram*

$$\begin{array}{ccc} K_{\mathrm{gd}}(\mathscr{D}_X) & \xrightarrow{\int_f} & K_{\mathrm{gd}}(\mathscr{D}_Y) \\ {\scriptstyle (\bullet)^{\mathrm{W}}}\big\downarrow & & \big\downarrow{\scriptstyle (\bullet)^{\mathrm{W}}} \\ K_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*X}) & \xrightarrow{\circ[\mathscr{K}^{\mathrm{loc}}]} & K_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*Y}). \end{array}$$

where the group homomorphisms $(\bullet)^{\mathrm{W}}$ are induced by (5.1). $\square$

We also need the following lemma.

Lemma 5.2. *Let X be a compact complex manifold, then the group homomorphism:*

$$K_{\mathrm{gd}}(\mathscr{D}_X) \xrightarrow{Car} K_{\mathrm{coh}}(\mathscr{O}_{T^*X})$$

is the composition of group homomorphisms:

$$K_{\mathrm{gd}}(\mathscr{D}_X) \xrightarrow{(\bullet)^{\mathrm{W}}} K_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*X}) \xrightarrow{\lambda} K_{\mathrm{coh}}(\mathscr{O}_{T^*X})$$

where λ is the group homomorphism of (4.7).

Proof. For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$ in $K_{\mathrm{gd}}(\mathscr{D}_X)$, we have $\mathscr{M} = \sum(-1)^i H^i$ $(\mathscr{M})$. Hence, it is sufficient to prove this for $\mathscr{M} \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_X)$. Since X is compact, $\mathscr{M}$ has a resolution

$$0 \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}_N \to \cdots \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}_0 \to \mathscr{M} \to 0$$

where $\mathscr{F}_i$ are coherent $\mathscr{O}_X$-modules. Hence it is enough to check for a good $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}$ and this is easy to check. $\square$

Hence from Theorem 4.10, Theorem 5.1 and Lemma 5.2, we obtain the following theorem.

Theorem 5.3. *Let $f : X \to Y$ be a proper morphism of compact complex manifolds and let $\mathscr{K} := \widehat{\mathscr{W}}_{T^*X \leftarrow T^*Y}(0)$. Then we have the following commutative diagram*

$$\begin{tikzcd}
K_{\mathrm{gd}}(\mathscr{D}_X) \arrow[r, "\int_f"] \arrow[d, "(\bullet)^{\mathrm{W}}"'] & K_{\mathrm{gd}}(\mathscr{D}_Y) \arrow[d, "(\bullet)^{\mathrm{W}}"] \\
K_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*X}) \arrow[r, "\circ[\mathscr{K}^{\mathrm{loc}}]"] \arrow[d, "\lambda"'] & K_{\mathrm{gd}}(\widehat{\mathscr{W}}_{T^*Y}) \arrow[d, "\lambda"] \\
K_{\mathrm{coh}}(\mathscr{O}_{T^*X}) \arrow[r, "\circ[\mathrm{gr}\mathscr{K}]"] & K_{\mathrm{coh}}(\mathscr{O}_{T^*Y})
\end{tikzcd}$$

where λ's denote the group homomorphisms of (4.7). In particular, since $\mathrm{gr}\mathscr{K} = \omega_X \otimes_{\mathscr{O}_X} \omega_Y^{\otimes -1} \otimes_{\mathscr{O}_Y} \mathscr{O}_{X \times_Y T^*Y}$, we recover the results of Theorem 3.3. □

6 Review on DQ-modules (after K-S)

6.1. Algebroid

In this section, we denote by X a topological space and $\mathbb{K}$ a commutative unital ring. If A is a ring, an A-module means a left A-module. Recall that the notion algebroid was first introduced by Kontsevich [15], see also [1] and [12].

Definition 6.1. A $\mathbb{K}$-algebroid $\mathscr{A}$ on X is a $\mathbb{K}$-linear stack such that

(1) $\mathscr{A}(U)$ is nonempty for $U \subseteq X$,

(2) two objects of $\mathscr{A}(U)$ are locally isomorphic for $U \subseteq X$.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X. In the sequel, we set $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_i \cap U_j \cap U_k$, etc.

Consider the data of

$$(6.1)\qquad \begin{cases} \text{a } \mathbb{K}\text{-algebroid } \mathscr{A} \text{ on } X \\ \sigma_i \in \mathscr{A}(U) \text{ and isomorphism } \varphi_{ij} : \sigma_j|_{U_{ij}} \to \sigma_i|_{U_{ij}}. \end{cases}$$

To these data, we associate:

- $\mathscr{A}_i = \mathscr{E}nd_{\mathbb{K}}(\sigma_i)$,

- $f_{ij} : \mathscr{A}_j|_{U_{ij}} \to \mathscr{A}_i|_{U_{ij}}$ the $\mathbb{K}$-algebra isomorphism $a \mapsto \varphi_{ij} \circ a \circ \varphi_{ij}^{-1}$,

- a_{ijk}, the invertible element of $\mathscr{A}_i(U_{ijk})$ given by $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ik}^{-1}$.

Then:

$$\begin{cases} f_{ij} \circ f_{jk} = \mathrm{Ad}(a_{ijk}) \circ f_{ik} \\ a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijl} \end{cases} \tag{6.2}$$

(Recall that $\mathrm{Ad}(a)(b) = aba^{-1}$)

Conversely, let $\mathscr{A}_i$ be sheaves of $\mathbb{K}$-algebras on U_i ($i \in I$), let $f_{ij} : \mathscr{A}_j|_{U_{ij}} \to \mathscr{A}_i|_{U_{ij}}$ ($i, j \in I$) be $\mathbb{K}$-algebra isomorphisms, and let a_{ijk} ($i, j, k \in I$) be invertible sections of $\mathscr{A}_i(U_{ij})$ satisfies (6.2). One calls:

$$(\{\mathscr{A}_i\}_{i\in I}, \{f_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I}) \quad . \tag{6.3}$$

a gluing datum for $\mathbb{K}$-algebroids on $\mathscr{U}$.

Theorem 6.2. *(See [4], [7]) Assume that the topological space X is paracompact. Considering a gluing datum* (6.3) *on $\mathscr{U}$. Then there exist an algebroid $\mathscr{A}$ on X and $\{\sigma_i, \varphi_{ij}\}_{i,j\in I}$ as in* (6.1) *to which this gluing datum is associated. Moreover, the data $(\mathscr{A}, \sigma_i, \varphi_{ij})$ are unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.*

In general, if a topological space X is not paracompact, for example for algebraic varieties, then Theorem 6.2 may be false. Hence we need another local description of such algebraic algebroids.

Definition 6.3. Let $\mathscr{A}$ and $\mathscr{A}'$ be two sheaves of $\mathbb{K}$-algebras. An $\mathscr{A} \otimes \mathscr{A}'$-module $\mathscr{L}$ is called *bi-invertible* if there exists locally a section ω of $\mathscr{L}$ such that $\mathscr{A} \ni a \mapsto (a \otimes 1)\omega \in \mathscr{L}$ and $\mathscr{A}' \ni a' \mapsto (a' \otimes 1)\omega \in \mathscr{L}$ give isomorphism of $\mathscr{A}$-modules and $\mathscr{A}'$-modules, respectively.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X. Consider the data of

$$\text{(6.4)} \qquad \begin{cases} \text{a } \mathbb{K}\text{-algebroid } \mathscr{A} \text{ on } X \\ \sigma_i \in \mathscr{A}(U). \end{cases}$$

To these data, we associate:

- $\mathscr{A}_i = \mathscr{E}nd_{\mathbb{K}}(\sigma_i)$,
- $\mathscr{L}_{ij} := \mathscr{H}om_{\mathscr{A}_i|_{U_{ij}}}(\sigma_j|_{U_{ij}}, \sigma_i|_{U_{ij}})$, hence $\mathscr{L}_{ij}$ is a bi-invertible $\mathscr{A}_i \otimes \mathscr{A}_j^{\mathrm{op}}$-module on U_{ij},
- the natural isomorphisms
  $$\text{(6.5)} \qquad a_{ijk} : \mathscr{L}_{ij} \otimes_{\mathscr{A}_j} \mathscr{L}_{jk} \xrightarrow{\sim} \mathscr{L}_{ik}.$$

Then the diagram below in $\mathrm{Mod}(\mathscr{A}_i \otimes \mathscr{A}_l^{\mathrm{op}}|_{U_{ijkl}})$ commutes:

$$\text{(6.6)} \qquad \begin{tikzcd} \mathscr{L}_{ij} \otimes \mathscr{L}_{jk} \mathscr{L}_{kl} \arrow[r, "a_{ijk}"] \arrow[d, "a_{jkl}"] & \mathscr{L}_{ik} \otimes \mathscr{L}_{kl} \arrow[d, "a_{ikl}"] \\ \mathscr{L}_{ij} \otimes \mathscr{L}_{jl} \arrow[r, "a_{ijl}"] & \mathscr{L}_{il}. \end{tikzcd}$$

Conversely, let $\mathscr{A}_i$ be sheaves of $\mathbb{K}$-algebras on U_i ($i \in I$), let $\mathscr{L}_{ij}$ be a bi-invertible $\mathscr{A}_i \otimes \mathscr{A}_j^{\mathrm{op}}$-module on U_{ij}, and let a_{ijk} be isomorphisms as in (6.5) such that the diagram (6.6) commutes. One calls

$$\text{(6.7)} \qquad (\{\mathscr{A}_i\}_{i\in I}, \{\mathscr{L}_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I})$$

an algebraic gluing datum for $\mathbb{K}$-algebroids on $\mathscr{U}$.

Theorem 6.4. *(See [13]) Consider an algebraic gluing datum (6.7) on $\mathscr{U}$. Then there exist an algebroid $\mathscr{A}$ on X and datum $\{\sigma_i, \varphi_{ij}\}_{i,j\in I}$ as in (6.1) to which this gluing datum is associated. Moreover, the data $(\mathscr{A}, \sigma_i, \varphi_{ij})$ are unique up to an equivalence* *of stacks, this equivalence being unique up to a unique isomorphism.*

Sketch of the proof. We define a category $\mathrm{Mod}(\mathscr{A}_X)$ as follows. As object $\mathscr{M} \in \mathrm{Mod}(\mathscr{A}_X)$ is defined as a family $\{\mathscr{M}_i, q_{ij}\}_{i,j\in I}$ with $\mathscr{M}_i \in \mathscr{A}_i$ and the $q'_{ij}s$ are isomorphisms

$$q_{ij} : \mathscr{L}_{ij} \otimes_{\mathscr{A}_j} \mathscr{M}_j \xrightarrow{\sim} \mathscr{M}_i$$

making the diagram below commutative:

$$\begin{tikzcd}
\mathscr{L}_{ij} \otimes \mathscr{L}_{jk} \otimes \mathscr{M}_k \arrow[r, "q_{jk}"] \arrow[d, "a_{jkl}"] & \mathscr{L}_{ij} \otimes \mathscr{M}_j \arrow[d, "q_{ij}"] \\
\mathscr{L}_{ik} \otimes \mathscr{M}_k \arrow[r, "q_{ik}"] & \mathscr{M}_i.
\end{tikzcd}$$

A morphism $\{\mathscr{M}_i, q_{ji}\}_{i,j\in I} \to \{\mathscr{M}'_i, q'_{ji}\}_{i,j\in I}$ in $\mathrm{Mod}(\mathscr{A}_X)$ is a family of morphisms $u_i : \mathscr{M}_i \to \mathscr{M}'_i$ satisfying the natural compatibility conditions. Replacing X with U open in X, we define a prestack $U \to \mathrm{Mod}(\mathscr{A}_U)$ and one easily check that this prestack is a stack and moreover that $\mathrm{Mod}(\mathscr{A}_{U_i})$ is equivalent to $\mathrm{Mod}(\mathscr{A}_i)$. We denote it by $\mathfrak{Mod}(\mathscr{A})$. Then we define the algebroid $\mathscr{A}_X$ as the substack of $(\mathfrak{Mod}(\mathscr{A}))^{\mathrm{op}}$ consisting of objects locally isomorphic to $\mathscr{A}_i$ on U_i.

For an algebroid $\mathscr{A}$, one defines the Grothendieck $\mathbb{K}$-linear abelian category $\mathrm{Mod}(\mathscr{A})$, whose objects are called $\mathscr{A}$-modules, by setting:

$$\mathrm{Mod}(\mathscr{A}) := \mathrm{Fct}_{\mathbb{K}}(\mathscr{A}, \mathfrak{Mod}(\mathbb{K}_X)).$$

Here $\mathfrak{Mod}(\mathbb{K}_X)$ is the $\mathbb{K}$-linear stack of sheaves of $\mathbb{K}$-modules on X, and $\mathrm{Fct}_{\mathbb{K}}$ is the category of $\mathbb{K}$-linear functors of stacks.

We have the well defined notion of tensor product for two $\mathbb{K}$-algebroids $\mathscr{C}, \mathscr{C}'$, say $\mathscr{C} \otimes_{\mathbb{K}} \mathscr{C}'$. For a $\mathbb{K}$-algebroid $\mathscr{A}$, $\mathrm{Mod}(\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}})$ has a canonical object given by

$$\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}} \ni (\sigma, \sigma'^{\mathrm{op}}) \mapsto \mathrm{H}om_{\mathscr{A}}(\sigma', \sigma) \in \mathfrak{Mod}(\mathbb{K}_X).$$

We denote this object by the same letter $\mathscr{A}$.

For $\mathbb{K}$-algebroids $\mathscr{A}_i$ ($i = 1, 2, 3$), we have functors:

$$\cdot \otimes_{\mathscr{A}_2} \cdot \colon \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_2^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{A}_2 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}) \to \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}})$$

and

$$\mathscr{H}om_{\mathscr{A}_1}(\cdot, \cdot) : \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_2^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}) \to \mathrm{Mod}(\mathscr{A}_2 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}).$$

In particular, we have

$$\cdot \otimes_{\mathscr{A}} \cdot \colon \mathrm{Mod}(\mathscr{A}^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{A}) \to \mathrm{Mod}(\mathbb{K}_X)$$

and

$$\mathscr{H}om_{\mathscr{A}}(\cdot, \cdot) : \mathrm{Mod}(\mathscr{A})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A}) \to \mathrm{Mod}(\mathbb{K}_X).$$

Let Y be another topological space and $f : X \mapsto Y$ be a continuous map, and let $\mathscr{A}$ be a $\mathbb{K}$-algebroid on Y. We denote by $f^{-1}\mathscr{A}$ the $\mathbb{K}$-linear stack associated with the prestack $\mathfrak{S}$ given by

$\mathfrak{S}(U) = \{(\sigma, V); V$ is an open subset of Y such that $f(U) \subset V$ and $\sigma \in \mathscr{A}(V)\}$ for any open subset U of X,

$$\mathrm{Hom}_{\mathfrak{S}(U)}((\sigma, V), (\sigma', V')) = \Gamma(U, f^{-1}\mathscr{H}om_{\mathscr{A}}(\sigma, \sigma')).$$

Then $f^{-1}\mathscr{A}$ is a $\mathbb{K}$-algebroid on X.

Notations: For the rest of this chapter, denote by X a complex manifold or a smooth variety and $\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]]$ the power series algebra.

6.2 Invertible $\mathscr{O}_X$-algebroids

Definition 6.5. A $\mathbb{C}$-algebroid $\mathscr{P}$ on X is called an invertible $\mathscr{O}_X$-algebroid if for any open subset U of X and any $\sigma \in \mathscr{P}(U)$, there is a $\mathbb{C}$-algebra isomorphism $\mathscr{E}nd_{\mathscr{P}}(\sigma) \simeq \mathscr{O}_U$.

We shall state some properties for invertible $\mathscr{O}_X$-algebroids.

Let $\mathscr{P}$ be an invertible $\mathscr{O}_X$-algebroid, then for $\sigma, \sigma' \in \mathscr{P}(U)$, $\mathscr{H}om(\sigma, \sigma')$ is an invertible $\mathscr{O}_U$-module.

For two invertible $\mathscr{O}_X$-algebroids $\mathscr{P}_1$ and $\mathscr{P}_2$, we denote by $\mathscr{P}_1 \otimes_{\mathscr{O}_X} \mathscr{P}_2$ the $\mathbb{C}$-linear stack associated with the prestack whose objects over an open set U is $\mathscr{P}_1(U) \times \mathscr{P}_2(U)$, and $\mathscr{H}om((\sigma_1, \sigma_2), (\sigma_1', \sigma_2')) = \mathscr{H}om(\sigma_1, \sigma_1') \otimes_{\mathscr{O}_X} \mathscr{H}om(\sigma_2, \sigma_2')$. Then $\mathscr{P}_1 \otimes_{\mathscr{O}_X} \mathscr{P}_2$ is an invertible $\mathscr{O}_X$-algebroid. Note also that the set of equivalence classes of invertible $\mathscr{O}_X$-algebroids has a structure of an additive group by the operation $\cdot \otimes_{\mathscr{O}_X} \cdot$, and this group is isomorphic to $H^2(X, \mathscr{O}_X^\times)$.

The following remark is due to Prof. Joseph Oesterlé and is crucial for the paper.

Remark 6.6. For a smooth algebraic variety X as Zariski topology over $\mathbb{C}$, the group $H^2(X, \mathscr{O}_X^\times)$ is trivial. Hence any invertible $\mathscr{O}_X$-algebroid $\mathscr{P}$ is equivalent to $\mathscr{O}_X$.

We sketch the proof of it. Let K be the field of rational functions on X, $K_X^\times$, the constant sheaf with stalk the abelian group $K^\times$ and denote by $X_1 = \{x \in X | \dim \mathscr{O}_{X,x} = 1\}$ (or the set of closed irreducible hypersurfaces of X). Let $x \in X_1$, since X is a variety, the ring $\mathscr{O}_{X,x}$ is a DVR with valuation v_x and quotient field K. Let $\mathbb{Z}_x = (i_x)_*(\mathbb{Z})$ where $i_x : x \to X$ and let $U \subset X$ be an open set, then $\mathbb{Z}_x(U) = 0$ if $x \notin U$ and $\mathbb{Z}_x(U) = \mathbb{Z}$ if $x \in U$.

Considering the sheaf $\bigoplus_{x \in X_1} \mathbb{Z}_x$, then $(\bigoplus_{x \in X_1} \mathbb{Z}_x)(U) = \bigoplus_{x \in X_1} \mathbb{Z}_x(U) = \mathbb{Z}^{U \cap X_1}$. Hence we can define a morphism of sheaves

$$\upsilon : K_X^\times \to \bigoplus_{x \in X_1} \mathbb{Z}_x$$

by: $\upsilon(f) = (\upsilon_x(f))_{x \in X_1 \cap U}$ where U is a nonempty open subset of X and $f \in K_X^\times(U) = K^\times$. Then one has an exact sequence

$$0 \to \mathscr{O}_X^\times \xrightarrow{u} K_X^\times \xrightarrow{\upsilon} \bigoplus_{x \in X_1} \mathbb{Z}_x \to 0$$

where u is the natural morphism. Since $K_X^\times$ is constant, it is a flabby sheaf for the Zariski topology. On the other hand the sheaf $\bigoplus_{x \in X_1} \mathbb{Z}_x$ is also flabby. It follows that $H^j(X, \mathscr{O}_X^\times)$ is zero for $j > 1$.

Let $f : X \to Y$ be a morphism of complex manifolds or smooth varieties. For an invertible $\mathscr{O}_Y$-algebroid $\mathscr{P}_Y$, we denote by $f^*\mathscr{P}_Y$ the $\mathbb{C}$-linear stack on X associated with the prestack whose objects on U are the objects of $(f^{-1}\mathscr{P}_Y)(U)$ and $\mathscr{H}om(\sigma, \sigma') = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{H}om_{f^{-1}\mathscr{P}_Y}(\sigma, \sigma')$. Then $f^*\mathscr{P}_Y$ is an invertible $\mathscr{O}_X$-algebroid.

6.3 DQ-algebroids

Definition 6.7. A DQ-algebroid $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebroid such that for each open set $U \subset X$ and each $\sigma \in \mathscr{A}(U)$, the $\mathbb{C}^\hbar$-algebra $\mathscr{H}om_{\mathscr{A}}(\sigma, \sigma)$ is a DQ-algebra on U.

Let $\mathscr{A}_X$ be a DQ-algebroid on X. For an $\mathscr{A}_X$-module $\mathscr{M}$, the local notions of being coherent or locally free, etc. make sense.

The category $\mathrm{Mod}(\mathscr{A}_X)$ is a Grothendieck category and we denote by $\mathrm{D}(\mathscr{A}_X)$ its derived category and by $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$ its bounded derived category. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$ consisting of objects with coherent cohomologies.

6.4 Graded modules

Let $\mathscr{A}_X$ be a DQ-algebroid on X. Let us denote by $\mathrm{gr}(\mathscr{A}_X)$ the $\mathbb{C}$-algebroid associated with the prestack $\mathfrak{S}$ given by

$$\mathrm{Ob}(\mathfrak{S}(U))=\mathrm{Ob}(\mathscr{A}_X(U)) \text{ for an open subset } U \text{ of } X,$$

$$\mathrm{Hom}_{\mathfrak{S}}(\sigma,\sigma')=\mathrm{Hom}_{\mathscr{A}_X}(\sigma,\sigma')/\hbar\mathrm{Hom}_{\mathscr{A}_X}(\sigma,\sigma') \text{ for } \sigma,\sigma' \in \mathscr{A}_X(U).$$

Then it is easy to see that $\mathrm{gr}(\mathscr{A}_X)$ is an invertible $\mathscr{O}_X$-algebroid and the left derived functor of the right exact functor $\mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathrm{gr}(\mathscr{A}_X))$ given by $\mathscr{M} \to \mathscr{M}/\hbar\mathscr{M}$ is denoted by $\mathrm{gr} : \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{gr}(\mathscr{A}_X))$.

The functor gr induces a functor (we keep the same notation):

$$\mathrm{gr} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)\to\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}(\mathscr{A}_X)). \tag{6.8}$$

We need the following lemma in [13].

Lemma 6.8. *The functor* gr *in (6.8) is conservative* (i.e., *a morphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ *is an isomorphism as soon as its image by* gr *is an isomorphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}(\mathscr{A}_X))$).

Set $\mathrm{D}^{\mathrm{b}}_{f}(\mathbb{C}^{\hbar}) := \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{C}^{\hbar})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\hbar})$ consisting of objects with finitely generated cohomologies and the same as the category $\mathrm{D}^{\mathrm{b}}_{f}(\mathbb{C})$.

Hence we have a well defined functor $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}} : \mathrm{D}^{\mathrm{b}}_{f}(\mathbb{C}^{\hbar}) \to \mathrm{D}^{\mathrm{b}}_{f}(\mathbb{C})$. As an application of Lemma 6.8, we have

Corollary 6.9. *The functor* $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar}\colon \mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar) \to \mathrm{D}^{\mathrm{b}}_f(\mathbb{C})$ *is conservative.*
Proof. Apply the functor gr in Lemma 6.8 to $X = \{\mathrm{pt}\}$.

The following proposition is in [13] which will be used in Theorem 8.2.

Proposition 6.10. *Let* $(X_i, \mathscr{A}_{X_i})$ *be complex manifolds or smooth varieties endowed with DQ*-algebroids $\mathscr{A}_{X_i}$ *(i=1,2,3).*

(i) *Let* $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}^a})$ $(i = 1, 2)$*. Then*

$$\mathrm{gr}(\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2} \mathscr{K}_2) \simeq \mathrm{gr}(\mathscr{K}_1) \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}(\mathscr{A}_2)} \mathrm{gr}(\mathscr{K}_2).$$

(ii) *Let* $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}})$ $(i = 1, 2)$*. Then*

$$\mathrm{gr}\mathrm{R}\mathscr{H}om_{\mathscr{A}_2}(\mathscr{K}_1, \mathscr{K}_2) \simeq \mathrm{R}\mathscr{H}om_{\mathrm{gr}(\mathscr{A}_2)}(\mathrm{gr}(\mathscr{K}_1), \mathrm{gr}(\mathscr{K}_2)).$$

6.5 Finiteness for DQ-kernels

Recall that we have the following Finiteness theorem.

Finiteness Theorem 6.11. *([13]) Let* $(X, \mathscr{A}_X)$ *be a compact complex manifold or a smooth projective variety endowed with a DQ-algebroid* $\mathscr{A}_X$ *and let* $\mathscr{M}$ *and* $\mathscr{N}$ *be two objects of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$*. Then the object* $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})$ *belongs to* $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$*.*

7 Analytization of a DQ-algebroid

In this chapter, we denote by X a smooth variety, X_{an} the complex manifold and $f : X_{\text{an}} \to X$ the continuous map induced by X.

Let $\mathscr{A}_X$ be a DQ-algebroid on X and let $\mathscr{U} = \{U_i\}_{i=1,\ldots,n}$ be a finite affine open covering of X. Consider the data:

$$\begin{cases} \text{a } \mathbb{C}\text{-algebroid } \mathscr{A}_X \text{ on } X \\ \sigma_i \in \mathscr{A}_X(U_i). \end{cases}$$

Then by Theorem 6.4, we have the following gluing data:

- $\mathscr{A}_i := \mathscr{E}nd_{\mathscr{A}}(\sigma_i) = (\mathscr{O}_{U_i}[[\hbar]], \star_i)$,
- $f_{ij} : \mathscr{A}_j|_{U_{ij}} \to \mathscr{A}_i|_{U_{ij}}$ the $\mathbb{C}^\hbar$-algebra isomorphism,
- a_{ijk}: invertible elements of $\mathscr{A}_i(U_{ijk})$

which satisfies:

$$\begin{cases} f_{ij} \circ f_{jk} = \mathrm{Ad}(a_{ijk}) \circ f_{ik} \\ a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl}. \end{cases}$$

Since $\mathscr{A}_i = (\mathscr{O}_{U_i}[[\hbar]], \star_i)$ is a star algebra for each $i = 1, \cdots, n$, by definition, we have

$$f_i \star_i g_i = f_i g_i + \sum_{j=1}^{\infty} \beta_j(f_i, g_i)\hbar^j \text{ for } f_i, g_i \in C_i := \Gamma(U_i, \mathscr{O}_X)$$

and $\beta_j : \mathscr{O}_{U_i} \times \mathscr{O}_{U_i} \to \mathscr{O}_{U_i}$ is a bi-differential operators for each j.

From the inclusion $C_i \hookrightarrow C_i^{\text{an}} := \Gamma(U_i, \mathscr{O}_{X_{\text{an}}})$, we can define a star product $\star_i^{\text{an}}$ on the analytic sheaf $\mathscr{O}_{U_i}$ by

$$f_i^{\mathrm{an}} \star_i^{\mathrm{an}} g_i^{\mathrm{an}} = f_i^{\mathrm{an}} g_i^{\mathrm{an}} + \sum_{j=1}^{\infty} \beta_j^{\mathrm{an}}(f_i^{\mathrm{an}}, g_i^{\mathrm{an}})\hbar^j \text{ for } f_i^{\mathrm{an}}, g_i^{\mathrm{an}} \in C_i^{\mathrm{an}},$$

where $\beta_j^{\mathrm{an}} : \mathscr{O}_{U_i} \times \mathscr{O}_{U_i} \to \mathscr{O}_{U_i}$ is a bi-differential operators on the analytic sheaf $\mathscr{O}_{U_i}$ for each j. Hence, we obtain the (analytic) star algebra $\mathscr{A}_i^{\mathrm{an}} = (\mathscr{O}_{U_i}[[\hbar]], \star_i^{\mathrm{an}})$ for each i.

Hence, we get the corresponding descent data on X_{an}:

- $\mathscr{A}_i^{\mathrm{an}} = (\mathscr{O}_{U_i}[[\hbar]], \star_i^{\mathrm{an}})$
- $f_{ij}^{\mathrm{an}} : \mathscr{A}_j^{\mathrm{an}}|_{U_{ij}} \to \mathscr{A}_i^{\mathrm{an}}|_{U_{ij}}$ the $\mathbb{C}^\hbar$-algebra isomorphism
- a_{ijk}^{an}, the invertible element of $\mathscr{A}_i^{\mathrm{an}}(U_{ijk})$

which satisfies the conditions for algebroid and we obtain the DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an} by Theorem 6.2. (Note that X_{an} is paracompact)

Hence for a DQ-algebroid $\mathscr{A}_X$ on X, we have the induced analytic DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an}.

Furthemore,

$$\mathscr{A}_{X_{\mathrm{an}}} \in \mathrm{Mod}(f^{-1}\mathscr{A}_X \otimes_{\mathbf{k}_0} \mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{op}}).$$

Hence for a DQ-algebroid $\mathscr{A}_X$ on a smooth variety X, we have the functor $f^* := \mathscr{A}_{X_{\mathrm{an}}} \otimes_{f^{-1}(\mathscr{A}_X)} f^{-1}(\cdot) : \mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}})$ which sends $\mathscr{M}$ to $\mathscr{A}_{X_{\mathrm{an}}} \otimes_{f^{-1}(\mathscr{A}_X)} f^{-1}(\mathscr{M})$. Denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ (resp. $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$) the category of $\mathscr{A}_X$ (resp. $\mathscr{A}_{X_{\mathrm{an}}}$) coherent sheaves. If $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then $f^*(\mathscr{M}) \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$.

8 The second main theorem

In this chapter, we prove the second main theorem of this thesis. Let $\mathscr{A}_X$ be a DQ-algebroid on a smooth algebraic variety X.

8.1 Flatness

Denote by X a smooth variety, X_{an} the complex analytic manifold and $f : X_{\mathrm{an}} \to X$ the continuous map induced by X. First, we need the following lemma. The following lemma over one point as a corollary of [13, Theorem 2.6].

Lemma 8.1. *The functor* $f^* : \mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}})$ *constructed in chapter 7 is exact.*

Proof. We may assume that $\mathscr{A}_X$ and $\mathscr{A}_{X_{\mathrm{an}}}$ are DQ-algebras. We need to show that $B := \mathscr{A}_{X_{\mathrm{an}},x}$ is flat over $R := \mathscr{A}_{X,x}$ for each $x \in X$. Note that:

(a) B has no $\hbar$-torsion,

(b) $B_0 := B/\hbar B = \mathscr{O}_{X_{\mathrm{an}},x}$ is a flat $R_0 := R/\hbar R = \mathscr{O}_{X,x}$-module,

(c) $B \simeq \varprojlim\limits_{n} B/\hbar^n B$.

Hence applying Theorem 2.6 in [13] to $X = \{\mathrm{pt}\}$, one gets the result. □

From Lemma 8.1, the functor $f^* : \mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}})$ induces a functor (we keep the same notation):

$$f^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}).$$

8.2 Fully faithfulness

Now we can prove the following theorem.

Theorem 8.2. *Let X be a smooth projective variety, then the functor $f^*\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is fully faithful.*
Proof. For $\mathscr{M}, \mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, we need to show that the morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)}(\mathscr{M}, \mathscr{N}) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})}(f^*(\mathscr{M}), f^*(\mathscr{N})) \tag{8.1}$$

is a bijection. In order to show that the morphism of (8.1) is a bijection, it is sufficient to show that the morphism

$$\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)}(\mathscr{M}, \mathscr{N}) \to \mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})}(f^*(\mathscr{M}), f^*(\mathscr{N})) \tag{8.2}$$

is an isomorphism. By Theorem 6.11, the two complexes

$$\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)}(\mathscr{M}, \mathscr{N}) \text{ and } \mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})}(f^*(\mathscr{M}), f^*(\mathscr{N}))$$

belong to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$. Hence, applying the functor $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar}$ to (8.2) and using Proposition 6.10, we get the morphism in $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C})$:

$$\begin{gathered}\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}(\mathscr{A}_X))}(\mathrm{gr}\mathscr{M}, \mathrm{gr}\mathscr{N}) \to \\ \mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}(\mathscr{A}_{X_{\mathrm{an}}}))}(f^*(\mathrm{gr}\mathscr{M}), f^*(\mathrm{gr}\mathscr{N})).\end{gathered} \tag{8.3}$$

Since X is a smooth projective variety and $\mathrm{gr}(\mathscr{A}_X)$ is an invertible $\mathscr{O}_X$-algebroid, $\mathrm{gr}(\mathscr{A}_X)$ is equivalent to $\mathscr{O}_X$ by Remark 6.6 and hence $\mathrm{gr}(\mathscr{A}_{X_{\mathrm{an}}})$ is equivalent to $\mathscr{O}_{X_{\mathrm{an}}}$. By the equivalence $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$ (Corollary 2.3), the morphism of (8.3) is an isomorphism. Moreover, the functor $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar} \cdot$ is conservative by Corollary 6.9. Therefore, the morphism of (8.2) is an isomorphism and the result follows. □

Corollary 8.3. *Let X be a smooth projective variety, then the natural functor $f^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is exact and* *fully faithful.*

For each $n > 0$, we denote by

$$\mathrm{Mod}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \text{ (resp. } \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}}))$$

the full subcategory of

$$\mathrm{Mod}(\mathscr{A}_X) \text{ (resp. } \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}}))$$

consisting of objects of $\mathscr{M}$ such that $\hbar^n : \mathscr{M} \to \mathscr{M}$ is the zero morphism.

Similarly, we denote by

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \text{ (resp. } \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}}))$$

the full subcategory of

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \text{ (resp. } \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}))$$

consisting of objects of $\mathscr{M}$ such that $\hbar^n : \mathscr{M} \to \mathscr{M}$ is the zero morphism for each $n > 0$. Therefore, we have a functor $f_n^* := f^*|_{\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)} : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}} /\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ for each $n > 0$.

Note that for $n = 1$, the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^1\mathscr{A}_X) \simeq \mathrm{Mod}_{\mathrm{coh}}\,(\mathscr{O}_X)$ is equivalent to the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^1\mathscr{A}_{X_{\mathrm{an}}}) \simeq \mathrm{Mod}_{\mathrm{coh}}\,(\mathscr{O}_{X_{\mathrm{an}}})$ by Theorem 1.1.

Corollary 8.4. *Let X be a smooth projective variety, then the functor $f_n^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ is exact and fully faithful for each $n > 0$.*

8.3 Essential surjectivity

Denote by X a smooth projective variety. Next, we shall prove that the functor $f^*\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is essentially surjective.

We first prove that the functor $f_n^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}\,(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ is essentially surjective for each $n > 0$. We need the following lemma.

Lemma 8.5. *Let $\mathcal{A}'$ and $\mathcal{B}'$ be thick subcategories of abelian categories $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be an exact functor that takes $\mathcal{A}'$ to $\mathcal{B}'$* and such that *the natural functor* (*we keep the same notation*) $\Phi : \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ *induced by* Φ *is fully faithful. Consider an exact sequence in $\mathcal{B}$*

$$(*) \qquad 0 \to \Phi(M') \to N \to \Phi(M'') \to 0,$$

with $M', M'' \in \mathcal{A}'$ and $N \in \mathcal{B}'$.
Then there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Phi(M') \arrow[r] \arrow[d, equal] & \Phi(M) \arrow[r] \arrow[d, "\wr"] & \Phi(M'') \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Phi(M') \arrow[r] & N \arrow[r] & \Phi(M'') \arrow[r] & 0
\end{tikzcd}$$

for some $M \in \mathcal{A}'$.(Note that the middle arrow is an isomorphism)

Proof. Since $(*)$ is an exact sequence, we get the morphism $v : \Phi(M'') \to \Phi(M')[1] = \Phi(M'[1])$ in $\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$. Since Φ is fully faithful, there exists a morphism $u : M'' \to M'[1]$ in $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ such that $v = \Phi(u)$. Consider the distinguished triangle

$$M'' \xrightarrow{u} M'[1] \longrightarrow L \xrightarrow{+1},$$

in $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ induced by u with $L \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$. Then from the long exact sequence

$$\cdots \longrightarrow H^i(M'') \longrightarrow H^i(M'[1]) \longrightarrow H^i(L) \longrightarrow H^i(M''[1]) \longrightarrow \cdots,$$

we get $H^i(L) = 0$ for $i \neq -1$. Hence $L[-1]$ is isomorphic to $H^0(L[-1]) \in \mathcal{A}'$ in $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$. Denote by $M = H^0(L[-1])$, then from the morphism of distinguished tirangles

$$\begin{array}{ccccccc}
\Phi(M') & \longrightarrow & \Phi(M) & \longrightarrow & \Phi(M'') & \longrightarrow & \Phi(M'[1]) \\
\| & & \downarrow & & \| & & \| \\
\Phi(M') & \longrightarrow & N & \longrightarrow & \Phi(M'') & \longrightarrow & \Phi(M'[1])
\end{array}$$

we get $N \simeq \Phi(M)$ and the result follows. $\square$

Set $\mathcal{A} = \mathrm{Mod}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$, $\mathcal{A}' = \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$, $\mathcal{B} = \mathrm{Mod}\ (\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ and $\mathcal{B}' = \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$. We shall apply Lemma 8.5.

Theorem 8.6. *The functor* $f_n^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}$ $(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ *is essentially surjective for each* $n > 0$.
Proof. We shall prove by induction.
When $n = 1$, it is Theorem 1.1.
We shall prove the theorem for $n > 1$.
For $\mathscr{M}^{\mathrm{an}} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$, consider the exact sequence

$$0 \to \hbar\mathscr{M}^{\mathrm{an}} \to \mathscr{M}^{\mathrm{an}} \to \mathscr{M}^{\mathrm{an}}/\hbar\mathscr{M}^{\mathrm{an}} \to 0$$

where $\hbar\mathscr{M}^{\mathrm{an}} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^{n-1}\mathscr{A}_{X_{\mathrm{an}}})$ and $\mathscr{M}^{\mathrm{an}}/\hbar\mathscr{M}^{\mathrm{an}}$ belongs to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$. Denote by $\mathscr{M}_1^{\mathrm{an}} = \hbar\mathscr{M}^{\mathrm{an}}$ and $\mathscr{M}_2^{\mathrm{an}} = \mathscr{M}^{\mathrm{an}}/\hbar\mathscr{M}^{\mathrm{an}}$. By induction hypothesis, there exists $\mathscr{M}_1 \in \mathrm{Mod}_{\mathrm{coh}}$ $(\mathscr{A}_X/\hbar^{n-1}\mathscr{A}_X)$ such that $f_{n-1}^*(\mathscr{M}_1) \simeq \mathscr{M}_1^{\mathrm{an}}$, on the other hand, by Theorem 1.1, there exists $\mathscr{M}_2 \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ such that $f_1^*(\mathscr{M}_2)$ $\simeq \mathscr{M}_2^{\mathrm{an}}$. Since $f^*|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is exact by Lemma 8.1 and the functor $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ induced by $f^*|_{\mathcal{A}}$ is fully faithful by the proof of Theorem 8.2, applying Lemma 8.5, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_n^*(\mathscr{M}_1) & \longrightarrow & f_n^*(\mathscr{M}) & \longrightarrow & f_n^*(\mathscr{M}_2) & \longrightarrow & 0 \\
& & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
0 & \longrightarrow & \mathscr{M}_1^{\mathrm{an}} & \longrightarrow & \mathscr{M}^{\mathrm{an}} & \longrightarrow & \mathscr{M}_2^{\mathrm{an}} & \longrightarrow & 0
\end{array}$$

for some $\mathscr{M} \in \mathcal{A}'$. Hence f_n^* is essentially surjective. □

From Corollary 8.4 and Theorem 8.6, we get

Theorem 8.7. *The functor* $f_n^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ *is an equivalence for each* $n > 0$.

Now we continue to prove that the functor $f^*\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is essentially surjective.

Denote by $\mathbb{N}$ the set of positive integers, viewed as a category defined by

$$\mathrm{Ob}(\mathbb{N}) = \mathbb{N}$$

$$\mathrm{Hom}_{\mathbb{N}}(i,j) = \begin{cases} \{\mathrm{pt}\} & \text{if } i \leq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Define prestacks $\mathfrak{S}, \mathfrak{S}_{\mathrm{an}}$ on $\mathbb{N}$ as follows:

- $\mathfrak{S}(n) := \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$ for $n \in \mathbb{N}$,
- $r_u : \mathfrak{S}(j) \to \mathfrak{S}(i)$ is the functor for $i \leq j$ and $u \in \mathrm{Hom}_{\mathbb{N}}(i,j)$.

Similarly,

- $\mathfrak{S}_{\mathrm{an}}(n) := \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ for $n \in \mathbb{N}$,
- $r_u : \mathfrak{S}_{\mathrm{an}}(j) \to \mathfrak{S}_{\mathrm{an}}(i)$ is the functor for $i \leq j$ and $u \in \mathrm{Hom}_{\mathbb{N}}(i,j)$.

We need the following theorem (cf §1.5).

Theorem 8.8. *We have the following equivalences:*

(1) $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n) \simeq \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$,

(2) $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}_{\mathrm{an}}(n) \simeq \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$.

Proof. We only need to prove (1) and similar to (2). Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then we obtain the family $\{(F_n, \varphi_u)\}$ where:

(i) $F_n := \mathscr{M}/\hbar^n\mathscr{M} \in \mathfrak{S}(n)$ for $n \in \mathbb{N}$,

(ii) $\varphi_u : r_u F_j \xrightarrow{\sim} F_i$ for $i \leq j$, $u \in \mathrm{Hom}_{\mathbb{N}}(i,j)$ and $r_u : \mathfrak{S}(j) \to \mathfrak{S}(i)$ is defined by sending $\mathscr{M}$ to $\mathscr{M}/\hbar^i\mathscr{M}$.

It is easy to check that $\{(F_n, \varphi_u)\}$ satisfies the cocycle condition (a) and hence $\{(F_n, \varphi_u)\} \in \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$.

For $\mathscr{M}, \mathscr{M}' \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then these define two objects $F = \{(F_n, \varphi_u)\}$ and $F' = \{(F'_n, \varphi_u)\}$ in $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$. Let $f : \mathscr{M} \to \mathscr{M}' \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then we have the set of families $\{f_n\}_{n\in\mathbb{N}}$ where $f_n : \mathscr{M}/\hbar^n\mathscr{M} \to \mathscr{M}'/\hbar^n\mathscr{M}' \in \mathrm{Hom}_{\mathfrak{S}(n)}(\mathscr{M}/\hbar^n\mathscr{M}, \mathscr{M}'/\hbar^n\mathscr{M}')$. It is easy to check that $\{f_n\}$ satisfies the commutative diagram for definition (b) and hence $\{f_n\}_{n\in\mathbb{N}} \in \mathrm{Hom}_{\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)}(F, F')$. Hence we can define a functor $\Phi : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$ by sending $\mathscr{M}$ to $\{(F_n, \varphi_u)\}$ and $f \in \mathrm{Hom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{M}')$ to $\{f_n\}_{n\in\mathbb{N}}$.

On the other hand, if $\{(F_n, \varphi_u)\} \in \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$, then by definition e.g (a), we have

(i) $F_n \in \mathfrak{S}(n)$ for $n \in \mathbb{N}$,

(ii) $\varphi_u : r_u F_j \xrightarrow{\sim} F_i$ for $i \leq j$, $u \in \mathrm{Hom}_{\mathbb{N}}(i,j)$ and $r_u : \mathfrak{S}(j) \to \mathfrak{S}(i)$.

Hence the system $\{F_n\}_{n\in\mathbb{N}}$ is a projective system and $\varprojlim_{n\in\mathbb{N}} F_n \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ by [13, Lemma 1.13].

For two objects $F = \{(F_n, \varphi_u)\}$ and $F' = \{(F'_n, \varphi'_u)\}$ in $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$. By definition e.g (b), $\mathrm{Hom}_{\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)}(F, F')$ is the set of families $f = \{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in \mathrm{Hom}_{\mathfrak{S}(n)}(F_n, F'_n)$ and the following diagram commutes for $u : i \to j$ and $i \le j$,

$$\begin{array}{ccc} r_u F_j & \xrightarrow{\varphi_u} & F_i \\ {\scriptstyle r_u(f_j)}\downarrow & & \downarrow{\scriptstyle f_i} \\ r_u F'_j & \xrightarrow{\varphi'_u} & F'_i. \end{array}$$

Hence the system $f = \{f_n\}_{n\in\mathbb{N}}$ is a projective system and we get that the morphism $\varprojlim_{n\in\mathbb{N}} f_n$ belongs to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$. Therefore, we can define a functor $\Psi : \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ by $\Psi(\{(F_n, \varphi_u)\}) = \varprojlim_{n\in\mathbb{N}} F_n$ and $\Psi(f = \{f_n\}_{n\in\mathbb{N}}) = \varprojlim_{n\in\mathbb{N}} f_n$. Now it is easy to check that $\Phi \circ \Psi \simeq \mathrm{id}_{\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)}$ and $\Psi \circ \Phi \simeq \mathrm{id}_{\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)}$. Therefore, the result follows. $\square$

Corollary 8.9. *The functor* $f^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is an equivalence.*

Proof. This follows from Theorem 8.7 and Theorem 8.8. $\square$

From Theorem 8.2, Corollary 8.9 and the proof of Lemma 1.2 for essentially surjectivity, we get what we want mentioned above.

Corollary 8.10. *The natural functor* f^*: $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is essentially surjective.*

Therefore, we get the second main theorem of this thesis.

Main Theorem 8.11.. *The natural functor* $f^*\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is an equivalence.*

Proof. This follows from Theorem 8.2 and Corollary 8.10. $\square$

References

[1] A. D'Agnolo and P. Polesello, *Deformation quantization of complex involutive submanifolds, in: Noncommutative geometry and physics*, World Sci. Publ., Hackensack, NJ p. 127-137 (2005).

[2] P. Berthelot and L. Illusie, Grothendieck, A. *Théorie des intersections et théorème de Riemann-Roch.* Lect. Notes Math **225**. Springer, Berlin Heidelberg New York (1971).

[3] R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in algebraic context*, Mosc. Math. J. **4** p. 559-592, (2004).

[4] L. Breen, *On the Classification of 2-gerbes and 2-stacks*, Astérisque-Soc. Math. France **225** (1994).

[5] A. Caldararu, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, Ph.D thesis, Cornell University (2000).

[6] Hou-Yi Chen, *GAGA for DQ-algebroids*, To appear in Rendiconti del Seminario Matematico dell'Universita' di Padova.

[7] J. Giraud, *Cohomologie non abélienne*, Grundlehren der Math. Wiss **179** Springer-Verlag (1971).

[8] A. Grothendieck, SGA 1 *Revêtements étales et Groupe Fondamental*, Lecture Notes in Math. **224**, Springer-Verlag, Heidelberg (1971).

[9] D. Huybrechts, *Fourier-Mukai transforms in algebraic variety, Oxford mathematical Monographs*, Oxford, (2006).

[10] R. Hotta, Kiyoshi. Takeuchi, Toshiyuki Tanisaki, *D-modules, Perverse sheaves, and Representation theory,*

Progress in Mathematics, Vol. **236**, Birkhauser Boston, Cambridge, MA, 2008.

[11] M. Kashiwara, *D-modules and Microlocal Calculus*, Iwanami series in Modern Mathematics, Iwanami Shorten, Tokyo, 2000 (in Japanese); Translations of Mathematical Monographs, Vol. **217**, American Mathematical Society, Providence, RI, 2003 (in English; translated by Mutsumi Saito.)

[12] ———, *Quantization of contact manifolds*, Publ. RIMS, Kyoto Univ. 32 p. 1-5 (1996).

[13] M. Kashiwara and P. Schapira, *Deformation quantization modules*, To appear.

[14] ———, *Categories and sheaves*, Grundlehren der Math. Wiss. **332**, Springer-Verlag (2006).

[15] M. Kontsevich, *Deformation quantization of algebraic varieties*, Euro-Conférence Moshé Flato, Part III (Dijon, 2000) Lett. Math. Phys. **56** (3) p. 271-294 (2001).

[16] G. Laumon, *Sur la catégorie dérivée des $\mathcal{D}$-modules filtrés*, Algebraic geometry (M. Raumaid amd T. Shioda eds) Lecture Notes in Math. Springer-Verlag **1016** pp. 151-237 (1983).

[17] P. Polesello and P. Schapira, *Stacks of quantization-deformation modules over complex symplectic manifolds*, Int. Math. Res. Notices **49** p. 2637-2664 (2004)

[18] P. Schapira, *From D-modules to deformation quantization modules*, course note given at Buenos Aires, July 2008. http://www.math.jussieu.fr/∼schapira/

[19] J-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1-42.

[20] A. Yekutieli, *Deformation quantization in algebraic geometry*, Adv. Math. **198** pp. 383-432 (2005).