國立臺灣大學理學院數學研究所

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master Thesis

一個描述獨異變換之下代數曲面上秩二的層的模空間動機變換公式

A Motivic Blowup Formula of Moduli Spaces of Rank 2 Sheaves on a Smooth Projective Algebraic Surface

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中華民國 112 年 6 月

June, 2023

doi:10.6342/NTU202301040

國立臺灣大學碩士學位論文
口試委員會審定書
MASTER'S THESIS ACCEPTANCE CERTIFICATE
NATIONAL TAIWAN UNIVERSITY

一個描述獨異變換之下代數曲面上秩二的層的模空間的動機變換公式

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本論文係 王 偉（R10221026）在國立臺灣大學數學系完成之碩士學位論文，於民國 112 年 6 月 9 日承下列考試委員審查通過及口試及格，特此證明。

The undersigned, appointed by the Department of mathematics on 9 June, 2023 have examined a Master's thesis entitled above presented by_Wei Wang (R10221026) candidate and hereby certify that it is worthy of acceptance.

口試委員 Oral examination committee:

（指導教授 Advisor）

系主任/所長 Director:

doi:10.6342/NTU202301040

Acknowledgements

感謝莊武諺老師的耐心指導。在過程中我學習到作數學的過程是如何，學到了許多非常有趣的數學，在人生及職涯方向上也得到許多寶貴的建議。

感謝齊震宇老師從我大四至進碩班前的指導，讓我明白學習做數學時應有的態度，也影響了我對於數學的視角與學習方式。

感謝口委員們對論文的建議，讓我能將論文的一些不足之處修正。

感謝家人近乎無條件的支持我，讓我能夠無後顧之憂的學習數學。

doi:10.6342/NTU202301040

doi:10.6342/NTU202301040

摘要

本篇論文主要宗旨係在推廣 [LQ99, LQ98] 中所做出來的一個從 Vafa 與 Witten 從 S-對偶猜想中預測出來的一個描述代數曲面上穩定秩二層的模空間的不變量在獨異變換之下的公式。

該兩篇論文所考慮的不變量為 virtual 霍奇多項式；我們想要將這些結果作到動機的版本。

在這篇論文中，我們驗證 [LQ99, LQ98] 當中的一些證明可以推廣到動機的設定之下，並在這樣的框架之下我們藉由 [Moz19] 中的一個 Quot 概型的動機生成函數公式回答了一個於 [LQ98] 中提出的帶有組合味道的猜想。我們也簡化了 [LQ98] 當中的一些計算。

關鍵字：代數曲面、獨異變換、穩定層的模空間、動機、S-對偶性猜想

doi:10.6342/NTU202301040

doi:10.6342/NTU202301040

Abstract

The purpose of this thesis is to generalize a collection of results in [LQ99, LQ98] concerning change of invariants of moduli space of rank-2 stable sheaves over an algebraic surface under blowup, which is a set of formulas predicted by Vafa and Witten in the context of the S-duality conjecture.

In these two papers, the invariants the authors considered are the virtual Hodge polynomials, and our goal is to refine these invariants to the settings of Grothendieck's motivic ring of varieties.

In this paper, we verified that some of the proofs given in [LQ99, LQ98] can be generalized to the motivic setting, and by working in the motivic ring of varieties, we are able to answer a conjectural combinatorial formula posed in [LQ98], by using a formulae concerning motivic generating series of Quot schemes given in [Moz19]. We also simplified some of the calculations in [LQ98].

doi:10.6342/NTU202301040

Keywords: Algebraic Surfaces, Monoidal Transformations, Moduli space of Stable Sheaves, Motives, S-Duality Conjecture

doi:10.6342/NTU202301040

Contents

Page

Verification Letter from the Oral Examination Committee **i**

Acknowledgements **iii**

摘要 **v**

Abstract **vii**

Contents **ix**

Chapter 1 Introduction **1**

1.1 Overview 1

1.2 Organization of Contents 2

1.3 Frameworks and Notations 3

1.4 Summary of Main Results 6

1.5 Outline of Proof 11

Chapter 2 The Motivic Ring of Varieties **13**

2.1 Definition and Generalities 13

2.2 Power Structures and Motivic Exponentials 15

Chapter 3 Moduli Spaces of Sheaves over Algebraic Surfaces **19**

3.1 Stability and Polarization 19

3.2 The Moduli Functor and the Moduli Space 21

doi:10.6342/NTU202301040

3.3 Change of Polarization - Chambers and Walls 25

3.4 Blowing-up and Stability 29

3.5 Compactification and Comparison 32

Chapter 4 Existence and Computation of Universal Functions 39

4.1 Existence of Universal Functions 39

4.2 Computation of Universal Functions (I) by Specialization 47

4.3 Computation of Universal Functions (II) by Recursion 54

References 63

Chapter 5 Appendix. A Survey on Related Results 67

doi:10.6342/NTU202301040

Chapter 1 Introduction

1.1 Overview

The purpose of this thesis is to generalize a collection of results in [LQ99, LQ98] concerning change of invariants of moduli space of rank-2 stable sheaves over an algebraic surface under blowup, which is a set of formulas predicted by Vafa and Witten in the context of the S-duality conjecture.

In these two papers, the invariants the authors considered are the virtual Hodge polynomials, and our goal is to refine these invariants to the settings of Grothendieck's motivic ring of varieties.

In this paper, we verified that some of the proofs given in [LQ99, LQ98] can be generalized to the motivic setting, and by working in the motivic ring of varieties, we are able to answer a conjectural combinatorial formula posed in [LQ98], by using a formulae concerning motivic generating series of Quot schemes given in [Moz19]. We also simplified some of the calculations in [LQ98].

doi:10.6342/NTU202301040

1.2 Organization of Contents

As mentioned in the previous section, our main goal is to prove things in the motivic context. This will be done in Chapter 4. Chapter 2 and 3 are essential background materials on these results.

- For Chapter 2, we give a brief overview on the construction of Grothendieck's motivic ring of varieties. We will review the concept of power structures, and mention a few results concerning generating series of Quot schemes given in [Moz19].
- For Chapter 3, we give a summary on some essential results and notions on the theory of moduli spaces of stable rank 2 sheaves over an algebraic surface.
  - In 3.1., we revisit the definition of stability, and recall some results concerning the existence of moduli spaces in 3.2..
  - In 3.3., we concern the change of moduli space under change of polarizations. Firstly, we revisit Zhen-Bo Qin's definition of chamber and wall structures, and mention Göttsche's result on a motivic decomposition of moduli spaces utilizing this chamber and wall structure in the case of ruled surfaces.
  - In 3.4., we look at preliminary results concerning how moduli spaces changes under blowing up, which justifies the technical definition of the space $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2)$.
  - In 3.5., we discuss how the Gieseker moduli spaces, Uhlenbeck compactifications, and Mumford-Takemoto spaces are related to each other under mild conditions.
- For Chapter 4, we generalize the main results given in [LQ99, LQ98] in the motivic

doi:10.6342/NTU202301040

setting. We show that a motivic universal function exists in 4.1.. In 4.2., 4.3., we compute explicit motivic formulas corresponding to the ones given in [LQ98].

- In the appendix, we will give a review on the relevant results given in some of the related works concerning change of moduli space under blowup.

1.3 Frameworks and Notations

For the rest of this paper, we will mainly work within the category **Sch**$/\mathbb{C}$ of $\mathbb{C}$-schemes of finite type. Product of schemes in this category will also be interpreted as product over Spec $\mathbb{C}$. The only exception is 3.18.

For the rest of this chapter, we take an algebraic surface X over $\mathbb{C}$ and an ample divisor H on it. We also take some $c_i \in H^{2i}(X, \mathbb{C})$ for $i = 1, 2$.

Notation 1.1. *We will use the following short hand notation for what properties X, c_1, H might posses:*

(A1) We say that X ***has property (A1)*** *if X is smooth, projective, and simply-connected.*

(A2) We say that X, c_1, H ***has property (A2)*** *if the intersection product $H.c_1$ is odd.*

(A3) We say that X ***has property (A3)*** *if the anticanonical divisor $-K_X$ is effective.*

For what follows, we will often always assume condition (A1). We list a few of the direct consequences of this assumption:

Remark. *When we assume (A1) on X (complex smooth projective simply-connected surface), we have the following simplications of some of the basic invariants of X:*

doi:10.6342/NTU202301040

- ***Betti Numbers and Euler Characteristic****. Since*

$$H^0(X,\mathbb{Z}) = H^4(X,\mathbb{Z}) \simeq \mathbb{Z},\ H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$$

and that $H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$ is free abelian, we have by Universal coefficient theorem that

$$H^2(X,\mathbb{Z}) \simeq \mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1(H_1(X,\mathbb{Z}),\mathbb{Z}) = \text{ torsion-free part of } H_2(X,\mathbb{Z})$$

In terms of Betti numbers $b^i := \dim H^i(X,\mathbb{C})$, we have $b^0 = b^4 = 1, b^1 = b^3 = 0$. In this case, the Euler characteristic is $e = 2 + b^2$.

- ***Hodge numbers****. For $h^{i,j} = \dim H^j(X,\Omega^i)$ with $i,j \in \{0,1,2\}$, as $b^k = \sum_{i+j=k} h^{i,j}$ and $h^{i,j} = h^{2-i,2-j}$ (Serre duality), the numbers are specified by*

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ p_g & & b^2 - 2p_g & & p_g \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

with p_g being the geometric genus.

- ***The geometric genus, arithmetic genus, irregularity, holomorphic Euler characteristic*** *p_g, p_a, q, χ. They can all be described by $h^{0,2}$:*

$$p_g = h^{0,2},\ q = h^{0,1} = 0,\ p_a = p_g - q = h^{0,2},\ \chi = p_g - q + 1 = h^{0,2} + 1$$

- ***The Picard group, the Neron-Severi group and the group of numerical equiva-***

doi:10.6342/NTU202301040

lence classes

$\text{Pic}(X), \text{NS}(X), \text{Num}(X)$. *We have an exact sequence (arising from the exponential sequence):*

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \text{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

As $H^i(X, \mathcal{O}_X)$ is the complex conjugate to $H^0(X, \Omega^i_X)$, $\text{Pic}(X)$ *can be identified with its image* $\text{NS}(X)$ *in $H^2(X, \mathbb{Z})$, and as $H^2(X, \mathbb{Z})$ is torsion-free,* $\text{Num}(X)$ *- being* $\text{NS}(X)$ *modulo torsion - is isomorphic to* $\text{NS}(X)$*. To summarize, we may think of this as an identification*

$$\text{Pic}(X) \subseteq \text{NS}(X) \simeq \text{Num}(X)$$

In this sense, when we talk about moduli spaces of sheaves parametrized by chern classes c_1, c_2 later, there is not ambiguity in specifying whether
$c_1 \in \text{Pic}(X), H^2(X, \mathbb{Z}), \text{NS}(X), \text{Num}(X)$ *(as some authors will use* $\text{Num}(X)$ *as the space of stability conditions when dealing with chamber structures, while some will work with* $\text{Pic}(X), H^2(X, \mathbb{Z})$ *when discussing general stability conditions).*

Notation 1.2. *Throughout this paper, by* ***blowing up****, we will only be considering blowing up along a single point, and by a* ***sheaf****, we will only be considering a coherent sheaf.*

Notation 1.3. *Suppose X satisfies (A1).*

- *We denote the symmetric products X^n/S_n of X as $X^{(n)}$.*
- *We denote the Hilbert schemes of zero-dimensional subschemes of length n as $X^{[n]}$.*
- *We denote* $\text{Quot}^n_{\mathcal{O}_X^{\oplus 2}/X}$ *- the Quot scheme of quotients of $\mathcal{O}_X^{\oplus 2}$ parametrized by the*

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constant polynomial n *(over the base scheme* Spec $\mathbb{C}$*) as* $X^{\{n\}}$.

- *We define the following generating series:*

$$\mathbf{S}(X;q) = \sum_n [X^{(n)}]q^n, \ \mathbf{H}(X;q) = \sum_n [X^{[n]}]q^n, \ \mathbf{Q}(X;q) = \sum_n [X^{\{n\}}]q^n$$

1.4 Summary of Main Results

Settings 1.4. *Consider an algebraic surface* X *satisfying (A1). Take a point* $* \in X$*, consider its blowup as a pullback square:*

$$\begin{array}{ccc} E & \xrightarrow{i} & \tilde{X} \\ \downarrow & & \downarrow \phi \\ * & \longrightarrow & X \end{array}$$

In this diagram, E *is the exceptional divisor,* ϕ *is the blowup map, and* $\tilde{X}$ *is the blown-up surface. If* X *has property (A1),* $\tilde{X}$ *will also have property (A1).*

Fix $c_1 \in H^2(X;\mathbb{Z})$ *and* $c_2 \in H^4(X;\mathbb{Z}) \simeq \mathbb{Z}$*. Take an ample divisor* H *on* X*. We may consider the following moduli spaces:*

- ***The moduli space of Gieseker-semistable rank*** 2 ***torsion sheaves*** *(or the* ***Gieseker space****) on* X *with chern classes* c_1, c_2*, denoted as* $\mathfrak{M}_H^G(c_1, c_2)$.
- ***The moduli space of Mumford-Takemoto rank*** 2 ***bundles*** *(or the* ***Mumford-Takemoto space****) on* X *with chern classes* c_1, c_2*, denoted as* $\mathfrak{M}_H^\mu(c_1, c_2)$.
- ***The Uhlenbeck compactification*** *(or the* ***Uhlenbeck space****) of* $\mathfrak{M}_H^\mu(c_1, c_2)$ *with chern classes* c_1, c_2*, denoted as* $\mathfrak{M}_H^U(c_1, c_2)$.

Corresponding to these spaces, we can define generating series:

$$\mathbf{M}^{\mu}(X,H,c_1;q) := \sum_{c_2}[\mathfrak{M}^{\mu}_{H}(c_1,c_2)]q^{\Delta(c_1,c_2)/4}$$

$$\mathbf{M}^{G}(X,H,c_1;q) := \sum_{c_2}[\mathfrak{M}^{G}_{H}(c_1,c_2)]q^{\Delta(c_1,c_2)/4}$$

$$\mathbf{M}^{U}(X,H,c_1;q) := \sum_{c_2}[\mathfrak{M}^{U}_{H}(c_1,c_2)]q^{\Delta(c_1,c_2)/4}$$

where we define $\Delta(c_1,c_2) = 4c_2 - c_1^2$*, and the square bracket indicates taking the motive of that space. The authors in [LQ99, LQ98] referred to* $\Delta(c_1,c_2)/4$ *as the* ***instanton numbers****.*

Now suppose X, H, c_1 *satisfies (A2). On the surface* $\tilde{X}$*, corresponding to* $\varepsilon \in \{0,1\}$*, we define:*

$$\tilde{c}_{1,\varepsilon} = \phi^* c_1 - \varepsilon E$$

then there exists $r_0 \in \mathbb{N}$ *such that for* $r \geq r_0$*, the moduli spaces* $\mathfrak{M}^{\mu}_{H_r}(\tilde{c}_{1,\varepsilon}, c_2)$ *(resp.* $\mathfrak{M}^{G}_{H_r}(\tilde{c}_{1,\varepsilon}, c_2), \mathfrak{M}^{U}_{H_r}(\tilde{c}_{1,\varepsilon}, c_2)$*) can be identified with each other, and that all* $H_r := r\phi^*H - E$ *are ample divisors (these facts will be visited in section 3.4). By choosing* r *with the correct parity, we may even find* $r \geq r_0$ *with* $H_r.\tilde{c}_{1,\varepsilon}$ *odd (hence* $\tilde{X}, H_r, \tilde{c}_{1,\varepsilon}$ *satisfies condition (A2)). For such* r*, we simply define:*

$$\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_1,c_2) = \mathfrak{M}^{\mu}_{H_r}(\tilde{c}_1,c_2),\ \mathfrak{M}^{G}_{H_\infty}(\tilde{c}_1,c_2) = \mathfrak{M}^{G}_{H_r}(\tilde{c}_1,c_2),\ \mathfrak{M}^{U}_{H_\infty}(\tilde{c}_1,c_2) = \mathfrak{M}^{U}_{H_r}(\tilde{c}_1,c_2)$$

and similarly, we may define generating series **M** *for these spaces.*

The main theorems of this paper are as follows:

Theorem 1.5 (Main Theorem)**.** *(Every power series considered below are to be regarded*

doi:10.6342/NTU202301040

as elements living in the ring $K(\mathbf{Var}/\mathbb{C})[\![q]\!]$.)

- *(Existence of Universal Functions) There are universal functions:*

$$\mathbf{Z}_{\varepsilon}^{\mu}(q),\ \mathbf{Z}_{\varepsilon}^{G}(q),\ \mathbf{Z}_{\varepsilon}^{U}(q)$$

  in the variables q, ε such that:

  – *For any X satisfying (A1), (A2) and any fixed c_1, ε, one has:*

$$\mathbf{M}^{\mu}(\tilde{X}, H_{\infty}, \tilde{c}_{1,\varepsilon}; q) = q^{1/12} \cdot \mathbf{Z}_{\varepsilon}^{\mu}(q) \cdot \mathbf{M}^{\mu}(X, H, c_1; q)$$

$$\mathbf{M}^{G}(\tilde{X}, H_{\infty}, \tilde{c}_{1,\varepsilon}; q) = q^{1/12} \cdot \mathbf{Z}_{\varepsilon}^{G}(q) \cdot \mathbf{M}^{G}(X, H, c_1; q)$$

  – *For any X satisfying (A1), (A2), (A3) with $\tilde{X}$ also satisfying (A3) and any fixed c_1, ε, one has:*

$$\mathbf{M}^{U}(\tilde{X}, H_{\infty}, \tilde{c}_{1,\varepsilon}; q) = q^{1/12} \cdot \mathbf{Z}_{\varepsilon}^{U}(q) \cdot \mathbf{M}^{U}(X, H, c_1; q)$$

- *(Relations between Universal Functions) These universal functions are related to each other by:*

$$\mathbf{M}_{\varepsilon}^{\mu}(X, H, c_1; q)\mathbf{S}(X; q) = \mathbf{M}_{\varepsilon}^{U}(X, H, c_1; q)$$

$$\mathbf{M}_{\varepsilon}^{\mu}(X, H, c_1; q)\mathbf{Q}(X; q) = \mathbf{M}_{\varepsilon}^{G}(X, H, c_1; q)$$

  Paired with the the formulas in chapter 2, we have:

$$\mathbf{Z}_{\varepsilon}^{\mu}(q)\,\mathrm{Exp}\,(\mathbb{L}q) = \mathbf{Z}_{\varepsilon}^{U}(q)$$

$$\mathbf{Z}_{\varepsilon}^{\mu}(q)\,\mathrm{Exp}\left(\frac{\mathbb{L}(1+\mathbb{L})q}{1-\mathbb{L}^2 q}\right) = \mathbf{Z}_{\varepsilon}^{G}(q)$$

doi:10.6342/NTU202301040

- *(Explicit Forms of Universal Functions) The universal function* $\mathbf{Z}^\mu_\varepsilon(q)$ *takes two equivalent forms:*

$$\mathbf{Z}^\mu_\varepsilon(q) = q^{-1/12}\Bigg(\left(\sum_{s\geq 0}\mathbb{L}^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)}{2}}q^{\frac{(2s+\varepsilon)^2}{4}}\prod_{j=1}^{2s+\varepsilon}\frac{1-\mathbb{L}^{2j-2}q^j}{1-\mathbb{L}^{2j}q^j}\right) + \left(\sum_{s\geq 1-\varepsilon}\mathbb{L}^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)-2}{2}}q^{\frac{(2s+\varepsilon)^2}{4}}\prod_{j=1}^{2s+\varepsilon-1}\frac{1-\mathbb{L}^{2j-2}q^j}{1-\mathbb{L}^{2j}q^j}\right)\Bigg)$$

The other given by:

$$\begin{aligned}\mathbf{Z}^\mu_\varepsilon(q) &= \mathrm{Exp}\left(\frac{2\mathbb{L}^2q}{1-\mathbb{L}^2q}-\frac{\mathbb{L}(1+\mathbb{L})q}{1-\mathbb{L}^2q}\right)q^{-1/12}\left(\sum_{n\in\mathbb{Z}}\mathbb{L}^{\frac{(2n+\varepsilon)^2-(2n+\varepsilon)}{2}}q^{\frac{(2n+\varepsilon)^2}{4}}\right)\\ &= \mathrm{Exp}\left(\frac{\mathbb{L}^2q-\mathbb{L}q}{1-\mathbb{L}^2q}\right)q^{-1/12}\left(\sum_{n\in\mathbb{Z}}\mathbb{L}^{\frac{(2n+\varepsilon)^2-(2n+\varepsilon)}{2}}q^{\frac{(2n+\varepsilon)^2}{4}}\right)\end{aligned}$$

originating form the formula (along with 2.4):

$$\mathbf{Z}^G_\varepsilon(q) = \mathrm{Exp}\left(\frac{2\mathbb{L}^2q}{1-\mathbb{L}^2q}\right)q^{-1/12}\left(\sum_{n\in\mathbb{Z}}\mathbb{L}^{\frac{(2n+\varepsilon)^2-(2n+\varepsilon)}{2}}q^{\frac{(2n+\varepsilon)^2}{4}}\right)$$

Remark. *It was conjectured in Remark 3.15 of [LQ98] that the following holds:*

$$\frac{\sum_{n\in\mathbb{Z}}(xy)^{\frac{(2n+\varepsilon)^2-(2n+\varepsilon)}{2}}q^{\frac{(2n+\varepsilon)^2}{4}}}{q^{1/12}(1-xyq)}\cdot\prod_{d\geq 1}\frac{1-(xy)^{2d-1}q^d}{1-(xy)^{2d}q^d}$$

$$= \frac{1}{q^{1/12}(1-xyq)}\left[\sum_{s\geq 0}(xy)^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)}{2}}q^{\frac{(2s+\varepsilon)^2}{4}}\prod_{i=1}^{2s+\varepsilon}\frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j} + \sum_{s\geq 1-\varepsilon}(xy)^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)-2}{2}}q^{\frac{(2s+\varepsilon)^2}{4}}\prod_{i=1}^{2s+\varepsilon-1}\frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}\right]$$

It turns out that this follows from the explicit descriptions of $\mathbf{Z}^\mu_\varepsilon(q)$ *theorem by taking the virtual Hodge polynomials of motives.*

Remark. *Under our present notations, the following are proved in [LQ99, LQ98] (see*

doi:10.6342/NTU202301040

Theorem A,B of [LQ99] and Theorem 1.2, 1.3 of [LQ98]):

- *Under assumption (A1), (A2), we have:*

$$\sum_n e(\mathfrak{M}^G_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2); x, y) q^{\Delta(\tilde{c}_{1,\varepsilon},c_2)/4} = q^{1/12} Z^G_\varepsilon(q; x, y) \cdot \sum_n e(\mathfrak{M}^G_H(c_1, c_2); x, y) q^{\Delta(c_1,c_2)/4}$$

for some universal function $Z^G_\varepsilon(q; x, y)$.

- *Under assumption (A1), (A2), (A3), we have:*

$$\sum_n e(\mathfrak{M}^U_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2); x, y) q^{\Delta(\tilde{c}_{1,\varepsilon},c_2)/4} = q^{1/12} Z^U_\varepsilon(q; x, y) \cdot \sum_n e(\mathfrak{M}^U_H(c_1, c_2); x, y) q^{\Delta(c_1,c_2)/4}$$

for some universal function $Z^U_\varepsilon(q; x, y)$.

- *The universal functions* $Z^G_\varepsilon(q; x, y), Z^U_\varepsilon(q; x, y)$ *are of the form:*

$$q^{1/12} Z^G_\varepsilon(q; x, y) = \frac{\sum_{n\in\mathbb{Z}} (xy)^{\frac{(2n+\varepsilon)^2-(2n+\varepsilon)}{2}} q^{(n+\varepsilon/2)^2}}{\left[\prod_{n\geq 1}(1-(xy)^{2n}q^n)\right]^2}$$

$$\begin{aligned}(1-xyq)q^{1/12} Z^U_\varepsilon(q; x, y) = &\sum_{s\geq 0} (xy)^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)}{2}} q^{\frac{(2s+\varepsilon)^2}{4}} \prod_{j=1}^{2s+\varepsilon} \frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j} \\ &+ \sum_{s\geq 1-\varepsilon} (xy)^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)-2}{2}} q^{\frac{(2s+\varepsilon)^2}{4}} \prod_{j=1}^{2s+\varepsilon-1} \frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}\end{aligned}$$

where $e(X; x, y)$ *is the virtual Hodge polynomial of* X.

Remark. *It is also remarked in Remark 3.22. of [NY03] that the above formula (which is a special case of Theorem 3.21. in that paper) holds true with* xy *replaced by* $\mathbb{L}$ *in the Grothendieck group of varieties using their method of proof. The present paper verifies that [LQ99, LQ98] generalizes to the motivic setting, and the universal functions can be rewritten as simple motivic exponential functions.*

doi:10.6342/NTU202301040

1.5 Outline of Proof

The outline of the overall proof of 1.5 can be schematically summarized as the following diagram:

$$\begin{array}{ccc}
\text{Uhlenbeck Spaces on } \tilde{X} & & \text{Uhlenbeck Spaces on } X \\
\updownarrow (2) & & (2) \updownarrow \\
\text{(4)Mumford-Takemoto Spaces on } \tilde{X} & \overset{(1)}{\longleftrightarrow} & \text{(4)Mumford-Takemoto Spaces on } X \\
\updownarrow (2) & & (2) \updownarrow \\
\text{(3)Gieseker Spaces on } \tilde{X} & & \text{(3)Gieseker Spaces on } X
\end{array}$$

Where respectively in (1)-(3) we verify the following:

(1) We show that the moduli space of a suitable surface and the moduli space of its blowup can be related by a universal function; this is done in section 4.1. of this paper.

(2) We show that the relation between Gieseker, Uhlenbeck, Mumford-Takemoto (with same base space) can be described by the motivic Macdonald's formula and Mozgovoy's formula; this is done in 3.17 and 3.16; see 4.5.

(3) We calculate the universal function for Gieseker spaces explicitly in the case where $X = \mathbb{F}_1$ using Göttsche's result, this would then give explicit formulas of all the universal functions; this is done in 4.10.

(4) Finally, we calculate in a recursive way the explicit description of a universal function on Mumford-Takemoto spaces; this is done in 4.13.

doi:10.6342/NTU202301040

doi:10.6342/NTU202301040

Chapter 2 The Motivic Ring of Varieties

In this chapter, we review the definition of Grothendieck's ring of varieties, the power structure, the motivic exponential function, along with some explicit power series that we shall use. We will only work with base field $\mathbb{C}$, although some of the definitions holds in more general contexts.

2.1 Definition and Generalities

Let $\mathrm{Sch}/\mathbb{C}$ be the category of schemes of finite type over $\mathbb{C}$.

Definition 2.1 (K-group of varieties)**.** *Let $K(\mathbf{Sch}/\mathbb{C})$ be the free abelian group generated by isomorphism classes of objects in $\mathbf{Sch}/\mathbb{C}$) modulo the **scissor relation**:*

Given $Z \subseteq X$ an inclusion of closed subscheme, we require $[Z] + [X - Z] = [X]$.

*Furthermore, we can promote $K(\mathbf{Sch}/\mathbb{C})$ to a commutative ring by letting $[X][Y] = [X \times Y]$; in this case, $[\mathrm{Spec}(\mathbb{C})]$ is the multiplicative unit. Given $X \in \mathbf{Sch}/\mathbb{C}$, we call $[X]$ the **motive** of X.*

Remark. *It is shown in [Bri12] that as a consequence of the scissor relations, the follow-*

doi:10.6342/NTU202301040

ing holds:

- *Given a bijective morphism $X \to Y$, then $[X] = [Y]$.*

- *Given X_1, X_2, we have $[X_1 \coprod X_2] = [X_1] + [X_2]$.*

*Also, one can define $K(\mathbf{Var}/\mathbb{C})$ starting from varieties; the resulting $K(\mathbf{Var}/\mathbb{C}) \to K(\mathbf{Sch}/\mathbb{C})$ arising from inclusion of categories is a ring isomorphism; see also [Bri12]. We also define the **Lefschetz motive**, denoted as $\mathbb{L}$, by $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$.*

Remark. *The product relation $[X \times Y] = [X][Y]$ can be generalized: given a Zariski fibration $Z \to X$ with general fiber Y, we have $[Z] = [X][Y]$; the proof is given by first stratify the base space so that Z is trivial over each stratum, then paste everything back together again.*

Many common spaces admits descriptions as polynomials in $\mathbb{L}$. For this paper, we will only need the following:

$$[\mathbb{P}^n] = 1 + \ldots + \mathbb{L}^n = (\mathbb{L}^{n+1} - 1)(\mathbb{L} - 1)^{-1}$$

Motives may specialize to other invariants. For example, there is an E-polynomial homormophism:

$$E : K(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}[x, y]$$
$$[X] \mapsto e(X; x, y) := \sum_{i,j} x^i y^j \sum_k (-1)^k \dim H^{i,j}(H^k_c(X))$$

that specifies a variety to its virtual Hodge; they are given by $\mathbb{L} \mapsto xy$. One can also specify further to the Poincáre polynomial of compactly supported Euler-characteristics

doi:10.6342/NTU202301040

by $\mathbb{L} \mapsto 1$.

2.2 Power Structures and Motivic Exponentials

In this section, we will review the definition of a power structure on $K(\mathbf{Var}/\mathbb{C})$, and the definition of the motivic exponential function, following [GZLMH04]; we regard these functions as maps on the ring of power series with coefficients in $K(\mathbf{Var}/\mathbb{C})$. Power structures in fact also exists for polynomial rings over $\mathbb{Z}$, and the E-polynomial homomorphism defined in the previous section is compatible with it; one may see the details in [GZLMH04].

Definition 2.2 (A Power Structure on $K(\mathbf{Var}/\mathbb{C})$)**.** *Given $X \in \mathbf{Var}/\mathbb{C}$ and*

$$A(t) = 1 + \sum_{n \geq 1} [A_i] t^n \in 1 + tK(\mathbf{Var}/\mathbb{C})[\![t]\!]$$

define:

$$A(t)^{[X]} = 1 + \sum_{n \geq 1} \left(\sum_{(n_i): \sum_i i n_i = n} \left[\left((\prod_i X^{k_i}) - \Delta \right) \times \prod_i A_i^{k_i} / \prod_i S_{k_i} \right] \right) t^n$$

where Δ is the diagonal consisting of $\sum_i k_i$-tuple of points with at least two coordinate being the same, and with S_{k_i} being the symmetric group acting on $X^{k_1} \times A_i^{k_i}$ by simultaneously permuting factors.

From this, we have:

$$(1-t)^{-[X]} = \left(\sum_{n \geq 0} t^n \right)^{[X]} = \sum_{n \geq 0} [X^{(n)}] t^n = \mathbf{S}(X; t)$$

doi:10.6342/NTU202301040

This formula is sometimes referred to as the **motivic Macdonald's formula**, which is the motivic version of the Macdonald's formula concerning generating series of betti numbers $X^{(n)}$ [Mac62].

The power structure satisfies the usual rules that one might expect, such as $A(t)^{xy} = (A(t)^x)^y$.

The motivic exponential is defined as follows.

Definition 2.3 (Motivic Exp for $K(\mathbf{Var}/\mathbb{C})$)**.** *Given*

$$A(t) = \sum_{n\geq 1}[A_i]t^n \in tK(\mathbf{Var}/\mathbb{C})[\![t]\!]$$

define:

$$\text{Exp}(A(t)) = \prod_{k\geq 1}(1-t^k)^{-[A_k]}$$

In particular, we have motivic Macdonald's formula:

$$\mathbf{S}(X;q) = \text{Exp}([X]q)$$

Remark. *Under the E polynomial homomorphism, one has* $E(\text{Exp}(\mathbb{L}^i q^j)) = (1-(xy)^i q^j)^{-1}$.

The generating series of Quot schemes of locally free sheaves with finite quotients are useful:

Theorem 2.4 (Mozgovoy, Ricolfi)**.** *Given a smooth complex projective surface X and a*

doi:10.6342/NTU202301040

rank r-bundle E on X, we have the following formula:

$$\sum_{n\geq 0}[\mathrm{Quot}^n_{E/X}]q^n = \mathrm{Exp}\left(\frac{[X][\mathbb{P}^{r-1}]q}{1-\mathbb{L}^r q}\right)$$

where Exp *is defined via the power structure given above.*

Remark (On the independence of the choice of bundle)**.** *Note that this proposition implies that only the rank of E is relevant when considering generating series of motives of Quot schemes.*

- *In [Ric20], it was shown that this phenomenon holds for higher dimensional X.*
- *We will see another result (Lemma 5.2. of [LQ99]) in the relative setting showing that this is in fact even true when we replace $\mathbb{C}$ by a general Noetherian scheme.*

Also, it is in [Moz19] that by quiver techniques, an explicit description of this generating series is shown.

Remark. *The explicit forms of 2.4 that we will use are the following two:*

$$\mathbf{H}(X;q) = \mathrm{Exp}\left(\frac{[X]q}{1-\mathbb{L}q}\right)$$
$$\mathbf{Q}(X;q) = \mathrm{Exp}\left(\frac{[X](1+\mathbb{L})q}{1-\mathbb{L}^2 q}\right) = \mathrm{Exp}\left([X]\left(\sum_{n\geq 1}\mathbb{L}^{2n-2}(1+\mathbb{L})q^n\right)\right)$$

where the case $E=\mathcal{O}_X$ is Göttsche's formula for Hilbert schemes (in the motivic setting given at [Gö00]); compare 1.5's statment on the relations between universal functions, and 3.16, 3.17.

doi:10.6342/NTU202301040

doi:10.6342/NTU202301040

Chapter 3 Moduli Spaces of Sheaves over Algebraic Surfaces

Throughout this chapter, X is a complex projective algebraic surface.

For a rank-2 sheaf F over X, we denote its Chern classes as $c_1(F), c_2(F)$; when the context is clear, we would simply denote it by c_1, c_2. Following [HL10], we denote its discriminant by

$$\Delta(F) = \Delta(c_1, c_2) = 4c_2 - c_1^2 \in H^4(X; \mathbb{Z}) \simeq \mathbb{Z}$$

3.1 Stability and Polarization

Stability is a crucial concept in the theory of moduli space of sheaves, and a key player in the theory of geometric invariant theory. The notion of stability also depends on the type of stability and the polarization we are concerned with:

- For the choice of a polarization, we fix an ample line bundle H on X for now.
- For the choice of definition of stability conditions, we will review both Gieseker stability (also known as *Marumuya stability* or just *stability*) and slope stability

doi:10.6342/NTU202301040

(also known as μ-*stability* or *Mumford-Takemoto stability*) in this section.

Definition 3.1 (Degree of a Coherent Sheaf)**.** *The **degree** of a coherent sheaf F over X is given by*

$$\deg(F) := c_1(F).H$$

via intersection theory on smooth projective algebraic surfaces.

Definition 3.2 (Slope of a sheaf)**.** *The **slope** of a sheaf F over X is given by*

$$\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)}$$

In this sense, the function μ is then used to define the appropriate notion of stability as follows:

Definition 3.3 (Slope (semi)stability)**.** *A vector bundle V is said to be **slope stable** (resp. **semistable**) if for any subsheaf $F \subset V$ with $\mathrm{rank}(F) < \mathrm{rank}(V)$, we have $\mu(F) < \mu(V)$ (resp. $\mu(F) \leq \mu(V)$).*

Definition 3.4 (Gieseker (semi)stability)**.** *A torsion-free sheaf G is said to be **Gieseker stable** (resp. **semistable**) if for any proper subsheaf F, we have*

$$\frac{P(F,n)}{\mathrm{rank}(F)} < \frac{P(G,n)}{\mathrm{rank}(G)} \left(\text{resp. } \frac{P(F,n)}{\mathrm{rank}(F)} \leq \frac{P(G,n)}{\mathrm{rank}(G)} \right) \text{ for } n \gg 0$$

Here, $P(F,n) = \chi(F(nH))$ is the Hilbert polynomial of F (with variable n) defined with respect to H.

Remark. *A few remarks on the definitions:*

- *The notion of degree, slope, stability are all dependent on the choice of the line bundle H.*

doi:10.6342/NTU202301040

- *The notion of degree and slope-stability can be defined also solely in terms of Hilbert polynomials as in 1.2.2. of [HL10]. However the intersection-theoretic definition is sufficient for our purposes.*

Proposition 3.5 (Comparing Stability)**.** *We have the implications:*

slope stable ⇒ Gieseker stable ⇒ Gieseker semistable ⇒ slope semistable

Nevertheless, the following will be useful for what follows:

Proposition 3.6. *Suppose X, H, c_1 satisfies $(A1), (A2)$, then slope semistability implies slope stability.*

For a short proof, see Lemma 1.2.14 of [HL10].

3.2 The Moduli Functor and the Moduli Space

The moduli spaces we will mainly be considering are the following three:

- The **moduli space of Mumford-Takemoto-stable rank-2 bundles**, written as $\mathfrak{M}^{\mu}_{H}(c_1, c_2)$.
- The **moduli space of Gieseker-semistable rank-2 torsion-free sheaves**, written as $\mathfrak{M}^{G}_{H}(c_1, c_2)$.
- The **Uhlenbeck compactification of** $\mathfrak{M}^{\mu}_{H}(c_1, c_2)$, written as $\mathfrak{M}^{U}_{H}(c_1, c_2)$.

Remark. *Remarks on terminologies and notations:*

- *We will sometimes also refer to $\mathfrak{M}^{\mu}_{H}(c_1, c_2), \mathfrak{M}^{G}_{H}(c_1, c_2), \mathfrak{M}^{U}_{H}(c_1, c_2)$ simply as the* ***Mumford-Takemoto, Gieseker, Uhlenbeck moduli spaces****.*

doi:10.6342/NTU202301040

- *We may assemble the parameters c_1, c_2 as well as the rank (which we fix to be 2) as a 3-tuple $(2, c_1, c_2)$; in the literature, this is often referred to as a Mukai vector. The space $\mathfrak{M}_H^G(c_1, c_2)$ is then in fact the special case of a moduli space with Mukai vector $(2, c_1, c_2)$.*

We will first define the spaces $\mathfrak{M}_H^\mu(c_1, c_2)$, $\mathfrak{M}_H^G(c_1, c_2)$ in terms of moduli functors in this section, and later look at $\mathfrak{M}_H^U(c_1, c_2)$ in section 2.4..

Definition 3.7 (The Moduli Functors $\mathcal{M}_H^\mu(c_1, c_2)$, $\mathcal{M}_H^G(c_1, c_2)$)**.** *We define a functor*

$$\mathcal{M}_H^\mu(c_1, c_2) : \ (\mathbf{Sch}/\mathbb{C})^{\mathrm{op}} \to \mathbf{Set}$$

as follows:

- *At the level of objects, given $Y \in \mathrm{Obj}(\mathbf{Sch}/\mathbb{C})$, we define $\mathcal{M}_H^\mu(c_1, c_2)(S)$ to be the set of equivalence classes of μ-stable family of rank 2 bundles with Chern classes c_1, c_2 on X indexed by S.*
  - *More explicitly, an S-**family of μ-stable rank 2 bundles with Chern classes** c_1, c_2 is a sheaf $\mathcal{V}$ on $S \times X$ such that the restriction to each $X_s := \{s\} \times X \simeq X$ is a 2-bundle.*
  - *Two S-families $\mathcal{V}, \mathcal{V}'$ are regarded to be in the same equivalence class if $\mathcal{V} \simeq \mathcal{V}' \otimes \left(\pi_X^{X,S}\right)^* L$ for some line bundle L on X; here $\pi_X^{X,S}$ is one of the standard coordinate projection map given below:*

$$X \xleftarrow{\pi_X^{X,S}} X \times S \xrightarrow{\pi_S^{X,S}} S$$

- *At the level of morphisms, given $f : S \to T$ in $\mathbf{Sch}/\mathbb{C}$, it sends an appropriate T-*

doi:10.6342/NTU202301040

family of bundles to an S-family of bundles by pulling back along $1 \times f : X \times S \to X \times T$.

The definition of the functor moduli functor for $\mathcal{M}_H^G(c_1, c_2)$ is similar, where the functor associates to each object the equivalence classes of families indexed by that object of Gieseker-semistable rank 2 torsion-free sheaves.

By the works of Maruyama and Gieseker [Mar75, Gie77, Mar77, Mar78], it is known that such moduli functors admits coarse moduli spaces.

Theorem/Definition 3.8 (Gieseker, Maruyama)**.** *The functors $\mathcal{M}_H^\mu(c_1, c_2)$, $\mathcal{M}_H^G(c_1, c_2)$ admits coarse moduli spaces $\mathfrak{M}_H^\mu(c_1, c_2)$, $\mathfrak{M}_H^G(c_1, c_2)$ with $\mathfrak{M}_H^G(c_1, c_2)$ being projective, containing $\mathfrak{M}_H^\mu(c_1, c_2)$ as an open subscheme.*

Here we briefly review what it means to be a coarse or a fine moduli space and spell out the universal property satisfied by such spaces. Suppose we are given a moduli functor, which associates to each $\mathbb{C}$-scheme a set of isomorphism classes of particular families of sheaves indexed by that scheme (here the definition of a class varies by context):

$$\mathcal{M} : (\mathbf{Sch}/\mathbb{C})^{\mathrm{op}} \to \mathbf{Set}$$

then we say that a $\mathbb{C}$-scheme $\mathfrak{M}$ is a **fine moduli space** for $\mathcal{M}$ if it represents the functor. In this case, the universal element yields a **universal family** V - which is a sheaf (or more precisely, the isomorphism class of this sheaf) - on $\mathfrak{M} \times X$. In this case, given any other $S \in \mathbf{Sch}/\mathbb{C}$ and an isomorphism class of an S-indexed family of sheaf $W \in \mathcal{M}(S)$ (which is a sheaf on $S \times X$), there exists a unique morphism $f : S \to \mathfrak{M}$ such that the pullback of

doi:10.6342/NTU202301040

V along $f \times 1 : S \times X \to \mathfrak{M} \times X$ is W. Conversely, by Yoneda lemma, this property also characterizes representability, and can be served as the definition of a fine moduli space.

On the other hand, when it is too much to expect that a fine moduli space to exist (mostly due to fact that we are considering classes of families of sheaves), the weaker notion of a coarse moduli space is also useful. Here we say that a $\mathbb{C}$-scheme $\mathfrak{M}$ is a **coarse moduli space** if, as above, a class of S-family of sheaf is given, then there exists a unique morphism $S \to \mathfrak{M}$ corresponding to it, and such $\mathfrak{M}$ is universal among those having this property.

In short, the main differences are as follows:

- Fine moduli spaces admits a universal family, while coarse moduli spaces do not.
- Fine moduli spaces represents the moduli functor, while coarse moduli spaces are the ones that best approximates the moduli functor at the level of objects when no fine moduli space exists.

However in the cases that we are interested, we do have fine moduli spaces, in which case a universal family exists (see Corollary 4.6.7. of [HL10]), and we will repeatedly make use of this fact for what follows:

Theorem 3.9. *Suppose X, c_1, H satisfies (A1), (A2), then there is a universal family on $X \times \mathfrak{M}_H^{G,s}(c_1, c_2)$; here $\mathfrak{M}_H^{G,s}(c_1, c_2)$ is the open subspace of $\mathfrak{M}_H^{G,s}(c_1, c_2)$ of H-Gieseker-stable torsion-free sheaves. In particular, there is a universal family on $X \times \mathfrak{M}_H^{\mu}(c_1, c_2)$ and $X \times \mathfrak{M}_H^{G,s}(c_1, c_2)$.*

Remark. *Note that in view of 3.6, $\mathfrak{M}_H^{G,s}(c_1, c_2) = \mathfrak{M}_H^{G}(c_1, c_2)$.*

doi:10.6342/NTU202301040

These facts will be crucial in our constructions of maps between moduli spaces in later chapters - namely, we will construct these maps by juggling around universal families of sheaves.

3.3 Change of Polarization - Chambers and Walls

Throughout this section, we assume that X, H, c_1 satisfies condition (A1), (A2).

The set of ample divisors can be thought as lying in the cone $\mathbf{C}_X$ defined inside the vector space $\text{Pic}(X) \otimes \mathbb{R}$ spanned by the classes defined by ample divisors; in the literature, this is sometimes referred to as the **ample cone**. Regarding the choices of different ample divisors as choices of different stability conditions, this ample cone can be thought of as the space of stability conditions.

In [Qin93] and in [Göt96], a wall and chamber structure is given on $\mathbf{C}_X$. We briefly sketch the geometric meaning of this wall and chamber structure as follows (for $\mathfrak{M}_H^{\mu}(c_1, c_2)$, see Theorem 2 of [Qin93]; for $\mathfrak{M}_H^{G}(c_1, c_2)$, see Theorem 2.9 of [Göt96]):

Heuristic 3.10. *The wall and chambers structure and the structure of the moduli spaces are roughly as follows:*

- *Suppose H_1, H_2 lie in the same chamber, then the Mumford-Takemoto (resp. Gieseker) moduli spaces can be identified with each other.*
- *Suppose H_1 lies in a chamber and H_2 lies on another chamber sharing a common wall, then passing from H_1 to H_2 will result in throwing away some constructible subsets and gaining back some constructible subsets on the moduli spaces.*

doi:10.6342/NTU202301040

- *In this sense, one can relate between moduli spaces of different polarizations by a series of wall-crossings.*

Here we give the definitions of polarizations, walls and chambers.

Definition 3.11 (Qin)**.** *For each* $\xi \in \text{Pic}(X)$*, let*

$$W^{\xi} := \mathbf{C}_X \cap \{x \in \text{Pic}(X) \otimes \mathbb{R} | x.\xi = 0\}$$

then W^{ξ} *is called the* ***wall of type*** (c_1, c_2) ***determined by*** ξ *if* $\xi + c_1 \in 2\,\text{Pic}(X)$ *and* $-\Delta(c_1, c_2) \leq \xi^2 < 0$*. On the other hand,* ***a chamber of type*** (c_1, c_2) *is a connected component in the subset of* $\mathbf{C}_X$ *defined by removing all such walls. A* ***polarization*** *is a choice of an ample divisor.*

Remark. *By looking at parity, condition (A2) would imply that* H *lies in a chamber.*

Remark. *For each two chambers, they can at most have a common wall. However, there might be multiple* ξ *defining the same wall.*

Remark. *Originally, these definitions are defined over* $\text{Num}(X)$*, but as noted in the remark after 1.1, we can also work entirely in* $\text{Pic}(X)$ *when assuming (A1).*

We briefly sketch how the walls and chamber structures arises in the case of rank 2 μ-stable bundles on algebraic surfaces. The crucial idea is that given any rank 2 bundle V with first chern classes c_1 on X, if μ-stable (with respect to some polarization), then it is indecomposable. In this case, one can express this bundle as a nontrivial extension:

$$0 \to \mathcal{O}_X(F) \to V \to I_Z(c_1 - F) \to 0$$

doi:10.6342/NTU202301040

for some zero-dimensional subscheme Z; for a proof, see the discussion before chapter 2, proposition 4, of [Fri12]. Nevertheless, for decomposable bundles, they will correspond to trivial extensions with Z being empty.

Now with walls and chambers as defined, note that each ξ defines a set $E_\xi(c_1, c_2)$ of non-trivial extensions of the form above, with $2F = \xi + c_1$ and $l(Z) = \frac{\Delta(c_1,c_2)+\xi^2}{4}$. A thorough analysis as in [Qin93] shows that heuristic 3.10 holds, and the wall-crossing behaviour are completed characterized by these $E_\xi(c_1, c_2)$; more explicitly, in the situation above, suppose H_1, H_2 are different polarizations in chambers sharing a common nonempty wall. We have:

$$\mathfrak{M}^{\mu}_{H_1}(c_1, c_2) = \left(\mathfrak{M}^{\mu}_{H_2}(c_1, c_2) - \coprod_{\xi} E_{-\xi}(c_1, c_2)\right) \coprod \left(\coprod_{\xi} E_\xi(c_1, c_2)\right)$$

where ξ runs over numerical classes where $\xi.H_1 = 0$ that defines this wall.

The case of rank 2 locally free sheaves can also be analyzed. As the inclusion from a locally free sheaf of rank 2 to its reflexive hull is an isomorphism away from a zero-dimensional subscheme, we can - by first expressing its reflexive hull (which is a rank 2-bundle) as an extension - express any torsion-free rank 2 sheaves W on X with first chern class c_1 as an extension:

$$0 \to I_{Z_1}(F) \to W \to I_{Z_2}(c_1 - F) \to 0$$

where Z_1, Z_2 are zero-dimensional subschemes. Using the same wall and chamber structures, Göttsche introduced in [Göt96] the following analogues of the $E_\xi(c_1, c_2)$ given above. He considered spaces of the form $E^{m,n}_\xi, V^{m,n}_\xi$, with:

doi:10.6342/NTU202301040

- The space $E_\xi^{m,n}$ being the space of extensions of the form $0 \to I_{Z_1}(F) \to W \to I_{Z_2}(c_1 - F) \to 0$ with $2F = \xi + c_1$, and $l(Z_1) + l(Z_2) = \frac{\Delta(c_1,c_2)+\xi^2}{4}$.

- and $V_\xi^{m,n} \subseteq E_\xi^{m,n}$ consisting of extension classes that does not lie in

$$\ker\left(\mathrm{Ext}^1(I_{Z_2}(c_1 - F), I_{Z_1}(F)) \to \mathrm{Ext}^1(I_{Z_2}(c_1 - F), \mathcal{O}_X(F))\right)$$

In this case, there is an analogous formula the change from $\mathfrak{M}_{H_1}^{G,s}(c_1, c_2)$ to $\mathfrak{M}_{H_2}^{G,s}(c_1, c_2)$ when H_1, H_2 share a common wall, where $\mathfrak{M}_{H_i}^{G,s}(c_1, c_2)$ is the subspace of H_i-Gieseker-stable rank 2 torsion-free sheaves. More explicitly, they are given as follows:

$$\mathfrak{M}_{H_1}^{G,s}(c_1, c_2) = \left(\mathfrak{M}_{H_2}^{G,s}(c_1, c_2) - \coprod_{\xi,m,n} V_{-\eta}^{m,n}\right) \coprod \left(\coprod_{\xi,m,n} V_\eta^{m,n}\right)$$

where in the disjoint union above, ξ is over numerical classes that defines the nonempty wall between H_1, H_2, with m, n satisfying the relation $m + n = \frac{\Delta(c_1,c_2)-\xi^2}{4}$.

What about the Gieseker-stable sheaves? Under the assumption (A1), (A2), $\mathfrak{M}_H^{G,s}(c_1, c_2)$ and $\mathfrak{M}_H^G(c_1, c_2)$ can be identified; see 3.6.

At the level of motives, $E_\eta^{m,n}$ can be explicitly described, and that the difference $V_\eta^{m,n} - V_{-\eta}^{m,n}$ can be described in terms of $E_\eta^{m,n}, E_{-\eta}^{m,n}$. From this, we have the following formula (originally stated as a result involving Hodge polynomials, but holds equally in the motivic setting as the proof is essentially motivic):

Proposition 3.12 (Göttsche)**.** *Suppose X has property (A1), (A2), (A3), and let H_1, H_2 be different polarizations not lying on any wall (in particular, if both $H_1.c_1, H_2.c_1$ are odd*

doi:10.6342/NTU202301040

this would hold), then we have the following decomposition of motives:

$$[\mathfrak{M}^G_{H_1}(c_1,c_2)]-[\mathfrak{M}^G_{H_2}(c_1,c_2)]=\sum_{\eta:H_1\to H_2}\left([\mathbb{P}^{w(\eta)}]-[\mathbb{P}^{w(-\eta)}]\right)\left(\sum_{m+n=l(\eta)}[X^{[m]}][X^{[n]}]\right)$$

where the summation $\sum_{\eta:H_1\to H_2}$ means summing over η that defines a nonempty wall between H_1, H_2 with $\eta.H_1<0$, and that

$$l(\eta)=\frac{\Delta(c_1,c_2)+\eta^2}{4},\ w(\eta)=\frac{\Delta(c_1,c_2)-\eta^2}{4}+\frac{\eta.K_X}{2}-(h^{0,2}+2)$$

We also define $l_0(\eta), w_0(\eta)$ to be $l(\eta)-c_2, w(\eta)-c_2$.

This proposition will serve as a basis for the explicit description of the universal function $\mathbf{Z}^G_\varepsilon(q)$ later; we will revisit this proposition later in section 4.2.

3.4 Blowing-up and Stability

For this section, we assume that X has property (A1). Recall that its blowup $\tilde{X}$ still has property (A1).

On the other hand, Pic$(\tilde{X})\simeq\mathbb{Z}E\oplus$ Pic(X) given by pulling back along ϕ. In terms of the intersection pairing on Pic$(\tilde{X})$, E and Pic(X) are orthogonal, with $E^2=-1$, and this pairing restricts to the original pairing on Pic(X) when restricted to Pic(X). The canonical divisor is given by $K_{\tilde{X}}=\phi^*K_X+E$.

For an ample divisor H, we consider divisors of the form $H_r:=r\phi^*H-E$ for $r\in\mathbb{N}$. For large r, these H_r will be ample by Nakai-Moishezon. For now, let us also define

doi:10.6342/NTU202301040

$\tilde{c}_{1,m} := \phi^* c_1 + mE \in \mathrm{Pic}(\tilde{X})$. We follow the proof given in [Nak93].

Lemma 3.13. *Fix $m \in \mathbb{Z}$. For sufficiently large r and fixed ε, the moduli spaces $\mathfrak{M}^{\mu}_{H_r}(\tilde{c}_{1,m}, c_2)$ are well-defined and can be identified with each other.*

Proof. We use the chambers and walls structure of the space of stability conditions. Take some $r_0 > 0$ with H_r ample. Take some $\xi \in \mathbf{C}_{\tilde{X}}$ such that ξ defines a wall; we can express in the form $\xi = s\phi^* H' + tE$ for some $s, t \in \mathbb{R}$. Define:

$$\mathbf{C}_{X,\xi,0} = \{G \in \mathbf{C}_X : G.\xi = 0\},\ \mathbf{C}_{X,\xi,+} = \{G \in \mathbf{C}_X : G.\xi > 0\},\ \mathbf{C}_{X,\xi,-} = \{G \in \mathbf{C}_X : G.\xi < 0\}$$

so in this sense $\mathbf{C}_{X,\xi,0}$ is the wall defined by ξ, and $\mathbf{C}_{X,\xi,\pm}$ are ”open half spaces” defined in the ample cone. The Lemma is proved if we can show the following claim:

Claim. There is a universal lower bound $r_1 > r_0$ (uniform in the choice of ξ) such that for any ξ that defines a wall, one of the following holds:

$$\{H_r\}_{r>r_1} \subset \mathbf{C}_{X,\xi,0},\ \{H_r\}_{r>r_1} \subset \mathbf{C}_{X,\xi,+},\ \{H_r\}_{r>r_1} \subset \mathbf{C}_{X,\xi,-}$$

To simplify the discussion, we can consider this statement with $r \in [r_0, +\infty) \subseteq \mathbb{R}$. In this case, we only need:

Claim’. There is a universal lower bound $r_1 > r_0$ (uniform in the choice of ξ) such that for any ξ that defines a wall, the ray $\{H_r\}_{r>r_1}$ either lies entirely in the wall $\mathbf{C}_{X,\xi,0}$ or is disjoint from the wall $\mathbf{C}_{X,\xi,0}$.

doi:10.6342/NTU202301040

We prove this claim as follows. Suppose H_r lies on some wall defined by some $\xi \in \mathbf{C}_{\tilde{X}}$. Expressing ξ as $s\phi^*H' + tE$ for some $s, t \in \mathbb{R}$ with $H' \in \text{Num}(X)$. As $\xi.H_r = 0$, we get

$$rsH.H' = t$$

- If $H.H' = 0$, $\xi.H_r = 0$ for any other r, so we may assume $H.H' \neq 0$.
- In this case, we have by the definition of a wall that

$$-\Delta(\tilde{c}_{1,m}, c_2) \leq \xi^2 < 0$$

and hence $-m^2 - \Delta(c_1, c_2) \leq s^2(H')^2 - t^2 < 0$. Plugging in the equality above, we have:

$$0 < r^2s^2(H.H')^2 - s^2(H')^2 \leq m^2 + \Delta(c_1, c_2)$$

so r is bounded above by a number only dependent on H, ξ, c_1, c_2, m. Therefore, for large enough r, H_r no longer lie on this wall. However, we want to have a uniform bound. By Hodge index theorem, we have:

$$r^2s^2H^2(H')^2 \leq r^2s^2(H.H')^2 \leq m^2 + \Delta(c_1, c_2) + s^2(H')^2$$

No matter $(H')^2 > 0$ or ≤ 0, we have the estimate:

$$r \leq \sqrt{m^2 + \Delta(c_1, c_2) + 1}$$

which concludes the proof.

□

Remark (Case of Gieseker Moduli Spaces)**.** *The same statement holds for* $\mathfrak{M}^G_{H_r}(\tilde{c}_{1,m}, c_2)$

doi:10.6342/NTU202301040

with the same proof for the chamber structure introduced in [Göt96]. If we assume (A2) and carefully choose r with the correct parity, we can also use 3.6.

Remark. *Note also that the bound r given above only depends on c_1, c_2, m.*

This lemma motivates the following definition:

Definition 3.14 (Moduli of Blown-up). *Given $\varepsilon \in \{0,1\}$, define $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2)$ (resp. $\mathfrak{M}^{G}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2)$, $\mathfrak{M}^{U}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2)$) as $\mathfrak{M}^{\mu}_{H_r}(\tilde{c}_{1,\varepsilon}, c_2)$ (resp. $\mathfrak{M}^{G}_{H_r}(\tilde{c}_{1,\varepsilon}, c_2)$, $\mathfrak{M}^{U}_{H_r}(\tilde{c}_{1,\varepsilon}, c_2)$) for $r \gg 0$; this is well-defined.*

Here we mention another useful technical result characterizing stability condition along a blowup given in [Bru90]. The characterization is based on pushing sheaves forward along ϕ and taking double duals.

Lemma 3.15 (Stability and Blowup). *Given a 2-bundle $\tilde{V}$ on $\tilde{X}$, an ample divisor H, and pick r so that H_r is ample. Then $\tilde{V}$ is H_r-stable iff $(\phi_*\tilde{V})^{**}$ is H-stable.*

Note that as $(\phi_*\tilde{V})^{**}$ is a reflexive sheaf on a smooth surface, it is locally free.

3.5 Compactification and Comparison

In this section, we assume that X has property (A1).

Recall that we have defined $\mathfrak{M}^{\mu}_{H}(c_1, c_2), \mathfrak{M}^{G}_{H}(c_1, c_2)$ with $\mathfrak{M}^{\mu}_{H}(c_1, c_2)$ being an open subscheme of the smooth projective scheme $\mathfrak{M}^{G}_{H}(c_1, c_2)$. By [FQ95], when we assume the conditions (A2), (A3) from 1.1, $\mathfrak{M}^{\mu}_{H}(c_1, c_2)$ is indeed dense in $\mathfrak{M}^{G}_{H}(c_1, c_2)$

doi:10.6342/NTU202301040

With origins from gauge theory and Donaldson's theory, one can construct an alternative compactification of $\mathfrak{M}^{\mu}_{H}(c_1, c_2)$, called the **Uhlenbeck compactification** $\mathfrak{M}^{U}_{H}(c_1, c_2)$. We will not explicitly say what this is, but rather mention the following decomposition proved in see Lemma 4.23. of [LQ98]:

Proposition 3.16 (Comparison of Compactifications (I)). *Under conditions (A1), (A2), (A3), we have a decomposition of motives:*

$$[\mathfrak{M}^{U}_{H}(c_1, c_2)] = \sum_i [\mathfrak{M}^{\mu}_{H}(c_1, i)][X^{(c_2-i)}]$$

On the other hand, the Mumford Takemoto space can also be compared with Gieseker compactifications.

Proposition 3.17 (Comparison of Compactifications (II)). *Under conditions (A1), (A2), we have a decomposition of motives:*

$$[\mathfrak{M}^{G}_{H}(c_1, c_2)] = \sum_i [\mathfrak{M}^{\mu}_{H}(c_1, i)][X^{\{c_2-i\}}]$$

Proof. There is a stratification of $\mathfrak{M}^{G}_{H}(c_1, c_2)$ by considering reflixive hulls. Explicitly, given $W \in \mathfrak{M}^{G}_{H}(c_1, c_2)$, we form the exact sequence

$$0 \to W \to W^{**} \to Q \to 0$$

then we have $W^{**} \in \mathfrak{M}^{\mu}_{H}(c_1, c_2 - h^0(X, Q))$ (stability follows from 3.6). The number $h^0(X, Q)$ is a finite number; call this i. If we define $\mathfrak{M}^{G}_{H}(c_1, c_2, i)$ to be the subscheme consisting of locally free sheaves W with $h^0(X, W^{**}/W) = i$, we obtain a decomposition:

$$[\mathfrak{M}^{G}_{H}(c_1, c_2)] = \sum_i [\mathfrak{M}^{G}_{H}(c_1, c_2, i)]$$

doi:10.6342/NTU202301040

We will be done if we can show the decomposition of motives:

$$[\mathfrak{M}_H^G(c_1, c_2, c_2 - i)] = [\mathfrak{M}_H^\mu(c_1, i)][X^{\{c_2 - i\}}]$$

We do this by juggling with universal sheaves. Let $\mathcal{V}$ be the universal sheaf on $X \times \mathfrak{M}_H^\mu(c_1, i)$. Corresponding to each point in $\mathfrak{M}_H^\mu(c_1, i)$, we want to consider all of its equivalence classes of quotients, and hence we would get the kernels - which would correspond to points in $\mathfrak{M}_H^G(c_1, c_2, c_2 - i)$. Therefore, we consider the Quot scheme:

$$\text{Quot} := \text{Quot}^{c_2 - i}_{\mathcal{V}/X \times \mathfrak{M}_H^\mu(c_1,i)/\mathfrak{M}_H^\mu(c_1,i)}$$

along with coordinate projections:

$$X \times \text{Quot} \simeq X \times \mathfrak{M}_H^\mu(c_1, i) \times_{\mathfrak{M}_H^\mu(c_1,i)} \text{Quot} \xrightarrow{p_1} X \times \mathfrak{M}_H^\mu(c_1, i)$$

which gives a universal quotient:

$$p_1^* \mathcal{V} \to \mathcal{Q} \to 0$$

Now let $\mathcal{K}$ be the kernel of this morphism, then $\mathcal{K}$ - a sheaf on $X \times$ Quot - will define a morphism to $\mathfrak{M}_H^G(c_1, c_2 - i)$ if its restriction to each fiber (which is X parametrized by some point in Quot) meets the stability requirements with the correct chern classes. So let us take restriction along a point $q \in$ Quot, which gives us an exact sequence:

$$0 \to \mathcal{K} \mid_{X \times \{q\}} \to \mathcal{V} \mid_{X \times \{\pi(q)\}} \to \mathcal{Q} \mid_{X \times \{q\}} \to 0$$

where $\pi : \text{Quot} \to \mathfrak{M}_H^\mu(c_1, i)$ is the canonical map used in the definition of Quot (in fact, $p_1 = 1 \times \pi$). It is easy to see from this sequence that $\mathcal{K} \mid_{X \times \{q\}} \in \mathfrak{M}_H^G(c_1, c_2, i)$ as follows.

doi:10.6342/NTU202301040

To simplify notations, rewrite this sequence as $0 \to K \to V \to Q \to 0$.

- By the definition of Quot, Q is a torsion free sheaf with $h^0(X, Q) = c_2 - i$.
- By the definition of $\mathcal{V}$, V lies in $\mathfrak{M}^{\mu}_H(c_1, i)$.
- By Whitney's product formula, K has the appropriate chern classes.
- As a subsheaf with the same first chern class as V, K is μ-stable and hence Gieseker-semistable.

Therefore, we get a morphism Quot $\to \mathfrak{M}^{\mu}_H(c_1, i)$. From the above discussion, it is also easy to see that this morphism is bijective. At the level of motives, we have:

$$[\text{Quot}] = [\mathfrak{M}^G_H(c_1, c_2, i)]$$

so the next thing is to see how to describe [Quot]. It turns out that we have:

$$\begin{aligned}
[\text{Quot}] &= [\text{Quot}^{c_2-i}_{\mathcal{V}/X\times\mathfrak{M}^{\mu}_H(c_1,i)/\mathfrak{M}^{\mu}_H(c_1,i)}] \\
&= [\text{Quot}^{c_2-i}_{\mathcal{O}^{\oplus 2}_{X\times\mathfrak{M}^{\mu}_H(c_1,i)}/X\times\mathfrak{M}^{\mu}_H(c_1,i)/\mathfrak{M}^{\mu}_H(c_1,i)}] \\
&= [\mathfrak{M}^{\mu}_H(c_1, i) \times X^{\{c_2-i\}}]
\end{aligned}$$

where in the above, the last equality follows from the universal properties of Quot-schemes, while the second equality is shown in the next lemma 3.18. This gives $[\text{Quot}] = [\mathfrak{M}^{\mu}_H(c_1, i)][X^{\{c_2-i\}}]$.

□

Lemma 3.18. *Suppose X is a projective scheme over a Noetherian base scheme S, then for any rank r bundle V on X and any constant n, we have:*

$$[\text{Quot}^n_{V/X/S}] = [\text{Quot}^n_{\mathcal{O}^{\oplus r}_X/X/S}]$$

doi:10.6342/NTU202301040

For the proof of this proposition, all the products are to be interpreted in the category of S-schemes.

Proof. Stratify X as $\coprod_{i=1}^m X_i$, and find for each X_i an open neighborhood U_i such that $\mathcal{V}$ is trivial on U_i. Let $\mathrm{Quot}^n_{V/X/S;X_i}$ be the constructible subset consisting of quotients of $\mathcal{V}$ with support in X_i. We claim that there is a bijective morphism:

$$\coprod_{\sum_i n_i = n} \left(\prod_{i=1}^m \mathrm{Quot}^{n_i}_{V/X/S;X_i} \right) \to \mathrm{Quot}^n_{V/X/S}$$

the idea being that we can write each quotient of V as a direct sum of quotients supported at different X_i; in fact, after constructing this morphism, bijectivity follows essentially from this observation. This morphism is constructed as follows:

- By universal property of disjoint unions, it suffices to construct for a fixed partition $\sum_i n_i = n$.

- Take coordinate projections:

$$X \overset{q_1}{\leftarrow} X \times \prod_{i=1}^m \mathrm{Quot}^{n_i}_{V/X/S;X_i} \overset{q_2}{\to} \prod_{i=1}^m \mathrm{Quot}^{n_i}_{V/X/S;X_i}$$

$$\prod_{i=1}^m \mathrm{Quot}^{n_i}_{V/X/S;X_i} \overset{\gamma_i}{\to} \mathrm{Quot}^{n_i}_{V/X/S;X_i}$$

  Let $p_{1,n_i}^* V \to Q_{n_i} \to 0$ be the universal quotient on $X \times \mathrm{Quot}^{n_i}_{V/X/S}$ where $p_{1,n_i} : X \times \mathrm{Quot}^{n_i}_{V/X/S} \to X$ is coordinate projection. Define $Q_{n_i,i}$ to be Q_{n_i} restricted along the map:

$$X \times \mathrm{Quot}^{n_i}_{V/X/S;S_i} \to X \times \mathrm{Quot}^n_{V/X/S}$$

doi:10.6342/NTU202301040

By pulling back along

$$X \times \prod_{i=1}^{m} \mathrm{Quot}^{n_i}_{V/X/S;X_i} \overset{1\times\gamma_i}{\to} X \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}$$

we get another exact sequence on $X \times \prod_{i=1}^{m} \mathrm{Quot}^{n_i}_{V/X/S;X_i}$:

$$q_1^* V \to (1 \times \gamma_i)^* Q_{n_i,i} \to 0$$

and hence, finally an exact sequence:

$$q_1^* V \to \bigoplus_i (1 \times \gamma_i)^* Q_{n_i,i} \to 0$$

where the morphism is canonically defined through universal property of $\bigoplus_i$, while surjectivity can be seen by looking at supports. By the definition of Quot schemes, we get a morphism:

$$\prod_{i=1}^{m} \mathrm{Quot}^{n_i}_{V/X/S;X_i} \to \mathrm{Quot}^{n}_{V/X/S}$$

This finishes the construction. Therefore, this proposition is proved if we can find a bijection:

$$[\mathrm{Quot}^{n_i}_{V/X/S;X_i}] = [\mathrm{Quot}^{n_i}_{\mathcal{O}_X^{\oplus r}/X/S;X_i}]$$

Let $p_{1,n_i}, Q_{n_i}, Q_{n_i,i}$ be as above and let $p_{1,n_i,i}$ be p_{1,n_i} restricted to $X \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}$. We want to construct a morphism:

$$\mathrm{Quot}^{n_i}_{V/X/S;X_i} \to \mathrm{Quot}^{n_i}_{\mathcal{O}_X^{\oplus r}/X/S}$$

doi:10.6342/NTU202301040

The strategy is again to juggle with universal families. We have the exact sequence:

$$p_{1,n_i,i}^* V \to Q_{n_i,i} \to 0$$

We construct a corresponding quotient of $p_{1,n_i,i}^* \mathcal{O}_X^{\oplus r}$ from this sequence

$$p_{1,n_i,i}^* \mathcal{O}_X^{\oplus r} \to Q_{n_i,i} \to 0$$

which would then induce the desired map by properties of Quot schemes. This map is constructed by the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}(p_{1,n_i,i}^* V, Q_{n_i,i}) &\simeq H^0(X \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}, (p_{1,n_i,i}^* V)^* \otimes Q_{n_i,i}) \\
&\simeq H^0(U_i \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}, \left((p_{1,n_i,i}^* V)^* \otimes Q_{n_i,i}\right)|_{U_i \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}}) \\
&\simeq H^0(U_i \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}, \left((p_{1,n_i,i}^* \mathcal{O}_X^{\oplus r})^* \otimes Q_{n_i,i}\right)|_{U_i \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}}) \\
&\simeq H^0(X \times \mathrm{Quot}^{n_i}_{V/X/S;X_i}, (p_{1,n_i,i}^* \mathcal{O}_X^{\oplus r})^* \otimes Q_{n_i,i}) \\
&\simeq \mathrm{Hom}(p_{1,n_i,i}^* \mathcal{O}_X^{\oplus r}, Q_{n_i,i})
\end{aligned}$$

Now the fact that the induced morphism $\mathrm{Quot}^{n_i}_{V/X/S;X_i} \to \mathrm{Quot}^{n_i}_{\mathcal{O}_X^{\oplus r}/X/S}$ is injective with image $\mathrm{Quot}^{n_i}_{\mathcal{O}_X^{\oplus r}/X/S;X_i}$ can also be seen from these isomorphisms. This concludes the proof. □

doi:10.6342/NTU202301040

Chapter 4 Existence and Computation of Universal Functions

Our goal in this chapter is to show that universal functions exists and compute the exact formula of these functions.

4.1 Existence of Universal Functions

We have an explicit relation between Mumford-Takemoto, Gieseker, Uhlenbeck spaces. In this section, we show that a universal function exists in the case of Mumford-Takemoto spaces, which implies the existence of universal functions in the cases of Gieseker, Uhlenbeck spaces.

Suppose we are given $V \in \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2)$. As all bundles on $\mathbb{P}^1$ are classified by Grothendieck's theorem, and that $c_1(V) = \tilde{c}_{1,0} = \phi^* c_1$, we have:

$$V \mid_E = \mathcal{O}_E(d) \oplus \mathcal{O}_E(-d),\ d \geq 0$$

doi:10.6342/NTU202301040

In this sense, we have the following preliminary stratification of $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2)$:

$$\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2) = \coprod_{d \geq 0} \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d)$$

with $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d)$ consisting of bundles that restricts to $\mathcal{O}_E(d) \oplus \mathcal{O}_E(-d)$ on E. Notice that we have a similar stratification for $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2)$:

$$\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2) = \coprod_{d \geq 0} \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2, d)$$

with $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2, d)$ consisting of bundles that restricts to $\mathcal{O}_E(d+1) \oplus \mathcal{O}_E(-d)$ on E.

Here we mention an observation. We know that every rank 2-bundle on X with chern class c_1, c_2 pulls back to a 2-bundle with chern classes $\tilde{c}_{1,0}, c_2$. It turns out that this construction is compatible with stability, and that all the stable 2-bundles that restricts to $\mathcal{O}_E^{\oplus 2}$ is a pullback of a stable bundle.

Lemma 4.1 (Identification of Moduli Space as a stratum under Blowup)**.** *For* $(\varepsilon, d) = (0, 0)$*, we have a bijective morphism:*

$$\mathfrak{M}^{\mu}_{H}(c_1, c_2) \to \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, 0)$$

This is shown in Proposition 2.3., Lemma 3.1. of [Nak93]. In fact, it was shown there that the map defines an open immersion into $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2)$ and that under certain circumstances the image is dense, but we won't need it here. This is the first step towards relating the motive of a moduli space to the motive of the moduli space of the blowup. Now see consider how to manage the motives of $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2, d)$ for $(\varepsilon, d) \neq (0, 0)$.

doi:10.6342/NTU202301040

Following [LQ99], the idea is to use Maruyama's elementary modifications (or sometimes called elementary transformations); for details, one may consult 5.2 of [HL10] or chapter 2 of [Fri12].

Suppose we take $V \in \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d)$ with $d > 0$. We have

$$\mathrm{Hom}_X(V, \mathcal{O}_E(-d)) \simeq \mathrm{Hom}_E(V\mid_E, \mathcal{O}_E(-d)) \simeq \mathbb{C}$$

so in this sense we obtained an exact sequence (unique up to $\mathbb{C}^\times$):

$$0 \to V' \to V \to \mathcal{O}_E(-d) \to 0$$

The theory of elementary modifications implies that V' is a 2-bundle. It is still H_∞ stable by 3.15, and that its chern classes can be explicitly described via Whitney's formula - namely, we have:

$$c_1(V') = c_1(V) - E,\ c_2(V') + c_1(V').E - \iota_* c_1(\mathcal{O}_E(-d)) = c_2(V)$$

where $\iota : E \to \tilde{X}$ is the inclusion. These simplifies to:

$$c_1(V') = c_1(V) - E = \tilde{c}_{1,1},\ c_2(V') = c_2(V) - d = c_2 - d$$

Also by dualizing, one can recover V by considering the exact sequence given by taking duals:

$$0 \to V^* \to (V')^* \to \mathcal{O}_E(d-1) \to 0$$

Therefore, we get the following:

doi:10.6342/NTU202301040

Lemma 4.2 (Elementary Modification of Stable Bundles)**.** *There are set-theoretic maps:*

$$\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2, d) \to \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1-\varepsilon}, c_2 - d)$$

given by elementary modification.

Proof. Only the case $\varepsilon = 1$ needs explanation. Its definition is given by first taking duals, then do elementary modifications. □

We can stratify $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d)$ further as:

$$\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d) = \coprod_{l \geq 0} \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$$

where $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$ is the subset of V with $V' \in \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)$. It turns out that we can relate $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$ and $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)$ by the following proposition.

Proposition 4.3. *We have a decomposition of motives:*

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)] = [U(d - l - 1, d + l)][\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)]$$

where $U(p, q)$ is the space parametrizing exact sequences of the form (up to $\mathbb{C}^\times$):

$$\mathcal{O}_E(-p) \oplus \mathcal{O}_E(-q) \to \mathcal{O}_E \to 0$$

for any two nonnegative integers p, q.

Remark. *One can explicitly describe $U(p, q)$. Firstly, by dualizing, we have the correspondence of exact sequences:*

$$[\mathcal{O}_E(-p) \oplus \mathcal{O}_E(-q) \to \mathcal{O}_E \to 0] \longleftrightarrow [0 \to \mathcal{O}_E \to \mathcal{O}_E(p) \oplus \mathcal{O}_E(q)]$$

doi:10.6342/NTU202301040

and hence corresponds to a subset of global sections of $\mathcal{O}_E \to \mathcal{O}_E(p) \oplus \mathcal{O}_E(q)$ modding out $\mathbb{C}^\times$. In this sense, we see that $U(p,q)$ is the set of homogeneous polynomials (f,g) each of degree p,q such that f,g are coprime (and we require two pairs to be the same if they differ by a constant in $\mathbb{C}^\times$); the latter description of $U(p,q)$ will be crucial to understand the motivic structure of this set. Here we adopt the convention that the 0 polynomial can take any degree.

Proof. Firstly, we know that we can construct each bundle V in $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$ as an extension of $\mathcal{O}_E(-d)$ with a bundle V' in $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)$. In this sense, V lies in:

$$\mathcal{E}xt^1_X(\mathcal{O}_E(-d), V')$$

To define a morphism between moduli spaces, we have to use universal families. There is a universal family $\mathcal{V}'$ on $\tilde{X} \times \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)$. Let p_1, p_2 be coordinate projections:

$$\tilde{X} \xleftarrow{p_1} \tilde{X} \times \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l) \xrightarrow{p_2} \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)$$

In this sense, we take the relative extension sheaf:

$$\mathcal{E} := \mathcal{E}xt^1_{p_2}(p_1^*\mathcal{O}_E(-d), \mathcal{V}')$$

This sheaf is good for doing universal constructions we shall see later by taking $\mathbb{P}(\mathcal{E}^*)$; other details can be found in [Lan83]. From a computational viewpoint, we can give an easier description of $\mathcal{E}$. There is a spectral sequence (see [BPS80]) converging to $H^{m+n}(A) = \mathcal{E}xt^{m+n}_{p_2}(p_1^*\mathcal{O}_E(-d), \mathcal{V}')$ with

$$E_2^{m,n} = R^m p_{2*}\mathcal{E}xt^n(p_1^*\mathcal{O}_E(-d), \mathcal{V}')$$

doi:10.6342/NTU202301040

From this, one has the five term exact sequence:

$$0 \to E_2^{1,0} \to H^1(A) \to E_2^{1,0} \to E_2^{2,0} \to H^2(A)$$

for which the first four terms expands to

$$0 \to R^1 p_{2*}(\mathcal{H}om(p_1^*\mathcal{O}_E(-d), \mathcal{V}')) \to \mathcal{E} \to p_{2*}(\mathcal{E}xt^1(p_1^*\mathcal{O}_E(-d), \mathcal{V}')) \to R^2 p_{2*}(\mathcal{H}om(p_1^*\mathcal{O}_E(-d), \mathcal{V}'))$$

Since $p_1^*\mathcal{O}_E(-d)$ is torsion while $\mathcal{V}'$ isn't, we get:

$$\mathcal{E} \simeq p_{2*}(\mathcal{E}xt^1(p_1^*\mathcal{O}_E(-d), \mathcal{V}')) \simeq p_{2*}(p_1^*\mathcal{O}_E(d-1) \otimes \mathcal{V}')$$

We verify that this is locally free using semicontinuity theorem, applied to p_2. Say given $V' \in \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)$, we calculate the zeroth-cohomology over the fiber at this point, which gives:

$$H^0(\tilde{X}, \mathcal{O}_E(d-1) \otimes V') \simeq H^0(\tilde{X}, \mathcal{O}_E(d-l-1) \oplus \mathcal{O}_E(d+l))$$

so h^0 is consistent among the fibers. Now we form the following space T with the canonical map γ:

$$T = \mathbb{P}(\mathcal{E}^*) \xrightarrow{\gamma} \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)$$

We take the open subset of $T_0 \subseteq T$ consisting of locally free extensions; what this means is that since the fiber of γ over a single V' is of the form

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_E(-d), V'))$$

we may fiberwise take the ones that corresponds to extensions classes of bundles, which

doi:10.6342/NTU202301040

would give us the space T_0. The proof will be complete if we can verify:

- The bundle projection $\gamma \mid_{T_0}$ is a Zariski locally-trivial fibration with fibers of the form $U(d-l-1, d+l)$.

- There is a bijective morphism $T_0 \to \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$.

We verify them separately as follows:

- We first show that the fibers consisting of locally free extensions can be identified as $U(d-l-1, d+l)$. A way to show this is to notice the fact that given an elementary modification

  $$0 \to V' \to V \to \mathcal{O}_E(-d) \to 0$$

  we can recover V as the kernel (by dualizing):

  $$0 \to V^* \to (V')^* \to \mathcal{O}_E(d-1) \to 0$$

  so we see have the following identifications (up to $\mathbb{C}^\times$):

  (i) A locally free V as an extension of V' and $\mathcal{O}_E(-d)$.

  (ii) A surjection $(V')^* \to \mathcal{O}_E(d-1)$.

  (iii) A surjection $\mathcal{O}_E(-l-1) \oplus \mathcal{O}_E(l) \simeq (V')^* \mid_E \to \mathcal{O}_E(d-1)$.

  (iv) A surjection $\mathcal{O}_E(l-d+1) \oplus \mathcal{O}_E(-l-d) \to \mathcal{O}_E$.

  As γ is readily a Zariski locally trivial fibration, the identification above exhibits the restriction of $\gamma \mid_{T_0}$ also as a Zariski locally trivial fibration.

- Our next thing is to construct a map from T_0 to $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$. We do this by

doi:10.6342/NTU202301040

constructing families of bundles. Define the following coordinate projection maps

$$\tilde{X} \overset{q_1}{\leftarrow} \tilde{X} \times T_0 \overset{q_2}{\rightarrow} T_0$$

By the universal property of $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$, we try to construct a T-family of bundles on $\tilde{X} \times T_0$ so that the bundle on $\tilde{X}$ corresponding to each $t \in T_0$ lies in $\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$. By Corollary 4.5. of [Lan83], we have an extension:

$$0 \to q_2^* \mathcal{O}_T(1) \otimes (1 \times \gamma)^* \mathcal{V}' \to \tilde{\mathcal{V}} \to q_1^* \mathcal{O}_E(-d) \to 0$$

This defines a morphism

$$T_0 \to \mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d, l)$$

which is the desired one.

□

Similarly, one can derive a formula when we make the change $\varepsilon \mapsto 1 - \varepsilon$.

Corollary 4.4. *We have the decomposition formula:*

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d)] = \sum_{l=0}^{d-1} [U(d-l-1, d+l)][\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)]$$

Similarly, one has:

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2, d)] = \sum_{l=0}^{d} [U(d-l, d+l)][\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2 - d, l)]$$

Theorem 4.5 (Existence of Universal Functions)**.** *The statement concerning the existence*

doi:10.6342/NTU202301040

of universal functions given in 1.5 is true.

Proof. For $\mathbf{Z}^{\mu}_{\varepsilon}(q)$, this follows from 4.1 and 4.4. For $\mathbf{Z}^{U}_{\varepsilon}(q), \mathbf{Z}^{G}_{\varepsilon}(q)$, this follows from 3.16 and 3.17. □

4.2 Computation of Universal Functions (I) by Specialization

The above discussion essentially says that the universal relations are essentially encoded in $U(p,q)$ - an open subspace of $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(q)))$. Conceptually speaking, the recursion relation in 4.4 and the identification 4.1 itself would be adequate for finding an expression of such universal function. This is given in [LQ98] for the computation of universal function for Mumford-Takemoto spaces. The resulting formula is not a closed form formula; we will revisit this later in section 4.3.

However, in this section, we will first visit the method used by the authors in [LQ98] for the computation of the universal function for Gieseker spaces.

Recall that in 3.12, an explicit formula given in [Göt96] relating change of polarizations and change of moduli spaces has been described. In fact, in the same paper, Göttsche was able to make this relative description absolute in the case where X is a ruled surface satisfying (A3). In this section, we will first describe the wall and chamber structures in this case, present Göttsche's result, and we will use the special case $X = \mathbb{F}_1$ considered in [LQ98] to compute the universal function - after all, since we already knew that a universal function exists, computing a particular case would be sufficient.

doi:10.6342/NTU202301040

Conditions 4.6. *To simplify discussion, for this section, we assume that X, H, c_1 has property (A1),(A2),(A3), and that X is ruled over a curve of genus 0 (this condition is forced as X is required to have property (A1)), and we introduce the following notations and setup:*

- *We choose S, F to be divisors corresponding to a section and a fiber of X; we have $F^2 = 0, F.S = 1$. We assume $c_1 = S$, $S^2 = -1$. It is also common to consider the invariant $e = -S^2$; in our case, $e = 1$.*

- *The Picard group of X is $\mathbb{Z}S + \mathbb{Z}F$. For $H = xS + yF \in \mathbb{Z}S + \mathbb{Z}F$, H is ample iff $\alpha > 0$ and $r_H > 1$ where we define $r_H = y/x$. Also, if H represents an irreducible curves on X, we will always have $r_H \geq 1$; see Proposition V.2.20 of [Har13].*

The wall and chamber structures can be described quite explicitly in this case; see [Qin92]. Before going into the closer description of these structures, we first remind the following facts that can be deduced from 3.11:

- By Bogomolov's inequality and the fact that $c_1^2 = S^2 = -1$, we will only consider the case $c_2 \geq 0$.

- By the parity condition, if η defines a wall, we have:

$$\eta = (2a - 1)S + 2bF$$

  with $-(c_2 + 1/4) = -\Delta(c_1, c_2) \leq \eta^2 < 0$. Note also that η^2 is an odd number, so there will be no walls when $c_2 = 0$. Therefore, we will only focus on the case $c_2 > 0$.

- The divisor F is nef, so all the chambers and walls are all contained in one of the

doi:10.6342/NTU202301040

half spaces in $\mathrm{Pic}(X) \otimes \mathbb{R}$ defined by F.

The chamber and wall structures near the ray spanned by F is in particular interesting; let us denote this ray by $[F]$. The following is a direct consequence of Lemma 1.11 and Proposition 2.3. of [Qin92]:

Proposition 4.7. *Given fixed* $c_1 = S$*, and fixed* $c_2 > 0$*, define* $\eta_0 = S - 2c_2F$*. Then:*

- η_0 *defines a nonempty wall of type* (c_1, c_2)*; we denote this wall as* W_0*.*
- *No wall of type* (c_1, c_2) *lies between* $[F]$ *and* W_0*. The chamber lying between* $[F]$ *and* W_0 *is written as* C_F *(note that this chamber is nonempty by considering* $S+nF$ *for* $n \gg 0$*).*
- *For any polarization* H *in* C_F*,* $\mathfrak{M}_H^G(c_1, c_2) = \emptyset$*.*

Remark. *If we consider the ample divisors* $S + nF$ *(for* $n > 0$*) - which lies in* C_F *for* $n \gg 0$ *- then it is easy to see that* F *is in the closure of* C_F*.*

Using such description, Göttsche is able to obtain explicit decomposition results on the Hodge polynomials of $\mathfrak{M}_H^G(c_1, c_2)$ in [Göt96] using 3.12. Similar techniques was also applied in [LQ98] to compute an explicit form of some universal functions.

Proposition 4.8. *Given a polarization* H*, let* $W(H)$ *be the set:*

$$W(H) := \{\eta \in S + 2\,\mathrm{Pic}(X) : \eta.H < 0 < \eta.F,\ \eta^2 < 0\}$$

then we get:

$$\mathbf{M}^G(X, H, c_1; q) = q^{-c_1^2/4}\mathbf{H}(X; \mathbb{L}q)^2 \left(\sum_{\eta \in W(H)} \left([\mathbb{P}^{w_0(\eta)}] - [\mathbb{P}^{w_0(-\eta)}] \right) (\mathbb{L}q)^{-l_0(\eta)} \right)$$

doi:10.6342/NTU202301040

with $l_0(\eta), w_0(\eta)$ defined as in 3.12.

Proof. For each $c_2 \geq 0$, define $W(H, c_2)$ to be the subset of $W(H)$ consisting of η satisfying the condition $-\Delta(c_1, c_2) \leq \eta^2 < 0$. By 4.7 and 3.12 (applied to H and a polarizations in C_F), we have:

$$[\mathfrak{M}_H^G(c_1, c_2)]q^{c_2} = \sum_{\eta \in W(H,c_2)} \left([\mathbb{P}^{w(\eta)}] - [\mathbb{P}^{w(-\eta)}]\right) \left(\sum_{m+n=l(\eta)} [X^{[m]}][X^{[n]}]\right) q^{c_2}$$

$$= \sum_{\eta \in W(H,c_2)} \left([\mathbb{P}^{w_0(\eta)}] - [\mathbb{P}^{w_0(-\eta)}]\right) (\mathbb{L}q)^{c_2 - l(\eta)} \left(\sum_{m+n=l(\eta)} [X^{[m]}][X^{[n]}]\right) (\mathbb{L}q)^{l(\eta)}$$

Since the condition for $\eta \in W(H)$ to lie in $W(H, c_2)$ is the same as $l(\eta) \geq 0$, we get the desired formula; note also that $c_2 - l(\eta) = -l_0(\eta)$. □

Remark. *It is also convenient to write the above formula in the following way:*

$$\mathbf{M}^G(X, H, c_1; q) = \mathbb{L}^{-(1+h^{0,2})} \mathbf{H}(X; \mathbb{L}q)^2 \left(\sum_{\eta \in W(H)} \left(\frac{\mathbb{L}^{\eta.K_X} - 1}{\mathbb{L} - 1}\right) \mathbb{L}^{-\frac{\eta^2 + \eta.K_X}{2}} q^{-\eta^2/4}\right)$$

A slightly more symmetric form is given as follows:

$$\mathbf{M}^G(X, H, c_1; q) = \frac{\mathbb{L}^{-(1+h^{0,2})}}{\mathbb{L} - 1} \mathbf{H}(X; \mathbb{L}q)^2 \left(\sum_{\eta \in W(H)} \left(\mathbb{L}^{\frac{\eta.K_X - \eta^2}{2}} - \mathbb{L}^{\frac{-\eta.K_X - \eta^2}{2}}\right) q^{-\eta^2}\right)$$

Now let us consider blowing up X via $\phi : \tilde{X} \to X$, with exceptional divisor E. The Picard group now has an extra dimension given by E. All the ample divisors lives in the half space of $\text{Pic}(\tilde{X}) \otimes \mathbb{R}$ given by those divisors $\tilde{H}$ with $\tilde{H}.E > 0$. The ample divisors closest to the plane containing $\text{Pic}(X) \oplus 0$ then lives in the $-E + \text{Pic}(X)$; note that all the H_r arising from an ample divisor H on X are all in this set.

doi:10.6342/NTU202301040

The wall and chamber structure of Pic$(\tilde{X})$ is not as well-behaved as Pic(X). However, an analogue statement of 4.7 still holds in this case as given below.

Proposition 4.9. *Given fixed $c_1 = S$, $c_2 > 0$, ε, define $\eta_0 = S - 2c_2F - \varepsilon E$. Then there exists a sufficiently big r_0, such that for any $r \geq r_0$ with the correct parity, the following holds:*

(1) For any ample divisor H on X, there are no walls between H_r and H in $\mathbf{C}_{\tilde{X}}$ with H_r ample.

(2) The line spanned by F in $\mathbf{C}_X$ still lies in the boundary of some chamber; call this chamber C_F.

(3) For some large n, $S + nF \in \partial C_F$.

(4) We have $\mathfrak{M}^{\mu}_{H'}(\tilde{c}_{1,\varepsilon}, c_2) = \emptyset$ for any polarization H' lying in C_F.

(5) Statement (4) holds true with μ replaced by G.

Proof. Let us first pick some r_0 as in 3.13. To show (1), note that if there is a wall between H and H_r, there will be a wall between H_r and H_{r+s} for some $s > 0$, but this will not happen when $r \geq r_0$. For (2), we see by (1) we have $\mathbf{C}_X \subset \partial \mathbf{C}_{\tilde{X}}$. For (3), say $\xi \in \tilde{c}_{1,\varepsilon} + 2\,\mathrm{Pic}(\tilde{X})$ that defines a wall of type $(\tilde{c}_{1,\varepsilon}, c_2)$, which is an element of the form $aS+bF+cE$, we consider how the wall W^{ξ} defined by ξ intersects with $\mathbf{C}_X$. By boundedness, we have:

$$0 > -a^2 + 2ab - c^2 \geq -c_2 - 1 - \varepsilon$$

or equivalently:

$$c^2 > a(2b - a) \geq c^2 - c_2 - 1 - \varepsilon$$

doi:10.6342/NTU202301040

Note that $W^\xi \cap \mathbf{C}_X$ is determined solely by b/a (we don't need to consider case where $a = 0$, as in this case, the wall would have empty intersection with $\mathbf{C}_{\tilde{X}}$). If we can bound b/a from below we are done, but by dividing the above inequality by a^2, we get:

$$\frac{2b}{a} \geq 1 + \frac{c^2 - c_2 - 1 - \varepsilon}{a^2} \geq 1 - \frac{c_2 + 2}{1^2} = -(c_2 - 1)$$

For (4), it suffices to show for some polarization in it. We may use (3) to choose some $H = S + nF$ so that there are no walls between F, H. By (1), there are also no walls between H_r and H, hence no walls between H_r, F, meaning $H_r \in C_F$. By the recursion formulas 4.4 and 4.1, it suffices to see that $[\mathfrak{M}^\mu_H(c_1, i)] = 0$ for $i \leq c_2$, but this is true by 4.7 (note that the chamber C_F in that proposition shrinks when c_2 increases). For (5), note that C_F shrinks when c_2 increases, so one may use (4) along with 3.17. □

Remark. *It was mentioned in [LQ98] that a "well-known" analogous statement corresponding to 4.9 holds true for general rational ruled surfaces, but the author of this thesis wasn't able to follow the reference supplied there to obtain a proof of the desired result. Here we give a full proof in the basic case where X is $\mathbb{F}_1$ and use the universal relations deduced before.*

Assume that $\tilde{X}$ also satisfies (A3) (this way 3.12 can be applied), then corresponding to 4.8, we have the following proposition:

Proposition 4.10. *Suppose $\tilde{X}$ also satisfies (A3), and let $W(H)$ be defined as in 4.8, then we have:*

$$\mathbf{Z}^G_\varepsilon(q) = \frac{\mathbf{M}^G(\tilde{X}, H_\infty, \tilde{c}_{1,\varepsilon}; q)}{q^{1/12}\mathbf{M}^G(X, H, c_1; q)} = q^{-1/12}\mathrm{Exp}\left(\frac{2\mathbb{L}^2 q}{1 - \mathbb{L}^2 q}\right)\left(\sum_{n\in\mathbb{Z}} \mathbb{L}^{\frac{(2n+\varepsilon)^2 - (2n+\varepsilon)}{2}} q^{\frac{(2n+\varepsilon)^2}{4}}\right)$$

Proof. By using the same proof as in 4.8 paired with the structures mentioned in 4.9's, we

doi:10.6342/NTU202301040

see that if we choose big enough r with correct parity, and define:

$$\tilde{W}(H_r) := \{\eta \in S + \varepsilon E + 2\,\mathrm{Pic}(\tilde{X}) :\ \eta.H_r < 0 < \eta.F,\ \eta^2 < 0\}$$

then we get:

$$\mathbf{M}^{\mu}(\tilde{X}, H_\infty, \tilde{c}_{1,\varepsilon}; q) = q^{-\tilde{c}_{1,\varepsilon}^2/4}\mathbf{H}(\tilde{X}; \mathbb{L}q)^2 \left(\sum_{\eta \in \tilde{W}(H_r)} \left([\mathbb{P}^{\tilde{w}_0(\eta)}] - [\mathbb{P}^{\tilde{w}_0(\eta)}]\right)(\mathbb{L}q)^{-\tilde{l}_0(\eta)} \right)$$

where $\tilde{w}_0, \tilde{l}_0$ are the functions w_0, l_0 corresponding to $\tilde{X}$. By 4.9's item (1), if we start with a big enough r, we would have the identification:

$$\tilde{W}(H_r) = \{\eta \in S + \varepsilon E + 2\,\mathrm{Pic}(\tilde{X}) :\ \eta.H < 0 < \eta.F,\ \eta^2 < 0\}$$

Notice that if we regard $W(H)$ as a subset in $\mathrm{Pic}(\tilde{X})$ via the inclusion $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(\tilde{X})$, we have:

$$\tilde{W}(H_r) = \varepsilon E + (W(H) + 2\mathbb{Z}E)$$

so let us write $\eta_{\varepsilon,n} = \eta + (2n + \varepsilon)E$ for $\eta \in W(H)$. By the remark following 4.8, we have:

$$\mathbf{M}^{\mu}(X, H, c_1; q) = \frac{\mathbb{L}^{-(1+h^{0,2})}}{\mathbb{L} - 1}\mathbf{H}(\tilde{X}; \mathbb{L}q)^2 \left(\sum_{\eta \in W(H)} \sum_{n \in \mathbb{Z}} \left(\mathbb{L}^{\frac{\eta_{\varepsilon,n}.K_{\tilde{X}} - \eta_{\varepsilon,n}^2}{2}} - \mathbb{L}^{\frac{-\eta_{\varepsilon,n}.K_{\tilde{X}} - \eta_{\varepsilon,n}^2}{2}} \right) q^{-\eta_{\varepsilon,n}^2/4} \right)$$

where we have used the fact that $h^{0,2}$ does not change undre blowup. Now since:

$$a_{\eta,\varepsilon,n} := \eta_{\varepsilon,n}^2 - \eta^2 = -(2n + \varepsilon)^2$$

$$b_{\eta,\varepsilon,n} := \eta_{\varepsilon,n}.K_{\tilde{X}} - \eta.K_X = -(2n + \varepsilon)$$

$$c_{\eta,\varepsilon,n,\pm} := (\pm\eta_{\varepsilon,n}.K_{\tilde{X}} - \eta_{\varepsilon,n}^2) - (\pm\eta.K_X - \eta^2)$$

doi:10.6342/NTU202301040

(here we used the fact that $K_{\tilde{X}} = K_X + E$), then under the change $n \mapsto -n - \varepsilon$, we get:

$$a_{\eta,\varepsilon,n} = a_{\eta,\varepsilon,-n-\varepsilon},\ b_{\eta,\varepsilon,n} = -b_{\eta,\varepsilon,-n-\varepsilon}$$

$$c_{\eta,\varepsilon,n,\pm} = \pm b_{\eta,\varepsilon,n} - a_{\eta,\varepsilon,n} = \mp b_{\eta,\varepsilon,-n-\varepsilon} - a_{\eta,\varepsilon,-n-\varepsilon} = c_{\eta,\varepsilon,-n-\varepsilon,\mp}$$

This gives:

$$\begin{aligned}
&\left(\sum_{\eta\in W(H)}\sum_{n\in\mathbb{Z}}\left(\mathbb{L}^{\frac{\eta_{\varepsilon,n}.K_{\tilde{X}}-\eta_{\varepsilon,n}^2}{2}} - \mathbb{L}^{\frac{-\eta_{\varepsilon,n}.K_{\tilde{X}}-\eta_{\varepsilon,n}^2}{2}}\right) q^{-\eta_{\varepsilon,n}^2/4}\right)\\
=&\left(\sum_{\eta\in W(H)}\sum_{n\in\mathbb{Z}}\left(\mathbb{L}^{\frac{\eta.K_X-\eta^2+c_{\eta,\varepsilon,n,+}}{2}} - \mathbb{L}^{\frac{-\eta.K_X-\eta^2+c_{\eta,\varepsilon,n,-}}{2}}\right) q^{-(\eta^2+a_{\eta,\varepsilon,n})/4}\right)\\
=&\left(\sum_{\eta\in W(H)}\sum_{n\in\mathbb{Z}}\left(\mathbb{L}^{\frac{\eta.K_X-\eta^2+c_{\eta,\varepsilon,n,+}}{2}} - \mathbb{L}^{\frac{-\eta.K_X-\eta^2+c_{\eta,\varepsilon,-n-\varepsilon,-}}{2}}\right) q^{-(\eta^2+a_{\eta,\varepsilon,n})/4}\right)\\
=&\left(\sum_{\eta\in W(H)}\sum_{n\in\mathbb{Z}}\left(\mathbb{L}^{\frac{\eta.K_X-\eta^2+c_{\eta,\varepsilon,n,+}}{2}} - \mathbb{L}^{\frac{-\eta.K_X-\eta^2+c_{\eta,\varepsilon,n,+}}{2}}\right) q^{-(\eta^2+a_{\eta,\varepsilon,n})/4}\right)\\
=&\left(\sum_{\eta\in W(H)}\left(\mathbb{L}^{\frac{\eta.K_X-\eta^2}{2}} - \mathbb{L}^{\frac{-\eta.K_X-\eta^2/4}{2}}\right) q^{-\eta^2}\right)\left(\sum_{n\in\mathbb{Z}}\mathbb{L}^{\frac{c_{\eta,\varepsilon,n,+}}{2}} q^{-a_{\eta,\varepsilon,n}/4}\right)
\end{aligned}$$

The rest follows from 2.4, the observation that when $X = \mathbb{F}_1$ it satisfies all the requirements, and the explicit formula for $a_{\eta,\varepsilon,n}$, $c_{\eta,\varepsilon,n,+}$. □

4.3 Computation of Universal Functions (II) by Recursion

In this section, we compute the universal function of Mumford-Takemoto spaces via recursion.

doi:10.6342/NTU202301040

We first need to understand the space $U(p,q)$. As mentioned before, $U(p,q)$ parametrizes pairs of homogeneous polynomials $f(x_0,x_1), g(x_0,x_1)$ with degree p,q up to $\mathbb{C}^\times$ that are not simultaneously 0 (here we use the convention that the 0 polynomial can be regarded to have any degree).

Lemma 4.11. *Assume $0 \le p \le q$. We have the following:*

$$[U(p,q)] = \begin{cases} \mathbb{L}^{q+1} + \delta_{p,q} & \text{if } p = 0 \\ \mathbb{L}^{p+q+1} - \mathbb{L}^{p+q-1} & \text{otherwise} \end{cases}$$

Proof. For a pair (f,g) representing an element in $U(p,q)$, by looking at $\deg(\gcd(f,g))$ and whether $f=0$ or not, we get:

$$[\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(q)))] - [\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)))]$$
$$= \left(\sum_{i=0}^{p} [\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)))] \Big([U(p-i,q-i)] - \delta_{p-i,q-i}\delta_{p-i,0} \Big) \right)$$

which expands to:

$$[\mathbb{P}^{p+q+1}] - [\mathbb{P}^q] + \delta_{p,q}[\mathbb{P}^p] = \sum_{i=1}^{p} [\mathbb{P}^i][U(p-i,q-i)]$$

When $p = 0,1,2$, $[U(p,q)]$ has the correct form.

For $p > 2$, we may first use the base case $p = 0$ and reduce this equation to

$$[\mathbb{P}^{p+q+1}] - [\mathbb{P}^q] - [\mathbb{P}^p]\mathbb{L}^{q-p+1} = [U(p,q)] + \sum_{i=1}^{p-1} [\mathbb{P}^i][U(p-i,q-i)] \qquad (*)$$

We can rewrite the summation $\sum$ as

$$\sum_{i=1}^{p-1} [\mathbb{P}^i][U(p-i,q-i)] = (\mathbb{L}+1) \left(\left(\sum_{i=1}^{p-1} \mathbb{L}^{p+q-i} \right) - \left(\sum_{i=1}^{p-1} \mathbb{L}^{p+q-2i-1} \right) \right) \qquad (**)$$

doi:10.6342/NTU202301040

using:

$$[\mathbb{P}^i][U(p-i,q-i)] = [\mathbb{P}^i](\mathbb{L}^2-1)\mathbb{L}^{p+q-2i-1} = (\mathbb{L}^{i+1}-1)(\mathbb{L}+1)\mathbb{L}^{p+q-2i-1}$$

Now we take $\Delta_{p,q} = [U(p,q)] - [U(p-2,q)]$. By induction, we only need to show:

$$\Delta_{p,q} = \mathbb{L}^{p+q+1} - 2\mathbb{L}^{p+q-1} + \mathbb{L}^{p+q-3}$$

By $(*)$, $(**)$, we have:

$$\mathbb{L}^{p+q+1}+\mathbb{L}^{p+q}-\mathbb{L}^{q-p+1}-\mathbb{L}^{q-p+2} = \Delta_{p,q}+(\mathbb{L}+1)\left(\mathbb{L}^{p+q-1}+\mathbb{L}^{p+q-2}-\mathbb{L}^{q-p+1}-\mathbb{L}^{p+q-3}\right)$$

which then simplifies to the desired equality for $\Delta_{p,q}$. □

Remark. *The first equation given in the proof given above is given in [LQ98], but thereafter we found other shortcuts to prove 4.11; namely the fact that we can induct on the symbol* $\Delta_{p,q}$.

Remark. *More succinctly,* $[U(p,q)] = \mathbb{L}^{p+q+1}(1-\mathbb{L}^{-2})^{1-\delta_{p,0}}(1+\delta_{q,0}\mathbb{L}^{-1})$ *(assuming* $0 \le p \le q$*).*

doi:10.6342/NTU202301040

Now we have the appropriate setup for finding a recurrence relation. We have the following formulas:

$$[\mathfrak{M}^{\mu}_{H}(c_1, c_2)] = [\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, 0)] \tag{1}$$

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2)] = \sum_{d \geq 0} [\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2, d)] \tag{2}$$

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d)] = \sum_{l=0}^{d-1} [U(d-l-1, d+l)][\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)] \tag{3}$$

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2, d)] = \sum_{l=0}^{d} [U(d-l, d+l)][\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2 - d, l)] \tag{4}$$

$$[U(p,q)] = \begin{cases} \mathbb{L}^{q+1} + \delta_{p,q} & \text{if } p = 0 \\ \mathbb{L}^{p+q+1} - \mathbb{L}^{p+q-1} & \text{otherwise} \end{cases} \tag{5}$$

Using (1)-(4), we get:

$$\begin{aligned}
[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2, d)] &= \sum_{0 \leq l < d} [\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,1}, c_2 - d, l)][U(d-l-1, d+l)] \\
&= \sum_{0 \leq k \leq l < d} [\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2 - d - l, k)][U(d-l-1, d+l)][U(l-k, l+k)] \\
&= \left(\sum_{0 \leq l < d} [\mathfrak{M}^{\mu}_{H}(c_1, c_2 - d - l)][U(d-l-1, d+l)][U(l-k, l+k)] \right) + \\
&\quad \left(\sum_{0 < k \leq l < d} [\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2 - d - l, k)][U(d-l-1, d+l)][U(l-k, l+k)] \right)
\end{aligned}$$

where the first equality is first two equalities are by (3), (4), and the third equality is by (1). If we focus on the component d in the above equation, we would notice that after one such iteration the value of the component d decreased. We can apply the same iteration to the terms $[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,0}, c_2 - d - l, k)]$.

doi:10.6342/NTU202301040

From this, one can derive a general formula. It would be convenient to introduce the following definition:

Definition 4.12 (Strings)**.** *Given $d \in \mathbb{N}$, define $\mathcal{S}_{\varepsilon,d}$ to be the set of sequences $(a_0, a_1, a_2, \ldots, a_s)$ with $s \equiv \varepsilon (\text{mod}\, 2)$ of the form:*

$$0 = a_0 \leq a_1 < a_2 \leq a_3 < \ldots a_{2i-1} < a_{2i} \leq a_{2i+1} \ldots a_s = d$$

*We call elements in $\mathcal{S}_{\varepsilon,d}$ as **strings**. Associated to each string $\mathbf{a} = (a_i)_{i=0}^{s}$, we define the motive:*

$$U_{\mathbf{a}} := \prod_{i=1}^{s} [U(a_i - a_{i-1} - \nu(i), a_i + a_{i-1})]$$

where we define

$$\nu(i) = \begin{cases} 0 & \textit{if } i \equiv 1 (\text{mod}\, 2) \\ 1 & \textit{otherwise} \end{cases}$$

and in the degenerate case where $d = 0, \mathbf{a} = (0)$, we define $U_{\mathbf{a}} = 1$. Corresponding to each $\mathbf{a}$, we define:

- *The **length** $l(\mathbf{a})$ to be s*
- *The **weight** $\sigma(\mathbf{a})$ to be the sum $\sum_{i=1}^{s} a_i$*
- *The **degeneracy** $\tau(\mathbf{a})$ to be the count of i with $a_i - a_{i-1} = \nu(i)$*

Utilizing this definition, we get:

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2, d)] = \sum_{k=-\infty}^{c_2} \left(\sum_{\substack{\mathbf{a} \in \mathcal{S}_{\varepsilon,d} \\ \sigma(\mathbf{a}) = c_2 - k}} U_{\mathbf{a}} \right) [\mathfrak{M}^{\mu}_{H}(c_1, k)] = \sum_{k \geq 0} \left(\sum_{\substack{\mathbf{a} \in \mathcal{S}_{\varepsilon,d} \\ \sigma(\mathbf{a}) = k}} U_{\mathbf{a}} \right) [\mathfrak{M}^{\mu}_{H}(c_1, c_2 - k)]$$

doi:10.6342/NTU202301040

and hence:

$$[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2)] = \sum_{k\geq 0}\left(\sum_{d\geq 0}\sum_{\substack{\mathbf{a}\in\mathcal{S}_{\varepsilon,d}\\ \sigma(\mathbf{a})=k}} U_{\mathbf{a}}\right)[\mathfrak{M}^{\mu}_{H}(c_1, c_2-k)]$$

so we get:

$$\sum_{c_2}[\mathfrak{M}^{\mu}_{H_\infty}(\tilde{c}_{1,\varepsilon}, c_2)]q^{c_2} = \left(\sum_{k\geq 0}\left(\sum_{d\geq 0}\sum_{\substack{\mathbf{a}\in\mathcal{S}_{\varepsilon,d}\\ \sigma(\mathbf{a})=k}} U_{\mathbf{a}}\right)q^k\right)\cdot\left(\sum_{c_2}[\mathfrak{M}^{\mu}_{H}(c_1, c_2)]q^{c_2}\right)$$

hence we have (by using the fact that $\Delta(\tilde{c}_1, c_2) = \Delta(c_1, c_2) + \varepsilon/4$):

$$\mathbf{Z}^{\mu}_{\varepsilon}(q) = q^{\varepsilon/4}q^{-1/12}\left(\sum_{n\geq 0}\left(\sum_{d\geq 0}\sum_{\substack{\mathbf{a}\in\mathcal{S}_{\varepsilon,d}\\ \sigma(\mathbf{a})=n}} U_{\mathbf{a}}\right)q^n\right) \tag{6}$$

Therefore the calculation boils down to the calculation of $U_{\mathbf{a}}$. By Formula (5), we have:

$$U_{\mathbf{a}} = (1+\delta_{a_1,0}\mathbb{L}^{-1})\mathbb{L}^{2\sigma(\mathbf{a})+\lfloor l(\mathbf{a})/2\rfloor}(1-\mathbb{L}^{-2})^{l(\mathbf{a})-\tau(\mathbf{a})} \tag{7}$$

Equation (6), (7) readily gives a rather explicit formula for $\mathbf{Z}^{\mu}_{\varepsilon}(q)$. However, a bit more can be done. We follow [LQ98] to give a simplified (but also non-closed) formula for $\mathbf{Z}^{\mu}_{\varepsilon}(q)$.

Theorem 4.13. *We have:*

$$q^{1/12}\mathbf{Z}^{\mu}_{\varepsilon}(q) = \left(\sum_{s\geq 0}\mathbb{L}^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)}{2}}q^{\frac{(2s+\varepsilon)^2}{4}}\prod_{j=1}^{2s+a}\frac{1-\mathbb{L}^{2j-2}q^j}{1-\mathbb{L}^{2j}q^j}\right)+$$
$$\left(\sum_{s\geq 1-\varepsilon}\mathbb{L}^{\frac{(2s+\varepsilon)^2+(2s+\varepsilon)-2}{2}}q^{\frac{(2s+\varepsilon)^2}{4}}\prod_{j=1}^{2s+\varepsilon-1}\frac{1-\mathbb{L}^{2j-2}q^j}{1-\mathbb{L}^{2j}q^j}\right)$$

where empty products are interpreted as 1.

doi:10.6342/NTU202301040

Proof. We will just do the case $\varepsilon = 0$; the same proof will show the corresponding case for $\varepsilon = 1$. This amounts to showing:

$$\sum_{n\geq 0}\left(\sum_{d\geq 0}\sum_{\substack{\mathbf{a}\in\mathcal{S}_{0,d}\\ \sigma(\mathbf{a})=n}} U_{\mathbf{a}}\right) q^n = \left(\sum_{s\geq 0}\mathbb{L}^{2s^2+s}q^{s^2}\prod_{j=1}^{2s}\frac{1-\mathbb{L}^{2j-2}q^j}{1-\mathbb{L}^{2j}q^j}\right)+\left(\sum_{s\geq 1}\mathbb{L}^{2s^2+s-1}q^{s^2}\prod_{j=1}^{2s-1}\frac{1-\mathbb{L}^{2j-2}q^j}{1-\mathbb{L}^{2j}q^j}\right)$$

Firstly, we have:

$$\sum_{n\geq 0}\left(\sum_{d\geq 0}\sum_{\substack{\mathbf{a}\in\mathcal{S}_{0,d}\\ \sigma(\mathbf{a})=n}} U_{\mathbf{a}}\right) q^n = 1+\sum_{n\geq 1}\left(\sum_{\substack{\mathbf{a}\in\bigcup_{d\geq 0}\mathcal{S}_{0,d}\\ \sigma(\mathbf{a})=n}} U_{\mathbf{a}}\right) q^n$$

Take $n \geq 0, s \geq 0$. Given $\mathbf{a} \in \mathcal{S}_{0,n}$ with $l(\mathbf{a}) = 2s$, define $b_i = a_i - a_{i-1} - \nu(i)$. The terms a_i and invariants $\sigma(\mathbf{a}), \tau(\mathbf{a})$ can be written in terms of these b_i by:

$$a_i = \lfloor i/2\rfloor + \sum_{j=1}^{i} b_i,\ \sigma(\mathbf{a}) = s^2 + \sum_{i=1}^{2s}(2s+1-i)b_i,\ \tau(\mathbf{a}) = \sum_{i=1}^{n}\delta_{0,b_i}$$

From this, instead of summing over strings, we can sum over sequences of non-negative integers. Let $\mathbb{Z}_+^s$ be the set of all nonnegative integers $\mathbf{b} = (b_1, b_2, \ldots, b_s)$. By (7), we get:

$$\begin{aligned}\sum_{n\geq 1}\left(\sum_{\substack{\mathbf{a}\in\bigcup_{d\geq 0}\mathcal{S}_{0,d}\\ \sigma(\mathbf{a})=n}} U_{\mathbf{a}}\right) q^n &= \sum_{s\geq 1}\sum_{\mathbf{b}\in\mathbb{Z}_+^{2s}}\mathbb{L}^{2s^2+s}q^{s^2}(1+\delta_{0,b_1}\mathbb{L}^{-1})\left((1-\mathbb{L}^{-2})^{2s-\sum_i\delta_{0,b_i}}\right)(\mathbb{L}^2q)^{\sum_i(2s+1-i)b_i}\\ &= \sum_{s\geq 1}\sum_{\mathbf{b}\in\mathbb{Z}_+^{2s}}\mathbb{L}^{2s^2+s}q^{s^2}(1+\delta_{0,b_{2s}}\mathbb{L}^{-1})\left((1-\mathbb{L}^{-2})^{2s-\sum_i\delta_{0,b_i}}\right)(\mathbb{L}^2q)^{\sum_i ib_i}\end{aligned}$$

Given $\mathbf{b} \in \mathbb{Z}_+^{2s}$, we can consider its support $\mathrm{Supp}(\mathbf{b})$, given by the set of indices i with $b_i \neq 0$. For each s, write $[s] = \{1, 2, \ldots, s\}$. In this way, we can further arrange the terms

doi:10.6342/NTU202301040

above:

$$
\begin{aligned}
&=\sum_{s\geq 1}\sum_{J\subseteq[2s]}\sum_{\substack{\mathbf{b}\in\mathbb{Z}_+^{2s}\\ \mathrm{Supp}(\mathbf{b})=J}}\mathbb{L}^{2s^2+s}q^{s^2}(1+\delta_{0,b_{2s}}\mathbb{L}^{-1})\left((1-\mathbb{L}^{-2})^{|J|}\right)(\mathbb{L}^2q)^{\sum_{i\in J}ib_i}\\
&=\sum_{s\geq 1}\sum_{J\subseteq[2s]}\sum_{\substack{\mathbf{b}\in\mathbb{Z}_+^{2s}\\ \mathrm{Supp}(\mathbf{b})=J}}\mathbb{L}^{2s^2+s}q^{s^2}(1+\delta_{0,b_{2s}}\mathbb{L}^{-1})\left((1-\mathbb{L}^{-2})^{|J|}\right)\prod_{i\in J}\left(\frac{1}{1-(\mathbb{L}^2q)^i}-1\right)\\
&=\sum_{s\geq 1}\sum_{J\subseteq[2s]}\mathbb{L}^{2s^2+s}q^{s^2}\left((1-\mathbb{L}^{-2})^{|J|}\right)\prod_{i\in J}\left(\frac{1}{1-(\mathbb{L}^2q)^i}-1\right)\\
&\quad+\sum_{s\geq 1}\sum_{J\subseteq[2s-1]}\mathbb{L}^{2s^2+s-1}q^{s^2}\left((1-\mathbb{L}^{-2})^{|J|}\right)\prod_{i\in J}\left(\frac{1}{1-(\mathbb{L}^2q)^i}-1\right)\\
&=\sum_{s\geq 1}\mathbb{L}^{2s^2+s}q^{s^2}\prod_{i=1}^{2s}\left(1+(1-\mathbb{L}^{-2})\left(\frac{1}{1-(\mathbb{L}^2q)^i}-1\right)\right)\\
&\quad+\sum_{s\geq 1}\mathbb{L}^{2s^2+s-1}q^{s^2}\prod_{i=1}^{2s-1}\left(1+(1-\mathbb{L}^{-2})\left(\frac{1}{1-(\mathbb{L}^2q)^i}-1\right)\right)
\end{aligned}
$$

which expands to the desired formula. □

Remark. *The idea of the proof given above is the same as the one in [LQ98]. The difference here is that we have introduced 4.12 and systematically collected formula (1)-(7) to simplify the presentation of the proofs and worked in the motivic context.*

doi:10.6342/NTU202301040

doi:10.6342/NTU202301040

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doi:10.6342/NTU202301040

Chapter 5 Appendix. A Survey on Related Results

In this chapter, we will give a survey on the historical contexts of some of the results in [LQ99, LQ98] and this paper.

In [VW94], Vafa and Witten predicted that as a result of the S-duality conjecture, the change of the generating series of the Euler characterisitcs moduli space of μ-stable rank 2 sheaves on X should be able to be given by some modular functions (modularity of the function) independent of the choice of surface (universality of the function). For example, if we specialize the function $\mathbf{Z}_{\varepsilon}^{G}(q) \mid_{\mathbb{L}\mapsto 1}$ in 1.5, we would have:

$$\frac{\sum_{n\in\mathbb{Z}} q^{(n+\varepsilon/2)^2}}{\left[q^{1/24}\prod_{n\geq 1}(1-q^n)\right]^2} = \frac{\theta_{\varepsilon}(q)}{\eta(q)^2}$$

where $\eta(q)$ is Dedekind's η-function, and θ_{ε} can be interpreted as θ-constants (when $\varepsilon = 0$, it is the Jacobi θ-function).

The modularity part is true in the case $X = \mathbb{P}^2$ according to [Yos94]. In the case where X is ruled with smooth moduli spaces, this is also true by [Yos96]. Nevertheless, in both papers, Yoshioka calculated the universal functions at the level of Hodge polynomials,

doi:10.6342/NTU202301040

and for the spaces $\mathfrak{M}^{\mu}_H(c_1, c_2), \mathfrak{M}^G_H(c_1, c_2)$. His method involves the Weil conjectures, which required the moduli spaces to be smooth. However, the method of using elementary modifications and recursions to calculate the blowup formula for $\mathfrak{M}^{\mu}_H(c_1, c_2)$, as well as the method of finding the Quot formula (over finite fields) and utilizing the chamber and wall structures to calculate the blowup formula for $\mathfrak{M}^G_H(c_1, c_2)$ are both present in these two papers.

In [LQ99, LQ98], they made the calculations for virtual Hodge polynomials, and significantly loosened up the constraints that the surface X should have, and considered in additional to $\mathfrak{M}^{\mu}_H(c_1, c_2), \mathfrak{M}^G_H(c_1, c_2)$ the spaces $\mathfrak{M}^U_H(c_1, c_2)$. They generalized the calculations by Yoshioka. They observed that the relation between the generating series of virtual Hodge of $\mathfrak{M}^{\mu}_H(c_1, c_2)$ and $\mathfrak{M}^G_H(c_1, c_2)$ can be given by a universal function involving Quot schemes, but the explicit form of it was not known.

Nakajima and Yoshioka proved the blowup formula of virtual Hodge of $\mathfrak{M}^G_H(c_1, c_2)$ in a different way similar to Yoshioka's calculation for $\mathbb{P}^2$, and remarked that their proofs works for motives in [NY03]. They considered other more general moduli spaces, such as framed ones.

Recently, categorifications of these results (at the level of derived categories) for $\mathfrak{M}^G_H(c_1, c_2)$ has been done by [Kos21] using Qing Yuan's Quot formula as proved in [Tod21].

The following table gives a summary.

doi:10.6342/NTU202301040

	References	Invariants	Moduli
Vafa, Witten	[VW94]	Euler Characteristics	μ
Yoshioka	[Yos94, Yos96]	Hodge (special cases)	μ
Li, Qin	[LQ99, LQ98]	Virtual Hodge	μ, G, U
Naka., Yos.	[NY03]	Motives	G
Toda, Koseki	[Tod21, Kos21]	Derived Categories	G

doi:10.6342/NTU202301040