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波茲曼方程及其邊界奇點綜述

A review on the Boltzmann equation and its boundary singularities.

馮晉知

Jin-Zhi Phoong

指導教授: 夏俊雄 教授

Advisor: Prof. Chun-Hsiung Hsia

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波茲曼方程及其邊界奇點綜述

A review on the Boltzmann equation and its boundary singularities.

本論文係馮晉知君（R08221015）在國立臺灣大學數學系完成之碩（博）士學位論文，於民國 111 年 06 月 08 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

夏俊雄（簽名）

（指導教授）

陳逸昆　　吳恭儉

系主任、所長 ______________（簽名）

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摘要

波茲曼方程是描述熱力學系統演化的重要數學模型。數學家已將其廣泛應用在各個科學領域，並對其進行了許多的研究。在本文中，我們回顧了波茲曼方程碰撞算子的性質。我們主要關注於 Milne 和 Kramers 問題的解的存在性，唯一性和漸近行為。我們也介紹近期線性波茲曼方程的宏觀變量的邊界奇點方面的工作。

關鍵字：偏微分方程、波茲曼方程、邊界奇點

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Abstract

The Boltzmann Equation is an important mathematical model that describes the evolution of a thermodynamic system. It has been studied and applied in various scientific areas. In this article, we review the properties of collision operator of Boltzmann equation. We will focus mainly on the well-posedness and the asymptotic behaviour of the Milne and Kramers problem as well as the recent work in the boundary singularity of macroscopic variables for linearized Boltzmann equation.

Keywords: Partial Differential Equations, Boltzmann Equation, Boundary Singularities

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Contents

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Chapter 1 Introduction

The Boltzmann equation provides a characterization of the motion of a collection of molecules. In 1872, Boltzmann introduced the fundamental equation of the kinetic theory of gases [3]. Thousands of works and analysis on this equation had been done over decades. As of our interest, we consider the linearized Boltzmann equation satisfies the angular cutoff assumption. Some related progress on linearized Boltzmann equation can be referred to [1, 9, 10, 12, 14, 15, 19]. Applications and analysis of the Boltzmann equation can be found in the books of Cercignani [4, 5]. We will not go through the physical interpretation in this review. In 2015, Chen and Hsia [6] [7] has established an asymptotic formula for the gradient of the moments of solution to the stationary Boltzmann equation for both hard-sphere potential and hard potential cases. They showed that the logarithmic singularity occurs near the boundary for macroscopic variables. Later, in 2020, Huang [11] extended the result to the soft potential case ($-\frac{3}{2} < \gamma < 0$). The challenges of the remaining cases is that the technique developed by Chen and Hsia is no longer applicable for soft potential cases as the L^2 norm can no longer be controlled by the weighted L^2 norm anymore, which is employed frequently for hard potential case ($0 \leq \gamma \leq 1$). By introducing some special functions, Huang is able to extend the result to ($-\frac{3}{2} < \gamma < 0$), however, the downside is that we cannot obtain a better estimate with Huang's technique.

doi:10.6342/NTU202202294

Now, we shall begin with the classical Boltzmann equation

$$\partial_t F + \xi \cdot \nabla_x F = Q(F, F), \tag{1.1}$$

where $F = F(t, x, \xi)$ is the distribution function of the gas particle at time $t \geq 0$, $x = (x, y, z) \in \Omega$ the position and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ is the velocity, where $\Omega \subset \mathbb{R}^3$. The functional Q is called a Boltzmann collision operator which depends only on the velocity ξ, and it takes the form

$$Q(F, G)(\xi) = \int_{S^2 \times \mathbb{R}^3} [F(\xi'_*)G(\xi') - F(\xi_*)G(\xi)] B d\omega d\xi_*, \tag{1.2}$$

where $B = B(|\xi_* - \xi|, |\cos\theta|)$ is the collision kernel. It is difficult to write down the explicit form of the collision kernel B. Grad suggested to make the following assumptions:

$$B(|\xi_* - \xi|, |\cos\theta|) = |\xi_* - \xi|^\gamma B(|\cos\theta|), \tag{1.3}$$

where $\gamma \in (-3, 1]$ and it satisfies the angular cutoff assumption

$$0 < B(|\cos\theta|) \leq C|\cos\theta|, \tag{1.4}$$

for all $0 \leq \theta \leq \pi$ and some constant $C > 0$.

Next, we consider linearized Boltzmann equation which the linearization is performed around the Maxwellian

$$M(\xi) = (2\pi)^{-\frac{3}{2}} e^{-\frac{1}{2}|\xi|^2}. \tag{1.5}$$

Substituting $F = M + M^{\frac{1}{2}} f$ into (1.1) and dropping the higher order terms of f gives the

doi:10.6342/NTU202202294

linearized Boltzmann equation

$$\partial_t f + \xi \cdot \nabla_x f + L(f) = 0, \tag{1.6}$$

where

$$L(f) = -M^{-\frac{1}{2}}(Q(M, M^{\frac{1}{2}} f) + Q(M^{\frac{1}{2}} f, M)). \tag{1.7}$$

Let us introduce the properties of the operator L. L acts on the function f which depends only on ξ, it is self-adjoint and non-negative. It has the domain $D(L) = \{f : (1+|\xi|)f \in L^2\}$, while the null space $N(L)$ of L is spanned by the following functions:

$$\psi_0 = M^{\frac{1}{2}}(\xi), \tag{1.8}$$

$$\psi_i = \xi_i M^{\frac{1}{2}}(\xi), \qquad \text{for } i = 1, 2, 3, \tag{1.9}$$

$$\psi_4 = |\xi|^2 M^{\frac{1}{2}}(\xi). \tag{1.10}$$

Moreover, its range $R(L)$ is equal to $N(L)^{\perp}$, the orthogonal complement of its null space. Also, we can write

$$L = \nu(\xi) + K, \tag{1.11}$$

where ν is a function of ξ only and K is a compact integral operator. A well-known fact about the function ν is it satisfies

$$\nu(\xi) = \nu(|\xi|), \tag{1.12}$$

doi:10.6342/NTU202202294

and

$$\nu_0(1+|\xi|)^\gamma \leq \nu(\xi) \leq \nu_1(1+|\xi|)^\gamma, \tag{1.13}$$

for some positive constant $0 < \nu_0 < \nu_1$ and for $-3 < \gamma \leq 1$. The following are the definitions for different value of γ

$$\begin{cases} \gamma = 1 & \text{, Hard-sphere model,} \\ 0 < \gamma < 1 & \text{, Hard potential model,} \\ \gamma = 0 & \text{, Maxwellian model,} \\ -3 < \gamma < 0 & \text{, Soft potential model.} \end{cases}$$

It is also useful to introduce the following fluid dynamic moments of f:

- Density

$$\rho = \int M^{\frac{1}{2}} f \, d\xi. \tag{1.14}$$

- Velocity

$$v_i = \int \xi_i M^{\frac{1}{2}} f \, d\xi. \tag{1.15}$$

- Temperature

$$T = \frac{1}{3} \int (\xi^2 - 3) M^{\frac{1}{2}} f \, d\xi. \tag{1.16}$$

Notice that, the density of Maxwellian is normalized to 1, so the mass flux and the velocity are equal, that is, the mass flux in the x-direction has the formula

$$m_f = \int \xi_1 M^{\frac{1}{2}} f \, d\xi. \tag{1.17}$$

In fact, m_f is constant in x which will be shown in the next chapter.

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Here, we recaptilate the mathematical setting of Bardos, Caflisch and Nicolaenko [2]. One can decompose the distribution function f into two parts, namely, the fluid dynamic part $q \in N(L)$ and the nonfluid dynamic part $w \in N(L)^{\perp}$, that is,

$$f = w + q, \tag{1.18}$$

$$q = (a + b_1\xi_1 + b_2\xi_2 + b_3\xi_3 + c\xi^2)M^{\frac{1}{2}}(\xi), \tag{1.19}$$

where a, b_1, b_2, b_3, c depend only on x. Furthermore, for any $f, g \in D(L)$,

$$\int fLf\, d\xi \geq \nu_0 \int (1 + |\xi|)w_f^2\, d\xi, \tag{1.20}$$

$$\int |fLg|\, d\xi \leq \nu_1 \int (1 + |\xi|)|w_f w_g|\, d\xi, \tag{1.21}$$

where the w_f and w_g are the nonfluid dynamic part of f and g respectively.

It is convenient to restrict the Boltzmann equation to the domain

$$\mathbb{R}^3_N = \{\xi : |\xi_1| > N^{-1}, |\xi| < N\}. \tag{1.22}$$

and define the following functions:

$$\chi_N = \begin{cases} 1 & \text{if } |\xi_1| > N^{-1} \text{ and } |\xi| < N, \\ 0 & \text{otherwise,} \end{cases} \tag{1.23}$$

$$\psi_{iN} = \chi_N \psi_i, \tag{1.24}$$

doi:10.6342/NTU202202294

$$P_N = \text{ Projection onto } \{\psi_{iN},\ i = 0, ..., 4\}. \tag{1.25}$$

Then the cutoff collision operator can be defined readily:

$$L_N = \chi_N(I - P_N)L(I - P_N)\chi_N = \nu\chi_N + K_N. \tag{1.26}$$

Notice that, L_N shares the same properties which mentioned above as L. Besides that, K_N is compact on $L^2(\mathbb{R}^3_N)$ and the null space $N(L_N)$ is spanned by ψ_{iN}, $i = 0, ..., 4$. Similarly, one can decompose $f \in L^2(\mathbb{R}^3_N)$ as $f = w + q$ with $w \in N(L_N)^\perp$ and $q \in N(L_N)$, where

$$q = \chi_N(a + b_1\xi_1 + b_2\xi_2 + b_3\xi_3 + c\xi^2)M^{\frac{1}{2}}(\xi). \tag{1.27}$$

Also, we have the following properties:

$$\int_{\mathbb{R}^3_N} fL_N f\, d\xi \geq \nu_0 \int_{\mathbb{R}^3_N} (1 + |\xi|)w_f^2\, d\xi, \tag{1.28}$$

$$\int_{\mathbb{R}^3_N} |fL_N g|\, d\xi \leq \nu_1 \int_{\mathbb{R}^3_N} (1 + |\xi|)|w_f w_g|\, d\xi, \tag{1.29}$$

for any $f, g \in L^2(\mathbb{R}^3_N)$ with w_f and w_g correspond to f and g, respectively.

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Chapter 2 The Milne Problem

The Milne problem is to solve stationary linearized Boltzmann equation for $0 \leq x < \infty$, with a given distribution g at $x = 0$ and f is required to be bounded at $x = \infty$. In [2], the Bardos, Caflisch and Nicolaenko has shown that this problem is well-posed if the average mass flux m_f as defined in the previous chapter is specified. The proof for the existence and uniqueness of the solution will be shown in this chapter.

We say distribution function f solves the Milne problem if it satisfies

$$\xi_1 \frac{\partial}{\partial x} f + Lf = 0, \text{ for } 0 < x < \infty, \tag{2.1}$$

$$f = g, \text{ for } x = 0, \xi_1 > 0, \tag{2.2}$$

$$\int \xi_1 M^{\frac{1}{2}} f \, d\xi = m_f, \tag{2.3}$$

Also, define the solution space $D = \{f : (1+|\xi|)^{\frac{\gamma}{2}} f \in L^\infty(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3)), f_x \in L^2_{loc}(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3))\}$, and assume that the function g satisfies

$$\int_{\xi_1 > 0} (1 + |\xi|)^\gamma g^2 \, d\xi = \kappa_g < \infty. \tag{2.4}$$

doi:10.6342/NTU202202294

Bardos, Calflisch and Nicolaenko have shown the following theorem in 1986 for the hard sphere model ($\gamma = 1$). [2]

Theorem 2.1. *(Existence) [2] Let* $\gamma = 1$. *For any* $m_f \in \mathbb{R}$ *and any* g *satisfying (2.4), there exists a solution* $f \in D$ *for the Milne problem (2.1)-(2.3).*

Theorem 2.2. *(Uniqueness) [2] Let* $\gamma = 1$. *For a specified* $m_f \in \mathbb{R}$ *and a given* g *satisfying (2.4), there is a unique solution* $f \in D$ *for the Milne problem (2.1)-(2.3).*

Theorem 2.3. *(Orthogonality and asymptotic properties) [2] Let* $\gamma = 1$. *Suppose* $f \in D$ *satisfies (2.1)-(2.3) with* $m_f \in \mathbb{R}$ *and* g *satisfying (2.4). Write* $f = w + q$ *as in chapter 1. Then we have the following properties:*

Orthogonality:

$$\frac{d}{dx}\int \xi_1 M^{\frac{1}{2}} f \, d\xi = 0, \tag{2.5}$$

$$\frac{d}{dx}\int \xi_1 \xi_i M^{\frac{1}{2}} f \, d\xi = 0, \quad i = 1, 2, 3, \tag{2.6}$$

$$\frac{d}{dx}\int \xi_1 \xi^2 M^{\frac{1}{2}} f \, d\xi = 0, \tag{2.7}$$

the first one implies that $b_1 = m_f$, *for all* x.

Asymptotics:

$$\lim_{x\to\infty} q = (a_\infty + m_f\xi_1 + b_{2\infty}\xi_2 + b_{3\infty}\xi_3 + c_\infty\xi^2)M^{\frac{1}{2}}, \tag{2.8}$$

exists, with

$$|a_\infty| + |b_{2\infty}| + |b_{3\infty}| + |c_\infty| < \kappa, \tag{2.9}$$

doi:10.6342/NTU202202294

$$\int (1+|\xi|)w^2 \, d\xi + |a-a_\infty|^2 + |b_2 - b_{2\infty}|^2 + |b_3 - b_{3\infty}|^2 + |c-c_\infty|^2 \le \kappa(\nu_0-\alpha)^{-2} e^{2\alpha x}, \tag{2.10}$$

for any $0 < \alpha < \nu_0$ *in which* $\kappa = \kappa(\kappa_g, m_f)$*. Furthermore, for any* $\delta > 0$ *and any* $0 < \alpha < \nu_0$*,*

$$\int_\delta^\infty \int (1+|\xi|) e^{2\alpha x} f_x^2 \, d\xi dx \le \kappa_\delta(\nu_0-\alpha)^{-3}, \tag{2.11}$$

where $\kappa_\delta = \kappa_\delta(\kappa_g, \delta, m_f)$*.*

We shall begin with the proof of Theorem 2.3 first. Following the argument by Bardos, Caflisch and Nicolaenko [2], the proof is divided into four parts:

Proof. (a) Let $f \in D$ be the solution of the Milne problem (2.1)-(2.3). We multiply the stationary linearized Boltzmann equation (2.1) by $\psi_i \in N(L)$ and integrate both sides with respect to ξ, we obtained

$$\begin{aligned} \int \psi_i \xi_1 \frac{\partial}{\partial x} f \, d\xi + \int \psi_i L f \, d\xi &= 0 \\ \frac{d}{dx} \int \psi_i \xi_1 f \, d\xi + \int (L\psi_i) f \, d\xi &= 0 \quad (\because L \text{ is self-adjoint}) \\ \frac{d}{dx} \int \psi_i \xi_1 f \, d\xi &= 0, \quad (\because \psi_i \in N(L)) \end{aligned} \tag{2.12}$$

for $i = 0, 1, 2, 3, 4$. Thus, (2.5)-(2.7) follow immediately. To see the identity $b_1 = m_f$, we observe that

$$\int \xi_1 M \, d\xi = \int \xi_1 \xi_i M \, d\xi = 0,$$

for $i = 2, 3, 4$ because we are integrating an odd function over a whole line. Then,

doi:10.6342/NTU202202294

by using spherical coordinates, we have that

$$\begin{aligned}
m_f &= \int \psi_1 f \, d\xi \\
&= \int \psi_1 q \, d\xi \\
&= a \int \psi_1 \psi_0 \, d\xi + b_1 \int \psi_1^2 \, d\xi + b_2 \int \psi_1 \psi_2 \, d\xi + b_3 \int \psi_1 \psi_3 \, d\xi + c \int \psi_1 \psi_4 \, d\xi \\
&= a \int \xi_1 M \, d\xi + b_1 \int \xi_1^2 M \, d\xi + b_2 \int \xi_1 \xi_2 M \, d\xi + b_3 \int \xi_1 \xi_3 M \, d\xi + c \int \xi_1 \xi^2 M \, d\xi \\
&= b_1 \int \xi_1^2 M \, d\xi \\
&= b_1.
\end{aligned}$$

Next, we subtract $m_f \xi_1 M^{\frac{1}{2}}$ from f and define

$$\tilde{f} := f - m_f \xi_1 M^{\frac{1}{2}} = w + \tilde{q}, \tag{2.13}$$

where

$$w = w_f = w_{\tilde{f}}, \tag{2.14}$$

$$\begin{aligned}
\tilde{q} = q_{\tilde{f}} &= q_f - m_f \xi_1 M^{\frac{1}{2}} \\
&= (a + b_2 \xi_2 + b_3 \xi_3 + c \xi^2) M^{\frac{1}{2}}.
\end{aligned} \tag{2.15}$$

Since m_f is a constant and $\xi_1 M^{\frac{1}{2}} = \psi_1 \in N(L)$, direct computation shows that $\tilde{f}$ solves the Milne problem as well:

$$\xi_1 \frac{\partial}{\partial x} \tilde{f} + L \tilde{f} = 0, \quad x > 0, \tag{2.16}$$

$$\tilde{f} = \tilde{g}, \quad x = 0, \quad \xi_1 > 0, \tag{2.17}$$

doi:10.6342/NTU202202294

where $\tilde{g} = g - m_f \xi_1 M^{\frac{1}{2}}$. Furthermore,

$$\int (1+|\xi|)\tilde{f}^2 \, d\xi \leq 2\Big[\int (1+|\xi|) f^2 \, d\xi + \int (1+|\xi|) m_f^2 \xi_1^2 M \, d\xi \Big]. \tag{2.18}$$

Clearly, $\tilde{f} \in D$. From (2.15), we see that

$$\int \xi_1 \psi_i \tilde{q} \, d\xi = 0, \quad (\because \text{ integrands are odd functions}) \tag{2.19}$$

for $i = 0, 2, 3, 4$. Immediately we have,

$$\int \xi_1 \tilde{q}^2 \, d\xi = 0. \tag{2.20}$$

By (2.12),

$$\frac{d}{dx} \int \psi_i \xi_1 w \, d\xi = 0, \tag{2.21}$$

for $i = 0, 2, 3, 4$. To proceed, we multiply (2.16) by $\tilde{f}$ and integrate the equation with respect to ξ. Then, by (1.20)

$$\begin{aligned} 0 &= \int \xi_1 \tilde{f} \frac{\partial}{\partial x} \tilde{f} + \tilde{f} L \tilde{f} \, d\xi \\ &= \frac{1}{2} \frac{d}{dx} \int \xi_1 \tilde{f}^2 \, d\xi + \int \tilde{f} L \tilde{f} \, d\xi \\ &\geq \frac{1}{2} \frac{d}{dx} \int \xi_1 \tilde{f}^2 \, d\xi + \nu_0 \int (1+|\xi|) w^2 \, d\xi. \end{aligned} \tag{2.22}$$

Since $\tilde{f} \in D$ implies $\int \xi_1 \tilde{f}^2 \, d\xi \in L^\infty(\mathbb{R}_x^+)$, integrating (2.22) with respect to x over 0 to ∞, we get

$$\frac{1}{2} \xi_1 \tilde{f}^2(x=\infty) - \frac{1}{2} \xi_1 \tilde{f}^2(x=0) + \nu_0 \int_0^\infty \int (1+|\xi|) w^2 \, d\xi dx \leq 0. \tag{2.23}$$

doi:10.6342/NTU202202294

This implies,

$$\int_0^\infty \int (1+|\xi|)w^2 \, d\xi dx \leq \infty \implies \int (1+|\xi|)w^2 \, d\xi \in L^1(\mathbb{R}_x^+). \tag{2.24}$$

Moreover,

$$\begin{aligned} |\int \xi_1 \psi_i w \, d\xi| &\leq (\int \xi_1 w^2 \, d\xi)^{\frac{1}{2}} (\int \xi_1 \psi_i^2 \, d\xi)^{\frac{1}{2}} \\ \implies \int_0^\infty |\int \xi_1 \psi_i w \, d\xi|^2 \, dx &\leq \infty \\ \implies \int \xi_1 \psi_i w \, d\xi &= 0 \quad \text{(by (2.21))} \\ \implies \int \xi_1 \tilde{q} w \, d\xi &= 0. \end{aligned} \tag{2.25}$$

Combining the results above, we have that

$$\int \xi_1 \tilde{f}^2 \, d\xi = \int \xi_1 w^2 \, d\xi. \tag{2.26}$$

(b) Continuing from (2.26), (2.22) can be reduced to

$$\frac{1}{2}\frac{d}{dx}\int \xi_1 w^2 \, d\xi + \nu_0 \int (1+|\xi|)w^2 \, d\xi \leq 0 \implies \frac{d}{dx}\int \xi_1 w^2 \, d\xi \leq 0. \tag{2.27}$$

This shows that $\int \xi_1 w^2 \, d\xi$ is decreasing and since it is in $L^1(\mathbb{R}_x^+)$, it must be converging to zero. Indeed, if the integral converges to some nontrivial constant, namely c, then

$$\iint \xi_1 w^2 \, d\xi dx \geq \int c \, dx = \infty. \tag{2.28}$$

doi:10.6342/NTU202202294

Further,

$$
\begin{aligned}
0 &\le \int \xi_1 w^2 \, d\xi \\
&\le \int \xi_1 w(x=0)^2 \, d\xi \\
&\le \int \xi_1 \tilde{f}(x=0)^2 \, d\xi \\
&\le \int_{\xi_1>0} \xi_1 \tilde{g}^2 \, d\xi \\
&\le 2(\int_{\xi_1>0} \xi_1 g^2 \, d\xi + m_f^2 \int_{\xi_1>0} \xi_1^3 M \, d\xi) \\
&\le C(\kappa_g + m_f^2) := \kappa.
\end{aligned} \tag{2.29}
$$

Next, multiply (2.16) by $e^{\alpha x}$ to obtain

$$
\begin{aligned}
0 &= \xi_1 \frac{\partial}{\partial x}[e^{\alpha x}\tilde{f}] - \xi_1 \alpha e^{\alpha x}\tilde{f} + L[e^{\alpha x}\tilde{f}] \\
&= \xi_1 e^{\alpha x}\tilde{f}\frac{\partial}{\partial x}[e^{\alpha x}\tilde{f}] - \xi_1 \alpha e^{2\alpha x}\tilde{f}^2 + e^{\alpha x}\tilde{f}L[e^{\alpha x}\tilde{f}] \\
&= \int \xi_1 e^{\alpha x}\tilde{f}\frac{\partial}{\partial x}[e^{\alpha x}\tilde{f}] \, d\xi - \int \xi_1 \alpha e^{2\alpha x}\tilde{f}^2 \, d\xi + \int e^{\alpha x}\tilde{f}L[e^{\alpha x}\tilde{f}] \, d\xi \\
&\ge \frac{1}{2}\frac{d}{dx}\int \xi_1 e^{2\alpha x}\tilde{f}^2 \, d\xi - \alpha e^{2\alpha x}\int \xi_1 w^2 \, d\xi + e^{2\alpha x}\int \nu_0(1+|\xi|)w^2 \, d\xi \\
&= \frac{1}{2}\frac{d}{dx}\int \xi_1 e^{2\alpha x} w^2 \, d\xi - e^{2\alpha x}\int (-\alpha\xi_1 + \nu_0(1+|\xi|))w^2 \, d\xi.
\end{aligned} \tag{2.30}
$$

Now observe that, if $0 < \alpha < \nu_0$

$$
\begin{aligned}
-\alpha\xi_1 + \nu_0(1+|\xi|) &\ge -\alpha|\xi| + \nu_0(1+|\xi|) \\
&\ge -\alpha(1+|\xi|) + \nu_0(1+|\xi|) \\
&\ge (\nu_0 - \alpha)(1+|\xi|) \\
&\ge 0.
\end{aligned}
$$

This implies that

$$
\frac{d}{dx}\int \xi_1 e^{2\alpha x} w^2 \, d\xi \le 0.
$$

That is, $\int \xi_1 e^{2\alpha x} w^2 \, d\xi$ is decreasing and nonnegative for all x, we can integrate

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(2.30) with respect to x over 0 to ∞.

$$
\begin{aligned}
-\frac{1}{2}e^{2\alpha x}\int \xi_1 w(x=0)^2\, d\xi + \int_0^\infty \int e^{2\alpha x}(-\alpha\xi_1 + \nu_0(1+|\xi|))w^2\, d\xi dx \leq 0 \\
\int_0^\infty \int e^{2\alpha x}(\nu_0 - \alpha)(1+|\xi|)w^2\, d\xi dx \leq \kappa \\
\int_0^\infty \int e^{2\alpha x}(1+|\xi|)w^2\, d\xi dx \leq (\nu_0-\alpha)^{-1}\kappa,
\end{aligned}
\tag{2.31}
$$

where $\kappa = \kappa(\kappa_g, m_f)$.

(c) We shall deduce the integrated decay of w_x first and then use it to obtain the point-wise decay of w. Notice that, $f_x = \tilde{f}_x$ and since $K\tilde{f}_x \in L^2_{loc}(\mathbb{R}^+_x, L^2(\mathbb{R}^3_\xi))$, we have that $(\xi_1(\frac{\partial}{\partial x}) + v)\tilde{f}_x \in L^2_{loc}(\mathbb{R}^+_x, L^2(\mathbb{R}^3_\xi))$ and therefore $\tilde{f}_x$ satisfies the equation

$$
\xi_1 \frac{\partial}{\partial x}\tilde{f}_x + L\tilde{f}_x = 0. \tag{2.32}
$$

Since we do not have any information about the boundary conditions for f_x, it can be remedied by introducing a cutoff function $\phi(x) \in C^\infty(\mathbb{R}^+ \cup \{0\})$ defined as

$$
\phi(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq \frac{1}{3}Y, \\ 1, & \text{for } Y \leq x < X, \\ 0, & \text{for } X+1 \leq x < \infty. \end{cases} \tag{2.33}
$$

Then, we observe the following, let $g(x) = e^{\alpha x}\phi(x)$.

$$
\begin{aligned}
\xi_1(g\tilde{f})_{xx} + L(g\tilde{f})_x &= \xi_1 g_{xx}\tilde{f} + 2\xi_1 g_x \tilde{f}_x + \xi_1 g\tilde{f}_{xx} + g_x L\tilde{f} + gL\tilde{f}_x \\
&= \xi_1 g_{xx}\tilde{f} - 2g_x L\tilde{f} - gL\tilde{f}_x + g_x L\tilde{f} + gL\tilde{f}_x \\
&= \xi_1 g_{xx}\tilde{f} - g_x L\tilde{f}.
\end{aligned}
$$

Next, we multiply the equation above by $(\phi e^{\alpha x}\tilde{f})_x$ and integrate with respect to ξ,

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we see that

$$L.H.S. = \frac{1}{2}\frac{d}{dx}\int \xi_1 (g\tilde{f})_x^2\, d\xi + \int (g\tilde{f})_x L (g\tilde{f})_x\, d\xi$$
$$\geq \frac{1}{2}\frac{d}{dx}\int \xi_1 (g\tilde{f})_x^2\, d\xi + \nu_0 \int (1+|\xi|)[(gw)_x]^2\, d\xi \tag{2.34}$$
$$\geq \frac{1}{2}\frac{d}{dx}\int \xi_1 (g\tilde{f})_x^2\, d\xi + (\nu_0-\alpha) \int (1+|\xi|)[(gw)_x]^2\, d\xi,$$

while for the right hand side,

$$R.H.S. = \int (g\tilde{f})_x [\xi_1 g_{xx}\tilde{f} - g_x L\tilde{f}]\, d\xi$$
$$= \int (gw)_x [\xi_1 g_{xx} w - g_x L w]\, d\xi \tag{2.35}$$
$$\leq \kappa (\int (1+|\xi|)(gw)_x^2\, d\xi)^{\frac{1}{2}} (\int (1+|\xi|) e^{2\alpha x} w^2\, d\xi)^{\frac{1}{2}}.$$

Integrate both sides with respect to x and using Hölder's inequality, we get

$$(\nu_0-\alpha)\int_0^\infty \int_0^\infty (1+|\xi|)(gw)_x^2\, d\xi dx$$
$$\leq \kappa \int_0^\infty \left(\int_0^\infty (1+|\xi|)(gw)_x^2\, d\xi\right)^{\frac{1}{2}} \cdot \left(\int_0^\infty (1+|\xi|) e^{2\alpha x} w^2\, d\xi\right)^{\frac{1}{2}} dx$$
$$\leq \kappa \left(\int_0^\infty \int_0^\infty (1+|\xi|)(gw)_x^2\, d\xi dx\right)^{\frac{1}{2}} \cdot \left(\int_0^\infty \int_0^\infty (1+|\xi|) e^{2\alpha x} w^2\, d\xi dx\right)^{\frac{1}{2}}, \tag{2.36}$$

Therefore, by (2.31),

$$(\nu_0-\alpha)^2 \int_0^\infty \int_0^\infty (1+|\xi|)(gw)_x^2\, d\xi dx \leq \kappa \left(\int_0^\infty \int_0^\infty (1+|\xi|) e^{2\alpha x} w^2\, d\xi dx\right)$$
$$\leq \kappa (\nu_0-\alpha)^{-1}$$
$$\int_0^\infty \int_0^\infty (1+|\xi|)(e^{\alpha x}\phi w)_x^2\, d\xi dx \leq \kappa (\nu_0-\alpha)^{-3}$$
$$\int_Y^X \int_0^\infty (1+|\xi|)(e^{\alpha x} w)_x^2\, d\xi dx \leq \kappa (\nu_0-\alpha)^{-3}$$
$$\int_Y^X \int_0^\infty (1+|\xi|) e^{2\alpha x} w_x^2\, d\xi dx \leq \kappa (\nu_0-\alpha)^{-3}. \tag{2.37}$$

doi:10.6342/NTU202202294

Since κ here doesn't depend on X, by taking the limit $X \to \infty$,

$$\int_Y^\infty \int_0^\infty (1+|\xi|)e^{2\alpha x} w_x^2 \, d\xi dx \le \kappa(\nu_0 - \alpha)^{-3}. \tag{2.38}$$

Now, we are ready to show the pointwise decay of ω.

$$\begin{aligned}
&e^{2\alpha X}\int (1+|\xi|)w(X)^2 \, d\xi \\
&= \int\int_0^X (1+|\xi|)\frac{\partial}{\partial x}(e^{\alpha x}\phi w)^2 \, dx d\xi \\
&= 2\int\int_0^\infty (1+|\xi|)(e^{\alpha x}\phi w)(e^{\alpha x}\phi w)_x \, dx d\xi \\
&\le 2\left(\int\int (1+|\xi|)(e^{\alpha x}\phi w)^2 \, dx d\xi\right)^{\frac{1}{2}}\left(\int\int (1+|\xi|)(e^{\alpha x}\phi w)_x^2 \, dx d\xi\right)^{\frac{1}{2}} \\
&\le \kappa(\nu_0 - \alpha)^{-2}
\end{aligned}$$

Thus,

$$\int (1+|\xi|)w(X)^2 \, d\xi \le \kappa e^{-2\alpha X}(\nu_0 - \alpha)^{-2}, \tag{2.39}$$

where $\kappa = \kappa(\kappa_g, m_f)$.

(d) To show the behavior of $\tilde{q} = (a + b_2\xi_2 + b_3\xi_3 + c\xi^2)M^{\frac{1}{2}}$ when x is large, we multiply (2.16) by the vector $\Xi_1 = \xi_1(1, \xi_2, \xi_3, \xi^2)^T M^{\frac{1}{2}}$ and integrating with respect to ξ. Let $A = (a, b_2, b_3, c)^T$, we have that

$$\begin{aligned}
\int \xi_1\Xi_1 \frac{d}{dx} f \, d\xi + \int \Xi_1 L f \, d\xi &= 0 \\
\int \xi_1\Xi_1 \frac{d}{dx}\tilde{q} \, d\xi + \int \Xi_1 L w \, d\xi &= -\int \xi_1\Xi_1 \frac{d}{dx} w \, d\xi \\
\int \xi_1\Xi_1 \frac{d}{dx}(a + b_2\xi_2 + b_3\xi_3 + c\xi^2)M^{\frac{1}{2}} \, d\xi + \int \Xi_1 L w \, d\xi &= -\frac{d}{dx}\int \xi_1\Xi_1 w \, d\xi \\
B_1 \frac{d}{dx} A + \int \Xi_1 L w \, d\xi &= -\frac{d}{dx}\int \xi_1\Xi_1 w \, d\xi,
\end{aligned} \tag{2.40}$$

doi:10.6342/NTU202202294

where

$$B_1 = \int \xi_1 \Xi_1 (1, \xi_2, \xi_3, \xi^2)^T \, d\xi$$
$$= \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 35 \end{pmatrix},$$

which can be calculated easily by odd and even function properties.

Now observe that,

$$\left| \int \xi_1 \Xi_1 w \, d\xi \right| \leq \int (1+|\xi|)|\Xi_1||w| \, d\xi$$
$$\leq \left(\int (1+|\xi|)|\Xi_1|^2 \right)^{\frac{1}{2}} \left(\int (1+|\xi|)w^2 \right)^{\frac{1}{2}} \tag{2.41}$$
$$\leq \kappa(\nu_0 - \alpha)^{-1} e^{-\alpha x}.$$

Similarly,

$$\left| \int \Xi_1 L w \, d\xi \right| \leq \nu_1 \int (1+|\xi|)|w_{\Xi_1}||w| \, d\xi$$
$$\leq \left(\int (1+|\xi|)|w_{\Xi_1}|^2 \right)^{\frac{1}{2}} \left(\int (1+|\xi|)w^2 \right)^{\frac{1}{2}} \tag{2.42}$$
$$\leq k(\nu_0 - \alpha)^{-1} e^{-\alpha x}.$$

Therefore, these two terms will go to 0 as $x \to \infty$, this implies that $\frac{d}{dx}A \to 0$ as $x \to \infty$. To ensure the boundedness of $(a_\infty, b_{2\infty}, b_{3\infty}, c_\infty)$, it suffices to show the bounededness of (a, b_2, b_3, c) at $x = 0$. Multiply (2.16) by the vector $\Xi_2 = (1, \xi_2, \xi_3, \xi^2)^T M^{\frac{1}{2}}$, integrate with respect to $\xi_1 > 0$ and evaluate at $x = 0$, we have that

$$\int_{\xi_1>0} q \Xi_2 \, d\xi + \int_{\xi_1>0} w \Xi_2 \, d\xi = \int_{\xi_1>0} f \Xi_2 \, d\xi$$
$$B_2 A(x=0) = \int_{\xi_1>0} (g - w) \Xi_2 \, d\xi, \tag{2.43}$$

doi:10.6342/NTU202202294

where

$$B_2 = \int \Xi_2(1, \xi_2, \xi_3, \xi^2)\, d\xi$$
$$= \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 15 \end{pmatrix}.$$

Now, observe that,

$$\begin{aligned} \int_{\xi_1>0} g\Xi_2\, d\xi &\le \left(\int (1+|\xi|)g^2\, d\xi\right)^{\frac{1}{2}} \left(\int (1+|\xi|)\Xi_2^2\, d\xi\right)^{\frac{1}{2}} \\ &\le k \left(\int (1+|\xi|)g^2\, d\xi\right)^{\frac{1}{2}} \\ &\le k. \end{aligned} \tag{2.44}$$

Similarly,

$$\begin{aligned} \left| - \int_{\xi_1>0} w\Xi_2\, d\xi \right| &\le \int |w||\Xi_2|\, d\xi \\ &\le \left(\int (1+|\xi|)w^2\, d\xi\right)^{\frac{1}{2}} \left(\int (1+|\xi|)\Xi_2^2\, d\xi\right)^{\frac{1}{2}} \\ &\le k \left(\int (1+|\xi|)w^2\, d\xi\right)^{\frac{1}{2}} \\ &\le k, \end{aligned} \tag{2.45}$$

for some constant k. Hence, we conclude that

$$B_2 A(x=0) \le k. \tag{2.46}$$

This shows that $A(x=0)$ is bounded. Notice that, A is converging at the rate of $e^{-\alpha x}$ which implies (2.10).

□

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To prove the existence and uniqueness theorems, Bardos, Caflisch and Nicolaenko [2] suggest that it is convenient to restrict the Boltzmann equation to the domain $\xi \in \mathbb{R}^3_N = \{\xi : |\xi_1| > N^{-1}, |\xi| < N\}$ and $x \in [0, l]$ with reflection boundary condition at the right end $x = l$. So the problem now reads

$$\xi_1 \frac{\partial}{\partial x} f + L_N f = 0 \quad 0 < x < l, \tag{2.47}$$

$$f = g \quad x = 0,\ \xi_1 > 0, \tag{2.48}$$

$$f(\xi_1, \xi_2, \xi_3) = f(-\xi_1, \xi_2, \xi_3) \quad x = l. \tag{2.49}$$

where L_N is an operator restricted to $\mathbb{R}^3_N$ as defined earlier. Theorem 2.3 can be rephrased as follows:

Proposition 2.1. *(Orthogonality and asymptotic properties for finite interval) [2] Let l and N be two positive constants and let*

$$f^{lN} \in L^2\{[0, l] \times \mathbb{R}^3_N\}, \tag{2.50}$$

be the solution of (2.47) - (2.49) with $g \in L^2(\mathbb{R}^3_N)$ satisfying (2.4). Decompose $f^{lN} = w + q$ as before with $q = \chi_N(a + b_2\xi_2 + b_3\xi_3 + c\xi^2)M^{\frac{1}{2}}$. Then we have the following properties:

$$m_f = \int_{\mathbb{R}^3_N} \xi_1 M^{\frac{1}{2}} f \, d\xi = 0, \tag{2.51}$$

there exist constants a_0, b_{20}, b_{30}, c_0 with

$$|a_0| + |b_{20}| + |b_{30}| + |c_0| < \kappa, \tag{2.52}$$

doi:10.6342/NTU202202294

$$\int (1+|\xi|)w^2\, d\xi + |a-a_0|^2 + |b_2-b_{20}|^2 + |b_3-b_{30}|^2 + |c-c_0|^2 \le \kappa(\nu_0-\alpha)^{-2}e^{2\alpha x}, \quad (2.53)$$

for any $0 < \alpha < \nu_0$ in which κ depends only on κ_g. Furthermore, for any $0 < \delta < l$ and any $0 < \alpha < \nu_0$,

$$\int_\delta^l \int (1+|\xi|)e^{2\alpha x} f_x^2\, d\xi dx \le \kappa_\delta(\nu_0-\alpha)^{-3}, \quad (2.54)$$

where κ_δ depends only on κ_g and δ. Notice that, $kappa$ and κ_g are independent of l, N and α.

Proof. The proof is similar to that one in Theorem 2.3 with a few exception:

(1) Since $f_x = -\xi_1^{-1} L_N f$, f is smooth in $\mathbb{R}^3_N$, thus, $(1+|\xi|)^{\frac{1}{2}} \in L^\infty([0,l], L^2(\mathbb{R}^3_N))$.

(2) Since we have the reflection boundary condition at $x = l$, both f and w are even function at $x = l$. Therefore, the derivation of (2.25) and (2.26) can be replaced by the following identities

$$\int \xi_1 \psi_1 w\, d\xi = 0, \text{ for } i = 0, 2, 3, 4, \quad (2.55)$$

$$\int \xi_1 w^2\, d\xi = \int \xi_1 f^2\, d\xi = 0, \text{ at } x = l. \quad (2.56)$$

(3) The values $(a_\infty, b_{2\infty}, b_{3\infty}, c_\infty)$ may be replaced by $(a_0, b_{20}, b_{30}, c_0)$ which are the value evaluated at $x = l$.

□

Now, we will prove the existence theorem for the finite interval.

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Proposition 2.2. *(Existence of solution for finite interval) [2] Let g satisfies the condition (2.4) and l, N are two positive constants. Then, there exist a solution $f = f^{lN} \in L^2([0,l] \times \mathbb{R}^3_N)$ for the problem*

$$\xi_1 \frac{\partial}{\partial x} f + L_N f = 0 \quad 0 \leq x \leq l, \tag{2.57}$$

$$f = g \quad x = 0,\ \xi_1 > 0, \tag{2.58}$$

$$f(\xi_1, \xi_2, \xi_3) = f(-\xi_1, \xi_2, \xi_3) \quad x = l. \tag{2.59}$$

$$m_f = 0. \tag{2.60}$$

Decompose $f^{lN} = w + q$ as before with $q = \chi_N(a + b_2\xi_2 + b_3\xi_3 + c\xi^2)M^{\frac{1}{2}}$. Then, there exist constants a_0, b_{20}, b_{30}, c_0 with

$$|a_0| + |b_{20}| + |b_{30}| + |c_0| < \kappa, \tag{2.61}$$

$$\int (1+|\xi|)w^2\, d\xi + |a-a_0|^2 + |b_2-b_{20}|^2 + |b_3-b_{30}|^2 + |c-c_0|^2 \leq \kappa(\nu_0-\alpha)^{-2}e^{2\alpha x}, \tag{2.62}$$

for any $0 < \alpha < \nu_0$ in which $\kappa = \kappa(\kappa_g)$. Furthermore, for any $0 < \delta < l$ and any $0 < \alpha < \nu_0$,

$$\int_\delta^l \int (1 + |\xi|)e^{2\alpha x} f_x^2\, d\xi dx \leq \kappa_\delta(\nu_0 - \alpha)^{-3}, \tag{2.63}$$

where $\kappa_\delta = \kappa_\delta(\kappa_g, \delta)$.

doi:10.6342/NTU202202294

Proof. Define

$$A_N = \xi_1 \frac{\partial}{\partial x} + L_N, \tag{2.64}$$

with the domain

$$D(A_N) = \{f : f, f_x \in L^2([0,l] \times \mathbb{R}^3_N) \text{ with } f = 0 \text{ for } x = 0,\ \xi_1 > 0$$
$$\text{and } f(\xi_1, \xi_2, \xi_3) = f(-\xi_1, \xi_2, \xi_3) \text{ for } x = l\}. \tag{2.65}$$

We claim that 0 is in the resolvent set of A_N, so that by Fredholm alternative, we know that for every $h \in L^2([0,l] \times \mathbb{R}^3_N)$, there exist a unique solution $f \in D(A_N)$ for the equation

$$A_N f = h. \tag{2.66}$$

Indeed, suppose 0 is an eigenvalue of A_N and suppose

$$A_N f = 0, \tag{2.67}$$

for some $f \in D(A_N)$. Then, we multiply the equation by f and integrating over ξ

$$\begin{aligned} A_N f &= 0 \\ \xi_1 \frac{\partial}{\partial x} f + L_N f &= 0 \\ \int \xi_1 f \frac{\partial}{\partial x} f \, d\xi + \int f L_N f \, d\xi &= 0 \\ \frac{1}{2} \frac{d}{dx} \int \xi_1 f^2 \, d\xi + \int f L_N f \, d\xi &= 0. \end{aligned} \tag{2.68}$$

Integrating over x again, and using the fact that the integrand in the first term is an odd

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function in ξ_1,

$$\begin{aligned}\frac{1}{2}\int \xi_1 f^2(x=l)\, d\xi - \frac{1}{2}\int \xi_1 f^2(x=0)\, d\xi + \int_0^l \int f L_N f\, d\xi dx = 0 \\ -\frac{1}{2}\int_{\xi_1<0} \xi_1 f^2(x=0)\, d\xi + \int_0^l \int f L_N f\, d\xi dx = 0.\end{aligned} \tag{2.69}$$

Since,

$$\int f L_N f\, d\xi \geq \nu_0 \int (1+|\xi|) w_f^2\, d\xi \tag{2.70}$$

all the terms in L.H.S. of (2.68) are positive and this implies that each terms on the left must be zero. Thus, we arrived at

$$\begin{aligned} & f(x=0) \equiv 0 \text{ and } L_N f \equiv 0 \\ \Longrightarrow\ & \xi_1 \frac{\partial}{\partial x} f \equiv 0 \\ \Longrightarrow\ & \frac{\partial}{\partial x} f \equiv 0 \\ \Longrightarrow\ & f \equiv 0. \end{aligned} \tag{2.71}$$

Therefore, 0 is not an eigenvalue of A_N. As a consequence, the system

$$A_N f = h, \tag{2.72}$$

possessed a unique solution for $h \in L^2([0,l] \times \mathbb{R}^3_N)$. Hence, the solution of (2.55) - (2.57) can be constructed as $\tilde{f} = f + \phi \cdot g$ where $\phi = \phi(x)$ is a smooth cutoff funtion with $\phi = 1$ for $x \leq \frac{l}{3}$, and $\phi = 0$ for $x > \frac{l}{2}$. f is the solution of $A_N f = h$, with

doi:10.6342/NTU202202294

$h = -\xi_1 \phi_x g - \phi L_N g \in L^2([0,l] \times \mathbb{R}^3_N)$. Clearly,

$$\begin{aligned}
A_N \tilde{f} &= A_N(f + \phi \cdot g) \\
(\xi_1 \frac{\partial}{\partial x} + L_N)\tilde{f} &= A_N f + (\xi_1 \frac{\partial}{\partial x} + L_N)(\phi \cdot g) \\
&= A_N f + \xi_1 \frac{\partial}{\partial x}(\phi \cdot g) + L_N(\phi \cdot g) \\
&= A_N f + \xi_1 g \phi_x + \phi L_N g \\
&= A_N f - h \\
&= 0.
\end{aligned} \tag{2.73}$$

□

Proposition 2.3. *(Existence of solution for zero mass flux) [2] Let g satisfies the condition (2.4). Then, there exist a solution $f \in D$ of the Milne problem with zero mass flux,*

$$\xi_1 \frac{\partial}{\partial x} f + Lf = 0, \textit{ for } 0 \leq x < \infty, \tag{2.74}$$

$$f = g, \textit{ for } x = 0, \xi_1 > 0, \tag{2.75}$$

$$\int \xi_1 M^{\frac{1}{2}} f \, d\xi = 0. \tag{2.76}$$

With proposition 2.3, theorem 2.1 can be established immediately. Bardos, Caflisch and Nicolaenko [2] found that one can simply construct the solution in the following way: Let $m_f \neq 0$ and g satisfy (2.4) denote $\tilde{g} = g - m_f \xi_1 M^{\frac{1}{2}}$ and solve the zero mass flux problem with this $\tilde{g}$ to obtain $\tilde{f}$. Then $f = \tilde{f} + m_f \xi_1 M^{\frac{1}{2}}$ will solves the Milne problem (2.1) - (2.3).

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Proof. (Proposition 2.3) Let f^{lN} solves (2.57) - (2.59) and define $q_0^{lN} = \chi_N(a_0 + b_{20}\xi_2 + b_{30}\xi_3 + c_0\xi^2)M^{\frac{1}{2}}$. Clearly, by (2.60) - (2.63), the sequence q_0^{lN}, f^{lN} and f_x^{lN} are bounded with bounds that does not depend on l or N. Therefore, by taking $l \to \infty$ and $N \to \infty$ there exists a subsequence with

(1) $q_0^{lN} \to q_\infty$ in $L^2(\mathbb{R}^3_\xi)$,

(2) $f_x^{lN} \rightharpoonup h$ weakly in $L^2_{loc}(\mathbb{R}^+_x, L^2(\mathbb{R}^3_\xi))$,

(3) $f^{lN} - q_0^{lN} \rightharpoonup f - q_\infty$ weakly in $L^2(\mathbb{R}^+_x \times \mathbb{R}^3_\xi)$.

Next, we will show $h = f_x$ and that f is the solution to the Milne problem (2.1) - (2.3). Firstly, we regard the Boltzmann equation as a first order ODE in x and solve for f^{lN} we have that

$$\xi_1 f^{lN}(x,\xi) = \begin{cases} \xi_1 e^{-\frac{x\nu}{\xi_1}} g(\xi) - \int_0^x e^{-\frac{y\nu}{\xi_1}} K_N f^{lN}(x-y,\xi)\, dy, & \text{for } \xi_1 > 0, \\ \xi_1 e^{-\frac{(l-x)\nu}{\xi_1}} f(l,\xi) + \int_0^{l-x} e^{-\frac{y\nu}{\xi_1}} K_N f^{lN}(x+y,\xi)\, dy, & \text{for } \xi_1 < 0. \end{cases} \tag{2.77}$$

Similarly, we can solve $\xi_1 f_x + \nu f = -Kf$ in an infinite strip

$$\xi_1 f(x,\xi) = \begin{cases} \xi_1 e^{-\frac{x\nu}{\xi_1}} g(\xi) - \int_0^x e^{-\frac{y\nu}{\xi_1}} K f(x-y,\xi)\, dy, & \text{for } \xi_1 > 0, \\ \int_x^\infty e^{-\frac{(y-x)\nu}{\xi_1}} K f(y,\xi)\, dy, & \text{for } \xi_1 < 0. \end{cases} \tag{2.78}$$

doi:10.6342/NTU202202294

Plugging in the expression for $f(l,\xi)$ we have

$$\xi_1 f^{lN}(x,\xi) = \begin{cases} \xi_1 e^{-\frac{x\nu}{\xi_1}} g(\xi) + \int_0^x e^{-\frac{y\nu}{\xi_1}} K_N f^{lN}(x-y,\xi)\, dy, & \text{,for } \xi_1 > 0, \\ e^{-\frac{(l-x)\nu}{\xi_1}} [\xi_1 e^{-\frac{l\nu}{\xi_1}} g(\tilde{\xi}) + \int_0^l e^{-\frac{y\nu}{\xi_1}} K_N f^{lN}(l-y,\tilde{\xi})\, dy] & \\ \qquad + \int_0^{l-x} e^{-\frac{y\nu}{\xi_1}} K_N f^{lN}(x+y,\xi)\, dy, & \text{,for } \xi_1 < 0, \end{cases} \tag{2.79}$$

where $\tilde{\xi} = (-\xi_1, \xi_2, \xi_3)$. Next, we subtract $\xi_1 q_0^{lN}$ from both sides and using the fact that $K_N q_0^{lN} = -\nu q_0^{lN}$, we get

$$\xi_1(f^{lN} - q_0^{lN})(x,\xi) = \xi_1 e^{-\frac{x\nu}{\xi_1}} (g - q_0^{lN})(\xi) - \int_0^x e^{-\frac{y\nu}{\xi_1}} K_N (f^{lN} - q_0^{lN})(x-y,\xi)\, dy, \tag{2.80}$$

for $\xi_1 > 0$ and

$$\xi_1(f^{lN} - q_0^{lN})(x,\xi) = e^{-\frac{(l-x)\nu}{\xi_1}} \{\xi_1 e^{-\frac{l\nu}{\xi_1}} (g - q_0^{lN})(\tilde{\xi}) + \int_0^l e^{-\frac{y\nu}{\xi_1}} K_N (f^{lN} - q_0^{lN})(l-y,\tilde{\xi})\, dy\} \\ + \int_0^{l-x} e^{-\frac{y\nu}{\xi_1}} K_N (f^{lN} - q_0^{lN})(x+y,\xi)\, dy, \tag{2.81}$$

for $\xi_1 < 0$. Now observe that, since $|\xi_1|^{\frac{1}{2}}(g - q_0^{lN}) \in L^2(\mathbb{R}^3_\xi)$, $K_N \to K$, $f^{lN} - q_0^{lN} \rightharpoonup f - q_\infty$, $\frac{\xi_1}{\nu}$ is finite, and both K and K_N are compact, (2.80) converges to

$$\begin{aligned} \lim_{l,N\to\infty} \xi_1(f^{lN} - q_0^{lN})(x,\xi) &= \xi_1 e^{-\frac{x\nu}{\xi_1}} (g - q_\infty)(\xi) - \int_0^x e^{-\frac{y\nu}{\xi_1}} K(f - q_\infty)(x-y,\xi)\, dy \\ &= \xi_1 f - \xi_1 q_\infty, \end{aligned} \tag{2.82}$$

and (2.81) converges to

$$\begin{aligned} \lim_{l,N\to\infty} \xi_1(f^{lN} - q_0^{lN})(x,\xi) &= \int_x^\infty e^{-\frac{(y-x)\nu}{|\xi_1|}} K(f - q_\infty)(y,\xi)\, dy \\ &= \xi_1 f - \xi_1 q_\infty. \end{aligned} \tag{2.83}$$

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This shows that the convergence is in $L^\infty(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3))$ and thus, $f \in L^\infty(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3))$ by the uniform boundedness of f^{lN}.

Next, we show f_x^{lN} converges to $h = f_x$. First, we introduce a smooth cutoff function to avoid the boundary values. Let $\phi(x)$ be a smooth cutoff function such that

$$\phi(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq \frac{1}{3}Y, \\ 1, & \text{for } Y \leq x < X, \\ 0, & \text{for } X+1 \leq x < \infty. \end{cases} \tag{2.84}$$

Then, let $X + 1 < l$, $(\phi f^{lN})_x$ satisfies

$$\begin{aligned} \xi_1 \frac{\partial}{\partial x}(\phi f^{lN})_x + L_N(\phi f^{lN})_x &= \xi_1(\phi_{xx} f^{lN} + 2\phi_x f_x^{lN} + \phi f_{xx}^{lN}) + \phi_x L_N f^{lN} + \phi L_N f_x^{lN}, \\ &= \xi_1 \phi_{xx} f^{lN} - 2\phi_x L_N f^{lN} - \phi L_N f_x^{lN} + \phi_x L_N f^{lN} + \phi L_N f_x^{lN}, \\ &= \xi_1 \phi_{xx} f^{lN} - \phi_x L_N f^{lN}, \end{aligned} \tag{2.85}$$

with the boundary conditions

$$(\phi f^{lN})_x = 0 \text{ for } x = 0,\ \xi_1 > 0 \text{ and for } x = l,\ \xi_1 < 0. \tag{2.86}$$

We can solve this equation in the similar fashion to obtain

$$\xi_1(\phi f^{lN})_x = \begin{cases} \int_0^x e^{-\frac{(x-y)\nu}{\xi_1}} [-K_N(\phi f^{lN})_x + \phi_{xx}\xi_1 f^{lN} - \phi_x L_N f^{lN}]\, dy, & \xi_1 > 0, \\ \int_x^l e^{-\frac{(y-x)\nu}{|\xi_1|}} [K_N(\phi f^{lN})_x - \phi_{xx}\xi_1 f^{lN} + \phi_x L_N f^{lN}]\, dy, & \xi_1 < 0. \end{cases} \tag{2.87}$$

Now, recall that $(\phi f^{lN})_x$ converges weakly, $K_N \to K$ and K is compact, both $\xi_1 f^{lN}$ and $K_N f^{lN}$ converge. Therefore, every term in (2.87) behave nicely except for the term νf^{lN}

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from $L_N f^{lN}$. To remedy the problem, we multiply (2.87) by $\xi_1\nu^{-1}$. Then

$$
\begin{aligned}
\xi_1^2\nu^{-1}(\phi f^{lN})_x &= \begin{cases} \xi_1\nu^{-1}\int_0^x e^{-\frac{(x-y)\nu}{\xi_1}}[-K_N(\phi f^{lN})_x + \phi_{xx}\xi_1 f^{lN} - \phi_x L_N f^{lN}]\,dy, & \xi_1 > 0, \\ \xi_1\nu^{-1}\int_x^l e^{-\frac{(y-x)\nu}{|\xi_1|}}[K_N(\phi f^{lN})_x - \phi_{xx}\xi_1 f^{lN} + \phi_x L_N f^{lN}]\,dy, & \xi_1 < 0, \end{cases} \\
&\to \begin{cases} \xi_1\nu^{-1}\int_0^x e^{-\frac{(x-y)\nu}{\xi_1}}[-K(\phi f)_x + \phi_{xx}\xi_1 f - \phi_x L f]\,dy, & \xi_1 > 0, \\ \xi_1\nu^{-1}\int_x^l e^{-\frac{(y-x)\nu}{|\xi_1|}}[K(\phi f)_x - \phi_{xx}\xi_1 f + \phi_x L f]\,dy, & \xi_1 < 0, \end{cases} \\
&= \xi_1^2\nu^{-1}(\phi f)_x.
\end{aligned}
\tag{2.88}
$$

This shows that $\xi_1^2\nu^{-1}f_x^{lN}$ converges to $\xi_1^2\nu^{-1}f_x$ in $L^2_{loc}(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3))$ and as a result $f_x = h \in L^2_{loc}(\mathbb{R}_x^+, L^2(\mathbb{R}_\xi^3))$. This concludes our proof. □

Finally, we turn our attention on the uniqueness theorem.

Proof. (Uniqueness) Suppose we have two solutions f_1 and f_2 to the Milne problem (2.1) - (2.3) with same boundary data g at $x = 0$ and mass flux m_f. Let $h = f_1 - f_2$. Clearly, we have

$$\xi_1 \frac{\partial}{\partial x} h + Lh = 0, \text{ for } x > 0, \tag{2.89}$$

$$h = 0 \text{ for } x = 0,\ \xi_1 > 0, \tag{2.90}$$

$$m_h = 0. \tag{2.91}$$

Decompose h like what we did before, $h = w(x,\xi) + \tilde{q}(x,\xi) + q_\infty(\xi)$ and assume that $\lim_{x\to\infty} h = q_\infty = (a_\infty + b_{2\infty}\xi_2 + b_{3\infty}\xi_3 + c_\infty\xi^2)M^{\frac{1}{2}}$, where $\tilde{q} = q - q_\infty$. Since

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$b_1 = m_h = 0$, we have

$$\int \xi_1 \tilde{q}^2 \, d\xi = \int \xi_1 q_\infty^2 \, d\xi = \int \xi_1 \tilde{q} q_\infty \, d\xi = 0, \tag{2.92}$$

since every integrand above are odd functions in ξ_1. Also, since $q_\infty \in N(L)$ and $Lh \in N(L)^\perp$, multiply (2.89) by q_∞ and integrate over ξ

$$\begin{aligned} \frac{d}{dx} \int \xi_1 q_\infty h \, d\xi + \int q_\infty Lh \, d\xi = 0, \\ \frac{d}{dx} \int \xi_1 q_\infty h \, d\xi = 0. \end{aligned} \tag{2.93}$$

This shows that the integral is a constant. Furthermore,

$$\lim_{x \to \infty} \int \xi_1 q_\infty h \, d\xi = \int \xi_1 q_\infty^2 \, d\xi = 0. \tag{2.94}$$

Thus,

$$0 = \int \xi_1 q_\infty h \, d\xi = \int \xi_1 q_\infty w \, d\xi. \tag{2.95}$$

After that, at $x = 0$

$$\begin{aligned} \int \xi_1 (w + \tilde{q})^2 \, d\xi &= \int \xi_1 (h - q_\infty)^2 \, d\xi \\ &= \int \xi_1 h^2 \, d\xi - 2 \int \xi_1 h q_\infty \, d\xi + \int \xi_1 q_\infty^2 \, d\xi \\ &= \int_{\xi_1 < 0} \xi_1 h^2 \, d\xi \\ &\leq 0. \end{aligned} \tag{2.96}$$

doi:10.6342/NTU202202294

On the other hand,

$$
\begin{aligned}
\xi_1 \frac{\partial}{\partial x} h + Lh &= 0 \\
\xi_1 \frac{\partial}{\partial x}(w+\tilde{q}) + Lw &= 0 \\
\xi_1 (w+\tilde{q}) \frac{\partial}{\partial x}(w+\tilde{q}) + (w+\tilde{q})Lw &= 0 \\
\xi_1 \int (w+\tilde{q}) \frac{\partial}{\partial x}(w+\tilde{q})\, d\xi + \int (w+\tilde{q})Lw \, d\xi &= 0 \qquad (2.97)\\
\frac{d}{dx} \int \xi_1 (w+\tilde{q})^2 \, d\xi &= -\int wLw \, d\xi \\
&\leq -\nu_0 \int (1+|\xi|) w^2 \, d\xi \\
&\leq 0.
\end{aligned}
$$

Since $\int \xi_1 (w+\tilde{q})^2$ has a limit 0 at $x \to \infty$, (2.96) and (2.97) implies that $\int \xi_1 (w+\tilde{q})^2 = 0$ for all x. Therefore, (2.97) shows that

$$
0 = \frac{d}{dx} \int \xi_1 (w+\tilde{q})^2 \, d\xi \leq -\nu_0 \int (1+|\xi|) w^2 \, d\xi \leq 0, \qquad (2.98)
$$

this implies $w \equiv 0$. Furthermore,

$$
\begin{aligned}
\xi_1 \frac{\partial}{\partial x}(w+\tilde{q}) + Lw &= 0 \\
\implies \frac{\partial}{\partial x} \tilde{q} &= 0.
\end{aligned} \qquad (2.99)
$$

So, $\tilde{q}$ is a constant and since $\lim_{x\to\infty} \tilde{q} = q_\infty$, this implies that $\tilde{q} = q_\infty$. Hence, at $x = 0$

$$
\begin{aligned}
h &= w + \tilde{q} + q_\infty \\
0 &= 0 + q_\infty + q_\infty \qquad (2.100)\\
q_\infty &= 0,
\end{aligned}
$$

and this is true for all x. Thus, $h = 0$ for all x. The proof for the uniqueness theorem is now complete. □

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Chapter 3 The Kramers Problem

The Kramers problem is similar to the Milne problem. However, this time we allowed the distribution to grows linearly at $x = \infty$. This time, if the average mass flux m_f, asymptotic gradient for velocity in both x_2 and x_3 direction, and temperature are specified, then we will have the existence and uniqueness theorem. [2]

We say f is the solution of the Kramers problem if

$$\xi_1 \frac{\partial}{\partial x} f + Lf = 0, \qquad 0 < x < \infty, \tag{3.1}$$

$$f = g, \qquad x = 0,\ \xi_1 > 0, \tag{3.2}$$

$$\int \xi_1 M^{\frac{1}{2}} f \, d\xi = m_f, \tag{3.3}$$

$$\lim_{x \to \infty} \frac{d}{dx} \int \xi_2 M^{\frac{1}{2}} f \, d\xi = v_2, \tag{3.4}$$

$$\lim_{x \to \infty} \frac{d}{dx} \int \xi_3 M^{\frac{1}{2}} f \, d\xi = v_3, \tag{3.5}$$

$$\lim_{x \to \infty} \frac{d}{dx} \frac{1}{3} \int (\xi^2 - 3) M^{\frac{1}{2}} f \, d\xi = \theta. \tag{3.6}$$

Following Bardos, Caflisch and Nicolaenko's [2] idea, one can expect for the solution

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that behaves like $f_0 + xf_1$ near $x = \infty$ with f_0 and f_1 depend only on ξ. Clearly,

$$\begin{aligned} \xi_1 \frac{\partial}{\partial x} f + Lf &= 0 \\ \xi_1 f_1 + L(f_0 + x f_1) &= 0 \\ \xi_1 f_1 + L f_0 + x L f_1 &= 0. \end{aligned} \tag{3.7}$$

Equating the coefficients of x to zero, we have

$$Lf_1 = 0, \tag{3.8}$$

$$Lf_0 = -\xi_1 f_1. \tag{3.9}$$

From previous section, we know that $f_1 \in N(L)$ implies $f_1 = (a + b_1\xi_1 + b_2\xi_2 + b_3\xi_3 + c\xi^2)M^{\frac{1}{2}}$. By multiplying (3.9) with ψ_i for $i = 0, ..., 4$ as defined in (1.8) - (1.10) and integrate over ξ, we see that $\int \psi_i \xi_1 f_1 \, d\xi = 0$. Now observe that,

$$\begin{aligned} \int \psi_0 \xi_1 f_1 \, d\xi &= 0 \\ \int \psi_0 \xi_1 (a + b_1\xi_1 + b_2\xi_2 + b_3\xi_3 + c\xi^2) M^{\frac{1}{2}} \, d\xi &= 0 \\ \int \xi_1 (a + b_1\xi_1 + b_2\xi_2 + b_3\xi_3 + c\xi^2) M \, d\xi &= 0 \\ b_1 \int \xi_1^2 M \, d\xi &= 0 \\ b_1 &= 0. \end{aligned} \tag{3.10}$$

doi:10.6342/NTU202202294

Also,

$$
\begin{aligned}
\int \psi_1 \xi_1 f_1 \, d\xi &= 0 \\
\int \psi_1 \xi_1 (a + b_2\xi_2 + b_3\xi_3 + c\xi^2) M^{\frac{1}{2}} \, d\xi &= 0 \\
\int \xi_1^2 (a + b_2\xi_2 + b_3\xi_3 + c\xi^2) M \, d\xi &= 0 \\
a \int \xi_1^2 M \, d\xi + c \int \xi_1^2 \xi^2 M \, d\xi &= 0 \\
a + 5c &= 0 \\
a &= -5c.
\end{aligned}
\tag{3.11}
$$

Furthermore,

$$
\begin{aligned}
v_2 &= \lim_{x\to\infty} \frac{d}{dx} \int \xi_2 M^{\frac{1}{2}} f \, d\xi \\
&= \lim_{x\to\infty} \int \xi_2 M^{\frac{1}{2}} f_1 \, d\xi \\
&= \int \xi_2 M^{\frac{1}{2}} f_1 \, d\xi \\
&= \int \xi_2 (a + b_2\xi_2 + b_3\xi_3 + c\xi^2) M \, d\xi \\
&= b_2 \int \xi_2^2 M \, d\xi \\
&= b_2.
\end{aligned}
\tag{3.12}
$$

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Similarly, we can show that $v_3 = b_3$, and

$$\begin{aligned}
\theta &= \lim_{x\to\infty} \frac{d}{dx}\frac{1}{3}\int (\xi^2-3)M^{\frac{1}{2}} f\, d\xi \\
&= \frac{1}{3}\int (\xi^2-3)(a+b_2\xi_2+b_3\xi_3+c\xi^2)M\, d\xi \\
&= \frac{1}{3}\int (\xi^2-3)(-5c+c\xi^2)M\, d\xi \\
&= \frac{c}{3}\int (\xi^2-3)(\xi^2-5)M\, d\xi \\
&= \frac{c}{3}\left(\int \xi^4 M\, d\xi - 8\int \xi^2 M\, d\xi + 15\int M\, d\xi\right) \\
&= \frac{c}{3}(15-8(3)+15(1)) \\
&= 2c.
\end{aligned} \tag{3.13}$$

Therefore, f_1 takes the form

$$f_1 = \left[v_2\xi_2 + v_3\xi_3 + \frac{\theta}{2}(\xi^2-5)\right] M^{\frac{1}{2}}. \tag{3.14}$$

and the equation $Lf_0 = -\xi_1 f_1$ has a unique solution $f_0 \in N(L)^\perp$. Now, we are all set to define the solution space for Kramers problem. The solution space takes the form $D_k = \{f : f - xf_1 \in D,\ f_1 = \left[v_2\xi_2 + v_3\xi_3 + \frac{\theta}{2}(\xi^2-5)\right] M^{\frac{1}{2}}\}$. Similar to the Milne problem, we have the following existence, uniqueness and asymptotic properties for Kramers problem.

Theorem 3.1. *(Existence) [2] Let* $\gamma = 1$. *For any* m_f, v_2, v_3, $\theta \in \mathbb{R}$ *and any* g *satisfying (2.4), there exist a solution* $f \in D_k$ *for the Kramers problem (3.1)-(3.6).*

Theorem 3.2. *(Uniqueness) [2] Let* $\gamma = 1$. *For a specified* m_f, v_2, v_3, $\theta \in \mathbb{R}$ *and a given* g *satisfying (2.4), there exist a unique solution* $f \in D_k$ *for the Kramers problem (3.1)-(3.6).*

Theorem 3.3. *(Orthogonality and asymptotic properties) [2] Let* $\gamma = 1$. *Suppose* $f \in D_k$

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satisfies (3.1)-(3.6) with m_f*,* v_2*,* v_3*,* $\theta \in \mathbb{R}$ *and* g *satisfying (2.4). Write* $f(x,\xi) = \tilde{f}(x,\xi) + x f_1(\xi)$ *where* $f_1(\xi) = \left[v_2\xi_2 + v_3\xi_3 + \frac{\theta}{2}(\xi^2 - 5)\right] M^{\frac{1}{2}}$*. Then* f *satisfies (2.5) - (2.7) while* $\tilde{f}$ *satisfies (2.8) - (2.11).*

Remark 3.1. We have shown the existence, uniqueness and the asymptotic behaviour for the hard-sphere model case. However, the technique used in proving the existence and uniqueness theorem may not work for other cases ($-3 < \gamma < 1$) as the term $\frac{\xi_1}{\nu}$ is not bounded anymore. Therefore, Golse and Poupaud modified the technique and used an suitable truncation to tackle the problem [10]. The detailed proof can be found in [10].

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Chapter 4 Boundary singularity

In this section, we shall discuss the boundary singularity of the gradient of macroscopic variables. Consider the following stationary linearized Boltzmann equation:

$$\xi_1 \frac{\partial}{\partial x} f(x,\xi) + Lf(x,\xi) = 0, \qquad \text{for } 0 < x < l,\ \xi \in \mathbb{R}^3, \tag{4.1}$$

with given boundary data:

$$f(0,\xi), \quad \text{given for } \xi_1 > 0, \tag{4.2}$$

$$f(l,\xi), \quad \text{given for } \xi_1 < 0. \tag{4.3}$$

Before we define the solution space for this problem, we first introduce the following definitions:

$$\|f\|_{L^{\infty,a}_\xi} := \sup_{\xi\in\mathbb{R}^3} (1+|\xi|)^a |f(\xi)|, \tag{4.4}$$

$$\|f\|_* := \left(\int f^2(\xi)\nu(\xi)\, d\xi \right)^{\frac{1}{2}}, \tag{4.5}$$

$$|||f||| := \sup_{0\le x\le l} \|f\|_*, \tag{4.6}$$

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and the space

$$L^*_\xi(\mathbb{R}^3) = \{f : \|f\|_* < \infty\}. \tag{4.7}$$

Now, we say that $f \in L^\infty_x\{[0,l], L^2_\xi(\mathbb{R}^3)\}$ is a solution to the problem (4.1) - (4.3) if it satisfies

$$f(x,\xi) = \begin{cases} e^{-\frac{x\nu}{|\xi_1|}} f(0,\xi) - \int_0^x \frac{1}{|\xi_1|} e^{-\frac{(x-y)\nu}{|\xi_1|}} Kf(y,\xi)\, dy, & \text{for } \xi_1 > 0, \\ e^{-\frac{(l-x)\nu}{|\xi_1|}} f(l,\xi) + \int_x^l \frac{1}{|\xi_1|} e^{-\frac{(y-x)\nu}{|\xi_1|}} Kf(y,\xi)\, dy, & \text{for } \xi_1 < 0, \end{cases} \tag{4.8}$$

which can be obtained by considering the equation (4.1) as a first order ODE in variable x. Notice that, the solution takes different form for different value of ξ_1. Indeed, $\xi_1 < 0$ means the particle is moving to the left, so we can see the right end as the starting point. Also, we can easily see that $L^\infty_x\{[0,l], L^2_\xi(\mathbb{R}^3)\}$ contains the solution space we discussed for Milne and Kramers problem. Indeed, by using (1.13)

$$\int (1+|\xi|)^\gamma f^2\, d\xi \le \int \nu f^2\, d\xi. \tag{4.9}$$

Now, we define α-moment.

$$\sigma_\alpha(x) = \int_{\mathbb{R}^3} f(x,\xi)\phi_\alpha(\xi)\, d\xi, \tag{4.10}$$

where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \alpha_i \ge 0 \tag{4.11}$$

and,

$$\phi_\alpha(\xi) := \xi^\alpha M^{\frac{1}{2}} = (2\pi)^{-\frac{3}{4}} \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3} e^{-\frac{|\xi|^2}{4}}. \tag{4.12}$$

doi:10.6342/NTU202202294

For different choices of α, this function will give us a different macroscopic variable. For instance, as what we have shown in chapter 2, $\rho = \sigma_{(0,0,0)}$ depicted the density, while $v_1 = \sigma_{(1,0,0)}$ is the velocity in x_1-direction and $T = \frac{1}{3}(\sigma_{(2,0,0)} + \sigma_{(0,2,0)} + \sigma_{(0,0,2)}) - \sigma_{(0,0,0)}$ is the temperature. In 2015, Chen and Hsia derived a formula to calculated the derivative of this moment function for hard potential case ($0 \leq \gamma \leq 1$) [7]. Later, Huang extended the result to soft potential case ($-\frac{3}{2} < \gamma < 0$) [11]. We summarize their results as follow:

Theorem 4.1. *[11] Consider the stationary linearized Boltzmann equation with cutoff (assuming (1.3) and (1.4)) and* $\gamma \in (-\frac{3}{2}, 1]$*. Let* $f \in L^\infty_x\{[0,l], L^*_\xi(\mathbb{R}^3)\}$ *be a solution to the problem (4.1) - (4.3) such that*

$$\nabla f(0,\cdot) \in L^p_\xi(\mathbb{R}^{3+}), \qquad \textit{for some } p \in (1,\infty), \tag{4.13}$$

$$f(0,\xi) \in L^\infty_\xi(\mathbb{R}^{3+}), \tag{4.14}$$

$$f(l,\xi) \in L^\infty_\xi(\mathbb{R}^{3-}), \tag{4.15}$$

then for small $x > 0$*, we have the formula*

$$\frac{\partial}{\partial x}\sigma_\alpha(x) = -\ln x \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_\alpha(x)(0,\xi_2,\xi_3) Lf(x=0,0^+,\xi_2,\xi_3)\, d\xi_2 d\xi_3 + O(\langle f\rangle), \tag{4.16}$$

where

$$Lf(x=0,0^+,\xi_2,\xi_3) := \lim_{\xi_1\to 0^+} Lf(x=0,\xi_1,\xi_2,\xi_3), \tag{4.17}$$

$$\langle f\rangle := 1 + |||f||| + \|f(0,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3+})} + \|f(l,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3-})} + \|\nabla f(0,\cdot)\|_{L^p_\xi(\mathbb{R}^{3+})}, \tag{4.18}$$

$$\mathbb{R}^{3+} := \{\xi \in \mathbb{R}^3 : \xi_1 > 0\}, \tag{4.19}$$

$$\mathbb{R}^{3-} := \{\xi \in \mathbb{R}^3 : \xi_1 < 0\}. \tag{4.20}$$

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This singularity was first discovered by Sone [16],[17], and Sone and Onishi [18],[13]. Later, the logarithmic singularity was presented by Chen, Liu and Takata [8] for the derivative of flow velocity. Another way to interpret this result is the macroscopic variables behave like $-x \ln x$ and approaching to zero near $x = 0$.

Since,

$$\frac{\partial}{\partial x}\sigma_\alpha(x) = \int \phi_\alpha \frac{\partial}{\partial x} f \, d\xi, \tag{4.21}$$

to derives the formula (4.16), we first differentiate the solution (4.8) to obtain

$$\frac{\partial}{\partial x} f(x,\xi) = -\frac{\nu}{|\xi_1|} e^{-\frac{x\nu}{|\xi_1|}} f(0,\xi) - \frac{1}{|\xi_1|} K f(x,\xi) + \int_0^x \frac{\nu}{|\xi_1|^2} e^{-\frac{(x-y)\nu}{|\xi_1|}} K f(y,\xi)\, dy \tag{4.22}$$

Observe that, when $x = 0$, the first term will not be integrable in ξ since it has a singularity at $\xi_1 = 0$. To remedy this issue, authors in [7] utilized the identity

$$\int_0^x e^{-\frac{(x-y)\nu}{|\xi_1|}} \, dy = \frac{|\xi_1|}{\nu}\left(1 - e^{-\frac{x\nu}{|\xi_1|}}\right), \tag{4.23}$$

to rewrite the equation (4.22) so that the problematic term $\frac{1}{|\xi_1|}$ will be associated with the integral operator which can helps to improve the regularity. Therefore,

$$\begin{aligned}\frac{\partial}{\partial x} f(x,\xi) = &-\frac{1}{|\xi_1|} e^{-\frac{x\nu}{|\xi_1|}} L f(0,\xi) - \frac{1}{|\xi_1|} e^{-\frac{x\nu}{|\xi_1|}} [K f(x,\xi) - K f(0,\xi)] \\ &- \int_0^x \frac{\nu}{|\xi_1|^2} e^{-\frac{(x-y)\nu}{|\xi_1|}} [K f(x,\xi) - K f(y,\xi)]\, dy,\end{aligned} \tag{4.24}$$

for $\xi_1 > 0$. Similarly, for $\xi_1 < 0$

$$\begin{aligned}\frac{\partial}{\partial x} f(x,\xi) = &\frac{1}{|\xi_1|} e^{-\frac{(l-x)\nu}{|\xi_1|}} L f(l,\xi) + \frac{1}{|\xi_1|} e^{-\frac{(l-x)\nu}{|\xi_1|}} [K f(l,\xi) - K f(x,\xi)] \\ &+ \int_x^l \frac{\nu}{|\xi_1|^2} e^{-\frac{(x-y)\nu}{|\xi_1|}} [K f(y,\xi) - K f(x,\xi)]\, dy.\end{aligned} \tag{4.25}$$

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Thus, both the second and the third terms can be treated by the Lipschitz-type continuity of the operator K, and it can be showed that both of them are uniformly bounded. Therefore, the logarithmic singularity we have concerned about is actually contributed by the first term. Before we derive the logarithmic singularity, let us adopt the following crucial estimates regarding the collision operator.

Lemma 4.1. *[7], [11] Let $-3 < \gamma \leq 1$, and $a \geq 0$, then we have the following estimate on the operator K*

$$\|K(f)\|_{L^{\infty,a+2-\gamma}} \leq C_{a,\gamma}\|f\|_{L^{\infty,a}}. \tag{4.26}$$

Furthermore, if $-\frac{3}{2} < \gamma \leq 1$, then we can improve the estimate

$$\|K(f)\|_{L^{\infty,\frac{3}{2}-\gamma}} \leq C_\gamma\|f\|_{L^2}, \tag{4.27}$$

$$\|K(f)\|_{L^{\infty,\frac{3}{2}-\frac{\gamma}{2}}} \leq C_\gamma\|f\|_{L^*}. \tag{4.28}$$

Lemma 4.2. *[7], [11] Let $-2 < \gamma \leq 1$, and $q \in [1, \infty]$. Then we have the bound*

$$\|\nabla_\xi K(f)\|_{L^q} \leq C_{q,\gamma}\|f\|_{L^q} \tag{4.29}$$

Lemma 4.3. *[7], [11] Let $-\frac{3}{2} < \gamma \leq 1$. Suppose $f \in L^\infty_x\{[0,l], L^*_\xi(\mathbb{R}^3)\}$ is the solution to (4.1) and that L satisfying (1.11), (1.13) and (4.24) - (4.27) such that*

$$f(0,\xi) \in L^\infty_\xi(\mathbb{R}^{3+}), \tag{4.30}$$

$$f(l,\xi) \in L^\infty_\xi(\mathbb{R}^{3-}). \tag{4.31}$$

doi:10.6342/NTU202202294

Then we have

$$\left|\int_{\xi_1>0} -\phi_\alpha \frac{1}{|\xi_1|} e^{-\frac{x\nu}{|\xi_1|}} Lf(0,\xi)\, d\xi\right| \leq C_{\alpha,\gamma}(|\ln x|+1)\langle f\rangle', \tag{4.32}$$

$$\left|\int_{\xi_1>0} -\phi_\alpha \frac{1}{|\xi_1|} e^{-\frac{x\nu}{|\xi_1|}} [Kf(x,\xi)-Kf(0,\xi)]\, d\xi\right| \leq C_{\alpha,\gamma}\langle f\rangle', \tag{4.33}$$

$$\left|\int_{\xi_1>0} -\phi_\alpha \int_0^x \frac{\nu}{|\xi_1|^2} e^{-\frac{(x-y)\nu}{|\xi_1|}} [Kf(x,\xi)-Kf(y,\xi)]\, dy\, d\xi\right| \leq C_{\alpha,\gamma}\langle f\rangle', \tag{4.34}$$

where

$$\langle f\rangle' := \|\|f\|\| + \|f(0,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3+})} + \|f(l,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3-})}. \tag{4.35}$$

A few remarks on the result above are as follow. Lemma 4.1 is crucial in proving the theorem 4.1 as well as Lemma 4.3. The proof for theorem 4.1 and Lemma 4.3 are identical for both cases, $-\frac{3}{2}<\gamma<0$ and $0\leq\gamma\leq 1$ with an exception that (4.28) has to be adopted for $-\frac{3}{2}<\gamma<0$. For the case when $0\leq\gamma\leq 1$, we do not need the estimation (4.28) since we have the inclusion $L^*\subset L^2$, however, the inclusion might not be true for the case $-\frac{3}{2}<\gamma<0$. The advantage of the estimations (4.27) and (4.28) allowed us to replace the collision term with a reciprocal function which can be further improved by a Gaussian functions.

$$\begin{aligned} \|K(f)\|_{L^{\infty,\frac{3}{2}-\frac{\gamma}{2}}} &\leq C_\gamma \|f\|_{L^*} \\ (1+|\xi|)^{\frac{3}{2}-\frac{\gamma}{2}} K(f) &\leq C_\gamma \|f\|_{L^*} \\ K(f) &\leq C_\gamma (1+|\xi|)^{\frac{\gamma}{2}-\frac{3}{2}} \|f\|_{L^*} \\ &\leq C_\gamma e^{\frac{|\xi|^2}{k}} \|f\|_{L^*}, \qquad \text{for any } k>0. \end{aligned} \tag{4.36}$$

Therefore,

$$K(f)e^{-\frac{|\xi|^2}{k}} \leq C_\gamma e^{-\frac{|\xi|^2}{k+1}} \|f\|_{L^*}, \qquad \text{for any } k>0. \tag{4.37}$$

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However, this formula is only valid for $-\frac{3}{2} < \gamma \leq 1$. As we have mentioned, (4.28) plays an important role of improving the results to soft potential case. The proof of Lemma 4.1 relies mainly on the pointwise estimate of kernel k_1 and k_2 which appear in the operator K,

$$K(f)(\xi) = \int_{\mathbb{R}^3} k(\xi, \xi_*) f(\xi_*)\, d\xi_* = \int_{\mathbb{R}^3} [k_1(\xi, \xi_*) - k_2(\xi, \xi_*)] f(\xi_*)\, d\xi_*. \tag{4.38}$$

The explicit form of the kernels can be found in [11] appendix A, and then using the formulas we will arrive at the following lemmas.

Lemma 4.4. *[11] Let* $-3 < \gamma \leq 1$. *We have*

$$|k_1(\xi, \xi_*)| \leq C \frac{e^{-\frac{1}{8}\left(\frac{|\xi|^2 - |\xi_*|^2}{|\xi - \xi_*|}\right)^2 - \frac{1}{8}|\xi - \xi_*|^2}}{|\xi - \xi_*|} (1 + |\xi| + |\xi_*|)^{\gamma - 1} w(|\xi - \xi_*|), \tag{4.39}$$

and

$$|\frac{\partial}{\partial \xi} k_1(\xi, \xi_*)| \leq C \frac{1 + |\xi|}{|\xi - \xi_*|^2} e^{-\frac{1}{16}\left(\frac{|\xi|^2 - |\xi_*|^2}{|\xi - \xi_*|}\right)^2 - \frac{1}{16}|\xi - \xi_*|^2} (1 + |\xi| + |\xi_*|)^{\gamma - 1} w(|\xi - \xi_*|), \tag{4.40}$$

where

$$w(t) = \begin{cases} t^{\gamma+1} \chi_{\{0 < t < \frac{1}{3}\}} + \chi_{\{t \geq \frac{1}{3}\}}, \\ |\ln t| \chi_{\{0 < t < \frac{1}{3}\}} + \chi_{\{t \geq \frac{1}{3}\}}, \\ 1, \end{cases} \leq \begin{cases} t^{\gamma+1} + 1, & \textit{if } \gamma < -1, \\ t^{-0.05} + 1, & \textit{if } \gamma = -1, \\ 1, & \textit{if } -1 < \gamma \leq 1. \end{cases} \tag{4.41}$$

Lemma 4.5. *[11] Let* $-3 < \gamma \leq 1$. *We have*

$$|k_2(\xi, \xi_*)| \leq C \frac{e^{-\frac{1}{8}\left(\frac{|\xi|^2 - |\xi_*|^2}{|\xi - \xi_*|}\right)^2 - \frac{1}{4}|\xi - \xi_*|^2}}{|\xi - \xi_*|} (1 + |\xi| + |\xi_*|)^{\gamma - 1} w(|\xi - \xi_*|), \tag{4.42}$$

doi:10.6342/NTU202202294

and

$$|\frac{\partial}{\partial\xi}k_2(\xi,\xi_*)| \le C\frac{1+|\xi|}{|\xi-\xi_*|^2}e^{-\frac{1}{16}\left(\frac{|\xi|^2-|\xi_*|^2}{|\xi-\xi_*|}\right)^2-\frac{1}{16}|\xi-\xi_*|^2}(1+|\xi|+|\xi_*|)^{\gamma-1}w(|\xi-\xi_*|), \quad (4.43)$$

where w as defined in Lemma 4.4.

Thus, using these two lemmas we can conclude that

$$|k(\xi,\xi_*)| \le \frac{e^{-\frac{1}{8}\left(\frac{|\xi|^2-|\xi_*|^2}{|\xi-\xi_*|}+|\xi-\xi_*|^2\right)}}{|\xi-\xi_*|}(1+|\xi|+|\xi_*|)^{\gamma-1}w(|\xi-\xi_*|). \quad (4.44)$$

The challenging part of this problem is that to improves the result beyond $-\frac{3}{2}$ we will need to improve the bound on the kernels, because the best the (4.44) could gives us is the estimate in Lemma 4.1. Another approach to this problem is to improve the result by Caflish in 1980, which states the following.

Lemma 4.6. *[9] Let $d \ge 3$ and $-\infty < \tau < d$. Then for any c_1, $c_2 > 0$, we have*

$$\int_{\mathbb{R}^d} \frac{1}{|\xi-\xi_*|^\tau} e^{-c_1\left|\frac{|\xi|^2-|\xi_*|^2}{|\xi-\xi_*|}\right|^2-c_2|\xi-\xi_*|^2} d\xi_* \le C_{\tau,c_1,c_2,d}(1+|\xi|)^{-1}. \quad (4.45)$$

Since we are dealing with $\mathbb{R}^3_\xi$, the reason why $\gamma > -\frac{3}{2}$ is the best we can obtain so far is because whenever we want to isolate the term

$$\frac{1}{|\xi-\xi'|^\tau} e^{-c_1\left|\frac{|\xi|^2-|\xi'|^2}{|\xi-\xi'|}\right|^2-c_2|\xi-\xi'|^2}, \quad (4.46)$$

in an integral, we need to apply Hölder inequality for $p = q = 2$ and in turn doubled the exponents.

Now, we run through again the calculations by Chen and Hsia [7] as well as Huang [11] to extract the logarithmic singularity. Firstly, rewrite the theorem 4.1 for $\xi_1 > 0$ (the

doi:10.6342/NTU202202294

case $\xi_1 < 0$ can be treated similarly).

Lemma 4.7. *[11] Consider the stationary linearized Boltzmann equation with cutoff (assuming (1.3) and (1.4)) and $\gamma \in (-\frac{3}{2}, 1]$. Let $f \in L^\infty_x\{[0,l], L^*_\xi(\mathbb{R}^3)\}$ be a solution to the problem (4.1) - (4.3) such that*

$$\nabla_\xi f(0,\xi) \in L^p_\xi(\mathbb{R}^{3+}), \qquad \text{for some } p \in (1,\infty), \tag{4.47}$$

$$f(0,\xi) \in L^\infty_\xi(\mathbb{R}^{3+}), \tag{4.48}$$

$$f(l,\xi) \in L^\infty_\xi(\mathbb{R}^{3-}). \tag{4.49}$$

Then for small $x > 0$, we have the formula

$$\frac{\partial}{\partial x}\sigma^+_\alpha(x) := -\ln x \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_\alpha(x)(0,\xi_2,\xi_3) Lf(x=0,0^+,\xi_2,\xi_3)\, d\xi_2 d\xi_3 + O(\langle f\rangle) \tag{4.50}$$

where

$$Lf(x=0,0^+,\xi_2,\xi_3) := \lim_{\xi_1\to 0^+} Lf(x=0,\xi_1,\xi_2,\xi_3), \tag{4.51}$$

$$\sigma^+_\alpha(x) = \int_{\xi_1>0} f(x,\xi)\phi_\alpha(\xi)\, d\xi \tag{4.52}$$

Proof. First we change everything into spherical coordinates

$$\xi = (\xi_1,\xi_2,\xi_3) = (\rho\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta\sin\varphi), \tag{4.53}$$

then we have

$$\frac{\partial}{\partial x}\sigma^+_\alpha(x) = \int_0^\infty \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\rho\cos\varphi} e^{-\frac{x\nu(\rho)}{\rho\cos\varphi}} Lf(0,\xi_1,\xi_2,\xi_3)\phi_\alpha(\xi)\rho^2\sin\varphi\, d\varphi d\theta d\rho. \tag{4.54}$$

doi:10.6342/NTU202202294

To simplify further, we let $z = \cos\varphi$, then

$$\begin{aligned} \frac{\partial}{\partial x}\sigma_\alpha^+(x) &= \int_0^\infty \int_0^{2\pi} \int_0^1 \frac{1}{z} e^{-\frac{x\nu}{\rho z}} Lf(0,\xi_1,\xi_2,\xi_3)\phi_\alpha(\xi)\rho \, dz d\theta d\rho \\ &= \int_0^\infty \int_0^{2\pi} \left(\int_0^1 \frac{1}{z} e^{-\frac{x\nu}{\rho z}} F(\rho, z, \theta)\, dz \right) e^{-\frac{\rho^2}{2}} \rho \, d\theta d\rho, \end{aligned} \tag{4.55}$$

where

$$F(\rho, z, \theta) = Lf(0,\xi_1,\xi_2,\xi_3)\xi^\alpha (2\pi)^{-\frac{3}{4}}. \tag{4.56}$$

Here, Chen and Hsia [7] suggest that the inner most integral of (4.55) can be handled with the following lemma.

Lemma 4.8. *Define the exponential integral*

$$E_1(x) := \int_0^1 \frac{1}{z} e^{-\frac{x}{z}} \, dz. \tag{4.57}$$

As $x \to 0^+$, we have the following asymptotic formula

$$E_1(x) = -\Upsilon - \ln x + \sum_{k=1}^\infty \frac{(-1)^{k+1} x^k}{k \cdot k!} = -\ln x + O(1), \tag{4.58}$$

where Υ is called the Euler-Mascheroni constant. Clearly, if $0 < \delta \le x$, then $E_1(x) \le C_\delta$, since it is a decreasing function.

Now observe that, if we let

$$G(z, x) := \int_1^z \frac{1}{u} e^{-\frac{x\nu}{\rho u}} \, du, \tag{4.59}$$

doi:10.6342/NTU202202294

the inner most integral of (4.55) becomes

$$\begin{aligned}\int_0^1 \frac{1}{z} e^{-\frac{x\nu}{\rho z}} F(\rho,z,\theta)\,dz &= \int_0^1 \left(\frac{\partial}{\partial z} G(z,x)\right) F(\rho,z,\theta)\,dz \\ &= E_1(\frac{x\nu}{\rho}) F(\rho,0,\theta) - \int_0^1 G(z,x)\left(\frac{\partial}{\partial z}F(\rho,z,\theta)\right) dz.\end{aligned} \tag{4.60}$$

Thus, we obtained two integrals

$$\frac{\partial}{\partial x}\sigma_\alpha^+(x) := I + II, \tag{4.61}$$

where

$$I := \int_0^\infty \int_0^{2\pi} E_1(\frac{x\nu}{\rho}) F(\rho,0,\theta) e^{-\frac{\rho^2}{2}} \rho\, d\theta d\rho, \tag{4.62}$$

$$II := \int_0^\infty \int_0^{2\pi} \int_0^1 G(z,x)\left(\frac{\partial}{\partial z}F(\rho,z,\theta)\right) dz e^{-\frac{\rho^2}{2}} \rho\, d\theta d\rho. \tag{4.63}$$

Since the first term has a singular point at $\rho = 0$, we will split the integral into two parts, namely, I_1 and I_2 which has the domain of ρ of $(0,\rho_0)$ and (ρ_0,∞), respectively. Also, ρ_0 is defined as

$$\rho_0 = \rho_0(x) := \sup\left\{\rho : \frac{x\nu(\rho)}{\rho} > 1\right\}. \tag{4.64}$$

ρ_0 exists since x is held fixed and $\nu(\rho)$ is a bounded function. Clearly, for $0 < \rho < \rho_0$

$$\frac{\nu(\rho)}{\rho} x \geq 1 > \frac{\nu_0}{\nu_1}. \tag{4.65}$$

On the other hand, for $\rho \geq \rho_0$ we have

$$\frac{\nu(\rho)}{\rho} x \leq 1. \tag{4.66}$$

Notice that, since for $\rho < \rho_0$, $\frac{\nu(\rho)}{\rho}x$ is uniformly bounded from below and the bound is

doi:10.6342/NTU202202294

independent of x, we shall expect the singularity comes from I_2.

Now, we turn out attention to I_2. By lemma 4.8

$$\begin{aligned}
I_2 &= \int_{\rho_0}^{\infty}\int_0^{2\pi} E_1(\frac{x\nu}{\rho})F(\rho,0,\theta)e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho \\
&= -\ln x\int_{\rho_0}^{\infty}\int_0^{2\pi} F(\rho,0,\theta)e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho + O(\langle f\rangle) \\
&= -\ln x\int_0^{\infty}\int_0^{2\pi} F(\rho,0,\theta)e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho + \ln x\int_0^{\rho_0}\int_0^{2\pi} F(\rho,0,\theta)e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho + O(\langle f\rangle).
\end{aligned} \tag{4.67}$$

Here, the second integral is bounded, because

$$\begin{aligned}
&\left|\int_0^{\rho_0}\int_0^{2\pi} F(\rho,0,\theta)e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho\right| \\
\leq& \int_0^{\rho_0}\int_0^{2\pi} |F(\rho,0,\theta)|e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho \\
\leq& C_\alpha\int_0^{\rho_0}\int_0^{2\pi} \rho^{\alpha+1}|L(f)(\rho,0,\theta)|e^{-\frac{\rho^2}{3}}\, d\theta d\rho \\
\leq& C_\alpha\int_0^{\rho_0}\int_0^{2\pi} \rho^{\alpha+1}|\nu(\rho)f(\rho,0,\theta))| + |K(f)(\rho,0,\theta)|e^{-\frac{\rho^2}{3}}\, d\theta d\rho \\
\leq& C_\alpha\int_0^{\rho_0}\int_0^{2\pi} \rho^{\alpha+1}\left[(1+\rho)^\gamma\|f(\rho,0,\theta)\|_{L^\infty} + (1+\rho)^{\frac{\gamma}{2}-\frac{3}{2}}|||f|||\right]e^{-\frac{\rho^2}{3}}\, d\theta d\rho \\
\leq& C_\alpha\langle f\rangle\int_0^{\rho_0}\int_0^{2\pi} e^{-\frac{\rho^2}{4}}\, d\theta d\rho \\
\leq& C_\alpha\langle f\rangle\int_0^{\rho_0}\int_0^{2\pi} d\theta d\rho \\
\leq& C_\alpha\langle f\rangle\rho_0.
\end{aligned} \tag{4.68}$$

Therefore, from (4.67),

$$\begin{aligned}
I_2 &= -\ln x\int_0^{\infty}\int_0^{2\pi} F(\rho,0,\theta)e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho + O(\langle f\rangle(1+\rho_0|\ln x)) \\
&= -\ln x\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_\alpha(0,\xi_2,\xi_3)Lf(x=0,0^+,\xi_2,\xi_3)\, d\xi_2 d\xi_3 + O(\langle f\rangle(1+\rho_0|\ln x)).
\end{aligned} \tag{4.69}$$

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Now, observe that for $-\frac{3}{2} < \gamma < 0$,

$$
\begin{aligned}
\frac{x}{\rho_0} &= \frac{1}{\nu(\rho_0)} && \text{(By definition of } \rho_0\text{)}, \\
&\geq \frac{1}{\nu_1(1+\rho_0)^\gamma} && \text{(By (1.13))}, \\
&\geq \frac{1}{\nu_1} \\
\therefore \rho_0 &\leq \nu_1 x.
\end{aligned}
\tag{4.70}
$$

Similarly, for $0 \leq \gamma \leq 1$, then

$$
\begin{aligned}
1 &= \frac{\nu(\rho_0)}{\rho_0} x && \text{(By definition of } \rho_0\text{)}, \\
&\leq \frac{\nu_1(1+\rho_0)^\gamma}{\rho_0} x && \text{(By (1.13))}.
\end{aligned}
\tag{4.71}
$$

Since this inequality is true for all $0 \leq \gamma \leq 1$, we can pick $\gamma = 0$, so we have

$$\rho_0 \leq \nu_1 x. \tag{4.72}$$

Thus, for small x,

$$\rho_0 |\ln x| \leq \nu_1 x |\ln x| \leq C. \tag{4.73}$$

Hence,

$$I_2 = -\ln x \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_\alpha(0, \xi_2, \xi_3) Lf(x = 0, 0^+, \xi_2, \xi_3) \, d\xi_2 d\xi_3 + O(\langle f \rangle). \tag{4.74}$$

For the remaining integrals, we will show that their contribution is of order $O(\langle f \rangle)$.

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Firstly, by lemma 4.8 and similar calculation as presented in (4.68)

$$\begin{aligned} I_1 &= \int_{\rho_0}^{\infty}\int_0^{2\pi} E_1(\frac{x\nu}{\rho})F(\rho,0,\theta)e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho \\ &\leq C\int_{\rho_0}^{\infty}\int_0^{2\pi} |F(\rho,0,\theta)|e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho \\ &\leq C\langle f\rangle. \end{aligned} \tag{4.75}$$

To estimate II, we need the close form of $\frac{\partial}{\partial z}F$. Clearly,

$$\begin{aligned} \frac{\partial}{\partial z}F(\rho,z,\theta) &= \frac{\partial}{\partial z}Lf(0,\xi)\xi^{\alpha}(2\pi)^{-\frac{3}{4}} \\ &= (2\pi)^{-\frac{3}{4}}\nu(\rho)\frac{\partial}{\partial z}\xi^{\alpha}f(0,\xi) + (2\pi)^{-\frac{3}{4}}\frac{\partial}{\partial z}\xi^{\alpha}Kf(0,\xi) \\ &=: (2\pi)^{-\frac{3}{4}}\nu(\rho)\frac{\partial}{\partial z}g_1(\xi) + (2\pi)^{-\frac{3}{4}}\frac{\partial}{\partial z}g_2(\xi), \end{aligned} \tag{4.76}$$

where

$$g_1(\xi) := \xi^{\alpha}f(0,\xi), \tag{4.77}$$

$$g_2(\xi) := \xi^{\alpha}Kf(0,\xi). \tag{4.78}$$

Therefore, we separate II into two integrals, namely, II_1 and II_2.

$$II_1 = \int_0^{\infty}\int_0^{2\pi}\int_0^1 G(z,x)\left((2\pi)^{-\frac{3}{4}}\nu(\rho)\frac{\partial}{\partial z}g_1(\xi)\right)\, dz\, e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho, \tag{4.79}$$

$$II_2 = \int_0^{\infty}\int_0^{2\pi}\int_0^1 G(z,x)\left((2\pi)^{-\frac{3}{4}}\frac{\partial}{\partial z}g_2(\xi)\right)\, dz\, e^{-\frac{\rho^2}{2}}\rho\, d\theta d\rho. \tag{4.80}$$

It is clear that $|G(z,x)| \leq |\ln z|$, and

$$\frac{\partial}{\partial z} = \rho\frac{\partial}{\partial \xi_1} - \frac{\rho z}{\sqrt{1-z^2}}\sin\theta\frac{\partial}{\partial \xi_2} - \frac{\rho z}{\sqrt{1-z^2}}\cos\theta\frac{\partial}{\partial \xi_3}, \tag{4.81}$$

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then we have

$$
\begin{aligned}
|II_1| &\le \int_0^\infty \int_0^{2\pi} \int_0^1 |G(z,x)| \left| \frac{\partial}{\partial z} g_1(\xi) \right| \, dz \, e^{-\frac{\rho^2}{2}} \rho \nu(\rho) \, d\theta d\rho \\
&\le \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| \left| \frac{\partial}{\partial z} g_1(\xi) \right| \, dz \, e^{-\frac{\rho^2}{3}} \rho \, d\theta d\rho \\
&\le \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| \left(\left| \frac{\partial}{\partial \xi_1} g_1(\xi) \right| + \frac{z}{\sqrt{1-z^2}} \left| \frac{\partial}{\partial \xi_2} g_1(\xi) \right| + \frac{z}{\sqrt{1-z^2}} \left| \frac{\partial}{\partial \xi_3} g_1(\xi) \right| \right) \, dz \, e^{-\frac{\rho^2}{3}} \rho^2 \, d\theta d\rho \\
&\le \int_{\xi_1>0} \left(\left| \frac{\partial}{\partial \xi_2} g_1(\xi) \right| + \left| \frac{\partial}{\partial \xi_3} g_1(\xi) \right| \right) e^{-\frac{|\xi|^2}{3}} |\xi|^2 \, d\xi + \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| \left| \frac{\partial}{\partial \xi_1} g_1(\xi) \right| \, dz \, e^{-\frac{\rho^2}{3}} \rho^2 \, d\theta d\rho \\
&=: II_{11} + II_{12}.
\end{aligned}
\tag{4.82}
$$

By the definition of $g_1(\xi)$ and product rule, we see that

$$
\begin{aligned}
II_{11} &\le C_\alpha \int_{\xi_1>0} e^{-\frac{|\xi|^2}{4}} \left(\left| \frac{\partial f}{\partial \xi_2}(0,\xi) \right| + \left| \frac{\partial f}{\partial \xi_2}(0,\xi) \right| + |f(0,\xi)| \right) d\xi \\
&\le C_{\alpha,p} \left(\|\nabla_\xi f(0,\xi)\|_{L^p_\xi(\mathbb{R}^{3+})} + \|f(0,\xi)\|_{L^p_\xi(\mathbb{R}^{3+})} \right) \\
&\le C_{\alpha,p} \langle f \rangle.
\end{aligned}
\tag{4.83}
$$

The last inequality can be manipulated by Hölder's inequality. On the other hand, apply Hölder's inequality again

$$
\begin{aligned}
II_{12} &= \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| \left| \frac{\partial}{\partial \xi_1} g_1(\xi) \right| \, dz \, e^{-\frac{\rho^2}{3}} \rho^2 \, d\theta d\rho \\
&\le \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| e^{-\frac{\rho^2}{8}} \left| \frac{\partial}{\partial \xi_1} g_1(\xi) \right| e^{-\frac{\rho^2}{8}} \, dz d\theta d\rho \\
&\le \left(\int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z|^{p'} e^{-\frac{p'\rho^2}{8}} \, dz d\theta d\rho \right)^{\frac{1}{p'}} \left(\int_0^\infty \int_0^{2\pi} \int_0^1 \left| \frac{\partial}{\partial \xi_1} g_1(\xi) \right|^p e^{-\frac{p\rho^2}{8}} \, dz d\theta d\rho \right)^{\frac{1}{p}} \\
&\le C_{\alpha,p} \left[\int_0^\infty \int_0^{2\pi} \int_0^1 \left(\left| \frac{\partial f}{\partial \xi_1}(0,\xi) \right|^p + |f(0,\xi)|^p \right) e^{-\frac{p\rho^2}{9}} \, dz d\theta d\rho \right]^{\frac{1}{p}} \\
&\le C_{\alpha,p} \left(\|\nabla_\xi f(0,\xi)\|_{L^p_\xi(\mathbb{R}^{3+})} + \|f(0,\xi)\|_{L^p_\xi(\mathbb{R}^{3+})} \right) \\
&\le C_{\alpha,p} \langle f \rangle.
\end{aligned}
\tag{4.84}
$$

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Similarly, we separate II_2 into two integrals.

$$II_2 \le \int_{\xi_1>0} \left(\left| \frac{\partial}{\partial \xi_2} g_2(\xi) \right| + \left| \frac{\partial}{\partial \xi_3} g_2(\xi) \right| \right) e^{-\frac{|\xi|^2}{3}} |\xi|^2 \, d\xi + \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| \left| \frac{\partial}{\partial \xi_1} g_2(\xi) \right| \, dz \, e^{-\frac{\rho^2}{3}} \rho^2 \, d\theta d\rho \quad (4.85)$$

$$=: II_{21} + II_{22}.$$

By the definition of $g_2(\xi)$ and product rule, we see that

$$\begin{aligned} |II_{21}| &\le C_\alpha \int_{\xi_1>0} (|Kf(0,\xi)| + |\nabla_\xi Kf(0,\xi)|) \, e^{-\frac{|\xi|^2}{4}} |\xi|^2 \, d\xi \\ &\le C_\alpha \int_{\xi_1>0} \left((1+|\xi|)^{\frac{\gamma}{2}-\frac{3}{2}} \|f(0,\cdot)\|_{L^*} + \|f(0,\xi)\|_{L^\infty_\xi} \right) e^{-\frac{|\xi|^2}{3}} |\xi|^2 \, d\xi \\ &\le C_\alpha \left(\|f(0,\xi)\|_{L^*_\xi} + \|f(0,\xi)\|_{L^\infty_\xi} \right) \\ &\le C_\alpha \langle f \rangle. \end{aligned} \quad (4.86)$$

The second inequality follows from lemma 4.1 and lemma 4.2. Finally,

$$\begin{aligned} II_{22} &= \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| \left| \frac{\partial}{\partial \xi_1} g_2(\xi) \right| \, dz \, e^{-\frac{\rho^2}{3}} \rho^2 \, d\theta d\rho \\ &\le \int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z| e^{-\frac{\rho^2}{8}} \left| \frac{\partial}{\partial \xi_1} g_2(\xi) \right| e^{-\frac{\rho^2}{8}} \, dz d\theta d\rho \\ &\le \left(\int_0^\infty \int_0^{2\pi} \int_0^1 |\ln z|^{p'} e^{-\frac{p'\rho^2}{8}} \, dz d\theta d\rho \right)^{\frac{1}{p'}} \left(\int_0^\infty \int_0^{2\pi} \int_0^1 \left| \frac{\partial}{\partial \xi_1} g_2(\xi) \right|^p e^{-\frac{p\rho^2}{8}} \, dz d\theta d\rho \right)^{\frac{1}{p}} \\ &\le C_{\alpha,p} \left[\int_0^\infty \int_0^{2\pi} \int_0^1 \left(\left| \frac{\partial}{\partial \xi_1} Kf(0,\xi) \right|^p + |Kf(0,\xi)|^p \right) e^{-\frac{p\rho^2}{9}} \, dz d\theta d\rho \right]^{\frac{1}{p}} \\ &\le C_{\alpha,p} \left(\|f(0,\xi)\|_{L^*_\xi} + \|f(0,\xi)\|_{L^\infty_\xi} \right) \\ &\le C_{\alpha,p} \langle f \rangle. \end{aligned} \quad (4.87)$$

To sum up, we have

$$|II| \le C_{\alpha,p} \langle f \rangle. \quad (4.88)$$

□

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As the proof shows, all the calculations are valid for all $-3 < \gamma \leq 1$ except for the parts where we need to employ the lemma 4.1 and lemma 4.2. The proof also shows that to obtain $\rho_0(x) \leq \nu_1 x$ as in (4.70) and (4.72), the idea in [7] can also be carried out instead of the small adjustment by [11], although, both technique can be adopted interchangeably.

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Chapter 5 Conclusion and Challenges

In this survey, we revised the existence and uniqueness theorem of the infamous Milne and Kramers problem. The proof provided by Bardos, Calflisch and Nicolaenko [2] was carried out for the case $\gamma = 1$. For the hard and soft potential model, we can refer to the work by Golse and Poupaud [10]. Therefore, the existence and uniqueness theorem is completed for both Milne and Kramers problems for finite interval.

On the other hand, more work has to be done to establish the formula of the derivative of moment function for soft potential model. Although Huang's astonishing idea [11] have extended the result to $-\frac{3}{2} < \gamma < 1$, the case $-3 < \gamma \leq -\frac{3}{2}$ is still remain unknown to us. There are several approaches which has a high potential of tackling the problem. We can improve the pointwise estimate for the kernels k_1 and k_2 or the estimate in lemma 4.6 by Caflish [9] or improving Lemma 4.6 by Caflish [9] as mentioned earlier.

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