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反應擴散對流方程的傳動波解

Travelling Wave Solutions for Reaction-Diffusion-Advection Equations

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本論文係裘愉生君（R96221030）在國立臺灣大學數學學系、所完成之碩（博）士學位論文，於民國 99 年 06 月 08 日承下列考試委員審查通過及口試及格，特此證明

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摘要：

本論文分成兩部份。

第一部份是關於反應擴散對流方程 $u_t - \Delta u + q(x,y)\cdot\nabla_{x,y}u = f(u)$ 的傳動波解。我們考慮具有週期性的對流場 $q(x,y)$、燃燒型態及半穩定的非線性反應項 f。我們主要整理 Berestycki 和 Hamel 文章[1]中的脈衝波的存在性、唯一性及單調性結果。第二部份主要是處理三種競爭物種 Lotka-Volterra 系統的確切傳動波解。

關鍵字：脈衝波、反應擴散對流方程、燃燒、週期、確切解、Lotka-Volterra 系統

Abstract

There are two parts in this paper. Part I is concerned with the travelling wave solutions for reaction-diffusion-advection equations $u_t - \triangle u + q(x, y) \cdot \nabla_{x,y} u = f(u)$. We consider periodic advection $q(x, y)$ and combustion, monostable nonlinear reaction term f. We mainly survey the results of existence, uniqueness, and monotonicity of pulsating waves from the paper by Berestycki and Hamel [1]. Part II deals with exact travelling wave solutions of competitive Lotka-Volterra systems of three species.

Keywords: pulsating travelling wave, reaction-diffusion-advection equations, combustion, periodic, exact solutions, Lotka-Volterra systems

Contents

List of Figures

Part I

Pulsating Travelling Wave Solutions

Chapter 1

Introduction

Consider the following reaction-diffusion-advection equation:

$$\begin{cases} u_t - \triangle u + q(x,y)\cdot \nabla_{x,y} u = f(u), & (t,x,y) \in \mathbb{R}\times\Sigma \\ \frac{\partial u}{\partial \nu} = 0, & (t,x,y) \in \mathbb{R}\times\partial\Sigma \end{cases} \tag{1.1}$$

where $\Sigma = \{(x,y) \in \mathbb{R}\times\Omega\}$, whose cross section $\Omega \subset \mathbb{R}^{n-1}$ is a bounded, smooth and connected open set; $\nu = \nu(x,y) = \nu(y)$ is the outward unit normal to $\partial\Sigma = \mathbb{R}\times\partial\Omega$. We assume that the nonlinearity f is of class $C^2[0,1]$. There are various types of f. For example,

- KPP type (monostable type): $0 < f(u) < 1$ in $(0,1)$, $f(0) = f(1) = 0$, $f'(0) > 0$,

  e.g. $f(u) = u(1-u)$.

- Bistable type: $f(u) < 0$ in $(0,\theta)$, $f(u) > 0$ in $(\theta,1)$,

  e.g. $f(u) = u(1-u)(u-\theta)$, $\theta < \frac{1}{2}$.

- Combustion type: $f(u) = 0$ in $[0,\theta]$, $f(u) > 0$ in $(\theta,1)$, $f(1) = 0$.

Throught this paper, we only consider the combustion and monostabel type.

The velocity field $q(x,y) = (q_1(x,y), q_2(x,y), ..., q_n(x,y))$ is assumed to be of class C^2 and satisfies the assumptions: there exists some $L > 0$ such that

$$
\begin{cases}
\operatorname{div} q = 0 & in\ \overline{\Sigma} \\
\forall (x,y) \in \overline{\Sigma}, & q(x+L,y) = q(x,y) \\
\forall (x,y) \in \overline{\Sigma}, & \int_{(0,L)\times\Omega} q_1(x,y)\, dx\, dy = 0 \\
q \cdot \nu = 0 & on\ \partial\Sigma
\end{cases}
\tag{1.2}
$$

The second assertion in (1.2) means that q is L-periodic in the x-variable.

We are interested in travelling wave solutions of (1.1), namely, the pulsating travelling wave.

Definition 1.1. A pulsating wave propagating with the speed $c \neq 0$ is a classical solution $u \in C^{1,2}\left(\mathbb{R} \times \overline{\Sigma}\right)$ of (1.1), if the following holds:

$$
u\left(t + \frac{L}{c}, x, y\right) = u(t, x+L, y)
$$

for all $(t,x,y) \in \mathbb{R} \times \overline{\Sigma}$, and has limiting conditions, for all $t \in \mathbb{R}$,

$$
\begin{cases}
u(t, -\infty, y) = 0 \\
u(t, \infty, y) = 1
\end{cases}
$$

uinformly with respect to y.

To sum up, a pulsating wave of (1.1) is a solution of the problem

$$
\begin{cases}
u_t - \triangle u + q(x,y) \cdot \nabla_{x,y} u = f(u), & (t,x,y) \in \mathbb{R} \times \Sigma \\
\frac{\partial u}{\partial \nu} = 0, & (t,x,y) \in \mathbb{R} \times \partial\Sigma \\
u\left(t + \frac{L}{c}, x, y\right) = u(t, x+L, y), & (t,x,y) \in \mathbb{R} \times \overline{\Sigma} \\
u(t, -\infty, y) = 0, & (t,y) \in \mathbb{R} \times \Omega \\
u(t, \infty, y) = 1, & (t,y) \in \mathbb{R} \times \Omega
\end{cases}
\tag{1.3}
$$

In the case of combustion type nonlinearity, Berestycki and Hamel [1] proved the following results:

Theorem 1.2. *Let q be a velocity field satisfying (1.2). Then there exists a unique classical solution (c, u) of (1.3) with combustion type. The function u is increasing in t and unique up to translation in t. Moreover, $c > 0$ and $0 < u < 1$.*

We will prove the properties of the pulsating travelling wave solutions and sketch the proof of existence in Chapter 2.

If nonlinear reaction term f is monostable type, we have the results as follows:

Theorem 1.3. *Let q be a velocity field satisfying (1.2) and nonlinear reaction term f is monostable type. Then there exists a $c^* > 0$ such that*

$$\begin{cases} \text{there exists a classical solution } (c, u) \text{ of (1.3)}, & if\ c \geq c^* \\ \text{there is no classical solution } (c, u) \text{ of (1.3)}, & if\ c < c^*. \end{cases}$$

Moreover, the solution u is increasing in t and $0 < u < 1$ if $c \geq c^*$.

We will sketch the proof of existence in Chapter 3.

Chapter 2

Proof of Theorem 1.2

In this chapter, we prove the Theorem 1.2. First, we prove the positivity of the speed c in section 2.1. Next, we prove the uniqueness of the speed c and monotonicity of the solution u in section 2.2. Finally, we sketch the proof of the existence of the solution in section 2.3.

We make the same change of variables as Xin [6]. Define $u(t,x,y) = \phi(s,x,y)$, where $s = x + ct$. The problem (1.3) is equivalent to

$$\begin{cases} \phi_{ss} + \phi_{sx} + \phi_{xs} + \triangle_{x,y}\phi - (q_1(x,y) + c)\,\phi_s - q(x,y)\cdot\nabla_{x,y}\phi + f(\phi) = 0, & in\ \mathbb{R}\times\Sigma \\ \phi_s\nu_1 + \nabla_{x,y}\phi\cdot\nu = 0, & on\ \mathbb{R}\times\partial\Sigma \\ \phi(s,x,y) = \phi(s,x+L,y), & in\ \mathbb{R}\times\overline{\Sigma} \\ \phi(-\infty,x,y) = 0 & \\ \phi(\infty,x,y) = 1 & \end{cases} \tag{2.1}$$

where $q(x,y) = (q_1(x,y), \widetilde{q}(x,y))$, $\widetilde{q}(x,y) = (q_2(x,y),...,q_n(x,y)) \in \mathbb{R}^{n-1}$ and ν_1 is the first component of ν. Note that the equation in (2.1) is the degenerate elliptic equation.

For simplicity, define the operator

$$\mathcal{L}\phi := \phi_{ss} + \phi_{sx} + \phi_{xs} + \triangle_{x,y}\phi - (q_1(x,y) + c)\,\phi_s - q(x,y) \cdot \nabla_{x,y}\phi. \qquad (2.2)$$

2.1 Positivity of the Speed

Theorem 2.1. *Suppose that (c,u) is a classical solution of (1.3). Then we have $c > 0$.*

Proof.

Integrating the first equation of (2.1) over $(-a,a) \times (0,L) \times \Omega$, for some $a > 0$, it follows that

$$\int_{(0,L)\times\Omega} [\phi_s(a,x,y) - \phi_s(-a,x,y)]\ dx\ dy + \int_{(0,L)\times\Omega} [\phi_x(a,x,y) - \phi_x(-a,x,y)]\ dx\ dy$$

$$- \int_{(0,L)\times\Omega} q_1(x,y)\,[\phi(a,x,y) - \phi(-a,x,y)]\ dx\ dy - c\int_{(0,L)\times\Omega} [\phi(a,x,y) - \phi(-a,x,y)]\ dx\ dy$$

$$- \int_{(-a,a)\times(0,L)\times\Omega} q \cdot \nabla_{x,y}\phi\ ds\ dx\ dy + \int_{(-a,a)\times(0,L)\times\Omega} f(\phi)\ ds\ dx\ dy = 0.$$

Here we have used integration by parts and the boudary condition in (2.1). Since the assumption of the velocity field q and limiting condition, one obtains that

$$\int_{(-a,a)\times(0,L)\times\Omega} q \cdot \nabla_{x,y}\phi\ ds\ dx\ dy = 0$$

and

$$\int_{(0,L)\times\Omega} q_1(x,y)\left[\phi\left(a,x,y\right)-\phi\left(-a,x,y\right)\right]\,dx\,dy=0$$

as $a\to\infty$. Lastly, since $\nabla_{s,x,y}\phi\to 0$ as $s\to\pm\infty$, we have

$$cL\left|\Omega\right|=\int_{\mathbb{R}\times(0,L)\times\Omega} f\left(\phi\right)\ ds\ dx\ dy$$

as $a\to\infty$. Then $c>0$ because $\int_{\mathbb{R}\times(0,L)\times\Omega} f\left(\phi\right)\ ds\ dx\ dy>0$. □

Actually, we have the bounds for the speeds of propagation [4].

Proposition 2.2. *Given $M>0$, there exist $\underline{c}$, $\bar{c}$ depending only on θ, f and M such that $0<\underline{c}<\bar{c}$ and $\|q\|_\infty\le M$. Then for all solution (c,u) of (1.3), we have the estimates on the speed*

$$\underline{c}\le c\le\bar{c}.$$

2.2 The Uniqueness of the Speed and Monotonicity of the Solution

In this section, we prove the uniqueness of the speed c and monotonicity of the solution u for the problem (1.3). We mainly use sliding method in infinite cylinders developed by Berestycki and Nirenberg [3]. We first prove the monotonicity of the solution u.

Theorem 2.3. *Let (c,u) be a classical solution of (1.3). Then the function u is increasing in t.*

Remark. The assertion of Theorem 2.3 is equivalent to the function ϕ is increasing in s. It turns out that this fact will be used in proving the uniqueness of the speed.

We want to use the sliding method so the following lemmas are useful. The following lemmas are maximum principle in unbounded domain [1].

Lemma 2.4. *Let $f(\phi)$ be a globally bounded and Lipschitz-continuous function defined on $\mathbb{R}$, and assume that f is nonincreasing with respect to ϕ in $(-\infty, \delta]$ for some $\delta > 0$. Let $h \in \mathbb{R}$ and define $\Sigma_h^- = (-\infty, h) \times \Sigma$. Let $c \neq 0$ and $\phi_1(s, x, y)$, $\phi_2(s, x, y) \in C^{1,2}\overline{(\Sigma_h^-)}$ such that*

$$\begin{cases} \mathcal{L}\phi_1 + f(\phi_1) \geq 0, & in\ \Sigma_h^- \\ \mathcal{L}\phi_2 + f(\phi_2) \leq 0, & in\ \Sigma_h^- \\ \partial_s (\phi_1 - \phi_2) \nu_1 + \nabla_{x,y} (\phi_1 - \phi_2) \cdot \nu \leq 0, & on\ (-\infty, h] \times \partial\Sigma \\ \lim\limits_{s_0 \to -\infty} \sup\limits_{\substack{s \leq s_0 \\ (x,y) \in \overline{\Sigma}}} (\phi_1 - \phi_2)(s, x, y) \leq 0, & \end{cases}$$

where $\mathcal{L}$ is defined as (2.2). If $\phi_1 \leq \delta$ in $\overline{\Sigma_h^-}$ and $\phi_1(h, x, y) \leq \phi_2(h, x, y)$ for all $(x, y) \in \overline{\Sigma}$, $\phi_1 \leq \phi_2$ in $\overline{\Sigma_h^-}$.

We replace s by $-s$ in Lemma 2.4, we have a similar result as follows:

Lemma 2.5. *Let $f(\phi)$ be a globally bounded and Lipschitz-continuous function defined on $\mathbb{R}$, and assume that f is nonincreasing with respect to ϕ in $[1-\delta, \infty)$ for some $\delta > 0$. Let $h \in \mathbb{R}$ and define $\Sigma_h^+ = (h, \infty) \times \Sigma$. Let $c \neq 0$ and $\phi_1(s, x, y)$, $\phi_2(s, x, y) \in C^{1,2}\overline{(\Sigma_h^+)}$ such that*

$$\begin{cases} \mathcal{L}\phi_1 + f(\phi_1) \geq 0, & in\ \Sigma_h^+ \\ \mathcal{L}\phi_2 + f(\phi_2) \leq 0, & in\ \Sigma_h^+ \\ \partial_s (\phi_1 - \phi_2) \nu_1 + \nabla_{x,y} (\phi_1 - \phi_2) \cdot \nu \leq 0, & on\ [h, \infty) \times \partial\Sigma \\ \lim\limits_{s_0 \to \infty} \sup\limits_{\substack{s \geq s_0 \\ (x,y) \in \overline{\Sigma}}} (\phi_1 - \phi_2)(s, x, y) \leq 0, & \end{cases}$$

where $\mathcal{L}$ is defined as (2.2). If $\phi_2 \geq 1 - \delta$ in $\overline{\Sigma_h^+}$ and $\phi_1(h, x, y) \leq \phi_2(h, x, y)$ for all $(x, y) \in \overline{\Sigma}$, $\phi_1 \leq \phi_2$ in $\overline{\Sigma_h^+}$.

Now, we prove the following unique result.

Theorem 2.6. *Suppose* (c_1, u_1) *and* (c_2, u_2) *are two classical solution of (1.3), then* $c_1 = c_2$ *and there exists* $h \in \mathbb{R}$ *such that* $u_1(t, x, y) = u_2(t + h, x, y)$ *for all* $(t, x, y) \in \mathbb{R} \times \overline{\Sigma}$.

Proof.

Let (c_1, u_1) and (c_2, u_2) be two classical solutions of (1.3). We assume that $c_1 \geq c_2 > 0$. Let $\phi_1(s, x, y) = u_1\left(\frac{s-x}{c_1}, x, y\right)$ and $\phi_2(s, x, y) = u_2\left(\frac{s-x}{c_2}, x, y\right)$. Then the functions ϕ_1 and ϕ_2 satisfy the same boundary, periodicity, and limiting condition. And ϕ_1 is a solution of

$$\partial_{ss}\phi_1 + \partial_{sx}\phi_1 + \partial_{xs}\phi_1 + \triangle\phi_1 - (q_1 + c_1)\,\partial_s\phi_1 - q \cdot \nabla\phi_1 + f(\phi_1) = 0. \tag{2.3}$$

On the other hand, ϕ_2 satisfies

$$\begin{aligned} & \partial_{ss}\phi_2 + \partial_{sx}\phi_2 + \partial_{xs}\phi_2 + \triangle\phi_2 - (q_1 + c_1)\,\partial_s\phi_2 - q \cdot \nabla\phi_2 + f(\phi_2) \\ = \;& (c_2 - c_1)\,\partial_s\phi_2 \\ \leq \;& 0, \end{aligned} \tag{2.4}$$

the last inequality holds since $\partial_s\phi_2 > 0$ from Theorem 2.3.

Now, we slide the function ϕ_2 with respect to ϕ_1. We use Lemma 2.4 and Lemma 2.5 to get that there exists a $\tau^* \in \mathbb{R}$ such that $\phi_2(s + \tau^*, x, y) = \phi_1(s, x, y)$ for all $(s, x, y) \in \mathbb{R} \times \overline{\Sigma}$. Putting that into (2.3) and (2.4) gives $(c_2 - c_1)\,\partial_s\phi_2 \equiv 0$. This implies that $\partial_s\phi_2 = 0$ then one has reached a contradiction. Hence, $c_1 = c_2 := c$, and from definition of ϕ_1 and ϕ_2, we have $u_1(t, x, y) = u_2(t + \frac{\tau^*}{c}, x, y)$ for all $(t, x, y) \in \mathbb{R} \times \overline{\Sigma}$. $\square$

2.3 Existence of the Pulsating Travelling Wave Solution

In this section, since the proof of the existence result in Berestycki and Hamel [1] is tedious, we only give the main idea of their proof. Recently, M. Bages and P. Martinez [4] gave the proof for the existence results by a new method.

We divide the proof into four steps:

Step 1: Elliptic regularization in finite cylinder

Recall that the first equation of (2.1) is the degenerate elliptic equation so we use elliptic regularization. Let $\varepsilon > 0$ is regularization parameter and define

$$\mathcal{L}_\varepsilon \phi := \varepsilon \phi_{ss} + \phi_{ss} + \phi_{sx} + \phi_{xs} + \triangle_{x,y}\phi - (q_1(x,y) + c)\,\phi_s - q(x,y)\cdot\nabla_{x,y}\phi.$$

Now, we consider the problem on the finite cylinder. Hence, the problem (2.1) becomes

$$\begin{cases} \mathcal{L}_\varepsilon \phi + f(\phi) = 0, & in\ \Sigma_a \\ \phi_s \nu_1 + \nabla_{x,y}\phi \cdot \nu = 0, & on\ (-a,a)\times\partial\Sigma \\ \phi(s,x,y) = \phi(s,x+L,y), & in\ \overline{\Sigma_a} \\ \phi(-a,x,y) = 0 & \\ \phi(a,x,y) = 1 & \end{cases} \tag{2.5}$$

where $\Sigma_a = (-a,a)\times\Sigma$, $a > 0$.

From Lax-Milgram theorem, we get a weak solution ϕ for the probelm (2.5), and then using regularity theory up to the boundary, hence the solution ϕ is a classical solution in $\widetilde{\Sigma_a} := \overline{\Sigma_a}\setminus(\{\pm a\}\times\partial\Sigma)$. Finally, we build a supersolution (see [1, 3]) to get the solution ϕ can be continuously extended on the corners $\{\pm a\}\times\partial\Sigma$ of the closed cylinder $\overline{\Sigma_a}$. Moreover, we use sliding method to get the uniquess and monotonicity of the solution, it's the same as Section 2.2. Hence, we have the results as the following:

Theorem 2.7. *For each $c \in \mathbb{R}$, there exists unique solution $\phi^c_{\varepsilon,a} \in C\left(\overline{\Sigma_a}\right) \cap C^2\left(\widetilde{\Sigma_a}\right)$ of (2.5) and the solution is increasing in s.*

For a large enough, we ensure that the existence of the nontrivial solution $\phi^c_{\varepsilon,a}$ and the speed $c_{\varepsilon,a}$. So we need bounds for the speed and the solution satisfies normalization condition.

Proposition 2.8. *There exists $a_0 > 0$ and $k > 0$ such that, $\forall a \geq a_0$ and $\forall \varepsilon \in (0,1]$, there exists a unique $c := c_{\varepsilon,a} \in \mathbb{R}$ such that the solution $\phi^c_{\varepsilon,a} \in C\left(\overline{\Sigma_a}\right) \cap C^2\left(\widetilde{\Sigma_a}\right)$ of (2.5) satisfies the normalization condition*

$$\max_{\overline{\Sigma}} \phi^c_{\varepsilon,a}(0,x,y) = \max_{[0,L]\times\overline{\Omega}} \phi^c_{\varepsilon,a}(0,x,y) = \theta;$$

moreover, $|c_{\varepsilon,a}| \leq k$. Here θ is defined in nonlinear function f.

Step 2: Eigenvalue problem of elliptic problem (2.5) in the half-cylinder $[-a,0]\times\Sigma$

Since the function $\phi^c_{\varepsilon,a}$ satisfies the normalization condition $\max_{\overline{\Sigma}} \phi^c_{\varepsilon,a}(0,x,y) = \theta$ and $\phi^c_{\varepsilon,a}$ is increasing in s, $\phi^c_{\varepsilon,a}(s,x,y) < \phi^c_{\varepsilon,a}(0,x,y) \leq \max_{\overline{\Sigma}} \phi^c_{\varepsilon,a}(0,x,y) = \theta$ for $s \in [-a,0]$. For $s \in [-a,0]$, we have $f(\phi^c_{\varepsilon,a}) = 0$; hence we must solve the problem

$$\begin{cases} \mathcal{L}_\varepsilon \phi = 0, & in\ (-a,0)\times\Sigma \\ \phi_s \nu_1 + \nabla_{x,y}\phi\cdot\nu = 0, & on\ (-a,0)\times\partial\Sigma\ \cdot \\ \phi(s,x,y) = \phi(s,x+L,y), & in\ [-a,0]\times\overline{\Sigma} \end{cases} \tag{2.6}$$

We want to build the solution of the exponential type $\phi(s,x,y) = e^{\lambda s}\psi(x,y)$ be L-periodic function for some $\lambda > 0$. Plug $\phi(s,x,y) = e^{\lambda s}\psi(x,y)$ into (2.6), we get the eigenvalue problem

$$\begin{cases} \mathcal{L}_{c,\lambda}\psi = (\varepsilon\lambda^2)\,\psi, & in\,\Sigma \\ \lambda\psi\nu_1 + \nabla_{x,y}\psi\cdot\nu = 0, & on\,\partial\Sigma \\ \psi(x,y) = \psi(x+L,y) & in\,\overline{\Sigma} \end{cases} \tag{2.7}$$

where $\mathcal{L}_{c,\lambda} := -\triangle_{x,y}\psi - 2\lambda\psi_x + q\cdot\nabla_{x,y}\psi + (q_1+c)\,\lambda\psi - \lambda^2\psi$.

For the eigenvalue problem (2.7), we have known that the existence and uniqueness of eigenvalue correspoding to the eigenfunction from Krein-Rutman theory [5].

Theorem 2.9. *For all $c > 0$ and $\varepsilon > 0$, there exists a uinque positive $\lambda = \lambda^{\varepsilon,c}$ and a positive function $\psi = \psi_c \in C^2\left(\overline{\Sigma}\right)$, unique up to multiplication such that the eigenvalue problem (2.7) is satisfied. Furthermore, $\lambda^{\varepsilon,c}$ is decreasing with respect to $\varepsilon > 0$ and increasing with respect to $c > 0$.*

Remark. This theorem is helpful to prove the limiting condition $\phi(-\infty,x,y) = 0$.

Step 3: Pass the limit $a \to \infty$ in the infinite cylinder

Letting $a \to \infty$, we need to ensure that the solution $\phi^c_{\varepsilon,a}$ with the speed $c = c_{\varepsilon,a}$, which converges to $\phi^{c_\varepsilon}_\varepsilon$ with the speed c_ε, up to extraction of some subsequence. In order to take subsequence which converges, we have the estimates for the speed.

Proposition 2.10. *There exists $a_0 > 0$ and $k > 0$, for all $\varepsilon > 0$, we have*

$$0 < c_\varepsilon := \liminf_{a\to\infty,a\geq a_0} c_{\varepsilon,a} \leq k.$$

Now, we consider a sequence $a_n \to \infty$ and let $\phi_n := \phi_{\varepsilon,a_n}$, Proposition 2.10 asserts that up to extraction of subsequence (still denoted by c_{ε,a_n}) $c_{\varepsilon,a_n} \to c_\varepsilon > 0$. On the other hand, up to extraction of subsequence (still denoted by ϕ_n) ϕ_n converges to a function ϕ_ε in $C^2_{loc}\left(\mathbb{R}\times\overline{\Sigma}\right)$ as $a_n \to \infty$.

Theorem 2.11. $(c_\varepsilon, \phi_\varepsilon)$ *is a solution of*

$$\begin{cases} \mathcal{L}_\varepsilon \phi + f(\phi) = 0, & in\, \mathbb{R} \times \overline{\Sigma} \\ \phi_s \nu_1 + \nabla_{x,y}\phi \cdot \nu = 0, & on\, \mathbb{R} \times \partial\Sigma \\ \phi(s,x,y) = \phi(s,x+L,y), & in\, \mathbb{R} \times \overline{\Sigma} \\ \phi(-\infty,x,y) = 0 \\ \phi(\infty,x,y) = 1 \end{cases} . \tag{2.8}$$

Furthermore, ϕ_ε *is increasing in* s *and satisfies the normalization condition*

$$\max_{\overline{\Sigma}} \phi_\varepsilon(0,x,y) = \max_{[0,L]\times\overline{\Omega}} \phi_\varepsilon(0,x,y) = \theta.$$

Let $u_\varepsilon(t,x,y) = \phi_\varepsilon(x + c_\varepsilon t, x, y)$ be the function defined for all $t \in \mathbb{R}$ and $(x,y) \in \Sigma$, where ϕ_ε is a solution of (2.8). In fact, the function u_ε satisfies the gradient estimate:

Proposition 2.12. *For any compact subset* Γ *of* $\overline{\Sigma}$*, there exists constant* K *depending only on* Γ*, such that, for all* $\varepsilon > 0$*, we have*

$$\int_{\mathbb{R}\times\Gamma} \left[\left(\frac{\partial}{\partial t} u_\varepsilon \right)^2 + |\nabla_{x,y} u_\varepsilon|^2 \right] dt\, dx\, dy \le K \left(\frac{1 + n\, \|q\|_\infty^2}{2} + F(1) \right)$$

where $F(1) = \int_0^1 f(\phi)\, d\phi$*.*

Proof.

For simplicity, we denote ϕ_ε by ϕ in this proof.

In Theorem 2.1, we have

$$c_\varepsilon L\, |\Omega| = \int_{\mathbb{R}\times(0,L)\times\Omega} f(\phi)\; ds\, dx\, dy.$$

Given $a > 0$, we multiply the first equation of (2.8) by ϕ over $(-a,a) \times (0,L) \times \Omega$, and then using integration by parts and boundary condition, it follows that

$$-\int_{(-a,a)\times(0,L)\times\Omega}\left[\varepsilon\phi_s^2+\phi_s^2+\phi_s\phi_x+\phi_x\phi_s+|\nabla_{x,y}\phi|^2\right]+\int_{(-a,a)\times(0,L)\times\Omega}f(\phi)\phi$$

$$+\int_{(0,L)\times\Omega}\left[\varepsilon\phi_s\phi+\phi_s\phi+\phi_x\phi-\frac{1}{2}\left(q_1+c_\varepsilon\right)\phi^2\right]_{-a}^{a}=0,$$

where $[\phi\left(\cdot\right)]_{-a}^{a}=\phi\left(a\right)-\phi\left(-a\right)$. Here we have used $\phi_{ss}\phi=\left(\phi_s\phi\right)_s-\phi_s^2$ and $\phi_{xs}\phi=\left(\phi_x\phi\right)_s-\phi_x\phi_s$. Letting $a\to\infty$, we have

$$\frac{1}{2}c_\varepsilon L\left|\Omega\right|+\int_{\mathbb{R}\times(0,L)\times\Omega}\left[\varepsilon\phi_s^2+\left|\nabla_y\phi\right|^2+\left(\phi_x+\phi_s\right)^2\right]=\int_{\mathbb{R}\times(0,L)\times\Omega}f(\phi)\phi$$

since $\nabla_{s,x,y}\phi\to 0$ as $s\to\pm\infty$ and the velocity field q satisfies (1.2). Indeed, we have

$$\int_{\mathbb{R}\times(0,L)\times\Omega}\left[\left|\nabla_y\phi\right|^2+\left(\phi_x+\phi_s\right)^2\right]=\int_{\mathbb{R}\times(0,L)\times\Omega}\left|\phi_s\cdot e+\nabla_{x,y}\phi\right|^2\leq\frac{1}{2}c_\varepsilon L\left|\Omega\right| \tag{2.9}$$

where $e=(1,0,0,..,0)\in\mathbb{R}^n$, since $\varepsilon>0$ and $\int_{\mathbb{R}\times(0,L)\times\Omega}f(\phi)\phi\leq\int_{\mathbb{R}\times(0,L)\times\Omega}f\left(\phi\right)=c_\varepsilon L\left|\Omega\right|$.

Now, we multiply the first equation of (2.8) by ϕ_s over $(-a,a)\times(0,L)\times\Omega$, and then using integration by parts, boundary condition and $\nabla_{s,x,y}\phi\to 0$ as $s\to\pm\infty$, we get

$$c_\varepsilon\int_{\mathbb{R}\times(0,L)\times\Omega}\phi_s^2=\int_{(0,L)\times\Omega}F(1)-\int_{\mathbb{R}\times(0,L)\times\Omega}q\cdot\left(\phi_s\cdot e+\nabla_{x,y}\phi\right)\phi_s$$

where $e=(1,0,0,..,0)\in\mathbb{R}^n$. Then

$$c_\varepsilon\int_{\mathbb{R}\times(0,L)\times\Omega}\phi_s^2=\int_{(0,L)\times\Omega}F(1)-\int_{\mathbb{R}\times(0,L)\times\Omega}q\cdot\left(\phi_s\cdot e+\nabla_{x,y}\phi\right)\phi_s$$

$$\leq \int_{(0,L)\times\Omega} F(1) + \int_{\mathbb{R}\times(0,L)\times\Omega} \frac{\|q\|_\infty}{2}\left(\alpha\,|\phi_s \cdot e + \nabla_{x,y}\phi|^2 + \frac{n}{\alpha}\phi_s^2\right).$$

We take $\alpha = \frac{n\|q\|_\infty}{c_\varepsilon} > 0$ and using (2.9), it obtains that

$$\frac{c_\varepsilon}{2}\int_{\mathbb{R}\times(0,L)\times\Omega} \phi_s^2 \leq \int_{(0,L)\times\Omega} F(1) + L\,|\Omega|\,\frac{n\,\|q\|_\infty^2}{4}. \tag{2.10}$$

Finally, we multiply the both sides of (2.10) by $2c_\varepsilon > 0$, one obtains that

$$\int_{\mathbb{R}\times(0,L)\times\Omega} (c_\varepsilon\phi_s)^2 \leq 2c_\varepsilon\int_{(0,L)\times\Omega} F(1) + c_\varepsilon L\,|\Omega|\,\frac{n\,\|q\|_\infty^2}{2}. \tag{2.11}$$

Lastly, combining (2.9) and (2.11) and using the fact that ϕ is L-periodic with respect to x. For any compact subset Γ of Σ, there exists a constant K depending only on Γ such that

$$\int_{\mathbb{R}\times[0,L]\times\Omega} \left[c_\varepsilon^2\phi_s^2 + |\nabla_y\phi|^2 + (\phi_x+\phi_s)^2\right] \leq K\left(\frac{1+n\,\|q\|_\infty^2}{2} + F(1)\right).$$

By using the change of variables $u_\varepsilon(t,x,y) = \phi_\varepsilon(x + c_\varepsilon t, x, y)$, we can get the desired result. □

Step 4: Regularization parameter $\varepsilon \to 0$

Finally, we need regularization parameter $\varepsilon \to 0$. For the speed $c_\varepsilon := \liminf_{a\to\infty, a\geq a_0} c_{\varepsilon,a}$, in order to take subsequence of c_ε as $\varepsilon \to 0$, we have the estimates as the following:

Proposition 2.13. *There exists $k > 0$, we have $0 < \liminf_{\varepsilon\to 0} c_\varepsilon \leq k$.*

From Theorem 2.11, we know that $(c_\varepsilon, \phi_\varepsilon)$ is a solution of

$$\begin{cases} \mathcal{L}_\varepsilon \phi + f(\phi) = 0, & in\, \mathbb{R} \times \overline{\Sigma} \\ \phi_s \nu_1 + \nabla_{x,y}\phi \cdot \nu = 0, & on\, \mathbb{R} \times \partial\Sigma \\ \phi(s,x,y) = \phi(s, x+L, y), & in\, \mathbb{R} \times \overline{\Sigma} \\ \phi(-\infty, x, y) = 0 & \\ \phi(\infty, x, y) = 1 & \end{cases}.$$

Recall that $u_\varepsilon(t,x,y) = \phi_\varepsilon(x + c_\varepsilon t, x, y)$, then $(c_\varepsilon, u_\varepsilon)$ is a classical solution of

$$\begin{cases} \frac{\varepsilon}{c^2} u_{tt} + \triangle_{x,y} u - u_t - q \cdot \nabla_{x,y} u + f(u) = 0, & in\, \mathbb{R} \times \overline{\Sigma} \\ \nabla_{x,y} u \cdot \nu = 0, & on\, \mathbb{R} \times \partial\Sigma \\ u\left(t + \frac{L}{c}, x, y\right) = u\left(t, x + L, y\right), & in\, \mathbb{R} \times \overline{\Sigma} \\ u\left(t, -\infty, y\right) = 0 & \\ u\left(t, \infty, y\right) = 1 & \end{cases} \tag{2.12}$$

and $0 < u_\varepsilon < 1$, u_ε is increasing in t. We observe $\frac{\varepsilon}{c_\varepsilon^2}$ in (2.12) and from Proposition 2.13, up to extraction of subsequence such that $\frac{\varepsilon}{c_\varepsilon^2} \to 0$ as $\varepsilon \to 0$. Then the equation (2.12) becomes the degenerate elliptic equation as $\varepsilon \to 0$.

Since the function u_ε satisfies the gradient estimate for all $\varepsilon > 0$ by Proposition 2.12, there exists a function $u \in H^1_{loc}(\mathbb{R} \times \Sigma)$ such that up tp extraction of subsequence $u_\varepsilon \to u$ almost everywhere in $\mathbb{R} \times \Sigma$ and $u_\varepsilon \rightharpoonup u$, $\nabla_{t,x,y} u_\varepsilon \rightharpoonup \nabla_{t,x,y} u$ in $L^2(\mathbb{R} \times \Gamma)$ for all compact subset $\Gamma \subset \overline{\Sigma}$ as $\varepsilon \to 0$. Moreover, we have $0 \le u \le 1$, $u_t \ge 0$ and u satisfies the gradient estimate

$$\int_{\mathbb{R} \times \Gamma} \left[\left(\frac{\partial}{\partial t} u \right)^2 + |\nabla_{x,y} u|^2 \right] dt\, dx\, dy \le K \left(\frac{1 + n \|q\|_\infty^2}{2} + F(1) \right)$$

for all compact subset $\Gamma \subset \overline{\Sigma}$, where $F(1) = \int_0^1 f(\phi)\, d\phi$.

Now, we must ensure that the function u is a classical solution of

$$\begin{cases} u_t - \triangle_{x,y} u + q \cdot \nabla_{x,y} u = f(u), & in\, \mathbb{R} \times \overline{\Sigma} \\ \nabla_{x,y} u \cdot \nu = 0, & on\, \mathbb{R} \times \partial\Sigma \end{cases}. \tag{2.13}$$

We take any test function $\phi \in C_c^2\left(\mathbb{R} \times \overline{\Sigma}\right)$, multiplying the first equation in (2.12) and integrating by parts, we get

$$\int_{\mathbb{R}\times\Sigma} \frac{\varepsilon}{c_\varepsilon^2} \frac{\partial u_\varepsilon}{\partial t} \phi_t - \frac{\partial u_\varepsilon}{\partial t} \phi - \nabla_{x,y} u_\varepsilon \cdot \nabla_{x,y} \phi - (q \cdot \nabla_{x,y} u_\varepsilon)\, \phi + f(u_\varepsilon)\phi = 0;$$

then letting $\varepsilon \to 0$, it follows that

$$\int_{\mathbb{R}\times\Sigma} u_t - \nabla_{x,y} u \cdot \nabla_{x,y} \phi + (q \cdot \nabla_{x,y} u)\, \phi - f(u)\phi = 0.$$

Hence, the function u is a classical solution of (2.13) by parabolic regularity theory.

Since the function u_ε satisfies the periodic condition

$$u_\varepsilon \left(t + \frac{L}{c_\varepsilon}, x, y\right) = u_\varepsilon\,(t, x + L, y)$$

and the gradient estimate in Proposition 2.12, we consider

$$\int_{(-a,a)\times\Gamma} \left[u_\varepsilon \left(t + \frac{L}{c}, x, y\right) - u_\varepsilon\,(t, x + L, y)\right]^2 dt\, dx\, dy$$

for all $a > 0$ and compact subset $\Gamma \subset \overline{\Sigma}$. It follows that

$$\int_{(-a,a)\times\Gamma} \left[u_\varepsilon \left(t + \frac{L}{c}, x, y\right) - u_\varepsilon\,(t, x + L, y)\right]^2 dt\, dx\, dy$$

$$= \int_{(-a,a)\times\Gamma} \left[u_\varepsilon \left(t + \frac{L}{c}, x, y \right) - u_\varepsilon \left(t + \frac{L}{c_\varepsilon}, x, y \right) \right]^2 dt\, dx\, dy$$

$$\leq \left(\frac{L}{c} - \frac{L}{c_\varepsilon} \right)^2 \int_{\mathbb{R}\times\Gamma} \left(\frac{\partial u_\varepsilon}{\partial t} \right)^2 dt\, dx\, dy$$

$$\leq \left(\frac{L}{c} - \frac{L}{c_\varepsilon} \right)^2 K \left(\frac{1 + n \|q\|_\infty^2}{2} + F(1) \right),$$

where $F(1) = \int_0^1 f(\phi)\ d\phi$. Letting $\varepsilon \to 0$, we have $u\left(t + \frac{L}{c}, x, y\right) = u\left(t, x + L, y\right)$ since u is continuous. This shows that the function u satisfies the periodic condition. Furthermore, from [1], the function u satisfies the normalization condition

$$\max_{\overline{\Sigma}} u \left(-\frac{x}{c}, x, y \right) = \max_{[0,L]\times\overline{\Omega}} u \left(-\frac{x}{c}, x, y \right) = \theta$$

and the limiting condition, for all $t \in \mathbb{R}$,

$$\begin{cases} u(t, -\infty, y) = 0 \\ u(t, \infty, y) = 1 \end{cases}.$$

Combining the above four steps, we get the existence of the pulsating travelling wave solution of (1.3).

Chapter 3

Proof of Theorem 1.3

In this chapter, we consider the nonlinear term f is monostable type. That is, f satisfies $0 < f(u) < 1$ in $(0, 1)$, $f(0) = f(1) = 0$, $f'(0) > 0$. First, we prove that the solution u is increasing with respect to t in Section 3.1. Second, we sketch the proof of the existence of travelling wave solutions (c, u) if $c \geq c^*$ in Section 3.2. Finally, we show that there is no solution (c, u) if $c < c^*$ in Section 3.3.

3.1 Monotonicity of the Solution

Proposition 3.1. *Let f be a function which satisfies $0 < f(u) < 1$ in $(0, 1)$, $f(0) = f(1) = 0$, $f'(0) > 0$. Suppose that (c, u) be a classical solution of (1.3). Then the function u is increasing in t.*

3.2 Existence of a Pulsating Travelling Wave Solution for $c \geq c^*$

We want to use a cutoff function such that monostable nonlinear term f becomes combustion type. We define the function $\chi \in C^1(\mathbb{R})$ such that

$$\begin{cases} \chi(u) = 0, & u \leq 1 \\ \chi(u) = 1, & u \geq 2 \\ 0 < \chi < 1, & 1 < u < 2 \\ \chi \text{ nondecreasing in } \mathbb{R} \end{cases}.$$

For all $\theta \in (0,1)$, let $\chi_\theta(u) = \chi\left(\frac{u}{\theta}\right)$, $\forall u \in \mathbb{R}$. Then we have

$$\begin{cases} \chi_\theta(u) = 0, & u \leq \theta \\ \chi_\theta(u) = 1, & u \geq 2\theta \\ 0 < \chi_\theta < 1, & \theta < u < 2\theta \\ \chi_\theta \text{ nonincreasing with respect to } \theta \end{cases}.$$

Hence, we define $f_\theta(u) = f(u)\chi_\theta(u)$, for all $u \in \mathbb{R}$. This function f_θ be a combustion type nonlinearity.

Now we consider the problem

$$\begin{cases} u_t - \triangle_{x,y} u + q(x,y)\cdot\nabla_{x,y} u = f_\theta(u), & in\, \mathbb{R}\times\overline{\Sigma} \\ \nabla_{x,y} u\cdot\nu = 0, & on\, \mathbb{R}\times\partial\Sigma \\ u\left(t+\frac{L}{c},x,y\right) = u\left(t,x+L,y\right), & in\, \mathbb{R}\times\overline{\Sigma} \\ u\left(t,-\infty,y\right) = 0 & \\ u\left(t,\infty,y\right) = 1 & \end{cases} . \tag{3.1}$$

From Theorem 1.2, there exists a uinque classical solution (c_θ, u_θ) for the problem (3.1). Furthermore, the function u_θ satisfies the gradient estimate

$$\int_{\mathbb{R}\times\Gamma}\left[\left(\frac{\partial}{\partial t}u\right)^2 + |\nabla_{x,y}u|^2\right] dt\, dx\, dy \le K\left(\frac{1+n\,\|q\|_\infty^2}{2} + F_\theta(1)\right)$$

for all compact subset $\Gamma\subset\overline{\Sigma}$, where $F_\theta(1) = \int_0^1 f_\theta(\phi)\,d\phi$ and K is a constant depending only on Γ.

Since the speed c_θ is nonincreasing with respect to θ (see [1]) and $\underline{c} < c_\theta < \overline{c}$ for some $0<\underline{c}<\overline{c}$ from Proposition 2.2, there exists $c^*>0$ such that $c_\theta \nearrow c^*$ as $\theta\searrow 0$. Consider a sequence $\theta_n\searrow 0$, one can assume that $u_{\theta_n}(0,x_0,y_0) = \frac{1}{2}$, where (x_0,y_0) is an arbitrarily chosen point in $\overline{\Sigma}$ since we can suitably shift in t. Up to extraction of some subsequence, the function $u_{\theta_n}\to u^*$ locally uinformly from parabolic regularity theory. Then the function u^* is a classical solution of

$$\begin{cases} u_t - \triangle_{x,y} u + q(x,y)\cdot\nabla_{x,y} u = f(u), & in\, \mathbb{R}\times\overline{\Sigma} \\ \nabla_{x,y} u\cdot\nu = 0, & on\, \mathbb{R}\times\partial\Sigma \\ u\left(t+\frac{L}{c},x,y\right) = u\left(t,x+L,y\right), & in\, \mathbb{R}\times\overline{\Sigma} \end{cases} \cdot$$

Furthermore, $u^*(0,x_0,y_0)=\frac{1}{2}$, $u_t^*\ge 0$ and u^* satisfies the gradient estimate. In [1], we can know that $u^*(-\infty,x,y)=0$, $u^*(\infty,x,y)=1$ and u^* is increasing with respect to t. Hence, (c^*,u^*) is a classical solution of (1.3) with the monostable nonlinearity f. We state the results as follows:

Theorem 3.2. *There exists (c^*,u^*) is a classical solution of (1.3). Moreover, $c^*>0$, $0<u^*<1$ and u^* is increasing with respect to t.*

Actually, here c^* is the minimal speed. Berestycki, Hamel and Nadirashvili [2] give a variational charaterization of this minimal speed c^*. In addition, we assume that nonlinearity f satisfies

$$0<f(u)\leq f'(0)u$$

for all $u\in(0,1)$. Then

$$c^*=\min_{\lambda>0}\frac{-k(\lambda)}{\lambda} \tag{3.2}$$

where $k(\lambda)$ is the principal eigenvalue of the operator

$$-\mathcal{L}_\lambda\psi:=-\triangle\psi-2\lambda\psi_x+q\cdot\nabla\psi+\left(q_1\lambda-\lambda^2-f'(0)\right)\psi$$

acting on the set $E=\left\{\psi\in C^2\left(\overline{\Sigma}\right):\psi \text{ is } L-\text{periodic with respect to } x \text{ and } \nabla\psi\cdot\nu=0 \text{ on } \partial\Sigma\right\}$.

In particular, when $\Sigma=\mathbb{R}^n$ and $q=0$, the formula (3.2) gives the well-known KPP formula $c^*=2\sqrt{f'(0)}$ for the minimal speed of planar fronts.

Now, we prove the existence of solutions if $c\geq c^*$.

Theorem 3.3. *For each $c\geq c^*$, there exists (c,u) is a classical solution of (1.3).*

Proof. The method for the proof is similar as Section 2.3 so we sketch the proof. We only consider the case $c>c^*$ because the case $c=c^*$ has been done in Theorem 3.2. We divide the proof into four steps:

Step 1: The estimate for u^*

In [1], we have known that for all $(s,x,y) \in \mathbb{R}\times\overline{\Sigma}$, $|\partial_{ss}\phi^*(s,x,y)| \le \frac{k}{c^*}\partial_s\phi^*(s,x,y)$, where k is a constant and $\phi^*(s,x,y) = u^*\left(\frac{s-x}{c^*},x,y\right)$.

Recall that the operator

$$\mathcal{L}_\varepsilon\phi = \varepsilon\phi_{ss} + \phi_{ss} + \phi_{sx} + \phi_{xs} + \triangle_{x,y}\phi - (q_1(x,y)+c)\,\phi_s - q(x,y)\cdot\nabla_{x,y}\phi,$$

for any $\varepsilon > 0$. From the definition of ϕ^*, one has $\mathcal{L}_\varepsilon\phi^* + f(\phi^*) = \varepsilon\phi^*_{ss} + (c^*-c)\,\phi^*_s$. Since $|\partial_{ss}\phi^*(s,x,y)| \le \frac{k}{c^*}\partial_s\phi^*(s,x,y)$ and $\phi^*_s > 0$, for ε small enough, we have

$$\mathcal{L}_\varepsilon\phi^* + f(\phi^*) = \varepsilon\phi^*_{ss} + (c^*-c)\,\phi^*_s \le \left(\varepsilon\frac{k}{c^*} + c^* - c\right)\phi^*_s < 0$$

for all $(s,x,y) \in \mathbb{R}\times\Sigma$.

Step 2: Solve the regularization problem in finite cylinder

Let $a > 0$, $\tau \in \mathbb{R}$ and $h_\tau := \min_{\overline{\Sigma}}\phi^*(-a+\tau,x,y) = \min_{[0,L]\times\overline{\Omega}}\phi^*(-a+\tau,x,y)$. Now, we consider the problem

$$\begin{cases} \mathcal{L}_\varepsilon\phi + f(\phi) = 0, & in\ \Sigma_a \\ \phi_s\nu_1 + \nabla_{x,y}\phi\cdot\nu = 0, & on\ (-a,a)\times\partial\Sigma \\ \phi(s,x,y) = \phi(s,x+L,y), & in\ \overline{\Sigma_a} \\ \phi(-a,x,y) = h_\tau \\ \phi\,(a,x,y) = \phi^*(a+\tau,x,y) \end{cases}, \tag{3.3}$$

where $\Sigma_a = (-a,a)\times\Sigma$. We can use the same method as in Section 2.3 to prove the existence of solution for the problem (3.3). Then there exists $\phi_\tau(s,x,y) \in C\left(\overline{\Sigma_a}\right)\cap C^2\left(\widetilde{\Sigma_a}\right)$ which is a solution of (3.3). Indeed, the function ϕ_τ is increasing in s, and ϕ_τ is increasing and continuous in τ (see [1]). Therefore, there exists unique $\tau(a) \in \mathbb{R}$ such that $\phi_{\varepsilon,a} := \phi_{\tau(a)}$ solves (3.3) and satisfies the normalization condition

$$\int_{(0,1)\times(0,L)\times\Omega} \phi_{\varepsilon,a}(s,x,y)\, ds\, dx\, dy = \frac{1}{2}L\,|\Omega|$$

after a suitable shift in s.

Step 3: Passage to the whole cylinder

Consider a sequence $a_n \to \infty$, up to extraction of some subsequce (still denoted by ϕ_{ε,a_n}) $\phi_{\varepsilon,a_n} \to \phi_\varepsilon$ in $C^2_{loc}\left(\mathbb{R}\times\overline{\Sigma}\right)$ as $a_n \to \infty$. Then the function ϕ_ε solves the problem

$$\begin{cases} \mathcal{L}_\varepsilon \phi + f(\phi) = 0, & in\, \mathbb{R}\times\overline{\Sigma} \\ \phi_s \nu_1 + \nabla_{x,y}\phi\cdot\nu = 0, & on\, \mathbb{R}\times\partial\Sigma \\ \phi(s,x,y) = \phi(s,x+L,y), & in\, \mathbb{R}\times\overline{\Sigma} \end{cases}$$

and $\frac{\partial}{\partial s}\phi_\varepsilon \geq 0$, $0 \leq \phi_\varepsilon \leq 1$. Furthermore, ϕ_ε satisfies the normalization condition

$$\int_{(0,1)\times(0,L)\times\Omega} \phi_\varepsilon(s,x,y)\, ds\, dx\, dy = \frac{1}{2}L\,|\Omega|\,.$$

Since $\phi_\varepsilon\,(s,x,y) \to \phi_\varepsilon^\pm\,(x,y)$ in $C^2_{loc}\left(\overline{\Sigma}\right)$ as $s \to \pm\infty$, $\phi_\varepsilon^\pm$ solves the equation

$$\begin{cases} \triangle_{x,y}\phi_\varepsilon^\pm - q\cdot\nabla_{x,y}\phi_\varepsilon^\pm + f(\phi_\varepsilon^\pm) = 0, & in\, \mathbb{R}\times\overline{\Sigma} \\ \nabla_{x,y}\phi_\varepsilon^\pm\cdot\nu = 0, & on\, \mathbb{R}\times\partial\Sigma \\ \phi_\varepsilon^\pm(x,y) = \phi_\varepsilon^\pm(x+L,y), & in\, \mathbb{R}\times\overline{\Sigma} \end{cases}$$

and $0 \leq \phi_\varepsilon^\pm \leq 1$. From [1], one can obtains that $\phi_\varepsilon^- = 0$ and $\phi_\varepsilon^+ = 1$. Therefore, ϕ_ε is a classical solution of

$$\begin{cases} \mathcal{L}_\varepsilon \phi + f(\phi) = 0, & in\ \mathbb{R} \times \overline{\Sigma} \\ \phi_s \nu_1 + \nabla_{x,y}\phi \cdot \nu = 0, & on\ \mathbb{R} \times \partial\Sigma \\ \phi(s,x,y) = \phi(s,x+L,y), & in\ \mathbb{R} \times \overline{\Sigma} \\ \phi(-\infty,x,y) = 0 \\ \phi(\infty,x,y) = 1 \end{cases}$$

and it satisfies the gradient estimate

$$\int_{\mathbb{R}\times\Gamma} \left[\left(\frac{\partial}{\partial t} u_\varepsilon \right)^2 + |\nabla_{x,y} u_\varepsilon|^2 \right] dt\, dx\, dy \leq K \left(\frac{1 + n\, \|q\|_\infty^2}{2} + F(1) \right) \tag{3.4}$$

for all compact subset $\Gamma \subset \overline{\Sigma}$, where $F(1) = \int_0^1 f(\phi)\, d\phi$, K is a constant depending only on Γ and $u_\varepsilon(t,x,y) = \phi_\varepsilon(x+ct,x,y)$.

Step 4: Regularization parameter $\varepsilon \to 0$

From (3.4), there exists $u \in H^1_{loc}(\mathbb{R} \times \Sigma)$ such that up tp extraction of subsequence $u_\varepsilon \to u$ almost everywhere in $\mathbb{R} \times \Sigma$ and $u_\varepsilon \rightharpoonup u$, $\nabla_{t,x,y} u_\varepsilon \rightharpoonup \nabla_{t,x,y} u$ in $L^2(\mathbb{R} \times \Gamma)$ for all compact subset $\Gamma \subset \overline{\Sigma}$ as $\varepsilon \to 0$. It is the same proof as in Section 2.3 so we can get the function u is a classical solution of (1.3).

Combining the above four steps, we prove the existence of solution for (1.3) with KPP type nonlinearity reaction term. □

3.3 Nonexistence of Solutions for $c < c^*$

Recall that (c_θ, u_θ) is a classical solution of (3.1) with f_θ for all $\theta \in (0,1)$. One knows that u_θ is increasing in t.

Theorem 3.4. *There is no solution (c,u) of (1.3) if $c < c^*$.*

Proof. Assume by contradition that there exists a solution (c, u) of (1.3) for $c < c^*$. Since speed c_θ is nonincreasing with respect to θ, there exists a $\theta > 0$ small enough so that $c < c_\theta$. Theorem 2.1 asserts that the speed $c > 0$. Let $\phi_\theta(s, x, y) = u_\theta\left(\frac{s-x}{c_\theta}, x, y\right)$, it satisfies

$$\begin{aligned} & \partial_{ss}\phi_\theta + \partial_{sx}\phi_\theta + \partial_{xs}\phi_\theta + \triangle\phi_\theta - (q_1 + c)\,\partial_s\phi_\theta - q \cdot \nabla\phi_\theta + f(\phi_\theta) \\ = \; & (c_\theta - c)\,\partial_s\phi_\theta + f(\phi_\theta) - f_\theta(\phi_\theta) \qquad (3.5) \\ \geq \; & 0, \end{aligned}$$

the last inequality holds since $c < c_\theta$, $\partial_s\phi_\theta > 0$ and $f(\phi_\theta) \geq f_\theta(\phi_\theta)$. On the other hand, let $\phi(s, x, y) = u\left(\frac{s-x}{c}, x, y\right)$ is a solution of

$$\partial_{ss}\phi + \partial_{sx}\phi + \partial_{xs}\phi + \triangle\phi - (q_1 + c)\,\partial_s\phi - q \cdot \nabla\phi + f(\phi) = 0.$$

Indeed, both function ϕ and ϕ_θ are L-periodic with respect to x and satisfy the same limiting condition and boudary condition. Now, we slide the function ϕ_θ with respect to ϕ. Then there exists a $\tau^* \in \mathbb{R}$ such that $\phi_\theta(s + \tau^*, x, y) = \phi(s, x, y)$ for all $(s, x, y) \in \mathbb{R} \times \overline{\Sigma}$. Putting that into (3.5) implies that

$$(c_\theta - c)\,\partial_s\phi_\theta + f(\phi_\theta) - f_\theta(\phi_\theta) = 0.$$

It contradicts to $\partial_s\phi_\theta > 0$ or $f(\phi_\theta) \geq f_\theta(\phi_\theta)$. $\square$

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Part II

Semi-Exact Travelling Wave Solutions for Systems of Three Competing Species

Chapter 4

Introduction

In this part, we are concerned with travelling wave solutions for the competitive Lotka-Volterra systems of three species $u = u(x,t)$, $v = v(x,t)$ and $w = w(x,t)$:

$$\begin{cases} u_t = d_1 u_{xx} + u\left(\lambda_1 - c_{11}u - c_{12}v - c_{13}w\right), \\ v_t = d_2 v_{xx} + v\left(\lambda_2 - c_{21}u - c_{22}v - c_{23}w\right), \quad x \in \mathbb{R}, \quad t > 0 \\ w_t = d_3 w_{xx} + w\left(\lambda_3 - c_{31}u - c_{32}v - c_{33}w\right), \end{cases} \tag{4.1}$$

where $u = u(x,t)$, $v = v(x,t)$, and $w = w(x,t)$ represent the density of the three species u, v and w, respectively; d_i, λ_i, c_{ii} $(i = 1,2,3)$, c_{ij} $(i,j = 1,2,3, i \neq j)$ are the diffusion rates, the intrinsic growth rates, the intra-specific competition rates, and the inter-specific competition rates, which are all assumed to be positive constants, respectively. This is a mathematical model frequently used in ecology to describe three species moving by diffusion and competing for the same resources [1].

We want to find travelling wave solutions of the form

$$(u(x,t), v(x,t), w(x,t)) = (U(z), V(z), W(z))$$

where $z = x - \theta t$ and θ is the wave speed. Then $(U(z), V(z), W(z))$ satisfies

$$\begin{cases} d_1 U_{zz} + \theta U_z + U\left(\lambda_1 - c_{11}U - c_{12}V - c_{13}W\right) = 0, \\ d_2 V_{zz} + \theta V_z + V\left(\lambda_2 - c_{21}U - c_{22}V - c_{23}W\right) = 0, \qquad z \in \mathbb{R}\,\cdot \\ d_3 W_{zz} + \theta W_z + W\left(\lambda_3 - c_{31}U - c_{32}V - c_{33}W\right) = 0, \end{cases} \tag{4.2}$$

In the case of the systems of two competing species,

$$\begin{cases} u_t = u_{xx} + u\left(1 - u - cv\right), \\ \qquad\qquad\qquad\qquad x \in \mathbb{R}, \quad t > 0 \\ v_t = dv_{xx} + v\left(a - bu - v\right), \end{cases} \tag{4.3}$$

after a suitable transformation, where the constants a, b, c, and d are positive. We look for travelling wave solutions of (4.3) of the form $(u(x,t), v(x,t)) = (U(z), V(z))$, where $z = x - \theta t$ and θ is the wave speed. Then $(U(z), V(z))$ satisfies

$$\begin{cases} U_{zz} + \theta U_z + U\left(1 - U - cV\right) = 0 \\ dV_{zz} + \theta V_z + V\left(a - bU - V\right) = 0 \end{cases}. \tag{4.4}$$

Rodrigo and Mimura [3, 4] give many exact solutions for (4.4). Indeed, Kan-on has proved the existence and uinqueness of the solution for (4.4) in [2]. We give an example of exact solutions for (4.4):

Example 4.1. Suppose that $d = \frac{1}{3c}$, $b = 2 + \frac{5a}{3} - ac$, $\theta = \frac{-2+ac}{\sqrt{2ac}}$. Then exact solution of (4.4) is of the form

$$\begin{aligned} U(z) &= \frac{1}{2}\left[1 + \tanh\left(\frac{\sqrt{2ac}}{4}z\right)\right], \\ V(z) &= \frac{1}{2}\left[1 - \tanh\left(\frac{\sqrt{2ac}}{4}z\right)\right]^2. \end{aligned}$$

We return to problem of the systems of three competing species. We will look for monotonic solutions $(U(z), V(z))$ and a pulse solution $W(z)$ of (4.2) satisfying U, V, $W \geq 0$ for all $z \in \mathbb{R}$. In next chapter, we will show these semi-exact solutions of (4.2).

Chapter 5

Semi-Exact Solutions

In this chapter, we show seven types of semi-exact solutions of (4.2). In order to find these semi-exact solutions, we introduce some anastz.

Let T be a solution of the initial value problem

$$\begin{cases} \frac{d}{dz}T(z) = T(z)\left(1 - T(z)\right)\left(a + T(z)\right), & z \in \mathbb{R} \\ T(0) = T_0 \end{cases} \tag{5.1}$$

where $T_0 \in (0, 1)$ be a constant and a is a determined constant. And we suppose that travelling wave solutions are of the form

$$\begin{cases} U(z) = k_1 T^i(z) \\ V(z) = k_2\left(1 - T(z)\right)^m \\ W(z) = k_3 T^n(z)\left(1 - T(z)\right)^2 \end{cases} \tag{5.2}$$

where i, m, n are positive integers, k_1, k_2, k_3 are positive constants and T is the solution of (5.1). We put (5.2) into (4.2) and use (5.1), then (4.2) becomes the polynomial of T. In order to balance the terms of the polynomial of T, we need to choose i, m, n appropriately. This will give a system of algebraic equations involving d_i, λ_i, c_{ii}

$(i=1,2,3)$, c_{ij} $(i,j=1,2,3, i\neq j)$, θ, a and k_i $(i=1,2,3)$. We use Mathematica to solve this system of algebraic equations so that we can get the restriction of parameters d_i, λ_i, c_{ii} $(i=1,2,3)$, c_{ij} $(i,j=1,2,3, i\neq j)$, θ, a and k_i $(i=1,2,3)$.

5.1 Type-1 Solutions $(i,m,n)=(2,4,1)$

The following restriction on parameters appearing in (4.2) is assumed so that an exact solution is allowed.

$$d_2=\frac{a(7+5a)d_1}{-2+a(11+a)}, d_3=\frac{(-1+3a)(7+5a)d_1}{-13+3a(8+3a)}, \theta=(-7-5a)d_1, \tag{5.3}$$

$$\lambda_1=2(1+a)(4+3a)d_1, \lambda_2=\frac{24a(7+5a)d_1}{-2+a(11+a)},$$

$$\lambda_3=\frac{(7+5a)\left(-15+a\left(32+a\left(25+6a\right)\right)\right)d_1}{-13+3a(8+3a)}, \tag{5.4}$$

$$c_{11}=\frac{2(1+a)(4+3a)d_1}{k_1}, c_{12}=\frac{8d_1}{k_2}, c_{13}=\frac{2(9+7a)d_1}{k_3}, \tag{5.5}$$

$$c_{21}=\frac{4(2+a)(7+5a)(-1+5a(2+a))d_1}{(-2+a(11+a))k_1}, c_{22}=\frac{24a(7+5a)d_1}{(-2+a(11+a))k_2},$$

$$c_{23}=\frac{44a(2+a)(7+5a)d_1}{(-2+a(11+a))k_3}, \tag{5.6}$$

$$c_{31}=\frac{(5+3a)(7+5a)(-9+a(17+12a))d_1}{(-13+3a(8+3a))k_1}, c_{32}=\frac{15(-1+3a)(7+5a)d_1}{(-13+3a(8+3a))k_2},$$

$$c_{33}=\frac{(-1+3a)(7+5a)(47+27a)d_1}{(-13+3a(8+3a))k_3}, \tag{5.7}$$

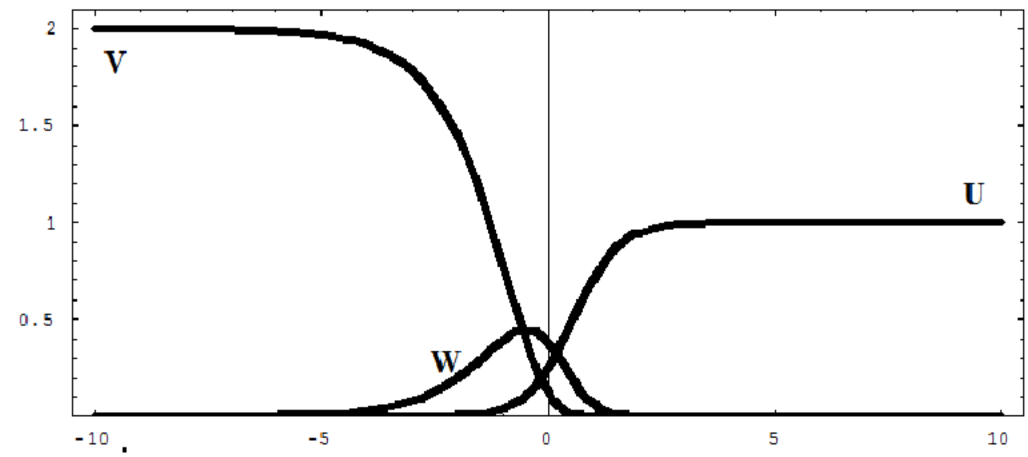

Figure 5.1: Profiles of U, V, W.

where k_1, k_2, k_3 are constants.

Under the conditions (5.3)-(5.7), (4.2) adimits a solution of the form

$$\begin{cases} U(z) = k_1 T^2(z) \\ V(z) = k_2 (1 - T(z))^4 \\ W(z) = k_3 T(z) (1 - T(z))^2 \end{cases} \tag{5.8}$$

where T is the solution of (5.1). Assume k_1, k_2, k_3, $d_1 > 0$, then the necessary and sufficient condition for d_i, λ_i, c_{ii} $(i = 1, 2, 3)$, c_{ij} $(i, j = 1, 2, 3, i \neq j) > 0$, and $a \notin [-1, 0]$ in (5.1) to be satisfied is given by

$$\frac{-11 + \sqrt{129}}{2} < a < \frac{1}{3} \quad \text{or} \quad a > \frac{-4 + \sqrt{29}}{3}.$$

Approximately, $0.178908 < a < 0.34$ or $a > 0.491722$.

Now, if one chooses $(a, k_1, k_2, k_3, d_1) = (1, 1, 2, 3, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = \frac{6}{5}$, $d_3 = \frac{6}{5}$, $\theta = -12$, $\lambda_1 = 28$, $\lambda_2 = \frac{144}{5}$, $\lambda_3 = \frac{144}{5}$, $c_{11} = 28$, $c_{12} = 4$, $c_{13} = \frac{32}{3}$, $c_{21} = \frac{1008}{5}$, $c_{22} = \frac{72}{5}$, $c_{23} = \frac{264}{5}$, $c_{31} = 96$, $c_{32} = 9$, and $c_{33} = \frac{148}{5}$ by (5.3)-(5.7). The resulting profiles of U, V, W and T are shown in Figure 5.1 and Figure 5.2, respectively.

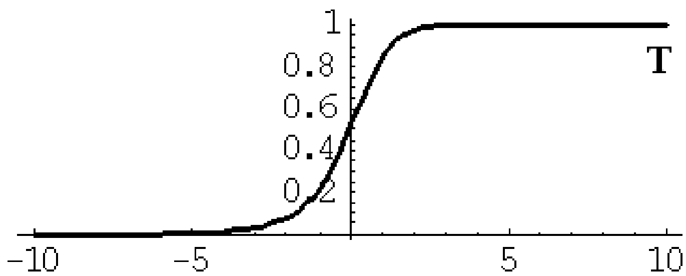

Figure 5.2: Profile of T with $a = 1$ and $T_0 = \frac{1}{2}$.

In particular, in the case of $a = 1$, we make the change of variable $T^2 = \frac{1}{2}(1 + v)$, where T is the solution of (5.1) with initial condition $T(0) = \frac{1}{2}$. Then v solves the equation

$$\begin{cases} \frac{d}{dz} v = 1 - v^2, & z \in \mathbb{R} \\ v(0) = -\frac{1}{2} \end{cases}.$$

It is easy to see $v(z) = \tanh z - \frac{1}{2}$. Hence, the semi-exact solution (5.8) can be rewritten in terms of $\tanh z$:

$$\begin{cases} U(z) = k_1 \left[\frac{1}{4}(1 + 2\tanh z)\right] \\ V(z) = k_2 \left(1 - \frac{1}{2}\sqrt{1 + 2\tanh z}\right)^4 \\ W(z) = k_3 \left[\frac{1}{2}\sqrt{1 + 2\tanh z}\left(1 - \frac{1}{2}\sqrt{1 + 2\tanh z}\right)^2\right] \end{cases}.$$

5.2 Type-2 Solutions $(i, m, n) = (3, 1, 2)$

The following restriction on parameters appearing in (4.2) is assumed so that an exact solution is allowed.

$$d_2 = \frac{(-1 + 4(-4 + a)a)d_1}{-3 + a(-3 + 2a)}, d_3 = \frac{2(-1 + 4(-4 + a)a)d_1}{-9 + a(-23 + 10a)}, \theta = (-1 + 4(-4 + a)a)d_1, \tag{5.9}$$

$$\lambda_1 = 3(1+9a)d_1, \lambda_2 = \frac{a(3+2a(25+a(9+4(-5+a)a)))d_1}{-3+a(-3+2a)},$$

$$\lambda_3 = \frac{2(-1+4(-4+a)a)(9+a(4+a(-17+10a)))d_1}{-9+a(-23+10a)}, \tag{5.10}$$

$$c_{11} = \frac{3(1+9a)d_1}{k_1}, c_{12} = \frac{3(-1+a)(-1+a(-9+4a))d_1}{k_2}, c_{13} = \frac{15d_1}{k_3}, \tag{5.11}$$

$$c_{21} = \frac{(4+5a)(-1+4(-4+a)a)d_1}{(-3+a(-3+2a))k_1}, c_{22} = \frac{a(3+2a(25+a(9+4(-5+a)a)))d_1}{(-3+a(-3+2a))k_2},$$

$$c_{23} = \frac{3(-1+4(-4+a)a)d_1}{(-3+a(-3+2a))k_3}, \tag{5.12}$$

$$c_{31} = \frac{44(1+2a)(-1+4(-4+a)a)d_1}{(-9+a(-23+10a))k_1}, c_{32} = \frac{2(-1+2a)(1+2a)(-9+5a)(-1+4(-4+a)a)d_1}{(-9+a(-23+10a))k_2},$$

$$c_{33} = \frac{48(-1+4(-4+a)a)d_1}{(-9+a(-23+10a))k_3}, \tag{5.13}$$

where k_1, k_2, k_3 are constants.

Under the conditions (5.9)-(5.13), (4.2) adimits a solution of the form

$$\begin{cases} U(z) = k_1 T^3(z) \\ V(z) = k_2 (1 - T(z)) \\ W(z) = k_3 T^2(z) (1 - T(z))^2 \end{cases}$$

where T is the solution of (5.1). Assume k_1, k_2, k_3, $d_1 > 0$, then the necessary and sufficient condition for d_i, λ_i, c_{ii} $(i = 1, 2, 3)$, c_{ij} $(i, j = 1, 2, 3, i \neq j) > 0$, and $a \notin [-1, 0]$ in (5.1) to be satisfied is given by

$$a > \frac{4 + \sqrt{17}}{2}.$$

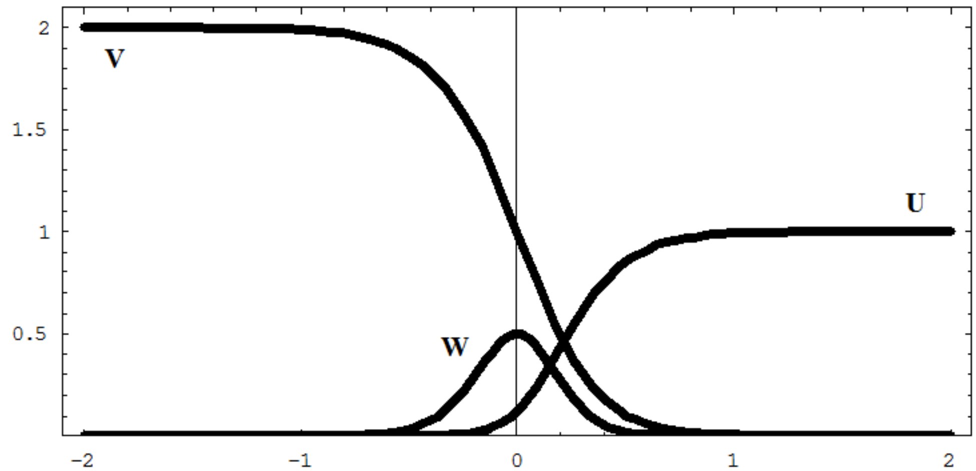

Figure 5.3: Profiles of U, V, W.

Approximately, $a > 4.06155$.

Now, if one chooses $(a, k_1, k_2, k_3, d_1) = (5, 1, 2, 8, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = \frac{19}{32}$, $d_3 = \frac{19}{63}$, $\theta = 19$, $\lambda_1 = 138$, $\lambda_2 = \frac{3515}{32}$, $\lambda_3 = \frac{2318}{9}$, $c_{11} = 138$, $c_{12} = 324$, $c_{13} = \frac{15}{8}$, $c_{21} = \frac{551}{32}$, $c_{22} = \frac{3515}{64}$, $c_{23} = \frac{57}{256}$, $c_{31} = \frac{4598}{63}$, $c_{32} = \frac{1672}{7}$, and $c_{33} = \frac{19}{21}$ by (5.9)-(5.13). The resulting profiles of U, V, W are shown in Figure 5.3 and the profile of T is similar to Figure 5.2.

5.3 Type-3 Solutions $(i, m, n) = (3, 2, 2)$

The following restriction on parameters appearing in (4.2) is assumed so that an exact solution is allowed.

$$d_2 = \frac{(2+a)(-2+(-25+a)a)d_1}{(-2+a)(1+a)(4+5a)}, d_3 = \frac{(3+2a)(-2+(-25+a)a)d_1}{(1+a)(-9+2a)(2+5a)}, \theta = (-26+a+\frac{24}{1+a})d_1, \tag{5.14}$$

$$\lambda_1 = 3d_1(1+9a), \lambda_2 = \frac{2a(-2+(-25+a)a)(-4+a(-2+3a))d_1}{(-2+a)(1+a)(4+5a)},$$

$$\lambda_3 = \frac{2(-2+(-25+a)a)(9+a(13+a(-1+6a)))d_1}{(1+a)(-9+2a)(2+5a)}, \tag{5.15}$$

$$c_{11} = \frac{3d_1(1+9a)}{k_1}, c_{12} = \frac{3(-1+a)(-1+a(-9+4a))d_1}{(1+a)k_2}, c_{13} = \frac{15d_1}{k_3}, \tag{5.16}$$

$$c_{21} = \frac{2(2+a)(6+7a)(-2+(-25+a)a)d_1}{(-2+a)(1+a)(4+5a)k_1}, c_{22} = \frac{2a(-2+(-25+a)a)(-4+a(-2+3a))d_1}{(-2+a)(1+a)(4+5a)k_2},$$

$$c_{23} = \frac{8(2+a)(-2+(-25+a)a)d_1}{(-2+a)(1+a)(4+5a)k_3}, \tag{5.17}$$

$$c_{31} = \frac{22(1+2a)(3+2a)(-2+(-25+a)a)d_1}{(1+a)(-9+2a)(2+5a)k_1}, c_{32} = \frac{2(-1+2a)(1+2a)(-9+5a)(-2+(-25+a)a)d_1}{(1+a)(-9+2a)(2+5a)k_2},$$

$$c_{33} = \frac{24(3+2a)(-2+(-25+a)a)d_1}{(1+a)(-9+2a)(2+5a)k_3}, \tag{5.18}$$

where k_1, k_2, k_3 are constants.

Under the conditions (5.14)-(5.18), (4.2) adimits a solution of the form

$$\begin{cases} U(z) = k_1 T^3(z) \\ V(z) = k_2 (1 - T(z))^2 \\ W(z) = k_3 T^2(z) (1 - T(z))^2 \end{cases}$$

where T is the solution of (5.1). Assume k_1, k_2, k_3, $d_1 > 0$, then the necessary and sufficient condition for d_i, λ_i, c_{ii} $(i = 1, 2, 3)$, c_{ij} $(i, j = 1, 2, 3, i \neq j) > 0$, and $a \notin [-1, 0]$

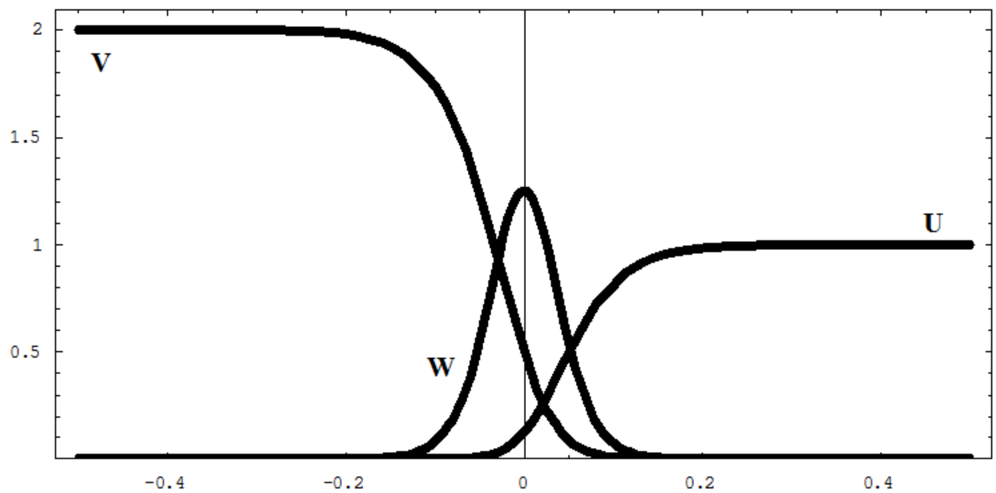

Figure 5.4: Profiles of U, V, W.

in (5.1) to be satisfied is given by

$$a > \frac{25 + \sqrt{633}}{2}.$$

Approximately, $a > 25.0797$.

Now, if one chooses $(a, k_1, k_2, k_3, d_1) = (26, 1, 2, 10, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = \frac{14}{1809}$, $d_3 = \frac{10}{1161}$, $\theta = \frac{8}{9}$, $\lambda_1 = 705$, $\lambda_2 = \frac{51272}{1809}$, $\lambda_3 = \frac{38228}{1161}$, $c_{11} = 705$, $c_{12} = \frac{20575}{6}$, $c_{13} = \frac{3}{4}$, $c_{21} = \frac{5264}{1809}$, $c_{22} = \frac{25636}{1809}$, $c_{23} = \frac{28}{9045}$, $c_{31} = \frac{11660}{1161}$, $c_{32} = \frac{19822}{387}$, and $c_{33} = \frac{4}{387}$ by (5.14)-(5.18). The resulting profiles of U, V, W are shown in Figure 5.4 and the profile of T is similar to Figure 5.2.

5.4 Type-4 Solutions $(i, m, n) = (3, 4, 1)$

The following restriction on parameters appearing in (4.2) is assumed so that an exact solution is allowed.

$$d_2 = \frac{2(1+2a)(-7+a+5a^2)d_1}{3(-1+2a)(4+(-1+a)a)}, d_3 = \frac{(1+6a)(-7+a+5a^2)d_1}{(4+a)(-1+2a)(-4+3a)}, \theta = \frac{-2(-7+a+5a^2)d_1}{-1+2a}, \tag{5.19}$$

$$\lambda_1 = \frac{3(-1+a)(1+a)(5+4a)d_1}{-1+2a}, \lambda_2 = \frac{-16(1+2a)(-7+a+5a^2)d_1}{(-1+2a)(4+(-1+a)a)},$$

$$\lambda_3 = \frac{(-7+a+5a^2)(15+58a+15a^2)d_1}{(4+a)(-1+2a)(-4+3a)}, \tag{5.20}$$

$$c_{11} = \frac{3(-1+a)(1+a)(5+4a)d_1}{(-1+2a)k_1}, c_{12} = \frac{15d_1}{k_2},$$

$$c_{13} = \frac{-3(6+a(1+a)(-17+4a))d_1}{(-1+2a)k_3}, \tag{5.21}$$

$$c_{21} = \frac{-8(2+a)(-7+a+5a^2)(-1+5a(2+a))d_1}{3(-1+2a)(4+(-1+a)a)k_1}, c_{22} = \frac{-16(1+2a)(-7+a+5a^2)d_1}{(-1+2a)(4+(-1+a)a)k_2},$$

$$c_{23} = \frac{8(2+a)(-7+a+5a^2)(-12+a(-12+5a))d_1}{3(-1+2a)(4+(-1+a)a)k_3}, \tag{5.22}$$

$$c_{31} = \frac{(5+3a)(-7+a+5a^2)(-9+17a+12a^2)d_1}{(4+a)(-1+2a)(-4+3a)k_1}, c_{32} = \frac{15(1+6a)(-7+a+5a^2)d_1}{(4+a)(-1+2a)(-4+3a)k_2},$$

$$c_{33} = \frac{-(-7+a+5a^2)(-92-251a-51a^2+36a^3)d_1}{(4+a)(-1+2a)(-4+3a)k_3}, \tag{5.23}$$

where k_1, k_2, k_3 are constants.

Under the conditions (5.19)-(5.23), (4.2) adimits a solution of the form

$$\begin{cases} U(z) = k_1 T^3(z) \\ V(z) = k_2 (1 - T(z))^4 \\ W(z) = k_3 T(z) (1 - T(z))^2 \end{cases}$$

where T is the solution of (5.1). Assume k_1, k_2, k_3, $d_1 > 0$, then the necessary and sufficient condition for d_i, λ_i, c_{ii} $(i = 1, 2, 3)$, c_{ij} $(i, j = 1, 2, 3, i \neq j) > 0$, and $a \notin [-1, 0]$

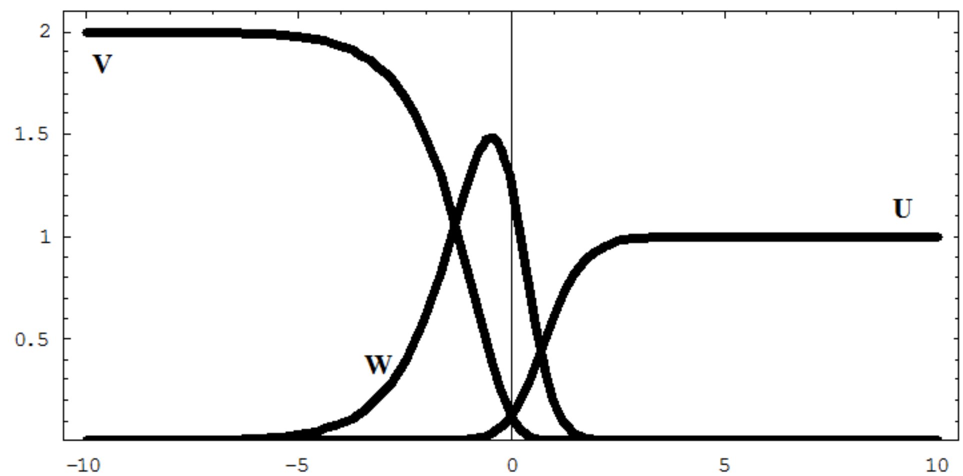

Figure 5.5: Profiles of U, V, W.

in (5.1) to be satisfied is given by

$$1 < a < \frac{-1+\sqrt{141}}{10}.$$

Approximately, $1 < a < 1.08743$.

Now, if one chooses $(a, k_1, k_2, k_3, d_1) = (1.05, 1, 2, 10, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = 0.20283$, $d_3 = 0.676391$, $\theta = 0.795455$, $\lambda_1 = 2.57182$, $\lambda_2 = 4.86793$, $\lambda_3 = 8.56492$, $c_{11} = 2.57182$, $c_{12} = \frac{15}{2}$, $c_{13} = 5.87782$, $c_{21} = 11.9835$, $c_{22} = 2.4436$, $c_{23} = 1.52363$, $c_{31} = 16.6737$, $c_{32} = 5.07293$, and $c_{33} = 3.4294$ by (5.19)-(5.23). The resulting profiles of U, V, W are shown in Figure 5.5 and the profile of T is similar to Figure 5.2.

5.5 Type-5 Solutions $(i, m, n) = (4, 1, 1)$

The following restriction on parameters appearing in (4.2) is assumed so that an exact solution is allowed.

$$d_2 = \frac{(-17+a(2+5a))d_1}{(4+a)(-1+2a)}, d_3 = \frac{3(-17+a(2+5a))d_1}{4(-6+a(8+3a))}, \theta = (-17+a(2+5a))d_1, \tag{5.24}$$

$$\lambda_1 = 24d_1, \lambda_2 = \frac{(-17 + a(2 + 5a))(-2 + a(1 + 2a(4 + a)))d_1}{(4 + a)(-1 + 2a)},$$

$$\lambda_3 = \frac{(-17 + a(2 + 5a))d_1}{4(-6 + a(8 + 3a))}, \tag{5.25}$$

$$c_{11} = \frac{24d_1}{k_1}, c_{12} = \frac{4(-1 + a)(-6 + a(11 + 5a))d_1}{k_2}, c_{13} = \frac{44(-1 + a)d_1}{k_3}, \tag{5.26}$$

$$c_{21} = \frac{3(-17 + a(2 + 5a))d_1}{(4 + a)(-1 + 2a)k_1}, c_{22} = \frac{(-17 + a(2 + 5a))(-2 + a(1 + 2a(4 + a)))d_1}{(4 + a)(-1 + 2a)k_2},$$

$$c_{23} = \frac{(34 - 89a + 25a^3)d_1}{(4 + a)(-1 + 2a)k_3}, \tag{5.27}$$

$$c_{31} = \frac{45(-17 + a(2 + 5a))d_1}{4(-6 + a(8 + 3a))k_1}, c_{32} = \frac{(-17 + a(2 + 5a))(-15 + a(-32 + 3a(37 + 12a)))d_1}{4(-6 + a(8 + 3a))k_2},$$

$$c_{33} = \frac{3(-13 + 27a)(-17 + a(2 + 5a))d_1}{4(-6 + a(8 + 3a))k_3}, \tag{5.28}$$

where k_1, k_2, k_3 are constants.

Under the conditions (5.24)-(5.28), (4.2) adimits a solution of the form

$$\begin{cases} U(z) = k_1 T^4(z) \\ V(z) = k_2 (1 - T(z)) \\ W(z) = k_3 T(z) (1 - T(z))^2 \end{cases}$$

where T is the solution of (5.1). Assume k_1, k_2, k_3, $d_1 > 0$, then the necessary and sufficient condition for d_i, λ_i, c_{ii} $(i = 1, 2, 3)$, c_{ij} $(i, j = 1, 2, 3, i \neq j) > 0$, and $a \notin [-1, 0]$ in (5.1) to be satisfied is given by

$$a > \frac{-1 + \sqrt{86}}{5}.$$

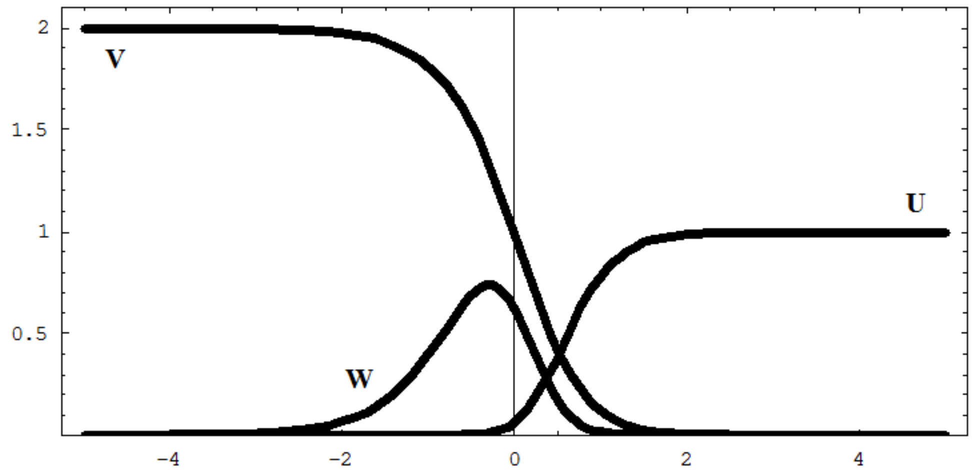

Figure 5.6: Profiles of U, V, W.

Approximately, $a > 1.65472$.

Now, if one chooses $(a, k_1, k_2, k_3, d_1) = (2, 1, 2, 5, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = \frac{7}{18}$, $d_3 = \frac{21}{88}$, $\theta = 7$, $\lambda_1 = 24$, $\lambda_2 = \frac{56}{3}$, $\lambda_3 = \frac{3255}{88}$, $c_{11} = 24$, $c_{12} = 72$, $c_{13} = \frac{44}{5}$, $c_{21} = \frac{7}{6}$, $c_{22} = \frac{28}{3}$, $c_{23} = \frac{28}{45}$, $c_{31} = \frac{315}{88}$, $c_{32} = \frac{4571}{176}$, and $c_{33} = \frac{861}{440}$ by (5.24)-(5.28). The resulting profiles of U, V, W are shown in Figure 5.6 and the profile of T is similar to Figure 5.2.

5.6 Type-6 Solutions $(i, m, n) = (4, 2, 1)$

The following restriction on parameters appearing in (4.2) is assumed so that an exact solution is allowed.

$$d_2 = \frac{(2+a)(-23+a(2+a))d_1}{(1+a)(-6+5a(3+a))}, d_3 = \frac{(5+3a)(-23+a(2+a))d_1}{5(1+a)(-7+a(8+3a))}, \theta = (1+a-\frac{24}{1+a})d_1, \tag{5.29}$$

$$\lambda_1 = 24d_1, \lambda_2 = \frac{2(-23+a(2+a)(-2+3a(1+a(4+a))))d_1}{(1+a)(-6+5a(3+a))},$$

$$\lambda_3 = \frac{3(-5+a+22a^2+6a^3)(-23+a(2+a))d_1}{5(1+a)(-7+a(8+3a))}, \tag{5.30}$$

$$c_{11} = \frac{24d_1}{k_1}, c_{12} = \frac{4(-18 + a + 5a^2 + \frac{24}{1+a})d_1}{k_2}, c_{13} = \frac{44(-1 + a)d_1}{k_3}, \tag{5.31}$$

$$c_{21} = \frac{8(2 + a)(-23 + a(2 + a))d_1}{(1 + a)(-6 + 5a(3 + a))k_1}, c_{22} = \frac{2(-23 + a(2 + a))(-2 + 3a(1 + a(4 + a)))d_1}{(1 + a)(-6 + 5a(3 + a))k_2},$$

$$c_{23} = \frac{2(2 + a)(-2 + 7a)(-23 + a(2 + a))d_1}{(1 + a)(-6 + 5a(3 + a))k_3}, \tag{5.32}$$

$$c_{31} = \frac{3(5 + 3a)(-23 + a(2 + a))d_1}{(1 + a)(-7 + a(8 + 3a))k_1}, c_{32} = \frac{(-23 + a(2 + a))(-15 + a(-32 + 3a(37 + 12a)))d_1}{5(1 + a)(-7 + a(8 + 3a))k_2},$$

$$c_{33} = \frac{(5 + 3a)(-13 + 27a)(-23 + a(2 + a))d_1}{5(1 + a)(-7 + a(8 + 3a))k_3}, \tag{5.33}$$

where k_1, k_2, k_3 are constants.

Under the conditions (5.29)-(5.33), (4.2) adimits a solution of the form

$$\begin{cases} U(z) = k_1 T^4(z) \\ V(z) = k_2 (1 - T(z))^2 \\ W(z) = k_3 T(z) (1 - T(z))^2 \end{cases}$$

where T is the solution of (5.1). Assume k_1, k_2, k_3, $d_1 > 0$, then the necessary and sufficient condition for d_i, λ_i, c_{ii} $(i = 1, 2, 3)$, c_{ij} $(i, j = 1, 2, 3, i \neq j) > 0$, and $a \notin [-1, 0]$ in (5.1) to be satisfied is given by

$$a > -1 + 2\sqrt{6}.$$

Approximately, $a > 3.89898$.

Now, if one chooses $(a, k_1, k_2, k_3, d_1) = (4, 1, 2, 10, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = \frac{3}{335}$, $d_3 = \frac{17}{1825}$, $\theta = \frac{1}{5}$, $\lambda_1 = 24$, $\lambda_2 = \frac{394}{335}$, $\lambda_3 = \frac{441}{365}$, $c_{11} = 24$, $c_{12} = \frac{705}{5}$, $c_{13} = \frac{66}{5}$, $c_{21} = \frac{24}{335}$, $c_{22} = \frac{197}{335}$, $c_{23} = \frac{78}{1675}$, $c_{31} = \frac{51}{365}$, $c_{32} = \frac{3937}{3650}$, and $c_{33} = \frac{323}{3650}$ by (5.29)-(5.33).

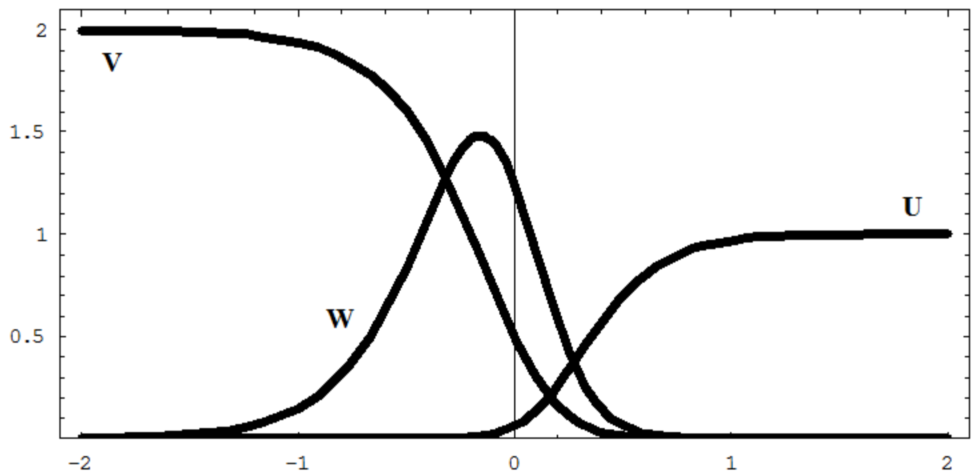

Figure 5.7: Profiles of U, V, W.

The resulting profiles of U, V, W are shown in Figure 5.7 and the profile of T is similar to Figure 5.2.

5.7 Type-7 Solutions $(i, m, n) = (4, 3, 1)$

The following restriction on parameters appearing in (4.2) is assumed so that an exact solution is allowed.

$$d_2 = \frac{(2+a)(-23+a(2+a))d_1}{(1+a)(-6+7a(3+a))}, d_3 = \frac{(5+3a)(-23+a(2+a))d_1}{5(1+a)(-7+a(8+3a))}, \theta = (1+a-\frac{24}{1+a})d_1, \tag{5.34}$$

$$\lambda_1 = 24d_1, \lambda_2 = \frac{3(-23+a(2+a)(-2+a(5+4a(4+a))))d_1}{(1+a)(-6+7a(3+a))},$$

$$\lambda_3 = \frac{3(-5+a+22a^2+6a^3)(-23+a(2+a))d_1}{5(1+a)(-7+a(8+3a))}, \tag{5.35}$$

$$c_{11} = \frac{24d_1}{k_1}, c_{12} = \frac{4(-18+a+5a^2+\frac{24}{1+a})d_1}{k_2},$$

$$c_{13} = \frac{4(-1+a)(5+a(22+5a))d_1}{(1+a)k_3}, \tag{5.36}$$

$$c_{21} = \frac{15(2+a)(-23+a(2+a))d_1}{(1+a)(-6+7a(3+a))k_1}, c_{22} = \frac{3(-23+a(2+a))(-2+a(5+4a(4+a)))d_1}{(1+a)(-6+7a(3+a))k_2},$$

$$c_{23} = \frac{3(-23+a(2+a))(-6+a(1+a)(21+4a))d_1}{(1+a)(-6+7a(3+a))k_3}, \tag{5.37}$$

$$c_{31} = \frac{3(5+3a)(-23+a(2+a))d_1}{(1+a)(-7+a(8+3a))k_1}, c_{32} = \frac{(-23+a(2+a))(-15+a(-32+3a(37+12a)))d_1}{5(1+a)(-7+a(8+3a))k_2},$$

$$c_{33} = \frac{4(-23+a(2+a))(-20+a(16+48a+9a^2))d_1}{5(1+a)(-7+a(8+3a))k_3}, \tag{5.38}$$

where k_1, k_2, k_3 are constants.

Under the conditions (5.34)-(5.38), (4.2) adimits a solution of the form

$$\begin{cases} U(z) = k_1 T^4(z) \\ V(z) = k_2 (1 - T(z))^3 \\ W(z) = k_3 T(z) (1 - T(z))^2 \end{cases}$$

where T is the solution of (5.1). Assume k_1, k_2, k_3, $d_1 > 0$, then the necessary and sufficient condition for d_i, λ_i, c_{ii} $(i = 1, 2, 3)$, c_{ij} $(i, j = 1, 2, 3, i \neq j) > 0$, and $a \notin [-1, 0]$ in (5.1) to be satisfied is given by

$$a > -1 + 2\sqrt{6} \quad \text{or} \quad a < -1 - 2\sqrt{6}.$$

Approximately, $a > 3.89898$ or $a < -5.89898$.

Now, if one chooses $(a, k_1, k_2, k_3, d_1) = (4, 1, 2, 10, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = \frac{3}{475}$, $d_3 = \frac{17}{1825}$, $\theta = \frac{1}{5}$, $\lambda_1 = 24$, $\lambda_2 = \frac{159}{95}$, $\lambda_3 = \frac{441}{365}$, $c_{11} = 24$, $c_{12} = \frac{708}{5}$, $c_{13} = \frac{1038}{25}$,

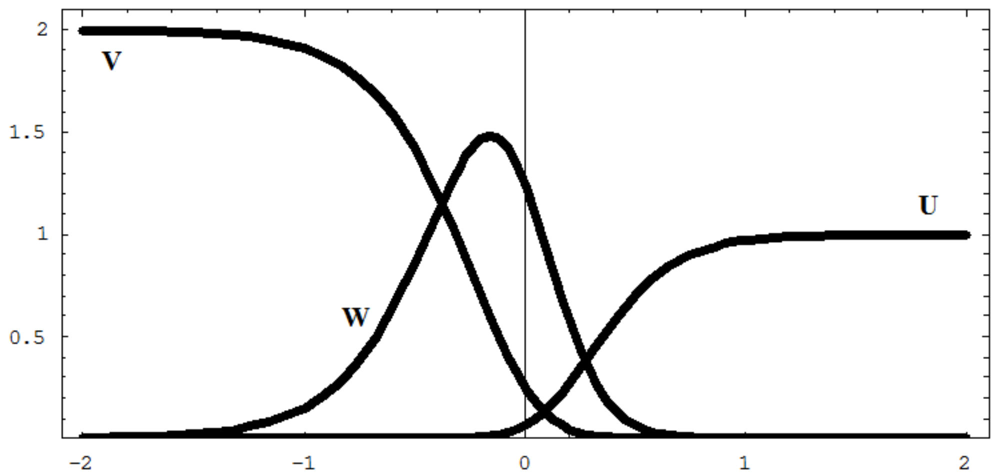

Figure 5.8: Profiles of U, V, W.

$c_{21} = \frac{9}{95}$, $c_{22} = \frac{159}{190}$, $c_{23} = \frac{1101}{4750}$, $c_{31} = \frac{51}{365}$, $c_{32} = \frac{3937}{3650}$, and $c_{33} = \frac{2776}{9125}$ by (5.34)-(5.38). The resulting profiles of U, V, W are shown in Figure 5.8 and the profile of T is similar to Figure 5.2.

If one chooses $(a, k_1, k_2, k_3, d_1) = (-6, 1, 2, 10, 1)$ and $T_0 = \frac{1}{2}$ in (5.1), then $d_2 = \frac{1}{150}$, $d_3 = \frac{13}{1325}$, $\theta = -\frac{1}{5}$, $\lambda_1 = 24$, $\lambda_2 = \frac{8}{5}$, $\lambda_3 = \frac{309}{265}$, $c_{11} = 24$, $c_{12} = \frac{1512}{5}$, $c_{13} = \frac{742}{25}$, $c_{21} = \frac{1}{10}$, $c_{22} = \frac{4}{5}$, $c_{23} = \frac{6}{125}$, $c_{31} = \frac{39}{265}$, $c_{32} = \frac{3603}{2650}$, and $c_{33} = \frac{664}{6625}$ by (5.34)-(5.38). The resulting profiles of U, V, W and T are shown in Figure 5.9 and Figure 5.10, respectively.

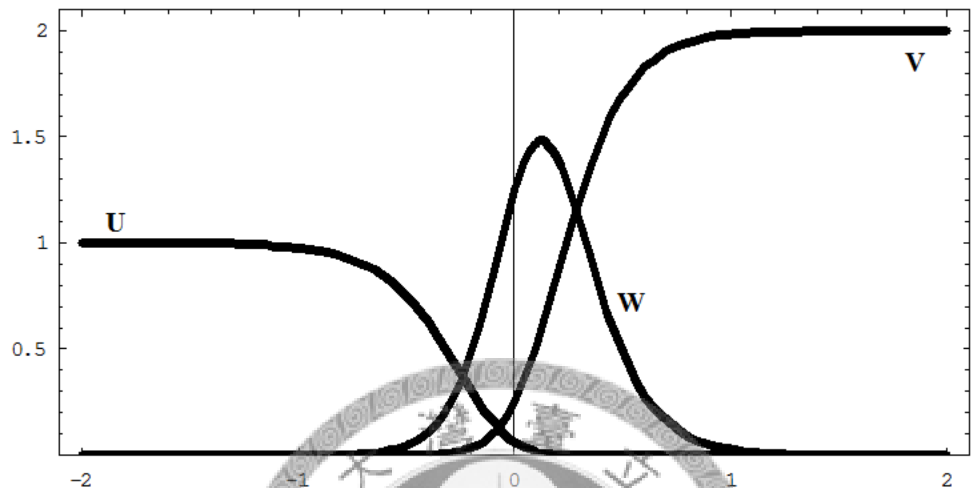

Figure 5.9: Profiles of U, V, W.

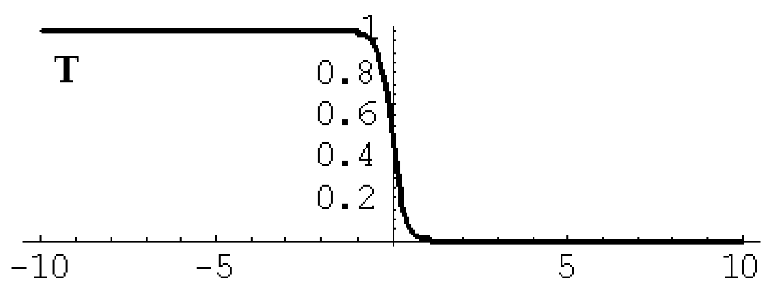

Figure 5.10: Profile of T with $a = -6$ and $T_0 = \frac{1}{2}$

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