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反應擴散方程的非平面傳動波

Nonplanar Traveling Wave Solutions of Reaction-Diffusion Equations

張菀庭

Yuan-Ting Chang

指導教授：陳俊全 教授

Advisor: Prof. Chiun-Chuan Chen

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國立臺灣大學（碩）博士學位論文
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反應擴散方程的非平面傳動波

Nonplanar Traveling Wave Solutions of Reaction-Diffusion Equations

本論文係張菀庭君（R96221023）在國立臺灣大學數學學系、所完成之碩（博）士學位論文，於民國 99 年 06 月 08 日承下列考試委員審查通過及口試及格，特此證明

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摘要

這篇論文主要處理的問題是反應擴散方程的傳動波$u_t = \Delta u + u_{zz} - f(u)$，其中$(x,z) \in \mathbb{R}^{n+1}$是空間變數。假定$f(u)$是一個雙穩定的非線性項，我們分別考慮平衡的狀態及非平衡的狀態。在平衡的狀態下，我們將描述多種連接兩個平衡點的傳動波，這些傳動波各有各的形狀。在非平衡的狀態下，我們將傳動波解限制為柱狀對稱的形式，接著證明這樣的解在$n \geq 2$時其形狀會近似於拋物面，而在$n = 1$時會近似於超餘弦函數。除此之外，我們也將證明單穩傳動波的存在性。本篇論文的主要參考文獻的作者有以下幾位：Y. Morita、H. Ninomiya、X.F. Chen、J-S Guo、F. Hamel 及 J-M Roquejoffre。

關鍵字：傳動波、反應擴散方程、雙穩定的非線性項、界面、雙穩、單穩

Abstract

We are dealing with traveling wave solutions of a reaction-diffusion equation $u_t = \Delta u + u_{zz} - f(u)$, where $(x,z) = (x_1, \cdots, x_n, z) \in \mathbb{R}^{n+1}$ is the space variable and Δ is the Laplacian in $\mathbb{R}^n$. Assume that $f(u)$ is a bistable nonlinearity, then we consider the balanced case and unbalanced case respectively. In the preceding case, we describe some types of traveling waves connecting two stable equilibria. In the case of latter, we want to find out the bistable-type traveling waves with the interfaces other than plane. If the solution is restricted to be cylindrically symmetric, then we can show that the interface is asymptotically a paraboloid as $n \geq 2$ and a hyperbolic cosine curve as $n = 1$. Besides, we prove the existence of the monostable-type traveling waves. The main references of this thesis are Y. Morita, H. Ninomiya, X.F. Chen, J-S GUO, F. Hamel and J-M Roquejoffre.

Keywords:

traveling wave, reaction-diffusion equation, bistable nonlinearity, interface, bistable, monostable

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Chapter 1

Introduction

The main list of references of this thesis is *Monostable-type traveling waves of bistable reaction-diffusion equations in the multi-dimensional space* written by Y. Morita and H. Ninomiya and *Traveling waves with paraboloid like interfaces for balanced bistable dynamics* written by X.F. Chen, J-S GUO, F. Hamel, H. Ninomiya and J-M Roquejoffre. In addition, [4], [5], [7], [8], and [9] offer a lot of valuable materials. Because of these articles, this thesis must be more intact and more abundant.

There are many phenomena are well studied by mathematical models. Reaction-Diffusion System is one of these useful models. It is a important and extensive application in the fields such as chemistry, biology, physics, material, etc.. For example, Hodgkin and Huxley won the Nobel Prize for medicine because of studying and deriving the Hodgkin-Huxley model which describing how action potentials in neurons are initiated and propagated. This equation is a special case of reaction-diffusion Systems.

In this paper, we focus on the following scalar reaction-diffusion equation, for $u = u\,(x, z, t)$,

$$u_t = \Delta u + u_{zz} - f(u), \tag{1.1}$$

where t is the time variable and $(x, z) = (x_1, \cdots, x_n, z) \in \mathbb{R}^{n+1}$ is the space variable and Δ is the Laplacian in $\mathbb{R}^n$. We assume that $f\,(u)$ is C^2 on an open interval containing

$[0,1]$ and satisfies

$$\begin{cases} f(0) = f(a) = f(1) = 0, \quad f'(0) > 0,\ f'(a) < 0,\ f'(1) > 0, \\ f(u) \neq 0 \quad \text{for} \quad u \in (0,a) \cup (a,1). \end{cases} \tag{1.2}$$

The condition (1.2) has the characteristic feature such that the dynamics generated by the diffusion-free equation of (1.1) adimts two stable equilibria $u = 0, 1$ and unstable one $u = a$ separating the basins of the two equilibria. Thus the equation (1.1) satisfying (1.2) is called a bistable reaction-diffusion equation or a reaction-diffusion equation with bistable nonlinearity. If f is cubic, it is also called the Allen-Cahn equation or the Nagumo equation. This equation is used as a simple model describing propagation of species in population biology or propagation of nerve excitation. Back to the viewpoint of mathematics, we consider two different cases. One is unbalanced case, and another one is balanced case.

In the unbalanced case, that is, with the assumption that

$$\int_0^1 f(u)\, du < 0,$$

we want to find some bistable-type traveling waves which connecting two stable equilibria $u = 0$ and $u = 1$. With the unbalanced condition, (1.1) with (1.2) allows many types of bistable traveling waves. Some details will be described in the Chapter 2. More specific, we are searching for solutions of the form $u(x,z,t) = U(x, z - ct)$ which satisfies the differential equation and the "boundary values"

$$\begin{cases} cU_z + U_{zz} + \Delta U = f(U) & \forall x \in \mathbb{R}^n,\ z \in \mathbb{R}, \\ \lim_{z \to \infty} U(x,z) = 1 & \forall x \in \mathbb{R}^n, \\ \lim_{z \to -\infty} U(x,z) = 0 & \forall x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

<table>
<tr><td colspan="3">Reaction-Diffusion Equations with Bistable Nonlinearities
$u_t = \Delta u + u_{zz} - f(u)$</td></tr>
<tr><td colspan="2">Bistable-type</td><td>Monostable-type</td></tr>
<tr><td>f is unbalanced</td><td>1. planar Ch2
2. conical
3. pyramidal</td><td rowspan="2">nonplanar Ch4
connect1 with v
and v with 0
v: standing wave</td></tr>
<tr><td>f is balanced</td><td>$n = 1$: Ch3
hyperbolic cosine
$n \geq 2$: paraboloid</td></tr>
</table>

Table 1.1: The classification of the interfaces

The solution satisfying (1.3) is called a bistable-type traveling wave.

In the chapter 3, we consider the balanced bistable nonlinearity. The same question as above, we want to know whether traveling waves connecting two stable equilibria exist or not. The answer is YES. Furthermore, we can find nonplanar traveling wave solutions.

Finally, we seek another type traveling wave solutions, that is, monostable-type traveling wave solutions. In the Chapter 4, we prove the exsitence of monostable-type traveling solutions which are also nonplaner. Monostable-type traveling wave solutions conncet one of the equilibria with a standing wave solution of (1.1). The proof in Chapter 3 and 4 are based on [6] and [3] respectively.

All of traveling waves we dealing with in this thesis are classified in Table 1.1.

Chapter 2

Bistable-Type Traveling Waves in the Unbalanced Condition

Bistable-type traveling waves connect two equilibria $u = 0$ and $u = 1$. The existence of traveling waves with different shapes of interfaces was proved. For example, planar solutions, conical solutions, and pyramidal solutions. Next, we will describe these types of bistable traveling waves.

It is well-known that (1.1) with (1.2) has a planar traveling wave with a monotone profile. That is, there is a solution $u = \hat{U}(z - c_0 t)$ satisfying

$$\begin{cases} \hat{U}_{zz} + c_0 \hat{U}_z - f\left(\hat{U}\right) = 0, \quad \hat{U}(z) > 0 \quad (z \in \mathbb{R}), \\ \lim_{z\to-\infty} \hat{U}(z) = 0, \quad \lim_{z\to\infty} \hat{U}(z) = 1. \end{cases} \tag{2.1}$$

Since $\hat{U}'(z) > 0$ holds for the solution $\hat{U}(z)$, we may call it a monotone planar traveling wave. Moreover the solution $\hat{U}$ is unique (up to translation).

2.1 Solutions with Conical Interfaces in Multi-Dimension

This subsection is based on [4, 5, 7, 8].

Let $u(x,z,t) = U(x, z-ct)$ as in (1.3). We look for solutions of (1.3) satisfying a conical asymptotic condition of angle α with respect to the direction $-e_{n+1} := (0,\cdots,0,-1)$. A natural condition, as in Figure 2.1, is the following:

$$\begin{cases} \limsup_{A\to+\infty,\, z\geq A-|x|cot\alpha} |U(x,z)-1| & = 0, \\ \limsup_{A\to-\infty,\, z\leq A-|x|cot\alpha} |U(x,z)| & = 0. \end{cases} \tag{2.2}$$

Note that, the planar front $\hat{U}(z)$ is the solution of (1.3) and (2.2) with $\alpha = \pi/2$. But the interesting case is $0 < \alpha < \pi/2$.

Since condition (2.2) is too strong for $n \geq 2$, the following less restrictive condition will be used:

$$\begin{cases} \limsup_{A\to+\infty,\, z\geq A+\eta(|x|)} |U(x,z)-1| & = 0, \\ \limsup_{A\to-\infty,\, z\leq A+\eta(|x|)} |U(x,z)| & = 0 \end{cases} \tag{2.3}$$

for some globally Lipschitz function η defined in $[0,+\infty)$. We will see that it automatically implies a weak conical condiotion with some given angle α.

Next, we state the theorm of exsitence of conical bistable case.

Theorem 2.1. (Existence result in dimension $n = 1$)

In dimension $n = 1$, for each $\alpha \in (0, \pi/2]$, there exists a unique-up to shift in the (x,z) variables - solutions of (c,U) of (1.3) and (2.2). Furthermore, $0 < U < 1$ in $\mathbb{R}^2$, c is given by $c = c_0/sin\alpha$, and up to shift, U is even in x and increasing in $|x|$. The fuction U is decreasing in any unit direction $\tau = (\tau_x, \tau_z) \in \mathbb{R}^2$ such that $\tau_z < -\cos\alpha$. For each $\lambda \in (0,1)$, the level set $\{U = \lambda\}$ is a globally Lipschitz graph $\{y = \eta_\lambda(x)\}$ whose Lipschitz norm is equal to $\cot\alpha$. Lastly, $U(x + x_n, z - |x_n|\cot\alpha) \to \hat{U}(\pm x\cos\alpha + z\sin\alpha)$ in $C^2_{loc}(\mathbb{R}^2)$, for any sequence $x_n \to \pm\infty$.

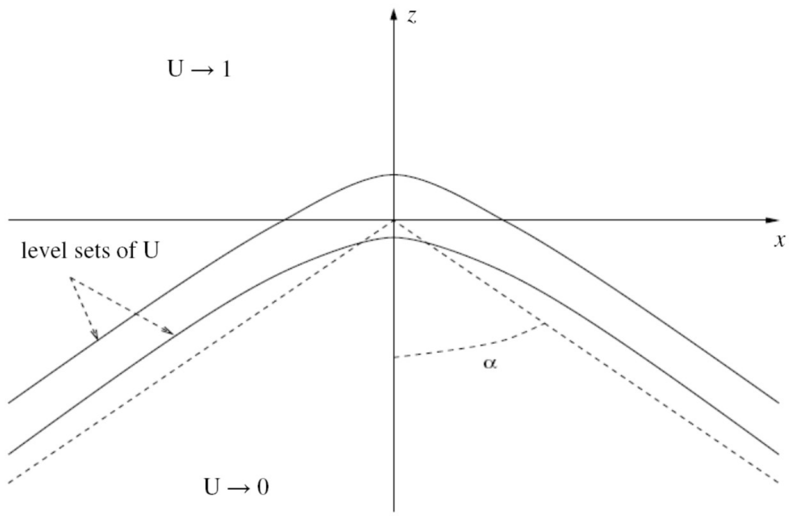

Figure 2.1: Level sets of a solution U satisfying (2.2) captured from [4]

If the dimension n is higher than one, we use the weaker condition (2.3).

Theorem 2.2. (Existence result in dimension $n \geq 2$)

In dimension $n \geq 2$, for each $\alpha \in (0, \pi/2]$, there exists a solution (c, U) of (1.3) such that

- $0 < U < 1$ *in* $\mathbb{R}^{n+1}$.

- $U(x, z) = \tilde{U}(|x|, z)$ *and the following monotonicity properties hold:* $\partial_{|x|}\tilde{U} \geq 0$, $\partial_z \tilde{U} \geq 0$.

- *the fuction U satisfies (2.3) with $\eta = \eta_\lambda$, for all $\lambda \in (0,1)$, where* $\{U(x,z) = \lambda\} = \{z = \eta_\lambda(x), x \in \mathbb{R}^n\}$.

- *there holds* $\hat{x} \cdot \nabla \eta_\lambda(x) \to -\cot\alpha$ *as* $|x| \to \infty$ *where* $\hat{x} = \frac{x}{|x|}$.

Moreover the function U is decreasing in any unit direction $\tau = (\tau_x, \tau_z) \in \mathbb{R}^n \times \mathbb{R}$ such that $\tau_z < -\cos\alpha$. Lastly, for any unique direction $e \in \mathbb{R}^n$, for any sequence $r_n \to \infty$ and for any $\lambda \in (0,1)$, $U(x + r_n e, z + \eta_\lambda(r_n e)) \to \hat{U}\left((x \cdot e)\cos\alpha + z\sin\alpha + \hat{U}^{-1}(\lambda)\right)$ *in* $C^2_{loc}(\mathbb{R}^{n+1})$.

2.2 Solutions with Pyramidal Interfaces in 3-Dimension

This subsection is based on [9] and [10]. Our aim is to construct three-dimensional traveling wave solutions for which the contour line has a pyramidal shape. More precisely, we rewrite $u = u(x, y, z, t)$, where t is the time variable and $(x, y, z) \in \mathbb{R}^3$ is the space variable and Δ is $\partial_{xx} + \partial_{yy}$. Let $u|_{t=0} = u_0$ be an initial condition, where u_0 is bounded and C^1. And for the nonlinear term f, we can consider a more general form than (1.2). We give the assumptions for f as follows:

(A1) f is $C^1[0,1]$ with $f(1) = f(0) = 0$, $f'(1) > 0$, and $f'(0) > 0$.

(A2) $\int_0^1 f(u)\,du < 0$.

(A3) There exists $\hat{U}$ which satisfies (2.1) for some $c_0 \in \mathbb{R}$.

Similar to above, we want to find traveling waves which travel upwards in the vertical z direction with a constant speed c. In this subsection, we assume $c > c_0$. If v is a traveling wave with speed c, then it satisfies

$$L(v) := -v_{xx} - v_{yy} - v_{zz} - cv_z + f(v) = 0 \text{ in } \mathbb{R}^3. \tag{2.4}$$

First, we construct a subsolution of (2.4).

Let $m \geq 3$ be a given integer. Set

$$\tau := \frac{\sqrt{c^2 - c_0^2}}{c_0} > 0.$$

Assume $(A_j, B_j) \in \mathbb{R}^2$ satisfies

$$A_j^2 + B_j^2 = 1 \quad \forall j = 1, \cdots, m$$

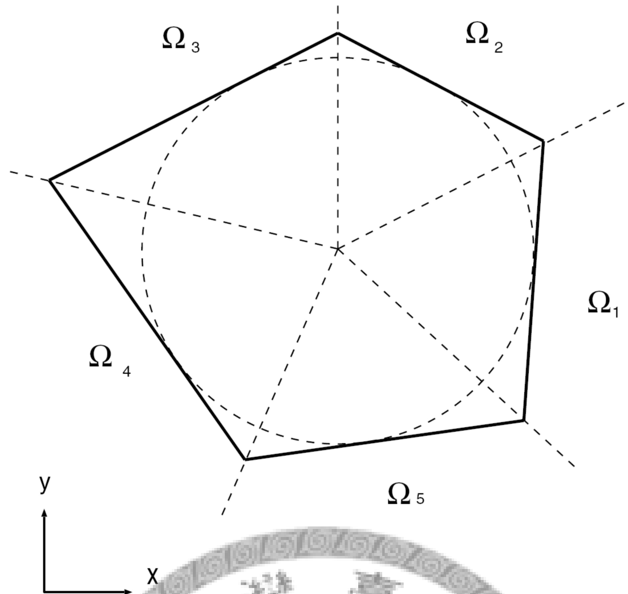

Figure 2.2: The decomposition of the $x-y$ plane by Ω_j for $m=5$ captured from [9]

and

$$A_j B_{j+1} - A_{j+1} B_j > 0, \quad \forall j = 1, \cdots, m$$
$$A_m B_1 - A_1 B_m > 0.$$

We also assume $(A_i, B_i) \neq (A_j, B_j)$ for all $i \neq j$. And consider

$$h_j(x,y) := \tau (A_j x + B_j y), \tag{2.5}$$
$$h(x,y) := \max_{1 \le j \le m} h_j(x,y) = \tau \max_{1 \le j \le m} (A_j x + B_j y). \tag{2.6}$$

Then $z = h(x,y)$ represents a pyramid in $\mathbb{R}^3$. If we set $\Omega_j := \{(x,y) | \, h(x,y) = h_j(x,y)\}$, then we can obtain $\mathbb{R}^2 = \cup_{j=1}^m \Omega_j$ and $\Omega_1, \cdots, \Omega_m$ locate counterwise as in Figure 2.2. For the lateral surfaces, we define

$$S_j := \{(x,y,z) | \, z = h_j(x,y), \ (x,y) \in \Omega_j\} \text{ for } j = 1, \cdots, m.$$

$$\Gamma_j := \begin{cases} S_j \cap S_{j+1} & if\ j = 1, \cdots, m-1, \\ S_m \cap S_1 & if\ j = m, \end{cases}$$

$$\Gamma := \cup_{j=1}^{m} \Gamma_j.$$

Then Γ_j represents an edge and Γ represents all edges of a pyramid.

For each (A_j, B_j) we defined above, there exists a planar wave $\hat{U}\left(\left(c_0/c\right)\left(z - h_j\left(x, y\right)\right)\right)$. Furthermore,

$$\hat{U}\left(\frac{c_0}{c}\left(z - h\left(x, y\right)\right)\right) = \max_{1 \le j \le m} \hat{U}\left(\frac{c_0}{c}\left(z - h_j\left(x, y\right)\right)\right)$$

becomes the subsolution of (2.4).

Next, we construct the supersolution of (2.4) carefully since a pyramidal wave is everywhere apart from a pyramid near the edges. Let $\tilde{\rho}\left(r\right) \in C^{\infty}\left[0, \infty\right)$ be a function with the following properties:

$$\begin{cases} \tilde{\rho}\left(r\right) > 0,\ \tilde{\rho}_r\left(r\right) \le 0 & \text{if } r > 0, \\ \tilde{\rho}\left(r\right) \equiv 1 & \text{if } 0 \le r \le \frac{1}{2}, \\ \tilde{\rho}\left(r\right) = e^{-r} & \text{if } r > 0 \text{ is large enough}, \\ \int_0^{\infty} \tilde{\rho}\left(r\right) r dr = \frac{1}{2\pi}. \end{cases}$$

Now let $\rho\left(x, y\right) := \tilde{\rho}\left(\sqrt{x^2 + y^2}\right)$ belong to $C^{\infty}\left(\mathbb{R}^2\right)$ and satisfies $\int_{\mathbb{R}^2} \rho = 1$. Then for a pyramid $z = h\left(x, y\right)$, we define a mollified pyramid $z = \chi\left(x, y\right)$ by $\chi\left(x, y\right) = \rho * h$. Consider the graph of

$$z = \frac{1}{\alpha}\chi\left(\alpha x, \alpha y\right)$$

where $\alpha \in (0, 1)$ will be chosen to be small enough. And then we rescaled coordinate

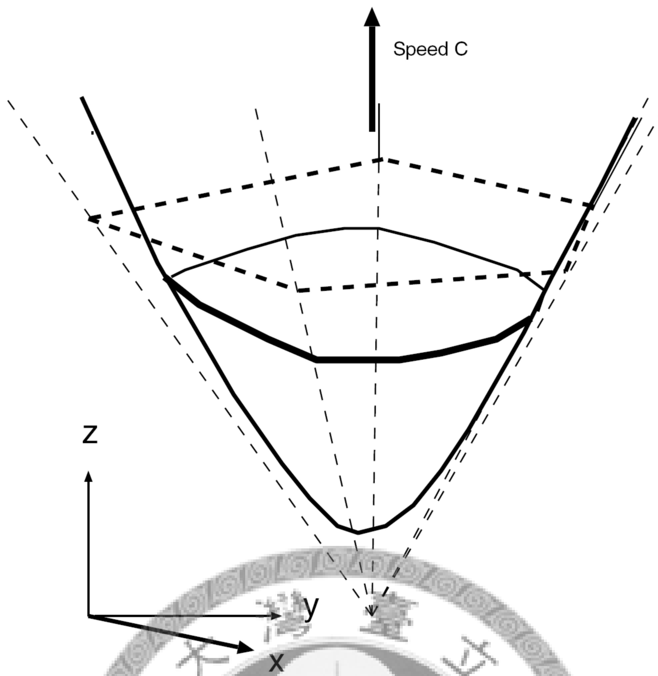

Figure 2.3: A pyramidal traveling wave captured from [9]

(ξ, η, ς) as

$$\xi = \alpha x,\ \eta = \alpha y,\ \varsigma = \alpha z$$

and obtain $\varsigma = \chi(\xi, \eta)$. Then we set

$$\begin{aligned}\hat{\mu} := & \frac{z - \frac{1}{\alpha}\chi(\alpha x, \alpha y)}{\sqrt{1 + \chi_\xi(\alpha x, \alpha y)^2 + \chi_\eta(\alpha x, \alpha y)^2}} \\ = & \frac{1}{\alpha}\frac{\varsigma - \chi(\xi, \eta)}{\sqrt{1 + \chi_\xi(\xi, \eta)^2 + \chi_\eta(\xi, \eta)^2}}.\end{aligned}$$

Finally, if we define

$$\bar{U}(x, y, z) := \chi(\hat{\mu}) + \sigma(x, y)$$

where

$$\sigma(x, y) := \varepsilon\left(\frac{c}{\sqrt{1 + \chi_x(\alpha x, \alpha y)^2 + \chi_y(\alpha x, \alpha y)^2}} - c_0\right)$$

with α and ε small enough, then $\bar{U}(x, y, z)$ is a supersolution of (2.4).

We have constructed the supersolution and subsolution of (2.4), then we have the traveling wave solutions of (2.4). That is, we have the following theorem:

Theorem 2.3. *Suppose $c > c_0$ and let $h(x, y)$ be given in (2.6). Under the assumptions (A1), (A2), and (A3), there exists $U(x, y, z)$ to (2.4) with*

$$\hat{U}\left(\frac{c_0}{c}(z - h(x, y))\right) < U(x, y, z) < 1 \quad in\ \mathbb{R}^3$$

and

$$\lim_{\gamma \to \infty} \sup_{(x,y,z) \in D(\gamma)} \left| U(x, y, z) - \hat{U}\left(\frac{c_0}{c}(z - h(x, y))\right) \right| = 0,$$

$$U_z(x, y, z) > 0 \quad for\ all\ (x, y, z) \in \mathbb{R}^3,$$

where $D(\gamma)$ is defined by

$$D(\gamma) := \{(x, y, z) \,|\, dist((x, y, z), \Gamma) > \gamma\}.$$

Chapter 3

Bistable-Type Traveling Waves in the Balanced Condition

Before we begin to study the question that whether traveling waves connecting two stable equilibria exist or not, we state the balanced bistable condition concretely. The forcing term f is the derivative of a double-equal-well, also called balanced bistable, potential. More precisely,

$$\begin{cases} f = F' \in C^2(\mathbb{R}), \\ F(0) = F(1) = 0 < F(s) \ \ \forall s \in (0,1), \\ F''(0) > 0, \ F''(1) > 0. \end{cases} \tag{3.1}$$

A typical example is the cubic function $f(u) = 2u(2u-1)(u-1)$ with potential $F = u^2(u-1)^2$.

We are interested in solutions with interfaces that travel upwards in the vertical z direction with a constant speed c. That is, we are searching for a solution for (1.3) in the balanced condition. Furthermore, we restrict our solutions in the cylindrically symmetric class. In other words, U depends only on z and $r = |x|$. Since we shall look for cylindrically symmetric solutions which are monotone decreasing along the radial direction, U must have the boundary value $\lim_{|x|\to\infty} U(x,z) = 0$.

Theorem 3.1. *Assume* (3.1). *For any* $c > 0$, (1.3) *admits a cylindrically symmetric solution* U *with the monotonicity property:*

$$U_z > 0 \quad \text{on}\, \mathbb{R}^{n+1} \quad \text{and} \quad U_r < 0 \quad \text{on}\, (\mathbb{R}^n / \{0\}) \times \mathbb{R}. \tag{3.2}$$

De Giorgi conjecture asserts that *when* $c = 0$ *and* $f(U) = U^3 - U$, *all* $z-$*monotonic solutions of* (1.3) *are planar* at least in dimension $n \leq 8$. In this conjecture, the radial symmetric in x is not assumed. The De Giorgi conjecture is ture if $f(U) = 2U(2U-1)(U-1)$. Thus, we want to ask whether planar solutions are the only solutions to the corresponding parabolic equation

$$u_t = \Delta u + u_{zz} + 2u(2u-1)(u-1), \quad (x,z) \in \mathbb{R}^n \times \mathbb{R}, \tag{3.3}$$

subject to the monotonicity conditions

$$\begin{cases} \lim_{z\to\infty} u(x,z,t) = 1, \\ \lim_{z\to-\infty} u(x,z,t) = 0, \quad \forall (x,z,t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}. \\ u_z(x,z,t) > 0 \end{cases} \tag{3.4}$$

The answer is NO since Theorem 3.1 provides an entire solution that satisfies (3.4) but not planar. Thus, for the elliptic equation (1.3) with $c \neq 0$ or for the parabolic equation (3.3), additional conditions are needed for an entire monotone solution to be planar.

The monotonicity condition (3.2) and the boundary values of U imply that the interface can be represented as a graph $z = H(|x|)$ or $|x| = R(z)$, where R is the inverse of H. We can describe the asymptotic shape of the interface as follows,

Theorem 3.2. *Assume (3.1). Let* (c, U) *be as in Theorem 3.1 and* Γ *be the* $\frac{1}{2}$*-level set of* U.

- *If* $n > 1$, Γ *is asymptotically a paraboloid, i.e.*

$$\lim_{z\to\infty, U(x,z)=\frac{1}{2}} \frac{|x|^2}{2z} = \frac{n-1}{c}.$$

- *If* $n = 1$, Γ *is asymptotically a hyperbolic cosine curve, i.e. for some* $A = A(f) > 0$,

$$\lim_{z\to\infty, U(x,z)=\frac{1}{2}} \frac{\cosh(2\mu x)}{\mu z} = \frac{A}{c}, \quad \mu := \sqrt{f'(1)}.$$

3.1 Preliminary

The condition (3.1) is assumed hereafter. It implies there exists $\alpha \in (1/2, 1)$ and $\hat{\alpha} \in (0, 1/2)$ satisfying

$$f' = F'' > 0 \quad \text{on} \quad [0, \hat{\alpha}] \cup [\alpha, 1], \quad F(\alpha) = F(\hat{\alpha}) < F(s) \ \forall s \in (\hat{\alpha}, \alpha). \tag{3.5}$$

Then α and $\hat{\alpha}$ are fixed. The wave speed c is also fixed.

Observe that all wells other than 0 and 1 lie either in $(\hat{\alpha}, \alpha)$ or in $(-\infty, 0) \cup (1, \infty)$ where the latter is not our concern at all. The depth of any well in $(-1, 1)$ is higher than $F(\alpha) > 0 = F(0) = F(1)$.

For convenience, we define the following notations:

$$x \in \mathbb{R}^n,\ z \in \mathbb{R},\ y = (x, z) \in \mathbb{R}^{n+1},\ r = |x|,\ \rho = |y| = \sqrt{|x|^2 + z^2}.$$

Stationary waves

Let $\Phi(x)$ be a one-dimensional stationary solution of (1.1), that is, $\Phi(x)$ is a standing wave solution of

$$\Phi'' = f(\Phi) \ \text{on} \quad \mathbb{R}, \quad \Phi(\infty) = 1, \quad \Phi(-\infty) = 0, \quad \Phi(0) = \alpha.$$

Through simple calculation, we can obtain

$$\Phi' = \sqrt{2F(\Phi)}, \quad \int_{\alpha}^{\Phi(\xi)} \frac{ds}{\sqrt{2F(s)}} = \xi \quad \forall \xi \in \mathbb{R}.$$

Traveling waves

Define

$$f_{\varepsilon}(u) := f(u) + \varepsilon\sqrt{2F(u)}, \quad F_{\varepsilon}(u) := \int_0^u f_{\varepsilon}(s)\, ds.$$

For any $\varepsilon > 0$, f_{ε} is unbalanced, thus F_{ε} attians its deepest well only at $u = 0$. And Φ is also a profile of a one-dimensional traveling wave of speed ε to

$$\varepsilon\Phi' + \Phi'' = f_{\varepsilon}(\Phi) \quad \text{on } \mathbb{R}. \tag{3.6}$$

In addition, we can assume that ε is small enough such that $f'(1) > 0$ and $f'(0) > 0$, and the profile of Φ is then a unique solution to (3.6) up to shift such that $\Phi(\infty) = 1$ and $\Phi(-\infty) = 0$. The family $\{f_{\varepsilon}\}_{\varepsilon>0}$ is an approximation of f and will be used to construct the approximating solution of (1.3).

Radially symmetric stationary waves

Let $\zeta \in C^3(\mathbb{R})$ be a fixed function satisfying

$$\zeta = 0 \text{ on } \{0\} \cup [\hat{\alpha}, 1], \quad \zeta > 0 \text{ on } (0, \hat{\alpha}), \quad \int_0^1 \left\{\zeta(s) - \sqrt{2F(s)}\right\} ds > 0.$$

And for each $\varepsilon > 0$, define

$$\begin{aligned} g_{\varepsilon}(s) &= f_{\varepsilon}(s) - \varepsilon\zeta(s) \\ &= f(s) + \varepsilon\sqrt{2F(s)} - \varepsilon\zeta(s) \quad \forall s \in [0, 1]. \end{aligned}$$

Note that for sufficiently small positive ε,

- Both wells 0 and 1 of g_ε are stable. i.e. $g'_\varepsilon(1) > 0 = g_\varepsilon(1)$ and $g'_\varepsilon(0) > 0 = g_\varepsilon(0)$.
- All wells of g_ε in $(0,1)$ lies in $(\hat{\alpha}, \alpha)$.
- 1 is the only deepest well of g_ε in $(0,1)$. i.e. $\int_1^s g_\varepsilon(u)\,du > 0\ \forall u \in [0,1)$.

Lemma 3.3. *For each sufficiently small positive ε, there exists a unique solution w^ε to*

$$\frac{n}{\rho} w^\varepsilon_\rho + w^\varepsilon_{\rho\rho} - g_\varepsilon\left(w^\varepsilon_\rho\right) = 0, \quad w^\varepsilon_\rho < 0 \, in \ (0,\infty), \ w^\varepsilon_\rho(0) = 0, \ w^\varepsilon(\infty) = 0. \tag{3.7}$$

The solution satisfies $w^\varepsilon(0) < 1 = \lim_{\varepsilon \searrow 0} w^\varepsilon(0)$.

Planar waves

For $\Psi = \Psi(\xi, z)$, $\xi \in \mathbb{R}$, $z \in \mathbb{R}$:

$$c\Psi_z + \Psi_{zz} + \Psi_{\xi\xi} = f(\Psi), \ 0 \leq \Psi \leq 1, \ \Psi_z \geq 0 \geq \Psi_\xi \text{ on } \mathbb{R}^2, \ \Psi(0,0) = \alpha. \tag{3.8}$$

Lemma 3.4. *Assume* (3.1) *and $c > 0$. Then $\Psi(\xi, z) = \Phi(-\xi)$, $(\xi, z) \in \mathbb{R}^2$, is the unique solution of* (3.8).

The result implies $\lim_{z\to\infty} ||U_z(\cdot, z)||_{L^\infty(\mathbb{R}^n)} = 0$. Thus, the interface is asmyptotically vertical.

Energy functionals

We know that (1.1) is a gradient flow of an energy functional with the density function $u_z^2 + |\nabla u|^2 + 2F(u)$. And for radial symmetric functions ψ, ψ_1, ψ_2 of $r = |x|$ and a cylindrically symmetric function W on $\mathbb{R}^n \times (-\infty, 0]$, we define

$$
\begin{aligned}
||\psi|| &:= \sqrt{\langle \psi, \psi \rangle} \quad \text{where } \langle \psi_1, \psi_2 \rangle = \int_0^\infty r^{n-1} \psi_1(r) \psi_2(r), \\
X(l) &:= \{\psi \in C(0,\infty) \,|\, \psi \geq \alpha \text{ on } (0,l], \; \psi(\infty) = 0\} \quad \forall l > 0, \\
E(\psi) &:= \int_0^\infty r^{n-1} \left\{ \frac{1}{2}\psi_r^2 + F(\psi) \right\} dr, \\
J(W) &:= \int_{-\infty}^0 \left\{ \frac{1}{2}||W_z||^2 + E(W) \right\} ce^{cz} dz.
\end{aligned}
$$

The function spaces are those that make the norms or functionals finite.

By the Euler-Lagrange equation for energy minimizers, we can obtain the following lemma:

Lemma 3.5. *Suppose for each* $z \in \mathbb{R}$, $U(\cdot, z + \cdot)$ *on* $\mathbb{R}^n \times (-\infty, 0]$ *is a minimizer of* J *subject to the boundary condition* $W(\cdot, 0) = U(\cdot, z)$ *on* $\mathbb{R}^n \times \{0\}$. *Then* $cU_z + U_{zz} + \triangle U = f(U)$ *in* $\mathbb{R}^{n+1}$.

Since the interface is asymptotically vertical, we can think for each large enough z, there is enough time t for $u(\cdot, z + ct, t)$ to tend towards an almost ideal shape that consumes energy as small as possible. That is, for $z \gg 1$, $U(\cdot, z)$ should be close to a minimizer $\phi(\cdot, l)$ of the energy E in the set $X(l)$ where $l = R(z)$. Thus, we want to look for the minimizer. In other words, we have to consider the following minimization problem:

$$
\phi \in X(l), \; E(\phi) = E(l) := \min_{\psi \in X(l)} E(\psi). \tag{3.9}
$$

Next, we state a property of the minimizer.

Lemma 3.6. *For each* $l > 0$, *(3.9) admits at least one solution. Any solution satisfies* $\phi_r < 0$ *in* $(0, l) \cup (l, \infty)$ *and* $\phi|_{r=l} = \alpha$. *In addition, for each* $\psi(\infty) = 0$,

$$
E(min\{\phi, \psi\}) \leq E(\psi). \tag{3.10}
$$

Furthermore, $\lim_{l \to \infty} \phi(\cdot, l) = 1$ *uniformly in any compact subset of* $[0, \infty)$.

3.2 The Existence of Cylindrically Symmetric Traveling Waves

We will prove the existence by two different methods.

The first one use the fact that existence of a one dimensional traveling wave with positive speed guarantee the existence of cylindrically symmetric traveling waves with any speed. We will construct a sequence of such waves to approximate a solution to (1.3).

The second one use the energy methods. The solution of (1.3) will be approximated by the vertically lifted energy J minimizers with the boundary values being energy minimizers of E in $X(l)$ as $l \to \infty$.

3.2.1 Approximation by Traveling Waves of Unbalanced Potentials

Existence

In the Section 3.1, we have known the existence of a one dimensional traveling wave Φ with positive speed ε satisfies

$$\varepsilon \Phi' + \Phi'' = f_\varepsilon(\Phi) \quad \text{on } \mathbb{R}$$

where the nonlinear term f_ε is defined below

$$f_\varepsilon(u) := f(u) + \varepsilon\sqrt{2F(u)}, \quad F_\varepsilon(u) := \int_0^u f_\varepsilon(s)\, ds.$$

Furthermore, Φ connects the two equilibrium 0 and 1 through proper translation.

Thus, according to [7] as $n+1=2$ and [4] as $n+1 \geq 3$, there exists a cylindrically symmetric traveling wave $U^{\varepsilon} = U^{\varepsilon}(x,z)$ with any given speed $c > 0$ satisfying

$$\begin{cases} cU_z^{\varepsilon} + U_{zz}^{\varepsilon} + \triangle U^{\varepsilon} = f_{\varepsilon}(U^{\varepsilon}) \ \ on \ \mathbb{R}^{n+1}, \\ U^{\varepsilon}(0,0) = \alpha, \ U^{\varepsilon}(\cdot,\infty) \equiv 1, \ U^{\varepsilon}(0,-\infty) \equiv 0 \\ U_z^{\varepsilon} \geq 0 \geq U_r^{\varepsilon} \text{ on } \mathbb{R}^{n+1}, \end{cases} \tag{3.11}$$

where $r = |x|$.

Since $0 \leq U^{\varepsilon} \leq 1$, $\{U^{\varepsilon}\}_{0<\varepsilon\ll 1}$ is a bounded family in $C^3(\mathbb{R}^{n+1})$, thus is a compact family in $C^2_{loc}(\mathbb{R}^{n+1})$. If we take the limit $\varepsilon \searrow 0$, U^{ε} will converge to a cylindrically symmetric solution U to

$$\begin{cases} cU_z + U_{zz} + \triangle U = f(U) \ \text{ on } \mathbb{R}^{n+1}, \\ 0 \leq U \leq 1 \text{ on } \mathbb{R}^{n+1}, \\ U_z \geq 0 \geq U_r \text{ on } \mathbb{R}^{n+1}, \\ U(0,0) = \alpha. \end{cases} \tag{3.12}$$

The solution given by (3.12) is almost the solution to (1.3). All we have to do is check the boundary values as $|x| \to \infty$ and $|z| \to \infty$.

The "boundary values"

To show the (3.12) has the right boundary values, we have to separate $n = 1$ from $n > 1$. As $n = 1$, we don't need to add any extra condition. However, as $n > 1$, we have to assume that either U is the limit of U^{ε} or the nonlinear term $f = 0$ has only one root in $(0,1)$. We only prove for the former case. Under the later assumption, please refer to [4] for more details.

Lemma 3.7.

1. *Suppose* $n = 1$. *Then any symmetric (about* x*) solution* U *to* (3.12) *satisfies*

$$\begin{cases} \lim_{z\to\infty} U(x,z) = 1,\ \lim_{z\to-\infty} U(x,z) = 0\ \forall x \in \mathbb{R}^n. \\ \lim_{|x|\to\infty} U(x,z) = 0\ \forall z \in \mathbb{R}. \end{cases} \tag{3.13}$$

2. *Suppose* $n > 1$. *Let* U *be a limit, along a sequence* $\varepsilon \searrow 0$, *of the cylindrically symmetric family* $\{U_\varepsilon\}$ *of solutions to* (3.11). *Then* U *has the boundary value* (3.13) .

Proof. We prove this lemma by five steps.

Step1. The limit equations.

Since we have the monotonicity $U_z \geq 0 \geq U_r$ and the bound $0 \leq U \leq 1$, there exists

$$\varphi^{\pm}(x) := \lim_{z\to\pm\infty} U(x,z)\ \forall x \in \mathbb{R}^n, \quad \varphi(z) := \lim_{|x|\to\infty} U(x,z)\ \forall z \in \mathbb{R}.$$

And we have $\lim_{z\to\pm\infty}(|U_z| + |U_{zz}|) = 0$ and $\lim_{|x|\to\infty} \triangle U = 0$. Thus,

$$\triangle\varphi^{\pm} - f\left(\varphi^{\pm}\right) = 0 \geq \varphi_r \text{ on } \mathbb{R}^n, \quad \varphi^+(0) \geq \alpha \geq \varphi^-(0),$$
$$\triangle\varphi - f(\varphi) = 0 \leq \varphi_z \text{ on } \mathbb{R}, \quad \varphi(0) \leq \alpha.$$

To complete the proof, we have to show $\varphi^+ \equiv 1$, $\varphi^- \equiv 0$, and $\varphi \equiv 0$. Indeed, the proof for the case $n = 1$ and $n \geq 2$ differ only at $\varphi^+ \equiv 1$.

Step2. The limit as $z \to \infty$ with $n = 1$.

We have already known $\varphi^+_{xx} - f(\varphi^+) = 0$. Multiply both sides by φ^+_x and then integrate over $[0,\infty)$. Then we have

$$\left[\frac{1}{2}\left(\varphi_x^+\right)^2 - F\left(\varphi^+\right)\right]|_0^\infty = 0$$
$$\Rightarrow F\left(\varphi^+(\infty)\right) = F\left(\varphi^+(0)\right) \text{ since } \varphi_x^+(0) = 0.$$

Furthermore, $\varphi^+(0) \geq \alpha$ implies $\varphi^+(\infty) \in [0,\hat{\alpha}] \cup [\alpha, 1]$. Note that the forcing term will be zero as $|x| + |z| = \infty$, we have $f(\varphi^+(\infty)) = 0$. Thus, $\varphi^+(\infty) = 1$ or $\varphi^+(\infty) = 0$.

But since F is a balanced potential with its deepest wells at 1 and 0, if we consider the following equation:

$$\psi_{xx} - f(\psi) = 0 \geq \psi_x \text{ } on \text{ } [0,\infty), \text{ } \psi_x(0) = 0, \text{ } \psi(\infty) = 0,$$

then there is an unique solution $\psi \equiv 0$. However, this is not what we want. Thus, $\varphi^+(\infty) = 1$. And because of $\varphi_x \leq 0$, we obtain that $\varphi^+ \equiv 1$ as $n = 1$.

Step3. The limit as $z \to \infty$ with $n \geq 2$.

We denote $\varphi^+(x)$ as $\varphi^+(r)$ where $r = |x|$.

Assume φ^+ is not equivalent to 1. Since $\varphi_r^+(0) \leq 0$, we have $\alpha \leq \varphi^+(0) < 1$. Let $\beta := \varphi^+(\infty)$, then $f(\beta) = f(\varphi^+(\infty)) = 0$.

$$\varphi_{rr}^+ + \frac{n-1}{r}\varphi_r^+ - f\left(\varphi^+\right) = 0$$
$$\Rightarrow \int_0^\infty \left\{\varphi_r^+\varphi_{rr}^+ + \frac{n-1}{r}\left(\varphi_r^+\right)^2 - f\left(\varphi^+\right)\varphi_r^+\right\} dr = 0$$
$$\Rightarrow \left[\frac{1}{2}\left(\varphi_r^+\right)^2 - F\left(\varphi^+\right)\right]|_\infty^0 = \int_0^\infty \frac{n-1}{r}\left(\varphi_r^+\right)^2 dr$$
$$\Rightarrow F\left(\varphi^+(\infty)\right) - F\left(\varphi^+(0)\right) = \int_0^\infty \frac{n-1}{r}\left(\varphi_r^+\right)^2 dr > 0$$
$$\Rightarrow F(\beta) > F\left(\varphi^+(0)\right)$$

Furthermore, $\varphi^+(0) \geq \alpha$ implies $\beta \in (\hat{\alpha}, \alpha)$.

Next, we consider w^ε in Lemma 3.7. Since $\lim_{\varepsilon \searrow 0} w^\varepsilon(0) = 1$, there exists $\varepsilon_0 > 0$ such that $w^{\varepsilon_0}(0) > \varphi^+(0)$. Also, since $w^{\varepsilon_0}(\infty) = 0$, there exists $R_0 > 0$ such that $w^{\varepsilon_0}(R_0) = \hat{\alpha}$. Then, let

$$\delta := \frac{1}{3} \min \left\{ w^{\varepsilon_0}(0) - \varphi^+(0), \beta - \hat{\alpha} \right\}.$$

Since $\lim_{z \to \infty} U(\cdot, z) = \varphi^+(|\cdot|)$ locally uniform, for the δ defined above, there exists $z_0 \in \mathbb{R}$ such that

$$|U(x,z) - \varphi^+(|x|)| \leq \delta \quad \text{if } |x| \leq R_0 \text{ and } z \geq z_0.$$

By the assumption, U_ε uniformly converges to U on any compact subset in $\mathbb{R}^{n+1}$ as $\varepsilon \searrow 0$. Thus, for the same δ, there exists $\varepsilon \in (0, \varepsilon_0)$ such that

$$|U^\varepsilon(x,z) - U(x,z)| \leq \delta \quad \text{if } |x| + |z - z_0| \leq 2R_0.$$

Finally, we obtain

$$|U^\varepsilon(x,z) - \varphi^+(|x|)| \leq 2\delta \quad \text{if } |x| \leq R_0 \text{ and } z_0 \leq z \leq z_0 + R_0.$$

Now we compare $U^\varepsilon((0,z) + \cdot)$ with $w^{\varepsilon_0}(|\cdot|)$ on $B(R_0) := \{y \in \mathbb{R}^{n+1} | |y| < R_0\}$. Since $\lim_{z \to \infty} U^\varepsilon(\cdot, z) = 1$ and $\lim_{z \to -\infty} U^\varepsilon(\cdot, z) = 0$ locally uniform, we can define

$$z^* := \min \left\{ z \in \mathbb{R} | U^\varepsilon((0,z) + y) \geq w^{\varepsilon_0}(|y|), \ \forall y \in \overline{B}(R_0) \right\}.$$

Note that $U^\varepsilon(0,z) \leq U^\varepsilon(0,z_0) \leq \varphi^+(0) + 2\delta < w^{\varepsilon_0}(0)$ for all $z \leq z_0$, thus $z^* > z_0$. Let

$y_0 \in \overline{B}(R_0)$ be the point such that

$$0 = U^{\varepsilon}((0,z^*)+y_0) - w^{\varepsilon_0}(|y_0|) = \min_{y\in\overline{B}(R_0)} \{U^{\varepsilon}((0,z^*)+y) - w^{\varepsilon_0}(|y|)\}$$

and $y_0 = (y_1, y_2)$ where $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}$. Then

$$w^{\varepsilon_0}(|y_0|) = U^{\varepsilon}((0,z^*)+y_0) \geq U^{\varepsilon}((0,z_0)+y_0) \geq \varphi^{+}(|y_1|) - 2\delta \geq \beta - 2\delta > \hat{\alpha} = w^{\varepsilon_0}(R_0).$$

Thus, y_0 is an interior point of $\overline{B}(R_0)$. Consequently, $(\partial_{zz} + \triangle) U^{\varepsilon}((0,z^*)+y_0) \geq (\partial_{zz} + \triangle) w^{\varepsilon_0}(|y_0|)$. And since $w^{\varepsilon_0}(|y_0|) > \hat{\alpha}$, $\zeta(w^{\varepsilon_0}(|y_0|)) = 0$. Hence

$$\begin{aligned} g_{\varepsilon_0}(w^{\varepsilon_0}(|y_0|)) = f_{\varepsilon_0}(w^{\varepsilon_0}(|y_0|)) &= \left(f + \varepsilon_0\sqrt{2F}\right)|_{w^{\varepsilon_0}(|y_0|)} \\ &> \left(f + \varepsilon\sqrt{2F}\right)|_{w^{\varepsilon_0}(|y_0|)} = f_{\varepsilon}(w^{\varepsilon_0}(|y_0|)) \\ &= f_{\varepsilon}(U^{\varepsilon}((0,z^*)+y_0)) \end{aligned}$$

Finally, we obtain

$$\begin{aligned} 0 &= cU^{\varepsilon}_z + (\partial_{zz} + \triangle) U^{\varepsilon} - f_{\varepsilon}(U^{\varepsilon})|_{(0,z^*)+y_0} \\ &> 0 + (\partial_{zz} + \triangle) w^{\varepsilon_0} - f_{\varepsilon_0}(w^{\varepsilon_0})|_{y_0} \\ &= (\partial_{zz} + \triangle) w^{\varepsilon_0} - g_{\varepsilon_0}(w^{\varepsilon_0})|_{y_0} = 0, \end{aligned}$$

which is a contradiction. Thus, $\varphi^{+} \equiv 1$.

Step4. The $z \to -\infty$ behavior.

We denote $\varphi^{-}(x)$ as $\varphi^{-}(r)$ where $r = |x|$.

Let $\beta := \varphi^{-}(\infty)$, we prove $\beta = 0$ first.

Suppose $\beta \neq 0$, then $\varphi^{-}(\infty) \leq \varphi^{-}(0) \leq U(0,0) = \alpha$ and $f(\beta) = 0$ imply $\beta \in$

$(\hat{\alpha}, \alpha)$. Furthermore,

$$U(x,z) \geq U(x,-\infty) = \varphi^-(|x|) \geq \varphi^-(\infty) = \beta \quad \forall (x,z) \in \mathbb{R}^{n+1}.$$

Let ε_0 and R_0 be defined as **Step3** such that $w^{\varepsilon_0}(0) > \alpha$ and $w^{\varepsilon_0}(R_0) = \hat{\alpha}$. Since $\lim_{z\to\infty} U(\cdot, z) = \varphi^+(|\cdot|) \equiv 1$ locally uniform, we can define

$$z^* := \min\left\{ z \in \mathbb{R} | U((0,z)+y) \geq w^{\varepsilon_0}(|y|),\ \forall y \in \overline{B}(R_0) \right\}.$$

Note that $U(0,z) \leq U(0,0) = \alpha < w^{\varepsilon_0}(0)$ for all $z \leq 0$, thus $z^* > 0$. Let $y_0 \in \overline{B}(R_0)$ be the point such that

$$0 = U((0,z^*)+y_0) - w^{\varepsilon_0}(|y_0|) = \min_{y \in \overline{B}(R_0)} \{U((0,z^*)+y) - w^{\varepsilon_0}(|y|)\}.$$

Then

$$w^{\varepsilon_0}(|y_0|) = U((0,z^*)+y_0) \geq \beta > \hat{\alpha} = w^{\varepsilon_0}(R_0).$$

Thus, y_0 is an interior point of $\overline{B}(R_0)$. Consequently, $(\partial_{zz} + \triangle) U((0,z^*)+y_0) \geq (\partial_{zz} + \triangle) w^{\varepsilon_0}(|y_0|)$. And since $w^{\varepsilon_0}(|y_0|) > \hat{\alpha}$, thus $\zeta(w^{\varepsilon_0}(|y_0|)) = 0$. Hence

$$\begin{aligned} g_{\varepsilon_0}(w^{\varepsilon_0}(|y_0|)) &= f_{\varepsilon_0}(w^{\varepsilon_0}(|y_0|)) = \left(f + \varepsilon_0\sqrt{2F}\right)|_{w^{\varepsilon_0}(|y_0|)} \\ &> f(w^{\varepsilon_0}(|y_0|)) = f(U((0,z^*)+y_0)) \end{aligned}$$

Finally, we obtain

$$\begin{aligned} 0 &= cU_z + (\partial_{zz} + \triangle) U - f(U)|_{(0,z^*)+y_0} \\ &> 0 + (\partial_{zz} + \triangle) w^{\varepsilon_0} - f_{\varepsilon_0}(w^{\varepsilon_0})|_{y_0} \\ &= (\partial_{zz} + \triangle) w^{\varepsilon_0} - g_{\varepsilon_0}(w^{\varepsilon_0})|_{y_0} = 0, \end{aligned}$$

which is a contradiction. Thus, $\varphi^-(\infty) = 0$.

Next, we prove $\varphi^- \equiv 0$.

$$
\begin{aligned}
&\varphi^-_{rr} + \frac{n-1}{r}\varphi^-_r - f(\varphi^-) = 0 \\
\Rightarrow &\varphi^-_r \left[\varphi^-_{rr} + \frac{n-1}{r}\varphi^-_r - f(\varphi^-)\right] = 0 \\
\Rightarrow &\int_0^\infty \left\{\varphi^-_r\varphi^-_{rr} + \frac{n-1}{r}(\varphi^-_r)^2 - f(\varphi^-)\varphi^-_r\right\} dr = 0 \\
\Rightarrow &\int_0^\infty \frac{n-1}{r}(\varphi^-_r)^2 dr = \left[\frac{1}{2}(\varphi^-_r)^2 - F(\varphi^-)\right]\Big|_\infty^0 \\
\Rightarrow &\int_0^\infty \frac{n-1}{r}(\varphi^-_r)^2 dr = F(\varphi^-(\infty)) - F(\varphi^-(0)) \\
\Rightarrow &\int_0^\infty \frac{n-1}{r}(\varphi^-_r)^2 dr = F(0) - F(\varphi^-(0)) \le 0 \\
\Rightarrow &\varphi^-_r = 0
\end{aligned}
$$

Thus, $\varphi^- \equiv 0$.

Step5. The limit as $|x| \to \infty$.

As $\varphi(z) \le U(x,z)$, $\varphi(-\infty) \le U(x,-\infty) = \varphi^-(x) \equiv 0$, hence $\varphi(-\infty) = 0$. Also $\varphi_z \ge 0$, we only have to show $\varphi(\infty) = 0$.

Suppose not. Let $\beta := \varphi(\infty) > 0$. We have already known that F is a balanced potential and φ satisfies $c\varphi_z + \varphi_{zz} = f(\varphi)$ with $c > 0$, we multiply both sides by φ_z and integrate it over $z \in (-\infty, \infty)$. Then we can get

$$
\begin{aligned}
&\left[\frac{1}{2}(\varphi_z)^2 + F(\varphi)\right]\Big|_\infty^{-\infty} = \int_{-\infty}^\infty (\varphi_z)^2 dz \\
\Rightarrow &F(\varphi(-\infty)) - F(\varphi(\infty)) > 0 \\
\Rightarrow &\varphi(\infty) \ne 1.
\end{aligned}
$$

thus $\beta < 1$. Furthermore, since $f(\beta)$ must be zero, $\beta \in (\hat{\alpha}, \alpha)$, i.e. $\varphi(\infty) \in (\hat{\alpha}, \alpha)$.

Then there exist z_0 such that $\varphi(z_0) > \hat{\alpha}$. We obtain that

$$U(x,z) \geq \varphi(z) > \hat{\alpha} \text{ if } (x,z) \in \mathbb{R}^n \times [z_0, \infty).$$

Let ε_0 and R_0 be defined as before such that $w^{\varepsilon_0}(0) > \alpha$ and $w^{\varepsilon_0}(R_0) = \hat{\alpha}$.

Also, $\lim_{z\to\infty} U(x,z) = 1$ locally uniformly, there exists $z^* > z_0 + R_0$ such that $U((0,z^*)+y) > w^{\varepsilon_0}(|y|)$ for all $|y| \geq R_0$. Then we define

$$R^* := \sup\left\{|x| \,|U((0,z^*)+y) > w^{\varepsilon_0}(|y|), \ \ \forall y \in \overline{B}(R_0)\right\}.$$

Since $\beta = \varphi(\infty) > \varphi(z^*) = \lim_{|x|\to\infty} U(x,z^*)$ and $w^{\varepsilon_0}(0) > \alpha > \beta$, R^* must be finite. Let $x_0 \in \mathbb{R}^n$ and $y_0 \in \overline{B}(R_0)$ be the points such that

$$0 = U((x_0,z^*)+y_0) - w^{\varepsilon_0}(|y_0|) = \min_{y\in\overline{B}(R_0)} \{U((x_0,z^*)+y) - w^{\varepsilon_0}(|y|)\}.$$

Then $w^{\varepsilon_0}(|y_0|) = U((0,z^*)+y_0) > U(0,z_0) > \hat{\alpha} = w^{\varepsilon_0}(R_0)$ implies y_0 is an interior point of $\overline{B}(R_0)$. A similar argument as before gives a contradiction. Thus, $\varphi(\infty) = 0$. And the monotonicity of φ guarantee $\varphi \equiv 0$.

We already have shown the existence of U which the boundary conditions are satisfied. The monotonicity of U can be proved by strong maximum principle. And then we complete the proof of Theorem 3.1. □

3.2.2 Approximation by Energy Minimizers

In this subsection, we use the energy method to prove the existence of a traveling wave claimed in Theorem 3.1.

Theorem 3.8. *Assume* (3.1)*. There exists a cylindrically symmetric solution* U *to* (1.3) *which satisfies the monotonicity* (3.2) *and the boundary condition* (3.13)*. Furthermore,* *U satisfies the minimum energy principle*

$$J\left(U\left(\cdot,z+\cdot\right)\right)=\min_{W(\cdot,0)=U(\cdot,z)}J\left(W\right). \tag{3.14}$$

Proof. We only show the existence, uniqueness, monotonicities.

Step1. Construction of a minimizer.

By Lemma 3.6, for each $l>0$, there exists a minimizer of the energy E in $X\left(l\right)$ which we denote it by $\phi\left(\cdot,l\right)$. Now consider another minimization problem

$$\min_{W(\cdot,0)=\phi(\cdot,l)}J\left(W\right).$$

It is easy to show that there exists a minimizer of this problem, and we denote the minimizer by U^l. For each W and negative constant h,

$$\begin{aligned}
J\left(W\right)&=\int_{-\infty}^{0}\left\{\frac{1}{2}\left\|W_z\right\|^2+E\left(W\right)\right\}ce^{cz}dz\\
&=\int^{h}\left\{\frac{1}{2}\left\|W_z\right\|^2+E\left(W\right)\right\}ce^{cz}dz+\int_{h}^{0}\left\{\frac{1}{2}\left\|W_z\right\|^2+E\left(W\right)\right\}ce^{cz}dz\\
&=\int_{-\infty}^{0}\left\{\frac{1}{2}\left\|W_z\right\|^2+E\left(W\left(\cdot,\cdot+h\right)\right)\right\}ce^{c(z+h)}dz+\int_{h}^{0}\left\{\frac{1}{2}\left\|W_z\right\|^2+E\left(W\right)\right\}ce^{cz}dz\\
&=e^{ch}J\left(W\left(\cdot,\cdot+h\right)\right)+\int_{h}^{0}\left\{\frac{1}{2}\left\|W_z\right\|^2+E\left(W\right)\right\}ce^{cz}dz,
\end{aligned}$$

and $J\left(U^l\left(\cdot,\cdot+h\right)\right)=\min_{W(\cdot,0)=U^l(\cdot,h)}J\left(W\right).$

In addition, the Euler-Lagrange equation shows that U^l is a solution to the following boundary condition problem

$$cU_z^l+U_{zz}^l+\triangle U^l=f\left(U^l\right)\ \ on\ \mathbb{R}^n\times\left(-\infty,0\right),\ U^l\left(\cdot,0\right)=\phi\left(\cdot,l\right).$$

Thus, we can do some calaulation

$$
\begin{aligned}
& cU_z^l U_z^l + U_{zz}^l U_z^l + \triangle U^l U_z^l = f\left(U^l\right) U_z^l \\
\Rightarrow & \int_0^\infty r^{n-1} \left\{ cU_z^l U_z^l + U_{zz}^l U_z^l + \triangle U^l U_z^l \right\} dr = \int_0^\infty r^{n-1} \left\{ f\left(U^l\right) U_z^l \right\} dr \\
\Rightarrow & c\left\|U_z^l\right\|^2 + \frac{1}{2}\frac{d}{dz}\left\|U_z^l\right\|^2 + \int_0^\infty \frac{d}{dz}\frac{-1}{2}\left(|\nabla U|^2\right) r^{n-1} dr = \int_0^\infty \frac{d}{dz} F\left(U^l\right) r^{n-1} dr \\
\Rightarrow & \frac{d}{dz} E\left(U^l\right) = c\left\|U_z^l\right\|^2 + \frac{1}{2}\frac{d}{dz}\left\|U_z^l\right\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dz}\left(E\left(U^l\right) e^{cz}\right) &= \frac{d}{dz} E\left(U^l\right) e^{cz} + cE\left(U^l\right) e^{cz} \\
&= c\left\|U_z^l\right\|^2 e^{cz} + \frac{1}{2}\frac{d}{dz}\left\|U_z^l\right\|^2 e^{cz} + cE\left(U^l\right) e^{cz} \\
&= \frac{1}{2}\frac{d}{dz}\left(\left\|U_z^l\right\|^2 e^{cz}\right) + \frac{c}{2}\left\|U_z^l\right\|^2 e^{cz} + cE\left(U^l\right) e^{cz} \\
&= \frac{1}{2}\frac{d}{dz}\left(\left\|U_z^l\right\|^2 e^{cz}\right) + ce^{cz}\left(\frac{1}{2}\left\|U_z^l\right\|^2 + E\left(U^l\right)\right).
\end{aligned}
$$

Hence, if we integrate over $z \in (-\infty, 0]$, we obtain

$$
\begin{aligned}
& E\left(U^l\right) e^{cz} \Big|_{-\infty}^0 = \frac{1}{2}\left(\left\|U_z^l\right\|^2 e^{cz}\right)\Big|_{-\infty}^0 + J\left(U^l\right) \\
\Rightarrow & J\left(U^l\right) = E\left(U^l(\cdot,0)\right) - \frac{1}{2}\left\|U_z^l(\cdot,0)\right\|^2.
\end{aligned}
$$

Step2. The Uniquness and the monotonicity of U^l.

If we rearrange in the $r = |x|$ direction, we can obtain $U_r^l \leq 0$.

If we want to show that $U_z^l \geq 0$, the rearrangement technique does nothing since the energy functional has e^{cz}. To show the monotonicity in z, we use another method. By Lemma 3.6, $J\left(\min\left(\phi, U^l\right)\right) \leq J\left(U^l\right)$ and the equality holds if and only if $\min\left(\phi, U^l\right) = U^l$. In addition, U^l is an energy minimizer, thus $U^l \leq \phi$ for all $z \leq 0$. For each $\varepsilon > 0$, define

$$U^{l,\varepsilon} := U^l(\cdot, \cdot - \varepsilon),$$
$$w_1 := \min\left\{U^l, U^{l,\varepsilon}\right\},$$
$$w_2 := \max\left\{U^l, U^{l,\varepsilon}\right\}.$$

Then $J(w_1) + J(w_2) = J\left(U^l\right) + J\left(U^{l,\varepsilon}\right)$. Since $w_1(\cdot, 0) = U^{l,\varepsilon}(\cdot, 0)$ and $w_2(\cdot, 0) = U^l(\cdot, 0)$, we have $J\left(U^{l,\varepsilon}\right) \leq J(w_1)$ and $J\left(U^l\right) \leq J(w_2)$. Thus, $U^{l,\varepsilon} = w_1$ and $U^l = w_2$. It implies that $U^l(\cdot, 0) \geq U^{l,\varepsilon}(\cdot, 0) = U^l(\cdot, -\varepsilon)$. That is, $U^l_z \geq 0$.

Next, we let $\varphi^- := \lim_{z\to-\infty} U^l(\cdot, z)$. Then

$$\varphi^-(\infty) = \lim_{z\to-\infty} U^l(\infty, z) \leq U^l(\infty, 0) = \varphi(\infty, l) = 0$$

implies that $\varphi^-(\infty) = 0$. Furthermore,

$$\begin{aligned}
&\varphi^-_{rr} + \frac{n-1}{r}\varphi^-_r - f\left(\varphi^-\right) = 0 \\
\Rightarrow &\varphi^-_r\left[\varphi^-_{rr} + \frac{n-1}{r}\varphi^-_r - f\left(\varphi^-\right)\right] = 0 \\
\Rightarrow &\int_0^\infty \left\{\varphi^-_r\varphi^-_{rr} + \frac{n-1}{r}\left(\varphi^-_r\right)^2 - f\left(\varphi^-\right)\varphi^-_r\right\} dr = 0 \\
\Rightarrow &\int_0^\infty \frac{n-1}{r}\left(\varphi^-_r\right)^2 dr = \left[\frac{1}{2}\left(\varphi^-_r\right)^2 - F\left(\varphi^-\right)\right]\Big|_\infty^0 \\
\Rightarrow &\int_0^\infty \frac{n-1}{r}\left(\varphi^-_r\right)^2 dr = F\left(\varphi^-(\infty)\right) - F\left(\varphi^-(0)\right) \\
\Rightarrow &\int_0^\infty \frac{n-1}{r}\left(\varphi^-_r\right)^2 dr = F(0) - F\left(\varphi^-(0)\right) \leq 0 \\
\Rightarrow &\varphi^-_r = 0
\end{aligned}$$

Thus, $\varphi^- \equiv 0$. Then the strong maximum principle implies that $U^l_z > 0$.

To show the minimizer U^l is unique. Suppose there exists a different minimizer $\tilde{U^l}$, then both $w_1 = \min\left\{U^l, \tilde{U^l}\right\}$ and $w_2 := \max\left\{U^l, \tilde{U^l}\right\}$ are minimizers. By Hopf

Lemma, we know that $w_{2z}(\cdot,0) \geq w_{1z}(\cdot,0) > 0$. Thus

$$J(w_1) = E(l) - \frac{1}{2}\|w_{1z}(\cdot,0)\|^2 > E(l) - \frac{1}{2}\|w_{2z}(\cdot,0)\|^2 = J(w_2)$$

contradicts with both w_1 and w_2 are minimizers.

Step3. Construct the approximating sequences of the solution.

For any $0 < l_1 < l_2$, we have $\phi(\cdot,l_1) < \phi(\cdot,l_2)$. If we set $w_1 = \min\left\{U^{l_1}, U^{l_2}\right\}$ and $w_2 = \max\left\{U^{l_1}, U^{l_2}\right\}$, then $J(w_1) + J(w_2) = J\left(U^{l_1}\right) + J\left(U^{l_2}\right)$.

Note that

$$\begin{aligned} w_1(\cdot,0) &= \min\left\{U^{l_1}(\cdot,0), U^{l_2}(\cdot,0)\right\} = \min\left\{\phi(\cdot,l_1), \phi(\cdot,l_2)\right\} \\ &= \phi(\cdot,l_1) = U^{l_1}(\cdot,0), \\ w_2(\cdot,0) &= U^{l_2}(\cdot,0), \end{aligned}$$

we have

$$\begin{aligned} J\left(U^{l_1}\right) &= \min_{W(\cdot,0)=\phi(\cdot,l_1)} J(W) \leq J(w_1), \\ J\left(U^{l_2}\right) &= \min_{W(\cdot,0)=\phi(\cdot,l_2)} J(W) \leq J(w_2). \end{aligned}$$

This implies $w_1 = U^{l_1}$ and $w_2 = U^{l_2}$. That is, $U^{l_1} < U^{l_2}$.

Observe that $U^l(\cdot,0) = \phi(\cdot,0)$, thus $U^l(0,0) = \phi(0,0) > \alpha$. Also, $U^l_z > 0$ implies that there exist a unique constant $H(l) > 0$ such that $U^l(0,-H(l)) = \alpha$. Since we have $U^{l_1} < U^{l_2}$ for any $0 < l_1 < l_2$, we obtain that $H(l)$ is monotonic increasing in $l > 0$. Next, we claim $\lim_{l\to\infty} H(l) = \infty$. To prove this, we define

$$E^c(h) = \min_{\psi(-h)\leq\alpha\leq\psi(0)} \int_{-\infty}^{0} \left\{\frac{1}{2}\psi_z^2 + F(\psi)\right\} ce^{cz}dz \quad \forall h > 0.$$

Note that $E^c(\cdot)$ is continuous and positive on $[0,\infty)$. In addition,

$$E(l) > J\left(U^l\right) \geq \int_0^l r^{n-1} E^c(H(l))\,dr = \frac{r^n}{n} E^c(H(l)).$$

Since $E(l) = O(1)\,r^{n-1}$, $\lim_{l\to\infty} H(l) = \infty$.

Finally, we set $U^{l,H(l)}(x,z) = U^l(x, z - H(l))$ and consider the family $\left\{U^{l,H(l)}\right\}_{l>0}$. This family is bounded in C^3, so there exists a subsequence of $\left\{U^{l,H(l)}\right\}_{l>0}$ converges to a limit U which is a solution of (1.3). It is cylindrically symmetric and has the monotonicity $U_z \geq o \geq U_r$. To prove the limit U has the right boundary conditions, we only to modify the proof in Section 3.2.1 by replacing U^ε in **Step3** by $U^{l,H(l)}$ and using the fact that

$$\lim_{l\to\infty} U^{l,H(l)}(\cdot, H(l)) = \lim_{l\to\infty} \phi(|\cdot|, l) = 1$$

uniformly in any compact subset of $\mathbb{R}^n$.

□

3.3 The Behavior of the Interfaces

In this section, we give a rough understanding of the profile of the level sets

$$\{U(x,z) = \alpha\} = \{|x| = R(z)\}.$$

It is well-known that the interface of solutions of (1.1) evolves according the motion by mean curvature flow. That is, the velocity of a point in the interface is given by the mean curvature of the interface. For a traveling wave solution of (1.3), we can shrink the space by $R(\hat{z})$ such that the interface near $\mathbb{R}^n \times \{\hat{z}\}$ is asymptotically a circular cylinder $\mathbb{S}(1) \times \mathbb{R}$ as $\hat{z} \to \infty$, where $S(1)$ is a sphere in $\mathbb{R}^n$ with radius 1 and center origin. Note that the mean curvature of the interface $\mathbb{S}(1) \times \mathbb{R}$ equals to $n-1$. Thus, when $n > 1$, the interface moves with a normal velocity equals to $n-1$. By an appropriate translation of coordinates, the motion promoted by mean curvature represents a constant vertical velocity c motion. It implies the approximation $cR' \sim \frac{n-1}{R}$, hence the behavior of the interface asymptotic to $\frac{cR^2}{2} \approx (n-1)\,z$.

However, as $n = 1$, the effect of mean curvature is insignificant since the shrinked interface is asymptotically two lines $\{\pm 1\} \times \mathbb{R}$. In this case, the dynamics have been discovered and researched thoroughly. Please see [1] for more detals. If initially there are two interfaces of distance d, the speed that two interfaces approach each other is $Ae^{-2\mu d/\varepsilon + o(1)}$ where $\varepsilon = 1/R(\hat{z})$ after an initiation that processes an arbitrary initial data into a special wave profile. Since the time which initiation needed is pretty short compared with the exponentially slow motion of the interface. If a vertical velocity c is produced by $Ae^{-2\mu d/\varepsilon + o(1)}$, the behavior of the interface should be asymptotically determined by $cR' \sim Ae^{-2\mu R}$ which implies the interface asymptotic to a hyperbolic cosine curve stated in Theorem 3.2.

There is another view point in the two dimension case. We have

$$cR'' = -2\mu A e^{-2\mu R} R' = o(1)\, R'$$

as z large. Thus we can dismiss the U_{zz} term in (1.3) since it does negligible effect. Now (1.3) becomes $cU_z + U_{xx} = f(U)$. If we change the variable by $s = \frac{z}{c}$, then we can obtain $U_s + U_{xx} = f(U)$ for $(x, s) \in \mathbb{R}^2$. Then from [2], we know there is a unique entire solution (up to translation) having two interfaces located asymptotically on the hyperbolice cosine curve stated in Theorem 3.2.

In conclusion, when $n > 1$, the curvature contributes to the vertical velocity c motion of the interface; when $n = 1$, the interaction of the two branches of the interface drives the principal effection of the vertical velocity c motion of the interface.

Chapter 4

Monostable-Type Traveling Waves

Let $v(x)$ be a stationary solution of (1.1), that is, $v(x)$ is a standing wave solution of

$$\Delta v - f(v) = 0 \quad (x \in \mathbb{R}^n), \qquad v(x) > 0, \quad \lim_{|x| \to \infty} v(x) = 0.$$

It is well-known that there is a one hump solution $v(x)$ for $n = 1$ or a radially symmetric solution $v(x) = \Phi(r)$ $(r = |x|)$

$$\begin{cases} \Phi_{rr} + \frac{n-1}{r}\Phi_r - f(\Phi) = 0, \quad \Phi(r) > 0 \ (0 < r < \infty) \\ \Phi_r(0) = 0, \quad \lim_{r \to \infty} \Phi(r) = 0 \end{cases} \tag{4.1}$$

for $n \geq 2$. Furthermore, this standing wave is unique up to translation and unstable. In fact, the linearized eigenvalue problem

$$\Delta \phi - f'(v(x))\phi = \mu\phi \quad (x \in \mathbb{R}^n) \tag{4.2}$$

has a positive eigenvalue μ and a corresponding eigenfunction $\phi(x)$ satisfying

$$\lim_{|x| \to \infty} \phi(x) = 0, \qquad \phi(x) > 0, \quad (x \in \mathbb{R}^n).$$

We have already known that there are traveling wave solutions of (1.1) for the monostabe nonlinearity. If we restrict $f(u)$ in $[0,a]$ or $[a,1]$, then we obtain a monostable nonlinearity. Thus, we can assume that there exists traveling wave solutions connecting $u=1$ (at $z=-\infty$) to $u=v(x)$ (at $z=\infty$) or connecting $u=v(x)$ (at $z=-\infty$) to $u=0$ (at $z=\infty$). That is, we want to find traveling wave solutions $u(x,z,t)=V(x,z-ct)$ and $u(x,z,t)=W(x,z-ct)$ which satisfying

$$\begin{cases} \Delta V + V_{zz} + cV_z - f(V) = 0 & (x,z) \in \mathbb{R}^{n+1} \\ \lim_{z\to-\infty} V(x,z) = 1, \quad \lim_{z\to\infty} V(x,z) = v(x), \end{cases} \tag{4.3}$$

and

$$\begin{cases} \Delta W + W_{zz} + cW_z - f(W) = 0 & (x,y) \in \mathbb{R}^{n+1} \\ \lim_{z\to-\infty} W(x,z) = v(x), \quad \lim_{z\to\infty} W(x,z) = 0 \end{cases} \tag{4.4}$$

respectively. Hence we have to show the following theorem.

Theorem 4.1. *Let μ_+ be a positive eigenvalue of (4.2). Then for each $c \geq \kappa$, there exists solutions $V(x,z-ct)$ and $W(x,z-ct)$ to (4.3) and (4.4) respectively, satisfying*

$$V_z(x,z) \leq 0, \quad W_z(x,z) \leq 0, \qquad (x,z) \in \mathbb{R}^{n+1}$$

where

$$\kappa := max\left\{\min_{0\leq u\leq 1} f'(u), \mu_+\right\}.$$

We can treat a more general reaction-diffusion equation than (1.1) and then prove the existence theorem for this case. Finally, we apply the result to (1.1).

4.1 The Existence of Nonplanar Traveling Waves

Consider the following equation:

$$u_t = \triangle u + u_{zz} + g(x, u), \tag{4.5}$$

where g and its derivatives in u up to the second order are continuous and bounded in $\{(x, u) : x \in \mathbb{R}^n, |u| \le K\}$ for a large constant K. Asume (4.5) has two stationary solutions v_- and v_+, that is,

$$\triangle v_- + g(x, v_-(x)) = 0, \quad \triangle v_+ + g(x, v_+(x)) = 0. \tag{4.6}$$

We want to find the traveling wave solutions of (4.5) connecting $v_-(x)$ at $z = -\infty$ and $v_+(x)$ at $z = +\infty$. Thus, let $u(x, z, t) = U(x, z - ct)$ and plug it to (4.5):

$$\begin{cases} \triangle U + U_{zz} + cU_z + g(x, U) = 0 & (x, z) \in \mathbb{R}^{n+1}, \\ \lim_{z \to -\infty} U(x, z) = v_-(x), \quad \lim_{z \to +\infty} U(x, z) = v_+(x). \end{cases} \tag{4.7}$$

Consider the eigenvalue problems of (4.6), let $\mu = \mu_\pm$ be the first eigenvalues and $\phi = \phi_\pm$ be the corresponding eigenfunctions. Namely,

$$\begin{cases} \triangle \phi + g_u(x, v_\pm(x))\phi = \mu\phi & x \in \mathbb{R}^n, \\ \lim_{|x| \to \infty} \phi(x) = 0, \ \phi(x) > 0 & x \in \mathbb{R}^n. \end{cases} \tag{4.8}$$

And then we divide into the following four situations:

1. $\mu_+ > 0$ and $v_-(x) \geq v_+(x)$ for $x \in \mathbb{R}^n$,

2. $\mu_+ > 0$ and $v_-(x) \leq v_+(x)$ for $x \in \mathbb{R}^n$,

3. $\mu_- > 0$ and $v_-(x) \geq v_+(x)$ for $x \in \mathbb{R}^n$,

4. $\mu_- > 0$ and $v_-(x) \leq v_+(x)$ for $x \in \mathbb{R}^n$.

Note that we only consider the case 1 and 2 since case 3 and 4 can be identified with case 2 and 1 by the change of variables $(z, c) \to (-z, -c)$ in the equation (4.7).

Theorem 4.2. *Let $v_\pm(x)$ be stationary solutions of* (4.5) *with the eigenvalues $\mu_\pm$ and the corresponding eigenfunctions $\phi_\pm$ of* (4.8).

(i) *Suppose that there are no other stationary solutions $v(x)$ sandwiched by $v_\pm(x)$ as $v_+(x) \leq v(x) \leq v_-(x)$. In addition, assume $v_+(x) + \varepsilon\phi_+(x) \leq v_-(x)$ for a small positive constant ε. Then for each $c \geq 2\sqrt{\kappa_g}$, there exists a solution U of* (4.7) *satisfying*

$$U_z \leq 0,$$

where

$$\kappa_g := \max\left\{ - \min_{x\in\mathbb{R}^n,\, v_+(x)\leq u\leq v_-(x)} g_u(x,u), \mu_+ \right\}.$$

(ii) *Suppose that there are no other stationary solutions $v(x)$ sandwiched by $v_\pm(x)$ as $v_-(x) \leq v(x) \leq v_+(x)$. In addition, assume $v_-(x) + \varepsilon\phi_-(x) \leq v_+(x)$ for a small positive constant ε. Then for each $c \geq 2\sqrt{\kappa_g}$, there exists a solution V of* (4.7) *satisfying*

$$V_z \geq 0,$$

where

$$\kappa_g := \max\left\{ - \min_{x\in\mathbb{R}^n,\, v_-(x)\leq u\leq v_+(x)} g_u(x,u), \mu_+ \right\}.$$

We will prove the theorem above by the comparison principle with an appropriate supersolution and a subsolution. We consider the case 1. at first.

4.1.1 A Subsolution and a Supersolution

Let $w(z)$ be a solution of

$$\begin{cases} w_{zz} + cw_z + \mu_+ w - w^2 = 0, \quad w(z) > 0, \quad w_z(z) < 0 \qquad z \in \mathbb{R} \\ \lim_{z\to-\infty} w(z) = \mu_+, \quad \lim_{z\to+\infty} w(z) = 0 \end{cases} \tag{4.9}$$

where $c \geq 2\sqrt{\mu_+}$. It is known that for each $c \geq 2\sqrt{\mu_+}$, $w(z)$ is unique up to translation. Let

$$W(z) := \sigma w(z).$$

And then we normalize $\phi_\pm$ of (4.8) such that

$$\max_{x\in\mathbb{R}^n} \phi_\pm(x) = 1.$$

Define

$$\mathfrak{F}(U) := -\triangle U - U_{zz} - cU_z - g(x,U)$$

and

$$\underline{U}(x,z) := v_+(x) + \phi_+(x) W(z).$$

And then we compute

$$\begin{aligned} \mathfrak{F}[\underline{U}] &= -\triangle(v_+ + \phi_+ W) - (v_+ + \phi_+ W)_{zz} - c(v_+ + \phi_+ W)_z - g(x, v_+ + \phi_+ W) \\ &= -\triangle v_+ - \triangle \phi_+ W - \phi_+ W_{zz} - c\phi_+ W_z - g(x, v_+ + \phi_+ W) \\ &= g(x, v_+) - (\mu_+ \phi_+ - g_u(x, v_+)\phi_+) W - \phi_+ W_{zz} - c\phi_+ W_z - g(x, v_+ + \phi_+ W) \\ &= -\phi_+(W_{zz} + cW_z + \mu_+ W) + g(x, v_+) - g(x, v_+ + \phi_+ W) + g_u(x, v_+)\phi_+ W \\ &\leq -\phi_+(W_{zz} + cW_z + \mu_+ W) + M_g \phi_+^2 W^2 \end{aligned}$$

where

$$M_g := \min_{x\in\mathbb{R}^n,\, v_+(x)\le u\le v_-(x)} g_{uu}(x,u).$$

Furthermore, since $W=\sigma w$, we have

$$\begin{aligned}\mathfrak{F}[\underline{U}] &\le -\phi_+\sigma\left\{w_{zz}+cw_z+\mu_+ w-\sigma M_g w^2\right\}.\\ &= -\phi_+\sigma w^2\left\{1-\sigma M_g\right\}\end{aligned}$$

Thus , if we let $\sigma \le 1/M_g$, then $\mathfrak{F}[\underline{U}]\le 0$. That is, $\underline{U}(x,z)=v_+(x)+\phi_+(x)W(z)$ is a subsolution of (4.7).

Next, we want to look for an appropriate supersolution.

Define

$$\tilde{g}^\delta(x,u) := g(x,u)+\delta$$

for a small positive constant δ and consider

$$\triangle v+\tilde{g}^\delta(x,v)=0. \tag{4.10}$$

This equation is an approximate equation of (4.6). With the assumption that $\mu_+\neq 0$, we know that there exists a solution $v=\tilde{v}_+^\delta(x)$ to (4.10) such that

$$\lim_{\delta\to 0}\tilde{v}_+^\delta(x)=v_+(x).$$

Note that

$$\begin{aligned}&-\triangle\left(v_++\frac{\delta}{\kappa_g}\right)-\tilde{g}^\delta\left(v_++\frac{\delta}{\kappa_g}\right)\\ =&g(x,v_+)-g\left(v_++\frac{\delta}{\kappa_g}\right)-\delta\\ \le&-\left(\min_{x\in\mathbb{R}^n,\, v_+(x)\le u\le v_+(x)+\delta/\kappa_g} g_u(x,u)+\kappa_g\right)\frac{\delta}{\kappa_g}.\end{aligned}$$

Thus, if we let $\kappa_g + \min g_u(x,u) \geq 0$, then $v_+(x) + \delta/\kappa_g$ is a subsolution of (4.10). Hence, we can obtain that

$$0 < v_+(x) + \frac{\delta}{\kappa_g} \leq \tilde{v}_+^{\delta}(x) \quad x \in \mathbb{R}^n.$$

Let $Q(z)$ is a solution of $Q_{zz} + cQ_z + \kappa_g Q(z) = 0$. So

$$Q(z) = \alpha \exp(\lambda z), \quad \lambda := -\frac{c - \sqrt{c^2 - 4\kappa_g}}{2}, \quad \alpha > 0.$$

Define

$$U^+(x,z) := \tilde{v}_+^{\delta}(x) + Q(z).$$

Then we can compute

$$\begin{aligned}
\mathfrak{F}\left[U^+\right] &= -\triangle\left(\tilde{v}_+^{\delta} + Q\right) - \left(\tilde{v}_+^{\delta} + Q\right)_{zz} - c\left(\tilde{v}_+^{\delta} + Q\right)_z - g\left(x, \tilde{v}_+^{\delta} + Q\right) \\
&= -\triangle\tilde{v}_+^{\delta} - Q_{zz} - cQ_z - g\left(x, \tilde{v}_+^{\delta} + Q\right) \\
&= \tilde{g}^{\delta}\left(x, \tilde{v}_+^{\delta}\right) + \kappa_g Q - g\left(x, \tilde{v}_+^{\delta} + Q\right) \\
&= \kappa_g Q + g\left(x, \tilde{v}_+^{\delta}\right) - g\left(x, \tilde{v}_+^{\delta} + Q\right) + \delta \\
&= \left[\kappa_g - g_u\left(x, \tilde{v}_+^{\delta} + \theta Q\right)\right] Q + \delta \\
&\geq \delta > 0
\end{aligned}$$

where $\theta \in (0,1)$. Hence

$$\overline{U}(x,z) := \min_{(x,z)\in\mathbb{R}^n\times\mathbb{R}} \left\{\tilde{v}_+^{\delta}(x) + Q(z), v_-(x)\right\}$$

is a supersolution of (4.7).

4.1.2 Proof of Theorem 4.2. (i)

We compare $\overline{U}$ and $\underline{U}$.

Consider $\overline{U} = \tilde{v}_+^{\delta}(x) + Q(z)$. Then

$$
\begin{aligned}
\overline{U}(x,z) - \underline{U}(x,z) &= \left\{\tilde{v}_+^{\delta}(x) + Q(z)\right\} - \left\{v_+(x) + \phi_+(x) W(z)\right\} \\
&\geq \tilde{v}_+^{\delta}(x) - v_+(x) - \phi_+(x) W(z) \\
&\geq \tilde{v}_+^{\delta}(x) - v_+(x) - W(z) \\
&= \tilde{v}_+^{\delta}(x) - v_+(x) - \sigma w(z) \\
&\geq \frac{\delta}{\kappa_g} - \sigma \mu_+.
\end{aligned}
$$

Thus, given any $\delta > 0$, if we let $\sigma \leq \delta / (\kappa_g \mu_+)$, then

$$
\underline{U}(x,z) \leq \overline{U}(x,z). \tag{4.11}
$$

On the other hand, when $\overline{U} = v_-(x)$, we can compute

$$
\begin{aligned}
\overline{U}(x,z) - \underline{U}(x,z) &= v_-(x) - \left\{v_+(x) + \phi_+(x) W(z)\right\} \\
&\geq v_+(x) + \varepsilon\phi_+ - v_+(x) - \phi_+ W(z) \\
&= \phi_+(\varepsilon - W(z)) \\
&= \phi_+(\varepsilon - \sigma w) \\
&\geq \phi_+(\varepsilon - \sigma\mu_+).
\end{aligned}
$$

Thus, if we take $\sigma \leq \varepsilon/\mu_+$, then we also can obtain (4.11). Finally, we only restrict $0 < \sigma \leq \min\{\delta/(\kappa_g \mu_+), \varepsilon/\mu_+\}$.

Define an operator

$$
\mathfrak{L}[u] := -\triangle u - u_{zz} - c u_z + \gamma u
$$

where γ is a positive constant satisfying

$$\gamma > \max\left\{\frac{c^2}{4}, -\min_{x\in\mathbb{R}^n,\, v_-(x)\le u\le v_+(x)} g_u(x,u)\right\}$$

and a sequence $\{u_n(x)\}_{n=0,1,2,\cdots}$ is given by

$$\begin{aligned}\mathfrak{L}[u_n] &= g(x,u_{n-1}) + \gamma u_{n-1}\\ u_0 &= \underline{U}.\end{aligned}$$

This sequence has the following property:

Lemma 4.3. *The sequence $\{u_n\}$ satisfies*

$$u_0 < u_1 < \cdots < u_n < u_{n+1} < \cdots < \overline{U},$$

$$\frac{\partial u_n}{\partial z} < 0 \quad (n = 0, 1, 2, \cdots).$$

Proof. We prove the former part of lemma first.

$$\begin{aligned}\mathfrak{L}[u_1 - u_0] &= \mathfrak{L}[u_1] - \mathfrak{L}[u_0]\\ &= g(x,u_0) + \gamma u_0 - \{-\triangle u_0 - u_{0zz} - cu_{0z} + \gamma u_0\}\\ &= \triangle\underline{U} + \underline{U}_{zz} + c\underline{U}_z + g(x,\underline{U}) \ge 0,\\ \mathfrak{L}[u_{n+1} - u_n] &= \mathfrak{L}[u_{n+1}] - \mathfrak{L}[u_n]\\ &= \{g(x,u_n) + \gamma u_n\} - \{g(x,u_{n-1}) + \gamma u_{n-1}\}\\ &= \{g_u(x, \theta u_n + (1-\theta)u_{n-1}) + \gamma\}(u_{n+1} - u_n)\end{aligned}$$

for some $\theta \in (0,1)$. Apply the strong maximum principle, we obtain the former part.

To prove the latter part, since

$$\begin{aligned}(u_0)_z &= (v_+(x) + \phi_+(x) W(z))_z \\ &= \phi_+(x) W'(z) < 0 \\ \mathfrak{L}[(u_{n+1})_z] &= \mathfrak{L}[u_{n+1}]_z \\ &= (g_u(x, u_n) + \gamma)(u_n)_z,\end{aligned}$$

we can use the inductive argument to the above equation with the inequality as $n = 0$. □

With (4.11) and Lemma 4.3, we can deduce that u_n converges as $n \to \infty$. We denote that

$$U(x, z) := \lim_{n\to\infty} u_n(x, z)$$

which satifying

$$u_n(x, z) \leq U(x, z) \leq \overline{U}(x, z).$$

By the Schauder estimate we can deduce that $U(x, z)$ is smooth and satisfies

$$\triangle U + U_{zz} + cU_z + g(x, U) = 0. \tag{4.12}$$

Hence $U(x, z)$ is a solution and the monotonicity of $U(x, z)$ in z follows from Lemma 4.3.

Finally, we want to check the boundary condition of (4.7).

For any $\delta > 0$, we take σ small enough that satisfies $0 < \sigma \leq \min\{\delta/(\kappa_g \mu_+), \varepsilon/\mu_+\}$, such that

$$\underline{U}(x, z) \leq U(x, z) \leq \overline{U}(x, z).$$

Then we can obtain that

$$\phi_+(x) W(z) \leq U(x,z) - v_+(x) \leq \tilde{v}_+^{\delta}(x) + Q(z) - v_+(x)$$

for any z. Hence

$$0 \leq \limsup_{z\to\infty} \{U(x,z) - v_+(x)\} = O(\delta).$$

Since U is independent of δ, we can take δ arbitrarily small. Hence

$$\limsup_{z\to\infty,\, x\in\mathbb{R}^n} \{U(x,z) - v_+(x)\} = 0,$$

i.e.

$$\lim_{z\to+\infty} U(x,z) = v_+(x).$$

Next we prove the behavior of U as $z \to -\infty$. Since U has the monotonicity in z, there exists a function $\varphi(x)$ such that

$$\varphi(x) := \lim_{z\to-\infty} U(x,z).$$

Because $v_+(x) < \underline{U}(x,z) \leq U(x,z) \leq \overline{U}(x,z) \leq v_-(z)$ and $U_z \leq 0$, we have

$$\lim_{z\to-\infty} U_z(x,z) = 0.$$

Note that U is bounded in $C^3(\mathbb{R}^{n+1})$. It follows from the above facts and the Schauder estimate that

$$\lim_{z\to-\infty} U_{zz}(x,z) = 0.$$

Hence if we take $z \to -\infty$ in (4.12),we have

$$\triangle\varphi + g(x,\varphi) = 0.$$

Observe that

$$v_+(x) < U(x,0) \le \varphi(x) \le v_-(x).$$

Thus φ must be $v_-(x)$, otherwise φ will be a stationary solution sandwiched by $v_\pm(x)$ which contradicts the assumption of Theorem 4.2. Hence, we obtain

$$\lim_{z\to-\infty} U(x,z) = v_-(x).$$

4.1.3 Proof of Theorem 4.2. (ii)

Define

$$\overline{V}(x,z) := v_+(x) - \phi_+(x) W(z).$$

And then we compute

$$\begin{aligned}
\mathfrak{F}\left[\overline{V}\right] &= -\triangle(v_+ - \phi_+ W) - (v_+ - \phi_+ W)_{zz} - c(v_+ - \phi_+ W)_z - g(x, v_+ - \phi_+ W) \\
&= -\triangle v_+ + \triangle\phi_+ W - +\phi W_{zz} + c\phi_+ W_z - g(x, v_+ - \phi_+ W) \\
&= g(x, v_+) + (\mu_+\phi_+ - g_u(x, v_+)\phi_+) W + \phi_+ W_{zz} + c\phi_+ W_z - g(x, v_+ - \phi_+ W) \\
&= \phi_+ (W_{zz} + cW_z + \mu_+ W) + g(x, v_+) - g(x, v_+ - \phi_+ W) - g_u(x, v_+)\phi_+ W \\
&\ge \phi_+ (W_{zz} + cW_z + \mu_+ W) - N_g \phi_+^2 W^2 \\
&\ge \phi_+ \sigma \left\{ w_{zz} + cw_z + \mu_+ w - \sigma N_g w^2 \right\} \\
&= \phi_+ \sigma w^2 \{1 - \sigma N_g\}
\end{aligned}$$

where

$$N_g := \min_{x\in\mathbb{R}^n,\, v_-(x)\le u\le v_+(x)} g_{uu}(x,u).$$

Thus , if we let $\sigma \le 1/N_g$, then $\overline{V}(x,z) := v_+(x) - \phi_+(x) W(z)$ is a supersolution. Define

$$\hat{g}^\delta(x,u) := g(x,u) - \delta$$

for a small positive constant δ and consider

$$\triangle v + \hat{g}^{\delta}(x, v) = 0.$$

We know there exists a solution $v = \hat{v}_{+}^{\delta}(x)$ and

$$\lim_{\delta \to 0} \hat{v}_{+}^{\delta}(x) = v_{+}(x).$$

Note that

$$\begin{aligned}
&- \triangle \left(v_{+} - \frac{\delta}{\kappa_g} \right) - \hat{g}^{\delta} \left(v_{+} - \frac{\delta}{\kappa_g} \right) \\
=& g(x, v_{+}) - g\left(v_{+} - \frac{\delta}{\kappa_g} \right) + \delta \\
=& \left(g_u \left(x, v_{+} - \vartheta \frac{\delta}{\kappa_g} \right) + \kappa_g \right) \frac{\delta}{\kappa_g} \geq 0
\end{aligned}$$

Hence, we can obtain that

$$\hat{v}_{+}^{\delta}(x) < v_{+}(x) - \frac{\delta}{\kappa_g} \quad x \in \mathbb{R}^n.$$

Define

$$V^{+}(x, z) := \hat{v}_{+}^{\delta}(x) - Q(z).$$

Then we can compute

$$\begin{aligned}
\mathfrak{F}\left[V^{+}\right] &= -\triangle \left(\hat{v}_{+}^{\delta} - Q \right) - \left(\hat{v}_{+}^{\delta} - Q \right)_{zz} - c \left(\hat{v}_{+}^{\delta} - Q \right)_{z} - g\left(x, \hat{v}_{+}^{\delta} - Q \right) \\
&= -\triangle \hat{v}_{+}^{\delta} + Q_{zz} + cQ_{z} - g\left(x, \hat{v}_{+}^{\delta} - Q \right) \\
&= \hat{g}^{\delta}\left(x, \hat{v}_{+}^{\delta} \right) - \kappa_g Q - g\left(x, \hat{v}_{+}^{\delta} - Q \right) \\
&= -\kappa_g Q + g\left(x, \hat{v}_{+}^{\delta} \right) - g\left(x, \hat{v}_{+}^{\delta} - Q \right) - \delta \\
&= -\left[\kappa_g - g_u\left(x, \hat{v}_{+}^{\delta} - \theta Q \right) \right] Q - \delta \leq -\delta < 0.
\end{aligned}$$

Thus we define the subsolution

$$\underline{V}(x,z) := \max\left\{\hat{v}_+^{\delta}(x) - Q(z), v_-(x)\right\}.$$

Let $\{v_n(x)\}_{n=0,1,2,\cdots}$ be a sequence given by

$$\begin{aligned}\mathfrak{L}[v_n] &= g(x, v_{n-1}) + \gamma v_{n-1} \\ v_0 &= \overline{U}\end{aligned}$$

and let

$$V(x,z) := \lim_{n\to\infty} v_n(x,z).$$

Then $V(x,z)$ is the solution of (4.7). The proof of boundary conditions is similar as before, thus we omit it.

4.2 Proof of Theorem 4.1

Recall that there exists a radially symmetric standing wave solution $v(x) = \psi(|x|)$ and it is unique up to translation. Consider the linearized eigenvalue problem of (4.2). Since ψ is a smooth function, thus

$$\left(\frac{\partial\psi}{\partial x_j}\right)_{rr} + \frac{r-1}{n}\left(\frac{\partial\psi}{\partial x_j}\right)_r - f'(\psi)\frac{\partial\psi}{\partial x_j} = \frac{\partial\left(\psi_{rr} + \frac{r-1}{n}\psi_r - f(\psi)\right)}{\partial x_j} = 0.$$

It implies (4.2) has zero eigenvalue and corresponding eigenvectors $\partial\psi/\partial x_j$ $(j = 1, \cdots, n)$. On the other hand, we know that (4.2) has a positive eigenvalue μ and a corresponding eigenfunction $\phi(x) = \tilde{\phi}(|x|)$.

Furthermore, since $v(x) \to 0$ as $|x| \to \infty$,

$$\psi_{rr} + \frac{n-1}{r}\psi_r - f'(0)\,\psi = 0$$
$$\tilde{\phi}_{rr} + \frac{n-1}{r}\tilde{\phi}_r - (f'(0) + \mu)\,\tilde{\phi} = 0$$

as $r \to \infty$, we notice that

$$\lim_{r\to\infty} \frac{\tilde{\phi}(r)}{\psi(r)} = 0.$$

Thus, $\tilde{\phi}$ is identical to $v(|x|)$ by translation.

Finally, we use Theorem 4.2 to complete the proof.

For (4.3), let

$$\mu_- = 0, \quad v_-(x) = 1,$$
$$\mu_+ = \mu, \quad v_+(x) = v(x).$$

Then it conform to the case 1.

And for (4.4), let

$$\mu_- = \mu, \quad v_-(x) = v(x),$$
$$\mu_+ = 0, \quad v_+(x) = 0.$$

Then it conform to the case 3. If we take $g(x,u)$ in Theorem 4.2 as $-f(x)$ in Theorem 4.1, then all the condition in Theorem 4.1 will be met in Theorem 4.2. Hence, we accomplish the proof.

Future work

First, we don't know the uniqueness of the traveling wave of (1.3) with condition (3.1) and (3.2) in Theorem 3.1. and (4.3), (4.4) for each $c \geq 2\sqrt{\kappa}$ in Theorem 4.1, which would be a future work. Second, the condition $c \geq 2\sqrt{\kappa}$ in Theorem 4.1 is a technical condition in our argument. It is an interesting problem to determine the minimum speed of the traveling waves.

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